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SOLUTION OF THE BOHR HAMLITONIAN FOR THE MORSE POTENTIAL IN THE γ~UNSTABLE REGION

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May 2023

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ABSTRACT

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Saeed Ayyed Jasim JASIM

Master of Science in Physics

Advisor: Prof. Prof. Dr. İlyas İNCİ

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Obtaining the properties of nuclei exhibiting structures around the critical point theoretically is one of the subjects that nuclear structure physicists have focused on in recent years. In this thesis, the excitation energy spectra of nuclei, which are located between the critical points U(5) and O(6), and exhibit structures around the critical point where the second-order shape phase transition takes place, denoted by E(5), were investigated. Since these nuclei deform their structures, the Collective Bohr Hamiltonian written depending on the deformation parameters was used. The Schrödinger equation, which is obtained by assuming that the potential energy of the nucleus is in the form of eksponential Morse potential, is solved analytically using the Nikiforov-Uvarov method. It is experimentally known that nuclei with mass numbers A =100−150 exhibit γ-unstable structure. Therefore, Xe isotopes with mass numbers between 118 and 134 were chosen to test the suitability of the Morse potential. Using the Mathematica program, potential free parameters were determined for each isotope that would give the best experimental data. By using these parameters, the entire excitation energy spectrum of the isotopes was created and compared with the experimental data. However, the existence of some situations that cannot be measured experimentally has been shown theoretically and a data set has been created for future experimental studies.

2023, 33 pages

Keywords: Morse potential, γ~unstable structure, Collective model, Nikiforov-Uvarov model

ÖZET

γ~KARARSIZ BÖLGEDE MORSE POTANSİYELİ İÇİN BOHR HAMİLTONİYENİNİN ÇÖZÜMÜ

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Kritik nokta civarında yapı sergileyen çekirdeklerin özelliklerinin teorik olarak elde edilmesi, son yıllarda nükleer yapı fizikçilerinin yoğunlaştığı konulardan biridir. Bu tezde, U(5) ile O(6) kritik noktaları arasında yer alan ve E(5) ile gösterilen ikinci-mertebe şekil faz geçişinin gerçekleştiği kritik nokta civarında yapı sergileyen çekirdeklerin uyarılma enerji spektrumları incelenmiştir. Bu çekirdeklerin yapılarını deforme olması nedeniyle deformasyon parametrelerine bağlı olarak yazılan Kollektif Bohr Hamiltoniyeni kullanılmıştır. Çekirdeğin potansiyel enerjisi eksponansiyel Morse potansiyeli formunda olduğu kabul edilerek elde edilen Schrödinger denklemi, Nikiforov-Uvarov medodu kullanılarak analitik olarak çözülmüştür. Kütle numaraları A =100−150 olan çekirdeklerin γ-kararsız yapı sergiledikleri deneysel olarak bilinmektedir. Bu nedenle Morse potansiyelinin uygunluğunu test etmek için, kütle numaraları 118 ile 134 arasında değişen Xe izotopları seçilmiştir. Mathematica programı kullanılarak, her bir izotop için deneysel verileri en iyi verecek potansiyel serbest parametreleri belirlenmiştir. Bu parametreler kullanılarak izotoplara ait tüm uyarılma enerji spektrumu oluşturulmuş ve deneysel verilerle karşılaştırılmıştır. Bununla birlikte deneysel olarak ölçülemeyen bazı durumların teorik olarak varlığı gösterilmiş ve ileride yapılacak deneysel çalışmalar için veri seti oluşturulmuştur.

2023, 33 sayfa

Anahtar Kelimeler: Morse potansiyeli, γ~kararsız yapı, Kollektif model, Nikiforov-Uvarov modeli

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Saeed Ayyed Jasim JASIM
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CONTENTS

LIST OF SYMBOLS

L	Angular momentum
$\langle\beta 2\rangle$	Average of $\beta 2$ over (β)
$J\nu$	Bessel function
β	Collective coordinate
γ	Collective coordinate
Rs	Energy rate value of Rs
B(E2)	Gamma transition ratio
B	Mass parameter
μ	Particle mass
h	Plank constant
$T^{(E2)}$	Quadrupole transition operator
ε	Reduced energy
xs, L	S of the Bessel function $J\nu(z)$.
t	Scale factor
C	Shielding constant
β_0	The point where the potential is minimum
θ_i	Three Euler angles = 1,2,3
Ψ	Wave function
$\mathcal{D}$	Wigner function of Euler angles (i = 1,2,3)
$u(\gamma)$	γ potential due to the variable
$u(\beta)$	β Potential due to the variable

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1. INTRODUCTION

The quantum phase transitions between different shapes of ground state depending on the change of nucleon number in atomic nuclei constitute an important branch within studies on the nuclear structure analysis. The quantum phase transitions occur as the result of deviation the number of nucleons from closed-shell configuration in which nucleus has equilibrium shape. While atomic nuclei around the closed-shell have almost a spherical shape, in the regions away from the nucleon numbers of the closed-shell, they undergo deformation on their shape (Bohr 1952).

In the deformations, the spectral structure and electric quadruple transitions (B (E2) between energy levels are important arguments for examining the Collective Model developed by Bohr and Mottelson in 1950's. In the beginning of 2000's the critical point symmetries developed by Iachello (Iachello 2000) triggered a lot of studies on the topic in the following years. The critical point symmetries labelled as E(5) and X(5) describe the shape phase transitions between vibrational and γ-unstable/prolate deformed rotational nuclei, respectively. In the both two cases, to reach analytical special solutions, the Bohr Hamiltonian is used as a kind of Schrödinger Equation explaining the collective behavior of the nucleus with collective variables γ and β, as well as the three Euler angles, in five-dimension. Using an infinite well potential depending on γ– variable for both cases, the separation of variables is achieved in the E(5) case by assuming, that the potential is independent of γ-variable, while the related X (5) case involves an harmonic oscillator like potential for γ-variable (Iachello 1987).

The underlying fact of using infinite well potential for both two cases is that the potential is expected to be flat around the point where shape phase transitions occur. In addition, to explain the spectral structure of medium mass and heavy even-even nuclei, many physicists have used different potentials, especially in the β-part of the Bohr Hamiltonian (Wilets and Jean 1956) in the critical point symmetries.

Investigating the corresponding potentials performed so far, one can see that these potentials have common mathematical properties, some of which are to have flat minimum and sloped walls.

2. LITERATURE REVIEW

In 2011, the eigenfunctions of the collective Bohr Hamiltonian with the Morse potential are obtained by using Asymptotic Iteration Method (AIM) for γ-unstable structures and rotated structures. B (E2) must calculate the conversion rate and compare it with the experimental data. Overall, internal transitions have achieved good coherence across countries, while some interband transitions appear to be systematically underestimated in γ-unstable nuclei and overestimated in spin nuclei (Inci *et al.* 2011).

In 2008, analytic solutions for the Bohr Hamiltonian can be obtained in the gamma-instability case as well as in the exact case The separable rotation case with $\gamma \approx 0$ is called the exact separable Morse (ES-M) solution closed expression. The energy eigenvalues are obtained by the asymptotic iterative method (AIM), and its effectiveness Proved by solving the related Bohr equations for the Davidson and Kratzer potentials. everything medium A two-parameter γ-instability solution using transition nuclei and a three-parameter ES-M fit for rotating nuclei with known β1 and γ-led masses and heavy nuclei. It shows that the band head and energy gap within the band are well reproduced for more than 50 nuclei each (Boztosun *et al.* 2008).

In 2007, bstract Exact solutions of the Bohr Hamiltonian with a five-dimensional square well potential, in isolation or coupled to a fermion by the five-dimensional spin–orbit interaction, are considered as examples of a new class of dynamical symmetry or Bose–Fermi dynamical symmetry. The solutions provide baselines for experimental studies of even–even [E (5)] and odd-mass [E (5|4)] nuclei near the critical point of the spherical to deformed γ -unstable phase transition (Caprio and Iachello 2007).

In 2019, an odd-mass nucleus has been thought as coupling of a single nucleon in the j = 3/2 single-particle orbit with the γ-unstable even-core. To get the properties of such a system, the Collective model Hamiltonian with the Morse potential has been used. Two cases, in which the interaction strength between the nucleon and the core are fixed or deformation dependent, are examined. The excitation energy spectrum and the electric

quadrupole transition ratios have been obtained. Then the results have been used to predict the experimental data of the some selected odd-mass nuclei (Inci 2019).

In 2020, the collective model with the Kratzer potential has been used to study the properties of the γ- unstable odd nuclei. An uncoupled nucleon in the odd nucleus has been assumed to move freely in the j = 1/2, 3/2, 5/2 orbits. The energy eigenvalues and eigenfunctions have been obtained in closed form from the analytical solution of the Hamiltonian for the odd-system. Excitation energy spectra of some selected W, Os, Pt and Hf isotopes have been predicted within this model. Due to lack of experimental B (E2) data, some intra-band and inter-band electric quadrupole transition rates were presented only for ^{179}Os isotope. Acceptable results have been obtained for these isotopes (Inci 2020).

In 2003, we present a new algebraic method for treating molecular vibrations in the Morse potential perturbed by an external field. We introduce a discrete normalizable basis, allowing to expand the dissociated states, as well. The Morse Hamiltonian is particularly simple in this basis, and the matrixelements of a realistic molecular dipole moment operator can also be expressed analytically. As a demonstration, we apply our method to calculate the dissociation probability of the NO molecule excited by appropriate series of chirped laser pulses (Molnar *et al*. 2003).

In 2000, the variational method is applied within the context of Supersymmetric Quantum Mechanics to provide information about the energy and eigenfunction of the lowest levels of a Hamiltonian. The approach is illustrated by the case of the Morse potential applied to several diatomic molecules and the results are compared with stabilished results (Filho and Ricotta 2000).

2.1 The Interacting Boson Model

Iachello and Arima created the IBM in 1974, and it has been successfully used to study a variety of nuclear collective phenomena (Arima and Iachello 1976). Nuclear features such as spins and energy of the lowest levels, decay probabilities for the emission of gamma quantas, probabilities (spectroscopic factors) of transfer processes, multipole moments, and so on must be able to be described by an atomic nucleus model.

The intermediate and heavy atomic nuclei can be well described by the interacting boson model (IBM). It can replicate the majority of the low-lying states of such nuclei by adjusting a small number of parameters.

The IBM is based on geometric collective models of the atomic nucleus and the well-known shell model. It is also of significant theoretical significance since it displays the Lie algebra-visible dynamical symmetries of a number of nuclei.

The fundamental concept is that proton and neutron bosons with spins of 0 and 2 can adequately explain the low energy collective degrees of freedom in nuclei. These structural elements function together. Various selections of the energies and interaction strengths of the L=0 (s-boson) and L=2 (d-boson) lead to various collective spectra (Pfeifer 1998).

The protons and neutrons in these bosons are thought to be coupled pairs in the valence shell. If the shell is less than halfway full, this interpretation limits the boson number, which is calculated by counting the number of particle pairs (separately for protons and neutrons). Additionally, if the shell is more than half filled, by counting the number of hole pairs. The most basic version of the interacting boson model, known as IBM-1, would exist if the bosons of neutrons and the bosons of neutrons were thought to be identical (Iachello and Arima 1987).

2.2 Collective Model

The Collective model, first proposed by Rainwater (1950), was developed by Bohr and Motelson (1969). The collective model was developed as a result of combining the liquid drop model and the shell model, which gives successful results in the explanation of the nuclear structure by examining the collective properties of the nucleus (Bohr and Mottelson 1969).

At the same time, this model is also successful in describing the properties of heavily deformed and even-even-nucleon nuclei and gives successful results especially about quadrupole moments of these nuclei (Bohr and Mottelson 1975).

In the collective model, it is accepted that the properties of the nucleus such as the excited state, magnetic and quadrupole moments are determined not only by the unpaired nucleons outside the closed shell, but also by the core and surrounding nucleons.

In short, in this model, the collective motions of all particles in the nucleus are taken into account, and the deformations of the nuclei that occur as a result of these collective motions are examined. As in the shell model, in this model, each nucleon in the nucleus moves freely within a potential created by nucleons other than itself. However, unlike the shell model potential, the potential in the collective model is not spherically symmetrical. This potential is due to the formation of core and non-core nucleons (Bohr and Mottelson 1975).

It is a potential that causes deformation as a result of movement (Arya 1966). In other words, it causes the core to lose its spherical symmetry and elongate (take an ellipsoidal shape) in the direction of orbital nucleons. The shape of the nucleus is determined by this potential (Arya 1966).

According to IBM, where microscopic interactions are taken into account in the collective model and which provides a completely algebraic U(6) (Arima and Iachello 1975) group structure, the harmonic oscillator structure U(5) (Arima and Iachello 1976) corresponds to the dynamic symmetry limit, while SU(3) and O(6) dynamic symmetry limits correspond to axisymmetric and asymmetric rotor structures, respectively and phase transitions take place between these limits (Arima and Iachello 1978).

2.3 Bohr Hamiltonian

Arima and Iachello (1974) proposed a nuclear model in which it was able to describe the characteristics of the energy levels in the even-even nuclei and positive parity (π^+) of medium and heavy mass numbers by the pairs of nucleons outside the closed shell that were treated as bosons (Otsuka *et al.*1978) so that it did not take into account the degrees of freedom for these bosons, they were called the first interacting boson model IBM-1. This model was developed by introducing degrees of freedom to bosons. As a result of this modification, new nuclear properties of the nuclei were revealed and it was called the second interacting boson model IBM-2 (Casten 1988).

The first interacting boson model does not distinguish between proton bosons (s_π, d_π) and neutron bosons (s_v, d_v). The total number of bosons (N) is calculated from the sum of (N_π proton bosons and N_v neutron bosons) as pairs of particles. Starting from the closest closed shell to the middle of the next shell, meaning that the number of bosons (N) is equal to the number of pairs of particles outside the closed shell, which is a fully preserved and constant quantity for each nucleus, but if it is more than half of the shell then (N_π) and (N_v) is taken as the number of Hole Pairs (Arima and Iachello 1978).

In this model, we find that the s and d bosons can interact with each other, and as a result, the general formula for the Hamiltonian effect for this system is written after the creation effects (s^+, d^+)and the annihilation effects (s, d) (Annihilation Operators) are defined (Iachello and Arima 1974).

Form the Bohr Hamiltonian, which describes the spectrum and wave function collective mode, first we describe Euler angles and collective coordinates as

$$q_1 = \theta_1; q_2 = \theta_2; q_3 = \theta_3; q_4 = \beta \tag{2.1}$$

The kinetic energy in the Bohr Hamiltonian is given by:

$$T = \frac{B}{2}\left(\frac{ds}{dt}\right)^2 \tag{2.2}$$

$$\nabla^2\Phi = \frac{1}{\sqrt{g}}\partial_i\sqrt{g}g^{ij}\,\partial_j\Phi \tag{2.3}$$

is expressed as. Square of displacement in Equation 2.3 $ds^2 = g_{ij}dq_i dq_j$ The Hamiltonian is obtained quantum mechanically by the Pauli-Podolsky rule (Podolsky 1928).

where the square of the displacement is $ds^2 = g_{ij}dq_i dq_j$. The Hamiltonian can be obtained as a quantum mechanical way using the Pauli-Podolsky formula (Podolsky 1928).

$$g_{ij} = \begin{pmatrix} g_{11} & g_{12} & g_{13} & 0 & 0 \\ g_{21} & g_{22} & 0 & 0 & 0 \\ g_{31} & 0 & g_{33} & 0 & 0 \\ 0 & 0 & 0 & g_{44} & 0 \\ 0 & 0 & 0 & 0 & g_{55} \end{pmatrix} \tag{2.4}$$

It is the determinant and inverse matrix of (Sitenko and Tartakovski 1975).

$$g = \frac{J_1 J_2 J_3}{B^3}\beta^2 \sin^2\theta_2 = 4\beta^8 \tag{2.5}$$

where the moments of inertia is

$$J_k = 4B\beta^2 \sin^2\left(\gamma - k\frac{2\pi}{3}\right) \quad (2.6)$$

Finally, from Equation (2.3), the Bohr-Hamilton equation in Equation (2.7) is obtained of any potential of the collective variables.

$$H = -\frac{h^2}{2\beta}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2}\frac{1}{\sin\ 3\gamma}\frac{\partial}{\partial\gamma}\sin\ 3\gamma\frac{\partial}{\partial\gamma^\gamma} - \frac{1}{4\beta^2}\sum_{k=1}^{3}\frac{Q_k^2}{\sin^2\left(\phi-\frac{2\pi k}{3}\right)}\right] + V(\beta,\gamma)$$

(2.7)

here , B_m is the mass parameter. Principal deformation coordinates β (deformation coordinate measuring the deviation from the spherical shape), γ (a measure of deviation from axial symmetry) and The Bohr Hamiltonian is five-dimensional, with $\hat{Q}(k = 1,2,3)$components of the angular momentum in the real structure.

$$\hat{Q}_1 = \hat{Q}_x = -i\left(-\frac{\cos\theta_3}{\sin\theta_2}\frac{\partial}{\partial\theta_1} + \sin\theta_3\frac{\partial}{\partial\theta_2} + \cot\theta_2\cos\theta_3\frac{\partial}{\partial\theta_3}\right) \quad (2.8)$$

$$\hat{Q}_2 = \hat{Q}_y = -i\left(-\frac{\sin\theta_3}{\sin\theta_2}\frac{\partial}{\partial\theta_1} + \cos\theta_3\frac{\partial}{\partial\theta_2} - \cot\theta_2\sin\theta_3\frac{\partial}{\partial\theta_3}\right) \quad (2.9)$$

$$\hat{Q}_3 = \hat{Q}_z = -i\frac{\partial}{\partial\theta_3} \quad (2.10)$$

collective wave function;

$$\Psi(\beta,\gamma,\theta_i) = f(\beta)\Phi^L_{M,K}(\gamma,\theta_i) \quad (2.11)$$

$\Phi^L_{M,K}$, to Wigner functions $D^L_{M,K}(\theta_i)$ It is defined as in .

$$\Phi^{L,\tau}_{M,K}(\gamma,\theta_i) = \sqrt{\frac{2I+1}{16\pi^2(1+\delta_{K,0})}}\mathrm{g}^{L,\tau}_K(\gamma)\left[D^L_{M,K}(\theta_i)+(-1)^L D^L_{M,-K}(\theta_i)\right] \quad (2.12)$$

Here θ_i(i = 1,2,3) Euler angle, $D^{L}_{M,K}(\theta_i)$ Wigner function of Euler angle, and Langular is the momentum quantum number. M and K are is a laboratory constant on the laboratory fixed z-axis and the body fixed z'-axis Figure 2.1.

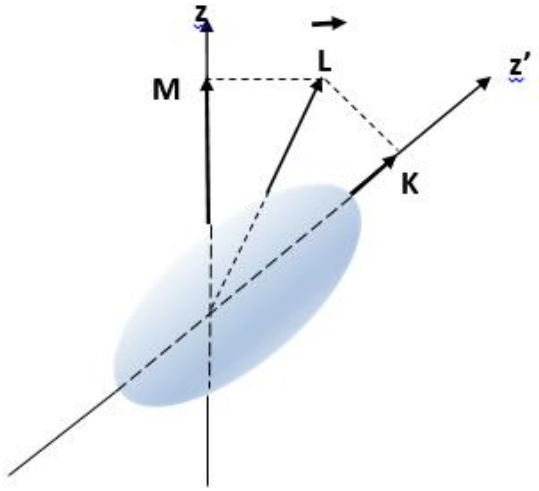

Figure 2.1 Total angular momentum vector projections on the laboratory fixed z-axis and the body fixed z'-axis (Sitenko and Tartakovski 1975).

Normalization condition and volume element;

$$1 = \int_{\beta=0}^{\infty} \int_{\gamma=0}^{\frac{\pi}{3}} \int_{\theta_1=0}^{2\pi} \int_{\theta_2=0}^{\pi} \int_{\theta_3=0}^{2\pi} \Psi^*(\beta,\gamma,\theta_1,\theta_2,\theta_3)\Psi(\beta,\gamma,\theta_1,\theta_2,\theta_3)dv$$

$$dv = \beta^4 d\beta |\sin\ 3\gamma| d\gamma d\theta_1 \sin\ \theta_2 d\theta_2 d\theta_3 \qquad (2.13)$$

2.4 The Gamma Unstable *O*(6) Symmetry

A third limiting symmetry of the IBM model occurs when bosonic interactions are governed by pairing forces (Arima and Iachello 1978). Similar to the SU(5) and SU(3) symmetries, Iachello and Arima diagonalize the IBM Hamiltonian generated by SU(6) Eq (2.61) by taking the SU(6) subgroup in which the Hamiltonian lies is constant. In this case, the subgroup is O(6), which also contains subgroups O(5) and O(3). With the

group chain SU(6) →O(6) O(5) →O(3) , the IBM Hamiltonian in the O(6) limit can be written as:

$$H = AP_6 + BC_5 + CC_3 \tag{2.14}$$

Where P_6 is a pairing operator in O(6) and C5 and C3 are Casimir operators in O(5) and O(3) respectively. A, B and C are the strengths of the different components. According to the IBM Hamiltonian of Eq (2.14), corresponding entry:

$$v_0 \left[(d^+d)^{(0)}(ss)^{(0)} + (s^+s^+)^{(0)}(dd)^{(0)}\right]^{(0)} \tag{2.15}$$

while C_5 and C_3 and correspond to the terms:

$$\varepsilon \sum_m d_m^+ d_m + \sum_{J=0,2,4} \frac{1}{2}(2J+1)^{1/2} C_J \left[(d^+d^+)^{(J)}(dd)^{(J)}\right]^{0)}. \tag{2.16}$$

A symmetric irreducible representation in O(6) is characterized by a quantum number σ (Arima and Iachello 1978):

$$\sigma = N, N-2, N-4, \ldots, 0 \text{ or } 1 \text{ for N} = \text{even or odd} \tag{2.16}$$

expectation value of the $Q(6)$ pairing operator, P_6, can be written in terms of σ as (Arima and Iachello 1978):

$$\langle P_6 \rangle = \frac{1}{4}(N-\sigma)(N+\sigma+4) \tag{2.17}$$

The quantum number τ is chosen to characterize the O(5) representation.

$$\tau = \sigma, \sigma-1, \ldots, 0 \tag{2.18}$$

The expected value of C_5 in the τ-representation of O(5) is given by (Arima and Iachello 1978):

$$\langle C_5 \rangle = \frac{1}{6}\tau(\tau+3) \tag{2.19}$$

Therefore, the eigenvalues of states subject to the Hamiltonian in Eq (2.20) are:

$$E([N]\sigma v_\Delta JM) = \frac{A}{4}(N-\sigma)(N+\sigma+4) + B\tau(\tau+3) + CJ(J+1). \tag{2.20}$$

The quantum number v_Δ is useful in labeling states: it is related to v_Δ, which counts the number of bosonic triplets coupled to zero angular momentum. The relationship between the quantum number τ and $v_\Delta$is $\tau = 3v_\Delta + \lambda$, the value of λ determines the angular momentum of the above state.

$$J = 2\lambda, 2\lambda-2, 2\lambda-3, \dots, \lambda+1, \lambda \tag{2.21}$$

Arima and Iachello also succeeded in obtaining analytical expressions for transition probabilities (Arima and Iachello 1978). Like the SU(5) and SU(3) symmetries, they require the E2 transformation operator T(E2) to be a generator of the underlying group structure, in this case O(6). T(E2) that satisfies this requirement is of the form(Arima and Iachello 1978):

$$T(E2) = \alpha(d^+ s + s^+ d)^{(2)} \tag{2.22}$$

T(E2) is an O(6) generator, so you can't connect states with different representations. Therefore, the selection rule is Δσ=0. Since O(6) contains the structure of O(5), the O(5) selection rule Δτ=±1 also applies. Note in particular that, like all IBM B(E2) values, the finite dimensionality of the system is automatically taken into account. By the form of the branching operator, the branching ratio occurring in the limit O(6) .

Within each σ group itself, the energy level separation is somewhat similar to the vibrational model described in Section A, but the energy separation is proportional to τ(τ+3), not just τ. This gives the energy ratio $\frac{E(4_1^+)}{E(2_1^+)} = 2.5$ instead of 2 as expected for the vibrational mode. In addition, branching ratios and absolute B (E2) values were significantly different from geometric regularity.

The O (6) limit (especially for large) seems to resemble most closely the γ-unstable model described by Wilets and Jean (1956).

In such a geometric description, the energy levels follow a τ(τ+3) energy dependence. Furthermore, the same levels and level distances that occur in the γ-instability n_β=0 group are repeated for the higher-level n_β≠0 group. In this sense, the role of nβ=0 is similar to that of various values.

However, in the O (6) system, the energy level degeneracy is no longer conserved, there are spin truncations and a certain number of different σ groups. By analogy with the corresponding relationship between the SU(5) oscillator and the SU(3) symmetric rotor, it is reasonable for the O (6) description to correspond to a γ-unstable geometric model. The Hamiltonian of a γ-unstable oscillator is characterized by the fact that the potential energy is independent of γ, although the γ-dependent term is included in the kinetic energy. Bohr-Mottelson image and coordinates of the IBM operator. The result proposed by Arima is that the γ-unstable potential corresponding to IBM O (6) limit will have the form:

$$V = -c\beta^2 + d\beta^4 \qquad (2.23)$$

where β is the deformation parameter and c and d are arbitrary constants. The potential of this form comes from the zero d boson and the two d boson number variables of the O (6) Hamiltonian. The γ-dependency in the potential will have the form $\beta^3 \cos 3\gamma$ which corresponds to the term changing the number of d-bosons not contained in the symmetry (Scholten *at el.* 1978). Currently trying to understand more explicitly the

analogy between O(6) symmetry and related geometric models. A convenient basis for describing wave functions of order O(6) is the vibrational limit, given by $J^{\pi}|n_d n_\beta n_\Delta\rangle$n_Δ ⟩, where n_d n_β n_Δ. In general, the number of d bosons is the number of pairs of d bosons coupled to zero angular momentum and the number of triplets of d bosons coupled to zero angular momentum.

The wave function is not pure on this basis, but can be easily described as a linear combination of ground states with different quantum numbers n_d and n_β. Using the vibrational limit example, the ground state a is pure $0^+ \mid 000 >$ state; in O(6), the ground state with $\sigma = \sigma_{\max}$ will be characterized by a 0^+ wave function. A suitable basis for describing wave functions of order O(6) is the oscillatory limit, given by $J^{\pi}|n_d n_\beta n_\Delta\rangle$, where $n_d n_\beta n_\Delta$.

In general, the number of d bosons is the number of pairs of d bosons coupled to zero angular momentum and the number of triplets of d bosons coupled to zero angular momentum. Although wave functions are not pure on this basis, they can be easily described as a linear combination of ground states with different quantum numbers n_d and n_β. For example, the ground state in the vibrational limit is pure $0^+ \mid 000 >$ state; in $O(6)$, the ground state, with $\sigma = \sigma_{\max}$ would be characterized by the 0^+ wave function $\alpha|000\rangle + \beta|210\rangle + \gamma|420\rangle + \varepsilon|NN/20\rangle$ Two types of perturbations may be added to the exact results of the $O(6)$ limit: one which does not change the forces of the symmetry, and one which introduces a force from outside the limit.

The former type can be accomplished, for example, by changing the boson energy from the value determined by B. This will alter the amplitudes of the nonzero components of all wave- functions, but will not add new components. The result will be to break the selection rule $\Delta\sigma = 0$, but to preserve the $\Delta\tau = \pm 1$ E2 Choose a rule. The second type of perturbation can be achieved, for example, by introducing quadrupole - quadrupole interaction forces. Since this interaction contains a d boson change term, all wavefunction components will be nonzero, albeit possibly small, and the effect will break the O(6) E2 selection rule and change all E2 branching ratios.

2.5 Standard Nikiforov-Uvarov Method

The Nikiforov-Uvarov (NU) method is based on the solutions of hypergeometric type second order linear differential equations by means of special orthogonal functions (Szego 1939).

It provides an exact solution of the non-relativistic Schrödinger equation for certain kinds of potentials. For a given potential, a coordinate transformation conforming to the Schrödinger equation in spherical coordinates or the non-relativistic Schrödinger equation $r \rightarrow s$ together with a generalized hypergeometric equation. The main equation is written as follows (Nikiforov and Uvarov 1988).

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0 \tag{2.24}$$

2.5.1 Parametric standard Nikiforov-Uvarov method

The Nikiforov-Uvarov (NU) method is based on solving second-order hyper geometric differential equations (Szego 1939) using specific orthogonal functions. For a given potential energy, the Schrödinger or Schrödinger-like equation in spherical coordinates reduces to a generalized hyper geometric equation using an appropriate coordinate transformation. It can be solved systematically to determine a precise or specific solution. According to Nikiforov and Uvarov (1988), the main equations closely related to this method are given in the following form, which is a general form of the Schrödinger equation, which can be obtained with different potentials (Nikiforov and Uvarov 1988).

2.6 Morse potential

The Morse potential (Morse 1929), named after physicist Philip M. Morse, is one of the useful models for a diatomic molecule's potential energy. It is a better approximation for

a molecule's vibrational structure than the harmonic oscillator model because it explicitly includes bond breaking effects, such as the existence of unbound states. For a diatomic molecular system of low mass (Morse 1929).

Even for polyatomic molecules, the Morse potential now plays a dominant role in model calculations in molecular spectroscopy. However, it is also widely used in several other branches of chemical physics, such as collision theory, intramolecular and intermolecular energy transfer theory, photodissociation theory, and so on. As a result, the Morse oscillator has been the subject of numerous detailed studies. These studies have almost entirely been conducted in the position space representation, but there has recently been an increase in interest in the use of the phase space representation, particularly in relation to time dependent problems. The phase space representation employs Wigner functions rather than wave functions, but a general analysis of these functions for the Morse oscillator is still lacking (Dahl 1983).

The Morse potential as an ideal and typical harmonic potential allows accurate Mathematical processing. In particular, the Morse potential in the harmonic limit is reduced to harmonic oscillators.

So far, the Morse potential has been widely used in the fields of physics and chemistry Undoubtedly, hundreds of articles have appeared in the literature since 1929. Considering its importance in different fields of physics and chemistry. In general, there are two ways to get the Eigen functions and eigenvalues of the Morse potential (Yukiya *et al*. 2013). The first method is the analytical method. That is to say, its exact solutions can be obtained by solving the Schrodinger equation as treated by Morse (Morse 1929). Landau and Lifshitz (Landau and Lifshitz 1977) and others. Another method is the susyqm method, which is also related to the algebraic method (Han *et. al*. 2005). For example, Panigrahi and Sukhatme presented the partner potentials of the ground state and the first excited state of the Morse potential. Filho and Ricotta (2000) derived the energy spectrum of the Morse potential using the variational method and the susyqm method (Filho and Ricotta 2000). Molnar, F¨oldi, Benedict and Bartha employed the susyqm technique to treat the NO molecular vibrations in the Morse potential perturbed

by an external field and calculated the dissociation probability of this molecule excited by appropriate series of chirped laser pulses (Molnar *et al*. 2003). Han, Song and Yang obtained the eigenfunctions and eigenvalues by the SUSYQM approach and found that the Morse potential is shape invariant (Han *et al.* 2005).

The main aim is to find a more suitable diatomic potential to describe the vibrational spectrum. For example, Deng and Fan proposed an electronic energy function for diatomic molecules in 1957 expressed as (Morse 1929):

$$V(r) = D\left[1 - \frac{e^{ar_e - 1}}{e^{ar-1}}\right]^2 \tag{2.25}$$

Which implies that this potential approaches infinite as the internuclear distance *r* tends to zero and the Schrodinger equation with this potential is exactly solvable.

Choosing the separated atoms limit as the zero of energy, the Morse potential has the following form (Morse 1929).

$$V(x) = V_0\left(e^{-2\beta x} - 2e^{-\beta x}\right) \tag{2.26}$$

3. MATERIALS AND METHODS

3.1 Solutions For $\gamma \approx 0$

The Bohr Hamiltonian is given by (Bohr1952):

$$H = -\frac{h^2}{2\beta}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2}\frac{1}{\sin\ 3\gamma}\frac{\partial}{\partial\gamma}\sin\ 3\gamma\frac{\partial}{\partial\gamma^\gamma} - \frac{1}{4\beta^2}\sum_{k=1}^{3}\frac{Q_k^2}{\sin^2\left(\phi-\frac{2\pi k}{3}\right)}\right] + V(\beta,\gamma)$$

(3.1)

This Hamiltonian is defined in a ficue-dimensional space. β and γ_1 are intrinsic variables and $\theta_i(i = 1,2,3)$ three Euler argles. Ween the potential depends only on β, momely $v(\beta,\gamma) = U(\beta)$, by wining $\psi(\beta,\gamma_1,\theta_i) = f(\beta)\cdot\Phi(\gamma_1,\theta_i)$ one can separate the Hamiltorian in Equation (3.1). In the

$$\left[-\frac{1}{\sin\ 3\gamma}\frac{\partial}{\partial\gamma}\sin\ 3\gamma\frac{\partial}{\partial\gamma} + \frac{1}{4}\sum_{k=1}^{3}\frac{Q_k^2}{\sin^2\left(\gamma-\frac{2\pi k}{3}\right)}\right]\Phi(\gamma,\theta_i) = \Lambda\Phi(\gamma,\theta_i) \quad (3.2)$$

where $\Lambda = \tau(\tau+3)$ with $\tau = 0,1,2,\ldots$ and

$$\left\{-\frac{h^2}{2\beta}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} - \frac{\Lambda}{\beta^2}\right] + U(\beta)\right\}f(\beta) = Ef(\beta) \quad (3.3)$$

If we introduce reduced potential $v(\beta) = \frac{2B}{\hbar^2}U(\beta)$ and reduced energy $\varepsilon = \frac{2B}{\hbar^2}E$ we can rite Equation (3.3) to be:

$$-\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} - \frac{\Lambda}{\beta^2}\right]f(\beta) + (U(\beta)f(\beta) = \varepsilon f(\beta) \quad (3.4)$$

$$-\frac{d^2f}{d\beta^2} - \frac{4}{\beta}\frac{df}{d\beta} + \frac{\Lambda}{\beta^2}f(\beta) + v(\beta)f(\beta) = \varepsilon f(\beta) \quad (3.5)$$

$\frac{d^2f}{d\beta^2} = f''(\beta)$ and $\frac{df}{d\beta} = f'(\beta)$, namely Equation (3.5) is written as:

$$f''(\beta) + \frac{4}{\beta}f'(\beta) + \left[\varepsilon - \frac{\Lambda}{\beta^2} - v\ \beta\right)\Big] f(\beta) = 0 \tag{3.6}$$

by using $f(\beta) = \beta^{-2}X(\beta)$ transformation:

$$f'(\beta) = -2\beta^{-3}x(\beta) + \beta^{-2}X'(\beta)$$
$$f''(\beta) = 6\beta^{-4}x(\beta) - 2\beta^{-3}x'(\beta) - 2\beta^{-3}x'(\beta) + \beta^{-2}X''(\beta)$$
$$f''(\beta) = \beta^{-2}x''(\beta) - 4\beta^{-3}x'(\beta) + 6\beta^{-4}x(\beta)$$

Then Equation (3.6) turns to be:

$$\frac{X''(\beta)}{\beta^2} - \frac{4}{\beta^3}X'(\beta) + \frac{6}{\beta^4}x(\beta) + \frac{4}{\beta}\left(\frac{x'}{\beta^2} - \frac{2}{\beta^3}x\right) + \left[\varepsilon - \frac{\Lambda}{\beta^2} - v(\beta)\right]\frac{x}{\beta^2} = 0$$
$$\frac{x''(\beta)}{\beta^2} + \frac{6}{\beta^4}x - \frac{8}{\beta^4}x + \left[\varepsilon - \frac{\Lambda}{\beta^2} - v(\beta)\right]\frac{x}{\beta^2} = 0$$
$$\frac{x''(\beta)}{\beta^2} - \frac{2}{\beta^4}x(\beta) + \left[\varepsilon - \frac{\Lambda}{\beta^2} - v(\beta)\right]\frac{x}{\beta^2} = 0$$

$$x''(\beta) + \left[\varepsilon - \frac{(\Lambda+2)}{\beta^2} - v(\beta)\right]x(\beta) = 0 \tag{3.7}$$

For the reduced potatial $Q(\beta)$. Let's use the Morse potential:

$$v(\beta) = e^{-2a(\beta-\beta e)} - 2e^{-a(\beta-\beta e)} \tag{3.8}$$

Then Equation (3.7) turns to be:

$$x''(\beta) + \left[\varepsilon - \frac{\Lambda+2}{\beta^2} - e^{-2a(\beta-\beta e)} + 2e^{-a(\beta-\beta e)}\right]x(\beta) = 0 \tag{3.9}$$

If we define new coriables:

$$\beta = \beta_e(1+x), \alpha = a\beta_e, \in = \beta_e^2\varepsilon \tag{3.10}$$

$\frac{\partial}{\partial\beta} = \frac{1}{\beta_E}\frac{\partial}{\partial x}$ and $\frac{\partial^2}{\partial\beta^2} = \frac{1}{\beta_e^2}\frac{\partial^2}{\partial x^2}$ (3.11)

By using the about variables:

$$\frac{1}{\beta e^2}x''(x) + \left[\varepsilon - \frac{\Lambda+2}{\beta_e^2(1+x)^2} - e^{-2\alpha x} + 2e^{-\alpha x}\right]x(x) = 0 \quad (3.12)$$

$$x''(x) + \left[\beta_e^2\varepsilon - \frac{\Lambda+2}{(1+x)^2} - \beta_e^2 e^{-2xx} + 2\beta_e^2 e^{-xx}\right]x(x) = 0$$

$$\Lambda + 2 = \mu$$

$$x''(x) + \left[\epsilon - \frac{\mu}{(1+x)^2} - \beta_e^2 e^{-2\alpha x} + 2\beta_e^2 e^{-\alpha x}\right]x(x) = 0 \quad (3.13)$$

If we renarame $\frac{\mu}{(1+x)^2} = U_L(x)$ and expanding this $U_L(x)$ x around egress gives.

$$U_L(x) = \mu(1 - 2x + 3x^2 - 4x^3 + \cdots) \quad (3.14)$$

In the exponential form, $U_L(x)$ con be rewritten as $\tilde{U}_L(x)$:

$$\tilde{U}_L(x) = \mu(c_0 + c_1 e^{-\alpha x} + c_2 e^{-2\alpha x} + \cdots) \quad (3.15)$$

and again expanding $x = 0$ gives.

$$\tilde{U}_L(x) = \mu\left(c_0 + c_1 + c_2 - [c_1 + 2C_2]\alpha x + \left[\frac{c_1}{2} + 2c_2\right]\alpha^2 x^2 + \cdots\right) \quad (3.16)$$

by compering Equations (3.16) and (3.14) we can define C_0, C_1, C_2 as follows:

$$C_0 = 1 - \frac{3}{\alpha} + \frac{3}{\alpha^2},\ C_1 = \frac{4}{\alpha} - \frac{6}{\alpha^2},\ C_2 = \frac{-1}{x} + \frac{3}{\alpha^2} \quad (3.17)$$

Now, turn back to Equation (3.13) and we con rewrite it by using Equation (3.15):

$$x''(x) + [e - \mu(c_0 + c_1 e^{-\alpha x} + c_2 e^{-2\alpha x}) - \beta e^2 e^{-2\alpha x} + 2\varepsilon_e e^{-\alpha x}]x(x) = 0 \qquad (3.18)$$

We can simplify Equation (3.18) by using new constants:

$$\begin{aligned} -g^2 &= \epsilon - \mu_0 c_0 \\ \gamma_1^2 &= 2\beta_e^2 - \mu c_1 \\ \gamma_2^2 &= \beta_c^2 + \mu c_2 \end{aligned} \qquad (3.19)$$

$$x''(x) + [-\rho^2 + \gamma_1^2 e^{-\alpha x} - \gamma_2^2 e^{-2\alpha x}]x(x) = 0 \qquad (3.20)$$

Now define a new variable $e^{-\alpha x} = y,\ e^{-2xx} = y^2$

$$-\alpha e^{-\alpha x} dx = dy \Rightarrow -\alpha dx = \frac{1}{y} dy$$
$$\frac{d}{dx} = \frac{d}{dy}\frac{dy}{dx} = (-\alpha y)\frac{d}{dy}$$

$$\begin{aligned} \frac{d^2}{dx^2} &= \left(\frac{dy}{dx}\frac{d}{dy}\right)\left(\frac{dy}{dx}\frac{d}{dy}\right) = \left(-\alpha y \frac{d}{dy}\right)\left(-\alpha y \frac{d}{dy}\right) \\ &= \alpha^2 \left(y \frac{d}{dy}\right)\left(y \frac{d}{dy}\right) = \alpha^2 \left[y \frac{d}{dy} + y^2 \frac{d^2}{dy^2}\right] \end{aligned} \qquad (2.21)$$

By using a new variable Equation (3.20) con be i rewrittenos:

$$\alpha^2 y^2 \frac{d^2x}{dy^2} + \alpha^2 y \frac{dx}{dy} + [-s^2 + x_1^2 y - \gamma_2'^2 y^2]x(y) = 0 \qquad (2.22)$$

$$x''(y) + \frac{1}{y}x'(y) + \left[\frac{-s^2}{\alpha^2 y^2} + \frac{x^2}{\alpha^2 y} - \frac{\gamma_2^2}{\alpha^2}\right]x(y) = 0 \qquad (2.23)$$

or more simply:

$$x''(y)+\frac{1}{y}x'(y)+\left[\frac{-3^2/x^2+x^2/x^2-y-\frac{x^2}{x^2}y^2}{y^2}\right]x(y)=0 \tag{2.24}$$

This for can be solved easily with Nikiforov-Uvarov Method:

According to this method;

$$\psi''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s)=0 \tag{3.25}$$

$\sigma(s)$ and $\sigma(s)$ are polynomials at most second-dgree and $\tilde{\tau}(S)$ is a first-degree polynomial.

Comparing equation (3.24) with (3.25) we ca verite that

$$\tilde{\tau}(y)=1$$
$$\sigma(y)=y$$

$$\tilde{\sigma}(y)=-\frac{\gamma_2^2}{\alpha_0^2}y^2+\frac{\gamma_1^2}{\alpha^2}y-\frac{\rho^2}{\alpha^2} \tag{3.26}$$

According to N.U. method :

$$\pi(y)=\frac{\sigma'(y)-\tilde{\tau}(y)}{2}\pm\sqrt{\left(\frac{(\sigma'(y)-\tilde{\tau}(y)}{2}\right)^2-\tilde{\sigma}(s)+k\sigma(s)} \tag{3.27}$$

for our case :

$$\pi(y)=\frac{\sigma'(y)-\tilde{\tau}(y)}{2}\pm\sqrt{\left(\frac{(\sigma'(y)-\tilde{\tau}(y)}{2}\right)^2-\tilde{\sigma}(y)+k\sigma(y)} \tag{3.28}$$

By using equation (3.26) in (3.28) we get.

$$\sigma(y) = 1$$
$$\tilde{\tau}(y) = 1 \qquad \pi(y) = \pm\sqrt{\frac{\gamma_2^2}{\alpha^2}y^2 - \frac{\gamma_1^2}{\alpha^2}y + \frac{\rho^2}{\alpha^2} + ky} \tag{3.29}$$

We con simplify $\pi(y)$ as follows:

$$\pi(y) = \pm\sqrt{\frac{\gamma_2^2}{\alpha^2}y^2 + \left(k - \frac{\gamma_1^2}{\alpha^2}\right)y + \frac{\rho^2}{\alpha^2}} \tag{3.30}$$

$$\pi(y) = \pm\frac{\gamma_2}{\alpha_2}\sqrt{y^2 + \left(\frac{k\alpha^2 - \gamma_1^2}{\gamma_2^2}\right)y + \frac{\rho^2}{\gamma_2^2}} \tag{3.31}$$

Setting the dacerimat of the quadratic equation equal to zero.

$$\Delta = \left(\frac{k\alpha^2 - y^2}{\gamma_2^2}\right)^2 - 4\frac{\rho^2}{\gamma_2^2} = 0$$

$$\left(\frac{k\alpha^2 - \gamma_1}{\phi_2^2}\right)^2 = \frac{4\rho^2}{\phi_2^2} \Rightarrow \frac{k\alpha^2 - \phi_1^2}{\gamma_2^2} = \pm\frac{2\rho}{\gamma_2} \tag{3.32}$$

$$k\alpha^2 - \gamma_1^2 = \pm 2\rho\gamma_2$$
$$k\alpha^2 = \gamma_1^2 \pm 2\rho\gamma_2$$

$$k_{\pm} = \frac{\gamma 1^2}{\alpha^2} \pm \frac{2\rho\gamma_2}{\alpha^2} \tag{3.33}$$

Then Equmatan $\pi(y)$ in Equation (3.31) is:

$$\pi(y) = \pm\frac{\gamma_2}{\alpha_2}\left(y + \frac{k\alpha^2 - \gamma_1^2}{\gamma_2^2}\right) \tag{3.34}$$

for $k+\Rightarrow \pi(y) = \pm\frac{\gamma_2}{\alpha_2}\left(y + \frac{k+\alpha^2 - \gamma_1^2}{\gamma_2^2}\right)$ (3.35)

And

for $k-\Rightarrow \pi(y) = \pm\frac{\gamma_2}{\alpha_2}\left(y + \frac{k-\alpha^2-\gamma_1^2}{\gamma_2^2}\right)$ (3.36)

From equation (3.33) $k+= \frac{\gamma_1^2}{\alpha^2} + \frac{2\rho\gamma_2}{\alpha^2}$ and $k-= \frac{\gamma_1^2}{\alpha^2} - \frac{2\rho\gamma_2}{\alpha^2}$

With these definitions equations (3.35) and (3.36) con be simplified as:

$$\pi(y) = \pm\frac{\gamma_2}{\alpha}\left(y + \frac{2\rho}{\gamma_2}\right) \quad (3.37)$$

$$\pi(y) = \pm\frac{\gamma_2}{\alpha}\left(y - \frac{2\rho}{\gamma_2}\right) \quad (3.38)$$

Accosting to N.U. meth ed:

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s) \quad (3.39)$$

From equation (3.39) re get our solution:

$\tau(y) = \tau(y) + 2\pi(y)$ and

$$\tau(y) = 1 \pm \frac{2\gamma}{\alpha}\left(y \pm \frac{2\rho}{\gamma_2}\right) \quad (3.40)$$

Here four passible $\tau(y)$ is present but we should select for $k\_$ and

$$\tau(y) = 1 - \frac{2\gamma_2}{\alpha^2}y + 4\frac{\rho}{\alpha^2} \quad (3.41)$$

And according to NU. method:

$\lambda = k + \pi'(s)$ and

$$\lambda = k - +\pi'(y) = \frac{\gamma_1^2}{\alpha^2} - \frac{2\rho\gamma_2}{\alpha^2} - \frac{\gamma_2}{\alpha} \tag{3.42}$$

$$\lambda_n = -n\tau' - \frac{n(n-1)}{2}\sigma'' \tag{3.43}$$

And

$\sigma''(y) = 0$ and $\tau'(y) = -\frac{2\gamma_2}{\alpha_2}$, then

$$\lambda_n = -n\left(-2\frac{\gamma_2}{\alpha}\right) \tag{3.44}$$

Since $\lambda = \lambda_1$ then

$$\frac{\gamma_1^2}{\alpha^2} - \frac{2\rho\gamma_2}{\alpha^2} - \frac{\gamma_2}{\alpha l} = 2n\frac{\gamma_2}{\partial t} \tag{3.45}$$

$$\frac{\gamma_1^2}{\alpha^2} - \frac{2\rho\gamma_2}{\alpha^2} = (2n+1)\frac{\gamma_2}{\alpha}$$

$$\frac{\gamma_1^2}{\alpha^2} - (2n+1)\frac{\gamma_2}{\alpha} = 2s\frac{\gamma_2}{\alpha^2} \tag{3.46}$$

$$\rho = \frac{\gamma_1^2}{2\gamma_2} - \left(n + \frac{1}{2}\right)\alpha \tag{3.47}$$

$$-\rho^2 = \epsilon - \mu_0 c_0 \Rightarrow E = \mu_1 c_0 - \rho^2$$

$$\epsilon = \mu_0 C_0 - \left[\frac{\gamma_1^2}{2\gamma_2} - \left(n + \frac{1}{2}\right)\alpha\right] \tag{3.48}$$

and $\epsilon = \beta^2\varepsilon$ then:

$$\varepsilon = \frac{1}{\beta_e^2} \in = \frac{\mu_0 c_0}{\beta_e^2} - \left[\frac{\gamma_1^2}{2\beta_c^2 \cdot \gamma_2} - \left(n + \frac{1}{r}\right)\frac{\alpha}{\beta_e}\right]^2 \quad (3.49)$$

This result is equaralent in Phys. Rev. C 77 (Bonatsos *et al.* 2007)

4. RESULTS AND DISCUSSION

Morse potential has three free parameters but the depth of the potential does not affect the results because the calculated energies are normalized. The energy eigenvalue equation corresponding to the Morse potential given in Eq.(3.49) is equal to the previous work resut. This shows that the Nikiforov-Uvarov method is a good method for solving second-order Schrödinger equation. To test the ability of the Morse potential in giving the experimental data of nuclei, we applied Eq.(3.49) to predict the experimental energy spectra of Xe isotopes. We have used Mathematica programme in this fit. First we have read the experimental data and then changing potential parameters in some range, we determined the best values of the parameters. In table 4.1 shows the hamiltonian parameters used to obtain experimental data for selected isotopes. In this table, β_e and a represent the position of the potential minimum and the slope of the inner potential wall, respectively. σ is the quality factor of the fit. From this table, the best fit is obtained for ^{120}Xe and ^{126}Xe isotopes, for which σ values are close to 0. In table 4.2, the theoretical results are compared with the experimental ones for the ground-state band up to $L=10^+$ state. As known, all parities of states are positive for even-even nuclei.

Table 4.1 Hamiltonian parameters used to obtain experimental data for selected isotopes.

Isotope	βe	a	σ
$^{118}_{54}$Xe	6.75	0.10	0.050
$^{120}_{54}$Xe	6.41	0.50	0.007
$^{122}_{54}$Xe	6.99	0.50	0.024
$^{124}_{54}$Xe	6.54	0.50	0.020
$^{126}_{54}$Xe	8.43	0.19	0.001
$^{128}_{54}$Xe	5.80	0.23	0.003
$^{130}_{54}$Xe	3.69	0.50	0.153
$^{132}_{54}$Xe	3.31	0.50	0.061
$^{134}_{54}$Xe	3.15	0.50	0.002

Table 4.2 Comparison of morse potential results for the lowest band (n = 0) with the experimental excitation energy data (Firestone 1996) for $^{118\text{-}134}$Xe isotopes.

Isotope	4^+		6^+		8^+		10^+	
	exp	Theo	exp	Theo	exp	Theo	exp	Theo
$^{118}_{54}$Xe	2.402	2.320	4.141	3.940	6.147	5.864	8.349	8.099
$^{120}_{54}$Xe	2.469	2.456	4.333	4.312	6.51	6.497	8.908	8.916
$^{122}_{54}$Xe	2.501	2.468	4.426	4.366	6.695	6.638	9.177	9.217
$^{124}_{54}$Xe	2.483	2.459	4.374	4.326	6.585	6.534	8.959	8.994
$^{126}_{54}$Xe	2.423	2.426	4.207	4.206	6.265	6.265	------	-----
$^{128}_{54}$Xe	2.332	2.336	4.922	3.912	5.672	5.673	7.594	7.599
$^{130}_{54}$Xe	2.247	2.442	3.626	3.626	5.030	4.877	5.544	6.134
$^{132}_{54}$Xe	2.157	2.206	3.16	3.505	----	4.877	----	6.316
$^{134}_{54}$Xe	2.043	2.201	-- --	3.522	----	4.962	-----	6.545

5. CONCLUSIONS

Analytical solutions of the Bohr-Hamilton collective operator with Morse potentials for the U(5)-O(6) and U(5)-SU(3) transition regions are obtained by a Mathematica program. The obtained energy eigenvalue equations were used to obtain the experimental excitation spectra of Xe isotopes. The results are in good agreement with the experimental data.

The analytical solution of the Bohr Hamiltonian is obtained in the γ-instability case and in the case of an exact separable rotation of γ≈0, called the exact separable Morse solution. Closed-loop expressions for the energy eigenvalues are obtained by Mathematica program the validity of which is demonstrated by solving the associated Bohr equations for the Morse potentials. All known medium and heavy cores

The γ heads were mounted using a two-parameter γ-unstable solution for transitional cores and a three-parameter for rotating cores. It shows that both the band head and the energy gap within the band are well reproduced for more than nuclei.

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