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SOLUTION OF KLEIN- GORDON EQUATION IN D-DIMENSION AND ITS APPLICATION IN PHYSICAL SYSTEMS

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Aram Bahroz BRZO

ABSTRACT

SOLUTION OF KG-EQUATION IN D-DIMENSION AND ITS APPLICATION IN PHYSICAL SYSTEMS

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The analytic and approximate solution of KG-equation in arbitrary dimension D are calculated by using Asymptotic Iteration Method. The system of some important type of potentials like harmonic oscillator potential, the Go’ldman-Krivchenkov potential and some exponential-type potentials such as Morse and Woods-Saxon potentials have been studied in the D-dimension. The corresponding eigenvalues are obtained. In addition, the total normalized eigenvectors are obtained after calculating normalization constant for each type of potentials. Furthermore, it is shown that application of Asymptotic Iteration Method in KG-equation gives the eigenvector solution in terms of the Gauss hypergeometric or its special cases using the wavefunction generator.

Key words: KG-equation, Asymptotic Iteration Method, Harmonic Oscillator, the Go’ldman-Krivchenkov potential, Morse potential, and Woods-Saxon potential

ÖZET

D-BOYUTLU KLEIN-GORDON DENKLEMININ ÇÖZÜMÜ VE FIZIKTEKI UYGUNLAMALARI

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D boyuttaki Klein-Gordon denkleminin analitik ve yaklaşık çözümleri Asimtotik Iterasyon Metodu ile hesaplandı. Harmonic oscillator, the Go'ldman-Krivchenkov potansiyeli ve Morse ve Woods-Saxon gibi exponansiyel potansiyellerin D boyuttaki sistemleri üzerinde çalışıldı. İlgili potansiyellerin özdeğerleri hesaplandı. Ek olarak, her bir potansiyel için normalize sabiti hesaplandıktan sonra potansiyelleri normalize edilmiş toplam özvektörleri eldi edildi. Ayrıca, dalgafonksiyonu üreteci kullanılarak, Klein-Gordon denkleminde AIM uygulamasının özvektör çözümlerini Gauss hypergeometric cinsinden veya Gauss hypergeometric fonksiyonun özel bir formunda verdiği gösterildi.

Anahtar Kelimeler: Klein-Gordon denklemi, Asimtotik Iterasyon Metodu, Harmonik Osilatör, Go'ldman-Krivchenkov potansiyeli, Morse potansiyeli ve Woods-Saxon potansiyeli.

To my mother

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$_2F_1$

LIST OF ABBRIVIATIONS

H	Hamiltonian
c	Speed of light
$\hbar$	The plank constant
M	Mass
E	Energy
p	Momentum
∇	Gradient Operator
l	Orbital Angular Momentum
n	Quantum Number
k	Iteration Number
$V(r)$	Vector Potential
$S(r)$	Scalar Potential
$U(r)$	Wavefunction
$R(r)$	Wavefunction Generator
$\psi(x)$	Eigenvector of the System
$_2F_1$	Gauss Hypergeometric Function
Δ_D	D-dimensional Laplacian Operator

CHAPTER 1

INTRODUCTION

1.1 Background Context

The Klein-Gordon (KG) equation is a relativistic extension of Schrödinger equation, which describes scalar (or pseudoscalar) particles with zero-spin such as pions and kaons. Furthermore, it is one of the earliest results of the attempts to enter special relativity in to quantum mechanics [1, 2]. In 1905 two important papers were published by Einstein. In the first paper, the well-known theory called special theory of relativity under two assumptions the constant speed of light c and the principle of relativity was introduced. In the second one, the equivalence relation between mass and energy was derived [3]. The relativistic expression can be derived in four-vector notation as of this relation was stated as

$$p^{\mu}p_{\mu} = M^2c^2 \tag{1.1}$$

where M is the rest mass of the particle, $p^{\mu} = \left[\frac{E}{c}, p_x, p_y, p_z\right]$, p_i is the momentum in i^{th} direction, and E is the energy of the particle. This two papers of Einstein created a great change in physics and their effects entered in all physical fields [1]. After that, in 1926 a series of important papers were published this time was by Schrödinger. In one of those series publications, he merged Max Plank's quantum assumptions with his wave mechanism. Thus, the quantum mechanic theory was born and Schrödinger derived his equation as

$$H\psi = E\psi \tag{1.2}$$

where H is the Hamiltonian which gives the total energy of the system, and ψ is the eigenvector of the system.

In the others, he suggested the interpretation of $\psi\psi^*$ in a way that this interpretation has a wide applications in quantum mechanics. Furthermore, he stated that the system could be viewed as being in every kinematically possible configuration at the same time, the function $\psi\psi^*$ deciding how "strongly" each configuration contributes to the overall state [3]. Indeed, in the same paper he tried to derive a new form of his equation which is a relativistic generalization for an electron that is known now as the KG-equation. Due to the simplifications that he did in the KG-equation, he rejected his results and he considered it is incomplete equation, because he couldn't make estimate his results in a correct way and also, he failed make them fit the data. The mistake that made Schrödinger regret form his calculation, is that during his study about electron, he didn't take into account the spin of the electron [4]. After that, in1926 paper of Gordon and in the 1927 paper of Klein, they completed the incomplete work of Schrödinger. They derived the equation bearing their names from the Schrödinger formulation of quantum mechanics Eq. (1.1) and combining it with Einstein's theory of relativity [3] as will done in this study in Sec. (2.1.1). In addition, and exactly in 1928 the British physicist Paul Dirac formulated a relativistic quantum mechanical wave equation which describes a relativistic particles with ½-spins. Furthermore, his equation is consistent with both quantum mechanic principles and the special theory of relativity [5]. The important point of views that should be noted is that the KG-equation differ from Schrödinger equation because of that it contains vector and scalar potentials. In addition, for the KG-equation, the two different energy solutions are defined to describe two different particles - the particle and the antiparticle. After all, due to those results there were a hot discussions about the physically acceptable conditions and reality of the negative energy i.e. anti-particles. But, finally the physical counterparts to these negative energy, antimatter particles, have turned out to be real [6].

As already stated, the relativistic version of the Schrödinger theory was done by Klein and Gordon, but there were others working on the same problem, including V. Fock, J. Kudar, T. de Donder, and H. van Dungen as stated in [7].

1.2 Literature Review

In nuclear and high energy physics [8, 9], one of the most important problem is to obtain the exact or approximate analytic solution of the relativistic wave equations like KG, Dirac and Saltpeter wave equations for mixed vector and scalar potential. The exact solutions of these wave equations (non-relativistic or relativistic) took much attention recent years because their importance. In the literature, analytical and approximate solutions are done for most of potentials with different methods. However, analytical solutions are possible only in some potentials such as the harmonic oscillator and some type of Anharmonic Oscillator potentials [10,11], Pseudoharmonic potential plus the Ring-shaped potentials [12,13], Makarov potential [14], Hydrogen atom [15], and The Mie-type potentials [16]. In contrast, there are several potentials that they cannot be solved without the approximations that offered to the centrifugal term as in the case of this thesis, and such potentials, Hulth`en potential [17], Rotating-Morse potential [18], Manning-Rosen potential [19], Yukawa potential [20], Eckart and Q-deformed Eckart potential [21], Exponential-Type potentials [22], and so on. In recent years, the D-dimensional solution of KG-equation has been developed by researcher and several methods have been tested and improved to solve this type of equations. Such most important methods that applied to D-dimensional KG-equation for different potentials are supersymmtric quantum mechanics (SUSYQM) [17], Nikiforov-Uvorov [23], Hellmann-Feynman theorem [24], Minimal length [25], Adomian's method, transformation and exponential function method [26], Ansatz method [27], and Asymptotic Iteration Method (AIM) [28]. Furthermore, there are a list of potentials that they are tested under the solution of KG-equation in arbitrary dimension D such as ring-shaped pseudoharmonic potential [12], Manning-Rosen potential [19], Kratzer potential and Coulomb potential [29], Hulthen-Type Potential [17], Hylleraas Potential [30], Energy-Dependent potential [31], Yukawa potential [32], Eckart potential [33], generalize Woods-Saxon potential [34], Pöschl–Teller potential [35], exponential-type potentials [22], Ring-Shaped Kratzer potential [36], and so on. Among those potentials and methods, this thesis is focused on different types of potentials, harmonic and exponential type because they are in a common form among potential types. Harmonic oscillator, Go'ldman-Krivchenkov, Morse, Woods-Saxon potential are tested in arbitrary dimension of KG-equation by AIM during this thesis.

First, the Harmonic Oscillator potential is among the most important example of explicit solvable problems, both in classical and quantum mechanics. It is the ideal model potential and it's one of the basic potentials, recently in literature it has been solved exactly in both non-relativistic and relativistic cases [10, 37] by AIM and Nikiforov-Uvarov methods, respectively. In addition, Go'ldman-Krivchenkov potential which is a special case of harmonic oscillator potential has been preferred with some other type of potentials as perturbation of Harmonic Oscillator potential or Anharmonic potential [11]. The Go'ldman-Krivchenkov potential in literature also has been studied exactly and gave good results for energy spectrum and corresponding eigenvector in nonrelativistic [38] or relativistically [39] by AIM and SUSYQM, respectively.

The Morse potential gives an excellent qualitative description of the interaction between the two atoms in a diatomic molecule and also it is a reasonable qualitative description of the interaction close to the surface [40]. It has been studied recently in Schrödinger equation and its relativistic version [41, 42] by Nikiforov-Uvarov method.

The Woods-Saxon potential has a great important in nuclear physics because of that it's a reasonable potential of nuclear shell model and a short range potentials [43]. Also, it has a wide applications in other fields of physics such as particle, atomic, chemical, and condensed matter physics [44]. Recently, the relativistic and non-relativistic solutions have been solved for Woods-Saxon potential with different methods [45-46].

The Morse and Woods-Saxon Potentials are in the most potentials among the exponential potentials and they cannot be solved in D-dimensional KG-equation exactly. Instead of that, the solution of this problem is solved numerically due to an approximation which preferred to the centrifugal term as in the case of this thesis in Chapter 4. During the applications of the corresponding potentials in KG-equation, it is considered that the vector potential $V(r)$ and the scalar potential $S(r)$ are equal to the each other except for the Woods-Saxon potential. When the potentials $V(r)$ and $S(r)$ are taken unequal, the results are obtained in same form with the case of equal form with different constant parameters. Since it is given a different form of equation in KG-equation, the relation between the $V(r)$ and $V(r)$ potential is considered unequal in Woods-Saxon potential.

1.3 Objective and Scope

The goal of this thesis is to show that the D-dimensional KG-equation is exactly and approximately solvable with some different type of potentials as mentioned before using AIM method.

This thesis is divided to 5 chapters and it is organized as follows: the derivation of KG-equation in one dimension and generalizing it to three and arbitrary dimensions D are outlined in Chapter 2. Chapter 3 is focused on AIM and formulation of this method from 1^{st} and 2^{nd} order homogeneous linear differential equations. Furthermore, calculating eigenenergy and eigenvector with AIM is highlighted in detail during same chapter.

The main part of this thesis is the Chapter 4. It includes all applications of AIM in to KG-equation with different potentials. The energy spectrum of those potentials are calculated and compared with the results in literature.

Ultimately, the last chapter is devoted to the discussion and conclusion of the main results.

CHAPTER 2

KLEIN-GORDON EQUATION

The KG-equation is a relativistic version of Schrödinger equation or it can be said that it is an alternative equation of Schrödinger equation in the relativistic case which deals with particles of zero spin (spin less particles) [47] like Boson particles, π mesons [48] and Higgs Boson H^o [49]. This equation was named after a great work that done by to physicists Oskar Klein and Walter Gordon after publishing their papers in 1927. They proposed this equation to describe electrons in the relativistic situation [9]. There are two objects that make it differ from Schrödinger equation, negative energy spectrum with the vector *V(x)* and scalar potential *S(x)* [50].

2.1 Formalism of KG-equation

There has been many different ways proposed to derive KG-equation and they started from different fields or different equations in physics such as did by authors in [51, 52]. They started the derivation from Maxwell's electric wave equation and solution of wave equation, respectively.

In order to derive the KG-equation from the relativistic way, the first step is starting from the well-known relation between energy and momentum which was stated by Einstein in special relativity and deriving KG-equation in (1+1) and generalizing it to (3+1) dimensions. The following step is the expanding of the derivation to the d-dimensional form of the KG-equation.

2.1.1 Derivation of One-Dimensional KG-equation

The starting point of the derivation of KG-equation is the fundamental energy relation used in special relations which is the most common and easy way in derivation

$$E^2 = p^2c^2 + M^2c^4. \tag{2.1}$$

By using quantum mechanical principles, the energy operator is expressed as

$$\hat{E} = i\hbar \frac{\partial}{\partial t}. \tag{2.2}$$

Similarly, the expression of momentum operator is in the form of

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}. \tag{2.3}$$

Taking the square of both sides of Eq. (2.2) and Eq. (2.3) leads to

$$\hat{E}^2 = -\hbar^2 \frac{\partial^2}{\partial t^2} \tag{2.4}$$

$$\hat{p}^2 = -\hbar^2 \frac{\partial^2}{\partial x^2}. \tag{2.5}$$

Then, by substituting results of Eq. (2.4) and Eq. (2.5) in to Eq. (2.1), the equation is transformed to

$$-\hbar^2 \frac{\partial^2}{\partial t^2} = -\hbar^2 c^2 \frac{\partial^2}{\partial x^2} + M^2c^4. \tag{2.6}$$

To give a physical taste to Eq. (2.6), the wavefunction of space and time $\Psi(x,t)$ must be applied as

$$-ħ^2 \ \frac{\partial^2 \Psi(x,t)}{\partial t^2} = -ħ^2 \ c^2 \frac{\partial^2 \Psi(x,t)}{\partial x^2} + M^2 c^4 \ \Psi(x,t). \tag{2.7}$$

When ħ =c=1(natural units) in Eq. (2.7) the KG-equation in cartesian coordinate system becomes

$$\frac{\partial^2 \Psi(x,t)}{\partial t^2} - \frac{\partial^2 \Psi(x,t)}{\partial x^2} + M^2 \ \Psi(x,t) = 0, \tag{2.8}$$

which is the one dimensional KG-equation. It can be generalized to three dimensional by defining (∇) operator which is the directional derivative operator;

$$\vec{\nabla} = \frac{\partial}{\partial x} \hat{\imath} + \frac{\partial}{\partial y} \hat{\jmath} + \frac{\partial}{\partial z} \hat{k}. \tag{2.9}$$

Then 3-dimensional KG-equation takes the form as follows;

$$\frac{\partial^2 \Psi(r,t)}{\partial t^2} - \nabla^2 \Psi(r,t) + M(r)^2 c^4 \ \Psi(r,t) = 0. \tag{2.10}$$

One important point that should be noted is that the KG-equation has some potential corrections due to the Stark and Zeeman Effect that they cause shifting and splitting spectral lines of atoms and molecules due to the presence of an external static electric and static magnetic field, respectively [47]. Hence, after taking in to account the effect of these two field, the energy and momentum operators have to be written in the form of

$$p = \ p \ + \ A \ \varphi \tag{2.11-a}$$

$$i \frac{\partial}{\partial t} = \ i \frac{\partial}{\partial t} - V(r) \tag{2.11-b}$$

where A is vector potential. The other correction is an additional coupling of the space-time scalar potential S(r, t) to the relativistic mass of a particle in a way that:

$$M \rightarrow M + S(r,t). \tag{2.11-c}$$

Using Poincare space-time symmetry group Gauge invariance of the vector coupling which is a unitary irreducible representation corresponding of the "four-vector" and "scalar" allows for the freedom to fix the gauge (remove the nonphysical gauge modes) without altering a physical content of the problem [53]. There are many choices of gauge fixing but, here the most commonly used conditions, the Lorentz gauge, $\partial . A = 0$ and the Coulomb gauge, $\nabla . A = 0$ are taking in to consideration. Adapting the later choice, it's time to write the time component of the four-vector potential as, g$A_o = V(r,t)$, where g is a quantity related to the magnetic moment of an electron, nucleus or other particle [1, 7]. Thus, consequently the two independent potential functions in KG-equation, vector potential and scalar potential are selected. Using the notation developed as ($\partial\mu\, \partial^{\mu} = \frac{\partial^2}{\partial t^2} - \nabla^2$), then Eq. (2.10) can be written in a compact style and it becomes:

$$(\partial\mu\, \partial^{\mu} + M(r)^2)\, \Psi(r,t) = 0. \tag{2.12}$$

The vector and scalar coupling potentials that introduced above in Eq. (2.11-b) and Eq. (2.11-c) will lead to potential interactions and Eq. (2.10) is rewrites as follows:

$$\{-(i\frac{\partial}{\partial t} - V(r))^2 - \nabla^2 + (M + S(r,t))^2\}\, \Psi(r,t) = 0. \tag{2.13}$$

After simplifying, above equation becomes in the most common using form as:

$$\nabla^2\Psi(r,t) + (E - V(r))^2\,\Psi(r,t) - (M + S(r,t))^2\,\Psi(r,t) = 0. \tag{2.14}$$

Again rearranging Eq. (2.14) with the effective potential $V_{eff}(r)$ yields:

$$\nabla^2\Psi(r,t) + V_{eff}(r)\,\Psi(r,t) = 0, \tag{2.15}$$

where,

$$V_{eff}(r) = E^2 - M^2 + (V^2(r) - S^2(r)) - (M\,S(r) + E\,V(r)).$$

It can be seen that Eq. (2.15) is the reduced form of KG-equation to Schrödinger equation with the energy dependent effective potential ($V_{eff}(r)$).

2.1.2 D-Dimensional KG-equation

The KG in Eq.(2.13) in higher dimension for spherically coordinate with the repulsive vector potential $V(r)$, the attractive scalar potential $S(r)$ and the rest mass M *(in units* $\hbar = c = 1$*)* reads [54],

$$\Delta_D\psi_{n,l,m}(r,\Omega_D) = \{\,(E_{n,l} - V(r))^2 - (M + S(r))^2\,\}\;\psi_{n,l,m}(r,\Omega_D) \tag{2.16}$$

with

$$\Delta_\mathrm{D} = \nabla^2{}_\mathrm{D} = \frac{1}{\mathrm{r}^{\mathrm{D}-1}}\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}^{\mathrm{D}-1}\frac{\partial}{\partial \mathrm{r}}\right) - \frac{\Lambda_\mathrm{D}^2\,(\Omega_\mathrm{D})}{\mathrm{r}^2} \tag{2.17}$$

where $E_{n,l}$, Δ_D, $\frac{\Lambda_\mathrm{D}^2\,(\Omega_\mathrm{D})}{\mathrm{r}^2}$ and Ω_D are the energy eigenvalues, the D-dimensional Laplacian, the generation of the centrifugal barrier for the D-dimensional space, and the angular coordinates, respectively.

The total wave function in D-dimension for separating variables is written as:

$$\psi_{n,l,m}(r,\Omega_D) = R_{n,l}(r)\ Y_l^m(\Omega_D). \qquad (2.18)$$

As mentioned above the term $\frac{\Lambda_D^2\ (\Omega_D)}{r^2}$ is the generalization of the centrifugal term for the higher dimensional space. The eigenvalues of $\Lambda_{\mathrm{D}}^2\ (\Omega_{\mathrm{D}})$ are defined by the relation,

$$\Lambda_{\mathrm{D}}^2\ (\Omega_{\mathrm{D}})\ Y_l^m(\Omega_D) = l\ (l + \mathrm{D} - 2)\ Y_l^m(\Omega_D), \mathrm{D} > 1 \qquad (2.19)$$

where $Y_l^m(\Omega_D)$, $R_{n,l}$(r) and l represent the hyperspherical harmonics, the hyperradial wavefunction and the orbital angular momentum quantum number, respectively. Suppose, the hyperradial wavefunction ansatz [54] as:

$$R_{n,l}(r) = \mathrm{r}^{-(\mathrm{D}-1)/2}\ U_{n,l}(r). \qquad (2.20)$$

After substituting the wavefunction ansatz and focusing on the hyperradial part of the KG-equation, Eq. (2.16) is transformed to:

$$\left\{\frac{d^2}{dr^2} + (E_{n,l} - V(r))^2 - \big(M + S(r)\big)^2 - \frac{(D+2l-1)(D+2l-3)}{4r^2}\right\} U_{n,l}(r) = 0 \qquad (2.21)$$

which is the basic equation in order to apply for some physical potentials.

CHAPTER 3

SOLUTION METHODOLOGY

In general, differential equations with different orders and their solutions play an important role to overcome problems in many branch of physics especially in quantum and mathematical physics. Among those differential equations, second-order differential equation has a special importance and it took much attention in the recent years in physics by researchers [35].

To solve second-order differential equations many methods were applied as mentioned in Sec. (1.2). AIM is one of those methods proposed to solve 2^{nd} order homogeneous linear differential equations with appropriate boundary conditions [55].

AIM has algorithm suitable for computer programming to get results directly and rapidly. It gives good results in both analytic and approximate solutions for both Schrodinger and Klein Gordon equations with different potentials [56, 57, 58, and 59].

In addition to the formalism of AIM, there are two different ways to derive the AIM as stated in the next two sections.

3.1 Derivation of AIM from 2^{nd} Order Linear Differential Equations

The AIM is proposed [35] to solve the 2^{nd} order homogeneous linear differential equations of the form

$$y''(x) = \lambda_o(x)\, y'(x) + s_o(x)\, y(x) \tag{3.1}$$

where $\lambda_o(x) \neq 0$ and $s_o(x)$ are functions defined in C^{∞}(a, b). And they have sufficiently many continuous derivatives. In order to find a general solution to this equation, the symmetric structure of the right hand side of Eq. (3.1) is differentiated with respect to x, and it gives

$$y'''(x) = \lambda_1(x)\, y'(x) + s_1(x)\, y(x) \tag{3.2}$$

where

$$\lambda_1(x) = \lambda_o{}'(x) + s_o(x) + \lambda_o{}^2(x),$$

and

$$s_1(x) = s_o{}'(x) + s_o(x)\, \lambda_o(x).$$

Again taking second derivative of Eq. (3.1) yields

$$y''''(x) = \lambda_2(x)\, y'(x) + s_2(x)\, y(x) \tag{3.3}$$

with

$$\lambda_2(x) = \lambda_1{}'(x) + s_1(x) + \lambda_o(x)\, \lambda_1(x),$$

and

$$s_2(x) = s_1{}'(x) + s_o\,(x)\, \lambda_1(x).$$

Thus, continuing with this successive derivative in the same procedure, and collecting them for $(k+1)^{th}$ and $(k+2)^{th}$ derivatives with $= 1, 2, \ldots$, then, it gives following results

$$y^{(k+1)}(x) = \lambda_{k-1}(x)\, y' + s_{k-1}(x)\, y, \tag{3.4}$$

and

$$y^{(k+2)}(x) = \lambda_k(x)\, y' + s_k(x)\, y, \tag{3.5}$$

while,

$$\lambda_k(x) = \lambda_{k-1}{}'(x) + s_{k-1}(x) + \lambda_o(x)\, \lambda_{k-1}(x), \tag{3.6a}$$

and,

$$s_k(x) = s_{k-1}{}'(x) + s_o\,(x)\, \lambda_{k-1}(x). \tag{3.6b}$$

Taking the ratio of the $(k+1)^{th}$ and $(k+2)^{th}$ derivatives, gives

$$\frac{d}{dx} l\, n\left(y^{(k+1)}(x)\right) = \frac{y^{(k+2)}(x)}{y^{(k+1)}(x)} = \frac{\lambda_k(x)\left(y'(x) + \frac{s_k(x)}{\lambda_k(x)} y(x)\right)}{\lambda_{k-1}\left(y'(x) + \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} y(x)\right)}. \tag{3.7}$$

It’s time to introduce the ‘asymptotic’ aspect of the iteration method. Hence, for sufficiently large k [60],

$$\frac{s_k(x)}{\lambda_k(x)} = \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} = \alpha(x). \tag{3.8}$$

Thus, some cancelation happens at the right hand side of Eq. (3.7) and it reduces to

$$\frac{d}{dx} l\, n\left(y^{(k+1)}(X)\right) = \frac{\lambda_k(x)}{\lambda_{k-1}(x)}. \tag{3.9}$$

After applying integration operator to both sides of Eq. (3.9), it yields the exact solution as

$$y^{(k+1)}(x) = C_1 \, exp\left(\int \frac{\lambda_k(t)}{\lambda_{k-1}(t)} dt\right) \quad (3.10)$$

where C_1 is the integration constant. Furthermore, by dividing both sides of Eq. (3.6a) by $\lambda_{k-1}(x)$ and looking to the Eq. (3.8) with substituting the result in Eq. (3.10) one can obtain

$$y^{(k+1)}(x) = C_1 \; \lambda_{k-1} \, exp\left(\int (\alpha + \lambda_o) \, dt\right). \quad (3.11)$$

In addition, by dividing both sides of Eq. (3.4) and Eq. (3.11) by λ_{k-1} and comparing results give the change for the form of Eq. (11) as

$$y'(x) + \alpha \, y(x) = C_1 \; \lambda_{k-1} \, exp\left(\int (\alpha + \lambda_o) \, dt\right). \quad (3.12)$$

Eq. (3.12) is 1^{st} order linear differential equation and it can be solved easily. Ultimately the general solution of Eq. (3.1) can be written as

$$y(x) = \exp\left(-\int \alpha \, dt\right) \left[C_2 + C_1 \int \exp\left(\int (2\alpha(\tau) + \lambda_o(\tau)) \, d\tau\right) dt \right]. \quad (3.13)$$

From the right hand side of Eq. (3.13), one can notice that the first part gives as the polynomial solution which is converge and physically acceptable while, the second part becomes divergence and physically unacceptable. Hence, to pass this problem it's convenient to put $C_1 = 0$ in order to find square integrable solutions [61]. Hence, the corresponding eigenfunctions can be derived from the following wave function generator

$$y(x) = C_2 \exp\left(-\int \alpha \, dt\right). \quad (3.14)$$

Hereafter, the calculation of finding energy eigenvalues and eigenfunctions with the aid of AIM will discuss in Sec (3.3) and Sec. (3.4), respectively. But before that, the derivation of AIM from 1^{st}order differential equation in the following section is calculated.

3.2 Derivation of AIM from 1^{st} Order Linear Differential Equations

AIM can be derived from the first order linear differential equation [62], considering two different first order differential equations in the form

$$\psi'_1(x) = \lambda_o(x)\,\psi_1(x) + s_o(x)\,\psi_1(x) \tag{3.15}$$

$$\psi'_2(x) = \omega_o(x)\,\psi_2(x) + \rho_o(x)\,\psi_2(x) \tag{3.16}$$

where ω_o and ρ_o are differentiable functions depend on x. Differentiating Eq. (3.15) and Eq. (3.16) yield

$$\psi''_1(x) = \lambda_1(x)\,\psi_1(x) + s_1(x)\,\psi_1(x), \tag{3.17}$$

$$\psi''_2(x) = \omega_1(x)\,\psi_2(x) + \rho_1(x)\,\psi_2(x), \tag{3.18}$$

where

$$\lambda_1(x) = \lambda_o'(x) + \lambda_o^2(x) + s_o(x)\,\omega_o(x) \tag{3.19}$$

$$s_1(x) = s_o'(x) + s_o(x)\,\lambda_o(x) + s_o(x)\,\rho_o(x) \tag{3.20}$$

$$\omega_1(x) = \omega_o'(x) + \lambda_o(x)\,\omega_o(x) + \rho_o(x)\,\omega_o(x) \tag{3.21}$$

$$\rho_1(x) = \rho_o'(x) + \rho_o^2(x) + s_o(x)\,\omega_o(x). \tag{3.22}$$

Taking $2^{nd}, 3^{rd}$ and so on derivatives of Eq. (3.15) and Eq. (3.16), and generalizing their derivatives for $(k+1)^{th}$ and $(k+2)^{th}$ then result will be as follows

$$\psi_1^{(k+1)}(x) = \lambda_k(x)\,\psi_1(x) + s_k(x)\,\psi_1(x) \tag{3.23}$$

$$\psi_2^{(k+1)}(x) = \omega_k(x)\,\psi_2(x) + \rho_k(x)\,\psi_2(x) \tag{3.24}$$

and

$$\psi_1^{(k+2)}(x) = \lambda_{k+1}(x)\,\psi_1(x) + s_{k+1}(x)\,\psi_1(x) \tag{3.25}$$

$$\psi_2^{(k+2)}(x) = \omega_{k+1}(x)\,\psi_2(x) + \rho_{k+1}(x)\,\psi_2(x), \tag{3.26}$$

while

$$\lambda_{k+1}(x) = \lambda_k'(x) + \lambda_k(x)\,\lambda_o(x) + s_k(x)\,\omega_o(x) \tag{3.27}$$

$$s_{k+1}(x) = s_k'(x) + \lambda_k(x)\,s_o(x) + s_k(x)\,\rho_o(x) \tag{3.28}$$

$$\omega_{k+1}(x) = \omega_k'(x) + \omega_k(x)\,\lambda_o(x) + \rho_k(x)\,\omega_o(x) \tag{3.29}$$

$$\rho_{k+1}(x) = \rho_k'(x) + \omega_k(x)\,s_o(x) + \rho_k(x)\,\rho_o(x). \tag{3.30}$$

From the ratio of $(k+1)^{th}$ and $(k+2)^{th}$ derivatives as done in Eq. (3.7), the result is transformed to

$$\frac{d}{dx} ln(\psi_1^{(k+1)}(x)) = \frac{\psi_1^{(k+2)}(x)}{\psi_1^{(k+1)}(x)} = \frac{\lambda_{k+1}(x)(\psi_1(x)+\frac{s_k+1(x)}{\lambda_{k+1}(x)}\psi_2(x))}{\lambda_k(x)(\psi_1(x)+\frac{s_k(x)}{\lambda_k(x)}\psi_2(x))}. \tag{3.31}$$

Applying the same procedure for $\psi_2(x)$ yields similar expression. For large value of k,

$$\frac{s_{k+1}(x)}{\lambda_{k+1}(x)} = \frac{s_k(x)}{\lambda_k(x)} = \alpha_k(x), \qquad k = 0, 1, 2, \dots . \tag{3.32}$$

Thus, by using relation of Eq. (3.32) and substituting in Eq. (3.31), and from Eq. (3.31) the expression of $\psi_1^{(k+1)}(x)$ is obtained as

$$\psi_1^{(k+1)}(x) = C_1 \exp\left(\int \frac{\lambda_{k+1}(t)}{\lambda_k(t)} dt\right) = C_1 \ \lambda_k(x) \exp\left(\int (\alpha\, \omega_o + \lambda_o)\, dt\right). \tag{3.33}$$

By dividing both sides of Eq. (3.23) by $\lambda_k(x)$ and comparing with Eq. (3.33), then Eq. (3.33) reduces to

$$\psi_1(x) + \alpha(x)\ \psi_2(x) = C_1 \exp\left(\int (\alpha\, \omega_o + \lambda_o)\, dt\right). \tag{3.34}$$

By using Eq. (3.18) and Eq. (3.34), the general solution of $\psi_2(x)$ is obtained in the following form [63]

$$\psi_2(x) = \exp\left(\int (\rho_o - \alpha\, \omega_o)\, dt\right)\{C_2 + C_1 \int [\exp\left(\int (\lambda_o - \rho_o + 2\alpha\, \omega_o)\, dt\right)]\} \tag{3.35}$$

This expression is the wavefunction generator for the first order differential equations.

3.3 Calculating Eigenvalues

Eq. (3.8) can be used to calculate energy eigenvalues. Rearranging this equation with the help of termination condition gives

$$\Delta_k(x) = \begin{vmatrix} s_k(x) & \lambda_k(x) \\ s_{k-1}(x) & \lambda_{k-1}(x) \end{vmatrix}, \; k = 1, 2, 3, \ldots$$

furthermore,

$$\Delta_k(x) = \lambda_{k-1}(x)\, s_k(x) - \lambda_k(x)\, s_{k-1}(x) = 0. \tag{3.36}$$

The energy eigenvalues can be obtained exactly from the roots of Eq. (3.36). There is some important points should take care about them before using this equation.

Initial conditions $\lambda_{k-1} = 1$ and $s_{k-1} = 0$ should be used while calculating eigenvalues [68]. Also λ_0 and s_0 are determined from Eq. (3.1) and λ_k and s_k from equations (3.6a) and (3.6b), respectively. If the problem is exactly solvable then Eq. (3.36) is enough to give all roots (eigenvalues). Otherwise, for a specific n principal quantum number a suitable x_0 point should be selected [35]. This point determined generally as the maximum value of the asymptotic wavefunction or the minimum value of the potential. At the end, for sufficiently large value of *k* the approximation energy eigenvalue will get from the roots of the Eq. (3.36) [59].

3.4 Calculating wavefunctions

The discovery of a common source for the entire set of a problems and their treatment in arbitrary dimension D by AIM are the most highlight points of this thesis. It should be noted that using AIM is related to an important aspects: the solution of second order differential equations with suitable boundary conditions. Therefore, of that during calculation wave functions of KG-equation with different potentials, different forms of solution will be seen. This calculation with more general type of second order homogeneous linear differential equation is in the form [35]

$$y''(x) = 2\left(\frac{a\,x^{N+1}}{1-b\,x^{N+2}} - \frac{(m+1)}{x}\right) y'(x) - \frac{w\,x^{N}}{1-b\,x^{N+2}}\, y\,(\mathrm{x}) \qquad (3.37)$$

where $0 \leq x < b^{\frac{1}{N+2}}$ if $b > 0$, and $0 \leq x < \infty$ if b ≤0. Here $N = -1, 0, 1, \ldots$, and the real numbers a, m, and w are specified according to the problems. It has to say some of the problems discussed during this thesis will be transformed to the form of above equation, otherwise they will be a special case of Eq. (3.37). Bender et al [64], recently has been studied on Eq. (3.37) and as a special case he showed that the exact solutions of the class of eigenvalue problem can be written in the follows form

$$-y''(x) + x^{2N+2}\, y(x) = E\, x^{N}\, y(x) y \quad (N = -1, 0, 1, \ldots,). \qquad (3.38)$$

This is helpful to improve and extend the solutions of Eq. (3.38) to provide analytic formula for the solutions of the eigenvalue problem as did by authors [55, 65] as

$$-y''(x) + \left(\frac{m(m+1)}{x^2} + x^{2N+2}\right) y(x) = E\, x^{N}\, y(x) y \quad (N = -1, 0, 1, \ldots,). \quad (3.39)$$

Hereafter, in this section the focus point is on the obtaining wavefunctions by applying AIM to investigate the second order linear homogeneous differential equation [65] in the form below

$$y''(x) = 2\left(a\,x\,p(x) - \frac{(m+1)}{x}\right) y'(x) - w\,p(x)\,\mathrm{y(x)} \qquad (3.40)$$

where $a, m,$ and w are arbitrary constants. To make Eq. (3.40) exactly solvable it's necessary to choose an appropriate polynomial function $p(x)$ [65]. Therefore, Eq. (3.40) is reduced to the most common form of Eq. (3.37)

$$y''(x) = 2\left(\frac{a\, x^{N+1}}{1-b\, x^{N+2}} - \frac{(m+1)}{x} \right) y'(x) - \frac{w\, x^{N}}{1-b\, x^{N+2}}\, y(\mathrm{x}).$$

Then, comparing last equation with Eq. (3.1) it works out that

$$\lambda_{\mathrm{o}}(x) = 2\left(\frac{a\, x^{N+1}}{1-b\, x^{N+2}} - \frac{(m+1)}{x} \right), \tag{3.41a}$$

and

$$\mathrm{s_o(x)} = -\frac{w\, x^{N}}{1-b\, x^{N+2}}. \tag{3.41b}$$

Now for calculating eigenvalues it must be apply the termination condition in Eq. (3.36) and it gives the eigenvalues result as

$$w_n^m(N) = \mathrm{b}(\mathrm{N}+2)^2\ \mathrm{n}\ (\mathrm{n}+\rho) \tag{3.42}$$

with $\rho = -\frac{(2m+1)b+2a}{(\mathrm{N}+2)^{b}}$, where $\mathrm{N} = -1, 0, 1, \ldots$ and $\mathrm{n} = 0, 1, 2, \ldots$. For obtaining $y_n(x)$ the generator of wavefunctions of exact solutions in Eq. (3.14) must be used

$$y_n(x) = \mathrm{C_2} \exp\left(-\int \alpha_k\, dt \right)$$

where $n = 0, 1, 2, \ldots$, and $k \geq n$ is the iteration step number. This leads to get wave functions [55, 65] as

- $y_0(x) = 1$, since $w_0^m(N)$=0 for N=-1, 0, 1, 2, … .

- $y_1(x) = -\,\mathrm{C_2(N{+}2)}\ \sigma(1- \frac{b(\rho+1)}{\sigma} x^{N+2})$

- $y_2(x) = \mathrm{C_2(N+2)}^{\,2}\ \sigma(\sigma+1)\ (1- \frac{2b(\rho+2)}{\sigma} x^{N+2} + \frac{b^{\,2}(\rho+2)(\rho+3)}{\sigma(\sigma+1)} x^{2(N+2)})$
- $y_3(x) = -\,\mathrm{C_2(N+2)^2}\ \sigma(\sigma+1)\ (1- \frac{3b(\rho+3)}{\sigma} x^{N+2}$

$$+\frac{3b^{\,2}(\rho+3)(\rho+4)}{\sigma(\sigma+1)} x^{2(N+2)} - \frac{b^{\,3}(\rho+3)(\rho+4)(\rho+5)}{\sigma(\sigma+1)(\sigma+2)} x^{3(N+2)}$$

… etc.

The general formula for calculating wavefunction $y_n(x)$ is obtained as follows

$$y_n(x) = (-1)^n\, \mathrm{C_n}\, (N+2)^n\, (\sigma)_n\ {}_2F_1\,(-\mathrm{n}, \rho+n;\ \sigma;\ b\,x^{N+2}), \tag{3.43}$$

where

$$(\sigma)_n \frac{\Gamma(\sigma+n)}{\Gamma(\sigma)} \quad , \quad \sigma = \frac{2m+N+3}{N+2}, \text{ and } \rho = \frac{(2m+1)b+2a}{(N+2)b}$$

and ${}_2F_1$ is the Gauss hypergeometric function and it's a special case of the below generalized hypergeometric function

$${}_\mathrm{p}F_\mathrm{q}(\mathrm{a_1}, \ldots, \mathrm{a_p}, \mathrm{c_1}, \ldots, \mathrm{c_q}; y) = \sum_{n=0}^{\infty} \frac{(\mathrm{a_1})_n, \ldots, (\mathrm{a_p})_n\, y^n}{(\mathrm{c_1})_n, \ldots, (\mathrm{c_q})_n\, n!}. \tag{3.44}$$

That is to say, a polynomial of degree n in y.

CHAPTER 4

APPLICATION OF KLEIN-GORDON EQUATION IN PHYSICAL SYSTEMS

During this chapter, in the framework of AIM, the solution of KG-equation in D-dimension is studied in order to calculate both of eigenvalues and eigenfunctions of some physical systems including different potentials with constant masses.

4.1 KG - Equation with Harmonic Oscillator Potential

The Harmonic Oscillator potential is among the most important example of explicit solvable problems, both in classical and quantum mechanics. The vector $V(r)$ and scalar $S(r)$ harmonic oscillator potentials can be written as the following form [38]

$$V(r) = V_o\, r^2\ , \text{ and } \ S(r) = S_o\, r^2. \tag{4.1}$$

The potentials $V(r)$ and $V(r)$ are considered as equal to each other because of giving the same results with unequal case. Substituting the potentials in to Eq. (2.21) gives

$$\left\{\frac{d^2}{dr^2} + \left[\, E_{n,l} - (V_o\, r^2)\,\right]^2 - \,[M + (V_o\, r^2)]^2 - \frac{(D+2l-1)(D+2l-3)}{4r^2}\right\} U_n(r) = 0 \qquad (4.\,2)$$

By expanding the above equation and with some algebraic steps it reduces to the form of

$$\left\{\frac{d^2}{dr^2} + \left(\, E_{n,l}^2 - M^2\,\right) - \,2\left(E_{n,l} + M\right) V_o\, r^2 - \,\frac{(D+2l-1)(D+2l-3)}{4r^2}\right\} U_{n,l}(r) = 0. \tag{4.3}$$

By introducing some parameters, the corresponding equation becomes in a simple form as

$$\left\{\frac{d^2}{dr^2} + \xi_{n,l} - \beta r^2 - \left(\frac{c(c+1)}{r^2}\right)\right\} U_{n,l}(r) = 0 \qquad (4.4)$$

where

$$\xi_{n,l} = E_{n,l}^2 - M^2, \quad \beta = 2\left(E_{n,l} + M\right)V_o, \quad \text{and} \quad c = \frac{1}{2}(D + 2l - 3). \qquad (4.5)$$

Now, to solve Eq. (4.4) by AIM, it has to transform this equation to the form of suitable equation like Eq. (3.1). To do this, the boundary conditions must be applied. Thus, it's clear that the Eq. (4.4) satisfies the condition $U_{n,l}(0) = 0$ which is known as the Dirichlet boundary condition. Therefore, the unnormalized generator wave function takes the form of

$$U_{n,l}(r) = r^{(c+1)}\, \mathrm{e}^{-\frac{r^2}{2}} R_{n,l}(r). \qquad (4.6)$$

Hence, substituting above wavefunction generator in to Eq. (4.4) gets the following result

$$R_{n,l}''(r) - 2\left(\mathrm{r} - \frac{(c+1)}{r}\right) R_{n,l}'(r) - (2\,c + 3 - \xi_{n,l})R_{n,l}(r) = 0. \qquad (4.7)$$

Furthermore, the above equation can read also in the form of

$$R''_{n,l}(r) = 2\left(\mathrm{r} - \tfrac{(c+1)}{r}\right) R'_{n,l}(r) + (2\,c + 3 - \xi_{n,l}\,)R_{n,l}(r) \qquad (4.8)$$

Eq. (4.8) is in compatible form with Eq. (3.1). Therefore, AIM now can be applied for calculating both eigenvalues and eigenfunctions as in the next two sections.

4.1.1 Solving for Eigenvalues

The application of AIM is now initiated by comparing Eq. (4.8) with Eq.(3.1). Ultimately, it gives that

$$\lambda_o(r) = 2\left(r - \tfrac{(c+1)}{r}\right),$$

and

$$s_o(r) = 2c + 3 - \xi_{n,l}$$

Taking *k*-time derivatives of the above two equations with respect to *r*, just like the steps that done in Sec. (3.3) Eq. (3.6-a) and Eq. (3.6-b), the high order functions $\lambda_k(y)$ and $\mathrm{s}_k(\mathrm{y})$ give the following functions

$$\lambda_1(r) = 5 + 2c + \tfrac{2}{r^2} + \tfrac{2c}{r^2} + (-\tfrac{2}{r} - \tfrac{2c}{r} + 2r)^2 - \xi$$

$$s_1(r) = (-\frac{2}{r} - \frac{2c}{r} + 2r)(3 + 2c - \xi)$$

$$\begin{aligned}\lambda_2(r) = &-\frac{4}{r^3} - \frac{4c}{r^3} + 2(2 + \frac{2}{r^2} + \frac{2c}{r^2})(-\frac{2}{r} - \frac{2c}{r} + 2r) + (-\frac{2}{r} - \frac{2c}{r} + 2r)(3 + 2 \\ &- \xi) \\ &+ \left(-\frac{2}{r} - \frac{2c}{r} + 2r\right)\left(5 + 2c + \frac{2}{r^2} + \frac{2c}{r^2} + \left(-\frac{2}{r} - \frac{2c}{r} + 2r\right)^2 - \xi\right)\end{aligned}$$

$$s_2(r) = (3 + 2c - \xi)\left(5 + 2c + \tfrac{2}{r^2} + \tfrac{2c}{r^2} + -\xi\right) + \left(2 + \tfrac{2}{r^2} + \tfrac{2c}{r^2}\right)(3 + 2c - \xi)$$

etc…

Hence, applying the termination condition Eq.(3.36) to the above and solving for $\xi_{n,l}$ for each value of *n*, it gives the following results,

$$\frac{\lambda_o(r)}{s_o(r)} = \frac{\lambda_1(r)}{s_1(r)} \Rightarrow \xi_{0,l} = 3 + 2c,$$

$$\frac{\lambda_1(r)}{s_1(r)} = \frac{\lambda_2(r)}{s_2(r)} \Rightarrow \xi_{1,l} = 7 + 2c,$$

$$\frac{\lambda_2(r)}{s_2(r)} = \frac{\lambda_3(y)}{s_3(y)} \Rightarrow \xi_{2,l} = 11 + 2\,c.$$

Respectively, that means

$$\xi_{n,l} = 4n + 2c + 3 \quad , n = 0, 1, 2, \dots \tag{4.9}$$

Substitution the value of c as in Eq. (4.5) in to the above Eq. (4.9), it gives the last result of the energy spectrum, namely

$$\xi_{n,l} = 4[n + \tfrac{1}{4}(D + 2l)] \quad , n = 0, 1, 2, \dots \tag{4.10}$$

This energy values are in a good agreement with the results of [66].

4.1.2 Solving for Eigenvector

Fortunately, Eq. (4.8) is the differential equation of the hypergeometric function ${}_2F_0$ with well-known eigenvector solution. Therefore, the general solution of Eq. (4.8) can carry out directly. But, before that it will be better to give some detail to get the last result. Hence, as mentioned before, the 2^{nd} order homogeneous linear differential eqaution, i.e. Eq. (4.8) is in suitable form with the Eq. (3.1). Thus, AIM has to apply for calculating corresponding eigenvector. Then, substituting each values of $\lambda_k(y)$ and $s_k(y)$ one by one in to Eq. (3.14) and obeying steps as described in Sec. (3.4) in detail. Eventually, it yields the solution of $R_{n,l}(r)$ after some straightforward computation as

$$R_{n,l}(r) = \sum_{m=0}^{n} (-1)^m 2^{(n-2m)} \frac{\Gamma(n+1)\Gamma(2c+2n+2)\Gamma(c+n-m+1)}{\Gamma(n+1)\Gamma(n-m+1)\Gamma(c+n+1)\Gamma(2n+2c-2m+2)} r^{(2n-2m)}. \tag{4.11}$$

It can be seen that Eq. (4.11) together with Eq. (4.6) gives the exact eigenvector for the Gol'dman and Krivchenkov potential. In addition, Eq. (4.11) has to proceed as follow. By using Pochhammer's identity

$$(a)_{-m} = \frac{\Gamma(\mathrm{a-m})}{\Gamma(\mathrm{a})} \tag{4.12}$$

Eq. (4.11) transforms to the form of

$$R_{n,l}(r) = 2^n r^{2n} \sum_{m=0}^{n} \ (-1)^m \ \frac{(c+n+1)_{-m}}{m!(n+1)_{-m}\ (2c+2n+2)_{-2m}} \left(\frac{1}{2r}\right)^{2m}. \tag{4.13}$$

To further simplifying above equation, the identity $[(a)_{-m} = \frac{(-1)^m}{(1-a)_m}]$ and the Gauss's duplication formula

$$(a)_{2n} = \left(\frac{a}{2}\right)_n \left(\frac{a+1}{2}\right)_n . \tag{4.14}$$

It has to apply to the Eq. (4.13), so the above equation changes to the new form as

$$R_{n,l}(r) = 2^n\ r^{2n} \sum_{m=0}^{n} (-1)^m\ \frac{(-n)_m\ (-n-c-\frac{1}{2})_m}{m!} \left(\frac{1}{r^2}\right)^m. \tag{4.15}$$

The finite sum in Eq. (4.15) is the series representation of the hypergeometric function ${}_2F_0$. Therefore

$$R_{n,l}(r) = 2^n\ r^{2n}\ {}_2F_0\left(-n, -n-c-\frac{1}{2}; \frac{1}{r^2}\right). \tag{4.16}$$

Above equation can be written in terms of a confluent hypergeometric function ${}_1F_1$ as

$$\begin{aligned} R_{n,l}(r) &= (-1)^n\ 2^n\ L_n^{c+\frac{1}{2}}(\ r^2) \\ &= (-1)^n\ 2^n\ \left(c+\frac{3}{2}\right)_n\ {}_1F_1\left(-n; c+\frac{3}{2};\ r^2\right). \end{aligned} \tag{4.17}$$

where $L_n^p(r)$ is Laguerre polynomials. At the end, to get the total unnormalized eigenvector, Eq. (4.17) has to substitute in to Eq. (4.6). Consequently, the unnormalized wavefunction takes the form of

$$U_{n,l}(r) = C_n\,(-1)^n\,2^n\,(c+\tfrac{3}{2})_n\,r^{(c+1)}\,\mathrm{e}^{-\frac{r^2}{2}}\;{}_1F_1(-n;c+\tfrac{3}{2};\,r^2). \qquad (4.18)$$

This eigenvector is close to the results of that obtained in [67]

4.1.3 Normalization Constant ($\boldsymbol{C_n}$)

A normalization constant $\mathrm{C_n}$ can be computed in the following way:

$$\int_0^\infty |U(r)|^2 dr = 1. \qquad (4.19)$$

Thus, substitution of Eq. (4.19) in to Eq.(4.18) gives

$$C_n^2\,2^{2n}\,{(c+\tfrac{3}{2})_n}^2 \int_0^\infty r^{2(c+1)}\,\mathrm{e}^{-\frac{r^2}{2}}\;{}_1F_1\left(-n;c+\tfrac{3}{2};\,r^2\right)\,{}_1F_1(-n;c+\tfrac{3}{2};\,r^2)dr = 1 \qquad (4.20)$$

Above equation faced some form of integral which in turn requires the computation definite integral [34, 35] in the following form

$$I_{n,m}(a) = \int_0^\infty \rho^a\,e^{-\rho}\;{}_1F_1(-n,a;\rho)\;{}_1F_1(-m,a;\rho). \qquad (4.21)$$

It can be seen that the computation of the above definite integral is depending on values both of n and m. Furthermore, by comparison between Eq. (4.21) and Eq. (4.20) it yields that this problem becomes interesting when $n = m$. Therefore, with the aid of the following lemma it's possible to compute the above integral for different values of n and m [36], as

$$\int_0^\infty \rho^a\, e^{-\rho^2}\ {}_1F_1\,(-\,n, a; \rho\,)\ \ {}_1F_1(-\,m, a; \rho\,) = \frac{\mathrm{n}!\,\Gamma(a)}{2(\mathrm{a})_n}, \text{where } n = m. \quad (4.22)$$

Thus, Eq. (4.20) becomes

$$C_n^2\ 2^{2n}\ (c+\tfrac{3}{2})_n{}^2\ \frac{\mathrm{n}!\,\Gamma\left(c+\frac{3}{2}\right)}{2\left(c+\frac{3}{2}\right)_n} = 1.$$

Ultimately, the normalization constant carried out as

$$C_n = \frac{1}{2^n\ (c+\frac{3}{2})_n}\sqrt{\frac{2\left(c+\frac{3}{2}\right)_n}{\mathrm{n}!\,\Gamma\left(c+\frac{3}{2}\right)}}. \quad (4.23)$$

Finally, the total normalized eigenvector by substituting Eq.(4.23) in to Eq.(4.18) is obtained as

$$U_{n,l}(r) = (-1)^n\sqrt{\frac{2\left(c+\frac{3}{2}\right)_n}{\mathrm{n}!\,\Gamma\left(c+\frac{3}{2}\right)}}\ r^{(c+1)}\ \mathrm{e}^{-\frac{r^2}{2}}\ {}_1F_1\left(-n; c+\tfrac{3}{2};\ r^2\right). \quad (4.24)$$

which is the required wavefunction.

4.2 KG - Equation with Gol'dman and Krivchenkov Potential

The Gol'dman and Krivchenkov Hamiltonian is the generalization of the Harmonic oscillator Hamiltonian in 3-dimensions [39], namely the corresponding potentials are

$$V(r) = K_1\, r^2 + \frac{\gamma(\gamma+1)}{r^2},\ S(r) = K_2\, r^2 + \frac{c(c+1)}{r^2}. \quad (4.25)$$

Starting in the case of equal vector and scalar potentials ($V(r) = S(r)$), and setting ($K_1 = K_2 = K$) with substitution of Eq. (4.25) in to the KG-equation in d-dimension give

$$\left\{\frac{d^2}{dr^2} + \left[E_{n,l} - \left(Kr^2 + \frac{\gamma(\gamma+1)}{r^2}\right) \right]^2 - \left[M + \left(Kr^2 + \frac{\gamma(\gamma+1)}{r^2}\right)\right]^2 - \frac{(D+2l-1)(D+2l-3)}{4r^2}\right\} U_{n,l}(r) = 0. \qquad (4.26)$$

After rearranging Eq. (4.26) with algebraic calculation, it gives

$$\left\{\frac{d^2}{dr^2} + \left(E_{n,l}^2 - M^2 \right) - 2\left(E_{n,l} + M\right)kr^2 - \frac{8\left(E_{n,l}+M\right) \gamma(\gamma+1)+(D+2l-1)(D+2l-3)}{4r^2}\right\} U_{n,l}(r) = 0. \qquad (4.27)$$

In a similar way, by defining the parameters $\xi_{n,l}$, β, and δ, the equation reduces to

$$\left\{\frac{d^2}{dr^2} + \xi_{n,l} - \beta r^2 - \left(\frac{\delta(\delta+1)}{r^2}\right)\right\} U_{n,l}(r) = 0 \qquad (4.28)$$

where

$$\xi_{n,l} = E_{n,l}^2 - M^2, \qquad \beta = 2\left(E_{n,l} + M\right)K, \qquad \text{and}$$

$$\delta = \frac{1}{2}\left(-1 \pm \sqrt{-2 + 4D - D^2 + 8l - 4Dl - 4l^2 - 4\beta\gamma - 4\beta\gamma^2} \right). \qquad (4.29)$$

Hereafter, Eq. (4.28) will obey the same way that presented before to solve the harmonic oscillator potential with little difference that appears just in parameters. It must be transform this equation to the form of suitable equation like Eq. (3.1). This transformation can happen relying on the boundary conditions. Thus, it has seen that Eq. (4.28) satisfies the boundary condition $U_{n,l}(0) = 0$ which is known as the Dirichlet boundary condition. The generalization lies in the parameter δ ranging over $[0, \infty)$ instead of the angular momentum quantum number l = 0, 1, 2 . . . Therefore, the unnormalized generator eigenvector may takes the form

$$U_{n,l}(r) = r^{(\delta+1)}\, e^{-\frac{r^2}{2}}\, R_{n,l}(r). \tag{4.30}$$

Hence, substituting above eigenvector generator in to Eq. (4.28), the result becomes

$$R_{n,l}''(r) - 2\left(\mathrm{r} - \frac{(\delta+1)}{r}\right) R_{n,l}'(r) - (2\,\delta + 3 - \xi_{n,l}\,)\, R_{n,l}(r) = 0. \tag{4.31}$$

Furthermore, the above equation will read in the form of

$$R_{n,l}''(r) = 2\left(\mathrm{r} - \frac{(\delta+1)}{r}\right) R_{n,l}'(r) + (2\,\delta + 3 - \xi_{n,l}\,)\, R_{n,l}(r). \tag{4.32}$$

AIM now can be applied to calculate eigenvalue and eigenfunction, after comparing Eq. (4.32) with Eq.(3.1) which they are in the same form. In the next sections the steps for calculation will be done as did before in the case of Harmonic Oscillator potential.

4.2.1 Solving for Eigenvalues

Comparing Eq. (4.32) with Eq. (3.1), it carries out value of $\lambda_o(r)$ and $s_o(r)$ as

$$\lambda_o(r) = 2\left(r - \frac{(\delta+1)}{r}\right),$$

and

$$s_o(r) = 2\,\delta + 3 - \xi_{n,l}.$$

By taking derivatives for above functions to calculate $\lambda_k(r)$ and s_k(r), it gives the following results

$$\lambda_1(r) = 5 + \frac{2}{r^2} + 2\delta + \frac{2\delta}{r^2} + (-\frac{2}{r} + 2r - \frac{2\delta}{r})^2 - \xi$$

$$s_1(r) = (-\frac{2}{r} + 2r - \frac{2\delta}{r})(3 + 2\delta - \xi)$$

$$\lambda_2(r) = -\frac{4}{r^3} - \frac{4\delta}{r^3} + 2(2 + \frac{2}{r^2} + \frac{2\delta}{r^2})(-\frac{2}{r} + 2r - \frac{2\delta}{r}) + (-\frac{2}{r} + 2r - \frac{2\delta}{r})(3 + 2\delta - \xi) + \left(-\frac{2}{r} + 2r - \frac{2\delta}{r}\right)\left(5 + \frac{2}{r^2} + 2\delta + \frac{2\delta}{r^2} + \left(-\frac{2}{r} + 2r - \frac{2\delta}{r}\right)^2 - \xi\right)$$

$$s_2(r) = (2 + \frac{2}{r^2} + \frac{2\delta}{r^2})(3 + 2\delta - \xi) + (3 + 2\delta - \xi)\left(5 + \frac{2}{r^2} + 2\delta + \frac{2\delta}{r^2} + \left(-\frac{2}{r} + 2r - \frac{2\delta}{r}\right)^2 - \xi\right)$$

and solving for $\xi_{n,l}$ for each value of *n*

$$\frac{\lambda_o(r)}{s_o(r)} = \frac{\lambda_1(r)}{s_1(r)} \Rightarrow \xi_{0,l} = 3 + 2\delta$$

$$\frac{\lambda_1(r)}{s_1(r)} = \frac{\lambda_2(r)}{s_2(r)} \Rightarrow \xi_{1,l} = 7 + 2\delta$$

$$\frac{\lambda_2(r)}{s_2(r)} = \frac{\lambda_3(y)}{s_3(y)} \Rightarrow \xi_{2,l} = 11 + 2\delta.$$

Respectively, that means

$$\xi_{n,l} = 4\mathrm{n} + 2\delta + 3, \qquad \mathrm{n} = 0, 1, 2, \ldots \quad . \tag{4.33}$$

These eigenenrgy values is in a good agreement with the results that obtained in [66].

4.2.2 Solving for Eigenvector

It seems that Eq. (4.32) has the same form as Eq. (4.8), therefore the same solution can be applied here in a way that the solution of Eq. (4.32) becomes

$$R_{n,l}(r) = \sum_{m=0}^{n} (-1)^m 2^{(n-2m)} \frac{\Gamma(n+1)\Gamma(2\delta+2n+2)\Gamma(\delta+n-m+1)}{\Gamma(n+1)\Gamma(n-m+1)\Gamma(\delta+n+1)\Gamma[2(n+\delta-m+1)]} r^{2(n-m)}. \tag{4.34}$$

Using Pochhammer's identity and the Gauss's duplication formula, the above equation transforms the new form as

$$R_{n,l}(r) = 2^n\, r^{2n} \sum_{m=0}^{n} (-1)^m \, \frac{(-n)_m \, (-\,n-\delta-\frac{1}{2})_m}{m!} \left(\frac{1}{r^2}\right)^m. \qquad (4.35)$$

Substituting the hypergeometric function ${}_2F_0$ instead of its finite series representation in the above equation gives

$$R_{n,l}(r) = 2^n\, r^{2n}\, {}_2F_0\left(-n, -n-\delta-\frac{1}{2}; \frac{1}{r^2}\right). \qquad (4.36)$$

Above equation can be written in terms of confluent hypergeometric function ${}_1F_1$ as

$$R_{n,l}(r) = (-1)^n\, 2^n \left(\delta + \frac{3}{2}\right)_n {}_1F_1\left(-n; \delta + \frac{3}{2};\, r^2\right). \qquad (4.37)$$

Substitution of Eq. (4.37) in to eigenvector generator in Eq. (4.30), leads radial wavefunction

$$U_{n,l}(r) \;= \mathrm{C}_n (-1)^n\, 2^n\, (\delta + \frac{3}{2})_{\,n} r^{(\delta+1)}\, \mathrm{e}^{-\frac{r^2}{2}}\; {}_1F_1\left(-n; \delta + \frac{3}{2};\, r^2\right). \qquad (4.38)$$

which is unnormalized wavefunction and it is in compatible form with the results of [66].

4.2.3 Normalization Constant (C_n)

As mentioned before the Gol'dman and Krivchenkov Hamiltonian is a special case of harmonic oscillator potential, as cleared during the calculation there is a lot of similarity between their calculations including eigenvalue and eigenvector calculations. Thus, comparing their total unnormalized radial eigenvectors Eq. (4.18) and Eq. (4.38), it seems that they are in the same form. Therefore, the same calculation for normalization constant can be used here with changing the parameter c there to δ here. Eventually C_n takes the same form of Eq. (4.23) as

$$C_n = \frac{1}{2^n\,(\delta+\frac{3}{2})_n}\sqrt{\frac{2\left(\delta+\frac{3}{2}\right)_n}{\mathrm{n}!\,\Gamma\left(\delta+\frac{3}{2}\right)}}. \tag{4.39}$$

At the latest, the eigenvector $U_{n,l}(r)$ is calculated by substituting Eq. (4.23) in to Eq. (4.18) yields

$$U_{n,l}(r) = (-1)^n\sqrt{\frac{2\left(\delta+\frac{3}{2}\right)_n}{\mathrm{n}!\,\Gamma\left(\delta+\frac{3}{2}\right)}}\; r^{(\delta+1)}\,\mathrm{e}^{-\frac{r^2}{2}}\;{}_1F_1\left(-n;\delta+\frac{3}{2};\,r^2\right). \tag{4.40}$$

4.3 KG - Equation with Morse Potential

The Morse potential is one of the most useful models to describe the interaction between two atoms in a diatomic molecule. Its vector and scalar potentials have the form of [42]

$$V(r) = A\,(e^{-2\alpha z} - 2e^{-\alpha z}), \text{ and } S(r) = B\,(e^{-2\alpha z} - 2e^{-\alpha z}) \tag{4.41}$$

with $z = [\frac{r-r_e}{r_e}]$ and $\alpha = a\,r_e$. Here, (A, B) and α denote dissociation energies and Morse parameter, respectively. r_e is the equilibrium distance (bound length) and a is a parameter to control the width of the potential. Due to the same reasons as stated in the previous applications, by taking equal case of potentials, the corresponding KG-equation reads

$$\left\{\frac{d^2}{dr^2} + \left[\,E_{n,l} - \left(A\,(e^{-2\alpha z} - 2e^{-\alpha z})\right)\right]^2 - \left[M + \left(A\,(e^{-2\alpha z} - 2e^{-\alpha z})\right)\right]^2 - \frac{(D+2l-1)\,(D+2l-3)}{4r^2}\right\} U_n(r) = 0. \tag{4.42}$$

It's well known that the KG-equation in D-dimension cannot be solved exactly for this potential for nonzero centrifugal term

$$(D + 2l - 1)(D + 2l - 3) \neq 0.$$

Therefore, an approximation has to be made to overcome this problem and transform it to some form like zero centrifugal term. There has been several approximations in literature that suggested to pass this problem. Here, the Pekeris approximation is used as a most convenient and powerful approximations among all approximations for this potential case. This approximation is based on the expansion of the centrifugal barrier in terms of exponential series form relying on the intermolecular distance, keeping terms up to second order. It should be noted that this approximation is valid only for low vibrational energy states. Thus, from coordinate relations (z $= [\frac{r-r_e}{r_e}]$) the centrifugal term is expanded in a series around $z = 0$,

$$\frac{\gamma}{(1+z)^2} = \gamma(1 - 2z + 3z^2 - 4z^3 + \cdots) \tag{4.43}$$

where ($\gamma = \frac{(D+2l-1)(D+2l-3)}{4{r_e}^2}$). Proceeding toward the second order degrees in upper series expansion and writing them in terms of exponential yield

$$\frac{\gamma}{(1+z)^2} = \gamma(c_o + c_1 e^{-\alpha z} + c_2 e^{-2\alpha z}). \tag{4.44}$$

In order to find out the constants ($c_o, c_1,$ and c_2), Eq. (4.44) also has to expand in a series of z, which it yields

$$\frac{\gamma}{(1+z)^2} = \gamma\left[c_o + c_1 + c_2 - (c_1 + 2c_2)\alpha z + \left(\frac{c_1}{2} + 2c_2\right)\alpha^2 z^2 + \cdots\right]. \tag{4.45}$$

Looking to the equal power of Eq. (4.43) and Eq. (4.45), the constants can be carried out as

$$c_o = 1 - \frac{3}{\alpha} + \frac{3}{\alpha^2}\ ,\ c_1 = \frac{4}{\alpha} - \frac{6}{\alpha^2}\ ,\ c_2 = -\frac{1}{\alpha} + \frac{3}{\alpha^2}.$$

Thus, instead of solving the radial KG-equation in arbitrary dimension D with Morse potential as in Eq. (4.42). The Eq. (4.45) can be expanded and transformed to the form including Pekeris approximation. Thus, with the aid of Eq. (4.44), Eq. (4.42) takes the new form as

$$\left\{\frac{d^2}{dz^2} + [\ (\ E_{n,l}^2 - M^2) - 2A\left(E_{n,l} + M\right)(\ e^{-2\alpha z} - 2e^{-\alpha z}\ ) \right.$$
$$\left. -\gamma(c_o + c_1 e^{-\alpha z} + c_2 e^{-2\alpha z}\ ]\right\} U_{n,l}(z) = 0. \qquad (4.46)$$

Above equation can be simplified further using some algebraic steps and some *ansatzs* then it will gives the form of

$$\left\{\frac{d^2}{dz^2} + \xi^2 + \beta^2\, e^{-\alpha z}\ - \mu^2\, e^{-2\alpha z}\right\} U_{n,l}(z) = 0 \qquad (4.47)$$

with

$$\xi^2 = (E_{n,l}^2 - M^2)\gamma c_o,$$

$$\beta^2 = 2A\left(E_{n,l} + M\right) - \gamma c_1 \qquad (4.48)$$

and

$$\mu^2 = 2A\left(E_{n,l} + M\right) + \gamma c_2.$$

Defining new variable in the form

$$y = e^{-\alpha z}. \tag{4.49}$$

and applying the well-known method for changing variables Chain Rule on the upper variable transformation, it yields

$$\frac{d^2 U_{n,l}(z)}{dz^2} = \alpha^2 y^2 \frac{d^2 U_{n,l}(y)}{dy^2} + \alpha^2 y \frac{dU_{n,l}(y)}{dy}. \tag{4.50}$$

Substitution of Eq. (4.49) and Eq. (4.50) in to Eq. (4.47), gives

$$\frac{d^2 U_{n,l}(y)}{dy^2} + \frac{1}{y}\frac{dU_{n,l}(y)}{dy} + \left\{\frac{\xi^2}{\alpha^2 y^2} + \frac{\beta^2}{\alpha^2 y} - \frac{\mu^2}{\alpha^2}\right\} U_{n,l}(y) = 0. \tag{4.51}$$

The reasonable physical eigenvector for this case is suggested as follows

$$U_{n,l}(y) = y^{\frac{\xi}{\alpha}} e^{-\frac{\mu}{\alpha}y} \; R_{n,l}(y). \tag{4.52}$$

After substituting this wavefunction generator in to the Eq. (4.50), Eq. (4.50) reduces to the second order homogeneous linear differential equation in the following form

$$\frac{d^2 R_{n,l}(y)}{dy^2} - \left\{\frac{2\mu\,\alpha y - 2\xi\alpha - \alpha^2}{y\alpha^2}\right\}\frac{dR_{n,l}(y)}{dy} - \left\{\frac{2\xi\mu + \alpha\mu - \beta^2}{y\alpha^2}\right\} R_{n,l}(y) = 0 \tag{4.53}$$

or, it can be written as

$$\frac{d^2R_{n,l}(y)}{dy^2} = \left\{\frac{2\mu\,\alpha y-2\xi\alpha-\alpha^2}{y\alpha^2}\right\}\frac{dR_{n,l}(y)}{dy} + \left\{\frac{2\xi\mu+\alpha\mu-\beta^2}{y\alpha^2}\right\}R_{n,l}(y). \tag{4.54}$$

By comparing the final second order differential equation with Eq. (3.1), it can be seen that the above equation is now amenable to an AIM solution. Thus, in the next sections all requirements for reaching the exact eigenvalue and eigenvector is calculated.

4.3.1 Solving for Eigenvalues

Eq. (4.54) gives the following results

$$\lambda_o(y) = \frac{2\mu\,\alpha\,y-2\xi\alpha-\alpha^2}{y\alpha^2}$$

$$s_o(y) = \frac{2\xi\mu+\alpha\mu-\beta^2}{y\alpha^2}.$$

With the help of Eq. (3.6), the following calculation can be done for finding $\lambda_k(y)$ and $s_k(y)$ with the results

$$\lambda_1(y) = -\frac{\beta^2}{y\alpha^2} + \frac{3\mu}{y\alpha} + \frac{2\xi\mu}{y\alpha^2} - \frac{-\alpha^2-2\alpha\xi+2y\alpha\mu}{y^2\alpha^2} + \frac{(-\alpha^2-2\alpha\xi+2y\alpha\mu)^2}{y^2\alpha^4}$$

$$s_1(y) = \frac{\beta^2}{y^2\alpha^2} - \frac{\mu}{y^2\alpha} - \frac{2\xi\mu}{y^2\alpha^2} + \frac{(-\alpha^2-2\alpha\xi+2y\alpha\mu)(-\frac{\beta^2}{y\alpha^2}+\frac{\mu}{y\alpha}+\frac{2\xi\mu}{y\alpha^2})}{y\alpha^2}$$

$$\lambda_2(y) = \frac{2\beta_1^2}{y^2\alpha^2} - \frac{6\beta_2}{y^2\alpha} - \frac{4\xi\beta_2}{y^2\alpha^2} + \frac{2(-\alpha^2-2\alpha\xi+2y\alpha\beta_2)}{y^3\alpha^2} + \frac{4\beta_2(-\alpha^2-2\alpha\xi+2y\alpha\beta_2)}{y^2\alpha^3} -$$
$$\frac{2(-\alpha^2-2\alpha\xi+2y\alpha\beta_2)^2}{y^3\alpha^4} + \frac{(-\alpha^2-2\alpha\xi+2y\alpha\beta_2)\left(-\frac{\beta_1^2}{y\alpha^2}+\frac{\beta_2}{y\alpha}+\frac{2\xi\beta_2}{y\alpha^2}\right)}{y\alpha^2} +$$
$$\frac{(-\alpha^2-2\alpha\xi+2y\alpha\beta_2)\left(-\frac{\beta_1^2}{y\alpha^2}+\frac{3\beta_2}{y\alpha}+\frac{2\xi\beta_2}{y\alpha^2}-\frac{-\alpha^2-2\alpha\xi+2y\alpha\beta_2}{y^2\alpha^2}+\frac{(-\alpha^2-2\alpha\xi+2y\alpha\beta_2)^2}{y^2\alpha^4}\right)}{y\alpha^2}$$

$$s_2(y) = -\frac{2\beta_1^2}{y^3\alpha^2} + \frac{2\beta_2}{y^3\alpha} + \frac{4\xi\beta_2}{y^3\alpha^2} + \frac{(-\alpha^2-2\alpha\xi+2y\alpha\beta_2)(\frac{\beta_1^2}{y^2\alpha^2}-\frac{\beta_2}{y^2\alpha}-\frac{2\xi\beta_2}{y^2\alpha^2})}{y\alpha^2} +$$
$$\frac{2\beta_2(-\frac{\beta_1^2}{y\alpha^2}+\frac{\beta_2}{y\alpha}+\frac{2\xi\beta_2}{y\alpha^2})}{y\alpha} - \frac{(-\alpha^2-2\alpha\xi+2y\alpha\beta_2)(-\frac{\beta_1^2}{y\alpha^2}+\frac{\beta_2}{y\alpha}+\frac{2\xi\beta_2}{y\alpha^2})}{y^2\alpha^2} + (-\frac{\beta_1^2}{y\alpha^2} + \frac{\beta_2}{y\alpha} + \frac{2\xi\beta_2}{y\alpha^2})(-\frac{\beta_1^2}{y\alpha^2} +$$
$$\frac{3\beta_2}{y\alpha} + \frac{2\xi\beta_2}{y\alpha^2} - \frac{-\alpha^2-2\alpha\xi+2y\alpha\beta_2}{y^2\alpha^2} + \frac{(-\alpha^2-2\alpha\xi+2y\alpha\beta_2)^2}{y^2\alpha^4})$$

etc…

Therefore applying the quantization condition given in Eq. (3.36), gives the following eigenvalues expression

$$\frac{\lambda_o(r)}{s_o(r)} = \frac{\lambda_1(r)}{s_1(r)} \Rightarrow \xi_{o,l} = \frac{\beta_1 - \alpha\mu}{2\mu}$$

$$\frac{\lambda_1(r)}{s_1(r)} = \frac{\lambda_2(r)}{s_2(r)} \Rightarrow \xi_{1,l} = \frac{\beta^2 - 3\alpha\mu}{2\mu}$$

$$\frac{\lambda_2(r)}{s_2(r)} = \frac{\lambda_3(y)}{s_3(y)} \Rightarrow \xi_{2,l} = \frac{\beta^2 - 5\alpha\mu}{2\mu}.$$

Generating the eigenvalues of the corresponding potential gives the general form of $\xi_{n,l}$ as

$$\xi_{n,l} = \frac{\beta^2 - (2n+1)\alpha\mu}{2\mu}. \tag{4.55}$$

These energy values are agree with the results that obtained in [67].

4.3.2 Solving for Eigenvector

After computing eigenvalues, and substituting them in to Eq. (3.14), the following eigenvector generators reads

$$R_{0,l}(y) = 1.$$

$$R_{1,l}(y) = (2\alpha\mu - \beta^2)\left(1 - \frac{2\mu y}{\alpha\left(\frac{\beta^2 - 3\alpha\mu}{\alpha\mu} + 1\right)}\right)$$

$$R_{2,l}(y) = (\beta^2 - 4\alpha\mu)(\beta^2 - 3\alpha\mu)\left(1 - \frac{4\mu y}{\alpha\left(\frac{\beta^2 - 5\alpha\mu}{\alpha\mu} + 1\right)} + \frac{2\mu^2 y^2}{\alpha^2\left(\frac{\beta^2 - 5\alpha\mu}{\alpha\mu} + 1\right)\left(\frac{\beta^2 - 5\alpha\mu}{\alpha\mu} + 2\right)}\right)$$

$$R_{3,l}(y) = (-4\alpha\mu + \beta^2)(\beta^2 - 5\alpha\mu)(\beta^2 - 6\,\alpha\mu)\ \left(1 - \frac{6\mu y}{\alpha\left(\frac{\beta^2 - 7\alpha\mu}{\alpha\mu} + 1\right)}\right.$$

$$\left. + \frac{12\,\mu^2 y^2}{\alpha^2\left(\frac{\beta^2 - 7\alpha\mu}{\alpha\mu} + 1\right)\left(\frac{\beta^2 - 7\alpha\mu}{\alpha\mu} + 2\right)} - \frac{8\,\mu^3 y^3}{\alpha^3\left(\frac{\beta^2 - 7\alpha\mu}{\alpha\mu} + 1\right)\left(\frac{\beta^2 - 7\alpha\mu}{\alpha\mu} + 2\right)\left(\frac{\beta^2 - 7\alpha\mu}{\alpha\mu} + 3\right)}\right)$$

etc…

The general solution of Eq. (4.52) using series representations becomes

$$R_{n,l}(y) = (-1)^n \left(\prod_{m=0}^{m=n} (\beta^2 - (m+1)\alpha\mu \right) \ {}_1F_1\left(-n, \frac{2\xi_{n,l}}{\alpha} + 1; \frac{2\mu y}{\alpha}\right). \quad (4.56)$$

Thus, the total radial unnormalized eigenvector can be determined by substituting Eq. (4.56) into Eq. (4.52) as follows

$$U_{n,l}(y) = (-1)^n \left(\prod_{m=0}^{m=n} (\beta^2 - (m+1)\alpha\mu \right) y^{\frac{\xi}{\alpha}} \, e^{-\frac{\mu}{\alpha}y} \ {}_1F_1\left(-n, \frac{2\xi}{\alpha} + 1; \frac{2\mu y}{\alpha}\right). (4.57)$$

The wavefunction van be written in the following form by using the definition of Laguerre polynomials

$$U_{n,l}(y) = C_n \, y^{\frac{\xi}{\alpha}} \, e^{-\frac{\mu}{\alpha}y} \, L_n^{\frac{2\xi_{n,l}}{\alpha}}\left(\frac{2\mu y}{\alpha}\right). \quad (4.58)$$

This eigenvector is close to the result that has been obtained in [67].

4.3.3 Normalization Constant (C_n)

C_n can be computed by applying Eq. (4.19) to the Eq. (4.58), then

$$C_n^2 \int_0^1 y^{\frac{2\xi}{\alpha}} \, e^{-\frac{2\mu}{\alpha}y} \left[L_n^{\frac{2\xi_{n,l}}{\alpha}}\left(\frac{2\mu y}{\alpha}\right)\right]^2 = 1 \quad (4.59)$$

$$C_n^2 \, (n!)^2 \, \left(\frac{2\mu}{\alpha}\right)^{\frac{1}{\varepsilon+1}} \frac{n!}{(n-\varepsilon)!} = 1 \quad (4.60)$$

Solving for normalization constant gives

$$C_n = \frac{1}{n!} \left(\frac{2\mu}{\alpha}\right)^{\frac{2\xi_{n,l}+\alpha}{2\alpha}} \sqrt{\frac{(n-\frac{2\xi_{n,l}}{\alpha})!}{n!}}. \tag{4.61}$$

By substituting Eq. (4.61) in to Eq. (4.58), $U_{n,l}(y)$ is obtained as

$$U_{n,l}(y) = \frac{1}{n!} \left(\frac{2\mu}{\alpha}\right)^{\frac{2\xi_{n,l}+\alpha}{2\alpha}} \sqrt{\frac{(n-\frac{2\xi_{n,l}}{\alpha})!}{n!}}\, y^{\frac{\xi}{\alpha}} e^{-\frac{\mu}{\alpha}y} L_n^{\frac{2\xi_{n,l}}{\alpha}} \left(\frac{2\mu y}{\alpha}\right). \tag{4.62}$$

4.4 KG - Equation with Woods – Saxon Potential

The potentials $V(r)$ and $S(r)$ for the standard form of Woods-Saxon potential [43] are defined as

$$V(r) = V_o \frac{e^{-\alpha r}}{1+e^{-\alpha r}}, \text{ and } \qquad S(r) = S_o \frac{e^{-\alpha r}}{1+e^{-\alpha r}} \tag{4.63}$$

For this potential, equal and unequal cases of Woods-Saxon potential are considered. Because of that for both cases, the KG-equation reads different energy spectrum.

4.4.1 Equal Vector and Scalar Potential ($V(r) = S(r)$) Case

Starting with the case of equal vector and scalar potentials and substituting them into D-dimensional KG-equation as stated in Sec. (2.1.2) with Eq. (2.21), yield

$$\left\{\frac{d^2}{dr^2} + (E_n - \frac{V_o e^{-\alpha r}}{1+e^{-\alpha r}})^2 - \left(M + \frac{V_o e^{-\alpha r}}{1+e^{-\alpha r}}\right)^2 - \frac{(K-1)(K-3)}{4r^2}\right\} U_n(r) = 0 \tag{4. 64}$$

where $k = D + 2l$ and both V_o and S_o are the depth of the vector and scalar Woods-Saxon potential, respectively. And $U_n(r)$ is the reduced radial wave function and it's satisfying the boundary conditions

$$U_n(\mathrm{r}=0) = 0 , \quad \text{and} \quad U_n(\mathrm{r}=\infty) = 0. \tag{4.65}$$

After some simplification, Eq. (4.63) reduces to

$$\left\{-\frac{d^2}{dr^2} - \frac{2(M+E_n)V_o\ e^{-\alpha r}}{1+e^{-\alpha r}} + \frac{(K-1)(K-3)}{4r^2}\right\} U_n(r) = \left(E_n{}^2 - M^2\right) U_n(r). \tag{4.66}$$

In order to obtain analytic solutions of this equation, there has been suggested an approximation as [25, 26]. In relativistic case same approximation can be used in non-relativistic cases. The centrifugal term that was used here, is defined as an appropriate solution to overcome this problem as in [25, 26, 27, and 28]. Thus, approximation of centrifugal term ($\frac{1}{r^2}$) is a main idea to pass this difficulty in a way that,

$$\frac{1}{r^2} \approx \frac{\alpha^2 e^{-\alpha r}}{(1-e^{-\alpha r})^2}. \tag{4.67}$$

Substituting above approximation in to Eq. (4.66) and after some mathematical steps Eq. (4.66) becomes

$$\left\{-\frac{d^2}{dr^2} - \frac{2(M+E_n)V_o\ e^{-\alpha r}}{1+e^{-\alpha r}} + \frac{\alpha^2(K-1)(K-3)\ e^{-\alpha r}}{4(1-e^{-\alpha r})^2}\right\} U_n(r) = \left(E_n{}^2 - M^2\right) U_n(r). \tag{4.68}$$

Eq. (4.68) can be further simplified using a new variable $y = e^{-\alpha r}$ (r ∈ [0,∞), y ∈ [1, 0)). Then by applying the most known method that was stated in every Calculus books

which is Chain Rule, the corresponding equation reduces to

$$\left\{\frac{d^2}{dy^2} + \frac{1}{y}\frac{d}{dy} - \frac{\xi^2}{y^2} + \frac{\eta}{y(1-y)} - \frac{\delta(\delta-1)}{y(1-y)^2}\right\} U_n(\mathrm{y}) = 0 \tag{4.69}$$

where the dimensionless parameters are

$$\xi^2 = \frac{(E_n{}^2 - M^2)}{\alpha}, \quad \eta = \frac{2(E_n+M)V_o}{\alpha^2}, \quad \text{and } \delta = \frac{3-k}{2} \; or \; \delta = \frac{1}{2}(-1+k). \tag{4.70}$$

Boundary conditions in Eq. (4.65) lead to look for a solution of Eq. (4.69) in the form of

$$U_n(y)) = \mathrm{y}^{\xi}(1 - \mathrm{y})^{\delta} R_n(y). \tag{4.71}$$

By substituting Eq. (4.71) in to Eq. (4.69), in this case Eq. (4.69) transforms to

$$R''_n(y) - \left(\frac{1+2\xi}{y} - \frac{2\delta}{1-y}\right) R'_n(y) - \left(\frac{-\delta^2-2\delta\xi+\eta}{(1-y)y}\right) R_n(y) = 0. \tag{4.72}$$

and also it can be written as

$$R''_n(y) = \left(\frac{1+2\xi}{y} - \frac{2\delta}{1-y}\right) R'_n(y) + \left(\frac{-\delta^2 - 2\delta\xi + \eta}{(1-y)y}\right) R_n(y) \qquad (4.73)$$

Eventually, the second order homogeneous linear differential equation is obtained. Now, Eq. (4.73) is absolutely ready to obey all requirements that they are necessary to calculate eigenvalue and eigenvector with the aid of AIM as in the following sections.

4.4.1.1 Solving for Eigenvalues

$\lambda_o(y)$ and $s_o(y)$ are determined by using of AIM for obtaining eigenvalues

$$\lambda_o(y) = \left(\frac{1+2\xi}{y} - \frac{2\delta}{1-y}\right),$$

$$s_o(y) = \left(\frac{-\delta^2 - 2\delta\xi + \eta}{(1-y)y}\right).$$

Hence, just like the steps that done in Sec. (3.3) Eq. (3.6-a) and Eq. (3.6-b), the high order functions for *k*-values can be get for $\lambda_k(y)$ and s_k(y) by taking derivatives, then it gives

$$\lambda_1(y) = -\frac{\delta^2}{(-1+y)y} + \frac{-1-2\delta-2\xi}{(-1+y)y} - \frac{2\delta\xi}{(-1+y)y} - \frac{1-y-2y\delta+2\xi-2y\xi}{(-1+y)y^2} - \frac{1-y-2y\delta+2\xi-2y\xi}{(-1+y)^2 y} + \frac{(1-y-2y\delta+2\xi-2y\xi)^2}{(-1+y)^2 y^2} + \frac{\eta}{(-1+y)y}$$

$$s_1(y) = \frac{\delta^2}{(-1+y)y^2} + \frac{\delta^2}{(-1+y)^2 y} + \frac{2\delta\xi}{(-1+y)y^2} + \frac{2\delta\xi}{(-1+y)^2 y} - \frac{\eta}{(-1+y)y^2} - \frac{\eta}{(-1+y)^2 y} + \frac{(1-y-2y\delta+2\xi-2y\xi)\left(-\frac{\delta^2}{(-1+y)y} - \frac{2\delta\xi}{(-1+y)y} + \frac{\eta}{(-1+y)y}\right)}{(-1+y)y}$$

…etc.

Combining the above functions in terms of termination conditions Eq. (3.36) and solving for ξ for each value of *n*, it can be found that

$$\frac{\lambda_o(y)}{s_o(y)} = \frac{\lambda_1(y)}{s_1(y)} \Rightarrow \xi_0^{(K)} = \frac{-\delta^2+\eta}{2\delta}$$

$$\frac{\lambda_1(y)}{s_1(y)} = \frac{\lambda_2(y)}{s_2(y)} \Rightarrow \xi_1^{(K)} = \frac{-1-2\delta-\delta^2+\eta}{2(1+\delta)}$$

$$\frac{\lambda_2(y)}{s_2(y)} = \frac{\lambda_3(y)}{s_3(y)} \Rightarrow \xi_2^{(K)} = \frac{-4-4\delta-\delta^2+\eta}{2(2+\delta)}$$

$$\frac{\lambda_3(y)}{s_3(y)} = \frac{\lambda_4(y)}{s_4(y)} \Rightarrow \xi_3^{(K)} = \frac{-9-6\delta-\delta^2+\eta}{2(3+\delta)}.$$

And the general form of $\xi_{n,l}$ is

$$\xi_{n,l} = \frac{-n^2-2n\delta-\delta^2+\eta}{2(n+\delta)}, \qquad n = 0, 1, 2, \ldots \tag{4.74}$$

Again, the energy calculation can proceed to find exact energy E_n by substituting values of dimensionless parameters ξ and η form Eq. (4.70) into Eq. (4.74) and solving for energy eigenvalue E_n, and after some simplification and mathematical steps the last result of energy spectrum is determined as follows

$$E_{n,l} = \frac{\mathrm{Vo} \pm \sqrt{\mathrm{Vo}^2 + 4M\,\mathrm{Vo}\,\delta + 4M^2\delta^2 - 2\delta^3}}{2\delta}. \tag{4.75}$$

This energy values obeys the results of [18].

4.4.1.2 Solving for Eigenvector

To find appropriate eigenvector, the steps that presented in Sec. (3.4) have been processed. Furthermore, Eq. (4.73) is in the suitable form with Eq. (3.37) and its differential equation of hypergeometric function ${}_2F_1$ with well-known solution as stated in Eq. (3.43). Thus, comparison of Eq. (4.73) with Eq. (3.4) gives that $b = 1$ and $N = -1$. Ultimately, the general unnormalized wanefunction gets as follows

$$R_n(y) = (-1)^n C_n (\sigma)_n \; {}_2F_1(-n, \rho + n; \sigma, y) \tag{4.76}$$

with dimensionless parameters

$$\sigma = 1 + 2\xi_{n,l}\,, \quad (\sigma)_n = \frac{\Gamma(2\xi_{n,l}+1+\mathrm{n})}{\Gamma(2\xi_{n,l}+1)}, \quad \text{and} \quad \rho = 2\xi_{n,l} + 2\,\delta.$$

In addition, Eq.(4.76) can be written without dimensionless parameter as

$$R_n(y) = (-1)^n\, C_n \left(1 + 2\xi_{n,l}\right)_n \, {}_2F_1\left(-n, 2\xi_{n,l} + 2\delta + n; 1 + 2\xi_{n,l}, y\right). \quad (4.77)$$

Using $R_n(y)$, the function $U_n(y)$ can be written as

$$U_n(y) = C_n y^{\xi_{n,l}} (1-y)^{\delta} \left(1 + 2\xi_{n,l}\right)_n \, {}_2F_1\left(-n, 2\xi_{n,l} + 2\,\delta + n; 1 + 2\xi_n^{(K)}, y\right) (4.78)$$

And $U_n(y)$ function can be written in terms of Jacobi polynomials as follows

$$U_n(y) = C_n y^{\xi_{n,l}} (1-\ y)^{\delta}\, P_n^{(2\xi_{n,l}\,,2\,\delta+1)}\,(1-2y) \quad (4.79)$$

where, the Jacobi polynomials $P_n^{(\alpha,\beta)}(y)$ has a deffiniton as

$$P_n^{(\alpha,\beta)}(y) = \frac{\Gamma(n+1+\alpha)}{n!\Gamma(1+\alpha)}\ {}_2F_1\left(-n, \alpha +\ \beta + 1 + n; 1 + \alpha, \frac{1-y}{2}\right). \quad (4.80)$$

Eq. (4.79) is close to the results of that obtained in [18].

4.4.1.3 Normalization Constant (C_n)

C_n is computed in the following way

$$\int_0^{\infty} |U(r)|^2 dr = -\int_1^0 |U(y)|^2 \frac{dy}{\alpha y} = 1. \qquad (4.81)$$

Thus, the determination of the normalization constant it will be done here in detail with the aid of method that used in [29]. By using Eq. (4.78) because of substitution $y = e^{-\alpha r}$and placing it in the above expression for computing normalization constant give

$$C_n^2 \int_0^1 y^{2\xi_{n,l}-1} (1 - y)^{2\delta} \left[{}_2F_1(-n, 2\xi_{n,l} + 2\,\delta + n;\ 1 + 2\xi_{n,l}, y)\right]^2 = \alpha. \qquad (4.82)$$

Using the series representation Eq. (3.44) of the hypergeometric function ${}_2F_1$, being a polynomial of degree n in y, then Eq. (4.82) transforms to

$$C_n^2 \ \sum_{i=0}^{n} \sum_{j=0}^{n} \frac{(-n)_i (\, 2\xi_{n,l}+2\,\delta+n)_i}{(1+2\xi_{n,l})_i \, i!} \frac{(-n)_j (\, 2\xi_{n,l}+2\,\delta+n)_j}{(1+2\xi_{n,l})_j \, j!} \int_0^1 y^{2\xi_{n,l}+i+j-1} (1 - y)^{2\delta} dy = \alpha \qquad (4.83)$$

The definite integral in Eq.(4.22) is just the integral representation of Incomplete Beta function [29],

$$B_q(x, y) = \int_0^q t^{x-1} (1 - t)^{y-1} \, dt,\ \Re(x) > 0, \Re(y) > 0. \qquad (4.84)$$

Therefore, Eq. (4.83) now reads

$$C_n^2 \ \sum_{i=0}^{n}\sum_{j=0}^{n}\frac{(-\mathrm{n})_i(\,2\xi_{n,l}+2\,\delta+n)_i}{(1+2\xi_{n,l})_i\; i!}\frac{(-\mathrm{n})_j(\,2\xi_{n,l}+2\,\delta+n)_j}{(1+2\xi_{n,l})_j\; j!}\, B_q(2\xi_{n,l}+i+j, 2\delta+1) = \alpha. \tag{4.85}$$

For $q = 1$ Incomplete Beta function B_q(x, y) reduces to Beta function B(x, y), then above equation becomes

$$C_n^2 \ \sum_{i=0}^{n}\sum_{j=0}^{n}\frac{(-\mathrm{n})_i(\,2\xi_{n,l}+2\,\delta+n)_i}{(1+2\xi_{n,l})_i\; i!}\frac{(-\mathrm{n})_j(\,2\xi_{n,l}+2\,\delta+n)_j}{(1+2\xi_{n,l})_j\; j!}\, B(2\xi_{n,l}+i+j, 2\delta+1) = \alpha. \tag{4.86}$$

By using the definition of Beta function [29] in terms of Gamma function

$$\mathrm{B(x, y)} = \frac{\Gamma(\mathrm{x})\,\Gamma(\mathrm{y})}{\Gamma(\mathrm{x+y})} \tag{4.87}$$

Eq. (4.86) can be written as

$$C_n^2 \ \sum_{i=0}^{n}\sum_{j=0}^{n}\frac{(-\mathrm{n})_i(\,2\xi_{n,l}+2\,\delta+n)_i}{(1+2\xi_{n,l})_i\; i!}\frac{(-\mathrm{n})_j(\,2\xi_{n,l}+2\,\delta+n)_j}{(1+2\xi_{n,l})_j\; j!}\,\frac{\Gamma\left(2\xi_{n,l}+i+j\right)\Gamma(2\delta+1)}{\Gamma(2\xi_{n,l}+i+j, 2\delta+1)} = \alpha. \tag{4.88}$$

Eq. (4.88) is simplified by the series representation of the hypergeometric function ${}_3F_2$

$$C_n^2 \ \sum_{i=0}^{n}\frac{(-\mathrm{n})_i(\,2\xi_n^{(K)}+2\,\delta+n)_i(2\xi_n^{(K)})_i}{(2\xi_n^{(K)}+2\,\delta+1)_i\; i!}\ {}_3F_2\left(\begin{matrix} -n & 2\xi_n^{(K)}+2\,\delta+n & 2\xi_n^{(K)}+i \\ 1+2\xi_n^{(K)} & 2\xi_n^{(K)}+i+2\,\delta+1 & \end{matrix};1\right)$$

$$= \frac{\alpha}{B(2\xi_n^{(K)}, 2\delta+1)}. \tag{4.89}$$

Now, Eq. (4.89) is ready to compute the normalization constant for $= 0, 1, 2, \ldots$. In particular, for the ground-state $n = 0$, it gives

$$C_o = \sqrt{\frac{\alpha}{B(2\xi_n^{(K)}, 2\delta+1)}}. \tag{4.90}$$

4.4.2 The Case of Unequal Vector and Scalar Potential ($\boldsymbol{V(r) \neq S(r)}$):

In this case by substituting both potentials which defined in Eq. (4.63) into Eq. (2.21), with this potential after some technical steps, it reduces to

$$\left\{-\frac{d^2}{dr^2} + \frac{(K-1)(K-3)}{4r^2} - \frac{(2MS_o + 2E_nV_o)e^{-\alpha r}}{1+e^{-\alpha r}} + \frac{(S_o{}^2 - V_o{}^2)\, e^{-2\alpha r}}{(1+e^{-\alpha r})^2}\right\} U_n(r) =$$

$$(E_n{}^2 - M^2)\, U_n(r). \tag{4.91}$$

With the centrifugal approximation the above equation becomes

$$\left\{-\frac{d^2}{dr^2} - \frac{(2MS_o + 2E_nV_o)e^{-\alpha r}}{1+e^{-\alpha r}} + \frac{(S_o{}^2 - V_o{}^2)\, e^{-2\alpha r}}{(1+e^{-\alpha r})^2} - \frac{\frac{\alpha^2}{4}(K-1)(K-3)e^{-\alpha r}}{(1-e^{-\alpha r})^2}\right\} U_n(r)$$

$$= (E_n{}^2 - M^2)\, U_n(r), \tag{4.92}$$

By defining a new variable $y = \mathrm{e}^{-\alpha\, \mathrm{r}}$, with some ansatzs above equation reads

$$\left\{\frac{d^2}{dy^2} + \frac{1}{y}\frac{d}{dy} - \frac{\xi^2}{y^2} + \frac{\eta}{y(1-y)} - \frac{\beta}{(1-y)^2} - \frac{\frac{1}{4}(K-1)(K-3)}{y(1-y)^2}\right\} U_n(\mathrm{y}) = 0 \tag{4.93}$$

with parameters

$$\xi_{n,l}{}^2 = \frac{(E_n{}^2 - M^2)}{\alpha}, \qquad \eta = \frac{2(MS_o + E_nV_o)}{\alpha^2} \quad \text{and} \quad \beta = \frac{S_o{}^2 - V_o{}^2}{\alpha^2}. \tag{4.94}$$

Boundary conditions in Eq. (4.65) leads to look for a solution of Eq. (4.93) in the form of

$$U_n(\mathrm{y}) = y^{\xi_{n,l}}(1 - y)^{\delta}\, R_n(\mathrm{y}) \tag{4.95}$$

By substituting Eq. (4.95) in to Eq. (4.93), in this case Eq. (4.7) changes to the form as follows

$$R''_n(\mathrm{y}) - (\frac{1+2\xi_{n,l}}{y} - \frac{2\delta}{1-y})R'_n(\mathrm{y}) - (\frac{((k-1)(k-3)+4(\delta(1+2\xi_{n,l})-\eta))}{4(1-y)y})\, R_n(\mathrm{y}) = 0 \tag{4.96}$$

where δ is chosen such that

$$\delta^2 - \delta - \beta - \frac{1}{4}(K-1)(K-3) = 0 \tag{4.97}$$

which yields

$$\delta = \frac{1}{2}\left(1 \pm \sqrt{-2+4k-k^2-4\beta}\right). \tag{4.98}$$

Then using this parameter, Eq. (4.96) is transformed to

$$R''_n(\mathrm{y}) = \left(\frac{1+2\xi_n}{y} - \frac{2\delta}{1-y}\right) R'_n(\mathrm{y}) + (\frac{((k-1)(k-3)+4(\delta(1+2\xi_{n,l})-\eta))}{4(1-y)y})\, R_n(\mathrm{y}) = 0. \tag{4.99}$$

At the end, the second order homogenous linear differential equation is obtained. Thus, AIM now can be applied as later sections.

4.4.2.1 Solving for Eigenvalues

Comparison of Eq. (4.99) with Eq. (3.1) gives

$$\lambda_o(y) = \left[\frac{1+2\xi_n}{y} - \frac{2\delta}{1-y}\right]$$

$$\mathrm{s_o}(\mathrm{y}) = [\frac{((k-1)(k-3)+4(\delta(1+2\xi_{n,l})-\eta))}{4(1-y)y}].$$

and the $\lambda_k(y)$ and $\mathrm{s}_k(\mathrm{y})$ values are

$$\lambda_1(y) = \frac{1}{4(-1+y)^2y^2}[\,8(1+3\xi+2\xi^2) + y(-13-4k+k^2-4\eta-48\xi-32\xi^2-12\delta(1+2\xi)) + y^2(5+4k-k^2+16\delta^2+4\eta+24\xi+16\xi^2+4\delta(5+6\xi)))]$$

$$s_1(y) = \frac{(-2(1+\xi)+y(3+2\delta+2\xi))(3-4k+k^2-4\eta+\delta(4+8\xi))}{4(-1+y)^2y^2}$$

$$\lambda_2(y) = \frac{1}{2(-1+y)^3y^3}\Big[4(1+\xi)(1+2\xi)(3+2\xi) - y^3(3+2\delta+2\xi)\big(1-(-4+k)k+4\eta+12\xi+8(\delta+\delta^2+\delta\xi+\xi^2)\big) - 2y(1+\xi)\Big(-(-4+k)k+4\eta+8\delta(1+2\xi)+3\big(5+8\xi(2+\xi)\big)\Big) + y^2\Big(21+34\delta+20\eta+120\xi+4k(5+2\delta+4\xi)-k^2(5+2\delta+4\xi)+8\Big(\delta^2(2+4\xi)+2\xi\big(\eta+3\xi(3+\xi)\big)+\delta\big(\eta+2\xi(7+4\xi)\big)\Big)\Big)\Big]$$

$$s_2(y) = \frac{1}{16(-1+y)^3y^3}[y(57-4k(-1+y)+k^2(-1+y)+20\delta+4\eta-y(41+4\delta(11+4\delta)+4\eta))+8y(11+3\delta-3y(2+\delta))\xi-16(-1+y)^2\xi^2-8(3+5\xi))(3+(-4+k)k-4\eta+\delta(4+8\xi)]$$

…etc.

The energy values for each n can be obtained as

$$\frac{\lambda_o(y)}{s_o(y)} = \frac{\lambda_1(y)}{s_1(y)} \Rightarrow \xi_0^{(K)} = \frac{-3+4k-k^2-4\delta+4\eta}{8\delta}$$

$$\frac{\lambda_1(y)}{s_1(y)} = \frac{\lambda_2(y)}{s_2(y)} \Rightarrow \xi_1^{(K)} = \frac{-7+4k-k^2-12\delta+4\eta}{8(1+\delta)}$$

$$\frac{\lambda_2(y)}{s_2(y)} = \frac{\lambda_3(y)}{s_3(y)} \Rightarrow \xi_2^{(K)} = \frac{-19+4k-k^2-20\delta+4\eta}{8(2+\delta)}$$

$$\frac{\lambda_3(y)}{s_3(y)} = \frac{\lambda_4(y)}{s_4(y)} \Rightarrow \xi_3^{(K)} = \frac{-39+4k-k^2-28\delta+4\eta}{8(3+\delta)}.$$

Generalizing this energy eigenvalues gives

$$\xi_n^{(K)} = \frac{-(4n^2+3)+k(4-k)-(8n+4)\delta+4\eta}{8(n+\delta)} \ , \qquad n = 0, 1, 2, \ldots \qquad (4.100)$$

4.4.2.2 Solving for Eigenvector

The corresponding eigenvectors $R_n(y)$ and $U_n(y)$ are calculated with similar procedure

$$R_n(y) = (-1)^n \, C_n \big(1 + 2\xi_{n,l}\big)_n \, {}_2F_1(-n, 2\xi_{n,l} + 2\delta + n; 1 + 2\xi_{n,l}, y) \qquad (4.101)$$

and

$$U_n(y) = C_n \, y^{\xi_{n,l}} (1-y)^{\delta} \big(1 + 2\xi_{n,l}\big)_n \, {}_2F_1\big(-n, 2\xi_{n,l} + 2\delta + n; 1 + 2\xi_{n,l}, y\big) (4.102)$$

4.4.2.3 Normalization Constant ($\boldsymbol{C_n}$)

Applying the general expression that is stated for obtaining normalization constant Eq. (4.81). It can be found and it is in the same form with the normalization constant that found for the case of equal potentials. Thus, here C_n has the same expression as normalization constant which found in Sec. (4.4.1.3) and exactly Eq. (4.89). Therefore, C_n can be written as

$$C_n^2 \sum_{i=0}^{n} \frac{(-\mathrm{n})_i (\, 2\xi_n^{(K)} + 2\,\delta + n)_i \big(2\xi_n^{(K)}\big)_i}{(2\xi_n^{(K)} + 2\,\delta + 1)_i \, i!} \; {}_3F_2\left(\begin{matrix} -n & 2\xi_n^{(K)} + 2\,\delta + n & 2\xi_n^{(K)} + i \\ 1 + 2\xi_n^{(K)} & 2\xi_n^{(K)} + i + 2\,\delta + 1 & \end{matrix}; 1\right)$$

$$= \frac{\alpha}{B(2\xi_n^{(K)}, 2\delta+1)} \qquad (4.104)$$

CHAPTER 5

CONCLUSION

During this study, by considering an alternative method i.e. AIM, the solution of KG-equation with some different type of potentials is analyzed in arbitrary dimensions D. The energy eigenvalues and corresponding eigenfunctions for each potentials are calculated. Furthermore, the normalization constants also calculated in each case of potentials. Except for Woods-Saxon potential, in all applications of KG-equation in physical systems, the potentials $V(r)$ and $S(r)$ are considered equal to each other. In the unequal case, the same energy spectrum is generated with slightly different parameters. But in Woods-Saxon potential, the energy spectrum is determined for both of cases because of giving different eigenvalues and eigenfunctions.

The type of potentials are chosen according to their groups and type of solutions. First, Harmonic Oscillator potential is tested, and it is shown that this potential is exactly solvable and it gives good results for energy spectrum. Gol'dman and Krivchenkov is one of the other potentials which is exactly solvable in non-relativistic and relativistic cases. The corresponding energy spectrum for both Harmonic Oscillator and Goldman-Krivchenkov potentials showed that they are in a good agreement with the results of [66], where the KG-equation in arbitrary dimension has been solved n D-dimension by Nikiforov–Uvarov method.

Other type of potentials include two important potentials among the exponential type potentials, Morse and Woods-Saxon potential. The energy eigenvalues and corresponding eigenfunctions of the spin-0 particles with the equal vector and scalar Morse potential by using Pekeris approximation are presented within the framework of AIM. And the results of eigenvalue and eigenvector is close to the results of [67] which KG-equation in D dimension with Morse potential has been solved using different way.

Additionally, the radial KG-equation is solved for the standard form of the Woods-Saxon potential with any angular momentum and any dimension (arbitrary dimension). The AIM method has been used. It should be reminded that this potential in its standard form had never been solved independently, and especially in the case of the unequal scalar and vector potentials. Hence, this form of the potential is close to the generalized Hulthen potential and the q-deformed Woods-Saxon potential. Therefore, the energy spectrum and the eigenfunctions of this potential can be compared with the same form potentials as mentioned before. Thus, by comparison the results of eigenvalues and eigenfunctions is in a good agreement with the results of [18] where the KG-equation with the generalized Hulthen potential has been solved in the framework of Nikiforov–Uvarov method.

It is concluded that AIM is easy, powerful, and stable than other algebraic methods to give the exact results. Furthermore, the importance of Gauss hypergemetric function is shown in a way that all eigenfunctions that get by AIM is this type of function, or they can be reduced to the special type of hypergemetric functions. This method can be applied for wide range of physical potentials both for constant masses and spatially dependent masses as a future work. It is hoped that it will give an interesting results with respect to other numerical and algebraical methods.

REFERENCES

[1] Greiner, W. (1990). Relativistic quantum mechanics (Vol. 3). Berlin: Springer.

[2] Baym, G. A. (1969). Lectures on quantum mechanics. Benjamin.

[3] Nolinder, A., & Sandberg, E. (2014). The KG-equation and Pionic Atoms.

[4] Yang, L. (2006). Numerical studies of the Klein-Gordon-Schrödinger equations (Doctoral dissertation, MSc thesis submitted to NUS, Singapore).

[5] Strange, P. (1998). Relativistic Quantum Mechanics: with applications in condensed matter and atomic physics. Cambridge University Press.

[6] Schaffer, D. L. (2007). The magnetic vector potential, KG-equation and Klein's paradox in relativistic quantum mechanics (Doctoral dissertation, Wichita State University).

[7] Ohlsson, T. (2011). Relativistic Quantum Physics: From Advanced Quantum Mechanics to Introductory Quantum Field Theory. Cambridge University Press.

[8] Diao, Y. F., Yi, L. Z., & Jia, C. S. (2004). Bound states of the Klein–Gordon equation with vector and scalar five-parameter exponential-type potentials. *Physics Letters A*, ***332*(3)**, 157-167.

[9] Yi, L. Z., Diao, Y. F., Liu, J. Y., & Jia, C. S. (2004). Bound states of the Klein–Gordon equation with vector and scalar Rosen–Morse-type potentials. *Physics Letters A*, ***333(3)***, *212-217.*

[10] Agboola, D. (2008). Solutions to some Molecular Potentials in D-Dimensions: Asymptotic Iteration Method. *arXiv preprint arXiv:**0812**.3776.*

[11] Min-Cang, Z., & Zhen-Bang, W. (2005). Exact Solutions of the Klein–Gordon Equation with a New Anharmonic Oscillator Potential. *Chinese Physics Letters*, ***22(12)***, *2994*

[12] Ikhdair, S., & Sever, R. (2008). Exact solutions of the D-dimensional Schrödinger equation for a ring-shaped pseudoharmonic potential. *Open Physics, **6(3)**, 685-696.*

[13] Shojaei, M. R., & Rajabi, A. A. (2011). Determination of energy levels of the Klein–Gordon equation, with pseudoharmonic potential plus the ring shaped potential. *Int. Phys. Sci, **6**, 7441-7446.*

[14] Chabab, M., & Oulne, M. (2010). Exact solutions of Klein Gordon equation for the Makarov potential with the asymptotic iteration method. *arXiv preprint arXiv:**1003**.4927.*

[15] Soylu, A., Bayrak, O., & Boztosun, I. (2006). The energy eigenvalues of the two dimensional hydrogen atom in a magnetic field. *International Journal of Modern Physics E, **15(06)**, 1263-1271.*

[16] Ikhdair, S., & Sever, R. (2007). Exact solutions of the radial Schrödinger equation for some physical potentials. *Open Physics, **5(4)**, 516-527.*

[17] Hassanabadi, H., Zarrinkamar, S., & Rahimov, H. (2011). Approximate solution of D-dimensional Klein–Gordon equation with Hulthen-type potential via SUSYQM. Commun. *Theor. Phys, **56**, 423.*

[18] Chabab, M., Jourdani, R., & Oulne, M. (2012). Exact solutions for vibrational states with generalized q-deformed Morse potential within the asymptotic iteration method. *International Journal of Physical Sciences, **7(8)**, 1150-1154.*

[19] Ikhdair, S. M., & Sever, R. (2008). Approximate l-state solutions of the D-dimensional Schrödinger equation for Manning-Rosen potential. *Annalen der Physik, **17(11)**, 897-910.*

[20] Hamzavi, M., Movahedi, M., Thylwe, K. E., & Rajabi, A. A. (2012). Approximate analytical solution of the Yukawa potential with arbitrary angular momenta. *Chinese Physics Letters, **29(8)**, 080302.*

[21] Zou, X., Yi, L. Z., & Jia, C. S. (2005). Bound states of the Dirac equation with vector and scalar Eckart potentials. *Physics Letters A, **346(1)**, 54-64.*

[22] Ikhdair, S. M. (2011). Bound states of the KG-equation in D-dimensions with some physical scalar and vector exponential-type potentials including orbital centrifugal term. *arXiv preprint arXiv:**1110**.0943.*

[23] Akpan, I. O., Antia, A. D., & Ikot, A. N. (2012). Bound-State Solutions of the KG-equation with-Deformed Equal Scalar and Vector Eckart Potential Using a Newly Improved Approximation Scheme. *ISRN High Energy Physics, 2012.*

[24] Agboola, D. (2010). Solutions to the Modified Pöschl–Teller potential in D-dimensions. *Chinese Physics Letters, **27(4)**, 040301.*

[25] Chargui, Y., Chetouani, L., & Trabelsi, A. (2010). Exact Solution of D-Dimensional Klein–Gordon Oscillator with Minimal Length. *Communications in Theoretical Physics, **53(2)**, 231.*

[26] Ebaid, A. (2009). Exact solutions for the generalized Klein–Gordon equation via a transformation and Exp-function method and comparison with Adomian's method. *Journal of Computational and Applied Mathematics, **223(1)**, 278-290.*

[27] Hassanabadi, H., Rahimov, H., & Zarrinkamar, S. (2011). Approximate Solutions of KG-equation with Kratzer Potential. *Advances in High Energy Physics, 2011.*

[28] Bayrak, O., Soylu, A., & Boztosun, I. (2010). The relativistic treatment of spin-0 particles under the rotating Morse oscillator. *Journal of Mathematical Physics, **51(11)**, 112301.*

[29] Saad, N., Hall, R. L., & Ciftci, H. (2008). The KG-equation with the Kratzer potential in d dimensions. *Central European Journal of Physics, **6(3)**, 717-729.*

[30] Ikot, A. N., Awoga, O. A., Antia, A. D., Hassanabadi, H., & Maghsoodi, E. (2013). Approximate Solutions of D-Dimensional KG-equation with modified Hylleraas Potential. *Few-Body Systems, **54(11)**, 2041-2051.*

[31] Hassanabadi, H., Zarrinkamar, S., Hamzavi, H., & Rajabi, A. A. (2012). Exact solutions of D-dimensional Klein–Gordon equation with an energy-dependent

potential by using of Nikiforov–Uvarov method. *Arabian Journal for Science and Engineering*, ***37(1)***, *209-215.*

[32] Hamzavi, M., Ikhdair, S. M., & Thylwe, K. E. (2013). Spinless particles in the field of unequal scalar—vector Yukawa potentials. *Chinese Physics B,* ***22(4)***, *040301*.

[33] Falaye, B. J., Ikhdair, S. M., & Hamzavi, M. (2015). Formula method for bound state problems. *Few-Body Systems,* ***56(1)***, *63-78.*

[34] ARDA, A., & Sever, R. (2009). Approximate ℓ-State Solutions to the Klein–Gordon Equation for Modified Woods–Saxon Potential with Position Dependent Mass. *International Journal of Modern Physics A,* ***24(20n21)***, *3985-3994.*

[35] Qiang, W. C., & Dong, S. H. (2008). Analytical approximations to the l-wave solutions of the Klein–Gordon equation for a second Pöschl–Teller like potential. *Physics Letters A,* ***372(27)***, *4789-4792.*

[36] Ikhdair, S. M., & Sever, R. (2007). Elativistic treatment in D-Dimensions to a spin-zero particle with noncentral equal scalar and vector ring-shaped Kratzer potential. *arXiv preprint arXiv:****0704****.0573.*

[37] Ikhdair, S. M. (2012). Exact solution of Dirac equation with charged harmonic oscillator in electric field: bound states. *arXiv preprint arXiv:****1203****.2458.*

[38] Ciftci, H., Hall, R. L., & Saad, N. (2003). Asymptotic iteration method for eigenvalue problems. *Journal of Physics A: Mathematical and General,* ***36(47)***, *11807.*

[39] Mikulski, D., Eder, K., & Molski, M. (2014). The algebraic approach for the derivation of ladder operators and coherent states for the Goldman and Krivchenkov oscillator by the use of supersymmetric quantum mechanics. *Journal of Mathematical Chemistry,* ***52(6)***, *1610-1623.*

[40] Berkdemir, C., & Han, J. (2005). Rotational Correction on the Morse Potential through the Pekeris Approximation and Nikiforov-Uvarov Method. *arXiv preprint quant-ph/****0502182***.

[41] Ikot, A. N., Awoga, O. A., Hassanabadi, H., & Maghsoodi, E. (2014). Analytical Approximate Solution of Schrödinger Equation in D Dimensions with Quadratic Exponential-Type Potential for Arbitrary l-State. *Communications in Theoretical Physics, **61(4)**, 457.*

[42] Berkdemir, C., & Han, J. (2005). Any l-state solutions of the Morse potential through the Pekeris approximation and Nikiforov–Uvarov method. *Chemical physics letters, **409(4)**, 203-207.*

[43] Woods, R. D., & Saxon, D. S. (1954). Diffuse surface optical model for nucleon-nuclei scattering. *Physical Review, **95(2)**, 577.*

[44] Arda, A., Aydoğdu, O., & Sever, R. (2010). Scattering of the Woods–Saxon potential in the Schrödinger equation. *Journal of Physics A: Mathematical and Theoretical, **43(42)**, 425204.*

[45] Ikot, A. N., & Akpan, I. O. (2012). Bound State Solutions of the Schrödinger Equation for a More General Woods—Saxon Potential with Arbitrary l-State. *Chinese Physics Letters, **29(9)**, 090302.*

[46] Jian-You, G., Zheng, F. X., & Fu-Xin, X. (2002). Solution of the relativistic Dirac-Woods-Saxon problem. *Physical Review A, **66(6)**, 062105.*

[47] W. Strauss (1989). Nonlinear Wave Equations. *CBMS Volume **73**, American Math. Soc.*

[48] Taylor, J. R., Dubson, M. A., & Zafiratos, C. D. (2004). Modern physics for scientists and engineers. *Prentice-Hall*.

[49] Sulaiman, R. I. (2014). Klein-Gordeon equation in 1+ 1-Dimensions Doctoral dissertation, Eastern Mediterranean University (EMU)-Doğu Akdeniz Üniversitesi (DAÜ)).

[50] McMahon, D. (2008). Quantum Field Theory Demystified. *McGraw Hill Professional.*

[51] K. G. Elgaylani, M. D. Abd Allah, K. A. Haroun, M. H. Eisa and A. S. Al Amer (2014). Derivation of KG-equation from Maxwell's electric wave equation. *International Journal of Physical Sciences Vol.* ***2(2)****, pp. 015-020.*

[52] Chambers, L. L. (1966). Derivation of solutions of the KG-equation from solutions of the wave equation. *Proceedings of the Edinburgh Mathematical Society (Series 2),* ***15(02)****, 125-129.*

[53] Alhaidari, A.D., Bahlouli,H. , Al-Hasan,A. (2006). Dirac and KG-equations with equal scalar and vector potentials. *Physics Letters A,* ***349****, 87-97.*

[64] Ikot, A. N., Obong, H. P., Hassanabadi, H., Salehi, N., & Thomas, O. S. (2014). Solutions of D-dimensional Klein–Gordon equation for multiparameter exponential-type potential using supersymmtric quantum mechanics. *Indian Journal of Physics,* ***1-8.***

[55] Ciftci, H., Hall, R. L., & Saad, N. (2005). Construction of exact solutions to eigenvalue problems by the asymptotic iteration method. *Journal of Physics A: Mathematical and General,* ***38(5)****, 1147.*

[56] Chabab, M., Lahbas, A., & Oulne, M. (2012). Analytic l-state solutions of the KG-equation for q-deformed Woods-Saxon plus generalized ring shape potential. *arXiv preprint arXiv:****1203****.5039.*

[57] Ciftci, H., Hall, R. L., & Saad, N. (2013). Exact and approximate solutions of Schrödinger's equation for a class of trigonometric potentials. *Central European Journal of Physics,* ***11(1)****, 37-48.*

[58] Olgar, E. (2008). Exact Solution of KG-equation by Asymptotic Iteration Method. *Chinese Physics Letters,* ***25(6)****, 1939.*

[59] Bayrak, O., & Boztosun, I. (2007). Analytical solutions to the Hulthén and the Morse potentials by using the asymptotic iteration method. *Journal of Molecular Structure: Theo.chem.* ***802(1)****, 17-21.*

[60] Rostami, A., & Motavali, H. (2008). Asymptotic iteration method: A powerful approach for analysis of inhomogeneous dielectric slab waveguides. *Progress in Electromagnetics Research B, **4**, 171-182.*

[61] Fernández, F. M. (2004). On an iteration method for eigenvalue problems. *Journal of Physics A: Mathematical and General, **37(23)***, 6173.

[62] Uslu, H. (2008). Özel Bir Potansiyel Sinifi Için Schrödinger Denkleminin Asimptotik Iterasyon Metodu Ile Çözümü.

[63] Ciftci, H., Hall, R. L., & Saad, N. (2005). Iterative solutions to the Dirac equation. *Physical Review A, **72(2)**, 022101.*

[64] Bender, C. M., & Wang, Q. (2001). A class of exactly-solvable eigenvalue problems. *Journal of Physics A: Mathematical and General, **34(46)**, 9835.*

[65] Boztosun, I., Karakoc, M., Yasuk, F., & Durmus, A. (2006). Asymptotic iteration method solutions to the relativistic Duffin-Kemmer-Petiau equation. *Journal of mathematical physics, **47(6)**, 062301.*

[66] Ita, B. I., Obong, H. P., Ehi-Eromosele, C. O., Edobor-Osoh, A., & Ikeuba, A. I. (2014). Solutions of the Klein-Gordon equation with equal scalar and vector harmonic oscillator plus inverse quadratic potential.

[67] Xie, X. J., & Jia, C. S. (2015). Solutions of the Klein–Gordon equation with the Morse potential energy model in higher spatial dimensions. *Physica Scripta, **90(3)**, 035207.*