ON SOME APPLICATIONS OF SINGULAR TWO-POINT BOUNDARY VALUE PROBLEMS

by

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

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ABSTRACT

In this thesis, we study the one-dimensional Schrödinger operator

$$L = -\frac{d^2}{dx^2} + q(x), \qquad -\infty < x < \infty \tag{1}$$

in $L_2(-\infty,\infty)$ with a complex-valued, periodic ($q(x+2\pi) = q(x)$) and Lebesgue integrable potential q(x). First, the spectrum of this operator is investigated by using the operator L_t ($t \in [0,1]$) generated by expression (1) in $L_2[0,2\pi]$ and the quasiperiodic boundary conditions. Then, the asymptotic formulas for eigenfunctions and eigenvalues of the operator L_t is obtained. Finally, using these asymptotic formulas, the spectral expansion for the operator L is constructed.

ÖZET

Bu tezde, $L_2(-\infty,\infty)$ uzayında, kompleks değerli, periyodik $(q(x+2\pi)=q(x))$ ve Lebesgue integrallenebilir q(x) potansiyele sahip olan bir boyutlu Schrödinger operatörü

$$L=-\frac{d^2}{dx^2}+q(x), \qquad -\infty<x<\infty \qquad (1)$$

çalışılmıştır. Önce, bu operatörün spektrumu, $L_2[0,2\pi]$ de, (1) ifadesi ve quasiperiyodik sınır koşulları ile genelleştirilmiş L_t $(t\in[0,1])$ operatöründen yararlanılarak incelenmiştir. Daha sonra, L_t operatörünün özfoksiyonları ve özdeğerleri için asimptotik formüller elde edilmiştir. Son olarak, bu asimptotik formüller kullanılarak L operatorünün spektral açılımı oluşturulmuştur.

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CHAPTER ONE
INTRODUCTION

The Schrödinger operator with a periodic potential describes the motion of a particle in a bulk matter. Therefore, it is interesting to have a detailed analysis of the spectral properties of this operator. The one-dimensional Schrödinger operator

$$L=-\frac{d^2}{dx^2}+q(x) \tag{1.1}$$

with periodic real-valued potential q(x) was investigated originally by the famous mathematicians H. Poincare, M. A. Lyapunov and G. W. Hill. They investigated the stability and unstability zones of Hill's equation

$$-y''+q(x)y=\lambda y \tag{1.2}$$

(Titchmarsh, 1958), (Dunford,1963), (Eastham, 1973). Then, it was proved (Eastham, 1973) that for a real-valued q(x) the conditional stability zones of Hill's equation (1.2) coincides with the spectrum of the operator L generated by (1.1) in Hilbert space $L_2(-\infty,\infty)$. The spectral expansion related to the self-adjoint operator L is constructed by I. M. Gelfand (Gelfand, 1950) and E. C. Titchmarsh (Titchmarsh,1958). They proved that the spectrum S(L) of the operator L is the union of the spectra S(L_t) of the operators L_t for $t\in[0,2\pi]$, which is generated by the operator (1.1) in $L_2[0,a]$ and the boundary conditions

$$\begin{aligned} y(a)&=e^{it}y(0)\\ y'(a)&=e^{it}y'(0) \end{aligned} \tag{1.3}$$

where a is the period of potential q(x), i.e. $q(x)=q(x+a)$.

The spectrum and some spectral properties of the nonself-adjoint operator L with complex-valued potential q(x) were examined by F. S. Rofe-Beketov (Rofe-

Beketov,1963), M. I Serov (Serov, 1960) and D. C. McGarvey (McGarvey, 1962,1965). In particularly, they showed that the relation

$$S(L) = \bigcup_{t \in [0,2\pi]} S(L_t)$$

holds for the nonself-adjoint case too.

O.A Veliev investigated the spectrum and the spectral singularity and constructed the spectral expansion for the operator L with the complex-valued continuous potential (Veliev, 1980,1983,1986). M. G. Gasymov studied the direct and inverse problem of L with the potential q(x) of the form

$$q(x) = \sum_{n=1}^{\infty} q_n e^{inx}$$

and he showed (Gasymov, 1980) that the operator L has infinitely many spectral singularities which complicate the construction of the spectral expansion for L.

In this thesis, we study the one-dimensional Schrödinger operator with an arbitrary Lebesgue integrable and complex-valued periodic potential ($q(x+2\pi) = q(x)$). So q(x) can have a finite number of singularities of order $\frac{1}{|x - x_k|^{\alpha}}$ where $\alpha < 1$.

In chapter two, we investigate the stability and conditional stability zones, spectra of L and L_t and so we prove that

$$S(L) = C(L) = \bigcup_{t \in [0,1]} S(L_t) = \overline{\bigcup_{k \geq 1} S(L(k))} \tag{1.4}$$

where $L(k)$ is the operator generated in $[0, 2\pi k]$ by (1.1) and the periodic boundary conditions. We prove this using the scheme in the book (Eastham, 1973) which was presented for a piecewise continuous and periodic function q(x). The relation (1.4) shows that the research for the spectrum of L is reduced to research for the spectra of L_t.

In chapter three, the asymptotic formulas for eigenvalues and eigenfunctions of L_t are obtained. These presented formulas are valid for the operator L_t with a generalized potential q(x) of the form

$$q(x) = \sum_{n=-\infty}^{\infty} q_n e^{inx}$$

where $q_n = O(1)$. Here, we followed the method given in the article (Veliev,1986).

Our object in chapter four is to establish the spectral expansion for L. For this reason, using the asymptotic formulas obtained in chapter three, we prove that the eigenfunctions of the operator L_t form a Riesz basis in $L_2[0,a]$, $a=1$. Then, we construct the spectral expansion for the operator L with integrable potential q(x) using the scheme of papers (Veliev, 1980, 1983,1986).

1.1 Quasiperiodic Boundary Conditions

Consider the nonselfadjoint differential operator L generated by differential expression

$$l(y) = -y'' + q(x)y, \qquad -\infty < x < \infty \tag{1.5}$$

in $L_2(-\infty,\infty)$, where $q(x)$ is a periodic with period 2π, Lebesgue integrable and complex-valued function which may have finite number of singularities, for instance $x_1, x_2, \ldots, x_n$, in the interval $[0,2\pi]$. Since the spectral property of the operator L with potential $q(x+b)$ are the same as the spectral property of L with the potential $q(x)$, without loss of generality it can be asssumed that $x_k \neq 0,2\pi$ for all k=1,2,...,n.

Now define the operator L_t generated by the differential expression (1.5) in $L_2[0,2\pi]$ and the following boundary conditions

$$\begin{aligned} U_1(y) &\equiv e^{i2\pi t} y'(0) - y'(2\pi) = 0 \\ U_2(y) &\equiv e^{i2\pi t} y(0) - y(2\pi) = 0 \end{aligned} \tag{1.6}$$

where t is a complex parameter. These conditions are termed by quasiperiodic boundary conditions.

As it is mentioned above, since the spectrum of L is connected with the spectrum of L_t with quasiperiodic boundary conditions (1.6), first of all we shall classify these boundary conditions.

Let denote $U_v(y)=0,\quad v=1,2,\dots,n$ which are the boundary conditions of a given linear differential equation of order n such that

$$U(y)=\alpha_0 y(a)+\alpha_1 y'(a)+\dots+\alpha_{n-1}y^{(n-1)}(a)+\beta_0 y(b)+\beta_1 y'(b)+\dots+\beta_{n-1}y^{(n-1)}(b)$$

A boundary condition such as U(y) has order k, if it contains the kth derivative $y^{(k)}(a)$ or $y^{(k)}(b)$ explicitly but $y^{(v)}(a)$ and $y^{(v)}(b)$ do not, for any $v>k$. If necessary to eliminate the order of boundary conditions, using equivalent linear combinations of the forms $U_v(y)$, we can reduce their number of orders to a minimum number. The obtained boundary conditions by this procedure are called normalized. Then the normalized boundary conditions take the form

$$U_v(y)\equiv U_{v0}(y)+U_{v1}(y)=0, \tag{1.7}$$

where

$$\begin{aligned} U_{v0}(y)&=\alpha_v y^{(k_v)}(a)+\sum_{j=0}^{k_v-1}\alpha_{vj}y^{(j)}(a),\\ U_{v1}(y)&=\beta_v y^{(k_v)}(b)+\sum_{j=0}^{k_v-1}\beta_{vj}y^{(j)}(b) \end{aligned} \tag{1.8}$$

$$n-1\ge k_1\ge k_2\ge\dots\ge k_n\ge 0,\quad k_{v+2}<k_v,$$

and for each value of the suffix v at least one of the numbers α_v,β_v is non-zero. (Naimark, 1968).

It is easy to see from (1.7) that we have normalized boundary conditions (1.6). If we apply (1.7) and (1.8) for our boundary conditions (1.6) with the orders $k_1=1$ and $k_2=0$, respectively

$$U_{10}(y) \equiv \alpha_1 y'(0) + \alpha_{10} y(0) = e^{i2\pi t} y'(0)$$
$$U_{11}(y) \equiv \beta_1 y'(2\pi) + \beta_{10} y(2\pi) = -y'(2\pi)$$
$$U_{20}(y) \equiv \alpha_2 y(0) = e^{i2\pi t} y(0)$$
$$U_{21}(y) \equiv \beta_2 y(2\pi) = -y(2\pi),$$

we get

$$\alpha_1 = \alpha_2 = e^{i2\pi t} \text{ and } \beta_1 = \beta_2 = -1. \tag{1.9}$$

The class of conditions which are regular defines a given differential operator. The definition of regular boundary conditions is given in two ways depending on whether its number of conditions n is even or odd. In particularly, we are dealing with the even case.

Definition 1.1.1 When n is even, n=2p, the normalized boundary conditions (1.7) are said to be regular if the numbers θ_{-1} and θ_1 defined by the following identity

$$\frac{\theta_{-1}}{s} + \theta_0 + \theta_1 s =$$
$$= \begin{vmatrix} \alpha_1\omega_1^{k_1} \cdots & \alpha_1\omega_{p-1}^{k_1} & (\alpha_1 + s\beta_1)\omega_p^{k_1} & (\alpha_1 + \frac{\beta_1}{s})\omega_{p+1}^{k_1} & \beta_1\omega_{p+2}^{k_1} \cdots & \beta_1\omega_n^{k_1} \\ \alpha_2\omega_1^{k_2} \cdots & \alpha_2\omega_{p-1}^{k_2} & (\alpha_2 + s\beta_2)\omega_p^{k_2} & (\alpha_2 + \frac{\beta_2}{s})\omega_{p+1}^{k_2} & \beta_2\omega_{p+2}^{k_2} \cdots & \beta_2\omega_n^{k_2} \\ \cdot \quad \cdots & \cdot & \cdot & \cdot & \cdot \quad \cdots & \cdot \\ \alpha_n\omega_1^{k_n} \cdots & \alpha_n\omega_{p-1}^{k_n} & (\alpha_n + s\beta_n)\omega_p^{k_n} & (\alpha_n + \frac{\beta_n}{s})\omega_{p+1}^{k_n} & \beta_{n2}\omega_{p+2}^{k_n} \cdots & \beta_{n1}\omega_n^{k_n} \end{vmatrix} \tag{1.10}$$

are different from zero. Here the numbers $\omega_1, \omega_2, \ldots, \omega_n$ denote the different nth roots of -1. (Naimark, 1968).

The numbers θ_{-1}, θ_0 and θ_{-1} can be derived from (1.9) and (1.10) while n=2, $\omega_1 = i$ and $\omega_2 = -i$ that

$$\frac{\theta_{-1}}{s}+\theta_0+\theta_1 s=\begin{vmatrix}(\alpha_1+s\beta_1)\omega_1^{k_1} & (\alpha_1+\frac{\beta_1}{s})\omega_2^{k_1}\\ (\alpha_2+s\beta_2)\omega_1^{k_2} & (\alpha_2+\frac{\beta_2}{s})\omega_2^{k_2}\end{vmatrix}$$

$$=\begin{vmatrix}(e^{i2\pi t}-s)i & -(e^{i2\pi t}-\frac{1}{s})i\\ (e^{i2\pi t}-s) & (e^{i2\pi t}-\frac{1}{s})\end{vmatrix}$$

$$=-2i(\frac{1}{s}e^{i2\pi t}-1-e^{i4\pi t}+se^{i2\pi t}).$$

Hence,

$$\theta_1=\theta_{-1}=-2ie^{i2\pi t}\neq 0 \text{ and } \theta_0=2i(1+e^{i4\pi t}). \tag{1.11}$$

This shows that the boundary conditions (1.2) are regular for all t. Moreover, the following condition which is known as strictly regular boundary condition

$$\theta_0^2-4\theta_1\theta_{-1}\neq 0 \tag{1.12}$$

holds if $t\neq\frac{k}{2}$ for all integers k.

Let U_1^* and U_2^* denote the boundary conditions of L_t^* which is the adjoint operator of L_t. Take y(x) and z(x) in their domain of definitions $D(L_t)$ and $D(L_t^*)$, respectively.

$$(L_t y,z)=\int_0^{2\pi}(-y''(x)+q(x)y(x))\overline{z(x)}dx=-y'(x)\overline{z(x)}+y(x)\overline{z'(x)}\Big|_0^{2\pi}+(y,L_t^*z)$$

Using the definition of an adjoint operator

$$(L_t y,z)=(y,L_t^*z),$$

it must be

$$-y'(x)\overline{z(x)}+y(x)\overline{z'(x)}\Big|_0^{2\pi}=0.$$

Equivalently,

$$-y'(0)(e^{i2\pi t}\overline{z(2\pi)}-\overline{z(0)})+y(0)(e^{i2\pi t}\overline{z'(\pi)}-\overline{z'(0)})=0.$$

Hence, L_t^* is generated by the following adjoint differential expression and boundary conditions

$$
\begin{aligned}
&l^*(z(x)) = -z''(x) + \overline{q(x)}z(x) \\
&U_1^*(z) \equiv e^{i2\pi\bar{t}}z'(0) - z'(2\pi) = 0 \\
&U_2^*(z) \equiv e^{i2\pi\bar{t}}z(0) - z(2\pi) = 0
\end{aligned}
\tag{1.13}
$$

Now consider the given boundary conditions for

$$l(y) = \lambda y \tag{1.14}$$

such as linear forms

$$U_i(y) \equiv a_{i1}y(0) + a_{i2}y'(0) + a_{i3}y(2\pi) + a_{i4}y'(2\pi) = 0, \quad \text{i=1,2.} \tag{1.15}$$

Let $\theta(x,\lambda)$ and $\varphi(x,\lambda)$ denote the fundamental solutions of (1.14) satisfying the initial conditions

$$
\begin{aligned}
\theta(0,\lambda) = \varphi'(0,\lambda) = 1, \\
\theta'(0,\lambda) = \varphi(0,\lambda) = 0.
\end{aligned}
$$

The general solution of (1.14) is

$$y(x,\lambda) = c_1\theta(x,\lambda) + c_2\varphi(x,\lambda)$$

where c_1 and c_2 are constants. Substituting this into boundary conditions (1.15), we have

$$
\begin{aligned}
c_1U_1(\theta) + c_2U_1(\varphi) = 0 \\
c_1U_2(\theta) + c_2U_2(\varphi) = 0
\end{aligned}
$$

There exists a nonzero solution of (1.14) and (1.15) if and only if

$$\Delta(\lambda) = \begin{vmatrix} U_1(\theta) & U_1(\varphi) \\ U_2(\theta) & U_2(\varphi) \end{vmatrix}$$

vanishes identically. Since the Wronskian $W(\theta,\varphi) = 1$ everywhere, the characteristic function becomes

$$\Delta(\lambda) = J_{12} + J_{34} + J_{13}\varphi(2\pi,\lambda) + J_{14}\varphi'(2\pi,\lambda) + J_{32}\theta(2\pi,\lambda) + J_{42}\theta'(2\pi,\lambda)$$

where

$$J_{mn} = a_{1m}a_{2n} - a_{2m}a_{1n} \qquad \text{(m,n=1,2,3,4).}$$

Definition 1.1.2 The boundary conditions (1.15) satisfying one of the following three conditons are called nondegenerate

(i) $J_{42} \neq 0$

(ii) $J_{42} = 0,\ J_{14} + J_{32} \neq 0$

(iii) $J_{42} = J_{14} + J_{32} = 0,\ J_{13} \neq 0$ (1.17)

(Marchenko, 1986).

The boundary conditions (1.6) of L_t are nondegenerate for all $t \in \mathbf{C}$, since they satisfy the condition (ii) in (1.17). This can be checked from (1.6) that

$$\begin{aligned} J_{42} &= a_{14}a_{22} - a_{24}a_{12} = 0 \\ J_{14} &= a_{11}a_{24} - a_{21}a_{14} = e^{i2\pi t} \\ J_{32} &= a_{13}a_{22} - a_{23}a_{12} = e^{i2\pi t}. \end{aligned}$$

Thus, the quasiperiodic boundary conditions are regular and nondegenerate for all complex value of t and also strictly regular for $t \neq \frac{k}{2},\ k \in \mathbf{Z}$.

At the end of this section, we should note that the boundary conditions (1.6) can be written in suitable form

$$y(x+2\pi) = e^{i2\pi t}y(x) \tag{1.18}$$

Indeed, putting x=0 in (1.18) and in the obtained equality after the differentiation of both sides of (1.18), it is seen that this implies (1.6). Conversely, suppose that y(x) is a solution of $l(y) = \lambda y$ satisfying (1.6). Since $y(x+2\pi)$ is a solution of this differential equation, $v(x) = y(x+2\pi) - e^{i2\pi t}y(x)$ is also a solution of this differential equation with $v(0) = v'(0) = 0$. Then, by the well-known theorem for the differential equations, v(x) vanishes identically. That is, (1.6) implies (1.18). Thus, (1.6) and (1.18) are equivalent.

1.2 Floquet Theory

We examine the solutions of the eigenvalue problem of the differential expression (1.5) rearranging it

$$y''(x)+(\lambda-q(x))y(x)=0 \tag{1.19}$$

where the potential function q(x) is a complex-valued, periodic and summable function, $q(x+2\pi)=q(x)$, which may have a finite number of singularities in the interval $[0,2\pi]$ but does not at the points 0 and 2π.

More generally, the differential equation (1.19) can be termed by Hill's equation.

Floquet theory given by the following two theorems states the form of the solutions for (1.19) (Eastham, 1973).

Theorem 1.2.1 There are a non-zero constant ρ and a non-trivial solution $y(x,\lambda)$ of the equation (1.19) such that

$$y(x+2\pi,\lambda)=\rho y(x,\lambda). \tag{1.20}$$

Proof. There exist linearly independent solutions $\theta(x,\lambda)$ and $\varphi(x,\lambda)$ of (1.19) satisfying the initial conditions at the regular point x=0

$$\begin{aligned}\theta(0,\lambda)=\varphi'(0,\lambda)=1,\\ \theta'(0,\lambda)=\varphi(0,\lambda)=0,\end{aligned} \tag{1.21}$$

because the Wronskian of $\theta(x,\lambda)$ and $\varphi(x,\lambda)$ does not vanish at the point zero. For each fixed value of x, the fundamental solutions $\theta(x,\lambda)$ and $\varphi(x,\lambda)$ are analytic functions of the parameter λ.

Any non-trivial general solution $y(x,\lambda)$ can be written as a linear combination of the fundamental solutions

$$y(x,\lambda)=c_1\theta(x,\lambda)+c_2\varphi(x,\lambda) \tag{1.22}$$

for which at least one of the constants c_1 and c_2 is non-zero.

Besides, the functions $\theta(x+2\pi,\lambda)$ and $\varphi(x+2\pi,\lambda)$ are linearly independent solutions of the equation (1.19), they can be expressed as a linear combination of the fundamental solutions

$$\begin{aligned}\theta(x+2\pi,\lambda) &= a_{11}\theta(x,\lambda)+a_{12}\varphi(x,\lambda),\\ \varphi(x+2\pi,\lambda) &= a_{21}\theta(x,\lambda)+a_{22}\varphi(x,\lambda)\end{aligned} \tag{1.23}$$

when its coefficient matrix is non-singular.

By the periodicity of the potential function q(x), the solution $y(x+2\pi,\lambda)$ holds for the differential equation (1.19). Using (1.23), (1.20) is satisfied if

$$\begin{aligned}(a_{11}-\rho)c_1+a_{21}c_2 &= 0,\\ a_{12}c_1+(a_{22}-\rho)c_2 &= 0\end{aligned} \tag{1.24}$$

where $a_{11}=\theta(2\pi,\lambda)$, $a_{12}=\theta'(2\pi,\lambda)$, $a_{21}=\varphi(2\pi,\lambda)$ and $a_{22}=\varphi'(2\pi,\lambda)$.

The equation (1.19) has a non-trivial solution if and only if the determinant of the coefficient matrix for (1.24) vanishes, that is

$$\rho^2-(\theta(2\pi,\lambda)+\varphi'(2\pi,\lambda))\rho+1=0. \tag{1.25}$$

Clearly, this quadratic equation has at least one non-zero root. This proves the theorem.

Here, the function $F(\lambda)=\theta(2\pi,\lambda)+\varphi'(2\pi,\lambda)$ is known as the Lyapunov function and is also analytic with respect to λ, being the analytic fundamental solutions and their derivatives.

Theorem 1.2.2 There are linearly independent solutions $y_1(x,\lambda)$ and $y_2(x,\lambda)$ of the equation (1.19) such that either

(i) $y_1(x,\lambda)=e^{m_1x}p_1(x,\lambda),\quad y_2(x,\lambda)=e^{m_2x}p_2(x,\lambda)$

where m_1 and m_2 are constants, not necessarily distinct, and $p_1(x,\lambda)$ and $p_2(x,\lambda)$ are periodic with period 2π ; or

(ii) $y_1(x,\lambda)=e^{mx}p_1(x,\lambda),\quad y_2(x,\lambda)=e^{mx}(xp_1(x,\lambda)+p_2(x,\lambda))$

where m is a constant and $p_1(x,\lambda)$ and $p_2(x,\lambda)$ are periodic with period 2π. (Eastham, 1973).

We should note that, ρ_k and so also m_k (k=1,2) depend on the value of λ.

Every solution of (1.25) can be expressed by an exponential number depending on the period 2π such that $\rho_k = e^{2\pi m_k}$ where $m_k = \alpha_k + i\beta_k$ for real numbers α_k and β_k (k=1,2). If there are two distinct solutions of (1.25), i.e. $\rho_1 \neq \rho_2$, then part (i) of Theorem (1.2.2) applies. For the case of double zeros, i.e. $\rho_1 = \rho_2$; if the rank of the coefficients matrix of the equations (1.24) vanishes then part (i) of Theorem (1.2.2) occurs, if the rank of the coefficients matrix of (1.24) is unity then part (ii) of Theorem (1.2.2) occurs.

CHAPTER TWO

ON THE SPECTRUM AND STABILITY ZONES OF NONSELF-ADJOINT HILL'S OPERATOR

2.3 Domain of Stability

The spectrum of L is closely related to the stability theory of the Hill's equation $l(y)=\lambda y$. The stability theory introduces the domain of stability of the differential equation with periodic coefficients. The domain of stability (stability zones) of the Hill equation (1.19) is the class of all parameters λ such that all of its non-trivial solutions are bounded on the whole of the real line.

Definition 2.1.1 The equation (1.19) is said to be

(i) stable if all solutions are bounded on the whole of the real line.

(ii) conditionally stable if there exists a non-trivial solution which is bounded on the whole of the real line.

(iii) unstable if all non-trivial solutions are unbounded on the whole of the real line.

Theorem 2.1.2 **(i)** If $|F(\lambda)|>2$, the equation (1.19) is unstable. (λ is in the unstability zones).

(ii) If $|F(\lambda)|<2$, the equation (1.19) is stable. (λ is in the stability zones).

Proof (i) Suppose that the Lyapunov function $F(\lambda)$ is complex-valued. Since $F^2(\lambda)-4$ is different from zero, there are distinct zeros of the characteristic equation (1.25). They satisfy $\rho_1\rho_2=1$, thus

$$\rho_1 = e^{2\pi m} \text{ and } \rho_2 = e^{-2\pi m} \tag{2.1}$$

with a number $m = \alpha + i\beta$. If their moduli were equal to unity, it would be

$$F(\lambda) = \rho_1 + \rho_2 = 2\cos 2\pi\beta,$$

but this contradicts being non-real $F(\lambda)$. Therefore it must be a number m with a non-zero real part. By Floquet's theorem (1.2.2), there are linearly independent solutions of (1.19) such that

$$y_1(x,\lambda) = e^{mx} p_1(x,\lambda) \text{ and } y_2(x,\lambda) = e^{-mx} p_2(x,\lambda) \tag{2.2}$$

where $p_1(x,\lambda)$ and $p_2(x,\lambda)$ are periodic functions with period 2π. This proves that all solutions of the equation (1.19) as non-trivial linear combinations of (2.2) are unbounded as $|x| \to \infty$ if the Lyapunov function is complex-valued.

Let F(λ) be real-valued. If F(λ)>2 then the zeros of (1.25) are real ,distinct and positive. Furthermore, the zeros are different from unity. Hence, by part(i) of Theorem (1.2.2), $y_1(x,\lambda)$ and $y_2(x,\lambda)$ are of the form (2.2) with a real non-zero number m.

If F(λ)<-2 then the zeros of (1.25) are real, distinct , negative and different from -1. Since they are negative, (2.1) and so (2.2) occur for which $2\pi m = 2\pi m' + i\pi$ with a real non-zero m'.

Consequently, every non-trivial linear combination of $y_1(x)$ and $y_2(x)$ is unbounded as $|x| \to \infty$ when F(λ) is real-valued and |F(λ)|>2 holds.

(ii) If -2<F(λ)<2 then there exist distinct and complex-valued zeros of (1.25). Since their multiplication equals to unity, their moduli are unity. Then,

$$\rho_1 = e^{i2\pi\beta}, \ \rho_2 = e^{-i2\pi\beta} \tag{2.3}$$

and by part (i) of Theorem (1.2.2), we have

$$y_1(x,\lambda) = e^{i\beta x} p_1(x,\lambda) \text{ and } y_2(x,\lambda) = e^{-i\beta x} p_2(x,\lambda) \tag{2.4}$$

with a real number β such that $0<\beta<\frac{1}{2}$ (or $-\frac{1}{2}<\beta<0$). Since every periodic function is bounded in $(-\infty,\infty)$, all linear combinations of $y_1(x,\lambda)$ and $y_2(x,\lambda)$ are also bounded on the whole of the real line. Thus, the theorem is proved.

On the other hand, there exist periodic and anti-periodic, i.e semi-periodic, solutions of the equation (1.19) if

$$F(\lambda)^2-4=0. \tag{2.5}$$

In this case, the equation (1.25) has double solutions, $\rho_1=\rho_2$. If both of $\varphi(2\pi,\lambda)$ and $\theta'(2\pi,\lambda)$ are zero then the rank of the system of equations (1.24) vanishes. It follows form the first part of Theorem (1.2.2) that all non-trivial solutions of (1.19) become periodic with period 2π while $F(\lambda)=2$ and anti-periodic with anti-period 2π (i.e. periodic with period 4π) while $F(\lambda)=-2$ .Thus, all solutions are bounded on the whole of the real line, namely the equation (1.19) is stable.

If $\varphi(2\pi,\lambda)$ and $\theta'(2\pi,\lambda)$ are not both zero then the rank of the system of the equation (1.24) does not vanishes, so the second part of Theorem (1.2.2) applies. Hence, only one of the linearly independent solutions becomes periodic with period 2π and anti-periodic with anti-period 2π while $F(\lambda)=2$ and while $F(\lambda)=-2$, respectively. Consequently, there exits a bounded solution of the equation (1.19) in $(-\infty,\infty)$, it means that (1.19) is conditionally stable.

Now, let us consider the differentiation operator $L(k)$ generated by the differential expression (1.5) in the interval $[0,2k\pi]$ and the boundary conditions

$$\begin{aligned} y(0)&=y(2k\pi) \\ y'(0)&=y'(2k\pi) \end{aligned} \qquad (k=1,2,...). \tag{2.6}$$

From these conditions, the periodic boundary value problem $L(k)y=\lambda y$ has nontrivial solution such that

$$y(x+2k\pi,\lambda)=y(x,\lambda) \tag{2.7}$$

Let the eigenvalues of $L(k)$ be denoted by $\Lambda_n(k)$. Then we have $\bigcup_{k\geq 1} S(L(k)) = \{\Lambda_n(k) : n,\ k \geq 1\}$. It is obvious from (2.7) that the eigenfunctions can be extended to $(-\infty,\infty)$ as a bounded function. Clearly, every eigenfunction forms as the linear combinations of the linearly independent functions in (2.4). Verifying (2.7) to the eigenfunction, we obtain

$$c_1(1-e^{i2k\pi\beta})y_1(x,\lambda)+c_2(1-e^{-i2k\pi\beta})y_2(x,\lambda)=0$$

Here, $y_1(x,\lambda)$ and $y_2(x,\lambda)$ are linearly independent if and only if their coefficients are both zero, so it must be

$$e^{i2k\pi\beta}=1.$$

This implies that there are some integers l such that $l = k\beta$.

Hence, from (2.3), the eigenvalues $\Lambda_n(k)$ of $L(k)$ satisfy the equation

$$F(\lambda)=\rho_1+\rho_2=2\cos(\frac{2\pi l}{k}). \tag{2.8}$$

Now, consider the differential operator L_t over $[0,\ 2\pi]$ recalling its boundary conditions

$$\begin{aligned} y(2\pi) &= e^{i2\pi t}y(0) \\ y'(2\pi) &= e^{i2\pi t}y'(0) \end{aligned} \tag{2.9}$$

where t is a complex parameter. From boundary conditions (2.9), this problem has a solution of the form

$$y(x+2\pi,\lambda)=e^{i2\pi t}y(x,\lambda). \tag{2.10}$$

If we verify the conditions (2.9) for the general solution of L_t, we obtain the system of the equations given by (1.24) such as

$$\begin{aligned} (\theta(2\pi,\lambda)-e^{i2\pi t})c_1+\varphi(2\pi,\lambda)c_2 &= 0 \\ \theta'(2\pi,\lambda)c_1+(\varphi'(2\pi,\lambda)-e^{i2\pi t})c_2 &= 0 \end{aligned} \tag{2.11}$$

As a result of (2.11), the number λ is an eigenvalue of L_t if and only if λ holds the equation

$$F(\lambda) = 2\cos 2\pi t\,. \tag{2.12}$$

Let the eigenvalues of the operator L_t be denoted by $\lambda_n(t)$ (n=1,2,...). Then the collection of all eigenvalues of the operators L_t for $0 \le t \le 1$ is

$$\bigcup_{t\in[0,1]} S(L_t) = \left\{\lambda_n(t) : n \ge 1,\ 0 \le t \le 1\right\}.$$

The eigenvalue $\lambda_n(t)$ is a continuous function of t. Indeed, the equation (2.12) can be expressed as an implicit function of λ and t. Since it is analytic and in the view of implicit function theorem, there exists a unique function $\lambda_n(t)$ which is analytic in an open neighborhoods of $t_0 \in (0,1)$ where $\frac{dF}{d\lambda}\Big|_{\lambda_n(t_0)} \ne 0$. So the union of spectrum of the operators L_t for $0 \le t \le 1$

$$\bigcup_{t\in[0,1]} S(L_t) = \bigcup_{n=1}^{\infty}\left\{\lambda_n(t) :\ 0 \le t \le 1\right\} = \bigcup_{n=1}^{\infty}\Gamma_n$$

consists of piecewise smooth arcs Γ_n.

Let *C(L)* be a set such that it includes the values of λ for which Hill's equation (1.19) has a non-trivial bounded solution on $(-\infty,\infty)$. In other words, *C(L)* is the conditional stability set of L, so that

$$C(L) = \left\{\lambda : |F(\lambda)| \le 2\right\}.$$

Now, we establish the relation to the spectrum of the operators $L(k)$ and the conditional stability set of L.

Theorem 2.1.3 The closure of $\bigcup_{k\ge1} S(L(k))$ is *C(L)*.

Proof. Take an eigenvalue $\Lambda_n(k) \in \bigcup_{k\ge1} S(L(k))$, that is

$$\Lambda_n(k) \in \{\lambda : F(\lambda) = 2\cos(2\pi \frac{l}{k}),\ k \geq 1,\ l \in \mathbf{Z}\}.$$

Then, $\Lambda_n(k) \in C(L)$. Hence,

$$\underset{k\geq 1}{\cup} S(L(k)) \subset C(L).$$

Since the function $F(\lambda)$ is continuous, for every point ω in the range of $F(\lambda)$ the set $C(L) = \{F^{-1}(\omega) : |\omega| \leq 2\}$ is closed, i.e. $\overline{C(L)} = C(L)$. Hence,

$$\overline{\underset{k\geq 1}{\cup} S(L(k))} \subset C(L). \tag{2.13}$$

For any $\lambda \in C(L)$, since $F(\lambda)$ is real, a real number r can be found satisfying $F(\lambda) = 2\cos(2\pi r)$. When r is a rational number there exist some integers p and q such that $F(\lambda) = 2\cos(2\pi \frac{p}{q})$. Therefore, $\lambda \in S(L(k))$. When r is not a rational number, a sequence of rational numbers converges to r, that is $\frac{p_n}{q_n} \to r$ as $n \to \infty$. By continuity of $F(\lambda)$, this implies that $2\cos(2\pi \frac{p_n}{q_n}) \to 2\cos(2\pi r)$ as $n \to \infty$. Therefore, $\lambda \in \overline{\underset{k\geq 1}{\cup} S(L(k))}$. So

$$C(L) \subset \overline{\underset{k\geq 1}{\cup} S(L(k))}. \tag{2.14}$$

As a result of (2.13) and (2.14), the set $\underset{k\geq 1}{\cup} S(L(k))$ is dense in $C(L)$

$$\overline{\underset{k\geq 1}{\cup} S(L(k))} = C(L).$$

Theorem 2.1.4 The sets $\bigcup_{t\in[0,1]} S(L_t)$ and $C(L)$ are identical.

Proof. By the reason of the previous theorem, we need to show that the closure of $\cup_{k\geq 1} S(L(k))$ and the set $\bigcup_{t\in[0,1]} S(L_t)$ are identical. Let us consider an eigenvalue $\lambda \in \bigcup_{t\in[0,1]} S(L_t)$ for $t=\frac{s}{k}$ with some integers s and k, i.e. $t \in Q$. Then

$$y(x+2\pi,\lambda)=e^{i2\pi\frac{s}{k}}y(x,\lambda)$$

for $x\in(-\infty,\infty)$. If we shift this eigenfunction on the real line by substituting $x=x'+2\pi$, we have

$$y(x'+4\pi,\lambda)=e^{i4\pi\frac{s}{k}}y(x',\lambda).$$

Applying this process (k-1) times, we obtain

$$y(x'+2k\pi,\lambda)=e^{i2\pi\frac{s}{k}k}y(x',\lambda)=y(x',\lambda).$$

That is, an eigenfunction of $L(k)$, $\lambda\in\cup_{k\geq 1} S(L(k))$. Hence,

$$\bigcup_{t\in Q} S(L_t)\subset\cup_{k\geq 1} S(L(k))\subset\overline{\cup_{k\geq 1} S(L(k))}.$$

Conversely, an eigenfunction of $L(k)$ associated with $\lambda\in\cup_{k\geq 1} S(L(k))$ coincides with an eigenfunction of L_t associated with $\lambda\in\bigcup_{t\in Q} S(L_t)$. Thus,

$$\bigcup_{t\in Q} S(L_t)=\cup_{k\geq 1} S(L(k)). \tag{2.15}$$

By virtue of the continuity of $\lambda_n(t)$, for the sequence of rational numbers $\frac{s_r}{k_r}$ converging to a real number in (0,1), this implies that $\lambda_n(\frac{s_r}{k_r})$ converges to $\lambda_n(t)$ as $r\to\infty$. Hence,

$$\overline{\bigcup_{t\in Q} S(L_t)}=\bigcup_{t\in[0,1]} S(L_t)$$

and so the closure of the sets in (2.15) gives $\overline{S(L(k))}=\overline{\bigcup_{t\in Q} S(L_t)}=\bigcup_{t\in[0,1]} S(L_t)$. This proves the theorem.

2.2 The Spectrum

Definition 2.2.1 The resolvent set $\rho(L)$ of L is the set of values of λ for which the resolvent operator $(L-\lambda I)^{-1}$ exists and is bounded. The spectrum $S(L)$ of L is defined by the complement of the resolvent set, i.e. $S(L) = \mathbf{C} \setminus \rho(L)$.

Theorem 2.2.2 The spectrum of L and the union of the spectrum of the operators L_t for $0 \le t \le 1$ are identical. That is, $S(L) = \bigcup_{t\in[0,1]} S(L_t)$.

Proof. Take any $\lambda \in \bigcup_{t\in[0,1]} S(L_t)$. If $t \neq 0, \frac{1}{2}$ then $\rho_1 \neq \rho_2$ where $\rho_1 = e^{i2\pi t}$ and $\rho_2 = e^{-i2\pi t}$. By Theorem (1.2.2) there are linearly independent solutions $y_1(x,\lambda)$ and $y_2(x,\lambda)$ of

$$(L-\lambda I)y = 0 \tag{2.16}$$

such that

$$y_1(x,\lambda) = e^{itx} p_1(x,\lambda), \quad y_2(x,\lambda) = e^{-itx} p_2(x,\lambda)$$

where $p_1(x,\lambda)$ and $p_2(x,\lambda)$ have period 2π. So the general solution is

$$c_1 e^{itx} p_1(x,\lambda) + c_2 e^{-itx} p_2(x,\lambda) \tag{2.17}$$

where c_1 and c_2 are constants. If (2.17) is a non-zero solution then $\exists\, x_0$ such that

$$\left|c_1 e^{itx} p_1(x_0,\lambda)\right| = \left|c_1 p_1(x_0,\lambda)\right| \neq \left|c_2 e^{-itx} p_2(x_0,\lambda)\right| = \left|c_2 p_2(x_0,\lambda)\right|.$$

By the periodicity of $p_1(x,\lambda)$ and $p_2(x,\lambda)$, we have

$$\left|c_1 p_1(x_0 + 2\pi n,\lambda)\right| \neq \left|c_2 p_2(x_0 + 2\pi n,\lambda)\right|$$

Hence, the solution (2.17) clearly does not have a finite norm in $L_2(-\infty,\infty)$, i.e. (2.17) is not an element of $L_2(-\infty,\infty)$.

If $t = 0,1$ then the general solution is of the form either

$$c_1 p_1(x,\lambda) + c_2 p_2(x,\lambda) \tag{2.18}$$

or

$$c_1 x p_1(x,\lambda) + c_2 p_2(x,\lambda). \tag{2.19}$$

where $p_i(x,\lambda)$ (i=1,2) are periodic functions. For the case of (2.18) repeating the above argument, we see that (2.18) does not belong to $L_2(-\infty,\infty)$. For the case of (2.19) the solution tends to infinity as $|x| \to \infty$, so it cannot be in $L_2(-\infty,\infty)$.

The case $t = \dfrac{1}{2}$ is similar.

As a result of the above argument, there is only trivial solution for (2.16) in $L_2(-\infty,\infty)$, so the resolvent of L exits.

Let $g(x)$ be chosen such that g(x) and *l(yg)* are bounded in $[0,2\pi]$ and g(x) has continuous second derivative satisfying

$$g(0) = 0,\ \ g(2\pi) = 1,\ \ g^{(n)}(0) = g^{(n)}(2\pi) = 0 \ \ (n=1,2).$$

In order for $q(x)g(x)$ to be finite at the singular points of *q(x)*, we assume that every singular point $x_i \in (0,2\pi)$ of q(x) is a zero of g(x), i.e g(x_i)=0.

Now define a sequence of functions by

$$f_n(x,\lambda) = y(x,\lambda)h_n(x), \qquad -\infty < x < \infty$$

where

$$h_n(x) = \begin{cases} 1, & |x| \le (n\text{-}1)2\pi \\ g(2\pi n - |x|), & (n-1)2\pi < |x| \le n2\pi \\ 0, & |x| > n2\pi \end{cases}$$

For $\lambda \in \bigcup\limits_{t\in[0,1]} S(L_t)$,

$$\int_{-\infty}^{\infty} |(L-\lambda I)f_n|^2 dx = \int_{-n2\pi}^{-(n-1)2\pi} |\ (L-\lambda I)(y(x)g(2n\pi - |x|)\ |^2 dx +$$

$$+\int_{-(n-1)2\pi}^{(n-1)2\pi}\left| (L-\lambda I)y(x)dx \right|^2 dx+\int_{(n-1)2\pi}^{n2\pi}\left| (L-\lambda I)(y(x)g(2n\pi-|x|)) \right|^2 dx$$

$$\leq M^2 4\pi,$$

from the definition of *g(x)*, a number M>0 can be found which does not depend on n. Hence, $\|(L-\lambda I)f_n\|$ is finite.

However, $\|f_n\|\to\infty$ as $n\to\infty$.

$$\|f_n\|^2=\int_{-n2\pi}^{-(n-1)2\pi}|y(x)g(2n\pi-|x|)|^2 dx+\int_{-(n-1)2\pi}^{(n-1)2\pi}|y(x)|^2 dx+\int_{(n-1)2\pi}^{n2\pi}|y(x)g(2n\pi-|x|)|^2 dx$$

$$\to\infty.$$

Thus,

$$\lim_{n\to\infty}\frac{\|(L-\lambda I)f_n\|}{\|f_n\|}=0.$$

Therefore, $L-\lambda I$ does not have a bounded inverse, i.e it is an unbounded resolvent, and so $\lambda\in S(L)$. Hence,

$$\bigcup_{t\in[0,1]} S(L_t)\subset S(L).$$

If $\lambda\notin\bigcup_{t\in[0,1]} S(L_t)$ then $|F(\lambda)|>2$ and the real parts of the characteristic exponents are nonzero, that is $\mathrm{Re}(m_k)\neq 0,\ (k=1,2)$. In this case, we shall show that the resolvent of L exist and is bounded, i.e. $\lambda\notin S(L)$. There are linearly independent solutions of $(L-\lambda I)y=0$ of the form

$$y_1(x,\lambda)=e^{(\alpha+i\beta)x}p_1(x,\lambda),\qquad y_2(x,\lambda)=e^{-(\alpha+i\beta)x}p_2(x,\lambda)\tag{2.20}$$

where $\alpha\neq 0$, α and β are real numbers, $p_1(x,\lambda)$ and $p_2(x,\lambda)$ are periodic functions. Let α .>0 then $y_1(x,\lambda)$ is defined in $L_2(-\infty,0)$ and $y_2(x,\lambda)$ is defined in. $L_2(0,\infty)$. Consider the differential equation

$$(L-\lambda I)y=f \tag{2.21}$$

for $f \in L_2(-\infty,\infty)$. The equality (2.20) shows that any linear combination of $y_1(x,\lambda)$ and $y_2(x,\lambda)$ are unbounded in $L_2(-\infty,\infty)$ so $(L-\lambda I)y=0$ only has a trivial solution in $L_2(-\infty,\infty)$. Therefore, the solution $y=(L-\lambda I)^{-1}f$ is of the form

$$y(x,\lambda)=c_1y_1(x,\lambda)+c_2y_2(x,\lambda) \tag{2.22}$$

Applying the method of variations of parameters to the differential equation (2.21), we have

$$\begin{aligned} c_1'y_1+c_2'y_2 &= 0 \\ c_1'y_1'+c_2'y_2' &= -f \end{aligned}$$

Solving this equations for c_1 and c_2 and replacing in (2.22), we obtain

$$((L-\lambda I)^{-1}f)(x)=\int_{-\infty}^{\infty}G(x,s,\lambda)f(s)ds$$

where Green's function is

$$G(x,s,\lambda)=\begin{cases}-y_1(x,\lambda)y_2(s,\lambda), & \mathrm{x<s} \\ -y_2(x,\lambda)y_1(s,\lambda), & \mathrm{x>s}\end{cases}$$

Thus, the resolvent is the integral operator with Green's kernel. When M is a boundary for $p_1(x,\lambda)$ and $p_2(x,\lambda)$,

$$\left|(L-\lambda I)^{-1}f\right| \le \left|\int_{-\infty}^{x}y_2(x,\lambda)y_1(s,\lambda)f(s)ds\right|+\left|\int_{x}^{\infty}y_1(x,\lambda)y_2(s,\lambda)f(s)ds\right|$$

$$\le \int_{-\infty}^{x}\left|e^{-(\alpha+i\beta)x}e^{(\alpha+i\beta)s}p_2(x,\lambda)p_1(s,\lambda)f(s)\right|ds+\int_{x}^{\infty}\left|e^{(\alpha+i\beta)x}e^{-(\alpha+i\beta)s}p_1(x,\lambda)p_2(s,\lambda)f(s)\right|ds$$

$$\le M^2[(T_1f)(x)+(T_2f)(x)]$$

where

$$(T_1f)(x)=\int_{-\infty}^{x}e^{\alpha(s-x)}\left|f(s)\right|ds,\qquad (T_2f)(x)=\int_{x}^{\infty}e^{-\alpha(s-x)}\left|f(s)\right|ds.$$

In order to show boundednees of the resolvent in the normed space $L_2(-\infty,\infty)$, it is enough to show boundedness of these operators. Integration by parts and using Schwarz inequality, it can be seen that

$$\|(T_1 f)(x)\| = [\int_{-\infty}^{\infty} | \int_{-\infty}^{x} e^{\alpha(s-x)} |f(s)| ds |^2 dx]^{1/2} \leq \frac{1}{\alpha} \|f\|$$

and similarly $(T_2 f)(x)$ is bounded. Thus, there exists a constant C such that

$$\frac{\|(L-\lambda I)^{-1} f\|}{\|f\|} = \frac{[\int_{-\infty}^{\infty} |L-\lambda I)^{-1} f|^2 dx]^{1/2}}{[\int_{-\infty}^{\infty} |f(x)|^2 dx]^{1/2}} \leq C$$

As a result, the operator $(L-\lambda I)^{-1}$ is bounded and so $\lambda \in \rho(L)$. In other words, $\lambda \notin S(L)$ if $\lambda \notin C(L)$, or equivalently

$$S(L) \subset C(L).$$

Hence,

$$C(L) = S(L)$$

and the previous theorem completes the proof immediately.

CHAPTER THREE
ASYMPTOTIC FORMULAS FOR EIGENFUNCTIONS AND EIGENVALUES

3.1 Introduction

In many applications of mathematics and physics, it is necessary to use discontinuous and singular functions, for instance, the step function and the delta function. A singular function which makes no sense in mathematics has a meaning whenever it corresponds to a functional. Under this consideration, it is possible to generalize the classical definition of functions. A generalized function is defined as a linear continuous functional on the space of test functions.

Since the potential q(x) has some singularities we shall take the periodic and singular function q(x) as a periodic generalized function whose Fourier series converges distributionally. Then, we shall seek the generalized eigenfunctions and eigenvalues of the differential operator L_t.

In this chapter, we obtain asymptotic formulas for eigenvalues and eigenfunctions of L_t with periodic generalized potential $q(x)=\sum_{n=-\infty}^{\infty} q_n e^{inx}$ which has Fourier coefficients q_n such that

$$q_n = O(1).$$

In particular, these asymptotic formulas remain true if the potential q(x) is a Lebesgue integrable function or if it is a generalized function as the delta distribution.

Now a brief note is needed about generalized functions before evaluating the asymptotic formulas for the eigenfunctions of L_t.

3.2 Generalized Functions

Definition 3.2.1 The support of a continuous function $\varphi(x)$ is the closure of the set $\{x:\ \varphi(x)\neq 0\}$, written as $\operatorname{supp}\varphi$.

The function $\varphi(x)$ has a compact support if its support is contained in a compact set, i.e. $\operatorname{supp}\varphi\subset[a,b]$ for some a, b $\in\mathbf{R}$.

Definition 3.2.2 Let K denote the set of functions infinitely differentiable in **R** with a compact support. A sequence φ_n in K converges to the function φ in K if there is a compact set Q=[a, b] such that $\operatorname{supp}\varphi_n\subset$ Q and for every k, the sequence of kth derivatives $\varphi_n^{(k)}(x)$ is uniformly convergent to the $\varphi^{(k)}(x)$ in Q. The set K with this sense of convergence is called the space of test functions.

Definition 3.2.3 A continuous linear functional *f* on K is called a generalized function. It is defined by the following conditions;

(i) there is a number *(f, φ)* associated with each $\varphi\in$K,

(ii) $(f,\alpha\varphi+\beta\psi)=\alpha(f,\varphi)+\beta(f,\psi)$ for all $\varphi,\psi\in$K and $\alpha,\beta\in\mathbf{C}$,

(iii) the generalized function *f* is a continuous functional on K, that is, if a sequence φ_n form K converges to φ in K then the sequence of complex numbers *(f, φ_n)* converges to *(f, φ)*. (Vladimirov, 1979).

The set of all generalized functions on K will be denoted by K$'$.

Example. If *f(x)* is an absolutely integrable function in any interval Q $\subset\mathbf{R}$ then *f(x)* defines a generalized function as

$$(f,\varphi)=\int_{-\infty}^{\infty} f(x)\varphi(x)dx, \qquad (3.1)$$

for every $\varphi \in \mathrm{K}$. Here, $f(x)\varphi(x)$ is integrable in **R** since $\varphi(x)=0$ outside of $\operatorname{supp}\varphi$. (3.1) is also linear and continuous. If $\sup_{x\in Q}|\varphi_n-\varphi|\to 0$, where $\operatorname{supp}\varphi\subset Q$, then

$$(f,\varphi_n-\varphi)=\int_{-\infty}^{\infty} f(x)(\varphi_n(x)-\varphi(x))dx \le \sup_{x\in Q}|\varphi_n-\varphi|\int_Q |f(x)|dx \to 0 \quad \text{as } n\to\infty .$$

Definition 3.2.4 A generalized function is said to be regular if it is generated by a locally integrable function. All others are called singular.

Example. The most important singular generalized function is the delta distribution defined by

$$(\delta(x),\varphi(x))=\varphi(0) \quad \text{for any } \varphi\in \mathrm{K}.$$

$\delta(x-x_0)$ shifts the value of $\varphi\in \mathrm{K}$ to x_0,

$$(\delta(x-x_0),\varphi(x))=\varphi(x_0).$$

Definition 3.2.5 The sequence of generalized functions f_n converges to $f\in \mathrm{K}'$ if $(f_k,\varphi)\to(f,\varphi)$ for all $\varphi\in \mathrm{K}$.

Definition 3.2.6 For $f\in \mathrm{K}'$ the functional $D^k f$ is called the kth generalized derivative if

$$(D^k f,\varphi)=(-1)^k(f,D^k\varphi)$$

for all $\varphi\in \mathrm{K}$. (Gelfand & Shilov, 1964).

Definition 3.2.7 A generalized function *f* is said to be periodic with period a if

$$(f(x+a),\varphi(x))\equiv(f(x),\varphi(x-a))=(f(x),\varphi(x))$$

for all $\varphi\in \mathrm{K}$.

Definition 3.2.8 Let S denote the space of infinitely differentiable functions which decrease as $|x| \to \infty$ together with their derivatives, faster than any power of $|x|^{-1}$. For $\varphi(x) \in \mathrm{S}$,

$$\left|x^N D^k \varphi(x)\right| \le C_{kN} \qquad (k = 0,1,2,\ldots,\ N = 1,2,3\ldots).$$

The set S' consisting of all linear continuous functionals on S is called the space of generalized functions of slow growth (tempered distribution). Clearly, $\mathrm{K} \subset \mathrm{S}$ and $\mathrm{S}' \subset \mathrm{K}'$.

Theorem 3.2.9 The Fourier series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{3.2}$$

converges to a periodic generalized function $f \in \mathrm{S}'$ if and only if

$$c_n = O(|n|^m)$$

for some integer m as $|n| \to \infty$.

The periodic generalized function f with the period 2π is determined uniquely by its Fourier coefficients such that

$$c_n = \frac{1}{2\pi}(f, e^{-inx}).$$

When f is an absolutely integrable function on $[0, a]$, the series in (3.2) converges to f in the sense of classical Fourier series where the coefficients c_n approach zero as n tends to infinity. However, the Fourier coefficients of a generalized function in S' may be polynomial growth.

Example. The singular distribution $\delta(x)$ has Fourier coefficients $c_n = \dfrac{1}{2\pi}$. Indeed, the series

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{inx}$$

does not converge in classical sense but it converges distributionally.

Therefore, $q(x)$ which is supposed to satisfy $q_n = O(1)$ behaves as the delta distribution

3.3 Asymptotic Formulas for Eigenfunctions and Eigenvalues

Definition 3.3.1 A nonzero function $y \in H_{1,t}[0,2\pi]$ is called generalized eigenfunction of the boundary value problem

$$-y'' + q(x)y = \lambda y$$

$$\begin{aligned} y(2\pi) &= e^{i2\pi t} y(0) \\ y'(2\pi) &= e^{i2\pi t} y'(0) \end{aligned} \qquad (3.3)$$

if there is a number λ corresponding to $y(x)$ such that the function $y(x)$ satisfies the identity

$$\int_0^{2\pi} \left(y'(x)\overline{v'(x)} + q(x)y(x)\overline{v(x)} - \lambda y(x)\overline{v(x)} \right) dx = 0 \qquad (3.4)$$

for all $v(x) \in H_{1,\bar{t}}[0,2\pi]$, where $H_{1,t}[0,2\pi]$ denotes the set of all functions satisfying the boundary conditions (3.3) and belonging to $L_2[0,2\pi]$ together with their first generalized derivatives.

Suppose that $\psi_{N,t}(x)$ is a generalized eigenfunction of L_t corresponding to the eigenvalue $\lambda_N(t)$. Taking $v(x) = e^{i(n+\bar{t})x}$, we have

$$\int_0^{2\pi} \left(\psi'_{N,t}(x)(e^{-i(n+t)x})' + q(x)\psi_{N,t}(x)e^{-i(n+t)x} - \lambda_N(t)\psi_{N,t}(x)e^{-i(n+t)x} \right) dx = 0 \quad (3.5)$$

The boundary conditions given in (1.18)

$$\psi_{N,t}(x+2\pi) = e^{i2\pi t}\psi_{N,t}(x)$$

shows that

$$\psi_{N,t}(x) = e^{itx} p(x)$$

where $p(x)$ is the 2π-periodic function

$$p(x) = \sum_{n=-\infty}^{\infty} p_n e^{inx} .$$

By definition of the space $H_{1,t}[0,2\pi]$, the generalized derivative of $\psi_{N,t}(x)$ belongs to $L_2[0,2\pi]$. It means that

$$\sum_{n=-\infty}^{\infty} inp_n e^{inx} \in L_2[0,2\pi],$$

that is

$$\sum_{n=-\infty}^{\infty} \left|np_n\right|^2 < \infty .$$

So,

$$\sum_n |p_n| = \sum_n |np_n| \frac{1}{|n|} \le \left(\sum_n |np_n|^2 \right)^{1/2} \left(\sum_n \frac{1}{|n|^2} \right)^{1/2} < \infty .$$

It is seen from this and the relation $q_n = O(1)$ that

$$q(x)\psi_{N,t}(x) = e^{itx} \sum_n \sum_k q_n p_k e^{i(n+k)x}$$
$$= \sum_m \sum_k q_{m-k} p_k e^{i(m+t)x} ,$$

$$\left| \int_0^{2\pi} q(x)\psi_{N,t}(x) e^{-i(n+t)x} dx \right| = \left| \int_0^{2\pi} \sum_m \sum_k q_{m-k} p_k e^{i(m+t)x} e^{-i(n+t)x} dx \right|$$
$$= 2\pi \left| \sum_n \sum_k q_{n-k} p_k \right|$$
$$= O(1) \sum_k |p_k| < \infty .$$

Hence, the integral $\int_0^{2\pi} q(x)\psi_{N,t}(x) e^{-i(n+t)x} dx$ exists. Therefore, (3.5) becomes

$$(\lambda_N(t) - (n+t)^2)(\psi_{N,t}(x), e^{i(n+t)x}) = (q(x)\psi_{N,t}(x), e^{i(n+t)x}) \qquad (3.6)$$

when we take $t \in \mathbf{R}$, i.e. $\bar{t} = t$, in order to simplify our notation.

Thus, we proved that

$$\left|(q(x)\psi_{N,t}(x), e^{i(n+t)x})\right| < C \tag{3.7}$$

Assume that C is independent on N and n. Here, we must note that if $q(x) \in L_1[0,2\pi]$ then the condition (3.7) is satisfied. (Marchenko, 1986).

Putting the distributionally convergent series (Fourier series of periodic generalized function of q(x))

$$q(x) = \sum_{n_1=-\infty}^{\infty} q_{n_1} e^{in_1 x} \ , \tag{3.8}$$

in (3.6), we get

$$(\lambda_N(t) - (n+t)^2)(\psi_{N,t}(x), e^{i(n+t)x}) = \sum_{n_1=-\infty}^{\infty} q_{n_1} (\psi_{N,t}(x), e^{i(n-n_1+t)x}). \tag{3.9}$$

Here, it can be assumed without loss of generality that $q_0 = 0$.

If we replace $n - n_1$ by n in the equation (3.6) to make an iteration and write the resulting expression in (3.9), then

$$(\lambda_N(t) - (n+t)^2)(\psi_{N,t}(x), e^{i(n+t)x}) = \sum_{n_1=-\infty}^{\infty} q_{n_1} \frac{(q(x)\psi_{N,t}(x), e^{i(n-n_1+t)x})}{\lambda_N(t) - (n-n_1+t)^2} \tag{3.10}$$

Setting

$$q(x) = \sum_{n_2=-\infty}^{\infty} q_{n_2} e^{in_2 x}$$

into (3.10) and using (3.6), we obtain

$$(\lambda_N(t) - (n+t)^2)(\psi_{N,t}(x), e^{i(n+t)x}) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} q_{n_1} q_{n_2} \frac{(\psi_{N,t}(x), e^{i(n-n_1-n_2+t)x})}{\lambda_N(t) - (n-n_1+t)^2} =$$

$$= [\sum_{\{n_1:\, n_1+n_2=0\}} \frac{q_{n_1} q_{-n_1}}{\lambda_N(t)-(n-n_1+t)^2}](\psi_{N,t}(x), e^{i(n+t)x}) +$$

$$+\sum_{n_1} \sum_{\{n_2:\, n_1+n_2\neq 0\}} q_{n_1} q_{n_2} \frac{(\psi_{N,t}(x), e^{i(n-n_1-n_2+t)x})}{\lambda_N(t)-(n-n_1+t)^2}$$

$$=[\sum_{\{n_1:\, n_1+n_2=0\}} \frac{q_{n_1} q_{-n_1}}{\lambda_N(t)-(n-n_1+t)^2}](\psi_{N,t}(x), e^{i(n+t)x}) +$$

$$+\sum_{n_1} \sum_{\{n_2:\, n_1+n_2\neq 0\}} q_{n_1} q_{n_2} \frac{(q(x)\psi_{N,t}(x), e^{i(n-n_1-n_2+t)x})}{[\lambda_N(t)-(n-n_1+t)^2][\lambda_N(t)-(n-n_1-n_2+t)^2]}. \tag{3.11}$$

Replacing (3.8) with index n_3 in the last sum of (3.11) and using (3.6), at the second step of iteration we get

$$(\lambda_N(t)-(n+t)^2)(\psi_{N,t}(x), e^{i(n+t)x}) = \{\sum_{\{n_1:\, n_1+n_2=0\}} \frac{q_{n_1} q_{-n_1}}{\lambda_N(t)-(n-n_1+t)^2} +$$

$$+\sum_{n_1} \sum_{\substack{\{n_2:\, n_1+n_2\neq 0,\\ n_1+n_2+n_3=0\}}} \frac{q_{n_1} q_{n_2} q_{-n_1-n_2}}{[\lambda_N(t)-(n-n_1+t)^2][\lambda_N(t)-(n-n_1-n_2+t)^2]}\}(\psi_{N,t}(x), e^{i(n+t)x}) +$$

$$+\sum_{n_1} \sum_{\{n_2: n_1+n_2\neq 0\}} \sum_{\{n_3:\, n_1+n_2+n_3\neq 0\}} \frac{q_{n_1} q_{n_2} q_{n_3}\ (q(x)\psi_{N,t}(x), e^{i(n-n_1-n_2-n_3+t)x})}{[\lambda_N(t)-(n-n_1+t)^2][\lambda_N(t)-(n-n_1-n_2+t)^2][\lambda_N(t)-(n-n_1-n_2-n_3+t)^2]}.$$

Continuing this iteration, the following general formula is obtained

$$(\lambda_N(t)-(n+t)^2)(\psi_{N,t}(x), e^{i(n+t)x}) = (\sum_{k=1}^{m} a_k)(\psi_{N,t}(x), e^{i(n+t)x}) + R_m \tag{3.12}$$

where

$$a_k = \sum_{n_1,n_2,\ldots,n_k} \frac{q_{n_1} q_{n_2} \cdots q_{n_k} q_{-n_1-n_2-\ldots-n_k}}{[\lambda_N(t)-(n-n_1+t)^2][\lambda_N(t)-(n-n_1-n_2+t)^2]\ldots[\lambda_N(t)-(n-n_1\ldots-n_k+t)^2]}$$

and

$$R_m = \sum_{n_1,n_2,\ldots,n_{m+1}} \frac{q_{n_1} q_{n_2} \cdots q_{n_m} q_{n_{m+1}} \,(q(x)\psi_{N,t}(x),\, e^{i(n-n_1\ldots-n_{m+1}+t)x})}{[\lambda_N(t)-(n-n_1+t)^2][\lambda_N(t)-(n-n_1-n_2+t)^2]\ldots[\lambda_N(t)-(n-n_1\ldots-n_{m+1}+t)^2]}$$

when $\sum_{j=1}^{m} n_j \neq 0$, (m=1,2,...).

Theorem 3.3.2 For each eigenvalue $\lambda_N(t)$ of L_t, where $|\lambda_N(t)| >> 1$, there is an integer n such that

$$\left|\lambda_N(t) - (n+t)^2\right| < C_1$$

and

$$\psi_{N,t}(x) = e^{i(n+t)x} + O(\frac{1}{n})$$

where C_1 is a constant. This asymptotic formula is uniform with respect to t in $Q_\varepsilon = \{t : t \in Q,\ \left|t - \frac{s}{2}\right| > \varepsilon,\ s = 0,1,2\}$ where Q is a bounded domain containing the interval [0,1] and ε is a sufficiently small positive number.

Proof. It follows from the equation (3.6) and (3.7) that

$$\left|\lambda_N(t) - (n+t)^2\right| \left|(\psi_{N,t}(x),\, e^{i(n+t)x})\right| < C. \tag{3.13}$$

The equation (3.6) can be written as

$$\left|(\psi_{N,t}(x),\, e^{i(n+t)x})\right| = \frac{\left|(q(x)\psi_{N,t}(x),\, e^{i(n+t)x})\right|}{\left|\lambda_N(t) - (n+t)^2\right|}. \tag{3.14}$$

Let $\lambda_N(t)$ be in the strip

$$(n_0+t)^2 \le \operatorname{Re}\lambda_N(t) \le (n_0+1+t)^2$$

for large integer $|n_0|$.

If $|k|>2$, $k\in\mathbf{Z}$, then

$$\left|\lambda_N(t)-(n_0+k+t)^2\right| > \left|(n_0+1+t)^2-(n_0+k+t)^2\right| > C_2|k||2(n_0+t)-k| \quad (3.15)$$

for a suitable constant C_2. Using (3.14), (3.15) and also (3.7), we get

$$\sum_{|k|>2}\left|(\psi_{N,t}(x),\, e^{i(n_0+k+t)x})\right|^2 < \frac{1}{C_2}\sum_{|k|>2}\frac{\left|(q(x)\psi_{N,t}(x),\, e^{i(n_0+k+t)x})\right|^2}{|k|^2|2(n_0+t)-k|^2}$$

$$< \frac{C^2}{C_2}\sum_{|k|>2}\frac{1}{k^2|2(n_0+t)-k|^2}. \quad (3.16)$$

Let substitute $a=2(n_0+t)$ where $t\ne 0,\frac{1}{2},1$. To simplify the sum in (3.16), we shall seperate some terms for which $k=a,a\pm1,a\pm2$. Hence,

$$\sum_{|k|>2}\frac{1}{k^2|2(n_0+t)-k|^2} = \sum_{\{k:\,|a-k|>2,\,|k|>2\}}\frac{1}{k^2|a-k|^2}+O\left(\frac{1}{{n_0}^2}\right)$$

Since $|a-k|\ne 0$, this can be expressed by a convergent integral.

$$\sum_{\{k:\,|a-k|>2,\,|k|>2\}}\frac{1}{k^2(a-k)^2} < \int_{-\infty}^{a-2}\frac{dx}{x^2(a-x)^2}+\int_{a+2}^{\infty}\frac{dx}{x^2(a-x)^2}$$

Let us calculate the last integral

$$\int_{a+2}^{\infty}\frac{dx}{x^2(a-x)^2} = \frac{1}{a^2}\int_{a+2}^{\infty}\left(\frac{2}{ax}+\frac{2}{a(a-x)}+\frac{1}{x^2}+\frac{1}{(a-x)^2}\right)dx$$

$$= \frac{1}{a^2}\lim_{u\to\infty}\left[\ \frac{2}{a}(\ln|x|-\ln(x-a))-\frac{1}{x}+\frac{1}{a-x}\ \right]_{a+2}^{u}$$

$$=\frac{1}{a^2}\{\lim_{u\to\infty}\left[\ \frac{2}{a}(\ln\frac{|u|}{u-a}-\ln\frac{|a+2|}{2})\ \right]+\frac{1}{a+2}+\frac{1}{2}\}$$

$$<\frac{1}{a^2}(\frac{\ln|a|}{|a|}+\frac{1}{|a+2|}+\frac{1}{2})$$

$$=O\left(\frac{1}{{n_0}^2}\right).$$

Similarly,

$$\int_{-\infty}^{a-2}\frac{dx}{x^2(a-x)^2}=O\left(\frac{1}{{n_0}^2}\right).$$

Thus,

$$\sum_{|k|>2}\left|(\psi_{N,t}(x),\ e^{i(n_0+k+t)x})\right|^2=O\left(\frac{1}{n_0^2}\right). \tag{3.17}$$

On the other hand, using Parseval's equality

$$\sum_{m=-\infty}^{\infty}\left|(\psi_{N,t}(x),\ e^{i(m+t)x})\right|^2=1$$

we have

$$\sum_{|k|\le 2}\left|(\psi_{N,t}(x),\ e^{i(n_0+k+t)x})\right|^2=1+O\left(\frac{1}{{n_0}^2}\right) \tag{3.18}$$

Therefore, there exists an integer n for which $n=n_0+k,\ |k|\le 2$ such that

$$\left|(\psi_{N,t}(x),\ e^{i(n+t)x})\right|>\frac{1}{2} \tag{3.19}$$

It follows from (3.19) and (3.13) that $\lambda_N(t)$ lies in the disc

$$\left|\lambda_N(t)-(n+t)^2\right|<C_1$$

where $C_1\ge 2C$.

For this reason, there exists a positive constant C_3 such that

$$\left|\lambda_N(t)-(n+k+t)^2\right|>C_3|n|$$

for $|k| \leq 2$ and $k \neq 0$. From this, (3.14) and (3.7), we obtain

$$\left|(\psi_{N,t}(x),\ e^{i(n+t)x})\right| = O\left(\frac{1}{n}\right)$$

Combining this with (3.17), we get

$$\sum_{\{k:k\neq 0\}} \left|(\psi_{N,t}(x),\ e^{i(n+k+t)x})\right|^2 = O\left(\frac{1}{n^2}\right). \qquad (3.20)$$

Hence, by Parseval's equality , for k=0 it is obtained

$$\left|(\psi_{N,t}(x),\ e^{i(n+t)x})\right|^2 = 1 + O\left(\frac{1}{n^2}\right),$$

and it yields

$$\left|(\psi_{N,t}(x),\ e^{i(n+t)x})\right| = 1 + O\left(\frac{1}{n^2}\right).$$

Hence, by (3.20), $\psi_{N,t}(x)$ has Fourier decomposition in the form

$$\psi_{N,t}(x) = e^{i(n+t)x} + g(x) + O\left(\frac{1}{n^2}\right)$$

where

$$\|g(x)\| = O\left(\frac{1}{n}\right).$$

As a result, we have the asymptotic formula

$$\psi_{N,t}(x) = e^{i(n+t)x} + O\left(\frac{1}{n}\right) \qquad (3.21)$$

Here, since the constants C_i and $O\left(\frac{1}{n}\right)$ does not depend on t, (3.21) is uniform with respect to $t \in Q_\varepsilon$.

Theorem 3.3.3 If the potential q(x) satisfies the condition

$$q_m = O(1)$$

then

$$\lambda_n(t) = (n+t)^2 + F_1 + F_2 + \ldots + F_{k-1} + O\left(\left(\frac{\ln n}{n}\right)^k\right) \qquad (3.22)$$

where

$$F_1 = \sum_{n_1} \frac{q_{n_1} q_{-n_1}}{(n+t)^2 - (n-n_1+t)^2} = O\left(\frac{\ln n}{n}\right)$$

$$F_2 = \sum_{n_1, n_2} \frac{q_{n_1} q_{n_2} q_{-n_1-n_2}}{[(n+t)^2 - (n-n_1+t)^2][(n+t)^2 - (n-n_1-n_2+t)^2]} = O\left((\frac{\ln n}{n})^2\right)$$

$$\vdots$$

$$F_s = O\left((\frac{\ln n}{n})^s\right)$$

and $\tilde{F}_s(x) = \sum_{k=1}^{s} F_k(x)$ is defined as following

$$\tilde{F}_s(x) = A_k(\tilde{F}_{s-1}(x) + x), \quad A_k(x) = \sum_{j=1}^{k-1} a_j(x),$$

$$a_j(x) = \sum_{n_1, n_2, \ldots n_j} \frac{q_{n_1} q_{n_2} \cdots q_{n_j} q_{-n_1-n_2 \ldots -n_j}}{[x-(n-n_1+t)^2][x-(n-n_1-n_2+t)^2 \ldots [x-(n-n_1-n_2 \ldots -n_j+t)^2]}$$

for which $x = (n+t)^2$.

Here, the asymptotic expression (3.22) is uniform with respect to $t \in Q_\varepsilon$.

Proof. $q_m = O(1)$ means that, there exists a constant C_4 such that

$$|q_m| \le C_4$$

for large integer $|m|$. By Theorem (3.3.2), we have

$$\left|\lambda_n(t) - (n+t)^2\right| < C_1.$$

Therefore, for any integer $k \neq 0$,

$$\left|\lambda_n(t) - (n+t+k)^2\right| > |k||2(n+t) - k| - C_1 > C_5 |k||2(n+t) - k|$$

holds for a suitable positive constant C_5. Then,

$$|a_1| \le \sum_{\{n_1 \in I:\, n_1 \ne 0\}} \frac{|q_{n_1} q_{n_{-1}}|}{|\lambda_n(t) - (n - n_1 + t)^2|} < \frac{C_4{}^2}{C_5} \sum_{\{k \in I:\, k \ne 0\}} \frac{1}{|k||2(n+t) - k|}$$

Let substitute $a = 2(n+t)$ where $t \in Q_\varepsilon$ and separate some terms of the above sum for which $|a - k| \le 2$. Then,

$$|a_1| = O\left(\frac{1}{n}\right) + \sum_{\{k:\, k \ne 0, |a-k| > 2\}} \frac{1}{|k||a-k|}.$$

Let us calculate this sum.

$$\sum_{\{k:\, k \ne 0, |a-k| > 2\}} \frac{1}{|k||a-k|} < \int_{-\infty}^{a-2} \frac{dx}{|x(a-x)|} + \int_{a+2}^{\infty} \frac{dx}{|x(a-x)|} \tag{3.23}$$

Here,

$$\int_{a+2}^{\infty} \frac{dx}{x(a-x)} = \frac{1}{a} \int_{a+2}^{\infty} \left(\frac{1}{x-a} - \frac{1}{x}\right) dx = \frac{1}{a} \lim_{u \to \infty} \Big[\ \ln|x-a| - \ln|x| \ \Big]_{a+2}^{u} =$$

$$= \frac{1}{a} \ln\left|\frac{a+2}{2}\right| < \frac{\ln|a|}{|a|} = O\left(\frac{\ln n}{n}\right).$$

Thus, together with a similar evaluation for the other integral in (3.23), we have

$$a_1 = O\left(\frac{1}{n}\right) + O\left(\frac{\ln n}{n}\right) \equiv O\left(\frac{\ln n}{n}\right).$$

Now, assume that

$$a_{k-1} = O\left(\left(\frac{\ln n}{n}\right)^{k-1} \right). \tag{3.24}$$

Then,

$$|a_k| = \sum_{n_1, n_2, \ldots, n_k} \frac{|\ q_{n_1} q_{n_2} \cdots q_{n_k} q_{-n_1 - n_2 \ldots - n_k}\ |}{|\ [\lambda_n(t) - (n - n_1 + t)^2][\lambda_n(t) - (n - n_1 - n_2 + t)^2] \ldots [\lambda_n(t) - (n - n_1 \ldots - n_k + t)^2]\ |}$$

$$< C_4^{k+1} \sum_{n_1,n_2,\ldots,n_k} \frac{1}{|\lambda_n(t)-(n-n_1+t)^2||\lambda_n(t)-(n-n_1-n_2+t)^2|\ldots|\lambda_n(t)-(n-n_1\ldots-n_k+t)^2|}$$

$$< C_4^{k+1} \sum_{n_1,n_2,\ldots,n_{k-1}} \left(\frac{1}{|\lambda_n(t)-(n-n_1+t)^2|\ldots|\lambda_n(t)-(n-n_1\ldots-n_{k-1}+t)^2|} f(n_1,n_2,\ldots,n_{k-1})\right) \tag{3.25}$$

where

$$f(n_1,n_2,\ldots,n_{k-1}) = \sum_{n_k} \frac{1}{|\lambda_n(t)-(n-n_1\ldots-n_{k-1}-n_k+t)^2|} .$$

For each value of $n_1,n_2,\ldots,n_{k-1}$ regarding as fix, the estimation of $f(n_1,n_2,\ldots,n_{k-1})$ with index n_k is the same as the estimation of a_1 with index n_1. Hence,

$$f(n_1,n_2,\ldots,n_{k-1}) = O\left(\frac{\ln n}{n}\right) .$$

Moreover, this estimation does not depend on $n_1,n_2,\ldots,n_{k-1}$, that is, there exists C_6 such that

$$|f(n_1,n_2,\ldots,n_{k-1})| \le C_6 \frac{\ln n}{n}$$

for all $n_1,n_2,\ldots,n_{k-1}$. Hence,

$$\sup_{\{n_1,n_2,\ldots,n_{k-1}\}} f(n_1,n_2,\ldots,n_{k-1}) = O\left(\frac{\ln n}{n}\right) . \tag{3.26}$$

From (3.25), (3.26) and the assumption (3.24), we obtain

$$a_k = O\left(\left(\frac{\ln n}{n}\right)^{k-1}\right) O\left(\left(\frac{\ln n}{n}\right)\right) \equiv O\left(\left(\frac{\ln n}{n}\right)^{k}\right)$$

Similarly, using (3.7), we can estimate

$$R_m = O\left(\left(\frac{\ln n}{n}\right)^{m+1}\right).$$

From the formula (3.12) and the inequality (3.19), we get

$$\left|\lambda_n(t)-(n+t)^2\right| < \sum_{k=1}^{m}|a_k| + 2|R_m|.$$

Therefore, the order of eigenvalues is determined by the order of a_1, that is

$$\lambda_n(t) = (n+t)^2 + O\left(\frac{\ln n}{n}\right).$$

So the formula (3.22) is proved for k=1.

To prove (3.22) by induction, assume that (3.22) is true for k=j, that is

$$\lambda_n(t) = (n+t)^2 + \tilde{F}_{j-1} + O\left((\frac{\ln n}{n})^j\right). \tag{3.27}$$

First, divide both sides of (3.12) by $(\psi_{N,t}(x), e^{i(n+t)x})$, recalling the new notation n=N and truncating the series (3.12) at k, and secondly, use the asymptotic formula for $\psi_{n,t}(x)$ given by (3.21). Then, it is obtained for k=j that

$$\lambda_n(t) = (n+t)^2 + \sum_{s=1}^{j} a_s + O\left((\frac{\ln n}{n})^{j+1}\right). \tag{3.28}$$

Putting (3.27) into a_s in (3.28), considering that a_s is defined in (3.12) with argument $\lambda_n(t)$, we get

$$\lambda_n(t) = (n+t)^2 + A_{j+1}((n+t)^2 + \tilde{F}_{j-1} + O\left((\frac{\ln n}{n})^j\right)) + O\left((\frac{\ln n}{n})^{j+1}\right)$$

$$= (n+t)^2 + A_{j+1}((n+t)^2 + \tilde{F}_{j-1}) + O\left((\frac{\ln n}{n})^{j+1}\right) +$$

$$+ [A_{j+1}((n+t)^2 + \tilde{F}_{j-1} + O\left((\frac{\ln n}{n})^j\right)) - A_{j+1}((n+t)^2 + \tilde{F}_{j-1})].$$

$$= (n+t)^2 + \tilde{F}_j + O\left((\frac{\ln n}{n})^{j+1}\right) +$$

$$+[A_{j+1}((n+t)^2+\tilde{F}_{j-1}+O\left((\frac{\ln n}{n})^j\right))-A_{j+1}((n+t)^2+\tilde{F}_{j-1})]$$

It is not difficult to see that the sum in the bracket is of order $O\left((\frac{\ln n}{n})^{j+1}\right)$. Thus, the theorem is proved.

CHAPTER FOUR

SPECTRAL EXPANSION FOR THE NONSELF-ADJOINT DIFFERENTIAL OPERATOR L

4.1 Eigenfunctions as a Riesz Basis

In this section, we prove that the system of eigenfunctions of the operator L_t with summable potential q(x) form a Riesz basis for almost all values of t. For this we shall use the following definitions and theorems.

Definition 4.1.1 An eigenvalue λ_0 is said to have multiplicity p if λ_0 is a root of the characteristic function $\Delta(\lambda)$ with multiplicity p.

Since $\Delta(\lambda)$ is an entire function of λ,

$$\frac{\partial^p \Delta(\lambda_0)}{\partial \lambda^p} \neq 0, \quad \frac{\partial^k \Delta(\lambda_0)}{\partial \lambda^k} = 0, \quad k = 0,1,\ldots,p-1.$$

An eigenvalue of multiplicity one is called simple. In this case, there is only one eigenfunction corresponding to a simple eigenvalue.

Definition 4.1.2 Let φ_0 be an eigenfunction of L_t corresponding to the eigenvalue λ_0. The functions $\varphi_1,\varphi_2,\ldots,\varphi_{p-1}$ are called associated functions with the eigenfunction φ_0 if

$$(L_t - \lambda_0 I)\varphi_k = \varphi_{k-1}$$
$$U_i(\varphi_k) = 0$$

for k=1,2,...,p-1 and i=1,2.

Theorem 4.1.3 The system of eigenfunctions and associated functions of the Strum-Liouville operator with nondegenerate boundary conditions and integrable potential is complete in the space $L_2[0,2\pi]$. (Marchenko, 1986).

Definition 4.1.4 Two sequences $\{\phi_j\}_1^\infty$ and $\{\psi_j\}_1^\infty$ are said to be biorthogonal if

$$(\phi_i, \psi_j) = \delta_{ij} \qquad \text{(i,j=1,2,...).}$$

Definition 4.1.5 A sequence $\{\phi_j\}_1^\infty$ in $L_2[0,2\pi]$, or generally in the Hilbert space, is called a basis if every function f in this space can be expanded in a series

$$f = \sum_{j=1}^{\infty} (f, \psi_j)\phi_j$$

which converges in the norm of $L_2[0,2\pi]$, where $(\phi_i, \psi_j) = \delta_{ij}$, $\quad \psi_j \in L_2[0,2\pi]$.

Definition 4.1.6 A basis $\{\psi_j\}_1^\infty$ which is obtained from an orthonormal basis by a linear bounded invertible transformation A is called a basis equivalent to an orthonormal basis, or a Riesz basis.

If $\{\phi_j\}_1^\infty$ is an orthonormal basis in $L_2[0,2\pi]$, $f \in L_2[0,2\pi]$ can be expanded as

$$f = \sum_{j=1}^{\infty} (f, \chi_j)\psi_j$$

such that

$$\psi_j = A\phi_j, \quad \chi_j = A^{*-1}\phi_j \qquad \text{j=1,2,....}$$

where $\{\psi_j\}_1^\infty$ is a Riesz basis in $L_2[0,2\pi]$ derived from $\{\phi_j\}_1^\infty$ by the transformation A. Here, $\{\chi_j\}_1^\infty$ is the biorthogonal system to $\{\psi_j\}_1^\infty$. (Goghberg&Krein,1965).

Theorem 4.1.7 The following assertions are equivalent.

(i) The sequence $\{\psi_j\}_1^\infty$ forms a Riesz basis.

(ii) The sequence $\{\psi_j\}_1^\infty$ is complete in Hilbert space, there corresponds to it a complete biorthogonal sequence $\{\chi_j\}_1^\infty$, and for any f in Hilbert space one has

$$\sum_{j=1}^{\infty}\left|(f,\psi_j)\right|^2 < \infty, \quad \sum_{j=1}^{\infty}\left|(f,\chi_j)\right|^2 < \infty .$$

(Goghberg & Krein, 1965).

In chapter one, it is proved that the boudary conditions (1.6) of L_t are nondegenerate for all $t \in \mathbf{C}$. Therefore, by Theorem (4.1.3) the system of eigenfunctions and associated functions of L_t is complete in $L_2[0,2\pi]$.

It is known form chapter one that the eigenvalues $\lambda_n(t)$ of L_t are the roots of

$$F(\lambda) = 2\cos 2\pi t .$$

If $\frac{dF}{d\lambda}\Big|_{\lambda=\lambda_n(t)} \neq 0$ then the eigenvalue $\lambda_n(t)$ is simple.

Let $\nu_1, \nu_2, \ldots$ be the roots of the entire function $\frac{dF}{d\lambda}$ and $t_1, t_2, \ldots$ be the points for which $\nu_k = \lambda_n(t_k)$. Then $\lambda_n(t)$ is simple for $t \neq t_1, t_2, \ldots$.

We wish to prove that if $t \neq t_1, t_2, \ldots, \overline{t_1}, \overline{t_2}, \ldots$ then $\{\psi_{n,t}(x)\}_1^\infty$ forms a Riesz basis. For this, let us find the biorthogonal system to $\{\psi_{n,t}(x)\}_1^\infty$.

Multiplying the inner product $(\psi_{n,t}, \psi_{m,t}{}^*)$ by $\lambda_n(t)$, we have

$$\lambda_n(t)(\psi_{n,t},\psi_{m,t}^{*}) = (L_t\psi_{n,t},\psi_{m,t}^{*}) = (\psi_{n,t},L_t^{*}\psi_{m,t}^{*}) = \overline{\mu_m(t)}(\psi_{n,t},\psi_{m,t}^{*})$$

where $\mu_m(t)$ the eigenvalue of the adjoint operator L_t^{*} corresponding to $\psi_{m,t}^{*}$. Thus,

$$(\lambda_n(t) - \overline{\mu_m(t)})(\psi_{n,t},\psi_{m,t}^{*}) = 0. \tag{4.1}$$

Since $\left\{\psi_{n,t}(x)\right\}_1^{\infty}$ is complete system, any $\psi_{m,t}^{*}$ cannot be orthogonal to $\psi_{n,t}(x)$ for all n. Hence, there exists $\psi_{n,t}$ such that

$$(\psi_{n,t},\psi_{m,t}^{*}) \neq 0. \tag{4.2}$$

Then, it is obtained from (4.1) that

$$\lambda_n(t) = \overline{\mu_m(t)}. \tag{4.3}$$

We denote by $\mu_n(t)$ the eigenvalues of L_t^{*} satisfying (4.3). Since for $t \neq t_1, t_2, \ldots, \overline{t_1}, \overline{t_2}, \ldots$ the eigenvalues $\mu_1(t), \mu_2(t), \ldots$ are distinct , (4.3) holds only for one m. So, we have

$$\mu_n(t) = \overline{\lambda_n(t)}$$

and

$$\alpha_n(t) \equiv (\psi_{n,t},\psi_{n,t}^{*}) \neq 0.$$

If $\mu_m(t) \neq \overline{\lambda_n(t)}$ then from (4.1),

$$(\psi_{n,t},\psi_{m,t}^{*}) = 0.$$

Hence, the systems $\left\{\chi_{n,t}\right\}_1^{\infty} = \left\{\dfrac{\psi_{n,t}^{*}}{\overline{\alpha_n(t)}}\right\}_1^{\infty}$ and $\left\{\psi_{n,t}\right\}_1^{\infty}$ are biorthogonal , i.e.

$$(\psi_{n,t},\chi_{m,t}) = \delta_{nm} \qquad \text{(n,m=1,2,...)}.$$

It is known from Theorem (3.3.2) that

$$\psi_{n,t} = e^{i(n+t)x} + O(\frac{1}{n}).$$

From here, considering the adjoint problem (1.13), we have

$$\psi^{*}_{n,t} = e^{i(n+\bar{t})x} + O(\frac{1}{n}). \tag{4.4}$$

Therefore,

$$\alpha_n(t) = 1 + O(\frac{1}{n}). \tag{4.5}$$

Hence, the biorthogonal system to $\{\psi_{n,t}(x)\}_1^\infty$ has the following asymptotic formula

$$\chi_{n,t} = e^{i(n+\bar{t})x} + O(\frac{1}{n}).$$

Theorem 4.1.8 If $t \neq t_1, t_2, \ldots, \overline{t_1}, \overline{t_2}, \ldots$, then $\{\psi_{n,t}(x)\}_1^\infty$ forms a Riesz basis of $L_2[0,2\pi]$. The sequence $\{t_k\}_1^\infty$ has only the limit point $\frac{s}{2}$ where s is an integer.

Proof. We proved that for $t \neq t_1, t_2, \ldots, \overline{t_1}, \overline{t_2}, \ldots$ $\{\psi_{n,t}(x)\}_1^\infty$ is complete system and it has complete biorthogonal system $\{\chi_{n,t}(x)\}_1^\infty$. For every $f(x) \in L_2[0,2\pi]$, the asymptotic formula for the eigenfunctions give us

$$\begin{aligned}\sum_{n=1}^{\infty} \left|(f,\psi_{n,t})\right|^2 &\leq 2\{\sum_{n=1}^{\infty} \left|(f,e^{i(n+t)x})\right|^2 + \sum_{n=1}^{\infty} O\left(\frac{1}{n^2}\right)\} \\ &= 2\{\sum_{n=1}^{\infty} \left|(fe^{-itx},e^{inx})\right|^2 + \sum_{n=1}^{\infty} O\left(\frac{1}{n^2}\right)\} \\ &= 2\{\left\|fe^{-itx}\right\|^2 + \sum_{n=1}^{\infty} O\left(\frac{1}{n^2}\right)\} < \infty\end{aligned}$$

since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Similarly,

$$\sum_{n=1}^{\infty} \left|(f,\chi_{n,t})\right|^2 < \infty.$$

As a result of Theorem (4.1.7), the system $\{\psi_{n,t}(x)\}_1^\infty$ forms a Riesz basis of $L_2[0,2\pi]$.

Now, assume that $t \in Q_\varepsilon$ and $\lambda_n(t)$ is a multiple eigenvalue for n>>1. Suppose that there are two eigenfunctions corresponding to the multiple eigenvalue $\lambda_n(t)$. Then, we can choose two eigenfunction $\psi_{n,t}$ and $\psi_{n',t}$ which are orthogonal

$$(\psi_{n,t}, \psi_{n',t}) = 0. \tag{4.6}$$

It follows from Theorem (3.3.2) that both $\psi_{n,t}$ and $\psi_{n',t}$ satisfy the asymptotic formula (3.21). Hence,

$$(\psi_{n,t}, \psi_{n',t}) = 1 + O\left(\frac{1}{n}\right).$$

This contradicts to (4.6), so there is only one eigenfunction $\psi_{n,t}$ corresponding to $\lambda_n(t)$. Similarly, the adjoint operator L_t^* has also only one eigenfunction $\psi_{n,t}{}^*$ correspoding to $\mu_n(t)$.

First, we show that there is an eigenvalue $\mu_n(t)$ of L_t^* such that $\overline{\mu_n(t)} = \lambda_n(t)$. Indeed, if $\overline{\mu_m(t)} \neq \lambda_n(t)$ for all m then (4.1) shows that $\psi_{n,t}$ is orthogonal to $\psi_{m,t}{}^*$ for all m. If $h_{m,t}{}^*$ is an associated function with the eigenfunction $\psi_{m,t}{}^*$, that is

$$(L_t^* - \mu_m(t)I)h_{m,t}{}^* = \psi_{m,t}{}^*,$$

then multiplying it from the rigth side by $\psi_{n,t}$, we obtain

$$\begin{aligned} 0 = (\psi_{m,t}{}^*, \psi_{n,t}) &= ((L_t^* - \mu_m(t)I)h_{m,t}{}^*, \psi_{n,t}) \\ &= (h_{m,t}{}^*, (L_t - \overline{\mu_m(t)}I)\psi_{n,t}) \\ &= (\overline{\lambda_n(t)} - \mu_m(t))(h_{m,t}{}^*, \psi_{n,t}) \end{aligned}$$

Hence, $\psi_{n,t}$ is orthogonal to the associated function $h_{m,t}{}^{*}$ too. In this way, we show that $\psi_{n,t}$ is othogonal to all eigenfunctions and associated functions of L_t^{*} which contradicts to its completeness. Thus,

$$\overline{\mu_n(t)} = \lambda_n(t).$$

Since $\lambda_n(t)$ is an multiple eigenvalue, there is an associated function $h_{n,t}$ with $\psi_{n,t}$, i.e.

$$(L_t - \lambda_n(t)I)h_{n,t} = \psi_{n,t}.$$

Multiplying this from the rigth side by $\psi_{n,t}{}^{*}$, we get

$$\begin{aligned}(\psi_{n,t}, \psi_{n,t}{}^{*}) &= ((L_t - \lambda_n(t)I)h_{n,t}, \psi_{n,t}{}^{*}) \\ &= (h_{n,t}, (L_t^{*} - \mu_n(t)I)\psi_{n,t}{}^{*}) \\ &= (\overline{\mu_n(t)} - \lambda_n(t))(h_{n,t}\ , \psi_{n,t}{}^{*}) \\ &= 0\end{aligned}$$

which contradicts to (4.5). Thus, for $t \in Q_\varepsilon$ and n>>1 the eigenvalue $\lambda_n(t)$ is simple. It means that, $t_k \notin Q_\varepsilon$ for large k. Hence, for each $\varepsilon > 0$ the point of Q_ε cannot be the limit point of sequence $\{t_k\}_1^\infty$. Therefore, $t_k \to \frac{s}{2}$ as $k \to \infty$ where $s \in \mathbf{Z}$.

We should note that, if the potential q(x) is a continuous function then the eigenfunctions of Strum-Liouville operator with strictly regular boundary conditions form a Riesz basis. (Naimark,1968). Since for $t \neq \frac{k}{2}$ the the boundary conditions (1.6) are strictly regular, which we have shown in chapter one, the system of eigenfunctions and associated eigenfunctions of L_t for $t \neq \frac{k}{2}$ form a Riesz basis.

4.2 Spectral Decomposition

Spectral decomposition corresponding to the self-adjoint different operator

$$L = -\frac{d^2}{dx^2} + q(x)$$

with a 1-periodic, continuous and real-valued potential q(x) is constructed in the following lemma by I. M. Gelfand.

Lemma 4.2.1 (Gelfand) Every continuous compactly supported function *f(x)* can be represented in the form

$$f(x) = \frac{1}{2\pi}\int_0^{2\pi} f_t(x)dt \tag{4.7}$$

where

$$f_t(x) = \sum_{k=-\infty}^{\infty} f(x+k)e^{-ikt} \tag{4.8}$$

satisfying

$$f_t(x+1) = e^{it} f_t(x). \tag{4.9}$$

Moreover,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi}\int_0^{2\pi} \|f_t(x)\|^2_{L_2[0,1]} dt. \tag{4.10}$$

If q(x) is real-valued, the eigenfunctions $\psi_{1,t}(x), \psi_{2,t}(x), \ldots$ of the self-adjoint differential operator L_t form orthonormal basis in $L_2[0,1]$. Therefore $f_t(x)$ has decomposition as

$$f_t(x) = \sum_{n=1}^{\infty} a_n(t)\psi_{n,t}(x)$$

and we have Parseval's equality

$$\int_0^1 |f_t(x)|^2 dx = \sum_{n=1}^{\infty} |a_n(t)|^2 \tag{4.11}$$

where

$$a_n(t) = (f_t(x), \psi_{n,t}(x))_{L_2[0,1]} = (f(x), \psi_{n,t}(x))_{L_2(-\infty,\infty)}.$$

It follows form (4.10) and (4.11) that

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_0^{2\pi} (\sum_{n=1}^{\infty} |a_n(t)|^2) dt .$$

Clearly, the sum $\sum_{n=1}^{\infty} |a_n(t)|^2$ can be integrated term by term so

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \sum_{n=1}^{\infty} \int_0^{2\pi} (|a_n(t)|^2) dt$$

from which the spectral expansion was obtained by I.M. Gelfand.

In the nonself-adjoint case , i.e. for the case when q(x) is complex-valued, the Parseval's equality does not hold. Therefore, we cannot use (4.11). Moreover, the function $a_n(t)$ has some sigularities and we cannot consider the integral

$$\int_0^{2\pi} |a_n(t)|^2 dt$$

or

$$\int_0^{2\pi} a_n(t) \psi_{n,t}(x) dt .$$

Remark. We shall construct the spectral expansion for the nonself-adjoint differential operator L by using the method of the papers (Veliev, 1980,1983,1986), in which the spectral expansion for L with a 1-periodic, complex-valued and contionuous potential q(x). Therefore, we shall consider the operator L_t generated by the differential expression (1.5) in $L_2[0,1]$ and the boundary conditions

$$y(1) = e^{it} y(0), \quad y'(1) = e^{it} y'(0) .$$

Theorem (4.1.8) shows that, there is a curve $l_\varepsilon \in Q_\varepsilon$ with the end points $-\pi + t_0$ and $\pi + t_0$ $(t_0 \neq t_k, 0, \pi)$ such that $\pi s, t_k \notin l_\varepsilon$ and the distance between any point $t \in l_\varepsilon$ and the real line less than 2ε.

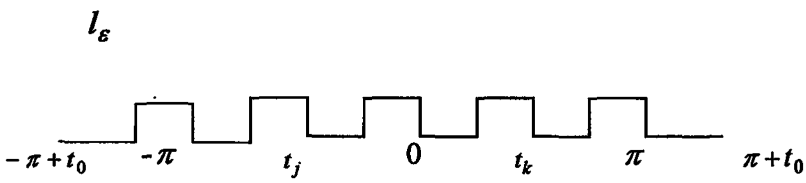

Figure 1

Hence, by Theorem (4.1.8), for every $t \in l_\varepsilon$ the sequence of eigenfunctions $\{\psi_{n,t}(x)\}_1^\infty$ of L_t form a Riesz basis of $L_2[0,1]$. Thus, we have the formula

$$f_t(x) = \sum_{n=1}^{\infty} a_n(t)\psi_{n,t}(x) \tag{4.13}$$

for all $t \in l_\varepsilon$, where

$$a_n(t) = (f_t(x), \chi_{n,t}(x)) = \frac{1}{\alpha_n(t)}(f_t(x), \psi_{n,t}^{\,*}(x)).$$

Using (4.8) which also holds for complex t and the relation

$$\chi_{n,t}(x+1) = e^{i\bar{t}}\chi_{n,t}(x)$$

we get

$$a_n(t) = \int_0^1 f_t(x), \overline{\chi_{n,t}}(x)dx,$$

$$= \int_0^1 \Big(\sum_{k=-\infty}^{\infty} f(x+k)e^{-ikt} \Big)\overline{\chi_{n,t}}(x)dx$$

$$= \sum_{k=-\infty}^{\infty} \int_0^1 (f(x+k)\overline{\chi_{n,t}(x)e^{ik\bar{t}}}dx$$

$$= \sum_{k=-\infty}^{\infty} \int_0^1 f(x+k)\overline{\chi_{n,t}}(x+k)dx$$

$$= (f(x), \chi_{n,t}(x))_{L_2(-\infty,\infty)}$$

Since for a compactly supported function *f(x)*, the function $f_t(x)$ defined by (4.8) is trigonometric polynomial with respect to t, $f_t(x)$ is an analytic function of t in any domain containing l_ε. Therefore, using Cauchy theorem and (4.9), we get

$$\int_0^{2\pi} f_t(x)dt = \int_{l_\varepsilon} f_t(x)dt$$

Now it follows from (4.7) and (4.13) that

$$f(x) = \frac{1}{2\pi}\int_{l_\varepsilon} f_t(x)dt = \frac{1}{2\pi}\int_{l_\varepsilon} \sum_{k=1}^{\infty} a_n(t)\psi_{n,t}(x)dt .$$

Repeating the proof which was given in the papers of O. A. Veliev (Veliev, 1980,1983,1986) and using the obtained asymptotic formula for $\psi_{n,t}(x)$ and $\chi_{n,t}(x)$, we obtain

$$f(x) = \frac{1}{2\pi}\sum_{k=1}^{\infty} \int_{l_\varepsilon} a_n(t)\psi_{n,t}(x)dt . \tag{4.14}$$

The series in (4.14) converges in the norm of $L_2[a,b]$ for arbitrary a and b.

It is not hard to check that

$$\psi_{n,t}(x) = \frac{\psi_1(x,\lambda_n(t))}{\|\psi_1(x,\lambda_n(t))\|}, \qquad \psi^*_{n,t}(x) = \frac{\overline{\psi_2(x,\lambda_n(t))}}{\|\psi_2(x,\lambda_n(t))\|} \tag{4.15}$$

where

$$\psi_1(x,\lambda) = \varphi(1,\lambda)\theta(x,\lambda) + (e^{it} - \theta(1,\lambda))\varphi(x,\lambda)$$

and

$$\psi_2(x,\lambda) = \varphi(1,\lambda)\theta(x,\lambda) + (e^{-it} - \theta(1,\lambda))\varphi(x,\lambda) .$$

From (4.15),

$$\alpha_n(t) = -\frac{\varphi(1,\lambda)}{\|\psi_1\|\|\psi_2\|}\int_0^1 (\theta'(1,\lambda)\varphi^2(x,\lambda) + (\theta(1,\lambda) - \varphi'(1,\lambda))\theta(x,\lambda)\varphi(x,\lambda) - \varphi(1,\lambda)\theta^2(x,\lambda))dx$$

$$=-\frac{\varphi(1,\lambda)}{\|\psi_1\|\|\psi_2\|}\frac{dF}{d\lambda}$$

(4.16)

On the other hand, from the relation

$$F(\lambda)=2\cos t$$

and by impilicit function theorem, we get

$$\frac{dt}{d\lambda}=\frac{dF}{d\lambda}\Big/2\sin t=\frac{dF}{d\lambda}\Big/\sqrt{4-F^2(\lambda)} \tag{4.17}$$

Denote by G_ε the domain with the boundary l_ε and $[-\pi+t_o,\pi+t_o]$. In the view of Theorem (4.1.8), we see that $a_n(t)\psi_{n,t}(x)$ has only finite number of singularities in G_ε for a fixed n. Indeed, if $t\neq t_k$ then

$$\alpha_n(t)=(\psi_{n,t},\psi_{n,t}{}^*)\neq 0.$$

Otherwise it contradicts to completeness of $\{\psi_{n,t}{}^*\}_1^\infty$. If $t=t_k$ and L_{t_k} has an associated function $h_{n,t}$ then in the proof of Theorem (4.1.8), we see that $\alpha_n(t)=0$ and $a_n(t)\psi_{n,t}(x)$ has singularity. Since $\nu_k\to\infty$ as $k\to\infty$ for fixed n the function $\alpha_n(t)$ can be vanished only for finite number of t_k. Therefore, if $\alpha_n(t)\neq 0$ in G_ε then we have

$$\int_{l_\varepsilon} a_n(t)\psi_{n,t}(x)dt=\int_0^{2\pi} a_n(t)\psi_{n,t}(x)dt.$$

Otherwise we have

$$\int_0^{2\pi} a_n(t)\psi_{n,t}(x)dt=\lim_{\varepsilon\to 0}\int_{l_\varepsilon} a_n(t)\psi_{n,t}(x)dt.$$

Now, using it and the relations (4.14), (4.16), (4.17), and changing the variable t to λ in the integral (4.14), we get the spectral expansion

$$f(x)=\sum_{n=1}^{\infty}[\frac{1}{\pi}\int_{\Gamma_n} B(\phi(x,\lambda))\frac{1}{p(\lambda)}d\lambda+\sum_{k\in K_n} M_k^n(x)]$$

where

$$\phi(x,\lambda)=\theta'(1,\lambda)h(\lambda)\varphi(x,\lambda)+\frac{1}{2}(\theta(1,\lambda)-\varphi'(1,\lambda))(\theta(x,\lambda)h(\lambda)$$
$$+g(\lambda)\varphi(x,\lambda))-\varphi(1,\lambda)g(\lambda)\theta(x,\lambda)$$

$h(\lambda)=\int\limits_{-\infty}^{\infty}\varphi(x,\lambda)f(x)dx,\quad g(\lambda)=\int\limits_{-\infty}^{\infty}\theta(x,\lambda)f(x)dx,\quad p(\lambda)=\sqrt{4-F^2(\lambda)}$. Here, $\Gamma_n=\{\lambda_n(t)\colon t\in[0,\pi]\}$ is a component of the spectrum of L and $M_k^n(x)$ corresponds to the spectral singularity λ_k

$$M_k^n(x)=\frac{1}{2\pi}\lim_{\varepsilon\to 0}\int\limits_{\Gamma_n}\sum_{j=0}^{i_k-1}B_{k,j}(\lambda)\phi(x,\lambda)^{(j)}(\lambda_k)\frac{d\lambda}{p(\lambda)}$$

where

$$B_{k,j}(\lambda)=\begin{cases}\dfrac{(\lambda-\lambda_k)^j}{j!} & if\ |\lambda-\lambda_k|<\delta\\ 0 & otherwise\end{cases}$$

and

$$B\phi(\lambda)=\phi(\lambda)-\sum_{j=0}^{i_k-1}B_{k,j}(\lambda)\phi(\lambda)^{(j)}(\lambda_k),\qquad K_n=\{k\colon\lambda_k\in\Gamma_n\}.$$

The following theorem which was proved for continuous potential q(x) in the paper (Veliev,1983) can be generalized without modifying for a Lebesgue integrable potential q(x).

Theorem 4.2.2 The number $\lambda=\lambda_n(t)$ is a spectral singularity of L if and only if $\alpha_n(t)=0$, that is, the operator L_t has an associated function at the point $\lambda_n(t)$.

Note. A point $\lambda_0\in S(L)$ is called the spectral singularity of the operator L if the system of the projections

$$P(\Delta) = \frac{1}{2\pi i} \int_{\gamma_\varepsilon} (L - \lambda I)^{-1} d\lambda$$

are not uniformly bounded in neighborhoods of λ_0, where $\gamma_\varepsilon \subset \rho(L)$, $\gamma_\varepsilon = \gamma_\varepsilon^1 \cup \gamma_\varepsilon^2$ and $\lim_{\varepsilon \to 0} \gamma_\varepsilon{}^i = \Delta$ for the curves γ_ε^1 and γ_ε^2 liying on opposite sides of Δ. Here Δ is a subset of spectrum $\Delta \subset \Gamma_n \subset S(L)$ and it contains only simple eigenvalues. In other words, $\lambda_0 \in S(L)$ is a spectral singularity of L if and only if there is a squence of $\{\Delta_n\}$ such that $\Delta_n \subset S(L)$ and $\|P(\Delta_n)\| \to \infty$ as Δ_n tends to λ_0.

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