A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF ÇANKIRI KARATEKİN UNIVERSITY

ON SOME BEST PROXIMITY POINT RESULTS FOR ĆIRIĆ TYPE CONTRACTIONS ON QUASI METRIC SPACE

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
MATHEMATICS

BY

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ÇANKIRI

2023

ON SOME BEST PROXIMITY POINT RESULTS FOR ĆIRIĆ TYPE CONTRACTIONS ON QUASI METRIC SPACE

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May 2023

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science

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Raghad Jabbar Sabir AL-OKBI

ABSTRACT

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Master of Science in Mathematics

Advisor: Assoc. Prof. Dr. Mustafa ASLANTAŞ

May 2023

This thesis is divided into four chapters. The first chapter discusses the motivation for the thesis and provides some background information on fixed point theory and best proximity point theory. In the second chapter, we recall the fundamentals definitions and notations, and some theorems related to our results about fixed point and best proximity point. We present our results in chapter three. We first introduce the concepts of d and d^{-1}- proximal Ćirić contraction mappings. Then, we prove some right and left best proximity point results for these mappings in o-complete quasi metric spaces. Hence, we obtain some generalization of famous results in the literature. Also, we present new definitions and notations by taking into account the lack of symmetry property of quasi metric. Moreover, we give some examples to support our definitions and notations. We provide our findings and suggestions in the fourth chapter.

2023, 35 pages

Keywords: Fixed point, Best proximity point, O-complete, O-continuous, Quasi metric space

ÖZET

QUASİ METRİK UZAYLARDA ĆİRİĆ TİPİ BÜZÜLMELER İÇİN BAZI EN İYİ YAKINLIK NOKTASI SONUÇLARI ÜZERİNE

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Mayıs 2023

Bu tez dört bölüme ayrılmıştır. İlk bölüm, tezin motivasyonunu tartışır ve sabit nokta teorisi ve en iyi yakınlık noktası teorisi hakkında bazı arka plan bilgileri sağlar. İkinci bölümde, sabit nokta ve en iyi yakınlık noktası ile ilgili temel tanımları ve notasyonları ve sonuçlarımızla ilgili bazı teoremleri hatırlıyoruz. Sonuçlarımızı üçüncü bölümde sunuyoruz. Önce d and d^{-1} -proksimal Ćirić kasılma eşlemeleri kavramlarını tanıtıyoruz. Ardından, yörüngesel olarak tamamlanmış yarı metrik uzaylarda bu eşlemeler için bazı sağ ve sol en iyi yakınlık noktası sonuçlarını kanıtlıyoruz. Bu nedenle, literatürdeki ünlü sonuçların bazı genellemelerini elde ederiz. Ayrıca, yarı metriklerin simetri eksikliğini dikkate alarak yeni tanımlar ve notasyonlar sunuyoruz. Ayrıca tanımlarımızı ve notasyonlarımızı desteklemek için bazı örnekler veriyoruz. Bulgu ve önerilerimizi dördüncü bölümde sunuyoruz.

2023, 35 sayfa

Anahtar Kelimeler: Sabit nokta, En iyi yakınlık noktası, Döngüsel tamlık, Döngüsel süreklilik, Quasi metrik uzaylar

PREFACE AND ACKNOWLEDGEMENTS

Before starting this, I would like to thank my family and all people who have helped and supported me throughout the completion of this work concludes my two years of university studies at the Çankırı Karatekin University. The Instructors, and the administrative team, for passing on their knowledge to me and for there involvement throughout my career to thank my thesis advisor, Assoc. Prof. Dr. Mustafa ASLANTAŞ, for his patience, guidance and understanding and special thanks for Mr. Mohammed Jasim for all help in my research.

Raghad Jabbar Sabir AL-OKBI
Çankırı-2023

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LIST OF SYMBOLS

$\geq$	Greatar than or equal
$\leq$	Less than or equal
$\subseteq$	Subset or equals
$\mathbb{N}$	The set of natural numbers
$\mathbb{Q}$	The set of rational
$\mathbb{R}$	The set of all real numbers

LIST OF ABBREVIATIONS

inf	Infimum
max	Maximum
min	Minimum
o-complete	o-complete
o-continuous	o-continuous
sup	Supremum

1. INTRODUCTION

In the study of topology, one of the key ideas is the notion of metric space. In his PhD thesis, French mathematician Maurice Fréchet (1906), who also contributed to the disciplines of Functional Analysis, Probability, and Calculus, originally articulated this idea. The term "distance" between any two elements in the space under study must be defined. In this way, the idea of metric space has served as the foundation for several research in a wide range of disciplines, including algebra, differential geometry, statistics, physics, and engineering. The equations $S(\check{s}) = 0$ and $H(\check{s}) = \check{s}$ are frequently encountered in analysis and functional analysis. We solve these equations in many ways. One such approach is fixed point theory. The existence theory of differential equations, partial differential equations, integral equations, and many other related topics make extensive use of the fixed point theorem. Assume that $H: \mathfrak{H} \rightarrow \mathfrak{H}$ is a function where $\mathfrak{H}$ is a non-empty set. The point $\check{s}_0$ is referred to as a fixed point of H if $H(\check{s}_0) = \check{s}_0$. Banach (1922), who is regarded as one of the most significant mathematicians of the 20th century, conducted one of the most notable investigations on defined functions on metric spaces. In his PhD thesis, he proposed the "Fixed Point Theorem in Metric Spaces" for the first time as follows

Let $(\mathfrak{H}, d)$ be a complete metric space and $H: \mathfrak{H} \rightarrow \mathfrak{H}$ be a mapping. If there exists k in $[0,1)$ such that

$$d(H\check{s}, H\breve{u}) \leq kd(\check{s}, \breve{u})$$

for all $\check{s}, \breve{u} \in \mathfrak{H}$, then H has a unique fixed point in $\mathfrak{H}$.

The Banach fixed point theorem ensures that the mapping has a fixed point, and it also demonstrates how to get at this point repeatedly. This conclusion has been broadly interpreted and expanded by numerous authors due to its applicability (Ahmad *et al.* 2015, Aslantas *et al.* 2021, Berinde 2004, Chen *et al.* 2009, Ćirić 1971, Joshi and Tomar 2021). One of the generalizations of these has been obtained by Ćirić (Ćirić 1974).

Let $H: \mathfrak{H} \to \mathfrak{H}$ be a mapping and $(\mathfrak{H}, d)$ be a H-o-complete metric space. If there is k in $[0,1)$ satisying

$$d(H\check{s}, H\breve{u}) \leq kM_d(\check{s}, \breve{u})$$

for all $\check{s}, \breve{u} \in \mathfrak{H}$ where

$$M_d(\check{s}, \breve{u}) = ma\check{s}\left\{d(\check{s}, \breve{u}), d(\check{s}, H\check{s}), d(\breve{u}, H\breve{u}), \frac{1}{2}(d(\check{s}, H\breve{u}) + d(\breve{u}, H\check{s}))\right\},$$

then H has a unique fixed point in $\mathfrak{H}$.

Recently, the idea of the best proximity point was introduced by Basha and Veeramani (1977). The best proximity point theory investigates the circumstances under which the optimization problem represented by the expression $itf_{\check{s}\in\Gamma} d(\check{s}, H\check{s})$ has a solution. Assume that Γ and B are non-empty subsets of $\mathfrak{H}$, $H: \Gamma \to B$ is a mapping, and $(\mathfrak{H}, d)$ is a metric space. The best proximity point of the mapping H is $a \in \Gamma$ if the statement $d(a, Ha) = d(\Gamma, B)$ holds. Since every best proximity point result becomes a fixed point result when $\Gamma = B$, numerous writers have studied this problem (Aslantas 2021, Aslantas *et al.* 2021, Aslantas 2022, Basha 2010, Basha *et al.* 2013, Jleli *et al.* 2013, Jleli and Samet 2013).

On the other hand, a real valued function d defined on $\mathfrak{H} \times \mathfrak{H}$ that satisfies the axioms of a metric with the exception that the distance between distinct points is nonzero is known as a pseudo metric on a non-empty set $\mathfrak{H}$. Numerous well-known results on metric space, such as the Baire category, Cantor intersection, and Banach fixed point theorems, are applicable in this situation and are readily extended to pseudo-metric contexts. But, these extensions are not as simple as in the pseudo metric spaces, when the symmetry condition is eliminated. Despite this, a number of authors have focused on ignoring the symmetry constraint since unsymmetric distance functions have a wide range of applications in both mathematics and many other fields (Chen *et al.* 2009, Künzi 2001, Waterman *et al.* 1976). Wilson (1931) introduced the idea of quasi metric in this context for the first time. Then, Kelly (1963) succeeded in generalizing a number of well-known conclusions, including the Baire category theorem and the Urysohn Lemma, by taking into consideration biotopological spaces, which are intimately related to quasi metric. The same paper also has been defined the Cauchy sequence for a quasi pseudo metric space. Reilly *et al.* (1982) pointed out that any convergent sequence might not be Cauchy in Kelly's definition. They suggested a variety of definitions of the Cauchy sequence in a quasi-metric space to address this drawback. By categorizing concepts of Cauchyness and completeness, Altun *et al.* (2017) recently established a few fixed point theorems on quasi-metric spaces. Many intriguing and notable results can be found in this path (Altun *et al.* 2016, Dağ *et al.* 2017, Hancer *et al.* 2019, Imdad *et al.* 2019, Karapınar *et al.* 2014, Karapınar *et al.* 2022, Marin *et al.* 2011).

This thesis is divided into four chapters. The first chapter discusses the motivation for the thesis and provides some background information on fixed point theory and best proximity point theory. In the second chapter, we recall the fundamentals definitions and notations, and some theorems related to our results about fixed point and best proximity point. We present our results in chapter three. We first introduce the concepts of d and d^{-1}- proximal Ćirić contraction mappings. Then, we prove some right and left best proximity point results for these mappings in o-complete quasi metric spaces. Hence, we obtain some generalization of famous results in the literature. Also, we present new definitions and notations by taking into account the lack of symmetry property of quasi metric. Moreover, we give some examples to support our definitions and notations. We provide our findings and suggestions in the fourth chapter.

2. PRELIMINARIES

In this section, we recall some definitions and theorems related to our main section.

2.1 Metric Spaces

Definition 2.1. Let $\mathfrak{H}$ be a nonempty set and $d: \mathfrak{H} \times \mathfrak{H} \to \mathbb{R}$ a mapping with the following properties:

(i) $d(\check{s}, \breve{u}) = 0$ if and only if $\breve{u} = \check{s}$.
(ii) $d(\check{s}, \breve{u}) = d(\breve{u}, \check{s})$ for all $\check{s}, \breve{u} \in \mathfrak{H}$.
(iii) $d(\check{s}, \breve{u}) + d(\breve{u}, z) \geq d(\check{s}, z)$ for all $\check{s}, \breve{u}, z \in \mathfrak{H}$.

Thus, we called d is a metric on $\mathfrak{H}$ and $(\mathfrak{H}, d)$ is a metric space.

Example 2.2. i) The function d: $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by, for all $\check{s}, \breve{u} \in \mathbb{R}$

$$d(\check{s}, \breve{u}) = |\check{s} - \breve{u}|$$

is a metric on $\mathbb{R}$. This metric is known as usual metric.

ii) The function d: $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$d((\check{s}_1, \check{s}_2), (\breve{u}_1, \breve{u}_2)) = \sqrt{(\check{s}_1 - \breve{u}_1)^2 + (\check{s}_2 - \breve{u}_2)^2}$$

is a metric on $\mathbb{R}^2$ called the euclidean metric on $\mathbb{R}^2$.

iii) Let $\mathfrak{H}$ be a non-empty set and d the function from $\mathfrak{H} \times \mathfrak{H}$ into $\mathbb{R}$ defined by

$$d(\check{s}, \breve{u}) = \begin{cases} 0, & if \quad \check{s} = \breve{u} \\ 1, & if \quad \check{s} \neq \breve{u} \end{cases}.$$

Then d is a metric on $\mathfrak{H}$ and is called the discrete metric.

Definition 2.3. Let $(\mathfrak{H}, d)$ be a metric space. The set

$$B_r(\check{s}_0) = \{\check{s} \in \mathfrak{H}: d(\check{s}, \check{s}_0) < r\}$$

is an open ball in $\mathfrak{H}$ with radius $r > 0$ centered $\check{s}_0$ and the set

$$\overline{B_r}(\check{s}_0) = \{\check{s} \in \mathfrak{H}: d(\check{s}, \check{s}_0) \leq r\}$$

is a closed ball in $\mathfrak{H}$ with radius $r > 0$ centered $\check{s}_0$

Definition 2.4. Let $(\mathfrak{H}, d)$ be a metric space and $\Gamma \subseteq \mathfrak{H}$.

i) If for every $\check{s} \in \Gamma$ there is a number $\varepsilon > 0$ such that $B_r(\check{s}) \subseteq \Gamma$, then the set Γ is called open set.

ii) If the set $\mathfrak{H} - \Gamma$ is open, the set Γ is called a closed set.

Theorem 2.5. In every metric space $(\mathfrak{H}, d)$, every $B_r(\check{s}_0)$ is an open set.

Theorem 2.6. In every metric space $(\mathfrak{H}, d)$, every $\overline{B_r}(\check{s}_0)$is a closed set.

Definition 2.7. Let $\{\check{s}_t\}$ be a sequence in a metric space $(\mathfrak{H}, d)$. We say that

i) the sequence $\{\check{s}_t\}$ converges to $\check{s} \in \mathfrak{H}$ (denoted by $\check{s}_t \rightarrow \check{s}$) if, for any $\varepsilon > 0$, there is natural number K such that $d(\check{s}, \check{s}_t) < \varepsilon$ for all $t > K$.

ii) the sequence $\{\check{s}_t\}$ is a Cauchy sequence if, for any $\varepsilon > 0$, there is a natural number K such that $d(\check{s}_m, \check{s}_t) < \varepsilon$ for all $m, t > K$.

iii) the sequence $\{\check{s}_t\}$ is bounded if there exists $M > 0$ such that $d(\check{s}_s, \check{s}_i) \leq M$ for all $s, i \in \mathbb{N}$.

Theorem 2.8. i) The limit of a convergent sequence in any metric space $(\mathfrak{H}, d)$ is unique**.**

ii) Every convergent sequence in a metric space $(\mathfrak{H}, d)$ is a Cauchy sequence.

Example 2.9. i) Let $(\mathbb{R}, d)$ be a usual metric space. Then, $(\mathbb{R}, d)$ is complete metric space.

ii) Let $(\mathfrak{H}, d)$ be a discrete metric space. Then, $(\mathfrak{H}, d)$ is complete metric space.

iii) Let $C([a, b])$ be the set of all continuous function and $d: C([a, b]) \times C([a, b]) \to \mathbb{R}$ be a function defined by

$$d(f, g) = \sup\{|(\check{s}) - g(\check{s})| : \check{s} \in [0,1]\}.$$

Then, $(C([a, b]), d)$ is a complete metric space.

iv) Let $(\mathbb{Q}, d)$ be a usual metric space. Then, $(\mathbb{Q}, d)$ is not complete metric space.

Definition 2.10. Let $(\mathfrak{H}, d_{\mathfrak{H}})$ and (Y, d_Y) be two metric spaces and $v \in \mathfrak{H}$. We say that a function $h: \mathfrak{H} \to Y$ is continuous at v if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_Y(h\check{s}, hv) < \varepsilon \text{ whenever } d_{\mathfrak{H}}(\check{s}, v) < \delta$$

Equivalently, a function $h: \mathfrak{H} \to Y$ is continuous at v if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$h\left(B_{d_{\mathfrak{H}}}(v, \delta)\right) \subseteq B_{d_Y}(hv, \varepsilon).$$

Theorem 2.11. Let $\left(\mathfrak{H}, d_{\mathfrak{H}}\right)$ and (Y, d_Y) be two metric spaces and $v \in \mathfrak{H}$. A function $h: \mathfrak{H} \to Y$ is said to continuous at $\in \mathfrak{H}$, if and only if $f(\check{s}_t) \to f(u)$ in Y, whenever $\check{s}_t \to u$ in $\mathfrak{H}$, for every sequence $\{\check{s}_t\}$ in $\mathfrak{H}$.

Definition 2.12. Let $h: \mathfrak{H} \to Y$ be a function and $\check{s}_0 \in \mathfrak{H}$ where $(\mathfrak{H}, d)$ is a metric space. If for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$h(\check{s}_0) - \varepsilon < h(\check{s})$$

for all $\check{s} \in B(\check{s}_0, \delta) \cap W$, then the function h is called lower semicontinuous function. Similarly, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$h(\check{s}_0) < h(\check{s}) + \varepsilon$$

for all $\check{s} \in B(\check{s}_0, \delta) \cap W$, then the function h is called upper semi-continuous function.

Theorem 2.13. Let $(\mathfrak{H}, d)$ be a metric space, $h: \mathfrak{H} \to Y$ be a function and $\check{s}_0 \in \mathfrak{H}$. For every sequence $\{\check{s}_t\}$ converging to $\check{s}_0$,

$$\underset{\check{s} \to \infty}{limitf}\, h(\check{s}_t) \geq h(\check{s}_0)$$

if and only if the function h is lower semi-continuous function. Similary, if for every sequence $\{\check{s}_t\}$ converging to $\check{s}_0$,

$$\lim_{š\to\infty}\inf h(š_t) \leq h(š_0)$$

if and only if the function h is upper semi-continuous function

2.2 Fixed Point Theory

In this section, we provide some fundamental definitions and theorems about fixed points of the mapping H on a metric space $(\mathfrak{H}, d)$, together with the prerequisites for their existence and uniqueness.

Definition 2.14. Let $H: \mathfrak{H} \to \mathfrak{H}$ be a mapping. Then, a point š is said to be a fixed point of H if it satisfies $Hš = š$.

Now, we give the following example

Example 2.15. i) Let $H: \mathbb{R} \to [0,1]$ be a mapping given as $Hš = sitš$. Then, it has a fixed point which is 0, but $(\pi/2)$ is not fixed of H.

ii) Let $H: (\mathfrak{H}, d) \to (\mathfrak{H}, d)$ be a mapping given as $Hš = š + a$ where $0 \neq a \in \mathbb{R}$. Then, H does not have a fixed point.

iii) Let $H: (\mathfrak{H}, d) \to (\mathfrak{H}, d)$ be a mapping given as $Hš = š$. The mapping has infinite fixed points.

Fixed point theory in full metric spaces was first studied by Banach in 1922.

Definition 2.16. Let $(\mathfrak{H}, d)$ be a metric space and $H: \mathfrak{H} \to \mathfrak{H}$ be a mapping. The mapping is called contraction mapping if there exists q in $[0,1]$ such that

$$d(Hš, Hŭ) \leq qd(š, ŭ)$$

for all $\check{s}, \breve{u} \in \mathfrak{H}$.

Theorem 2.17. Let $H: \mathfrak{H} \to \mathfrak{H}$ be a mapping on complete metric space. Then, if H is a contraction mapping, then H has a unique fixed point**.**

The Banach fixed point theorem is an existence and uniqueness theorem for the fixed points of particular mappings. The proof will show us that it also gives us a useful method for steadily improving fixed point approximations. We begin by choosing an arbitrary $\check{s}_0$ from a given set, and then compute a sequence recusively by letting

$$\check{s}_{t+1} = H\check{s}_t \text{ for all } t \in \mathbb{N}$$

This process is known as iteration. Many branches of applied mathematics use such iteration techniques, and Banach's Fixed Point Theorem is routinely used to guarantee the convergence of the scheme and the uniqueness of the solution. One of the novel analysis findings is the fundamentals of Banach contraction mapping. As the foundation for metric fixed point theory, it is widely used. Its significance is also based on the wide range of mathematics and disciplines to which it is applicable.

The solution might not be original in nonlinear systems, which are frequently used to address problems in real life. Theorems that do not guarantee the uniqueness of the fixed point are thus important to consider in practice. Using this concept, Ćirić (1974) obtained a non-unique fixed point result for self mapping H on a complete metric space. Before giving Ćirić's result, we give some definitions.

Definition 2.18. Let $H: \mathfrak{H} \to \mathfrak{H}$ be a mapping where $\mathfrak{H}$ is nonempty set. The set

$$O(\check{s}) = \{\check{s}, H\check{s}, H^2\check{s}, \cdots, H^t\check{s} \cdots\}$$

is called an orbit of $\check{s}$.

Definition 2.19. Let $H: \mathfrak{H} \to \mathfrak{H}$ be a mapping where $(\mathfrak{H}, d)$ is a metric space. The mapping H is called an o-continuous mapping if

$$\lim_{i\to\infty} H^{t_i}\check{s} = u \text{ implies } \lim_{i\to\infty} HH^{t_i}\check{s} = Hu$$

for each $\check{s} \in \mathfrak{H}$.

Remark 2.20. It is clear that if $H: \mathfrak{H} \to \mathfrak{H}$ is a o-continuous, then for $m \in \mathbb{N}$, H^m:X $\to \mathfrak{H}$ is a o-continuous mapping.

Definition 2.21. Let $(\mathfrak{H}, d)$ be a metric space and $H: \mathfrak{H} \to \mathfrak{H}$ be a mapping. A metric space $(\mathfrak{H}, d)$is called H-o-complete if for each Cauchy sequence of the form $\{H^{t_i}\check{s}\}$ $\check{s} \in \mathfrak{H}$ converges in $\mathfrak{H}$.

Now, we give Ćirić's result

Theorem 2.22. Let $H: (\mathfrak{H}, d) \to (\mathfrak{H}, d)$ be an arbitrary continuous mapping and $(\mathfrak{H}, d)$ be H-o-complete. If H satisfies the following condition

$$mit\{d(H\check{s}, H\check{u}), d(\check{s}, H\check{s}), d(\check{u}, H\check{u})\} - mit\{d(\check{s}, H\check{u}), d(\check{u}, H\check{s})\} \leq q.d(\check{s}, \check{u})$$

for some $q < 1$ and $\check{s}, \check{u} \in \mathfrak{H}$ then for all $\check{s} \in \mathfrak{H}$, the sequence $\{H^t\check{s}\}_{t=1}^{\infty}$ convergent to a fixed point of H.

2.3 Best Proximity Point Theory

Now, we present some notations, definitions and theorems about best proximity point theory. Consider the following sets

$$\Gamma_0 = \{\check{s} \in \Gamma: d(\check{s}, \check{u}) = d(\Gamma, \Lambda) \; for\ some \; \check{u} \in \Lambda\}$$

and

$$\Lambda^0 = \{\check{s} \in \Lambda : d(\check{s}, \check{u}) = d(\Gamma, \Lambda) for\ some\ \check{u} \in \Gamma\}.$$

The following definition is a version of contraction mapping in the best proximity point theory.

Definition 2.23. Let $\emptyset \neq \Gamma, \Lambda$ be subsets of $(\mathfrak{H}, d)$ where $(\mathfrak{H}, d)$ is a metric space. If there is q in [0,1) satisfying

$$\left.\begin{aligned} d(\check{s}_1, Hu_1) = d(\Gamma, \Lambda) \\ d(\check{s}_2, Hu_2) = d(\Gamma, \Lambda) \end{aligned}\right\} \Rightarrow d(\check{s}_1, \check{s}_2) \leq qd(u_1, u_2)$$

for all $\check{s}_1, \check{s}_2, u_1, u_2 \in \Gamma$, then $H: \Gamma \to \Lambda$ is called a proximal contraction.

Sahin *et al.* (2021) introduced the following definitions.

Definition 2.31. Let $\emptyset \neq \Gamma, \Lambda$ be subsets of $(\mathfrak{H}, d)$ where $(\mathfrak{H}, d)$ is a metric space, $\check{s} \in \Gamma$ and $H: \Gamma \to \Lambda$ be a mapping. The set

$$O^H(\check{s}) = \left\{\{\check{s}_s\} : \check{s} = \check{s}_0\ atd\ d(\check{s}_{s+1}, H\check{s}_s) = d(\Gamma, \Lambda)\ for\ all\ s \in \mathbb{N}\right\}$$

is said to be a orbit of š

Definition 2.32. Let $(\mathfrak{H}, d)$ be a metric space, $\emptyset \neq \Gamma, \Lambda \subseteq \mathfrak{H}$ and $H: \Gamma \to \Lambda$ be a mapping. If every Cauchy subsequence $\{\check{s}_{s_i}\}$ of $\{\check{s}_s\}$ in $O_H(\check{s})$ converges to a point in Γ_0, then we say that Γ is H −best o-complete if for all $\check{s} \in \Gamma$.

Note that when $\Gamma = \Lambda = \mathfrak{H}$, H-best o-completeness of $\mathfrak{H}$ becomes H-o-completeness in the sense of Ćirić.

Definition 2.33. Let $\emptyset \neq \Gamma, \Lambda$ be subsets of $(\mathfrak{H}, d)$ where $(\mathfrak{H}, d)$ is a metric space and $H: \Gamma \to \Lambda$ be a mapping. If for all š $\in \Gamma$ and $\{š_s\}$ in $O_H(š)$ the following statement is true.

$$š_{s_i} \to š \; as \; i \to \infty \;\Rightarrow H š_{s_i} \to H š \; as \; i \to \infty$$

for all subsequence $\{š_{s_i}\}$ of $\{š_s\}$, then we say that H is best o-continuous at a point š in Γ

Note that when $\Gamma = \Lambda = \mathfrak{H}$, best o-continuity of H turns into o-continuity of H in the sense of Ćirić.

Definition 2.34. Let $\emptyset \neq \Gamma, \Lambda$ be subsets of $(\mathfrak{H}, d)$ where $(\mathfrak{H}, d)$ is a metric space and $H: \Gamma \to \Lambda$ be a mapping. Suppose that $h: \Gamma \to \mathbb{R}$ be function. Then, h is called best o-lower semicontinuous at š in Γ if for all š $\in \Gamma$ and $\{š_s\}$ in $O_H(š)$ the following statement is true.

$$š_{t_i} \to š \; as \; i \to \infty \;\Rightarrow g(š) \leq \lim_{i \to \infty} itf g(š_{s_i})$$

for all subsequence $\{š_{s_i}\}$ of $\{š_s\}$. function.

2.4 Quasi Metric Spaces

Definition 2.35. Let $\mathfrak{H} \neq \emptyset$ and $d: \mathfrak{H} \times \mathfrak{H} \to [0, \infty)$ be a function. If the following statements hold: for all š, ŭ, $z \in \mathfrak{H}$,

(i) $d(š, š) = 0$,
(ii) $d(š, z) \leq d(š, ŭ) + d(ŭ, z)$,

then the function d is said to be a quasi-pseudo metric on X. If the following condition hold instead of i)

$$d(\check{s}, \breve{u}) = d(\breve{u}, \check{s}) = 0 \Leftrightarrow \check{s} = \breve{u},$$

then the function d is said to be a quasi metric space.

Let $(\mathfrak{H}, d)$ be a quasi metric space, $d^{-1}: \mathfrak{H} \times \mathfrak{H} \to [0, \infty)$ and $d^{s}: \mathfrak{H} \times \mathfrak{H} \to [0, \infty)$ be mappings defined by

$$d^{-1}(\check{s}, \breve{u}) = d(\breve{u}, \check{s})$$

and

$$d^{s}(\check{s}, \breve{u}) = max\{d(\check{s}, \breve{u}), d^{-1}(\check{s}, \breve{u})\}$$

for all $\check{s}, \breve{u} \in \mathfrak{H}$. Then d^{-1} is also a quasi metric (called conjugate of d) and d^{s} is an ordinary metric on $\mathfrak{H}$.

Example 2.36. Let $\mathfrak{H} = \mathbb{R}$ and $d: \mathfrak{H} \times \mathfrak{H} \to \mathbb{R}$ be a function defined by

$$d(\check{s}, \breve{u}) = \max\{\breve{u} - \check{s}, 0\}$$

for all $\check{s}, \breve{u} \in \mathfrak{H}$. Then, $(\mathfrak{H}, d)$ is a quasi metric space. It can be seen that it is not metric space.

Example 2.37. Let $\mathfrak{H} = \mathbb{R}$ and $d: \mathfrak{H} \times \mathfrak{H} \to \mathbb{R}$ be a function defined by

$$d(\check{s}, \breve{u}) = \begin{cases} \check{s}, & \check{s} \neq \breve{u} \\ 0, & \check{s} = \breve{u} \end{cases}$$

for all $\check{s}, \breve{u} \in \mathfrak{H}$. Then, $(\mathfrak{H}, d)$ is a quasi metric space.

Definition 2.38. Let $(\mathfrak{H}, d)$ be a quasi metric space, $\check{s}_0 \in \mathfrak{H}$ and $r > 0$. The set

$$B_d(\check{s}_0, r) = \{\breve{u} \in \mathfrak{H}: d(\check{s}_0, \breve{u}) < r\}$$

is called d-open ball with centre $\check{s}_0$ and radius r. The set

$$B_{d^{-1}}(\check{s}_0, r) = \{\breve{u} \in \mathfrak{H}: d(\breve{u}, \check{s}_0) < r\}$$

is called d^{-1}-open ball with centre $\check{s}_0$ and radius r.

If τ_d and $\tau_{d^{-1}}$ denote the family of all d-open subsets of $\mathfrak{H}$ and the family of all d^{-1}-open subsets of $\mathfrak{H}$, respectively, then τ_d and $\tau_{d^{-1}}$ are T_0-topologies on $\mathfrak{H}$. If d is a T_1-quasi metric, then τ_d is a H_1-topology on $\mathfrak{H}$.

Definition 2.39. Let $\{\check{s}_t\}$ be a sequence in a quasi metric space $(\mathfrak{H}, d)$.

(i) The sequence $\{\check{s}_t\}$ is said to be a d-convergent to $\check{s}$ if for all $\varepsilon > 0$, there exists $t_0 \in \mathbb{N}$ such that $d(\check{s}, \check{s}_t) < \varepsilon$ for all $t \geq t_0$.

(ii) The sequence $\{\check{s}_t\}$ is said to be a d^{-1}-convergent to $\check{s}$ if for all $\varepsilon > 0$, there exists $t_0 \in \mathbb{N}$ such that $d(\check{s}_t, \check{s}) < \varepsilon$ for all $t \geq t_0$.

The closure of a nonempty subset Γ of $\mathfrak{H}$ with respect to τ_d and $\tau_{d^{-1}}$ will be denoted by $\bar{\Gamma}^{\tau_d}$ and $\bar{\Gamma}^{\tau_{d^{-1}}}$, respectively,

Theorem 2.40. Let $(\mathfrak{H}, d)$ be a quasi metric space and $\Gamma \subseteq \mathfrak{H}$. Then, the following statements are true.

(i) $\check{s} \in \bar{\Gamma}^{\tau_d}$ if and only if $d(\check{s}, \Gamma) = \inf\{d(\check{s}, a) : a \in \Gamma\} = 0$.

(ii) $\check{s} \in \bar{\Gamma}^{\tau_{d^{-1}}}$ if and only if $d(\check{s}, \Gamma) = \inf\{d(a, \check{s}) : a \in \Gamma\} = 0$.

These expansions are more difficult in quasi metric spaces than in metric spaces due to the absence of the symmetry criterion. Furthermore, Cauchyness and completeness in quasi metric spaces are defined in a wide variety of ways. We can think back to various definitions of completeness and Cauchyness.

Definition 2.36. Let $\{\check{s}_t\}$ be a sequence in a quasi metric space $(\mathfrak{H}, d)$.

(i) The sequence $\{\check{s}_t\}$ is said to be a left K-Cauchy sequence if for every $\varepsilon > 0$ there exists $t_0 \in \mathbb{N}$ such that $d(\check{s}_t, \check{s}_m\}) < \varepsilon$ for all $m \geq t \geq t_0$.

(ii) The sequence $\{\check{s}_t\}$ is said to be a right K-Cauchy sequence if for every $\varepsilon > 0$ there exists $t_0 \in \mathbb{N}$ such that $d(\check{s}_m, \check{s}_t\}) < \varepsilon$ for all $m \geq t \geq t_0$.

Definition 2.37. Let $(\mathfrak{H}, d)$ be a quasi metric space.

(i) The space $(\mathfrak{H}, d)$ is left d-complete if every left K-Cauchy sequence is convergent w.r.t. τ_d.

(ii) The space $(\mathfrak{H}, d)$ is left d^{-1}-complete if every left K-Cauchy sequence is convergent w.r.t. $\tau_{d^{-1}}$.

(iii) The space $(\mathfrak{H}, d)$ is right d-complete if every right K-Cauchy sequence is convergent w.r.t. τ_d.

The space $(\mathfrak{H}, d)$ is right d^{-1}-complete if every right K-Cauchy sequence is convergent w.r.t. $\tau_{d^{-1}}$.

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3. SOME BEST PROXIMITY POINT RESULTS ON O-COMPLETE QUASI METRIC SPACES

In this section, we prove a left best proximity point results for d-proximal Ćirić type contractions. Now, we present the following definitions.

Definition 3.1. Let $(\mathfrak{H}, d)$ be a quasi metric space, $H : \Gamma \to \Lambda$ be a mapping where $\emptyset \neq \Gamma, \mathrm{V} \subseteq \mathfrak{H}$ and $\check{s} \in \Gamma$. Then, the set

$$O_0^{\rho}(\check{s}) = \{\{\check{s}_t\} \subseteq \Gamma : \check{s}_0 = \check{s}\, atd\, \rho(\check{s}_{t+1}, H\check{s}_t) = \rho(\Gamma, \Lambda)\ for\ all\ t \in \mathbb{N}\}$$

is called the $\rho(d\ or\ d^{-1})$−orbit of š.

Definition 3.2. Let $(\mathfrak{H}, d)$ be a quasi metric space, $H : \Gamma \to \Lambda$ be a mapping where $\emptyset \neq \Gamma, \mathrm{V} \subseteq \mathfrak{H}$. We say that Γ is

i) H−best ρ−o-right d−complete if for all $\check{s} \in \Gamma$ and $\{\check{s}_t\}$ in $O_H^{\rho}(\check{s})$, every right K−Cauchy subsequence $\{\check{s}_{t_i}\}$ of $\{\check{s}_t\}$ converges to $\check{s} \in \Gamma_0$ with respect to d.

ii) H−best ρ−o-left d−complete if for all $\check{s} \in \Gamma$ and $\{\check{s}_t\}$ in $O_H^{\rho}(\check{s})$, every left K−Cauchy subsequence $\{\check{s}_{t_i}\}$ $of \{\check{s}_t\}$ converges $\check{s} \in \Gamma_0$ with respect to d.

iii) H−best ρ−o-right d^{-1}−complete if for all $\check{s} \in \Gamma$ and $\{\check{s}_t\}$ in $O_H^{\rho}(\check{s})$, every right K−Cauchy subsequence $\{\check{s}_{t_i}\}$ of $\{\check{s}_t\}$ converges to $\check{s} \in \Gamma_0$ with respect to d^{-1}.

iv) H−best ρ−o-left d^{-1}−complete if for all $\check{s} \in \Gamma$ and $\{\check{s}_t\}$ in $O_H^{\rho}(\check{s})$, every left K−Cauchy subsequence $\{\check{s}_{t_i}\}$ of $\{\check{s}_t\}$ converges to $\check{s} \in \Gamma_0$ with respect to d^{-1}.

Definition 3.3. Let $(\mathfrak{H}, d)$ be a quasi metric space, $H : \Gamma \to \Lambda$ be a mapping where $\emptyset \neq \Gamma, \mathrm{V} \subseteq \mathfrak{H}$. Then, a function $g : \Gamma \to \mathbb{R}$ is called

i) $d-$best $\rho-$orbitally lower semicontinuous at a point $š^* \in \Gamma$ if for each $š \in \Gamma$ and $\{š_t\}$ in O_H^ρ, the implication

$$š_{ti} \overset{d}{\rightarrow} š^* \Rightarrow g(š^*) \leq \lim_{i\rightarrow\infty} itf\ g\ (š_t)$$

holds for any subsequence$\{š_{t_i}\}\ of \{š_t\}$.

ii) $d^{-1}-$best $\rho-$orbitally lower semicontinuous at a point $š^* \in \Gamma$ if for each $š \in \Gamma$ and $\{š_t\}$ in O_H^ρ, the implication

$$š_{ti} \overset{d^{-1}}{\rightarrow} š^* \Rightarrow g(š^*) \leq \lim_{i\rightarrow\infty} itf\ g\ (š_t)$$

holds for any subsequence$\{š_{t_i}\}\ of \{š_t\}$.

After that, we give the following definitions.

Definition 3.4. Let $(\mathfrak{H}, d)$ be a quasi metric space, $\emptyset \neq \Gamma, \Lambda \subseteq \mathfrak{H}$. If there is $k \in [0,1)$ such that

$$\left.\begin{array}{c} d(Hš_2, u_1) = d(\Lambda, \Gamma) \\ d(Hš_2, u_2) = d(\Lambda, \Gamma) \end{array}\right\}$$

implies

$$d(u_2, u_1) \leq kM_{d^{-1}}(š_1, š_{2'} u_1, u_2)$$

for all $u_1, u_2, š_{1'} š_2 \in \Gamma$ where

$$M_{d^{-1}}(š_1, š_2, u_1 u_2) = maš \begin{Bmatrix} d(š_2, š_1), d(u_1, š_1), d(u_2, š_2) \\ \frac{1}{2}\big(d(u_2, š_1) + d(u_1, š_2)\big) \end{Bmatrix},$$

then $H: \Gamma \rightarrow \Lambda$ is called d^{-1}-proximal Ćirić type contraction.

Definition 3.5. Let $(\mathfrak{H}, d)$ be a quasi metric space, $\emptyset \neq \Gamma, \Lambda \subseteq \mathfrak{H}$. Γ mapping $H: \Gamma \to \Lambda$ is said to be d-proximal Ćirić type contraction if there exist $k \in [0,1)$ such that

$$\left.\begin{array}{l} d(u_1, H\check{s}_2) = d(\Gamma, \Lambda) \\ d(u_2, H\check{s}_2) = d(\Gamma, \Lambda) \end{array}\right\}$$

implies

$$d(u_1, u_2) \leq k M_d(\check{s}_1, \check{s}_2, u_1, u_2) \tag{3.1}$$

for all $u_1, u_2, \check{s}_1, \check{s}_2 \in \Gamma$ where

$$M_d(\check{s}_1, \check{s}_2, u_1 u_2) = ma\check{s}\left\{\begin{array}{c} d(\check{s}_1, \check{s}_2), d(\check{s}_1, u_1), d(\check{s}_2, u_2) \\ \frac{1}{2}\big(d(\check{s}_1, u_2) + d(\check{s}_2, u_1)\big) \end{array}\right\}.$$

Now, we present the following theorem which is first our main result.

Theorem 3.6. Let $(\mathfrak{H}, d)$ be a quasi metric space $\emptyset \neq \Gamma, \Lambda \subseteq \mathfrak{H}, \Gamma_0^r \neq \emptyset$ and $H: \Gamma \to \Lambda$ d^{-1}-proximal Ćirić type contraction satisfying $H(\Gamma_0^r) \subseteq \Lambda_0^r$. If Γ is H $-$best d^{-1}-o-right $d^{-1}-$ complete and a function $g: \Gamma \to \mathbb{R}$ given as $g(\check{s}) = d(H\check{s}, \check{s})$ is d^{-1}-best d^{-1}-o-lower semicontinuous on Γ, then H has a right best proximity point in Γ.

Proof. Consider any point $\check{s}_0 \in \Gamma_0^r$. Since $H\check{s}_0 \in H(\Gamma_0^r) \subseteq \Lambda_0^r$, there exists $\check{s}_1 \in \Gamma_0^r$ such that

$$d(H\check{s}_0, \check{s}_1) = d(\Lambda, \Gamma).$$

Similarly, there exists $\check{s}_2 \in \Gamma_0^r$ such that

$$d(H\check{s}_1, \check{s}_2) = d(\Lambda, \Gamma).$$

By carrying out this procedure again, we may create a sequence $\{\check{s}_t\}$ satisfying

$$d(H\check{s}_t, \check{s}_{t+1}) = d(\Lambda, \Gamma) \tag{3.2}$$

for all $t \in \mathbb{N}$, that is, $\{\check{s}_t\} \in O_H^{d^{-1}}(\check{s}_0)$. If there exists $t_0 \in \mathbb{N}$ such that $\check{s}_{t_0} = \check{s}_{to+1}$, then the proof is complete. Then, suppose that $\check{s}_t \neq \check{s}_{t+1}$ for all $t \geq 1$. Due to the fact that H is a d^{-1}-proximal Ćirić-type contraction, we have

$$d(\check{s}_{t+1}, \check{s}_t) \leq kM_d \to (\check{s}_t - 1, \check{s}_t, \check{s}_t, \check{s}_{t+1})$$

where

$$M_{d^{-1}}(\check{s}_1, \check{s}_2, u_1 u_2) = ma\check{s}\begin{Bmatrix} d(\check{s}_t, \check{s}_{t-1}), d(\check{s}_t, \check{s}_{t-1}), d(\check{s}_{t+1}, \check{s}_t) \\ \frac{1}{2}\left(d(\check{s}_{t+1,}\check{s}_{t-1}) + d(\check{s}_{t,}\check{s}_t)\right) \end{Bmatrix}$$

$$= ma\check{s}\{d(\check{s}_t, \check{s}_{t-1})\,, d(\check{s}_{t+1}\check{s}_t), \tfrac{1}{2} d(\check{s}_{t+1}, \check{s}_{t-1})\}$$

$$\leq ma\check{s}\begin{Bmatrix} d(\check{s}_t, \check{s}_{t-1}), d(\check{s}_{t+1}, \check{s}_t) \\ \frac{1}{2}\left(d(\check{s}_{t+1,}\check{s}_t) + d(\check{s}_{t,}\check{s}_{t-1})\right) \end{Bmatrix}$$

$$= ma\check{s}\{d(\check{s}_{t,}\check{s}_{t-1})\, d(\check{s}_{t+1}, \check{s}_t)\}$$

for all $t \geq 1$. If $M_{d^{-1}}(\check{s}_{t-1}, \check{s}_t, \check{s}_{t,}\check{s}_{t+1}) = d(\check{s}_{t+1}, \check{s}_t)$, then we have

$$\begin{aligned} d(\check{s}_{t+1}, \check{s}_t) &\leq k\, d\left(\check{s}_{t+1,}\check{s}_t\right) \\ &< d(\check{s}_{t+1}, \check{s}_t) \end{aligned}$$

which is a contradiction. Then, we get $M_{d^{-1}}(\check{s}_{t-1}, \check{s}_t, \check{s}_t, \check{s}_{t+1}) = d(\check{s}_{t+1}, \check{s}_t)$ for all $t \geq 1$.

$$d(\check{s}_{t+1}, \check{s}_t) \leq k\, d\left(\check{s}_{t,}\check{s}_{t-1}\right)$$

for all $t \geq 1$. Therefore, we have

$$
\begin{aligned}
d(\check{s}_{t+1}, \check{s}_t) &\leq k\, d\big(\check{s}_{t,}\check{s}_{t-1}\big) \\
&\leq k^2\, d\big(\check{s}_{t-1,}\check{s}_{t-2}\big) \\
&\ \ \vdots \\
&\leq k^t\, d\big(\check{s}_{1,}\check{s}_0\big)
\end{aligned}
$$

for all $t \in \mathbb{N}$. Then, we have

$$
\lim_{t\to\infty} d(\check{s}_{t+1}, \check{s}_t) = 0 \tag{3.3}
$$

Hence, for all $m > t$, we obtain

$$
\begin{aligned}
d(\check{s}_m, \check{s}_t) &= d(\check{s}_m, \check{s}_{m-1}) + d\big(\check{s}_{m-1}, \check{s}_{m_{-2}}\big) + \cdots + d(\check{s}_{t+1}, \check{s}_t) \\
&\leq k^t\, d(\check{s}_1, \check{s}_0) + k^{t+1}\, d(\check{s}_1, \check{s}_0) + \cdots + k^{m-1}\, d(\check{s}_1, \check{s}_0) \\
&= k^t d(\check{s}_0, \check{s}_1)(1 + k + \cdots + k^{m-t-1}) \\
&= k^t d(\check{s}_1, \check{s}_0)\, \tfrac{1-k^{m-t}}{1-K} \\
&= \tfrac{k^t\, d(\check{s}_1,\check{s}_0)}{1-k},
\end{aligned}
$$

and so $\{\check{s}_t\}$ is a right K-Cauchy sequence in Γ. Since Γ is H-best $d^{-1}-$o-right $d^{-1}-$complete, there exist $\check{s}* \in \Gamma_0^{\ell}$ such that $d(\check{s}_t, \check{s}*) \to 0$ as $t \to \infty$. On the other hand, from (3.2) and (3.3) we have

$$
\begin{aligned}
d(\Lambda, \Gamma) &\leq d(H_{\check{s}_t}, \check{s}_t) \\
&\leq (H_{\check{s}_t}, \check{s}_{t+1}) + d(\check{s}_{t+1}, \check{s}_t) \\
&= d(\Lambda, \Gamma) + d(\check{s}_{t+1}, \check{s}_t)
\end{aligned}
$$

for all $t \in \mathbb{N}$. For the limit as $t \to \infty$ we have

$$\lim_{t\to\infty} d(H_{\check{s}_t}, \check{s}_t) = d(\Lambda, \Gamma).$$

Then, since g is a d^{-1} −best d^{-1} − orbitally lower semicontinuous on Γ, we have

$$\begin{aligned} d(\Lambda,\Gamma) &\leq d(H\check{s}*, \check{s}*) \\ &= g(\check{s}*) \\ &\leq \lim_{t\to\infty} itf\ g(\check{s}_t) \\ &= \lim_{t\to\infty} itf\ d(H_{\check{s}_t}, \check{s}_t) \\ &= d(\Lambda,\Gamma), \end{aligned}$$

and so we get $d(H\check{s}^*, \check{s}^*) = d(\Lambda,\Gamma)$. Hence, $\check{s}^*$ is a right best proximity point of H.

The following example is given to show the effectiveness of Theorem (3.6).

Example 3.7. Let $\mathfrak{H} = [0,\infty) \times [0,\infty)$ and $d: \mathfrak{H} \times \mathfrak{H} \to \mathbb{R}$ be a function defined by

$$d(\check{s},\breve{u}) = \begin{cases} 0 & ,\check{s} = \breve{u} \\ 2\check{s}_1 + \breve{u}_1 + |\check{s}_2 - \breve{u}_2|, & \check{s} \neq \breve{u} \end{cases}$$

for all $\check{s} = (\check{s}_1, \check{s}_2), \breve{u} = (\breve{u}_1, \breve{u}_2) \in \mathfrak{H}$. Hence, $(\mathfrak{H}, d)$ is a quasi metric space. Consider the sets $\Gamma = \{0\} \times [0,\infty), \Lambda = \{1\} \times [0,\infty)$ and a mapping $H: \Gamma \to \Lambda$ given as

$$H(0,\check{s}) = \left(1, \frac{\check{s}}{3}\right).$$

Then, we have $d(\Lambda,\Gamma) = 2$, and so we get

$$O_H^{d^{-1}}(\check{s}) = \left\{\{(0,\check{s}_t)\}\ in\ \Gamma : \begin{array}{c}(0,\check{s}) = (0,\check{s}_0) atd \\ d\big(H(0,\check{s}_t),(0,\check{s}_{t+1})\big) = d(\Lambda,\Lambda)\ for\ all\ t \in \mathbb{N}\end{array}\right\}$$

$$= \left\{\{(0,\check{s}_t)\}\ in\ \Gamma : \begin{array}{c}\check{s} = \check{s}_0\ atd \\ d\left(\left(1,\frac{\check{s}_t}{3}\right),(0,\check{s}_{t+1})\right) = 2\ for\ all\ t\ \in\ \mathbb{N}\end{array}\right\}$$

$$= \left\{\{(0,\check{s}_t)\}\ in\ \Gamma : \check{s} = \check{s}_0 at\ d\ \check{s}_{t+1} = \frac{\check{s}_t}{3}\ for\ all\ t\ \in\ \mathbb{N}\right\}$$

$$= \left\{\left\{\left(0,\frac{\check{s}}{3^t}\right)\right\} in\ \Gamma : \check{s} \in \Gamma\right\}$$

for all $t \in \mathbb{N}$. Also, we obtain $\Gamma_0^r = \Gamma$, $\Lambda_0^r = \Lambda$ and $H(\Gamma_0^r) \subseteq \Lambda_0^r$. Let for all $\check{s} \in \Gamma$ and $\{\check{s}_t\}$ in $O_H^{d^{-1}}(\check{s})$, $\{\check{s}_{t_i}\}$ be right K-Cauchy subsequence of $\{\check{s}_t\}$. Since every sequence in $O_H^{d^{-1}}(\check{s})$ is convergent with respect to d^{-1}, the subsequences $\{\check{s}_{t_i}\}$ is convergent to $(0,0)$ with respect to d^{-1}. Hence, we have $\check{s}_{t_i} \overset{d^{-1}}{\to} (0,0) \in \Gamma$. Then, Γ is a $H - besi\ d^{-1}$- o-right d^{-1} complete. It is clear that a mapping $g : \Gamma \to \mathbb{R}$ given as $g(\mathfrak{H}) = d(H\check{s},\check{s})$ is a d^{-1} orbitally lower semicontinuous on Γ. Now, we will show that H is a d^{-1}-proximal Ćirić type contraction for $k = \frac{1}{3}$. Let $\check{s}_1, \check{s}_2, u_1, u_2 \in \Gamma$ satisfying

$$\begin{array}{c} d(H\check{s}_1,\ u_1) = d(\Lambda,\Gamma) \\ d(H\check{s}_2,\ u_2) = d(\Lambda,\Gamma) \end{array}$$

Hence, we have $\check{s}_1 = (0,a), \check{s}_2 = (0,b), u_1 = \left(0,\frac{a}{3}\right), u_2 = \left(0,\frac{b}{3}\right)$ where $a,b \in [0,\infty)$. Then,

$$\begin{aligned} d(u_1,u_2) &= \left|\frac{a}{3} - \frac{b}{3}\right| \\ &= \frac{1}{3} d(\check{s}_1,\check{s}_2) \\ &\le \frac{1}{3} M_d(\check{s}_1,\check{s}_2,u_1,u_2). \end{aligned}$$

Therefore, all the conditions of Theorem 3.1 hold. Hence, H is a right best proximity point š $*$ in Γ which is š $*= \ (0,0)$.

Now, we give the following definition.

Definition 3.8. i) Let $(\mathfrak{H}, d)$ be a quasi metric space and $H : \Gamma \ \rightarrow \Lambda$ be a mapping where $\emptyset \neq \Gamma, \Lambda \subseteq \mathfrak{H}$. We say that H is $d - d$ -best ρ $(d \ or \ d^{-1})$-o-continuous at $š^* \in \Gamma$ if for each š Є Γ and $\{š_t\}$ in $O_H^P(š)$ the implication

$$š_{ti} \overset{d}{\rightarrow} š^* \Rightarrow Hš_{t_i} \overset{d}{\rightarrow} Hš^*, as \ \ i \rightarrow \infty$$

satisfies for $\{š_{ti}\}$ of $\{š_t\}$.

ii) Let $(\mathfrak{H}, d)$ be a quasi metric space and $H : \Gamma \ \rightarrow \Lambda$ be a mapping where $\emptyset \neq \Gamma, \Lambda \subseteq \mathfrak{H}$. We say that H is $d - d^{-1}$ -best p $(d \ or \ d^{-1})$-o-continuous at $š^* \in \Gamma$ if for each š Є Γ and $\{š_t\}$ in $O_H^P(š)$ the implication

$$š_{ti} \overset{d}{\rightarrow} š^* \Rightarrow Hš_{t_i} \overset{d^{-1}}{\rightarrow} Hš^*, as \ \ i \rightarrow \infty$$

holds for $\{š_{ti}\}$ of $\{š_t\}$.

iii) Let $(\mathfrak{H}, d)$ be a quasi metric space and $H : \Gamma \ \rightarrow \Lambda$ be a mapping where $\emptyset \neq \Gamma, \Lambda \subseteq \mathfrak{H}$. We say that H is $d^{-1} - d$ -best p $(d \ or \ d^{-1})$ -o-continuous at a point $š^* \in \Gamma$ if for each š Є Γ and $\{š_t\}$ in $O_H^P(š)$ the implication

$$š_{ti} \overset{d^{-1}}{\rightarrow} š^* \Rightarrow Hš_{t_i} \overset{d}{\rightarrow} Hš^*, as \ \ i \rightarrow \infty$$

holds for any subsequence $\{š_{ti}\}$ of $\{š_t\}$.

iv) Let $(\mathfrak{H}, d)$ be a quasi metric space and $H : \Gamma \to \Lambda$ be a mapping where $\emptyset \neq \Gamma, \Lambda \subseteq \mathfrak{H}$. We say that H is $d^{-1} - d^{-1}$ -best p (d or d^{-1})-o-continuous at a point $\check{s}^* \in \Gamma$ if for each $\check{s} \in \Gamma$ and $\{\check{s}_t\}$ in $O_H^p(\check{s})$ the implication

$$\check{s}_{ti} \overset{d^{-1}}{\to} \check{s}^* \Rightarrow H\check{s}_{t_i} \overset{d^{-1}}{\to} H\check{s}^*, as \ \ i \to \infty$$

holds for any subsequence $\{\check{s}_{ti}\}$ of $\{\check{s}_t\}$.

The following example is important to better understand Definition 12.

Example 3.9. Let $\check{s} = [0, \infty) \times [0, \infty)$ and $d: \check{s} * \check{s} \to \mathbb{R}$ be a function defined as

$$d(\check{s}, \check{u}) = ma\check{s}\{\check{u}_1 - \check{s}_1, 0\} + |\check{s}_2 - \check{u}_2|$$

for all $\check{s} = (\check{s}_1, \check{s}_z), \check{u} = (\check{u}_1, \check{u}_2) \in \check{s}$. Then, $(\mathfrak{H}, d)$ is a quasi metric space. Consider the sets

$$\Gamma = \left\{\left(1 + \frac{1}{t}, 0\right) : t \in \mathbb{N}\right\} \cup \{(1,0)\},$$

$$\Lambda = \left\{\left(\frac{1}{t}, 1\right) : t \ \in \mathbb{N}\right\}$$

And a mapping $H : \Gamma \to \Lambda$ given as

$$H\check{s} = \begin{cases} \left(\frac{1}{2}, 1\right) & , \check{s} = (1,0) \\ \left(\frac{1}{t+1}, 1\right) & , \check{s} = \left(1 + \frac{1}{t}, 0\right) \end{cases}$$

Then, we have $d(\Gamma, \Lambda) = 1$, and so $O_H^d(\check{s})$ is the set of all sequences in Γ. Also, it can be seen that the sequence $\{\check{s}_t\}$ in $O_H^d(\check{s})$ converges to (1,0) with respect to d^{-1}. Also, we have

$$\lim_{t\to\infty} d(H\check{s}, H\check{s}_t) = \lim_{t\to\infty} d\left(\left(\frac{1}{2},1\right),\left(\frac{1}{t+1},1\right)\right)$$
$$= \lim_{t\to\infty} \max\left\{\frac{1}{t+1}-\frac{1}{2},0\right\}$$
$$= 0$$

Therefore, H is $d^{-1} - d$-best d-o-continuous on Γ. But, we have

$$\lim_{t\to\infty} d(,H\check{s}_t, H\check{s}) = \lim_{t\to\infty} d\left(\left(\frac{1}{t+1},1\right),\left(\frac{1}{2},1\right)\right)$$
$$= \lim_{t\to\infty} \max\left\{\frac{1}{2}-\frac{1}{t+1},0\right\}$$
$$= \frac{1}{2}.$$

Hence, H is not $d^{-1} - d^{-1}$-best d-o-continuous on Γ.

Proposition 3.10. Let $(\mathfrak{H}, d)$ be a quasi metric space and $\emptyset \neq \Gamma, \Lambda \subseteq \mathfrak{H}$. Assume that $H : \Gamma \to \Lambda$ is a mapping and $g: \Gamma \to \mathbb{R}$ is a function given as $g(\check{s}) = d(\check{s}, H\check{s})$. Then, the following statements are true.

i) If H is $d - d^{-1}$ best ρ-o-continuous at a point $\check{s}* \in \Gamma$ then g is a $d -$ best ρ-o-lower semicontinuous at point $\check{s}* \in \Gamma$.

ii) If H is $d^{-1} - d$ best ρ-o-continuous at a point $\check{s}* \in \Gamma$ then g is a d^{-1} −best ρ-o-lower semicontinuous at point $\check{s}* \in \Gamma$.

Proof. i) Let $\check{s} \in \Gamma$, $\{\check{s}_t\}$ be a sequence in $O_H^d(\check{s})$ and $\{\check{s}_{t_i}\}$ be a subsequence of $\{\check{s}_t\}$ such that $d(\check{s}^*, \check{s}_{t_i}) \to 0$ as $i \to \infty$. Since H is $d - d^{-1}$-best ρ-o-continuous on Γ, we have that $d(H\check{s}_{t_i}, H\check{s}^*) \to 0$ as $i \to \infty$. Hence, we get

$$g(\check{s}^*) = d(\check{s}^*, H\check{s}^*)$$
$$\leq d(\check{s}^*, \check{s}_{ti}) + d\left(\check{s}_{t_i}, H\check{s}_{ti}\right) + d(H\check{s}_{ti}, H\check{s}^*)$$

Taking the limit inferior as $i \to \infty$; we obtain

$$g(\check{s}^*) \leq \lim_{i\to\infty} inf\, d\left(\check{s}_{t_i}, H\check{s}_{t_i}\right) = \lim_{i\to\infty} itf\, g\left(\check{s}_{t_i}\right),$$

and so g is d -best ρ-orbitally lower semicontinuous on Γ.

ii) Let $\check{s} \in \Gamma$, $\{\check{s}_t\}$ be a sequence in O_H^d (š) and $\{\check{s}_{ti}\}$ be a subsequence of $\{\check{s}_t\}$ such that $d(\check{s}_{t_i}, \check{s}^*) \to 0$ as $i \to \infty$. Since H is $d^{-1} - d$-best ρ-o-continuous on Γ. Then, we have $d\left(H\check{s}^*, H\check{s}_{t_i}\right) \to 0$ as $i \to \infty$. Hence, we get

$$g(\check{s}^*) = d(H\check{s}^*, \check{s}^*)$$
$$\leq d(H\check{s}^*, H\check{s}_{ti}) + d\left(H\check{s}_{t_i}, \check{s}_{ti}\right) + d(\check{s}_{ti}, \check{s}^*)$$

Taking the limit inferior as $i \to \infty$; we obtain

$$g(\check{s}^*) \leq \lim_{i\to\infty} itf\, d\left(H\check{s}_{t_i}, \check{s}_{t_i}\right) = \lim_{i\to\infty} itf\, g\left(\check{s}_{t_i}\right),$$

and so g is d^{-1} -best ρ-o-lower semicontinuous on Γ.

The converse of Proposition 1 may not be true. We give an example to demonstrate this fact.

Example 3.11. Let $\mathfrak{H} = [0,\infty) \times [0,\infty)$ and $d : \mathfrak{H} * \mathfrak{H} \to \mathbb{R}$ be a function defined by

$$d(\check{s}, \breve{u}) = \begin{cases} 0 & , \quad \check{s} = \breve{u} \\ \check{s}_1 + |\check{s}_2 - \breve{u}_2| & , \quad \check{s} \neq \breve{u} \end{cases}$$

for all $\check{s} = (\check{s}_1, \check{s}_z), \check{u} = (\check{u}_1, \check{u}_2) \in \mathfrak{H}$. Then, $(\mathfrak{H}, d)$ is a quasi metric space. Consider the sets $\Gamma = \{0\} \times [0, \infty)$, $\Lambda = \{1\} \times [0, \infty)$ and a mapping $H : \Gamma \to \Lambda$ given as $H(0, \check{s}) = \left(1, \frac{\check{s}}{2}\right)$. Then, we have that $d(\Gamma . \Lambda) = 0$, and so we get

$$O_H^d(\check{s}) = \left\{ \{(0, \check{s}_t)\}\ it\ \Gamma : \begin{matrix} (0, \check{s}) = (0, \check{s}_0) atd \\ d\left((0, \check{s}_{t+1}), \left(1, \dfrac{\check{s}_t}{2}\right)\right) = d(\Gamma, \Lambda)\ for\ all\ t\ \in\ \mathbb{N} \end{matrix} \right\}$$

$$= \left\{ \{(0, \check{s}_t)\}\ it\ \Gamma : \begin{matrix} \check{s} = \check{s}_0\ atd \\ d\left((0, \check{s}_{t+1}), \left(1, \dfrac{\check{s}_t}{2}\right)\right) = 0\ \ for\ all\ t\ \in\ \mathbb{N} \end{matrix} \right\}$$

$$= \left\{ \{(0, \check{s}_t)\}\ it\ \Gamma : \check{s} = \check{s}_0\ atd\ \ \check{s}_{t+1} = \frac{\check{s}_t}{2}\ for\ all\ t \in \mathbb{N} \right\}$$

$$= \left\{ \left\{\left(0, \frac{\check{s}}{2^t}\right)\right\} it\ \Gamma : \check{s} \in \Gamma \right\}$$

Now, we will show that a mapping $g : \Gamma \to \mathbb{R}$ given as $g(\check{s}) = d(\check{s}, H\check{s})$ is a d -best d -orbitally lower semicontinuous. Let $\{\check{s}_{t_i}\}$ be a convergent subsequence of $\{\check{s}_t\}$in $O_H^d(\check{s})$ with respect to d. From the definition of d, it can be seen that every sequence in $O_H^d(\check{s})$ converges to $(0, 0)$ with respect to d. Thus, we have that $\check{s}_{t_i} \xrightarrow{d} (0,0)$ Hence, we have

$$\begin{aligned} g\big((0,0)\big) &= d\big((0,0), H(0,0)\big) \\ &= 0 \\ &= \lim_{t\to\infty} inf\ g(\check{s}_t), \end{aligned}$$

and hence g is a d -best d-o-lower semicontinuous. Also, we can show that g is a d^{-1}-best d-orbitally lower semicontinuous. However, H is not $d - d^{-1}$-best d-o-continuous. Indeed, for $\check{s} = (0\ ,1) \in \Gamma$, we have

$$O_H^d(\check{s}) = \left\{\left(0, \frac{1}{2^t}\right)\right\} : t \in \mathbb{N})\}$$

Hence, if we take $\{\check{s}_t\}$ = $\left\{\left(0, \frac{1}{2^t}\right)\right\}$ IN $O_H^d(\check{s})$, then although we have $\check{s}_t \xrightarrow{d} 0$, we get

$$d(H_{\check{s}_t}, H_{\check{s}}) = d\left(\left(1, \frac{\check{s}}{2^{t+1}}\right), (1,0)\right)$$
$$= 1 + \frac{\check{s}}{2^{t+1}}$$

which implies that

$$\lim_{t\to\infty} d\ (H\check{s}_t, H\check{s}) = 1$$

Hence, the sequence $\{H_{\check{s}_t}\}$ is not convergent to $H\check{s}$ with respect to d^{-1} and So H is not $d - d^{-1}$ best d-o-continuous on Γ. It can be seen that H is not d-d-best d-o-continuous, d^{-1} -d-best d-o-continuous and d^{-1} d^{-1}best d-o-continuous.

Now, we present the following result by using Proposition 1.

Corollary 3.12. Let $(\mathfrak{H}, d)$ be a quasi metric space, $\emptyset \neq \Gamma, \Lambda \subseteq \mathfrak{H}$,$\Gamma_0^r \neq \emptyset$ and $H : \Gamma \to \Lambda$ be a d^{-1}-proximal Ćirić type contraction satisfying $H(\Gamma_0^r) \subseteq \Lambda_0^r$. Then, H has a right best proximity point in Γ provided that Γ is H-best d^{-1}-o-right d^{-1}-complete and H is d^{-1}-d-best d^{-1}- o-continuous on Γ.

Theorem 3.13. Let $(\mathfrak{H}, d)$ be a quasi metric space, $\emptyset \neq \Gamma, \Lambda \subseteq \mathfrak{H}$,$\Gamma_0^l \neq \emptyset$ and $H : \Gamma \to \Lambda$ be a d -proximal Ćirić-type contraction satisfying $H(\Gamma_0^\ell) \subseteq \Lambda_0^\ell$ If U is H-best d -o-left d^{-1}-complete and a function $g : \Gamma \to \mathbb{R}$ given as $g(\check{s}) = d(\check{s}, H_{\check{s}})$ is d^{-1}-best d-o-lower semicontinuous on Γ, then H has a left best proximity point in Γ.

Proof. Consider any point $\check{s}_0 \in \Gamma_0^{\ell}$. Because $H\check{s}_0 \in H(\Gamma_0^{\ell}) \subseteq \Lambda_0^{\ell}$, there exists $\check{s}_1 \in \Gamma_0^{\ell}$ such that

$$d(\check{s}_1, H\check{s}_0) = d(\Gamma, \Lambda).$$

Similarly, there exists $\check{s}_2 \in \Gamma_0^{r}$ such that

$$d(\check{s}_2, H\check{s}_1) = d(\Gamma, \Lambda).$$

By carrying out this procedure again, we may create a sequence $\{\check{s}_t\}$ satisfying

$$d(\check{s}_{t+1}, H\check{s}_t) = d(\Gamma, \Lambda) \tag{3.4}$$

for all $t \in \mathbb{N}$, that is, $\{\check{s}_t\}$ in $O_H^d(\check{s}_0)$. If there exists $t_0 \in \mathbb{N}$such that $\check{s}_{t_0} = \check{s}_{t_0+1}$, then $\check{s}_{t_0}$ is a right best proximity point of H. Therefore, we suppose $\check{s}_t \neq \check{s}_{t+1}$ for all $t \in \mathbb{N}$. Similar to the proof of Theorem 1, we can obtain that

$$\lim_{t\to\infty} d\ (\check{s}_t, \check{s}_{t+1}) = 0 \tag{3.5}$$

and the sequence $\{\check{s}_t\}$ is a left K-Cauchy sequence. There is $\check{s}^* \in \Gamma_0^{l}$ such that $d(\check{s}_t, \check{s}^*) \to 0$ as n $\to \infty$ because Γ is H-best d- o-left d^{-1} complete. On the other hand, from Equality (3.4)

$$\begin{aligned} d(\Gamma, \Lambda) &\leq d(\check{s}_t, H\check{s}_t) \\ &\leq d(\check{s}_t, \check{s}_{t+1}) + d(\check{s}_{t+1}, H\check{s}_t) \\ &\leq d(\check{s}_t, \check{s}_{t+1}) + d(\Gamma, \Lambda) \end{aligned}$$

for all $t \in \mathbb{N}$. Taking the limit as $t \to \infty$, from Equality (3.5) we get

$$\lim_{t\to\infty} d\,(\check{s}_t, H\check{s}_t) = d(\Gamma, \Lambda)$$

Then, since g is a d^{-1}-best d -orbitally lower semicontinuous on Γ, we have

$$\begin{aligned} d(\Gamma, \Lambda) &\leq d(\check{s}^*, H\check{s}^*) \\ &= g(\check{s}^*) \\ &\leq \lim_{t\to\infty} \inf g(\check{s}_t) \\ &= \lim_{t\to\infty} \inf d(\check{s}_t, H\check{s}_t) \\ &= d(\Gamma, \Lambda) \end{aligned}$$

and so we get $d(\check{s}^*, H\check{s}^*) = d(\Gamma, \Lambda)$. Hence, $\check{s}^*$ is a left best proximity point of H.

Corollary 3.14. Let $(\mathfrak{H}, d)$ be a quasi metric space, $\emptyset \neq \Gamma, \Lambda \subseteq \mathfrak{H}, \Gamma_0^{\ell} \neq 0$and $H : \Gamma \to \Lambda$ be a d -proximal Ćirić-type contraction satisfying $H(\Gamma_0^{\ell}) \subseteq \Lambda_0^{\ell}$. If Γ is H -best d-o-left d^{-1}-complete and H is d^{-1}-d-best d-o-continuous on Γ, then H has a right best proximity point in Γ.

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4. CONCLUSIONS

This thesis is divided into four chapters. The first chapter discusses the motivation for the thesis and provides some background information on fixed point theory and best proximity point theory. In the second chapter, we recall the fundamentals definitions and notations, and some theorems related to our results about fixed point and best proximity point. We present our results in chapter three. We first introduce the concepts of d and d^{-1}- proximal Ćirić contraction mappings. Then, we prove some right and left best proximity point results for these mappings in o-complete quasi metric spaces. Hence, we obtain some generalization of famous results in the literature. Also, we present new definitions and notations by taking into account the lack of symmetry property of quasi metric. Moreover, we give some examples to support our definitions and notations.

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