A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF ÇANKIRI KARATEKİN UNIVERSITY

SOME PROBLEMS ON VECTOR VARIATIONAL-LIKE INEQUALITIES

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
MATHEMATICS

BY

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ÇANKIRI

2023

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February 2023

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science

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ABSTRACT

SOME PROBLEM ON VECTOR VARIATIONAL- LIKE INEQUALITIES

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February 2022

The summary theory for vector variational inequalities has found its use in several branches, whether applied sciences or pure. It gives a unifiet formula to understand and study a large amount of linear-non-linear mathematical issues. This thesis dealt with the study and discuss of some results for the solutions existed to mixed vector variational-like problems (for shortly, MVVLP). Firstly, a new MVVLP involving generalized monotone operator in Hausdorff topological linear spaces (for shortly, HTLS) is introduced. By employing KKM technique and using some relevant hypotheses on the considering non-linear mappings the existence of a result for a recent class of MVVLP in the setting of $(HTLS)$ are presented. Secondly, new definition of invexity is called $\mathrm{F}(\mathrm{T},\Theta)$ $-$invex function and new idea of inequality variational and additional problems to vector of topology space are given. Finally, some findings of nonstandard vector variational inequality in H-spaces are generalized. The solutions provided in this study renew and make better some solutions to many researchers.

2023, 64 pages

Keywords: Vector variational-like problem, Solution existence, Hausdorff metric, KKM theory

ÖZET

VEKTÖR VARYASYONEL BENZERİ ESİTSİZLİKLER ÜZERİNDE BAZI PROBLEMLER

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Vektör varyasyonel eşitsizlikler için toplam teorisi uygulamalı veya soyut bilimlerde kullanılmaktadır. Bu teori bize lineer ya da lineer olmayan matematiksel ifadelerin geniş bir alanını anlamak için birleşik bir formül verir. Bu tez karışık vektör varyasyonu benzeri problemlerin (kısaca KVBP) çözümü icin bir çalışmadır. İlk olarak, Hausdorff toplojik lineer uzaylarda (kısaca HTLU) genelleştirilmiş monoton operatorleri içeren yeni bir KVBP tanıtılmıstır. KKM tekniklerini çalışarak ve ilgili HTLU kurulumunda son bir KVBP sınıfı icin lineer olmayan dönüşümleri düşünerek çözümün varlığı gösterilmiştir. İkinci olarak $F(T,\Theta)-$ olarak adlandırılan inveks fonksiyonu, eşitsizlik varyasyonel fikri ve topolojik vektör uzaylarında ilave problemler verilmiştir. Son olarak, H-uzaylarındaki standart olmayan vektör varyasyonel eşitsizlikler genelleştirilmiştir. Bu çalismadaki çözümler yenidir ve birçok araştırmacıya çözümleri daha iyi anlamak için olanak sunar.

2022, 64 sayfa

Anahtar Kelimeler: Vektör varyasyonel benzeri problem, Çözüm varlığı, Hausdorff metrik, KKM teorisi

PREFACE AND ACKNOWLEDGEMENTS

First and foremost, I'd like to offer my heartfelt gratitude and gratitude to the almighty ALLAH (May He be glorified and exalted) for my success and inspiration in earning this wonderful opportunity to advance my professional life. My heartfelt gratitude goes to the Ministry of Education and the Turkish Government for providing me with the opportunity to pursue a master's degree of ÇANKIRI KARATEKIN UNIVERSITY. I would like to take this opportunity to extend my sincere gratitude to my advisor, Prof. Dr. Faruk POLAT, for all of the essential assistance, advice, collaboration, and support he provided me while I was working on my thesis. To make my thanks and gratitude to (Co-Advisor: Asst. Prof. Dr. Ayed Eleiwis HASHOOSH), because he has a distinguished role, good opinions and good touches on every page of my thesis. And my deep gratitude for all my professors and colleagues in Mathematic Department of Çankırı Karatekin University Graduate School of Natural and Applied Sciences who helped through the years of my studyAdditionally, I appreciate my family's support and encouragement. Lastly, I would want to convey my gratitude to everyone who assisted me in finishing my thesis.

Suhair AbdulFattah AJEEL
Çankırı-2023

CONTENTS

LIST OF SYMBOLS

$\mathbb{R}$	The set of Real numbers.
$\mathbb{N}$	The set of Natural numbers.
$\in$	Belonge to.
diam (A)	Diameter of a set A.
Conv(A)	Convex hull of set A.
$\mathbf{2}^{\boldsymbol{E}}$	The family of all a nonempty set of E.
$\langle \quad \rangle$	Canonical inner product
$\geq$	The greater than or equal
$\leq$	The less than or equal
$\exists$	There exists.
$\forall$	For all
$\mathbf{2}^{\mathbf{X}}$	The set of all subsets of X.

LIST OF ABBREVIATIONS

usc	Upper Semi Continuous
lsc	Lower Semi Continuous
IPS	Inner product space
H.C	Hemicontinuous
resp.	Respectively
w.r.t	With respect to
et al.	And others
B.S	Banach spaces
V.E	Vector equilibrium
T.V.S	Topological vector space
H.T.S	Hausdorff topological space
HTLS	Hausdorff topological linear space
VVLIP	Vector variational-like inequality problem
GMVVLI	Generalized mixed vector variational-Like inequalities
OTLS	Ordered topological linear space
iff	If, and only if

1. INTRODUCTION

The vector variational problems have undergone great advance development in hypotheses and algorithms across fields of pure and applied sciences. It is worth mentioning, Giannessi was the first to present (VVI) in the spaces of limited dimensional Euclidean. Consequently, a framework to analyses a broad types of problems arising linear equilibrium problems, linear complementarity problems, linear inequalities of ariation and linear optimization issues was broadly used in a general composition by different researchers see (Nguyen and Phan 1998), (Rais and Akram 2014), (Hashoosh *et al*. 2016), (Rahaman *et al*. 2014) and (Noor and Oettli 1994).

1.1 Background

Equilibrium problems studied by (Blum and Oettli 1994) has many applications in the different branches of sciences. These problems have many mathematical issues in special cases, such as issues in programming in mathematics problems, complementary, variational inequality, and fixed point. The notion of vector of inequality variation problem reported by (Giannessi 1980). However, in the year of 1990 the equilibrium problem was used to cover the problem of V.E (Chen and Yang 1990).

The V.E problem involves the vector optimization problem, vector variational inequality problems, vector complementarity problems, and vector of inequality variational issues as a special instance. They can be stated as follows: after making a closed convex subset K of a real Hausdorff T.V.S and a bifunction $F(x, y)$ valued in a real ordered locally convex space, one must locate $x \in K$ in which $F(x, y) \not< 0$ for all $y \in K$. M. Bianchi *et al*. established existence of results for the standardized form involving that F is either a quasimonotone or pseudomonotone bifunction (Bianchi *et al*. 1997). By using Fan's Lemma, Noor introduced new concepts of a $F_{(C,D)}$-pseudo-monotone mapping and of (C, D)-pseudo-monotone pair of mapping in (1999). Also he established numerous existence results that are gained from universal problems of V.E (Konnov and Yao 1999). Through aplying this dual formulations (Li *et al* 2001).

Presented new concepts of dual formulations for problems of generalized V.E involving generalized pseudo-monotonicity conditions. By using KKM-Fan theorem (Li *et al*. 2003) introduced new a class which has many problems of implicit equilibrium, implicit inequalities of variation, and problems of implicit complementarity as specific cases in Hausdorff topological vector spaces. As (Fakhar, M. and Zafarani, J. 2005) used a novel kind of the Nirenberg, Brézis and Stampacchia result by using pseudo-monotonicity and few coercivity assumptions to prove existence findings solution for problems of universal V.E of multivalued bifunctions. Ansari introduced a form of vector for equilibrium version for Ekeland-type principle of variational (Ansari 2007) presented similar results to our principle of variation. As he derived the existence of the solutions of a V.E issue in constructing complete quasi-metric spaces, he proved the existence of the results. (Li and Li 2009) were able to make a qualitative leap in their investigation and presentation of two different forms of Levitin–Polyak well-posedness of V.E issues with changing domination structures. This was made possible by the qualitative leap. In addition to that, they demonstrated it using Criteria and characterizations for two different forms of Levitin–Polyak well-posedness of V.E issues. (Balaj 2010) established a common fixed points for of set-valued mappings with applications by the Brouwer fixed points. (Hardy and Kuijlaars 2012) established lower semi-continuity and strict convexity of the energy functional for a huge type of V.E problems in setting a theory of logarithmic potential. (Anh *et al*. 2014) considered lexicographic V.E problems in metric spaces. Further, they established the relevant conditions for a set of such problems to be (uniquely) well posed at the reference point. On the other hand (Han and Huang 2016) introduced a secularization result and a density theorem focused with the sets for weakly efficient and efficient approximate results to a generalized V.E problem. (Van Su 2018) established Fritz John necessary contexts for local weak efficient solutions of V.E problems with constraints in kinds of contingent derivatives. (Anh 2021) considered V.E problems in real B.S and studied their regularized problems, Taking into account the cone continuity and generalized convexity qualities of vector-valued mappings, the general conditions that guarantee the existence of solutions to such problems under monotonicity and nonmonotonicity are demonstrated. Last but not least, (Hung 2022) presented innovative error bounds for a class of V.E problems with partial order given by a polyhedral cone constructed from a matrix.

1.2 Organization of the Thesis

The goal of this thesis is to present developed type of VVLIP in HTLS. By employing FKKM technique and according to relevant suppositions on the study of nonlinear mappings, In the framework of MVVLP, we will demonstrate that there is an existence of a results involving a class of monotone operator (HTLS). The findings of numerous scholars that relate to them are improved thanks to the solutions that are presented in this thesis.

This thesis consists of four chapters, The first chapter is devoted to the general introduction (background and Organization) to the thesis. This chapter gave a survey of the literature on vector variational inequality issues; now we introduce the following chapters: Chapter 2, This chapter contains four sections: after introduction, in Section two, we recalled basic concepts of Topological vector spaces and we investigated in section three some notions convex analysis that needful in next chapters. Finally, we mention fact concerning of vector variational inequality issues. The study in third chapter is ordered into the three sections. In section two, we outline definitions and preliminary theorems that impose to prove the main results. In section three, we present new findings by using the FKKM–Mapping, under compact and non-compact hypotheses on underlying convex sets. Fourth chapter is organized as follows: After the introduction and the basic priorities that we need us in the next sections, in section three, we presented new definition of invexity is called $\mathrm{F}(\mathrm{T},\Theta)$-invex function. Further, some finding complementarity problems are introduced. In section four, some existence solutions of (FVVI). Employing the H-KKM theorem and H-compact sets, we present an existence result for our problem on H spaces. Finally, chapter five contains some conclusions and future work.

2. PRELIMINARIES AND FUNDAMENTAL CONCEPTS

2.1 Introduction

This chapter reviews certain definitions and initial findings. These findings are intended to assist us in identifying the key findings of our study, which will be discussed in the subsequent chapters. We begin with the most fundamental explanation of topological vector spaces and conclude with a quick introduction to vector variational inequality problems. We studied certain convex analytic concepts that would be necessary in subsequent chapters, and we highlight some topological space concepts.

2.2 Some Concepts Of B.S

Definition 2.1 (Arsove 1960) A vector space is defined as a non-empty set X in which the individual components are referred to as vectors. It also has a scalar field F with its elements called scalars, given with two operations one named addition ($+: X \times X \to X$) and the other named multiplication ($. : F \times X \to X$) that are relevant to features.

i. $\varrho + \tau \in X \ \forall \ \varrho, \tau \in X$;

ii. $\varrho + \tau = \tau + \varrho \ \forall \ \varrho, \tau \in X$;

iii. $\varrho + (\tau + z) = (\varrho + \tau) + z \ \ \forall \ \varrho, \tau, z \in X$.

iv. $\exists \ 0 \in X$ such that $\varrho + 0 = 0 + \varrho = \varrho$ for all $\varrho \in X$ and 0 is the zero vector or the origin.

v. $\forall \ \varrho \in X, \exists \ -\varrho \in X$ such that $\varrho + (-\varrho) = (-\varrho) + \varrho = 0$

vi. $\lambda . \varrho \in X \ \forall \ \lambda \in F$ and $\forall \ \varrho \in X$

vii. $\lambda.(\varrho+\tau)=\lambda.\varrho+\lambda.\tau\ \forall\ \lambda \in$ F and $\forall\ \varrho, \tau\ \in$ X

viii. $(\alpha+\beta).\varrho=\alpha.\varrho+\beta.\varrho\ \forall\ \alpha, \beta\ \in$ F and $\forall\ \varrho\ \in$ X

ix. $(\alpha.\beta).\varrho=\alpha.(\beta.\varrho)\ \forall\ \alpha, \beta\ \in$ F and $\forall\ \varrho\ \in$ X

x. $I.\varrho=\varrho\ \forall\ \varrho \in$ X and I is the unity element of the field F.

Definition 2.2 (Arsove 1960) Let M represent a non-empty subset of the linear space X over the field F. M is a subspace of X if M is a linear space over F By depend to same operations as X. Then, M of X over F is a subspace of X iff

i. $\varrho+\tau \in M, \forall\ \varrho, \tau \in M$;

ii. $\lambda\varrho \in M, \forall\ \lambda\ \in F$ and $\forall\ \varrho \in M$;

or equivalently , $\alpha\varrho+\beta\tau \in M, \forall\ \alpha, \beta \in F$ and $\forall\ \varrho, \tau \in M$.

Definition 2.3 (Milgram 1940) Consider (X, T) and (Y, Γ) to be topological spaces. A function $f: X \rightarrow Y$ is named to be continuous if, for any an open $U \subseteq Y, f^{-1}(U)$ is an open in X.

Definition 2.4 Consider (X, T) is a topological space. A mapping $\Gamma: A \longrightarrow R$ is considered to be

i. Lower semi-continuous (for short, lsc) at $a_0 \in$ X if

$$\Gamma(a_0) \leq \operatorname{Lim\,inf}\ \Gamma(a_n).$$

ii. Upper semi-continuous (for short , usc) at $a_0 \in$ X if

$$\Gamma(a_0) \geq \text{Lim sup}\,\Gamma(a_n).$$

For any sequence a_n of X such that $a_n \longrightarrow a_0$.

Definition 2.5 (Milgram 1940) Let (X, T) represent a topological space, and let A represent a subset of X. Therefore,

(1) A is a closed set of $X\ if\ X - A$ is an open set and vice versa.

(2) The closure set of A in X is the intersection of all closed sets of X that include A and are designated by $\bar{A}$.

(3) The interior set of A in X is the union of all open sets of X, contained in A and denoted by $Int\ A$.

(4) $\varrho \in X$ is a limit point of A if $\varrho \in \overline{A - \{\varrho\}}$.

(5) $\bar{A} = A$ iff A is closed, and $Int\ A = A$ iff A is an open.

(6) A subset A's boundary set within a space X, indicated by the sign ∂A and defined as

$$\partial \mathrm{A} = \overline{\mathrm{A}} - \mathrm{Int\ A}.$$

Definition 2.6 (Milgram 1940) A space X is called Hausdorff topological if it is T_1 and for any two distinct points $\varrho, \tau \in X$ there is a disjoint open sets U and V where $\varrho \in$ U and $\tau \in$ V.

Definition 2.7 (Sutherland 2009) Considering X be a topological space. A collection of open sets $U_\alpha, \alpha \in A$ is an open covering of X if $X = \cup\ U_\alpha$.

Definition 2.8 (Sutherland 2009). If (X,T) is a topological space, then X is a compact space only if every open cover of X has a finite subcover.

Definition 2.9 (Sutherland 2009) A topology T of a vector space X over a field F is a named a vector topology if the functions $+: X \times X \to X$ and $*: F \times X \to X$ are continuous. A vector space endowed with a vector topology is called a topological vector space (for shortly, T.V.S). A T.V.S is called complex or real as a vector space. The zero element of a topological vector is also called the origin.

Definition 2.10 (Sutherland 2009) Given a nonempty set X. A function $d : X \times X \to$ R is termed metric on X if $\forall\, \varrho, \tau, z \in X$ satisfy this following conditions:

(D_1) $d(\varrho, \tau) \geq 0$, and $d(\varrho, \tau) = 0$ if and only if $\varrho = \tau$.

(D_2) $d(\varrho, \tau) = d(\tau, \varrho)$

(D_3) $d(\varrho, \tau) \leq d(\varrho, z) + d(z, \tau)$.

The set X which equipped by metric is going to be called metric space, and characterized by (X, d).

Definition 2.11 (Sutherland 2009) Considering the topological space (X,T) and a point $\varrho_0 \in X$ and a real numbe $\iota > 0$, then the open and closed ball define as following

(a) $B(\varrho_0; \iota) = \{\varrho \in X : d(\varrho, \varrho_0) < \iota\}$ is called open ball

(b) $\bar{B}(\varrho_0; \iota) = \{\varrho \in X : d(\varrho, \varrho_0) \leq \iota\}$ is called closed ball

In all two cases, ϱ_0 is called the center and ι the radius.

The radius open ball ι is defined as the collection of all points in X whose distance from the center of an open ball is less than ι.

Example 2.12 R is a metric space with $d(\varrho, \tau) = |\varrho - \tau|$.

Definition 2.13 (Sutherland 2009) Let X be a metric space and S $\subseteq X$. S is bounded if, there $\varrho_o \in S$ and $\iota \in R$ in which $d(\varrho, \varrho_o) < \iota$, for each $\varrho \in S$.

Definition 2.14 (Sutherland 2009) A bounded set if, there is a real number $M > 0$ in which $|\varrho_n| \leq M$ for each $n \in N$.

Proposition 2.15 (Sutherland 2009) Any bounded monotone sequence of the real numbers is convergent sequence.

Definition 2.16 (Sutherland 2009) Consider metric spaces (X, d) and $(\mathbb{R}, d^*), A \subset X$. The function $f: A \to \mathbb{R}$ is called to be continuous at a point $c \in A$ if for each $\varepsilon > 0$, $\exists$ a $\delta > 0$ in which $d^*\big(f(\varrho), f(c)\big) < \varepsilon$, whenever $\varrho \in A, d(\varrho, c) < \delta$. While, f is continuous on A if at each point $c \in A$ it is continuous.

Definition 2.17 (Tarski 1955) Considering $T: X \to X$ is a function operating on the set X. A point $\varrho_0 \in X$ is named a fixed point of transformation T if $T_{\varrho_0} = \varrho_0$; it mens that, if the point remains unchanged under the transformation. T is the name of T's fixed point.

Definition 2.18 (Kreyszig 1991) Considering V is a vector space over the field of the real number $\mathbb{R}$. A norm on V is a real-valued function $\|.\| : V \to [0, \infty)$, in which for any $u, v \in V$ and $\alpha \in \mathbb{R}$. the following statements are true

i. $\|u\| \geq 0$, and $\|u\| = 0$, iff $u = 0$;

ii. $\|\alpha u\| = |\alpha|\|u\|, \forall u \in V$ and $\alpha \in \mathbb{R}$;

iii. $\|u+v\| \leq \|u\| + \|v\|, \forall u, v \in V$.

Example 2.19 Any normed linear space is a metric space by $d(\varrho, \tau) = \|\varrho - \tau\|$.

Definition 2.20 (Kreyszig 1991) Assume that U and V are two normed linear spaces. An operator $\mathrm{A}: \mathrm{U} \rightarrow \mathrm{V}$ is continuous at $\mathrm{u}_0 \in \mathrm{U}$, if $\forall \epsilon > 0$, there is a $\delta = \delta(\epsilon, \mathrm{u}_0)$ such that $\|\mathrm{A}_{\mathrm{U}} - \mathrm{A}_{\mathrm{u}_0}\| < \epsilon$, whenever $\|\mathrm{u} - \mathrm{u}_0\| < \delta$.

Definition 2.21 (Kreyszig 1991) A sequence $\{U_n\}_{n=1}^{\infty}$ in a normed linear space $(R^n, \|.\|)$ converges to a point $u \in R^n$, if and only if, $\forall$ real number $\epsilon > 0, \exists$ an integer N in which $\|U_n - u\| < \epsilon, n > N$.

Definition 2.22 (Kreyszig 1991) A normed vector space $(X, \|.\|)$ is said to be B.S, if any Cauchy sequence is convergent in X.

Example 2.23 Let $\|.\|: X = F \rightarrow \mathbb{R}$ be a function described by $\| \varrho \| = |\varrho| \; \forall \varrho \in F$

i. Let $\varrho \in X$ then $\| \varrho \| = 0$ iff $|\varrho| = 0$ iff $\varrho = 0$

ii. Let $\varrho \in X$ and $\beta \in F$ then $\| \beta\varrho \| = |\beta||\varrho|$;

iii. Let $\varrho, \tau \in X$, then

$$\| \varrho + \tau \| = |\varrho + \tau| \leq |\varrho| + |\tau| = \| \varrho \| + \| \tau \|$$

so that X is a normed space, τsince R or C is complete space , then F is complete space.

Hence proved X is a B.S.

Definition 2.24 (Kreyszig 1991) Consider that X is a normed space. The set of all bounded linear functions on X then defines a normed space with the corresponding norm.

$$\|f\| = Sup_{\substack{\varrho\in X\\ \varrho\neq 0}} \frac{|f(\varrho)|}{\|\varrho\|} = Sup_{\substack{\varrho\in X\\ \|\varrho=1\|}} |f(\varrho)|.$$

And it is named the dual space of X and it is denoted by X' .

Theorem 2.25 (Kreyszig 1991) A Banach space is the dual space X' of a normed space X.

Definition 2.26 (Kreyszig 1991) An inner product (pre-Hilbert space) is a vector space X over a field K (ℝ or C) if there is a function $< \varrho, \tau >$ from $X \times X$ to K that is defined as follows:

(1) $< \varrho, \varrho > \geq 0 \ \forall \ \varrho \in X$;

(2) $< \varrho, \varrho >= 0$ if and only if $\varrho = 0$;

(3) $< \varrho, \tau > = \overline{< \tau, \varrho >}$;

(4) $< \lambda\varrho + \rho\tau, z > = \lambda < \varrho, z > +\rho < \tau, z > \ \forall \ \varrho, \tau, z \in X$, and $\lambda, \rho \in K$.

The value $< \varrho, \tau >$ is named the inner product of vectors ϱ, τ in case $K = \mathbb{R}$, so,

$< \varrho, \tau > =< \tau, \varrho > \forall \ \varrho, \tau \in \ X.$

If an inner product has been defined on the space X in a vector space, then X is said to be an inner product (for shortly IPS). The IPS is called a Hilbert space if $(X, \|.\|)$ is complete.

2.3 Convex Analysis

Definition 2.27 (Fenchel and Blackett 1953)

1) For two distinct points $\varrho, \tau \in \mathbb{R}^n$, A subset $M \subseteq \mathbb{R}^n$ is called an affine set if $(1 - \lambda)\varrho + \lambda\tau \in M$ $f$or every $\varrho \in M, \tau \in M$, and $\lambda \in \mathbb{R}$.

2) A set $C \subseteq \mathbb{R}^n$ is called to be convex if $(1 - \lambda)\varrho + \lambda\tau \in C$ for any ϱ, $\tau \in C$.

3) Let $S \subseteq \mathbb{R}^n$. The intersection of all the convex sets that contain S is the convex hull of S, which is written as conv(S).

4) A vector sum $\lambda_1\varrho_1 + \cdots + \lambda_m\varrho_m$ is called a convex combination of $\varrho_1, \ldots, \varrho_m$, if the coefficients λ_i are all nonnegative and $\lambda_1 + \cdots + \lambda_m = 1$.

Remark 2.28. (Fenchel and Blackett 1953) Assume that X is a vector space, then

i. Convex sets include both empty sets and singleton sets.

ii. The intersection of all convex sets yields a convex set.

iii. The convex set's closure is also convex.

Definition 2.29. (Fenchel and Blackett 1953) A nonempty set $X \subseteq \mathbb{R}^n$ is invex at $u \in X$ if there is a vector function $\eta: X \times X \to \mathbb{R}^n$ in which if $\forall$ $\varrho \in X$ and $\lambda \in [0,1]$ then

i. $u + \lambda\eta(\varrho, u) \in X$.

ii. A set X is called invex if (i) holds $\forall$ $u \in X$.

Definition 2.30 (Clarke 1990) Consider that $T: K \to L(X, Y)$ and $\eta: K \times K \to K$ are two functions. We claim that T is η −hemi continuous if for any given $u, v \in K$,

$t \in (0,1)$, the mapping $t \longmapsto \langle T(u + t(v - u)), \eta(v, u) \rangle$ is continuous at 0^+.

Definition 2.31 (Fenchel and Blackett 1953) The function $f\colon D \to \mathbb{R}$, where D is a convex set in $\mathbb{R}^m$ and R denotes the real line, is convex on D provided that ∀ ϱ and τ in D and $\forall\ \lambda \in [0,1]$

$$f(\lambda\varrho + (1-\lambda)\tau) \leq \lambda f(\varrho) + (1-\lambda) f(\tau).$$

Definition 2.32 (Fenchel and Blackett 1953). Consider X is a T.V.S. A nonempty subset C of X is said to be convex cone if $C + C \subseteq C$ and $\lambda C \subseteq C$, for any $\lambda \geq 0$. A cone C is said to be pointed if $C \cap (-C) = \{0\}$, where 0 denotes the zero vector.

Example 2.33 The set $\mathbb{R}^n_+ = \{\varrho = (\varrho_1, \varrho_2, \dots, \varrho_n) \in \mathbb{R}^n\colon \varrho_i \geq 0, \forall\ i = \{1,2, \dots, n\}$ is a pointed cone.

Definition 2.34 (Fenchel and Blackett 1953) Consider that $f\colon X \to \mathbb{R}$ be a differentiable function on $X \subset \mathbb{R}^n$ and $u \in \mathbb{R}^n$ If, ∃ a vector-valued function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that, $\forall \varrho \in X$, the following inequality

$$f(\varrho) - f(u) \geq \nabla f(u)\eta(\varrho, u) \tag{2.1}$$

If the above inequality applies, therefore f is an invex (strictly invex) function w.r.t $\eta\ at\ u$ on X. If (2.1) holds at each member $u \in \mathbb{R}^n$,then f is named to be an invex (strictly invex) function w.r.t η on $\mathbb{R}^n$.

2.4 Some Concepts Of Vector Variational Inequality Problem

Lemma 2.35. (Chen and Hou 2000) Suppose that (Y, C) be an ordering topological linear space by a convex cone pointed and closed with $intC \neq \emptyset$. Therefore, for any $v, w \in Y$ one can have

i. $v - w \in intC$ and $v \notin intC$ imply $w \notin intC$.

ii. $v - w \in C$ and $v \notin intC$ imply $w \notin intC$.

iii. $v - w \in -intC$ and $v \notin -intC$ imply $w \notin -intC$.

iv. $v - w \in -C$ and $v \notin -intC$ imply $w \notin -intC$.

Lemma 2.36. (Kim *et al*. 2018) Considering a nonemty set M is a convex, and closed subset of a H.T.S, and $G: M \to 2^M$ a multivalued mapping, if any finite set $\{u_1, u_2, \dots, u_n\} \subset M$, one have $conv\,\{u_1, u_2, \dots, u_n\} \subset \bigcup_{i=1}^{n} G(u_i)$.

It means that, F is a KKM-mapping and $G(u)$ is closed for each $u \in M$ and compact for a few $u \in M$, where $conv$ refer to the convex hull operator. Then $\bigcap_{u \in M} G(u) \neq \emptyset$.

Lemma 2.37. (Kim *et al*. 2018) Consider that X is a H.T.S, $B_1, B_2, \dots, B_n$ be nonempty convex and closed compact subsets of X. Therefore, $conv \bigcup_{i=1}^{n} G(B_i)$ is compact.

Lemma 2.38. (Murphy 2008) Consider two topological spaces X and Y. If $B: X \to 2^Y$ is usc by closed values. Therefore, B is closed.

Theorem 2.39.(Yuan 1999) Consider that K is any nonempty set in a HTLS X. Consider the set-valued mapping $\theta : K \to 2^X$ to be a KKM map where

(1) $\theta(\varrho)$ is closed set $\forall\ \varrho \in K$.

(2) The set $\theta(\varrho)$ is compact for at least one $\varrho \in K$.

Therefore, $\cap\ \{\theta\ (\varrho) : \varrho \in K\} \neq \emptyset$.

Lets shall show our results using the notations listed below.

Remark 2.40. (Fan 1984) For simplicity, we apply the below terminologies:

1. $\nu \notin -intP$ iff $\nu \geq_P 0$

2. $\nu \in intP$ iff $\nu >_P 0$

3. $\nu \notin intP$ iff $\nu \leq_P 0$

4. $\nu \in -intP$ iff $\nu <_P 0$

5. $\nu - w \notin -intP$ iff $\nu - w \geq_P 0, i.e\ (\nu \geq_P w)$

6. $\nu - w \notin intP$ iff $\nu - w \leq_P 0, i.e\ (\nu \leq_P w)$

7. $\nu - w \notin (-intP \cup intP)$ iff $\nu - w =_P 0, i.e\ (\nu =_P w)$

We apply the below terminologies when necessary:

1. $\nu - w \notin -P$ and $w \notin -intP$ imply $\nu \notin -intP$

2. $\nu - w \notin -intP$ and $w \notin -intP$ imply $\nu \notin -intP$

3. $\nu - w \notin -P$ and $\nu \in -intP$ imply $w \in -intP$

4. $\nu - w \notin -intP$ and $\nu \in -intP$ imply $w \in -intP$

5. $\nu - w \in -intP$ and $w \in -intP$ imply $\nu \in -intP$

6. $\nu \notin -intP$ iff $-\nu \notin intP$

7. $\nu \notin -intP$ and $w \notin -intP$ imply $\nu + w \notin -intP$.

Definition 2.41 (Das 2006) Consider X is topological space. Let $(X, \{\Gamma_A\})$ represent a set of nonempty and contractible sets that are indexed by finite subsets of X.

i. $(X, \{\Gamma_A\})$ is called to be a H-space iff $A \subset B$ then, $\Gamma_A \subset \Gamma_B$.

ii. $\Gamma_A \subset D$ holds for any finite set $A \subset D$ iff a set $D \subset X$ is named to be H-convexity.

iii. For all finite set $A \subset D$, $\Gamma_A \cap D$ is nonempty and contractible iff A set $D \subset X$ is called weakly H-convex. This is comparable to asserting that the pair $\{D, \{\Gamma_A \cap D\})$ is an H-space.

iv. There is a compact and weakly H $-$convex set $D \subset X$ such that $K \cup A \subset D$ f for any finite subset $A \subset X$ iff a subset $K \subset X$ is said to be H-compact.

v. A set-valued mapping $\mathrm{G} : \mathrm{X} \to 2^{\mathrm{X}}$ is considered to be an $\mathrm{H} - \mathrm{KKM}$ mapping if, for any finite subset $\mathrm{A} \subset \mathrm{X}$. After that, we get $\Gamma_{\mathrm{A}} \subset \bigcup_{\varrho \in \mathrm{A}} \mathrm{G}(\varrho)$.

Theorem 2.42. (Das 2006) Consider$(X, \{\Gamma_A\})$ is an H-space and that $F : X \to 2^X$ an $H - KKM$ multifunction in which

i. For all $\varrho \in X$, $F(\varrho)$ is a compactly and closed set, $\beta \cap F(\varrho)$ is closed in β, for any compact $\beta \subset X$.

ii. There existing a compact set $L \subset X$ and an H −compact set $K \subset X$, in which for any weakly H-convex set D accompanied by $K \subset D \subset X$ in which

$$\cap \{F(\varrho) \cap D : \varrho \in D\} \subset L.$$

Thus, the set $\cap \{F(\varrho) : \varrho \in X\} \neq \emptyset$.

Definition 2.43. (Das 2006) Let K be a subset of T.V.S X. A mapping $\mathrm{T} : \mathrm{K} \to 2^{\mathrm{X}}$ is named KKM mapping. In the case that $\{\varrho_1, \varrho_2, \cdots, \varrho_n\} \subset K$ is a nonempty finite subset, then we have $co\{\varrho_1, \varrho_2 \cdots, \varrho_n\} \subset \bigcup_{i=1}^{n} T(\varrho_i)$.

Definition 2.44. (Browder 1968) Let X be a linear space, and T be a mapping, if

$< T(u) - T(v), u - v > \geq 0$, for any $u, v \in X$;

Then, T is said to be monotone operator.

Definition 2.45. (Park *et al.*1998) Consider that X and Y are two real HTLS and the set of all linear continuous functions $L(X, Y)$ from X to Y. Considering P is a closed convex cone of Y. Let $T: X \to L(X, Y)$ and $\eta: K \times K \to K$. Thus

(1) T is said to be η −monotone, if

$< T_\varrho - T_\tau, \eta(\varrho, \tau) > \in P$, $\forall\, \varrho, \tau \in K$

(2) VVLIP: Find $\varrho_0 \in K$ in which for any $z \in K, \lambda \in (0,1]$.

$$< T\big(\lambda_{\varrho_0} + (1-\lambda)z\big), \eta(\tau, \varrho_0) > \notin -intP, for\ all\ \tau \in K \tag{2.2}$$

Special cases of (2.2)

A. if $\lambda = 1$, then problem (2.2) minimizes to the VVLIP provided and investigated by (Siddiqi *et al*. 1997).

Find $\varrho_0 \in K$ in which

$$< T\varrho_0, \eta(\tau, \varrho_0) > \notin -intP, \forall\ \tau \in K$$

B. If $\eta(\tau, \varrho_0) = \tau - \varrho_0$, then problem (2.2) is similar to the below VVLIP given and presented by article in (Yu *et al*. 2001).

Considering $\varrho_0 \in K$ in which for each $z \in K, \gamma \in (0,1]$

$$< T\big(\lambda_{\varrho_0} + (1-\gamma)z\big), \tau - \varrho_0 > \notin -intP, \forall\ \tau \in K.$$

3. NEW RESULTS OF MIXED VECTOR VARIATIONAL-LIKE INEQUALITIES INVOLVING $\beta - \eta$ – MONOTONE OPERATR

3.1 Introduction

A comprehensive investigation into the process of establishing the existence of new types of VVLIP can be found in this chapter. These are inequalities in a variety of disciplines play novel roles in area of science, engineering, and economics and so on. In recent years, monotone operator and it generalization has proved a very valuable tool to solve problems of vector variational-like problem. It is obvious that vector variational-like inequalities extend us a framework to study a many types of linked problems in a united setting for instance (Chen 2000), (Ceng and Huang 2010) and (Li *et al*. 2010). The study in this chapter is classified into the many sections. In section two, we outline explanations preliminary theorems that impose to prove the findings. In section thee, we introduced a new concept of monotone operator. Also, we investigate some findings by using the FKKM–Mapping, under compact and non-compact hypotheses on underlying convex sets. The solutions shows in this work advance and generalized a few famous results mentioned for some authors (see (Hashoosh and Khisbag 2021) and (Jayswal *et al*. 2016).

3.2 Preliminaries

Let's assume that Y and X are two different (HTLS) sets, and that $C \subset Y$ is a convex, closed, pointed cone by its point at the origin such that $\emptyset \neq K \subset X$ is a convex, closed subset of X. Recall that Y is (HTLS) and is named to be an ordered HTLS, which is represented with (Y, C). As shown in Y, ordering relations can be exemplified as follows:

$$\forall\, \mu, v \in Y, \mu \leq v \iff v - \mu \in C;$$

$$\forall\, \mu, v \in Y, \mu \nleq v \iff v - \mu \notin C.$$

if intC is not equal to zero, then the weak ordering relations in Y are what is commonly referred to as

$$\forall\, \mu, \upsilon \in Y, \mu < \upsilon \Leftrightarrow \upsilon - \mu \in \text{intC}$$

$$\forall\, \mu, \upsilon \in Y, \mu \nless \upsilon \Leftrightarrow \upsilon - \mu \notin \text{intC}.$$

Consider that $\Pi(X, Y)$ is the space of all continuous vector maps from X to Y, and that $T: X \to \Pi(X, Y)$, then $(\imath, \mu)$ is used to talk about the value of $\imath \in \Pi(X, Y)$ on $\mu \in X$.

During this work, we'll say that a family of Y cones that are convex, closed, and pointed is denoted by $C(\mu): \mu \in K$ where $\text{intC}(\mu) \neq \emptyset$ for all $\mu \in K$, let $\Theta: K \times K \to X$ and $g: K \times K \to Y$ be two functions. We will look at the two types of linear variational inequalities listed below:

Generalized mixed vector variational-like inequalities $(\text{for brief, GMVVLI})$:

For all $\upsilon \in K$, find $\mu \in K$, in which

$$< T(\mu), \Theta(\upsilon, \mu) > +\mathcal{H}(\upsilon) - \mathcal{H}(\mu) + g(\upsilon, \mu) \notin -\text{intC}(\mu) \qquad (3.1)$$

Generalized strong vector variational-like inequalities (for brief, GSVVLI):

$\forall\, \upsilon \in K$, find $\mu \in K$, where

$$< T(\mu), \Theta(\upsilon, \mu) > +\mathcal{H}(\upsilon) - \mathcal{H}(\mu) + g(\upsilon, \mu) \notin -C(\mu)\backslash\{0\} \qquad (3.2)$$

In addition to solving problem (3.1), we suppose that the following hypotheses have been fulfilled:

$H_1: \dfrac{\beta(t\upsilon + (1-t)\mu, \mu)}{t} = 0;$

H_2: The function $T: K \to \Pi\,(X, Y)$ is an Θ −H.C;

H_3: $C = \bigcap_{\mu \in K} C(\mu) \neq \emptyset$ and T is $\beta -\ \Theta$ −monotone operator in C;

H_4: $\mathcal{H}: K \to Y$ is convex on K and lsc on K such that $K \cap \operatorname{dom} \mathcal{H} \neq \mathcal{H}$, in which $\operatorname{dom} \mathcal{H} = \{\varrho \in \mathbb{R}^n : \mathcal{H}(\varrho) < +\infty\}$ is the effective domain of $\mathcal{H}$.

3.3 Main Results

Below we introduce the equivalence theory which we need to prove our principal outcomes.

Theorem 3.1 Consider that X is (HTLS), and let K be a nonempty, convex, and closed subset of X. Let $(Y, C(u))$ be a OTLS with $\operatorname{int} C(\mu) \neq \emptyset$ for every μ in K. Considering $\Theta: K \times K \to X$, $g: K \times K \to X$ are affine bifunctions, and let $\beta: X \times X \to Y$ be a bifunction that fulfills the conditions $(H_1 - H_4)$. So, the following issues are all the equivalent.

(A) Let $\mu \in K$, $\forall\, \upsilon \in K$ in which

$$\langle T(\mu), \Theta(\upsilon, \mu)\rangle + \mathcal{H}(\upsilon) - \mathcal{H}(\mu) + g(\upsilon, \mu) \notin -\operatorname{int} C(\mu).$$

(B) Let $\mu \in K$, $\forall\, \upsilon \in K$ in which

$$\langle T(\mu), \Theta(\upsilon, \mu)\rangle + \beta(\upsilon, \mu) + \mathcal{H}(\upsilon) - \mathcal{H}(\mu) + g(\upsilon, \mu) \notin -\operatorname{int} C(\mu).$$

Proof Assume that (A) holds, One can deduce $\mu \in K$,

$$\langle T(\mu), \Theta(\upsilon, \mu)\rangle + \mathcal{H}(\upsilon) - \mathcal{H}(\mu) + g(\upsilon, \mu) \notin -\text{int } c(\mu), \forall\, \upsilon \in K$$

By virtue of that T is $\beta - \Theta$ −monotone, one can obtain

$$\langle\, T(\upsilon) - T(\mu), \Theta(\upsilon, \mu)\rangle + \beta(\upsilon, \mu) \in C \quad \forall\, \mu, \upsilon \in K.$$

That is

$$\langle T(\mu), \Theta(\upsilon, \mu)\rangle - \langle T(\upsilon), \Theta(\upsilon, \mu)\rangle - \beta(\upsilon, \mu) \in -C\,, \forall\, \upsilon \in K.$$

Since $C = \bigcap_{\mu \in K} C(\mu)$, then $\forall\, \upsilon \in K$

$$[\langle T(\mu), \Theta(\upsilon, \mu)\rangle + \mathcal{H}(\upsilon) - \mathcal{H}(\mu) + g(\upsilon, \mu)] - [\langle T(\upsilon), \Theta(\upsilon, \mu)\rangle +$$

$$\mathcal{H}(\upsilon) - \mathcal{H}(\mu) + \beta(\upsilon, \mu) + g(\upsilon, \mu)] \in -C \subset -C(\mu)\,.$$

Through the use Remark (2.40) then $\forall\, \upsilon \in K$,

$$\langle T(\upsilon) + \Theta(\upsilon, \mu)\rangle + \mathcal{H}(\upsilon) - \mathcal{H}(\mu) + g(\upsilon, \mu) + \beta(\upsilon, \mu) \notin -\text{int } C(\mu).$$

As a direct consequence of this, μ is the solution to (B).

If, on the other hand, (B) holds, then exists. $\mu_0 \in K$, $\forall\, \upsilon \in K$. Then

$$\langle T(\upsilon), \Theta(\upsilon, \mu_0)\rangle + \mathcal{H}(\upsilon) - \mathcal{H}(\mu_0) + \beta(\upsilon, \mu_0) + g(\upsilon, \mu_0) \notin -\text{int}C(\mu_0),$$

for all $\upsilon \in K$, $t \in (0,1)$, let $\upsilon_t = t\upsilon + (1-t)\mu_o$, it is clear that $\upsilon_t \in K$, then

$$\langle T(\upsilon_t), \Theta(\upsilon_t, \mu_0)\rangle + \mathcal{H}(\upsilon_t) - \mathcal{H}(\mu_0) + \beta(\upsilon_t, \mu_0) + g(\upsilon_t, \mu_0) \notin -\text{int}C(\mu_0).$$

Since g and Θ are affine and H_1, we get

$$\langle T\,(t\upsilon + (1-t)\mu_0), t\Theta(\upsilon, \mu_0)\rangle + \mathcal{H}(t\upsilon + (1-t)\mu_0) - \mathcal{H}(\mu_0)$$

$$+ \beta(t\upsilon + (1-t)\mu_0\,, \mu_0) + tg(\upsilon, \mu_0)$$

$$= t\left[\langle T\,(t\upsilon + (1-t)\mu_0), \Theta(\upsilon, \mu_0)\rangle + \mathcal{H}(\upsilon) - \mathcal{H}(\mu_0) + \frac{\beta(t\upsilon + (1-t)\mu_0\,, \mu_0)}{t} \right.$$

$$\left. + g(\upsilon, \mu_0)\right] \notin -\mathrm{int}C(\mu_0).$$

Regarding the Θ −H.C of T and taking $t \to 0^+$, we get

$$\langle T(\mu_0), \Theta(\upsilon, \mu_0)\rangle + \mathcal{H}(\upsilon) - \mathcal{H}(\mu_0) + g(\upsilon, \mu_0) \notin -\mathrm{int}C(\mu_0), \forall\, \upsilon \in K.$$

This finish the proof.

Remark 3.2 If $\mathcal{H} \equiv 0$, for each μ , $\upsilon \in K$, then Theorem 3.1 is decreases to Theorem 3.1 of (Hashoosh and Khisbag 2021).

Let us consider that K is a convex closed subset of T.V.S X, and that $\{C(\mu) : \mu \in K\}$ is a family of convex closed and pointed cones of a topological space Y, with the condition that $\mathrm{int}C(\mu) \neq \emptyset\ \forall\, \mu \in K$. We will be referring to a set-valued map throughout the completion of this assignment.

$\bar{C}: K \to 2^Y$ as shown: $\bar{C}(\mu) = Y \setminus \{-\mathrm{int}C(\mu)\}$, for each $\mu \in K\}$.

Theorem 3.3 Consider that X is (HTLS) and a nonempty subset K is compact closed and convex of X and $(Y, C(\mu))$ an OTLS by $\mathrm{int}C(\mu) \neq \emptyset$ for any $\mu \in K$. Assume that both $\Theta: K \times K \to X$ and $g: K \times K \to X$ are affine bifunctions in which $\Theta(\mu, \mu) = 0 =$

$g(\mu,\mu)$, for any $\mu \in K$ and a mapping $\beta: X \times X \to Y$ be continuous. In addition the conditions $(H_1 - H_4)$ are hold. If the following condition be satisfied

(A) A set-valued mapping $\bar{C}: K \to 2^Y$ is an upper semi-continuous.

Then, there exists $\mu_0 \in K$, in which

$$< T(\mu_0), \Theta(\upsilon, \mu_0) > +\mathcal{H}(\upsilon) - \mathcal{H}(\mu) + g(\upsilon, \mu_0) \notin -\mathrm{int}C(\mu_0) \quad \forall\, \upsilon \in K.$$

Proof For each $\upsilon \in K$, we define

$$F_1(\upsilon) = \{\mu \in K: < T(\mu), \Theta(\upsilon, \mu) > +\mathcal{H}(\upsilon) - \mathcal{H}(\mu) + g(\upsilon, \mu) \notin -\mathrm{int}\, C(\mu)\};$$

$$F_2(\upsilon) = \{\mu \in K: < T(\upsilon), \Theta(\upsilon, \mu) > +\mathcal{H}(\upsilon) - \mathcal{H}(\mu) + \beta(\upsilon, \mu) + g(\upsilon, \mu) \\ \notin -\mathrm{int}\, C(\mu)\}.$$

If this is the case, then F_1 and F_2 are not empty because $\upsilon \in F_1$ and $\upsilon \in F_2$ resp. Our proof is broken down into three various steps:

Step one. At firstly, we will prove following exemplify: F_1 is a KKM mapping. Let us suppose that F_1 is not a KKM mapping, so there exists $\varrho_1, \varrho_2, \dots, \varrho_n \in K$ and $\lambda_1 \geq 0$, $\lambda_2 \geq 0, \dots, \lambda_m \geq 0$ accompanied by $\sum_{i=1}^{m} \lambda_i = 1$ and $\omega = \sum_{i=1}^{m} \lambda_i \varrho_i$ in which $\omega \notin \bigcup_{i=1}^{m} F_1(\varrho_i)$. This means that, $\forall i = 1,2, \dots, m$

$$< T(\omega), \Theta(\varrho_i, \omega) > +\mathcal{H}(\varrho_i) - \mathcal{H}(\omega) + g(\varrho_i, \omega) \in -\mathrm{int}\, C(\omega),$$

for each $\omega \in K$, since Θ and g affine, and semicontinuity of $\mathcal{H}$, we obtain

$$< T(\omega), \Theta(\omega, \omega) > +\mathcal{H}(\omega) - \mathcal{H}(\omega) + g(\omega, \omega)$$

$$= < T(\omega), \Theta(\sum_{i=1}^{m} \lambda_i \varrho_i, \omega) > + \mathcal{H}\left(\sum_{i=1}^{m} \lambda_i \varrho_i\right) - \mathcal{H}(\omega) + g(\sum_{i=1}^{m} \lambda_i \varrho_i, \omega)$$

$$= \sum_{i=1}^{m} \lambda_i [< T(\omega), \Theta(\varrho_i, \omega) > + \mathcal{H}(\varrho_i) - \mathcal{H}(\omega) + g(\varrho_i, \omega)] \in -\mathrm{int}C(\omega).$$

From other side, since $\Theta(\omega, \omega) = g(\omega, \omega) = 0$. One can have

$$0 =< T(\omega), \Theta(\omega, \omega) > + \mathcal{H}(\omega) - \mathcal{H}(\omega) + g(\omega, \omega) \in -\mathrm{int}C(\omega),$$

Due to the impossibility, $F_1: K \to 2^K$ is a KMM mapping.

Step two, our claim that

$$\bigcap_{\upsilon \in K} F_1(\upsilon) = \bigcap_{\upsilon \in K} F_2(\upsilon).$$

Actually, if $\mu \in F_1(\upsilon)$ then

$< T(\mu), \Theta(\upsilon, \mu) > + \mathcal{H}(\upsilon) - \mathcal{H}(\mu) + g(\upsilon, \mu) \notin -\mathrm{int}C(\mu)$, from H_3

we get

$$< T(\upsilon) - T(\mu), \Theta(\upsilon, \mu) > + \beta(\upsilon, \mu) \in C,$$

Thus,

$$[< T(\mu), \Theta(\upsilon, \mu) > + \mathcal{H}(\upsilon) - \mathcal{H}(\mu) + g(\upsilon, \mu)]$$

$$-[< T(\upsilon), \Theta(\upsilon, \mu) > - \mathcal{H}(\upsilon) + \mathcal{H}(\mu) + \beta(\upsilon, \mu) + g(\upsilon, \mu)] \in -C \subset -C(\mu).$$

This is possible because we are using Remark (2.40).

$$< T(\upsilon), \Theta(\upsilon, \mu) > +\mathcal{H}(\upsilon) - \mathcal{H}(\mu) + \beta(\upsilon, \mu) + g(\upsilon, \mu) \notin -\text{int } C(\mu)$$

Along with, we acquire

$\mu \in F_2(\upsilon)$ for each $\upsilon \in K$. It means that,

$F_1(\upsilon) \subset F_2(\upsilon)$. Therefore,

$$\bigcap_{\upsilon \in K} F_1(\upsilon) \subset \bigcap_{\upsilon \in K} F_2(\upsilon).$$

Conversely, assume that $\mu \in \bigcap_{\upsilon \in K} F_2(\upsilon)$. Then

$$< T(\upsilon), \Theta(\upsilon, \mu) > +\mathcal{H}(\upsilon) - \mathcal{H}(\mu) + \beta(\upsilon, \mu) + g(\upsilon, \mu) \notin -\text{int}C(\mu) \ , \forall\, \upsilon \in K$$

On can just follow of Theorem 3.1 that

$$< T(\mu), \Theta(\upsilon, \mu) > +\mathcal{H}(\upsilon) - \mathcal{H}(\mu) + g(\upsilon, \mu) \notin -\text{int}C(\mu) \ , \forall\, \upsilon \in K$$

Therefore, $\mu \in \bigcap_{\upsilon \in K} F_1(\upsilon)$ and so $\bigcap_{\upsilon \in K} F_2(\upsilon) \subset \bigcap_{\upsilon \in K} F_1(\upsilon)$, which proves that

$$\bigcap_{\upsilon \in K} F_1(\upsilon) = \bigcap_{\upsilon \in K} F_2(\upsilon).$$

Step three, our claim that

$\bigcap_{\upsilon \in K} F_2(\upsilon) \neq \emptyset$, due to the fact that F_1 is KKM mapping we are aware that for each finite set $\{\upsilon_1, \upsilon_2, \dots, \upsilon_n\} \in K$, one must have

$$\mathrm{conv}\{\upsilon_1, \upsilon_2, \dots, \upsilon_n\} \subset \bigcup_{i=1}^{n} F_1(\upsilon_i) \subset \bigcup_{i=1}^{n} F_2(\upsilon_i).$$

In this case, it is clear that F_2 is a KKM map, too.

Now, our claim to prove $F_2(\upsilon)$ is closed for each $\upsilon \in K$.

Suppose that there is a net $\{\mu_\alpha\} \subset F_2(\upsilon)$ by $\mu_\alpha \to \mu \in K$. Therefore,

$$< T(\upsilon), \Theta(\upsilon, \mu_\alpha) > +\mathcal{H}(\upsilon) - \mathcal{H}(\mu_\alpha) + \beta(\upsilon, \mu_\alpha) + g(\upsilon, \mu_\alpha) \notin -\mathrm{int}C(\mu_\alpha).$$

Through the use the definition of $\bar{C}$, one can get

$$< T(\upsilon), \Theta(\upsilon, \mu_\alpha) > +\mathcal{H}(\upsilon) - \mathcal{H}(\mu_\alpha) + \beta(\upsilon, \mu_\alpha) + g(\upsilon, \mu_\alpha) \notin \bar{C}(\mu_\alpha).$$

By the continuum property for both Θ and g, it follows that

$< T(\upsilon), \Theta(\upsilon, \mu_\alpha) > +\mathcal{H}(\upsilon) - \mathcal{H}(\mu_\alpha) + \beta(\upsilon, \mu_\alpha) + g(\upsilon, \mu_\alpha)$ is approach to

$$< T(\upsilon), \Theta(\upsilon, \mu) > +\mathcal{H}(\upsilon) - \mathcal{H}(\mu_\alpha) + \beta(\upsilon, \mu) + g(\upsilon, \mu).$$

Because $\bar{C}$ is a lsc that has values that are close. When we utilize Lemma 2.36, we are able to determine that C is closed; as a result,

$$< T(\upsilon), \Theta(\upsilon, \mu) > +\mathcal{H}(\upsilon) - \mathcal{H}(\mu) + \beta(\upsilon, \mu) + g(\upsilon, \mu) \in \bar{C}(\mu).$$

This proves that

$$< T(\upsilon), \Theta(\upsilon, \mu) > + \mathcal{H}(\upsilon) - \mathcal{H}(\mu) + \beta(\upsilon, \mu) + g(\upsilon, \mu) \notin \mathrm{int}C(\mu),$$

hence $F_2(\upsilon)$ is closed. Due to the compactness of K and the closeness of $F_2(\upsilon) \subset K$. One observe that $F_2(\upsilon)$ is compact. Using Theorem 2.39, it is possible to obtain

$$\bigcap_{\upsilon \in K} F_2(\upsilon) \neq \emptyset.$$

And it follows that

$$\bigcap_{\upsilon \in K} F_1(\upsilon) \neq \emptyset.$$

This indicates that $\exists\ \mu_0 \in K$, where

$$< T(\mu_0), \Theta(\upsilon, \mu_0) > + \mathcal{H}(\upsilon) - \mathcal{H}(\mu_0) + g(\upsilon, \mu_0) \notin -\mathrm{int}C(\mu_0), \forall \upsilon \in K.$$

Thus, (GMVVLI) is solvable. This end the proof.

It is worth mentioning that the several researchers recalled condition (A) in Theorem 3.3 see, e.g. (Ding 1999) and (Krause and Zorin-Kranich 2018).

Remark 3.5 While in Theorem 3.3, K imposed compact set, in the Theorem that follows, with a few proper conditions and without the compactness of K, we will prove that there is a newly discovered existence solution of solutions for (GMVVLI).

Theorem 3.6 Let's say that X is a (HTLS), K is a nonempty, closed, and convex subset of X, and $(Y, C(\mu))$ an OTLS that is ordered by $\mathrm{int}C(\mu) \neq \emptyset$ for each $\mu \in K$. Let $\Theta: K \times K \to X$ and $g: K \times K \to X$ be affine bifunctions with the property that $\Theta(\mu, \mu) = 0 = g(\mu, \mu)\ \forall\ \mu \in K$ and a mapping $\beta: X \times X \to Y$ is continuous that fulfills the conditions $(H_1 - H_3)$ and. If the following assumptions are true, let $T: K \to \Pi(X, Y)$ be a mapping.

(A) A set-valued map $\bar{C}: K \to 2^{Y}$ is an semi-continuous.

(B) There is a non-empty convex and compact subset E of K and for

all $\mu \in K/_{E}$, $\exists\ \upsilon_0 \in E$ such that

$$< T(\upsilon_0), \Theta(\upsilon_0, \mu) > +\mathcal{H}(\upsilon_0) - \mathcal{H}(\mu) + \beta(\upsilon_0, \mu) + g(\upsilon_0, \mu) \in -\mathrm{int}C(\upsilon_0).$$

Then, there is $\mu_0 \in E$ in which

$$< T(\mu_0), \Theta(\upsilon, \mu_0) > +\mathcal{H}(\upsilon) - \mathcal{H}(\mu_0) + g(\upsilon, \mu_0) \notin -\mathrm{int}C(\mu_0), \forall\, \upsilon \in K.$$

Proof We can see, with the help of Theorem 3.1, that the problem that was presented in 3.1 is the equivalent as the one that is presented below:

Find $\mu \in K$, in which

$$< T(\upsilon), \Theta(\upsilon, \mu) > +\mathcal{H}(\upsilon) - \mathcal{H}(\mu) + \beta(\upsilon, \mu) + g(\upsilon, \mu) \notin -\mathrm{int}\, C(\mu), \forall\, \upsilon \in K.$$

Let $H: K \to 2^{E}$ be define as shown $\forall\, \upsilon \in K$.

$$G(\upsilon) = \{\mu \in E: < T(\upsilon), \Theta(\upsilon, \mu) > +\mathcal{H}(\upsilon) - \mathcal{H}(\mu) + \beta(\upsilon, \mu) + g(\upsilon, \mu) \notin -\mathrm{int}C(\mu)\}.$$

Obviously, $\forall\, \upsilon \in K$

$$G(\upsilon) = \{\mu \in K: < T(\upsilon), \Theta(\upsilon, \mu) > +\mathcal{H}(\upsilon) - \mathcal{H}(\mu) + \beta(\upsilon, \mu) + g(\upsilon, \mu) \notin -\mathrm{int}C(\mu)\} \cap E.$$

We can show that G(υ) is a closed subset of E by using the proof of Theorem 3.3. From the compactness of E and closeness of G(υ), then G(υ) is compact. Following, we demonstrate that for any finite set $\{\upsilon_1, \upsilon_2, \dots, \upsilon_n\} \subset K$, one must have

$\bigcap_{i=1}^{n} G(\upsilon_1) \neq \emptyset$.

Assuming that $Y_n = \bigcup_{i=1}^{n}\{\upsilon_i\}$. Because Y is a real HTLS, $\forall\, \upsilon_i \in \{\upsilon_1, \upsilon_2, \dots, \upsilon_n\}$, where $\{\upsilon_i\}$ is convex and compact suppose $S = \mathrm{conv}(E \cup Y_n)$. Using Lemma 2.36, we know that S is a compact and convex subset of K.

Let $F_1, F_2 : S \to 2^S$ be a mappings define as shown for any $\upsilon \in S$:

$F_1(\upsilon) = \{\mu \in K : < T(\mu), \Theta(\upsilon, \mu) > + \mathcal{H}(\upsilon) - \mathcal{H}(\mu) + g(\upsilon, \mu) \notin -\mathrm{int}C(\mu)\}$,

$F_2(\upsilon) = \{\mu \in K : < T(\upsilon), \Theta(\upsilon, \mu) > + \mathcal{H}(\upsilon) - \mathcal{H}(\mu) + \beta(\upsilon, \mu) + g(\upsilon, \mu) \notin -\mathrm{int}C(\mu)\}$.

According to Theorem 3.3, we can conclude that

$$\bigcap_{\upsilon \in S} F_1(\upsilon) = \bigcap_{\upsilon \in S} F_2(\upsilon) \neq \emptyset,$$

and so, there is $\upsilon_0 \in \bigcap_{\upsilon \in S} F_2(\upsilon)$.

Here, we show that $\upsilon_0 \in E$. In fact, if $\upsilon_0 \in {}^{S}/_{E}$, then the assumption proves that $\exists\, \varrho \in E$ in which

$$< T(\upsilon_0), \Theta(\varrho, \upsilon_0) > + \mathcal{H}(\upsilon_0) - \mathcal{H}(\mu) + \beta(\varrho, \upsilon_0) + g(\varrho, \upsilon_0) \in -\mathrm{int}\, C(\varrho).$$

That is contradicts $\upsilon_0 \in F_2(\varrho)$ and so $\upsilon_0 \in E$.

Since $\{\upsilon_1, \upsilon_2, \dots, \upsilon_n\} \subset S$, and $G(\upsilon_i) = F_2(\upsilon_i) \cap E$ for each $\upsilon_i \in \{\upsilon_1, \upsilon_2, \dots, \upsilon_n\}$ it follows that $\upsilon_0 \in \bigcap_{i=1}^{n} H(\upsilon_i)$. As a result, for any finite set $\{\upsilon_1, \upsilon_2, \dots, \upsilon_n\} \subset K$, we have $\bigcap_{i=1}^{n} G(\upsilon_i) \neq \emptyset$. Taking into account the compactness of $H(\upsilon)$ for each $\upsilon \in K$, we know that $\exists\ \mu_0 \in E$ such that

$$\mu_0 \in \bigcap_{\upsilon \in K} G(\upsilon_i) \neq \emptyset.$$

Therefore, the problem (3.1) is solved. This end the proof.

Theorem 3.7. Assume that X is a (HTLS), that $K \subset X$ is a nonempty closed and convex subset, and that $(Y, C(\mu))$ is an OTLS with $\mathrm{int}C(\mu) \neq \emptyset$ for any $\mu \in K$ .Suppose that $\forall\ \upsilon \in K$, $\mu \to \Theta(\mu, \upsilon)$ and $\mu \to g(\mu, \upsilon)$ are affine, $\Theta(\mu, \upsilon) + \Theta(\upsilon, \mu) = 0$, $g(\mu, \upsilon) + g(\upsilon, \mu) = 0$, $\forall\ \mu \in K$. Let $T: K \to \Pi(X, Y)$ shall be a mapping such that

A)$\forall\ \upsilon \in K$, the set $\{\mu \in K: < T(\mu), \Theta(\upsilon, \mu) > + \mathcal{H}(\upsilon) - \mathcal{H}(\mu) + g(\upsilon, \mu) \in -C(\mu)\backslash\{0\}\}$ is an open in K.

B) $\exists$ a non-empty convex and compact subset E of K and $\forall\ \mu \in K/_E\ \exists\ \varrho \in E$, in which

$$< T(\mu), \Theta(\varrho, \mu) > + \mathcal{H}(\varrho) - \mathcal{H}(\mu) + g(\varrho, \mu) \in -C(\mu)\backslash\{0\}.$$

There exists $\mu_0 \in K$, such that

$$< T(\mu_0), \Theta(\upsilon, \mu_0) > + \mathcal{H}(\mu) - \mathcal{H}(\mu_0) + g(\upsilon, \mu_0) \notin -C(\mu_0)\backslash\{0\}, \forall\ \upsilon \in K$$

Proof. Suppose that $H: K \to 2^E$ be defined as shown:

$$H(\upsilon) = \{\mu \in E: < T(\mu), \Theta(\upsilon, \mu) > + \mathcal{H}(\upsilon) - \mathcal{H}(\mu) + g(\upsilon, \mu) \notin -C(\mu)\backslash\{0\}\}, \forall\ \upsilon \in K$$

Obviously, for all $\upsilon \in K$

$$H(\upsilon) = \{\mu \in K: < T(\mu), \Theta(\upsilon, \mu) > + \mathcal{H}(\upsilon) - \mathcal{H}(\mu) + g(\upsilon, \mu) \notin -C(\mu)\backslash\{0\}\} \cap E,$$

$H(\upsilon)$ is compact because it is a closed subset of E, taking into account the compactness of E and the closeness of $H(\upsilon)$. In addition, we show that for any finite set

$\{\upsilon_1, \upsilon_2, \dots, \upsilon_n\} \subset K$, one has

$$\bigcap_{i-1}^{n} H(\upsilon_i) \neq \emptyset.$$

Suppose $Y_n = \bigcup_{i=1}^{n}\{\upsilon_i\}$, since Y is a real HTLS $\forall\, \upsilon_i \in \{\upsilon_1, \upsilon_2, \dots, \upsilon_n\}$, $\{\upsilon_i\}$ is a convex and compact subset of K. Suppose $M: S \to 2^S$ be defined as shown $\forall\, \upsilon \in S$:

$$M(\upsilon): \{\mu \in S: < T(\mu), \Theta(\upsilon, \mu) > + \mathcal{H}(\upsilon) - \mathcal{H}(\mu) + g(\upsilon, \mu) \notin -C(\mu)\backslash\{0\}\}.$$

We suppose that M is a KKM map, in fact, suppose that M is not a KKM map

Thus $\exists\ \varrho_1, \varrho_2, \dots, \varrho_m \in K,\ t_1 \geq 0, t_2 \geq 0, \dots, t_m \geq 0$ by $\sum_{i=1}^{m} t_i = 1$and $\omega = \sum_{i=1}^{m} t_i \varrho_i$,

in which

$\omega \notin \bigcup_{i=1}^{m} M(\varrho_i)$, $i = 1,2, \dots, m$.

Exactly $\forall\ i = 1,2, \dots, m$, we have

$$< T(\omega), \Theta(\varrho_i, \omega) > + \mathcal{H}(\varrho_i) - \mathcal{H}(\omega) + g(\varrho_i, \omega) \in -C(\omega)\backslash\{0\}.$$

Since Θ and g are affine, one can get

$< \mathrm{T}(\omega), \Theta(\omega, \omega) > +\mathcal{H}(\omega) - \mathcal{H}(\omega) + \mathrm{g}(\omega, \omega)$

$$=< \mathrm{T}(\omega), \Theta\left(\sum_{i=1}^{m} t_i \varrho_i , \omega\right) > +\mathcal{H}\left(\sum_{i=1}^{m} t_i \varrho_i\right) - \mathcal{H}(\omega) + \mathrm{g}\left(\sum_{i=1}^{m} t_i \varrho_i , \omega\right)$$

$$= \sum_{i=1}^{m} t_i \left(< \mathrm{T}(\omega), \Theta(\varrho_i, \omega) > +\mathcal{H}(\varrho_i) - \mathcal{H}(\omega) + \mathrm{g}(\varrho_i, \omega)\right) \in -\mathrm{C}(\omega)\backslash\{0\}$$

From other side, since $\Theta(\omega, \omega) = \mathrm{g}(\omega, \omega) = 0$. Therefore,

$$0 =< \mathrm{T}(\omega), \Theta(\omega, \omega) > +\mathcal{H}(\omega) - \mathcal{H}(\omega) + \mathrm{g}(\omega, \omega) \in -\mathrm{C}(\omega)\backslash\{0\},$$

which is impossible. Thus, $\mathrm{M}: \mathrm{S} \to 2^{\mathrm{S}}$ is a KKM mapping. Since $\mathrm{M}(\upsilon)$ is a closed subset of S, it following that $\mathrm{M}(\upsilon)$ is compact with Lemma 2.36, we have

$\bigcap_{\upsilon \in \mathrm{S}} \mathrm{M}(\upsilon) \neq \emptyset.$

Thus, $\exists\ \upsilon_0 \in \bigcap_{\upsilon \in \mathrm{S}} \mathrm{M}(\upsilon)$. Then, we show that $\upsilon_0 \in \mathrm{E}$. In fact, if $\upsilon_0 \in {}^{\mathrm{S}}/_{\mathrm{E}}$. Therefore, the condition (B) proves that there is $\varrho \in \mathrm{E}$, such that

$$< \mathrm{T}(\upsilon_0), \Theta(\varrho, \upsilon_0) > +\mathcal{H}(\varrho) - \mathcal{H}(\upsilon_0) + \mathrm{g}(\varrho, \upsilon_0) \in -\mathrm{C}(\upsilon_0)\backslash\{0\}$$

which contradicts $\upsilon_0 \in \mathrm{M}(\varrho)$, and so $\upsilon_0 \in \mathrm{E}$.

Remark 3.8 Based on what was presented above. One can proof the solution of (GSVVLI) exists if we extend the assumptions for Θ , g and β. Taking this into account, if we impose some conditions, it will ensure that the problem (3.2) admits a solution.

4. NOVEL SOLUTIONS OF VARIATIONAL INEQUALITIES IN H-SPACES

4.1 Introduction

Studying the Variational-type inequality of H-spaces in the now days become an important subject for the field of scientific research (Das 2006) and (Behera and Das 2006). Since then, The fundamental ideas, including but not limited to weakly convex sets, convex sets, the fixed point theorem, and KKM theory in B.S are successively replaced with H-weakly convex sets, H-convex sets, and H-KKM mapping in H spaces. Some examples of these fundamental ideas include: convex sets and fixed point theorem. The following outline was used in the development of this chapter: Following the brief introduction and the essential considerations that will be covered in the subsequent parts, in section three, the new definition of invexity is called $(F_{T\Theta}, \beta)$-invex function are introduced, as we present some findings of complementarity problems. In Section four, we establish and investigate several existence solutions for (FVVI). Using the H-KKM theorem and H-compact sets, we demonstrate the existence of our problem in H-spaces.

4.2 Preliminary

In chapter four, let us suppose that X is a T.V.S and (Y, P) be a OTLS equipped by a closed-convex pointed cone P in which $\mathrm{int}P \neq \emptyset$. Considering K is every subset of X and , $\Pi(X, Y)$ be a set of all continuous and linear functional from X to Y. Assume that $\Theta: K \times K \to X$ and $\beta: X \times X \to Y$ be functions, $F(.,.,.)$ be arbitrary trifunction and $T: K \to \Pi(X, Y)$ is an arbitrary function. Here, we consider the equation, there exists $\upsilon_0 \in K$ in which

$$F(\upsilon_0, T(\upsilon_0), \Theta(\upsilon, \upsilon_0)) + \beta(\upsilon, \upsilon_0) \notin -\mathrm{int}C, \forall\, \upsilon \in K \tag{4.1}$$

To illustrate the main findings, we discuss the following definitions of our requirements for them.

Definition 4.1 (Mititelu 2001) Suppose that X be a T.V.S and a nonempty set $K \subset X$, K is called an Θ −invexity if, there is a function $\Theta: K \times K \to X$, in which $\tau + t\Theta(\varrho, \tau) \in K$, for all $\varrho, \tau \in K$ and for all $t \in (0,1)$.

Definition 4.2 (Behera and Das 2006) (condition C_0) Assume that X is T.V.S and that $K \subset X$ is not empty. If the next conditions are met, a function $\Theta : K \times K \to X$ is named to meet condition C_0:

$$\Theta(\tau + \Theta(\varrho, \tau), \tau) + \Theta(\tau, \tau + \Theta(\varrho, \tau)) = 0$$

$$\Theta(\tau + t\Theta(\varrho, \tau), \tau) + t\Theta(\varrho, \tau) = 0.$$

Moreover, $\beta: X \times X \to Y$ is satisfy condition C_0, if the following two equations are satisfied:

$$\beta(\tau + \beta(\varrho, \tau), \tau) + \beta\big(\tau, \tau + \beta(\varrho, \tau)\big) = 0;$$

$$\beta(\tau + t\beta(\varrho, \tau), \tau) + t\beta(\varrho, \tau) = 0.$$

for any ϱ, $\tau \in K$ and for any $t \in (0,1)$.

4.3 F-Vector Complementarity Problems

In this section, a novel class of vector complementarity problems in topological vector spaces is described. A new concept of $(F_{T\Theta}, \beta)$-invex in OTLS that is generalized of definition (4.2) is given. The novel kind of vector variational inequality issues (FVVI) is named F- vector variational inequality is studied. Also, new class of vector complementarity problem (FVCP) is called F- vector complementarity problem, Which is known as:

(FVVI): Find $\upsilon_0 \in K$ such that

$$F(\upsilon_0, T(\upsilon_0), \Theta(\omega, \upsilon_0)) + \beta(\omega, \upsilon_0) \notin -\text{int}C, \text{ for each } \omega \in K \quad (4.2)$$

(FVCP): Find $\upsilon_0 \in K$ such that

$$F(\upsilon_0, T(\upsilon_0), \Theta(\omega, \upsilon_0)) + \beta(\omega, \upsilon_0) \in Y - (-\text{int}C \cup \text{int}C), \text{ for each } \omega \in K \quad (4.3)$$

Theorem 4.3: Considering K is nonempty Θ-invex set of T.V.S X, and let that $T: K \to \Pi(X, Y)$ be a continuous map, furthermore let the functions $\Theta: K \times K \to X$ and $\beta: X \times X \to Y$ be continuous. If the subsequent conditions are met:

i. $F(\omega, T(\omega), \Theta(\omega, \omega)) + \beta(\omega, \omega) =_C 0$, for each $\omega \in K$;

ii. $\forall \tau \in K$, the set $B(\tau) = \{\omega \in K: F(\tau, T(\tau), \Theta(\omega, \tau)) + \beta(\omega, \tau) \in -\text{int}C\}$ is an Θ −invex set

iii. Both Θ and β satisfies condition C_0.

Then, the issue (4.2) admit a solution.

The next lemma is required in order to demonstrate that Theorem 4.3 is true.

Lemma 4.4: Let *K* be non-empty Θ-invex subset of T.V.S X. Let $F: K \times \Pi(X, Y) \times X \to Y$ is continuous trifunction, a map $T: K \to \Pi(X, Y)$ is a continuous, furthermore let $\Theta: K \times K \to X$ and $\beta: X \times X \to Y$ be continuous functions. If

i. $F(\upsilon, T(\upsilon), \Theta(\upsilon, \upsilon)) + \beta(\upsilon, \upsilon) \notin -\text{int}C, \forall \upsilon \in K$;

ii. $S(\varrho) = \{\upsilon \in K: F(\varrho, T(\varrho), \Theta(\upsilon, \varrho)) + \beta(\upsilon, \varrho) \in -\text{int}C\}$ is an $\Theta-$ invex set, $\forall \varrho \in K$;

iii. A map $\varrho \longmapsto F\big(\varrho, T(\varrho), \Theta(u, \varrho)\big) + \beta(u, \varrho)$ of K into Π(X,Y) is H.C;

iv. The set $\big\{\varrho \in K: F\big(\varrho, T(\varrho), \Theta(\upsilon, \varrho)\big) + \beta(\upsilon, \varrho) \notin -\text{int}C\big\}$ is considered compact if and only if there is at least one $\upsilon \in K$.

Thus, there is $\upsilon_0 \in K$ in which

$$F(\upsilon_0, T(\upsilon_0), \Theta(\upsilon, \upsilon_0)) + \beta(\upsilon, \upsilon_0) \notin -\text{int}C, \forall \upsilon \in K.$$

Proof: Following is the definition of the set-valued mapping $\gamma: K \to 2^P$, in which

$$\gamma(\upsilon) = \big\{\varrho \in K: F\big(\varrho, T(\varrho), \Theta(\upsilon, \varrho)\big) + \beta(\upsilon, \varrho) \notin -\text{int}C\big\}, \text{ for each } \upsilon \in K.$$

Our claim $\gamma(\upsilon)$ is closed $\forall \upsilon \in K$. Let $\{\varrho_n\}$ be a sequence in $\gamma(\upsilon)$ such that it satisfies the condition that $\varrho_n \to \varrho$. This means that we can find the value of the following problem

$$F(\varrho_n, T(\varrho_n), \Theta(\upsilon, \varrho_n)) + \beta(\upsilon, \varrho_n) \notin -\text{int}C, \text{ for each } \upsilon \in K.$$

By using $\varrho_n \in \gamma(\upsilon)$.

This rewards to

$$F(\varrho_n, T(\varrho_n), \Theta(\upsilon, \varrho_n)) + \beta(\upsilon, \varrho_n) \in (Y - (-\text{int}C)), \text{ for each } \upsilon \in K.$$

By continuity of F, T, Θ and β, one can get

$$F(\varrho_n, T(\varrho_n), \Theta(\upsilon, \varrho_n)) + \beta(\upsilon, \varrho_n) \to F(\varrho, T(\varrho), \Theta(\upsilon, \varrho)) + \beta(\upsilon, \varrho).$$

By the closeness of $(Y - (-\mathrm{int}C))$, one can get

$F(\varrho, T(\varrho), \Theta(\upsilon, \varrho)) + \beta(\upsilon, \varrho) \in (Y - (-\mathrm{int}C)), \forall\, \upsilon \in K.$

So,

$F(\varrho, T(\varrho), \Theta(\upsilon, \varrho)) + \beta(\upsilon, \varrho) \notin -\mathrm{int}C, \forall\, \upsilon \in K.$

Thus, $\varrho \in \gamma(\upsilon)$ and $\gamma(\upsilon)$ is closed.

Our claim γ is a KKM mapping, we prove it by contradiction, so there is a finite subset $\{\upsilon_1, \upsilon_2, \dots, \upsilon_n\}$ of K in which

$\mathrm{con}(\{\upsilon_1, \upsilon_2, \dots, \upsilon_n\}) \not\subset \cup \{F(\upsilon) : \upsilon \in \{\upsilon_1, \upsilon_2, \dots, \upsilon_n\}\}.$

It means that $\mathrm{con}(\{\upsilon_1, \upsilon_2, \dots, \upsilon_n\}) \not\subset \gamma(\upsilon)$, for any $\upsilon \in \{\upsilon_1, \upsilon_2, \dots, \upsilon_n\}$. Now, assume that $h \in \mathrm{con}(\{\upsilon_1, \upsilon_2, \dots, \upsilon_n\})$ such that $h \notin \cup \{\gamma(\upsilon) : \upsilon \in \{\upsilon_1, \upsilon_2, \dots, \upsilon_n\}\}$. Then, $h \notin \gamma(\upsilon)$, for any $\upsilon \in \{\upsilon_1, \upsilon_2, \dots, \upsilon_n\}$. Note that $h \in K$. Thus $F(h, T(h), \Theta(\upsilon, h)) + \beta(\upsilon, h) \in -\mathrm{int}P$ for each $\upsilon \in \{\upsilon_1, \upsilon_2, \dots, \upsilon_n\}$. That is

$\upsilon \in \{\upsilon_1, \upsilon_2, \dots, \upsilon_n\} \subset \mathrm{con}(\{\upsilon_1, \upsilon_2, \dots, \upsilon_n\}) \subset B(h).$

Hence, $h \in B(h)$. Then,

$$F(h, T(h), \Theta(h, h)) + \beta(h, h) \in -\mathrm{int}C.$$

Which contradicts first condition. Therefore, γ is a KKM mapping. Thus, according to Theorem 4.3, then

$\cap \{\gamma(\upsilon) : \upsilon \in K\} \neq \emptyset$.

It means that, $\exists\ \upsilon_0 \in K$, such that

$F(\upsilon_0, T(\upsilon_0), \Theta(\upsilon, \upsilon_0)) + \beta(\upsilon, \upsilon_0) \notin -\mathrm{int}C$, for each $\upsilon \in K$.

Then, the equation (FVVI) has solution.

In the following example, the authors show the importance of Lemma 4.4.

Example 4.5. Let $X = \{\omega_i : \omega \in \mathbb{R}\}$, $K = \{\omega_i : \omega \in [0, \infty)\}$ and $Y = \mathbb{R}$, $C = [0, \infty)$. Consider that $\Theta : K \times K \to X$ be defined by $\Theta(\varrho, \tau) = \varrho - \tau$, for each ϱ, $\tau \in K$. Let $\beta : X \times X \to Y$ be defined by $\beta(\varrho, \tau) = \varrho - \tau$, for each ϱ, $\tau \in K$ and $T : K \to \Pi(X, Y)$ defined as $T(\upsilon) = -\upsilon$, for each $\upsilon \in K$ such that $F(\varrho, z, \tau) = z.\tau$ for any $\varrho \in K$ and $\tau \in X$. One can get

(1) for each $\upsilon \in K, F\big(\upsilon, T(\upsilon), \Theta(\upsilon, \upsilon)\big) + \beta(\upsilon, \upsilon) = \upsilon(\upsilon - \upsilon) + (\upsilon - \upsilon) =_c 0$.

(2) Let $c, d \in B(\varrho)$. We show $d + h\Theta(c, d) \in B(\varrho)$.

Firstly, we show that

$hF\big(\varrho, T(\varrho), \Theta(c, \varrho)\big) + h\beta(c, \varrho) + (1 - h)F\big(\varrho, T(\varrho), \Theta(d, \varrho)\big) + (1 - h)\beta(d, \varrho)$

$-(F(\varrho, T(\varrho), \Theta(d + h\Theta(c, d), \varrho)) + \beta(d + h\Theta(c, d), \varrho)) = 0$, for each c, $d \in B(\varrho)$ and for each $h \in (0,1)$, one can obtain

$$h\, F(\varrho, T(\varrho), \Theta(c, \varrho)) + h\beta(c, \varrho) + (1 - h)F(\varrho, T(\varrho), \Theta(d, \varrho)) + (1 - h)\beta(d, \varrho) - F(\varrho, T(\varrho), \Theta(d + h\Theta(c, d), \varrho)) - \beta(d + h\Theta(c, d), \varrho)$$

$= h(-\varrho)(c-\varrho) + h(c-\varrho) + (1-h)((-\varrho)(d-\varrho)) + (1-h)(d-\varrho) - ((-\varrho)(d + h(c-d) - \varrho)) - (d + h(c-d) - \varrho)$.

$$= -h\varrho c + h\varrho^2 + hc - h\varrho - \varrho d + \varrho^2 + h\varrho d - h\varrho^2 + d - \varrho - hd + h\varrho + \varrho d + h\varrho c - h\varrho d - \varrho^2 - d - hc + hd + \varrho = 0.$$

Therefore, for any $\varrho \in K$

$hF\big(\varrho, T(\varrho), \Theta(c,\varrho)\big) + h\beta(c,\varrho) + (1-h)F\big(\varrho, T(\varrho), \Theta(d,\varrho)\big) + (1-h)\beta(d,\varrho) - (F(\varrho, T(\varrho), \Theta(d + h\Theta(c,d),\varrho)) + \beta(d + h\Theta(c,d),\varrho)) = 0 \notin -\mathrm{int}C$, for each $c, d \in B(\varrho)$.

As, $c, d \in B(\varrho), F(\varrho, T(\varrho), \Theta(c,\varrho)) + \beta(c,\varrho) \in -\mathrm{int}C$,

and

$$F\big(\varrho, T(\varrho), \Theta(d,\varrho)\big) + \beta(d,\varrho) \in -\mathrm{int}C.$$

Now, for any $h \in (0,1)$, and for each $c, d \in B(\varrho)$. At that moment, we get

$$F\big(\varrho, T(\varrho), \Theta(d + h\Theta(c,d),\varrho)\big) = hF\big(\varrho, T(\varrho), \Theta(c,\varrho)\big) + (1-h)F\big(\omega, T(\omega), \Theta(d,\varrho)\big) \in -\mathrm{int}C.$$

Hence, $d + h\Theta(c,d) \in B(\varrho)$ for each $c, d \in B(\varrho)$ and for each $h \in (0,1)$.

(3) Obviously, the map $\varrho \longmapsto F(\varrho, T(\varrho), \Theta(\tau,\varrho)) + \beta(\tau,\varrho)$ is continuous.

(4) At $\varrho = \tau$, and for each $\varrho \in K$, we have

$F\big(\varrho, T(\varrho), \Theta(\tau, \varrho)\big) + \beta(\tau, \varrho) = T(\varrho).(\tau - \varrho) + (\tau - \varrho) = 0,$

Illustrating that the following set is compact if, there exists at least $\tau = 0$ in K, the neϱt set is compact

$\{\varrho \in K : F\big(\varrho, T(\varrho), \Theta(\tau, \varrho)\big) + \beta(\tau, \varrho) \notin -\mathrm{int}C\}.$

Now, all the assumptions of the Lemma 4.4 are fulfilled. So, $\exists\ \tau_0 = 0$

solving the problem

$F(\tau_0, T(\tau_0), \Theta(\tau, \tau_0)) + \beta(\tau, \tau_0) \notin -\mathrm{int}C$,for each $\tau \in K$.

Proof Theorem 4.3, if we take into consideration the Lemma 4.4, $\exists\ \upsilon_0 \in K$ such that

$$F\big(\upsilon_0, T(\upsilon_0), \Theta(\omega, \upsilon_0)\big) + \beta(\omega, \upsilon_0) \notin -\mathrm{int}C.$$

This rewards

$F\big(\upsilon_0, T(\upsilon_0), \Theta(\omega, \upsilon_0)\big) + \beta(\omega, \upsilon_0) \geq_c 0$,for each $\omega \in K$.

Because K is a $\Theta -$ inveϱ cone, we get that $\upsilon_0 + t\Theta(\omega, \upsilon_0) \in K$ for every $\omega \in K$.

Replacing ω by $\upsilon_0 + t\Theta(\omega, \upsilon_0)$. One can get

$F\big(\upsilon_0, T(\upsilon_0), \Theta(\upsilon_0 + t\Theta(\omega, \upsilon_0), \upsilon_0)\big) + \beta(\upsilon_0 + t\beta(\omega, \upsilon_0), \upsilon_0) \geq_C 0$, for each $\omega \in K$.

Thus,

$$0 \leq_c F\big(\upsilon_0, T(\upsilon_0), \Theta(\upsilon_0 + t\Theta(\omega, \upsilon_0), \upsilon_0)\big) + \beta(\upsilon_0 + t\beta(\omega, \upsilon_0), \upsilon_0)$$

$$= F(\upsilon_0, T(\upsilon_0), -t\Theta(\omega, \upsilon_0)) + (-t\beta(\omega, \upsilon_0))$$

$$= -tF(\upsilon_0, T(\upsilon_0), \Theta(\omega, \upsilon_0)) - t(\beta(\omega, \upsilon_0))$$

$$= -t(F(\upsilon_0, T(\upsilon_0), \Theta(\omega, \upsilon_0)) + \beta(\omega, \upsilon_0)).$$

Hence, $-t\left(F(\upsilon_0, T(\upsilon_0), \Theta(\omega, \upsilon_0)) + \beta(\omega, \upsilon_0)\right) \notin -\mathrm{int}C$, for each $\omega \in K$ since $t > 0$

we get

$$F(\upsilon_0, T(\upsilon_0), \Theta(\omega, \upsilon_0)) + \beta(\omega, \upsilon_0) \notin \mathrm{int}C.$$

So,

$F(\upsilon_0, T(\upsilon_0), \Theta(\omega, \upsilon_0) > +\beta(\omega, \upsilon_0) \notin (-\mathrm{int}C \cup \mathrm{int}C)$, for each $\omega \in K$

Showing, for each $\omega \in K$

$$F(\upsilon_0, T(\upsilon_0), \Theta(\omega, \upsilon_0) > +\beta(\omega, \upsilon_0) \in Y - (-\mathrm{int}C \cup \mathrm{int}C),$$

Then, υ_0 is a solution of the problem (4.2).

4.4 New Results Of Variational-like inequality In H-spaces.

This section deals with New Results of Variational-like inequality in H-spaces. By employing the ideas of H-convexity, H-compactness and H-KKM maps, we are able to show some existence findings for (FVV I). It is noteworthy, the findings of this section shows an important and refinement and advancement of outcomes that have already been reported (Behera and Das 2006), (Das 2006), and (Behera and Das 2006). (Bonet et al. 2020).The $T - \Theta$ convex function is defined and developed as follows

Definition 4.6. Consider that $g: K \to Y$ be a $F(T,\Theta)$-invex function in K if, it satisfies the condition.

$$g(\upsilon) - g(\omega) - F\big(\omega, T(\omega), \Theta(\upsilon,\omega)\big) \notin -\mathrm{int}C, \text{for each } \upsilon, \omega \in K.$$

If set $F\big(\omega, T(\omega), \Theta(\upsilon,\omega)\big) = \langle T(\omega), \Theta(\upsilon,\omega)\rangle$, then the Definition 4.6 minimizes to the Definition 2.8 in (Behera and Das 2006).

Next, we introduce new definition of invexity as follows

Definition 4.7. A map $g: K \to Y$ is named to be $(F_{T\Theta}, \beta)$-invex function in K if,

$$g(\upsilon) - g(\omega) - F\big(\omega, T(\omega), \Theta(\upsilon,\omega)\big) - \beta(\upsilon,\omega) \notin -\mathrm{int}C, \text{ for each } \upsilon, \omega \in K.$$

Example 4.8. Let $X = \mathbb{R}$, $K = \mathbb{R}_+$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$. Assume $g: K \to Y$ is that defined as $g(\varrho) = \binom{2\varrho^2}{0}$, for each $\varrho \in K$ and $T: K \to \Pi(X, Y)$ is defined as following $T(\varrho) = \binom{-2\varrho}{0}$, for each $\varrho \in K$ where, $F(\varrho, \tau, z) = \tau.z$ for each $\varrho \in K$,$z \in X$. Defined a function $\Theta: K \times K \to X$ by $\Theta(\varrho,\tau) = \varrho + \tau$, for each $\varrho, \tau \in K$ and $\beta: X \times X \to Y$ be defined by $\beta(\varrho,\tau) = \binom{\varrho^2 - \tau^2}{0}$, for each $\varrho, \tau \in X$. Now for each $\varrho, \tau \in K$, we have

$$g(\varrho) - g(\tau) - F\big(\tau, T(\tau), \Theta(\varrho,\tau)\big) - \beta(\varrho,\tau)$$

$$= \begin{pmatrix} 2\varrho^2 \\ 0 \end{pmatrix} - \begin{pmatrix} 2\tau^2 \\ 0 \end{pmatrix} - \begin{pmatrix} -2\tau \\ 0 \end{pmatrix}(\varrho + \tau) - \begin{pmatrix} \varrho^2 - \tau^2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2\varrho^2 - 2\tau^2 \\ 0 \end{pmatrix} + \begin{pmatrix} 2\varrho\tau + 2\tau^2 \\ 0 \end{pmatrix} - \begin{pmatrix} \varrho^2 - \tau^2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2\varrho^2 + 2\varrho\tau \\ 0 \end{pmatrix} - \begin{pmatrix} \varrho^2 - \tau^2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} (\varrho + \tau)^2 \\ 0 \end{pmatrix} \notin -\mathrm{int}C.$$

One the other hand,

$$g(\tau) - g(\varrho) - F\big(\varrho, T\varrho, \Theta(\tau, \varrho)\big) - \beta(\tau, \varrho)$$

$$= \begin{pmatrix} 2\tau^2 \\ 0 \end{pmatrix} - \begin{pmatrix} 2\varrho^2 \\ 0 \end{pmatrix} - \begin{pmatrix} -2\varrho \\ 0 \end{pmatrix} (\tau + \varrho) - \begin{pmatrix} \tau^2 - \varrho^2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2\tau^2 - 2\varrho^2 \\ 0 \end{pmatrix} + \begin{pmatrix} 2\varrho\tau + 2\varrho^2 \\ 0 \end{pmatrix} - \begin{pmatrix} \tau^2 - \varrho^2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2\tau^2 + 2\varrho\tau \\ 0 \end{pmatrix} - \begin{pmatrix} \tau^2 - \varrho^2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} (\varrho + \tau)^2 \\ 0 \end{pmatrix} \notin -\mathrm{int}C.$$

Showing g is $(F_{T\Theta}, \beta)$ $-$invex in K. Using the example from above, one can demonstrate that g is$(F_{T\Theta}, \beta)$ $-$invex in K. While, if one takes take $\varrho < -\tau$, then g cannot be said to be $F(T, \Theta)$-invex.

Proposition 4.9 Consider that $(X, \{\Gamma_A\})$ is an H-space and $K \subset X$. Considering $T: K \to \Pi(X, Y)$ represent a mapping, $\Theta : K \times K \to X$, $\beta : X \times X \to Y$ represent functions $F: K \times \Pi(X, Y) \times X \to Y$ represent continuous trifunction and a function $\varsigma: K \to Y$ be continuous . Assuming that these requirements are fulfilled

(1)$\varsigma(\varrho) \notin -\mathrm{int}C \ \forall \ \varrho \in K$;

(2)$\varsigma: K \to Y$ is $(F_{T\Theta}, \beta)$ −invex in K;

(3) $\varpi_u = \{\varrho \in X: \varsigma(u) - \varsigma(\varrho) \in -\mathrm{int}C\}$ is either H-convex or empty set for each $u \in K$. Thus, $\varsigma(\varrho') - F(\varrho', T(\varrho'), \Theta(\varrho, \varrho')) - \beta(\varrho, \varrho') \notin -\mathrm{int}C, \forall \ \varrho, \varrho' \in K$.

Proof. Consider that $\pi: K \to 2^X$ is a set valued mapping, that is defined as for each $\varrho \in K$

$$\pi(\varrho) = \{\varrho' \in X: \varsigma(\varrho') - F(\varrho', T(\varrho'), \Theta(\varrho, \varrho')) - \beta(\varrho, \varrho') \notin -\mathrm{int}C\}.$$

Notice that $\pi(\varrho)$ is nonempty for any $\varrho \in K$. Our claim to prove that π s an H-KKM map. We prove it by contradiction, then $\exists$ a finite set $A \subset K$ in which $\Gamma_A \not\subseteq \bigcup\{\pi(\varrho): \varrho \in A\}$.

Taking $z \in \Gamma_A$ in which $z \notin \bigcup\{\pi(\varrho): \varrho \in A\}$. Therefore, $z \notin \pi(\varrho)$. It means that

$$\varsigma(z) - F(z, T(z), \Theta(\varrho, z)) - \beta(\varrho, z) \in -\mathrm{int}C, \forall \ \varrho \in A. \quad (4.4)$$

According to $(F_{T\Theta}, \beta)$ −invexity of ς in X at z, one can get

$$\varsigma(\varrho) - \varsigma(z) - F(z, T(z), \Theta(\varrho, z)) - \beta(\varrho, z) \notin -\mathrm{int}C, \forall \ \varrho \in A$$

By (4.4) and $\varsigma(z) \notin -\mathrm{int}C$, so by Remark (2.40) we get

$$\varsigma(\varrho) - F(z, T(z), \Theta(\varrho, z)) - \beta(\varrho, z) \notin -\mathrm{int}C, \ \forall \ \varrho \in A \quad (4.5)$$

Thus, $\varsigma(\varrho) - F(z, T(z), \Theta(\varrho, z)) - \beta(\varrho, z) \in \mathrm{int}C$.

It means that

$$F(z, T(z), \Theta(\varrho, z)) + \beta(\varrho, z) - \varsigma(\varrho) \in -\mathrm{int}C, \forall\, \varrho \in A. \qquad (4.6)$$

The equations (4.4) and (4.6) give us

$\varsigma(z) - \varsigma(\varrho) \in -\mathrm{int}C,\ \forall\, \varrho \in A.$

So, $\varrho \in \varpi_z$, for any $\varrho \in A$.Therefore, given that $A \subset \varpi_z$. Taking into account the H-convexity of ϖ_z then, $\Gamma_A \subset \varpi_z$.

Because $z \in \Gamma_A$,we have $z \in \varpi_z$. So, $\varsigma(z) - \varsigma(z) \in -\mathrm{int}C$. This rewards to $0 \in -\mathrm{int}C$.

This means that we have reached a contradiction, from pointedness condition of C, $0 \in -C \cap C$, we get $0 \notin -\mathrm{int}C \cup \mathrm{int}C$. Therfore, the mapping π is an H-KKM.

Theorem 4.10 Considering $(X, \{\Gamma_A\})$ is an H-space and $K \subset X$. Let $T: K \to \Pi(X, Y)$ be a mapping, $\Theta : K \times K \to X$, $\beta: X \times X \to Y$ be two vector functions, $F: K \times \Pi(X, Y) \times X \to Y$ be continuous trifunction and $\varsigma : K \to Y$ is continuous function if the preceding hypotheses are correct,

(1)$\varsigma(\varrho) \notin -\mathrm{int}C\ \forall\, \varrho \in K$;

(2)$\varsigma: K \to Y$ is $(F_{T\Theta}, \beta) -$ invex in K;

(3)$\forall\ u \in K$, the set $\varpi_u = \{\varrho \in X : \varsigma(u) - \varsigma(\varrho) \in -\mathrm{int}C\}$ either has a H-convex structure or is empty.

(4)The map $\beta(\varrho, .)$of K into a space Y is continuous;

(5)$\exists$ a set $L \subset X$ that is compact in which $H -$compact $C \subset X$ for any weakly H-convex set D by $C \subset D \subset X$, one can get

$$\bigcap\{\varrho \in D: \varsigma(\varrho') - F(\varrho', T(\varrho'), \Theta(\varrho, \varrho')) - \beta(\varrho, \varrho') \notin -\mathrm{int}C\} \subset L.$$

Then, there is $\varrho_0 \in K$ where,

$$\varsigma(\varrho_0) - F\big(\varrho_0, T(\varrho_0), \Theta(\varrho, \varrho_0)\big) - \beta(\varrho, \varrho_0) \notin -\mathrm{int}C, \forall\, \varrho \in K.$$

Proof: As shown in Proposition 4.4's proof, the set valued mapping

$$\pi(\varrho) = \{\varrho' \in X: \varsigma(\varrho') - F(\varrho', (T(\varrho'), \Theta(\varrho, \varrho')) - \beta(\varrho, \varrho') \notin -\mathrm{int}C\}$$

is an $H - KKM$ map, we demonstrate that

$$\cap\,\{\pi(\varrho): \varrho \in X\} \neq \emptyset.$$

We will show that π is closed by using Theorem (2.42).

Consider that $\{z_n\} \subset \pi(\varrho)$ be a sequence in $\pi(\varrho)$ wherever $z_n \to z$ then we demonstrate that $z \in \pi(\varrho)$, by continuity of the map $z \mapsto \varsigma(z) - F(z, T(z), \Theta(\varrho, z)) - \beta(\varrho, z)$, we get

$$\varsigma(z_n) - F\big(z_n, T(z_n), \Theta(\varrho, z_n)\big) - \beta(\varrho, z_n) \to \varsigma(z) - F\big(z, T(z), \Theta(\varrho, z)\big) - \beta(\varrho, z),$$

Considering the fact that

$$\varsigma(z_n) - F(z_n, T(z_n), \Theta(\varrho, z_n)) - \beta(\varrho, z_n) \notin -\mathrm{int}C.$$

Thus,

$$\varsigma(z_n) - F\big(z_n, T(z_n), \Theta(\varrho, z_n)\big) - \beta(\varrho, z_n) \in Y - (-\text{int}C).$$

Since, $Y - (-\text{int}C)$ is closed. Then,

$$\varsigma(z) - F\big(z, T(z), \Theta(\varrho, z)\big) - \beta(\varrho, z) \in Y - (-\text{int}C).$$

Letting

$\varsigma(z) - F(z, T(z), \Theta(\varrho, z)) - \beta(\varrho, z) \notin -\text{int}C.$

Hence, $z \in \pi(\varrho)$, Thus, there is $\varrho_0 \in \bigcap\{\pi(\varrho): \varrho \in X\}$, in which $\forall\ \varrho \in K$

$$\varsigma(\varrho_0) - F\big(\varrho_0, T(\varrho_0), \Theta(\varrho, \varrho_0)\big) - \beta(\varrho, \varrho_0) \notin -\text{int}C.$$

Theorem 4.11 Consider that $(X, \{\Gamma_A\})$ is an $H-$spaces and $K \subset X$, $T: K \to \Pi(X, Y)$ is a map and $\Theta : K \times K \to X$, $\beta : X \times X \to Y$ are two functions, $F: K \times \Pi(X, Y) \times X \to Y$ be continuous trifunction and afunction $\varsigma : K \to Y$ be continuous. If the below hypotheses are correct,

(1)$\varsigma(\varrho) \notin -\text{int}C\ \forall\ \varrho \in K$;

(2)$\varsigma: K \to Y$ is $(F_{T\Theta}, \beta) -$invex in K;

(3)for any $u \in K$ the set $\varpi_u = \{\varrho \in X: \varsigma(u) - \varsigma(\varrho) \in -\text{int}C\}$ is either $H-$convexity or empty;

(4) $\beta(\varrho, .)$ of K into Y is continuous map;

(5) there is a compact set known as $L \subset X$ and an H-compact set known as $C \subset X$, and in both of these sets, for every weakly H-convex set D with $C \subset D \subset X$, we have

$$\cap \{\varrho \in D: \varsigma(\varrho') - F(\varrho', T(\varrho'), \Theta(\varrho, \varrho')) - \beta(\varrho, \varrho') \notin -\mathrm{int}C\} \subset L.$$

(6)$F\big(\varrho', T(\varrho'), \Theta(\varrho, \varrho')\big) + \beta(\varrho, \varrho') \geq_C 0, \forall\, \varrho, \varrho' \in K, \varrho \neq \varrho'$. Then, there is $\varrho_o \in K$, where $\varsigma(\varrho) - \varsigma(\varrho_0) \notin -\mathrm{int}C, \forall\, \varrho \in K$.

Proof. The existence of $\varrho_0 \in K$ is taken into consideration in Theorem (4.10), because ς is $(F_{T\Theta}, \beta)$ $-$invex in K. So

$$\varsigma(\varrho) - \varsigma(\varrho_0) - F\big(\varrho_0, T(\varrho_0), \Theta(\varrho, \varrho_o)\big) - \beta(\varrho, \varrho_0) \notin -\mathrm{int}C.$$

It denotes that, $\forall\, \varrho \in K$

$$\varsigma(\varrho) - \varsigma(\varrho_0) - F\big(\varrho_0, T(\varrho_0), \Theta(\varrho, \varrho_0)\big) - \beta(\varrho, \varrho_0) \geq_C 0, \tag{4.7}$$

Putting $\varrho' = \varrho_0$ in condition (6), we have

$$F(\varrho_0, T(\varrho_0), \Theta(\varrho, \varrho_0)) + \beta(\varrho, \varrho_0) \geq_C 0. \tag{4.8}$$

From (4.7) and (4.8), we get

$\varsigma(\varrho) - \varsigma(\varrho_0) \geq_C 0,$

So, there exists $\varrho_0 \in K$ in which

$$\varsigma(\varrho) - \varsigma(\varrho_0) \notin -\mathrm{int}C, \forall\, \varrho \in K.$$

Corollary 4.12 Assume $(X, \{\Gamma_a\})$ is $H-$space and $K \subset X$. Considering $T: K \to \Pi(X, Y)$ is a mapping, $\Theta : K \times K \to X$, $\beta : X \times X \to Y$ be two functions, $F: K \times \Pi(X, Y) \times X \to Y$ be acontinuous trifunction and $\varsigma : K \to Y$be continuous function. Under the following conditions:

(1)$\varsigma(\varrho) \notin -\mathrm{int}C \ \forall \ \varrho \in K$;

(2)$\varsigma: K \to Y$ is $(F_{T\Theta}, \beta) -$ invex in K;

(3)For each $u \in K$ the set $\varpi_u = \{\varrho \in X: \varsigma(u) - \varsigma(\varrho) \in -\mathrm{int}C\}$ is either H $-$convex or empty;

(4) $\beta(\varrho, .)$ of K into Y is continuous map;

(5) $\exists$ $L \subset X$ is a compact set and an H $-$compact $P \subset X$ such that for each weakly

H-convex set D accompanied by $P \subset D \subset X$,we get

$$\bigcap\{\varrho \in D: \varsigma(\varrho') - F(\varrho', T(\varrho'), \Theta(\varrho, \varrho')) - \beta(\varrho, \varrho') \notin -\mathrm{int}C\} \subset L.$$

(6)$F(\varrho', T(\varrho'), \Theta(\varrho, \varrho')) + \beta(\varrho, \varrho') \geq_C 0, \forall \ \varrho, \varrho' \in K, \varrho \neq \varrho'$

(7)$F\big(\varrho', T(\varrho'), \Theta(\varrho, \varrho')\big) - F\big(\varrho', T(\varrho'), \Theta(\varrho, \varrho')\big) \geq_C 0, \forall \ \varrho, \varrho' \in K, \varrho \neq \varrho'$

Then, there is $\varrho_0 \in K$ in which

$$\varsigma(\varrho) - F\big(\varrho, T(\varrho), \Theta(\varrho_0, \varrho)\big) - \beta(\varrho_0, \varrho) \notin -\mathrm{int}C, \forall \ \varrho \in K.$$

Proof. The Theorem (4.10) considers existence of $\varrho_0 \in K$, under the condition (7) at $\varrho' = \varrho_0$, we have $F(\varrho_0, T(\varrho_0), \Theta(\varrho, \varrho_0)) - F(\varrho, T(\varrho), \Theta(\varrho_0, \varrho)) \notin -\mathrm{int}C$. So,

$$F\big(\varrho_0, T(\varrho_0), \Theta(\varrho, \varrho_0)\big) - F\big(\varrho, T(\varrho), \Theta(\varrho_0, \varrho)\big) \geq_C 0, \forall \ \varrho \in K \tag{4.9}$$

since ς is $(F_{T\Theta}, \beta)$ $-$invex in $K, \forall \ \varrho \in K$, then

$$\varsigma(\varrho)-\varsigma(\varrho_0)-F\big(\varrho_0,T(\varrho_0),\Theta(\varrho,\varrho_0)\big)-\beta(\varrho,\varrho_0)\geq_C 0 \tag{4.10}$$

Adding the inequalities, (4.9) and (4.10) we get

$$\varsigma(\varrho)-\varsigma(\varrho_0)-F(\varrho,T(\varrho),\Theta(\varrho_0,\varrho))-\beta(\varrho_0,\varrho)\geq_C 0.$$

It means that

$$\varsigma(\varrho)-F\big(\varrho,T(\varrho),\Theta(\varrho_0,\varrho)\big)-\beta(\varrho_0,\varrho)\geq_C \varsigma(\varrho_0)\geq_C 0,\forall\ \varrho\in K.$$

Therefore, the equation holds,

$$\varsigma(\varrho)-F\big(\varrho,T(\varrho),\Theta(\varrho_0,\varrho)\big)-\beta(\varrho_0,\varrho)\notin -\mathrm{int}C,\forall\ \varrho\in K.$$

Corollary 4.13 Considering K is subset of H-space $(X,\{\Gamma_A\})$. Let $T: K\rightarrow \Pi(X,Y)$ be a mapping, $\Theta: K\times K\rightarrow X$,$\beta: X\times X\rightarrow Y$ be two a vector functions and $\varsigma: K\rightarrow Y$ be continuous function. So if this hypotheses are hold, then

(1)$\varsigma(\varrho)\notin -\mathrm{int}C\ \forall\ \varrho\in K$;

(2)$\varsigma: K\rightarrow Y$ is $(F_{T\Theta},\beta)-$ invex in K;

(3) For any $u\in K$ the set $\varpi_u=\{\varrho\in X: \varsigma(u)-\varsigma(\varrho)\in -\mathrm{int}C\}$ is either H-convex or empty;

(4) $\beta(\varrho,.)$ of K into Y is continuous in second argument;

(5) For any weakly H-convex set D with PDX, there is compact set $L\subset X$ and an H-compact set $P\subset D\subset X$, we get

$$\cap\{\varrho \in D: \varsigma(\varrho') - F(\varrho', T(\varrho'), \Theta(\varrho, \varrho')) - \beta(\varrho, \varrho') \notin -\mathrm{int}C\} \subset L.$$

$$(6) F\big(\varrho', T(\varrho'), \Theta(\varrho, \varrho')\big) + \beta(\varrho, \varrho') \geq_c 0, \forall\, \varrho, \varrho' \in K, \varrho \neq \varrho'.$$

Then, there is $\varrho_0 \in K$ in which

$$(a) F\big(\varrho_0, T(\varrho_0), \Theta(\varrho, \varrho_0)\big) + \beta(\varrho, \varrho_0) - F\big(\varrho, T(\varrho), \Theta(\varrho_0, \varrho)\big) - \beta(\varrho_0, \varrho) \notin -\mathrm{int}C, \forall\, \varrho \in K$$

$$(b)\{\varsigma(\varrho) - F(\varrho, T(\varrho), \Theta(\varrho_0, \varrho)) - \beta(\varrho_0, \varrho)\} - \{\varsigma(\varrho_0) - F(\varrho_0, T(\varrho_0), \Theta(\varrho, \varrho_0)) - \beta(\varrho, \varrho_0\} \notin -\mathrm{int}C, \forall\, \varrho \in K$$

Proof: The existence of $\varrho_0 \in K$ is shown by the Theorem (4.11), with $\varsigma(\varrho) - \varsigma(\varrho_0) \notin -\mathrm{int}C$. So,

$$\varsigma(\varrho) - \varsigma(\varrho_0) \geq_C 0, \qquad \forall\, \varrho \in K \tag{4.11}$$

(a) Given that ς is $(F_{T\Theta}, \beta)$ $-$invex in K, it follows that

$$\varsigma(\varrho_0) - \varsigma(\varrho) - F(\varrho, T(\varrho), \Theta(\varrho_0, \varrho)) - \beta(\varrho_0, \varrho) \notin -\mathrm{int}C$$

It means that

$$\varsigma(\varrho_0) - \varsigma(\varrho) - F\big(\varrho, T(\varrho), \Theta(\varrho_0, \varrho)\big) - \beta(\varrho_0, \varrho) \geq_C 0, \forall\, \varrho \in K \tag{4.12}$$

From (4.11) and (4.12), one can get

$$-F(\varrho, T(\varrho), \Theta(\varrho_0, \varrho)) - \beta(\varrho_0, \varrho) \geq_C 0.$$

From condition(6), at $\varrho' = \varrho_0$, we have $F(\varrho_0, T(\varrho_0), \Theta(\varrho, \varrho_0)) + \beta(\varrho, \varrho_0) \geq_C 0$.

Hence, $\forall\ \varrho \in K$

$$F\big(\varrho_0, T(\varrho_0), \Theta(\varrho, \varrho_0)\big) + \beta(\varrho, \varrho_0) - F\big(\varrho, T(\varrho), \Theta(\varrho_0, \varrho)\big) - \beta(\varrho_0, \varrho) \notin -\mathrm{int}C, \qquad (4.13)$$

(b) From (4.11) and (4.13), one can have

$$\varsigma(\varrho) - \varsigma(\varrho_0) + F(\varrho_0, T(\varrho_0), \Theta(\varrho, \varrho_0)) + \beta(\varrho, \varrho_0) - F(\varrho, T(\varrho), \Theta(\varrho_0, \varrho)) - \beta(\varrho_0, \varrho) \geq_C 0$$

This means, $\forall\ \varrho \in K$, one can get

$$\left\{ \varsigma(\varrho) - F\big(\varrho, T(\varrho), \Theta(\varrho_0, \varrho)\big) - \beta(\varrho_0, \varrho)\right\} - \left\{\varsigma(\varrho_0) - F\big(\varrho_0, T(\varrho_0), \Theta(\varrho, \varrho_0)\big) - \beta(\varrho, \varrho_0)\right\} \geq_C 0.$$

For that reason,

$$\left\{ \varsigma(\varrho) - F\big(\varrho, T(\varrho), \Theta(\varrho_0, \varrho)\big) - \beta(\varrho_0, \varrho)\right\} - \left\{\varsigma(\varrho_0) - F\big(\varrho_0, T(\varrho_0), \Theta(\varrho, \varrho_0)\big) - \beta(\varrho, \varrho_0)\right\}$$

$\notin -\mathrm{int}C, \forall\ \varrho \in K.$

5. CONCLUSIONS AND FUTURE WORK

5.1 Conclusions

As mentioned in the introduction, the aim of the study is to presented a novel type of VVLIP involving generalized monotone operator in HTLS is introduced. On the other hand, by employing FKKM technique and following reliable hypotheses on the nonlinear mappings, the existence of a result for a new type of vector variational-like problems in involving (HTLS) are presented. Moreover, we presented new definition of invexity is called ($\mathbf{F_{T\Theta}}$,β)-invex function and new idea of variational inequality and complementary problems in OTLS are given. Finally, There are various generalizations made to the conclusions of nonstandard vector-variational inequality in H-spaces. The findings given in this study develop and advance some interesting solutions for researcher.

5.2 Future work

1- Finding the new types of open set for VVLIP and investigate about major operations of these inequalities, in addition to study the generalized implicit V.E problems.

2- Studying well-posedness of a generalized new type of these problem.

3- Using well-known Nadler's theorem, one can applying of vector variational and V.E problems in all areas for social, industrial, pure, physical, regional, and engineering sciences.

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