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By Rahmah Nazar NOAH NOAH

February 2023

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science

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ABSTRACT

SOME RESULTS OF EQUILIBRIUM PROBLEMS INVOLVING PSEUDO-MONOTONE BIFUNCTION

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The hypothetical of Equilibrium Problems (EP) has found its applications in different offshoot of non-linear sciences. The subgradient projection method, which is a contin-uation of the steepest descent projection method in smooth optimization, is a crucial strategy for resolving equilibrium problems. It is common knowledge that when the bi-fun f is convex subdifferentiable due to the 2nd argument and lipschitz, strongly monotone on K, one can select regularization parameters such that this technique linearly converges. The primary goal of this thesis is to determine some outcomes for new class of equilibrium problem (for short EP) involving pseudomonotonicity bi-fun by using KKM technique, whose outcome is sought in a subset K of a Banach space X. This thesis consists of five chapters, chapters two and three contain a set of basic approach along with some theories. The aim of these two chapters is to help the reader to get the gettings we will reach in this thesis. In the fourth chapter, we discuss a novel idea known as pseudo-monotone and present some existence results of solutions for the problem (IEP). In addition to this, we investigate the implications of applying (IEP) to hemivariational inequalities and well-posedness. Among the main outcomes that were introduced in this thises: Assume that $f: K \times K \to \mathbb{R}$ be relaxed α-β-η-pseudo-monotone and let K be a non-empty subset of Banach space X, under some assumptions. Then, the (IEP) is equivalent to the $f\big(\vartheta, \eta(\bar{v}, \vartheta)\big) \leq \alpha\big(\eta(\vartheta, \bar{v})\big) + \beta(\vartheta, \bar{v})$.

2022, 65 pages

Keywords: Equilibrium problems, Solution existence, KKM mapping, Pseudo-monotonicity

ÖZET

PSÖDOMONOTON ÇİFT FONKSİYONU İÇEREN DENGE PROBLEMLERİNİN BAZI SONUÇLARI

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Denge problemleri (EP'ler) varsayımı, lineer olmayan bilimlerin farklı dallarında uygulamalarını bulmuştur. Denge sorununu çözmek için önemli bir yaklaşım, düz optimizasyonda en dik iniş projeksiyonu yönteminin bir uzantısı olarak kabul edilebilecek alt gradyan projeksiyon yöntemidir. İki işlevli f fonksiyonunun, ikinci bağımsız değişken nedeniyle alt türevlenebilir bombeli olduğu ve K üzerinde güçlü bir şekilde monoton olan Lipchitz olduğu zaman, bu yöntemin doğrusal olarak yakınsadığı şekilde düzenlileştirme parametrelerinin seçilebileceği iyi bilinmektedir. Bu tezin birincil amacı, eşitlemesi bir Banach alanı X'in bir K alt sınıfında aranan KKM tekniğini kullanarak, psödomonotonisite iki işlevli yeni denge sorunu sınıfı (kısaca EP için) için bazı sonuçları belirlemektir. Bu tez beş bölümden oluşmaktadır, ikinci ve üçüncü bölümler bazı teorilerle birlikte bir dizi temel yaklaşım içermektedir. Bu iki bölümün amacı, okuyucunun bu tezde ulaşacağımız bulguları elde etmesine yardımcı olmaktır. Dördüncü bölümde, α-β-η-psödo-monoton olan yeni bir kavramı ele alıyoruz ve (IEP) için eşitlemelerin bazı mevcudiyet sonuçlarını gösteriyoruz. Ayrıca, eğer (IEP)'nin hemivaryasyonel eşitsizliklere ve iyi pozlamaya uygulandığını inceliyoruz. Bu tezde tanıtılan ana sonuçlar arasında: f: K×K→R'nin α-β-η-psödo-monoton olduğunu varsayıp ve bazı varsayımlar altında K'nin Banach alanı X'in boş olmayan bir alt sınıfı olmasına izin verelim. O zaman (IEP) $f\big(\vartheta, \eta(\bar{\upsilon}, \vartheta)\big) \leq \alpha\big(\eta(\vartheta, \bar{\upsilon})\big) + \beta(\vartheta, \bar{\upsilon})$. ile eşdeğerdir.

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Anahtar Kelimeler: Denge Problemleri, Çözüm varlığı, KKM-Teoremi, Psödomonotonluk

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Rahmah Nazar NOAH NOAH
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CONTENTS

LIST OF SYMBOLS

$=$	Equality.
$\mathbb{R}$	The set of real numbers.
$\in$	Belongs to
$\forall\, n$	For all n
$\leq$	Less than or equal.
$\geq$	Greater than or equal.
2^X	Family of all X-sets that are not empty.
$\lVert \quad \rVert$	Norm.
$\langle .\,,. \rangle$	Inner product.
X^*	The dual of X.
Ø	Empty set.
İnf	Infimum.
Sup	Supremum.
Lim	Limit.
diam(S)	Diameter of a set S.
Conv(x)	Convex hull of a set X.
2nd	second.

LIST OF ABBREVIATIONS

B.S	Banach space
EP	Equilibrium problems.
Fun	Function.
Bi-fun	Bifunction.
PM	Pseudomonotone.
u.s.c	Upper Semi Continuous.
l.s.c	Lower Semi Continuous.
Diag	Diagonally.
TVS	Topological vector space.
IEP	Invex Equilibrium Problems.
HVI	Hemi-Varitional İnequality.
IHEP	Invex Hemi-Equilibrium Problems.
OPIPEC	Optimization Problem With Invex Parametric Equilibrium Constraint.
et al.	And others.
i.e.,	In other words.
Appr. Seq	Approximating sequence.
(w-p)	Well-posed
wpgs	Well-posed in the generalized sense.
Swpgs	Strongly well-posed in a generalized sense

1. INTRODUCTION

One of the most important problems is termed EP, which were introduced by (Blume and Oettli 1994) they presented some proof discussing the underlying cohesiveness of EP and gived some basic approach and gettings. EP offer us a unified, innovative, natural, and overall structure to study a broad class of problems uprising in economics, finance, transportation, optimization, network analysis and elasticity. Also, it contributed to studying and expanding are other sciences like economics, engineering science, and mechanics. For instance, it may mention to chemical processes, physical or mechanical designs, the appropriation of traffic over PC or over open streets. O. In an effort to standardize and generalize the different new findings that have been achieved in the current field of research for the existence of different solutions for EP, (Chadli et al. 2000) aims to do just that. In same year, (Konnov and Schaible 2000) studied a duality of the abstract equilibrium problem, which contain s variational inequalities problems and optimization problems. (Bianchi and Schaible 2004) reviewed existence results for EP found under generalized convexity and generalized monotonicity. Existence results for an arrangement of vector EP under summed up convexity are graphed which have requesting to an arrangement of vector variational inequalities problems. All through the assessment showd that the outcomes can be accomplished without the inflexible suspicions of convexity and monotonicity. (Noor 2005) presented a new class of EP, recognized as IEP in the setting of invexity. This class of EP contains variational inequalities, EP and variational-like inequalities as special cases.

During the last decades, the concept of monotone has shown itself, due to its importance in many spaces. Many mathematicians have been conducted research to study a class of mixed EP which contain s variational inequalities as well as, convex optimization, saddle point-problems, complementarity problems, Nash equilibria problems, and problems of getting a zero of a maximal monotone operator, as special cases, see (Moudafi 2002). On closed and convex (unbounded or bounded) sets in Banach spaces, (Hashoosh et al. 2016) presented various existence results of EP.

Moreover, a solution to a differential problem is applied to exist on the basis of a request.

Recently, (Hadjisavvas 2003), defined an equivalence relation on the set of pseudo-monotone operators. (While and Quy 2015) studied the EP with strongly pseudo-monotone bi-fun. Liu and Zeng (2016) introduced IEP. Schauder's fixed point theorem will produce an existence result for a solution at the time when the constraint set is convex and compact. They focused on the concept of relaxed $\boldsymbol{\alpha} - \boldsymbol{\eta} -$ pseudo-monotone mappings and proved some existence results of solutions for the constraint set in the event that the constraint set is closed and invex (IEP). In few past years, a substantial number of papers on hemivariational inequalities (for short HVI), see (Panagiotopoulos 1983), (Naniewicz and Panagiotopoulos 1994), (Repovs and Varga 2011), (Hashoosh 2017). (Costea and Radulescu 2009) introduced some existence results for hemivariational inequalities with relaxed η-α monotone mapping on bounded, closed and convex subset in reflexive Banach space.

There are in the writing an exceptionally huge number of papers managing the well-posedness (w-p) idea that was first presented by (Tikhonov 1966), and that implies the existence of a minimizer and the convergence of a subsequenc of each approximating sequence to a solution. Consequently, a few methodology of (w-p) have been read up and introduced for optimization problems. For additional subtleties, see (Zolezzi 1995), (Dontchev and Zolezzi 2006), (Crespi et al. 2007), Fang and (Hu 2007). In addition, (Fang et al.2008) explored the methodology of (w-p) to EP and to optimization problems with equilibrium constraints. They decide a few metric portrayals of (w-p) for EP and for optimization problems with equilibrium constraints.

The aim of this thesis is to give and advance results on the EP that contain pseudo-monotone bi-fun. We will show the existence of a result for α-β-η- pseudomonotone bi-fun by using the KKM mapping.

We now present a brief of the thesis. It consists of five chapters. The first chapter is a general introduction of EP. Chapter two recall a set of basic definitions with some theorems, the aim of this chapter is to support the reader understand the gettings we will reach in this thesis. This chapter divided into six parts, starting with preliminary definitions of normed space, in the 2nd part remember the basic approach of the convex analysis tools, in the third part we refer to the basic approach of EP, while the fourth part presents the initial definitions of the monotone bi-fun types, the fifth part of the chapter discusses some approach from hemivariational inequalities. In the last part, we refer to the basic concept of the well-posed. Chapter three contains a series of previous outcomes for a group of researchers who studied EP which are essential for the presentation of the outcomes in subsequent chapters. Chapter four contains three parts, after the introduction, in the 2nd part we try to reach a new concept of pseudo-monotone which is α-β-η- pseudomonotone with mentioning the necessary definitions along with some results in the literature. In the third part we refer to some applications about our study which are hemivariational inequalities (for short (HVI)) and well-posed of (IEP) with metric characterizations and W-P for optimization issues by invex parametric Equilibrium constraints (denoted by (OPIPEC)) and we get some outcomes in literature. Finally, in chapter five we give a short conclusion of our work contain in this thesis. We then state some open problems to give a future research direction for the interested researchers.

2. PRELIMINARY MATERIAL

2.1 Introduction

In this chapter, we refer to many basic definitions as well as basic hypotheses. The aim of providing these approach and hypotheses is to help us understand realize the main outcomes that have been reached in this thesis, which will be presented in the next chapters. This chapter serves as the prerequisite for the study of subsequent chapters and we shall keep on referring back to it as and when required.

In the meantime, we basing our research on some of the convex analysis ideas needed for our next chapters. Moreover, we present some of approach for EP, additionally, we refer the definitions of a monotone bi-fun, pseudomonotone, hemivariational inequalities and well-posed.

2.2 Normed Spaces

Definition 2.1. (Jain et al. 1995) Consider that X is a vector space. A mapping $\|.\|: X \to \mathbb{R}$ is said to be a norm if for all $\upsilon, \vartheta \in X \ \wedge \ \rho \in \mathbb{R}$ we have

i. $\|\upsilon\| \geq 0$;
ii. $\|\upsilon\| = 0 \Leftrightarrow \upsilon = 0$;
iii. $\|\varsigma\upsilon\| = |\varsigma|\|\upsilon\|$;
iv. $\|\upsilon + \vartheta\| \leq \|\upsilon\| + \|\vartheta\|$.

The pair $(X, \|.\|)$ is termed a normed, and that $\|\upsilon\|$ is named the norm of υ.

Example 2.2. (Jain et al. 1995) Each of the linear space $\mathbb{R}$ and $\mathbb{C}$ with defined by $\|\upsilon\| = |\upsilon|$ is a normed space.

Definitions 2.3. (Jain et al. 1995) Let $(X, \|.\|)$ be a normed space.

i. The set $\{\upsilon \in X: \|\upsilon - \upsilon_0\| < r\}$, denoted by $S(\upsilon_0, r)$ is termed the open ball of radius r with center υ_0.

ii. The set $\{\upsilon \in X: \|\upsilon - \upsilon_0\| \leq r\}$, denoted by $S[\upsilon_0, r]$, is termed the closed ball of radius r with center υ_0.

iii. A sequence $\{\upsilon_n\} \subset X$ is let to be convergent if there exists $\upsilon \in X$ in which

$$\lim_{n\to\infty} \|\upsilon_n - \upsilon\| = 0.$$

iv. A sequence $\{\upsilon_n\} \subset X$ is let to be a Cauchy, if for a given, $\mu > 0$, there exists a positive integer N in which

$$\|\upsilon_m - \upsilon_n\| < \mu, \quad \forall m, n \geq N.$$

v. If every Cauchy sequence included in X can be converge to an element in X, then the space X is considered to be complete.

Examples 2.4. (Kreyszig 1991)

i. Space l^p

$$\|\upsilon\| = \left(\sum_{i=1}^{\infty} |\xi_i|\right).^{1/p}$$

ii. Space l^∞

$$\|\upsilon\| = \sup|\xi_i|.$$

iii. Space $c[a, b]$

$$\|\upsilon\| = \max_{\lambda \in i} |\upsilon(\lambda)|.$$

Definition 2.5. (Kreyszig 1991) A normed space in which the set of Cauchy sequences that is equal to the set of convergent sequences is termed a Banach space. We also termed it a complete normed space.

Example 2.6. (Malkowsky and Rakocevic 2019) Considering $X = F$, and the function $\|.\|: X \to \mathbb{R}$ with $\|\upsilon\| = |\upsilon|$ $\forall \upsilon \in F$.

i. Since $|\upsilon| \geq 0 \; \forall \; \upsilon \in F$, then $\|\upsilon\| \geq 0$;
ii. Let $\upsilon \in X$, then $\|\upsilon\| = 0 \Leftrightarrow |\upsilon| = 0 \Leftrightarrow \upsilon = 0$;
iii. Let $\upsilon \in X, \rho \in F$, then $\|\rho\upsilon\| = |\rho|\|\upsilon\|$;
iv. Let $\upsilon, \vartheta \in X$, then $\|\upsilon + \vartheta\| = |\upsilon + \vartheta| \leq |\upsilon| + |\vartheta| = \|\upsilon\| + \|\vartheta\|$.

X is a normed space, since $\mathbb{R}$ or $\mathbb{C}$ is complete space, then F is complete space, so

X is B.S.

Corollary 2.7. (Malkowsky and Rakocevic 2019) The dual X^* of each normed space X is a B.S.

Definition 2.8. (Malkowsky and Rakocevic 2019) Let X and Y be two normed space over the field C and $\Theta: X \to Y$ a linear operator. Θ is let to be bounded on X, if there exists a real number $t > 0$ in which

$$\|\Theta\upsilon\|_Y \leq t\|\upsilon\|_X$$

If Θ is not bounded, then it is let to be undounded.

Theorem 2.9. (Kreyszig 1991) Let K be a subspace of B.S X, then K is Banach if and only if it is closed in X.

Definition 2.10. (Malkowsky and Rakocevic 2019) Let X and Y be two normed space. A Fun $f: X \to Y$ is let to be continuous at a point $\upsilon_0 \in X$, if for each $\mathcal{L} > 0$, there exists a $\varphi > 0$ in which $\upsilon \in X$

$\|\upsilon - \upsilon_0\| < \varphi$ implies $\|f(\upsilon) - f(\upsilon_0)\| < \mathcal{L}$.

Definition 2.11. (Kreyszig 1991) A Fun $\langle .,. \rangle : X \times X \to \mathbb{R}$, then $\langle .,. \rangle$ termed an inner product on X if:

i. $\forall \upsilon \in X, \langle \upsilon, \upsilon \rangle \geq 0, if\ \upsilon \in X\ and\ \langle \upsilon, \upsilon \rangle = 0$ if and only if $\upsilon = 0$,
ii. $\forall\ \upsilon, \vartheta, z \in X, \langle \upsilon + \vartheta, z \rangle = \langle \upsilon, z \rangle + \langle \vartheta, z \rangle$,
iii. $\forall \upsilon, \nu \in X\ , \beta \in K, \langle \beta \upsilon, \nu \rangle = \beta \langle \upsilon, \nu \rangle$,
iv. $\forall \upsilon, \nu \in X, \langle \upsilon, \nu \rangle = \langle \nu, \upsilon \rangle^*$

Where $*$: mean complex conjugation.

Lemma 2.12. (Malkowsky and Rakocevic 2019) Let X be an inner product space and $\upsilon_a, \upsilon_b \in X$. If $\langle \upsilon_a, \vartheta \rangle = \langle \upsilon_b, \vartheta \rangle$ for all $\vartheta \in X$, then $\upsilon_a = \upsilon_b$. In particular, if $\langle \upsilon_a, \vartheta \rangle = 0$ for all $\vartheta \in X$, then $\upsilon_a = 0$.

2.3 Convex Analysis Tools

Definition 2.13. (Grunbaum et al. 1967) A set K of $\mathbb{R}^n$ is let to be convex set if

$\lambda \vartheta + (1 - \lambda)\upsilon \in K$ whenever $\upsilon, \vartheta \in K\ \ and\ 0 < \lambda < 1$.

Definition 2.14. (Kreyszig 1991) Considering X is a B.S. A mapping $\omega: X \to \mathbb{R}$ is termed:

i. (usc) at the point $\upsilon_0 \in X$ if $\omega(\upsilon_0) \geq \lim_n \sup \omega(\upsilon_n)$.

For any net $\{\upsilon_n\}$ of X which converges to υ_0 weakly.

ii. (lsc) at the point $\upsilon_0 \in X$ if $\omega(\upsilon_0) \leq \lim_n \inf \omega(\upsilon_n)$

For any net $\{\upsilon_n\}$ of X which converges to the point υ_0 weakly.

Proposition 2.15. (Borwein and Lewis 2006)

i. The interpart $\bigcap_{i \in I} K_i$ of any collection $\{K_i : i \in I\}$ of convex sets is convex.

ii. The vector sum $K_1 + K_2$ of two convex sets K_1 and K_2 is convex

iii. The set αK is convex for any convex set K and scalar α. Moreover, if K is a convex set and α_1, α_2 are positive scalars $(\alpha_1 + \alpha_2)K = \alpha_1 K + \alpha_2 K$.

iv. The conclusion and the inside of a convex set are convex.

v. The picture and the opposite picture of a convex set under a relative capability are convex.

Definition 2.16. (Kreyszig 1991) Considering K be a convex set of $\mathbb{R}^n$. A Fun $\omega: K \to \mathbb{R}$ is termed convex Fun if

$$\omega(\lambda v + (1-\lambda)\vartheta \leq \lambda\omega(v) + (1-\lambda)\omega(\vartheta), \;\; \forall v, \vartheta \in K, \lambda \in [0,1].$$

Definition 2.17. (Grunbaum et al. 1967) The convex hull of a set X, denoted (conv(X)), is the interpart of all convex sets containing X, if X consiste of a finite number of vectors $v_1, v_2, \dots, v_n$, its convex hull is

$$conv\{v_1, \dots, v_n\} = \left\{ \sum_{i=1}^{n} \alpha_i v_i : \; \alpha_i \geq 0, i = 1, \dots, n, \sum_{i=1}^{n} \alpha_i = 1 \right\}$$

Definition 2.18. (Grunbaum et al. 1967) Considering K be a nonempty convex set and $v_1, \dots, v_n \in K$. Let $\nu \in K$ is a convex combination of the points $v_1, \dots, v_n$ if there exists $\alpha_1, \dots, \alpha_n \in [0,1]$, with $\sum_{i=1}^{n} \alpha_i = 1$, in which $\nu = \sum_{i=1}^{n} \alpha_i v_i$.

Proposition 2.19. (Borwein and Lewis 2006) The convex hull of a compact set is compact.

Definition 2.20. (Browder 1963) A function with real-valued arguments that is defined on a convex subset of X is let to be hemicontinuous, if

$$\lim_{t \to 0^+} \omega(tv + (1-t))\vartheta = \omega(\vartheta) \quad \forall v, \vartheta \in K.$$

Definition 2.21. (Rockafellar 1970) Considering X be a real Banach space with dual X^*. A proper convex Fun. on X is a Fun ω from X to $(-\infty, +\infty]$ not identically $+\infty$, in which

$$\omega\big((1-\lambda)\upsilon + \lambda\vartheta\big) \leq (1-\lambda)\omega(\upsilon) + \lambda\omega(\vartheta).$$

Whenever, $\upsilon, \vartheta \in X$ and $0 < \lambda < 1$. To define the subdifferential of a Fun ω, one uses the mapping $\partial\omega : X \to X$ defined by

$$\partial\omega(\upsilon) =: \{\upsilon^* \in X^* : \omega(\vartheta) \geq \omega(\upsilon) + \langle \vartheta - \upsilon, \upsilon^* \rangle, \forall \vartheta \in X\}.$$

Definition 2.22. (Knaster et al. 1929) Let K be a nonempty convex subset of a Hausdorff TVS E. A mapping $A: K \to 2^E$ (by 2^E is the family of all subset of E) is let to be a KKM. If for every finite subset $\{u_1, u_2, \dots, u_n\}$ of K we have $co\{u_1, u_2, \dots, u_n\} \subset \bigcup_{i=1}^{n} A(u_i)$ where $co\{u_1, u_2, \dots, u_n\}$ denotes the convex hull of $\{u_1, u_2, \dots, u_n\}$.

Lemma 2.23. (Fang and Huang 2003) Let K be a nonempty subset of a Hausdorff TVS E and let $A : K \to 2^E$ be a KKM- mapping. If $A(\upsilon)$ is closed in E for every $\upsilon \in K$ and compact for some $u_0 \in K$, then $\bigcap_{u \in K} A(u) \neq \emptyset$.

Definition 2.24. (Chadli et al. 2002). A bi-fun $\omega : K \times K \to \mathbb{R}$ is let to be vector 0-Diag convex if, for any finite subset $\{u_1, \dots, u_n\}$ of K and $\alpha_i \geq 0$ $(i = 1,2, \dots, n)$ with $x = \sum_{i=1}^{n} \alpha_i u_i$ and $\sum_{i=1}^{n} \alpha_i = 1$ one can have

$$\sum_{i=1}^{n} \alpha_i \omega(u, u_i) \geq 0 .$$

2.4 Equilibrium Problems

By Blum and Oettli (1994) EP it has been displayed that mathematical programming problems can be observed as special understanding of the abstract EP. EP have many of applications, mechanics, finance, traffic analysis, circuit network analysis, game theory, and economics.

In this section, we will discuss one more category of equilibrium issue, which is known as an invex equilibrium problem in an invexity environment. There is evidence to suggest that invex EP include not only EP but also variational-like inequalities and variational inequalities as extreme cases (Browder 1965).

Definition 2.25. (Weir and Mond 1988) Let K subset of B.S X. Then the set K is let to be invex at υ due to $\eta : K \times K \to K$, if for all $\upsilon, \vartheta \in K, t \in [0,1]$,

$$\upsilon + t\eta(\vartheta, \upsilon) \in K.$$

Definition 2. 26. (Weir and Mond 1988) The Fun $\omega: K \to \mathbb{R}$ is let to be pre-invex due to $\eta : K \times K \to K$, if for all $\upsilon, \vartheta \in K, t \in [0,1]$,

$$\omega\big(\upsilon, t\eta(\vartheta, \upsilon)\big) \leq (1 - t)\omega(\upsilon) + t\omega(\vartheta).$$

Definition 2.27. (Hanson 1981) The differentiable Fun $\omega: K \to \mathbb{R}$ is let to be an invex Fun due to $\eta : K \times K \to K$, if for all $\upsilon, \vartheta \in K$,

$$\omega(\vartheta) - \omega(\upsilon) \geq \langle \omega'(\upsilon), \eta(\vartheta, \upsilon) \rangle,$$

Where $\acute{\omega}(\upsilon)$ is the differential of ω at υ.

Definition 2.28. (Noor 2005) A Fun ω is let to be strongly pre-invex Fun on K due to the Fun η with modulus ℓ, if, for all $\upsilon, \vartheta \in K, t \in [0,1]$,

$$\omega\big(\upsilon + t\eta(\upsilon,\vartheta)\big) \leq (1-t)\omega(\upsilon) + t\omega(\vartheta) - t(1-t)\ell\|\eta(\vartheta,\upsilon)\|^2.$$

Clearly, the differentiable strongly pre-invex Fun ω is strongly invex Fun with module constant ℓ, that is,

$$\omega(\vartheta) - \omega(\upsilon) \geq \langle \acute{\omega}(\upsilon), \eta(\vartheta,\upsilon)\rangle + \ell\|\eta(\vartheta,\upsilon)\|^2,$$

For a given continuous bi-fun Fun $\omega : K \times K \to \mathbb{R}$, let the problem of getting $\upsilon \in K$ in which

$$\omega(\upsilon,\vartheta) \geq 0 \quad \forall\, \vartheta \in K \tag{2.1}$$

Which is termed an invex Equilibrium problem. Note that the set K is an invex set in $\mathbb{R}$.

Remark 2.29. (Noor 2005) Considering $\omega(\upsilon,\vartheta) \equiv \langle T\upsilon, \eta(\vartheta,\upsilon)\rangle$, where

$T: \mathbb{R} \to \mathbb{R}$, and $\eta : K \times K \to \mathbb{R} \cup \infty$, then problem (2.1) is reward to getting $\upsilon \in K$ in which

$$\langle T\upsilon, \eta(\vartheta,\upsilon)\rangle \geq 0 \quad \forall\vartheta \in K \tag{2.2}$$

Which is known as the variational-like inequalities problem.

Definition 2.30. (Noor 2005) The Fun $\omega : K \times K \to \mathbb{R}$ is let to be :

i. Pseudomonotone, if

$$\omega(\upsilon,\vartheta) \geq 0 \Longrightarrow -\omega(\vartheta,\upsilon) \geq 0, \quad \forall \upsilon,\vartheta \in K.$$

ii. To some extent, untense if there is a constant $\ell > 0$ in which, then the function is said to be Partially relaxed strongly monotone, if

$$\omega(\upsilon, \vartheta) + \omega(\vartheta, z) \leq \ell \|\eta(z, \upsilon)\|^2, \qquad \forall \upsilon, \vartheta, z \in K.$$

iii. Hemicontinuous, if for all $\upsilon, \vartheta \in K$, $t \in [0,1]$, the mapping $\omega(\upsilon + t\eta(\vartheta, \upsilon), \vartheta)$ is continuous.

Note that for $z = \upsilon$, partially relaxed strongly monotone minimize to

$$\omega(\upsilon, \vartheta) + \omega(\vartheta, \upsilon) \leq 0, \quad \forall \upsilon, \vartheta \in K,$$

Lemma 2.31. (Noor 2005) Assume that Fun $\omega(.,.)$ be pseudo-monotone and hemicontinuous. If the Fun $\omega(.,.)$ is preinvex in the 2nd argument, the issue (2.1) is reward to getting $\upsilon \in K$ in which

$$\omega(\vartheta, \upsilon) \leq 0, \quad \forall \vartheta \in K.$$

Lemma 2.32. (Noor 2005) Assume that T be η-pseudo-monotone and η-hemicontinuous. If Assumption

$\omega(\upsilon, \vartheta) \geq 0 \quad \forall \vartheta \in K$ holds,

then problem $\langle T\upsilon, \eta(\vartheta, \upsilon)\rangle \geq 0 \ \ \forall \vartheta \in K$ is reward to getting $\upsilon \in K$ in which

$$\langle T\vartheta, \eta(\upsilon, \vartheta)\rangle \leq 0\,, \ \ \forall \vartheta \in K.$$

2.5 Some Monotone Types

The theory of monotone operators has a large body of published work. In this article, we will review several examples of the generalized monotone operators that have garnered a lot of interest over the past three decades for their applications in various areas of mathematics.

(Browder 1965) first presented the definition of monotone as the following: Considering X be a real B.S, and K is a closed convex subset of X, X^* its conjugate space, (w, υ) the pairing among w in X^* and υ in X. A mapping T of K into X^* is let to be monotone if

$$\langle T\upsilon - T\vartheta, \upsilon - \vartheta \rangle \geq 0 \quad \forall \upsilon, \vartheta \in K.$$

By (Chadli et al. 2000) the monotoncity was well-defined

$$\omega(\upsilon, \vartheta) + \omega(\vartheta, \upsilon) \leq 0 \quad \forall \upsilon, \vartheta \in K.$$

(Aoyama et al. 2008) deal with multi-valued monotone operators and two variable monotone Funs for EP in B.S s.

Let $\omega: X \times X \rightarrow [-\infty, \infty]$ and K a nonempty subset of X. We say that ω is monotone due to K if

$$\omega(\upsilon, \vartheta) \leq -\omega(\vartheta, \upsilon) \qquad \forall \upsilon, \vartheta \in K.$$

(Hashoosh 2016) introduced a new definition of monotone bi-fun that is,

Let $\omega, \alpha : K \times K \rightarrow \mathbb{R}$, ω is termed α-monotone if

$$\omega(\upsilon, \vartheta) + \omega(\vartheta, \upsilon) + \alpha(\upsilon, \vartheta) \leq 0 \quad \forall \upsilon, \vartheta \in K.$$

Remark 2.33. (Bianchi and Schaible 2004).

ω strongly monotone $\Longrightarrow \omega$ strictly monotone $\Longrightarrow \omega$ is monotone.

Definition 2.34. (Quoc et al. 2008) Let ω is strictly monotone on K if for all distinct $\upsilon, \vartheta \in K$, we have

$$\omega(\upsilon, \vartheta) + \omega(\vartheta, \upsilon) < 0.$$

Definition 2.35. (Quoc et al. 2008) Let ω be strongly monotone on K with constant $\ell > 0$ if for each pair $\upsilon, \vartheta \in K$, we have

$$\omega(\upsilon, \vartheta) + \omega(\vartheta, \upsilon) \leq -\ell\|\upsilon - \vartheta\|^2.$$

Definition 2.36. (Bianchi and Schaible 2004) Let ω be pseudo-monotone (pm) if for all $\upsilon, \vartheta \in K$

$$\omega(\upsilon, \vartheta) \geq 0 \Longrightarrow \omega(\vartheta, \upsilon) \leq 0.$$

Definition 2.37. (Bianchi and Schaible 2004) Let ω be strictly pseudo-monotone (s.pm) if for all $\upsilon, \vartheta \in K, \upsilon \neq \vartheta$,

$$\omega(\upsilon, \vartheta) \geq 0 \Longrightarrow \omega(\vartheta, \upsilon) < 0.$$

Definition 2.38. (Bianchi and Schaible 2004) Let ω is quasimonotone (qm) if for all $\upsilon, \vartheta \in K$

$$\omega(\upsilon, \vartheta) > 0 \Longrightarrow \omega(\vartheta, \upsilon) \leq 0.$$

Definition 2.39. (Vuong and Strodiot 2020) A bi-fun $\omega: K \times K \to \mathbb{R}$ is named to be: strongly pseudo-monotone with modulus $\ell > 0$ on K , if

$$\omega(\upsilon,\vartheta) \geq 0 \Longrightarrow \omega(\vartheta,\upsilon) \leq -\ell\|\upsilon-\vartheta\|^2 \ \forall\, \upsilon,\vartheta \in K$$

Remark 2.40. (Vuong and Strodiot 2020) Every strongly monotone bi-fun is monotone. And every strongly pseudo-monotone is pseudo-monotone gived is an equilibrium bi-fun, that is

$$\omega(\upsilon,\upsilon) = 0 \quad \forall\, \upsilon \in K.$$

2.6 Some Concept Of Hemivariational Inequalities

Hemivariational inequalities arise in engineering sciences, mechanics and economics in connection with nonconvex energy functional which was initially studied by (Panagiotopoulos 1993).

Definition 2.41. (Clarke 1976) A Fun $J : X \to \mathbb{R}$ is let to be locally lipschitz if every point $u \in X$ there exists a neighborhood w such that

$$|J(a) - J(b)| \leq M_u \|a-b\|_X \qquad \forall a, b \in w.$$

For a constant $M_u \geq 0$ which depends on w.

Definition 2.42. (Clarke 1981) Considering $J : X \to \mathbb{R}$ is a locally lipschitz Fun. The generalized derivative of J at $u \in w$ in the direction $v \in E$, designated as $J^o(u,v)$, this concept is explained by

$$J^0(u,v) = \lim_{\substack{a \to u \\ \lambda \downarrow 0}} \sup \frac{J(a+\lambda v) - J(a)}{\lambda}.$$

Definition 2.43. (Clarke 1981) Considering X is a B.S and the Fun $J : X \to \mathbb{R}$ is locally lipschitz. We recall that J is regular (in the sense of Clarke) at $u \in X$ if for every $v \in X$ the one sided directional derivative $J'(u,v)$ exists and $J^0(u,v) = J'(u,v)$. We say that J is regular where J is regular at every point $u \in X$.

Proposition 2.44. (Clarke 1981) Considering $J : X \to \mathbb{R}$ is a Fun on a B.S X, which is locally lipschitz for rank M_u near the point $v \in X$, then

i. the Fun $J^0(u,.)$ is finite-positively-homogeneous-subadditive and satisfies

$$|J^0(u,v)| \leq M_u \|v\|_X.$$

ii. $J^0(u,v)$ is u.s.c as a Fun of (u,v).

Theorem 2.45. (Berger 1977) Considering K be a convex-compact set in a B.S X and a mapping $g : K \to K$ is a continuous. Hence, g has a fixed point in set K.

2.7 Some Concept Of Well-Posed

(w-p) is a significant issue in the concentrate on EP, variational inequalities, fixed point issue and other related problems. Tykhonov (1966) first characterized the classical concep of (w-p) for a minimization issue, which was notable as Tykhonov (w-p) . It depends on two fundamental fixings: the convergence of any approximate sequence to the minimizer and the existence and uniqueness of the minimizer. Because of the cozy relationship with optimization issues, the idea of (w-p) have been introduced to variational inequalities, EP, fixed point issues and saddle point issues. Hence, Lucchetti and patrone (1981) first presented the (w-p) for variational inequalities and showd a few related results through Ekeland's variational guideline. Goeleven and Mentagui (1995) presented one more significant generalization of (w-p) concept is for hemivariational inequalities.

Assuming that X and E are both B.S s, and that K is a nonempty, convex, and closed subset of a B.S. X, and that X^* is the topological dual space of X, where we will represent the norm in X^*. We retermed the following invex equilibrium issue

$$\omega\big(\upsilon,\eta(\vartheta,\upsilon)\big) \geq 0 \ \ \forall\vartheta \in K \tag{2.3}$$

in which $\omega : K \times K \to \mathbb{R}$, $\eta : K \times K \to K$ bi-funs and ω is a invex equilibrium Fun with $\eta(\upsilon,\upsilon) = 0$.

Let $h: T \times K \to \mathbb{R}$ and $\omega: T \times K \times K \to \mathbb{R}$ be two Funs, in which $T \subset X$ is a nonempty set. The optimization issue with invex Parametric Equilibrium constraint (symbolized by OPIPEC) is formulated as:

$$\min(t,b) \ such\ that\ (t,b) \in T \times K\ and\ b \in M(t),$$

where $M(t)$ is the solution set of the parametric invex equilibrium problem (IEP(t)) defined by, $\upsilon \in M(t)$ if and only if

$$\omega\big(t,\upsilon,\eta(\vartheta,\upsilon)\big) \geq 0, \quad \forall\vartheta \in K \tag{2.4}$$

Instead of writing $\{(IEP(t) : t \in T\}$ for the group of invex EP i.e., the parametric issue, we will basically compose (IEP) in the continuation.

To feature the consensus of the issue (IEP) we review beneath a few extraordinary cases:

i. if $\eta(\vartheta,\upsilon) = \vartheta$ then issue (2.4) is minimizes to the parametric Equilibrium (for short , EP(t)), see (Fang al et. 2008). Find $\upsilon \in K$ in which

$\omega(t,\upsilon,\vartheta) \geq 0 \ \ \forall\vartheta \in K$.

ii. if $\eta(\vartheta,\upsilon) = \vartheta\ \ and\ \ \omega(t,\upsilon,\vartheta) = -h(t,\upsilon,\upsilon-\vartheta)\ \ \forall\vartheta \in K$, then issue (2.4) is minimize to the parametric quasivariational inequalities (for short ,QVI (t)), see Zhang al et. (2009).

Definition 2.46. (Kuratowski 1968) Let A, B be nonempty subset of X. The Hausdorff metric $H(.,.)$ among A and B is defined by

$$H(A,B) = \max\{e(A,B), e(B,A)\}$$

where $e(A,B) = sup_{a\in A} d(a,B),\ d(a,B) = inf_{b\in B}\|a-b\|$.

Definition 2.47. (Kuratowski 1968) Let A be a nonempty subset of X. The measure of noncompactness ξ of the set A is defined as

$$\xi(A) = inf\left\{\lambda > 0 : A \subset \bigcup_{j=1}^{n} A_j,\ \ diam\, A_j < \lambda\ , j = 1,2,\dots,n\right\},$$

In which (diam) means the diameter of a set.

3. SOME CONCEPTS OF THEOREMS, COROLLARİES, LEMMAS AND DEFINITIONS.

A number of theorems, corollaries, lemmas, and definitions are included in this chapter. which is going to be used as the topic develops. Additionally, a collection of theorems needed to develop the topic of this thesis are offered in this section.

3.1 EP With Generalized Monotone Bi-funs

Definition 3.1. (Chaldi et al. 2000) Assume X is TVSs by its dual X^* and that K is a nonempty subset of X. In case f is a real bi-function defined on KK, the equilibrium issue is represented by

$$\text{(EP) find } \bar{\vartheta} \in K \text{ in which } f(\upsilon, \bar{\vartheta}) \leq 0, \quad \forall \upsilon \in K. \tag{3.1}$$

We deal with the following equilibrium issue using bi-fun of the generalized monotone type:

$$\text{(EP) find } \bar{\upsilon} \in K \text{ in which } f(\bar{\upsilon}, \vartheta) \geq 0, \ \forall \vartheta \in K. \tag{3.2}$$

Several examples, both old and new, are used to illustrate the concepts we've been exploring in EP.

i. Minimization. Let $\Delta : K \to \mathbb{R}$ and let the minimization issue
(M) find $\bar{\upsilon} \in K$ in which $\Delta(\bar{\upsilon}) \leq \Delta(\vartheta), \qquad \forall \vartheta \in K.$
ii. Saddle point. Let $X \times Y \to \mathbb{R}$ be a real bi-fun, and let the saddle-point issue (SP) find $(\bar{\upsilon}, \bar{\vartheta}) \in X \times Y$ in which

$$f(\bar{\upsilon}, \vartheta) \leq f(\bar{\upsilon}, \bar{\vartheta}) \leq f(\upsilon, \bar{\vartheta}), \qquad \forall (\upsilon, \vartheta) \in X \times Y.$$

iii. Variational inequalities. Let $T: K \to X^*$ be an operator in which the following hold:

(VIP) Find $\bar{\upsilon} \in K$ in which $\langle T(\bar{\upsilon}), \vartheta - \bar{\upsilon} \rangle \geq 0, \quad \forall \vartheta \in K$.

Let $f(\upsilon, \vartheta) = \langle T(\upsilon), \vartheta - \upsilon \rangle$ then, the issues (VIP) and $(EP)'$ are equal.

iv. Complementarity issues. Let $T : K \to X^*$ be an operator. Then

(CP) Find $\bar{\upsilon} \in K$ in which $T(\bar{\upsilon}) \in K$ and $\langle T(\bar{\upsilon}), \bar{\upsilon} \rangle = 0$, where

$$K^* = \{\upsilon^* \in X^* : \langle \upsilon^*, \vartheta \rangle \geq 0, \forall \vartheta \in K\}.$$

Definition 3.2. (Chaldi et al. 2000) Consider that K be a closed convex subset of X. A real bi-fun $f : K \times K \to \mathbb{R}$ is let to be upper hemi-continuous if for all $\upsilon, \vartheta, z \in K$, the Fun $t \in [0,1] \to f(t\vartheta + (1-t)\upsilon, z)$ is upper semi-continuous, where $t \in [0,1]$.

Lemma 3.3. (Chaldi et al. 2000) Consider that X be a TVS, let K be a closed convex nonempty subset of K, and let $f: K \times K \to \mathbb{R}$ be a real bi-fun satisfying $f(\upsilon, \upsilon) \geq 0$ for each $\upsilon \in K$. If f is upper hemicontinuous and convex in the 2-argument, then f is maximal.

Lemma 3.4.(Chaldi et al.2000) (Fan lemma) Assume X is a Hausdorff TVS, and let K be a nonempty closed convex subset. Let there be two real bifunctions φ and τ defined on $K \times K$ with the following properties:

i. For each $\upsilon, \vartheta \in K$, if $\tau(\upsilon, \vartheta) \leq 0$, then $\varphi(\upsilon, \vartheta) \leq 0$.
ii. For each fixed $\upsilon \in X$, the Fun $\varphi(\upsilon, .)$ is l.s.c on every compact subset of K.
iii. For each finite subset A of K, one has

$$\sup_{y \in conv(A)} \min_{x \in A} \tau(\upsilon, \vartheta) \leq 0.$$

iv. The compactness hypothesis. Exists a compact convex subset C of K satisfying either (i) or (ii):

i. For all $\vartheta \in K \setminus C$, there exists $\upsilon \in C$ in which

$$\varphi(\upsilon, \vartheta) > 0$$

ii. There exists $\upsilon_0 \in C$ in which for all $\vartheta \in K \setminus C$,

$$\tau(\upsilon_0, \vartheta) > 0.$$

Then, there exists an equilibrium point $\bar{\vartheta} \in C$ i.e., $\varphi(\upsilon, \bar{\vartheta}) \leq 0$ for each $\upsilon \in K$.

As a result, the set of possible solutions is compact.

3.2 The Tikhonov Regularization For EP

Definition 3.5. (Hung 2011) To define a "regularized issue" for $EP(K, f)$, we first take a "regularization bi-fun," which is a strongly monotone equilibrium bi-fun $g: K \times K \to \mathbb{R}$. Then, we link the well-defined, regularized issue $EP(K, f)$ to the issue

$$EP(K, f_X) \begin{cases} Find\ \upsilon \in K\ such\ that \\ f_X(\upsilon, \vartheta) = f(\upsilon, \vartheta) + \varepsilon g(\upsilon, \vartheta) \geq 0 \quad \forall \vartheta \in K, \end{cases}$$

in which $\varepsilon > 0$ is a regularization parameter, $\upsilon(\varepsilon)$ is a solution of the regularized issue.

In this section, they show that the Tikhonov regularization approach can be extended to equilibrium issues if the function f is monotone on K. This is because monotonicity is a necessary condition for the method $EP(K, f)$. Always, they will make the assumption that the mathematical formalism equilibrium bifunction g is highly monotone on K and meets the criterion.

$$\text{There exists } \delta > 0 : \ |g(\upsilon,\vartheta)| \leq \delta \|\upsilon - \upsilon^g\| \|\vartheta - \upsilon\| \quad \text{for all } \upsilon, \vartheta \in K, \tag{3.3}$$

Where $\upsilon^g \in K$ is given. One example that fulfills the criteria for the bi-fun satisfying condition (3.3) is

$$g(\upsilon,\vartheta) = \langle F(\upsilon) - F(\upsilon^g), \vartheta - \upsilon \rangle.$$

Where f is lipschitz operator on K. the interval bi-fun given as

$$g(\upsilon,\vartheta) = \langle \upsilon - \upsilon^g, \vartheta - \upsilon \rangle.$$

Remark 3.6. (Hung 2011) For a bi-fun, the following presumptions will be used $h: K \times K \to \mathbb{R}$:

(A1): $h(.,\vartheta)$ is u.s.c for each $\vartheta \in K$;

(A2): $h(\upsilon,.)$ is l.s.c function and convex for each $\upsilon \in K$;

(A3): There is a compact subset that is not empty $B \subset \mathbb{R}^n$ and vector $\vartheta_0 \in B \cap K$ in which

$$h(\upsilon,\vartheta_0) < 0, \ \ \forall \upsilon \in K \setminus B.$$

Theorem 3.7. (Hung 2011) Assume that

i. It is assumed (A1) and (A2) that f is monotone on K.
ii. g is a strongly monotonic bi-fun of equilibrium on K with modulus γ satisfying assumption (A1),(A2) and (A3).

Thus, for every $\varepsilon > 0$, the issue $\mathrm{EP}(K, f_\varepsilon)$ has a unique solution $\upsilon(\varepsilon)$ for which the is made up of the following three assertions:

a) $\lim_{\varepsilon \to 0^+} \upsilon(\varepsilon)$ exists.

b) $\lim_{\varepsilon \to 0^+} \sup\|\upsilon(\varepsilon)\| < \infty$.

c) $SEP(K, f)$ is nonempty.

Furthermore, if any of these claims is true, then $\lim_{\varepsilon \to 0^+} \upsilon(\varepsilon)$ is equal to the unique solution υ^* of equilibrium issue $EP(\widetilde{K}, g)$ where $\widetilde{K} = SEP(K, f)$. In addition, if g is the bi-fun of interval, then υ^* is the Euclidean projection of υ^g onto the solution set of $EP(K, f)$.

3.3 EP With Generalized Monotone Mapping

Definition 3.8. (Liu and Zeng 2016) Assume that K is a nonempty subset of X and that X is a real B.S. The bi-fun f is let to be a relaxed η-α pseudo-monotone, if for any $\upsilon, \vartheta \in K$,

$$f\big(\upsilon, \eta(\vartheta, \upsilon)\big) \geq 0 \Longrightarrow f\big(\vartheta, \eta(\upsilon, \vartheta)\big) \geq \alpha(\vartheta - \upsilon) \tag{3.4}$$

in which $\eta: K \times K \to X$ and $\alpha: X \to \mathbb{R}$.

Remark 3.9. (Liu and Zeng 2016) If we define a bi-fun f with

$$f\big(\upsilon, \eta(\vartheta, \upsilon)\big) = \langle A\upsilon, \eta(\vartheta, \upsilon)\rangle.$$

where $A : K \to X^*$. The mapping A, which is a relaxed η-α pseudomonotone at that moment, leads one to believe that f is also a relaxed η-α pseudomonotone. By meaning of the relaxed η-α monotonicity of A, one has

$$\langle A\vartheta - A\upsilon, \eta(\vartheta - \upsilon)\rangle \geq \alpha(\vartheta - \upsilon) \quad \text{for all } \upsilon, \vartheta \in K.$$

Then,

$$\langle A\vartheta, \eta(\vartheta, \upsilon)\rangle \geq \langle A\upsilon, \eta(\vartheta, \upsilon)\rangle + \alpha(\vartheta - \upsilon).$$

We define $f\big(\upsilon, \eta(\vartheta, \upsilon)\big) = \langle A\upsilon, \eta(\vartheta, \upsilon)\rangle \geq 0$, then

$$f\big(\vartheta, \eta(\vartheta, \upsilon)\big) = \langle A\vartheta, \eta(\vartheta, \upsilon)\rangle$$

$$\geq \langle A\upsilon, \eta(\vartheta, \upsilon)\rangle + \alpha(\vartheta - \upsilon)$$

$$\geq \alpha(\vartheta - \upsilon),$$

Which infers that a bi-fun f is a relaxed η-α pseudomonotone.

Definition 3.10 (Liu and Zeng 2016) A Fun. $A: K \rightarrow X^*$ is let to be as follows:

i. Pseudomonotone, if for each $\upsilon, \vartheta \in K$,

$$\langle A\upsilon, \vartheta - \upsilon\rangle \geq 0 \Longrightarrow \langle A\vartheta, \vartheta - \upsilon\rangle \geq 0.$$

ii. α-pseudomonotone, if for each $\upsilon, \vartheta \in K$,

$$\langle A\upsilon, \vartheta - \upsilon\rangle \geq 0 \Longrightarrow \langle A\vartheta, \vartheta - \upsilon\rangle \geq \alpha(\vartheta - \upsilon).$$

iii. Stably α-pseudo-monotone due to the set $X \subset X^*$, if A and $A(.) - \xi$ both are α-pseudo-monotone for every $\xi \in X$.

Remark 3.11 (Liu and Zeng 2016) From the above-mentioned definition, we have the accompanying ramifications (the converse of each and every ramifications isn't accurate). Every Stably α-pseudomonotonicity due to X is α-pesudomonotonicity.

Let $J : L^P(\Omega, \mathbb{R}^K) \rightarrow \mathbb{R}$ be a locally lipschitz capability and $i: X \rightarrow (\Omega, \mathbb{R}^K)$ be a linear compact operator.

Here, let us assume that the following HVIS issues.

Find $\upsilon \in K$ in which

$$\langle A\upsilon - \phi, \vartheta - \upsilon\rangle + J^0(i\upsilon, i\vartheta - i\upsilon) \geq 0, \quad \forall \vartheta \in E. \tag{3.5}$$

And f is denoted by

$$f(\upsilon, \vartheta - \upsilon) = \langle A\upsilon - \phi, \vartheta - \upsilon\rangle + J^0(i\upsilon, i\vartheta - i\upsilon). \tag{3.6}$$

Theorem 3.12 (Liu and Zeng 2016) Consider K to be a nonempty, closed, convex subset of a real B.S X. Assume that $A : K \to X^*$ is hemicontinuous stable and α-pseudo -monotone because of the set $U(J, i)$, and $\alpha : X \to \mathbb{R}$ fulfills.

$$\lim_{t \to 0^+} \frac{\alpha(t\upsilon)}{t} = 0, \quad \forall \upsilon \in K.$$

Then, one can have the accompanying cases:

Claim 1. The issue (3.5) is reward to the issue.

Find $\upsilon \in K$ in which

$$\langle A\vartheta - \phi, \vartheta - \upsilon\rangle + J^0(i\vartheta, i\vartheta - i\upsilon) \geq \alpha(\vartheta - \upsilon), \quad \forall \vartheta \in K.$$

Claim 2. The bi-fun f in (3.6) is a relaxed η-α pseudo-monotone with $\eta(\vartheta, \upsilon) = \vartheta - \upsilon$, and the same a Fun. α of A.

3.4 Existence Results For Some EP Involving α-Monotone Bi-fun

In this section, we will review some of the findings that were established by the researchers (Hashoosh et al. 2016). These findings suggest that some existence results of (EP) can be found on convex and closed sets in B.S by utilizing a new form of monotonous function technique.

Definition 3.13 (Hashoosh et al. 2016) Assume that K be a nonempty subset of a real reflexive B.S. Let $f, \psi, \alpha : K \times K \to \mathbb{R}$ be three real-valued Funs. f is equilibrium issue that is, $f(\upsilon, \upsilon) = 0$.

Let the accompanying overview of the equilibrium issue to be presented (for short, (EP_{ψ})) is to find $\upsilon \in K$ in which

$$f(\upsilon, \vartheta) + \psi(\upsilon, \vartheta) \geq 0, \quad \forall \vartheta \in K. \tag{3.7}$$

Using the following formula, one can determine the special case:

i. If $\psi \equiv 0$. Then the issue (3.7) is minimized to the classical equilibrium issue (EP), which is to find $\upsilon \in K$ such that $f(\upsilon, \vartheta) \geq 0, \ \forall \vartheta \in K$.
ii. If $\psi(\upsilon, \vartheta) = \psi(\vartheta) - \psi(\upsilon) \ \forall \vartheta \in K$ then the issue (3.7) is diminished to the blended equilibrium (MEP).
iii. If $\psi(\upsilon, \vartheta) = \langle A\upsilon, \vartheta - \upsilon \rangle \ \forall \vartheta \in K$ then the issue (3.7) is minimized to the generalized equilibrium issue (for short, GEP).

Definition 3.14 (Hashoosh et al. 2016) A bi-fun $f : K \times K \to \mathbb{R}$ is termed α-monotone if

$$f(\upsilon, \vartheta) + f(\vartheta, \upsilon) + \alpha(\upsilon, \vartheta) \leq 0, \quad \forall \upsilon, \vartheta \in K. \tag{3.8}$$

Definition 3.15 (Hashoosh et al. 2016) Let X is a B.S and $f : E \to \mathbb{R} \cup \{+\infty\}$ a Fun. a $\upsilon^* \in X^*$ is α-subdifferential of f in $\upsilon \in dom\, f = \{\upsilon: f(x) < \infty\}$, if

$$\partial_\alpha f(\upsilon) = \left\{\upsilon^* \in X^* : f(\vartheta) - \frac{\alpha(\vartheta,\upsilon)}{2} \geq f(\upsilon) + \langle \upsilon^*, \vartheta - \upsilon\rangle \ \ (\forall \vartheta \in K)\right\}.$$

Remark 3.16 (Hashoosh et al. 2016) Let $\forall\, t \in [0,1]$:

$$\lim_{t\to 0} \frac{\alpha(\upsilon, \upsilon_t)}{t} = 0,$$

$$\alpha(\upsilon,\vartheta) \leq \lim_{t\to 0} \frac{t-1}{t}[\psi(\upsilon,\upsilon) + \alpha(\upsilon,\upsilon)].$$

Theorem 3.17 (Hashoosh et al. 2016) Assume that $f : K \times K \to \mathbb{R}$ is α-monotone bi-fun, hemicontinuous in the first argument and convex in the 2nd argument. Let $\psi, \alpha : K \times K \to \mathbb{R}$ be convex in the 2nd argument, then generalized equilibrium issue (EP_ψ) is reward to the following issue. Find $\upsilon \in K$ in which

$$f(\vartheta,\upsilon) + \alpha(\upsilon,\vartheta) \leq \psi(\upsilon,\vartheta) \qquad \forall \vartheta \in K.$$

Theorem 3.18 (Hashoosh et al. 2016) Consider that K be a nonempty closed bounded convex subset of a real reflexive B.S X. Consider to be that

i. $f : K \times K \to \mathbb{R}$ is α-monotone bifunction bi-fun, 0- diagonal convex, hemi-continuous in first contention and lsc convex in 2nd contention
ii. $\psi : K \times K \to \mathbb{R}$ is convex in 2nd argument, usc in first argument, and 0-diagonal convex
iii. $\alpha : K \times K \to \mathbb{R}$ is l.s.c in first contention and convex in 2nd contention.

Then, (EP_ψ) has a solution.

Corollary 3.19 (Hashoosh et al. 2016) Considering $0 \in K$ is a nonempty convex subset of a B.S X and assumption (i-iii) in theorem 3.18 hold.

i. $\psi(v, 0) = 0$

ii. $\zeta_1 = \{v \in K : \alpha(v, 0) \leq 0\}$ and $\zeta_2 = \{v \in K : f(0, v) \leq 0\}$ are relative compact sets.

Then, (EP_ψ) has a solution.

4. OUTCOMES OF EQUILIBRIUM PROBLEMS INVOLVING PSEUDO-MONOTONE BIFUNCTION WITH APPLICATIONS

4.1 Introduction

This chapter's objective is to introduce new defintion of monotonicity is called relaxed α-β-η-Pseudo-monotone operator, and we establish some findings of the existence and uniqueness for invex equilibrium issue (IEP).

It is worth noting that we strengthened our results by mentioning two important applications, one about Hemivaritional Inequalities, where we studied a special type of them called invex Hemi-Equilibrium issue and we got some important results and the other about well-posedness (w-p), where derive a helpful characterization of metric projection as well as adequate conditions for these types of w-p.

Finally, we introduce a novel notion of w-p for optimization issues by constraints specified by a parametric form of (IEP).

4.2 Existence Results

In this section, we give a new type of monotonous function is called relaxed α-β-η-Pseudo-monotone operator on a nonempty set K of real reflexive B.S X, and $\eta: K \times K \to K$ be a Fun. Among the results that we obtained in this chapter, we establish some existence results of (IEP), using KKM-mapping.

Moreover, we research the existence of adjustments for the accompanying (IEP):

$$(IEP) \text{ Find } \bar{\upsilon} \in K, \text{in which } f\big(\bar{\upsilon}, \eta(\vartheta, \bar{\upsilon})\big) \geq 0, \forall \vartheta \in K, \tag{4.1}$$

Following is a new definition that we have developed for the pseudo-monotone operator that contributed significantly to the solutions of our main findings.

Definition4.1. A map $f: K \times K \to \mathbb{R}$ is to be termed relaxed α-β-η-Pseudo-monotone if, there is a Fun. $\eta: K \times K \to K$, $\alpha: X \to \mathbb{R}$ and $\beta: K \times K \to \mathbb{R}$ in which for each $\upsilon, \vartheta \in K$, and

$$f\big(\upsilon, \eta(\vartheta, \upsilon)\big) \geq 0 \Longrightarrow f\big(\vartheta, \eta(\upsilon, \vartheta)\big) \leq \alpha\big(\eta(\vartheta, \upsilon)\big) + \beta(\vartheta, \upsilon) \tag{4.2}$$

where

$$\lim_{t \to 0^+} \inf \left[\frac{\alpha(\eta(\upsilon,\vartheta))}{t} + \frac{\beta(\upsilon, t\vartheta + (1-t)\upsilon)}{t} \leq 0\right].$$

Remark 4.2. If $\eta(\vartheta, \upsilon) = \vartheta$ for all $\upsilon, \vartheta \in K$, then (4.1) becomes $f(\upsilon, \vartheta) \geq 0$ and f is let to be Equilibrium issue.

Theorem 4.3. Considering $f: K \times K \to \mathbb{R}$ is a relaxed α-β-η- Pseudo-monotone and let that K be a nonempty subset of B.S X, assume the following:

i. $f(\upsilon_t, \eta(\upsilon_t, \upsilon_t)) = 0 \quad \forall \upsilon \in K$;
ii. $f(.\,, \eta(\vartheta, .))$ is hemicontinuous $\forall \vartheta \in K$;
iii. $f(\vartheta, \eta(.\,, \vartheta))$ is convex $\forall \vartheta \in K$.

Then, the (IEP) is comparable to the accompanying issue :

Find $\bar{\upsilon} \in K$ in which

$$f\big(\vartheta, \eta(\bar{\upsilon}, \vartheta)\big) \leq \alpha\big(\eta(\vartheta, \bar{\upsilon})\big) + \beta(\vartheta, \bar{\upsilon}) \tag{4.3}$$

Proof. Let $\bar{\upsilon} \in K$ is a solution of (4.1), that is

$$f\big(\bar{\upsilon}, \eta(\vartheta, \bar{\upsilon})\big) \geq 0, \forall \vartheta \in K.$$

And, we get from the relaxed α-β-η-Pseudo-monotone of f

$$f\big(\vartheta,\eta(\bar{\upsilon},\vartheta)\big)\le\alpha\big(\eta(\vartheta,\bar{\upsilon})\big)+\beta(\vartheta,\bar{\upsilon}). \tag{4.4}$$

Hence, $\bar{\upsilon}\in K$ is a solve of the issue that defined via (4.3).

On the other hand, suppose that $\bar{\upsilon}\in K$ is a solve of the equation (4.3). After that, with regard to any $\vartheta\in K,$ let us assume that $\upsilon_t=t\vartheta+(1-t)\bar{\upsilon},t\in(0,1]$. So, $\upsilon_t\in K$ and

$$f\big(\upsilon_t,\eta(\bar{\upsilon},\upsilon_t)\big)\le\alpha\big(\eta(\upsilon_t,\bar{\upsilon})\big)+\beta(\upsilon_t,\bar{\upsilon}). \tag{4.5}$$

Since $\eta(.,\vartheta)$ is convex and (iii), we obtain

$$0=f\big(\upsilon_t,\eta(\upsilon_t,\upsilon_t)\big)\le tf\big(\upsilon_t,\eta(\vartheta,\upsilon_t)\big)+(1-t)f\big(\upsilon_t,\eta(\bar{\upsilon},\upsilon_t)\big). \tag{4.6}$$

The equation (4.5) and (4.6) imply that

$$f\big(\upsilon_t,\eta(\bar{\upsilon},\upsilon_t)\big)-f\big(\upsilon_t,\eta(\vartheta,\upsilon_t)\big)\le\frac{\alpha\big(\eta(\upsilon_t,\bar{\upsilon})\big)}{t}+\frac{\beta(\upsilon_t,\bar{\upsilon})}{t}\qquad\forall\vartheta\in K$$

Since, $\eta(\vartheta,.)$ is hemicontinuous and from the definition of Pseudo-monotonicity of f, choosing $t\to0^+$ to have that

$$f\big(\bar{\upsilon},\eta(\bar{\upsilon},\bar{\upsilon})\big)-f\big(\bar{\upsilon},\eta(\vartheta,\bar{\upsilon})\big)\le0.$$

Note that, $f(\bar{\upsilon},\eta(\bar{\upsilon},\bar{\upsilon})=0$, then $\forall\vartheta\in K,$

$$f\big(\bar{\upsilon},\eta(\vartheta,\bar{\upsilon})\big)\ge0$$

At that moment, $\bar{\upsilon}\in K$ is a solve of the issue (4.1).

Theorem 4.4. Considering K is a nonempty closed convex and bounded subset of a real reflexive B.S X. Considering $f: K \times K \rightarrow \mathbb{R}$ is a relaxed α-β-η-Pseudomonotone convex in the 2nd argument by $f(v, v) = 0$, and $\alpha : X \rightarrow \mathbb{R}$ and $\beta: K \times K \rightarrow \mathbb{R}$ is weakly usc in the 2nd argument. So, the issue $f(\bar{v}, \eta(\vartheta, \bar{v}) \geq 0$ has a solution.

Proof. Let $F: K \rightarrow 2^X$ be a multivalued map characterized by:

$$F(\vartheta) = \{v \in K: f(v, \eta(\vartheta, v)) \geq 0\}, \quad \forall \vartheta \in K \tag{4.7}$$

Obviously, $\bar{v} \in K$ is a solution of $f(\bar{v}, \eta(\vartheta, \bar{v})) \geq 0$ iff $\bar{v} \in \bigcap_{\vartheta \in K} F(\vartheta)$. Our claim to show that $\bigcap_{\vartheta \in K} F(\vartheta) \neq \emptyset$.

Presently, to prove F is a KKM map. Running against the norm, F isn't KKM map, then there is finite set $\{v_1, v_2, \dots, v_n\}$ of K in which $co\{v_1, v_2, \dots, v_n\} \not\subseteq \bigcup_{i=1}^{n} F(v_i)$. As a result of which there is $v_0 \in co\{v_1, v_2, \dots, v_n\}$ where for all $i \in \{1, \dots, n\}, v_0 \notin F(v_i)$. So, for $i = 1,2, \dots, n$ one can get

$$f\big(v_0 , \eta(v_i, v_0)\big) < 0. \tag{4.8}$$

As a result of which, there is $\lambda_i \geq 0$ $(i = 1,2, \dots, n)$ by $\sum_{i=1}^{n} \lambda_i = 1$ in which $v_0 = \sum_{i=1}^{n} \lambda_i v_i$

By multiplying both sides of the relationship (4.8) by λ_i and adding the results, we arrive at

$$0 = f(v_0, \eta(v_0, v_0)) = \sum_{i=1}^{n} \lambda_i \big[f\big(v_0, \eta(v_i, v_0)\big)\big] < 0.$$

This led us to a conclusion, which is the contradiction $0 < 0$. As a result of which, the multi-valued mapping F is a KKM.

Currently, Define multivalued mapping. $G: K \to 2^E$ by accompanying

$$G(\vartheta) = \{\upsilon \in K: f(\vartheta, \eta(\upsilon, \vartheta)) \leq \alpha(\eta(\vartheta, \upsilon)) + \beta(\vartheta, \upsilon)\}, \text{ for all } \vartheta \in K. \quad (4.9)$$

We will show that $F(\vartheta) \subset G(\vartheta)\ \forall \vartheta \in K$. Letting $\bar{\upsilon} \in F(\vartheta)$, so

$$f(\bar{\upsilon}, \eta(\vartheta, \bar{\upsilon})) \geq 0. \quad (4.10)$$

A relaxed α-β-η-Pseudo-monotone for f requires that $F(\vartheta) \subseteq G(\vartheta)$ for all $\vartheta \in K$. Hence, the mapping G is a KKM. Since $\alpha(\eta(\vartheta, .))$ and $\beta(\vartheta, .)$ are weakly usc in the 2nd argument, As a result, it makes sense

$$\alpha(\eta(\vartheta, \upsilon)) + \beta(\vartheta, \upsilon) \geq \lim_n \sup \langle \alpha(\eta(\vartheta, \upsilon_n)) + \beta(\vartheta, \upsilon_n) \rangle.$$

Ergo, $G(\vartheta)$ is weakly closed mapping for all $\vartheta \in K$. Yet, K is bounded, closed and convex of reflexive B.S X. So, it is weakly-compact and consequently $G(\vartheta)$ is weakly-compact in K for all $\vartheta \in K$. As a result, it stands to reason (lemma of KKM mapping) then,

$$\bigcap_{\vartheta \in K} F(\vartheta) \neq \emptyset.$$

And

$$\bigcap_{\vartheta \in K} F(\vartheta) = \bigcap_{\vartheta \in K} G(\vartheta).$$

Then,

$$f(\bar{\upsilon}, \eta(\vartheta, \bar{\upsilon})) \geq 0, \quad \forall \vartheta \in K. \quad (4.11)$$

So the issue (4.11) has a solution.

Corollary 4.5. It is assumed here that $0 \in K$ is a nonempty convex subset of B.S. X, and that The assumptions that were stated in Theorem (4.4) are correct. If

i. $\alpha\big(\eta(0,\upsilon)\big) + \beta(0,\upsilon) = 0;$

ii. $\varphi = \{\upsilon \in K : f(0, \eta(\upsilon, 0)) \leq 0\}$ is relative compact set.

Then (IEP) has a solution.

Proof. It suffices to show $G(\vartheta)$ is compact for some $\vartheta \in K$.

Since

$$G(\vartheta) := \big\{\upsilon \in K : f\big(\vartheta, \eta(\upsilon, \vartheta)\big) \leq \alpha\big(\eta(\vartheta, \upsilon)\big) + \beta(\vartheta, \upsilon)\big\}.$$

Then,

$$G(0) \subset \big\{\upsilon \in K : f\big(0, \eta(\upsilon, 0)\big) \leq (\alpha(\eta(0, \upsilon)) + \beta(0, \upsilon))\big\}$$

$$= \big\{\upsilon \in K : f\big(0, \eta(\upsilon, 0)\big) \leq 0\big\}.$$

From conditions (i and ii), the relative compact set contains the subset denoted by $G(0)$. This means that $G(\vartheta)$ is compact for some values for some $\vartheta \in K$.

Theorem 4.6. Assume that similar speculations as in hypothesis (4.4) fulfill without the supposition of boundedness of K. Take it for granted that the following requirements are met:

i. $\eta(\upsilon, \upsilon) = 0$ for all $\upsilon \in K$.

ii. there exist $\upsilon_0 \in K$ in which

$$f(\upsilon, \eta(\upsilon_0, \upsilon)) \leq 0,$$

where $\upsilon \in K$ and $\|\upsilon\|$ is large enough . Therefore, the issue (IEP) admits a solution.

Proof. Set $K_n = \{\vartheta \in K: \|\vartheta\| \leq n, n > 0\}$. Let us consider that the following issue. Find $\upsilon_n \in K_n$ in which for any $\vartheta \in K_n$

$$f(\upsilon_n, \eta(\vartheta, \upsilon_n)) \geq 0. \tag{4.12}$$

From the boundness of K_n and the theorem 4.3 we get that the issue (IEP) has at least one solution. Choose $\|\upsilon_{n_0}\| \leq n_0$. From (4.12) we get

$$f\left(\upsilon_{n_0}, \eta(\upsilon_0, \upsilon_{n_0})\right) \geq 0\ , \tag{4.13}$$

so $\|\upsilon_{n_0}\| \leq n_0$ since $\upsilon_{n_0} \in K_{n_0}$.

Let us choose $\|\upsilon_{n_0}\| = n_0$, and that n_0 large enough so that from condition (ii), one can have

$$f\left(\upsilon_{n_0}, \eta(\upsilon_0, \upsilon_{n_0})\right) < 0,$$

which contradicts (4.13). Hence $\|\upsilon_{n_0}\| < n_0$, for each $\vartheta \in K$. One can take $\lambda \in (0,1)$ small enough in which $z = \lambda\vartheta + (1-\lambda)\upsilon_{n_0} \in K_{n_0}$. Now, from (4.12) and for each $\vartheta \in K$, one can obtain

$$0 \leq f\left(\upsilon_{n_0}, \eta(z, \upsilon_{n_0})\right)$$

$$\leq f\left(\upsilon_{n_0}, \eta(\lambda\vartheta + (1-\lambda)\upsilon_{n_0}, \upsilon_{n_0})\right)$$

$$\leq \lambda f\left(\upsilon_{n_0}, \eta\left(\vartheta, \upsilon_{n_0}\right)\right)$$

$$\leq f\left(\upsilon_{n_0}, \eta\left(\vartheta, \upsilon_{n_0}\right)\right).$$

Therefore, the issue (IEP) admits a solution.

4.3 Applications

4.3.1 Hemivaritional Inequalities (HVI)

In this subpart we applied our study using (HVI), a new class of variational inequalities, hemivariational inequalities have important applications in engineering and mechanics, particularly in nonsmooth analysis and optimization.

In the spin-off except if expressed in any case, Consider that X is B.S and X^* is dual B.S of E, though $\langle .,. \rangle$ and $\|.\|$ denoted to a duality pairing among X and X^* and norm in X^*.

Now, we let the following issue, which is termed by (Invex Hemi-Equilibrium issue (IHEP)), if for any $\upsilon, \vartheta \in K$,

$$f\big(\upsilon, \eta(\vartheta, \upsilon)\big) + J^0(\upsilon, \vartheta - \upsilon) \geq 0. \tag{4.14}$$

Where $\eta : K \times K \to K$ is a Fun and f is positive bi-fun.

Lemma 4.7. Let K represent a nonempty subset of B.S. X, and let $J : X \to \mathbb{R}$ be a locally Lipschitz Fun. Take it for granted that the following requirements are met:

i. $f(\upsilon, \eta(.,\upsilon))$ is convex $\forall \upsilon \in K$.
ii. $f(., \eta(\vartheta, .))$ is hemicontinuous $\forall \vartheta \in K$.
iii. $\alpha(\eta(.,\upsilon))$ and $\beta(.,\upsilon)$ are convex $\forall \upsilon \in K$.

The issue (4.14) is reward to the following issue:

Find $\upsilon \in K$ in which

$$f(\vartheta, \eta(\upsilon, \vartheta)) \leq \alpha\big(\eta(\vartheta, \upsilon)\big) + \beta(\vartheta, \upsilon) + j^0(\upsilon, \vartheta - \upsilon) \tag{4.15}$$

Proof. Let's suppose υ is indeed a solution to (4.14) and with the definition of α-β-η-Pseudo-monotone , one can have

$$f\big(\vartheta, \eta(\upsilon, \vartheta)\big) - \alpha\big(\eta(\vartheta, \upsilon)\big) - \beta(\vartheta, \upsilon) \leq 0 \tag{4.16}$$

$$f\big(\vartheta, \eta(\upsilon, \vartheta)\big) \leq \alpha\big(\eta(\vartheta, \upsilon)\big) + \beta(\vartheta, \upsilon) + j^0(\upsilon, \vartheta - \upsilon).$$

On the other hand, let us assume that $\upsilon \in K$ is the solution to equation 4.15, and then we will fix $\vartheta \in K$. For $t \in]0,1[$, we let $\upsilon_t = \upsilon - t(\upsilon - \vartheta)$, then $\upsilon_t \in K$. Since K is a convex, so we have:

$$f\big(\upsilon_t, \eta(\upsilon, \upsilon_t)\big) \leq \alpha\big(\eta(\upsilon_t, \upsilon)\big) + \beta(\upsilon_t, \upsilon) + j^0(\upsilon, \upsilon_t - \upsilon). \tag{4.17}$$

So,

$$f\big(\upsilon_t, \eta(\upsilon, \upsilon_t)\big) - \alpha\big(\eta(\upsilon_t, \upsilon)\big) - \beta(\upsilon_t, \upsilon) \leq j^0(\upsilon, \upsilon_t - \upsilon)$$

$$= tj^0(\upsilon, \vartheta - \upsilon).$$

Since $f(\upsilon_t, \eta(.\,, \upsilon_t))$ is convex.

$$0 = f\big(\upsilon_t, \eta(\upsilon_t, \upsilon_t)\big) \leq f\big(\upsilon_t, \eta(\upsilon, \upsilon_t)\big) - t\big[f\big(\upsilon_t, \eta(\upsilon, \upsilon_t)\big) - f\big(\upsilon_t, \eta(\vartheta, \upsilon_t)\big)\big]$$

$$t\left[f\left(v_t,\eta(v,v_t)\right)-f(v_t,\eta(\vartheta,v_t))\right]\leq f\left(v_t,\eta(v,v_t)\right) \tag{4.18}$$

By the convexity of $\alpha\left(\eta(.,v)\right)$ *and* $\beta(\,.,v)$.

$$\alpha\left(\eta(v_t,v)\right)\leq\alpha\left(\eta(v,v)\right)-t\left[\alpha\left(\eta(v,v)\right)-\alpha\left(\eta(\vartheta,v)\right)\right] \tag{4.19}$$

$$\beta(v_t,v)\leq\beta(v,v)-t[\beta(v,v)-\beta(\vartheta,v)] \tag{4.20}$$

From (4.17), (4.18), (4.19) and (4.20)

$$t\left[f\left(v_t,\eta(v,v_t)\right)-f\left(v_t,\eta(\vartheta,v_t)\right)+\alpha(\eta(v,v)-\alpha\left(\eta(\vartheta,v)\right)+\beta(v,v)-\beta(\vartheta,v)\right]$$

$$\begin{aligned}\leq f\left(v_t,\eta(v,v_t)\right)+\alpha\left(\eta(v,v)\right)-\alpha\left(\eta(v_t,v)\right)+\beta(v,v)-\beta(v_t,v)+tj^0(v,\vartheta-v)\\ +\ \alpha\left(\eta(v_t,v)\right)+\beta(v_t,v)+\alpha\left(\eta(v,v)\right)-\alpha\left(\eta(v_t,v)\right)+\beta(v,v)\\ -\beta(v_t,v).\end{aligned}$$

$$\leq tJ^0(v,\vartheta-v)+\alpha\left(\eta(v,v)\right)+\beta(v,v).$$

Since $f(.,\eta(\vartheta,.))$ is hemicontinuous.

$$t\left[-f\left(v,\eta(\vartheta,v)\right)-J^0(v,\vartheta-v)\right]\leq\alpha\left(\eta(v,v)\right)+\beta(v,v)-t\alpha\left(\eta(v,v)\right)+ t\alpha\left(\eta(\vartheta,v)\right)-t\beta(v,v)+t\beta(\vartheta,v)$$

$$\leq(1-t)\left[\alpha\left(\eta(v,v)\right)+\beta(v,v)\right]+t\alpha(\eta(\vartheta,v)+t\beta(\vartheta,v)$$

$$f(v,\eta(\vartheta,v)+J^0(v,\vartheta-v)\geq\frac{t-1}{t}[\alpha(\eta(v,v)+\beta(v,v)]-\alpha\left(\eta(\vartheta,v)\right)-\beta(\vartheta,v).$$

From (Remark 2.45) one can get

$$f(\upsilon, \eta(\vartheta, \upsilon)) + J^0(\upsilon, \vartheta - \upsilon) \geq 0 \qquad (\forall \vartheta \in K).$$

Therefore, the issue (4.14) is reward to (4.15).

Theorem 4.8. It is assumed that K is a nonempty, closed, convex subset of a real reflexive B.S. X. Take it for granted that the following requirements are met:

i. $f\big(\upsilon, \eta\,(.\,, \upsilon)\big)$ *is* α-β-η-Pseudo-monotone and convex .
ii. $f(\vartheta, \eta(.\,, \vartheta\,))$ is l.s.c
iii. $\alpha\big(\eta(\vartheta, .)\big)$ and $\beta(\vartheta, .)$ are u.s.c

Then issue (4.14) admits a solution.

Proof. We think about two set valued map $F, G : K \multimap K$ defined by

$$F(\vartheta) = \big\{f\big(\upsilon, \eta(\vartheta, \upsilon)\big) + J^0(\upsilon, \vartheta - \upsilon) \geq 0 \quad \forall \vartheta \in K\big\}$$

$$G(\vartheta) = \big\{f(\vartheta, \eta(\upsilon, \vartheta) \leq \alpha\big(\eta(\vartheta, \upsilon)\big) + \beta(\vartheta, \upsilon) + j^0(\upsilon, \vartheta - \upsilon) \quad \forall \vartheta \in K\big\}.$$

Then, the issue (4.14) has a solution if and only if $\bigcap_{\vartheta \in K} F(\vartheta) \neq \emptyset$ .Now we proof that F is KKM mapping.

In actuality, F isn't a KKM map. Then, at that point, there exists a bounded subset$\{\upsilon_1, \upsilon_2, \dots, \upsilon_n\}$ of K and $\rho_i \geq 0$ for every $i = \{1,2, \dots, n\}$ together with $\sum_{i=1}^{n} \rho_i = 1$ in which

$$\upsilon_0 = \sum_{i=1}^{n} \rho_i \upsilon_i \notin \bigcup_{i=1}^{n} F(\upsilon_i).$$

Then,

$f\big(v_0, \eta(v_i, v_0)\big) + J^0(v_0, v_i - v_0) < 0.$

By convexity of $f(v_0, \eta(\,.\,, v_0))$, and $J^0(v_0,\ .)$ is a subadditive,

$$0 = f\big(v_0, \eta(v_0, v_0)\big) + J^0(v_0, v_0 - v_0)$$

$$= f\big(v_0, \eta(\textstyle\sum_{i=1}^{n} \rho_i v_i\ , v_0)\big) + J^0(v_0, \textstyle\sum_{i=1}^{n} \rho_i v_i\ - v_0)$$

$$\leq \textstyle\sum_{i=1}^{n} \rho_i \big[f\big(v_0, \eta(v_i, v_0)\big) + J^0(v_0, v_i - v_0)\big]$$

$$< 0.$$

For each $i = \{1,2, \dots, n\}$.

This is an inconsistency. Thusly, F is a KKM map. From lemma 4.7 $F(\vartheta) \subset G(\vartheta)\ \ \forall \vartheta \in K$. Therefore $G(\vartheta)$ is a KKM mapping, Since $f(\vartheta, \eta(., \vartheta))$ is l.s.c, $\alpha\big(\eta(\vartheta, .)\big)$ *and* $\beta(\vartheta, .)$ are u.s.c, we have

$$f(\vartheta, \eta(v, \vartheta)) \leq \liminf_{n}\ [f(\vartheta, \eta(v_n, \vartheta)]$$

$$\leq \limsup_{n} \big[\alpha\big(\eta(\vartheta, v_n)\big) + \beta(\vartheta, v_n) + j^0(v_n, \vartheta - v_n)\big]$$

$$\leq \alpha\big(\eta(\vartheta, v)\big) + \beta(\vartheta, v) + j^0(v, \vartheta - v).$$

Thus, for all $\vartheta \in K$, $G(\vartheta)$ is weakly closed. If X is real and reflexive, then K is weakly compact if and only if K is nonempty, bounded, closed, and convex. Since for any $\vartheta \in K$, $G(\vartheta)$ is only weakly compact.

It follows from lemma 2.23 and lemma 4.7 that

$$\bigcap_{\vartheta \in K} F(\vartheta) = \bigcap_{\vartheta \in K} G(\vartheta) \neq \emptyset.$$

Therefore, any member of this interpart is a solution, so equation (4.14) has a solution.

Theorem 4.9. Assume that K is a nonempty compact convex subset of a B.S X and if

i. $f\big(.,\eta(\vartheta,.)\big)$ is u.s.c
ii. $f(\upsilon,\eta(.,\upsilon)$ is Convex

Then the issue (4.14) admits at least one solution.

Proof. Contending by logical inconsistency, let us consider to be that (4.14) has no adjustment. Then, at that point, for each $\upsilon \in K$ there exists $\vartheta \in K$ with the end goal that

$$f\big(\upsilon,\eta(\vartheta,\upsilon)\big) + J^0(\upsilon,\vartheta-\upsilon) < 0. \tag{4.21}$$

Allow us to characterize the set - valued map $\Lambda: K \multimap K$ as

$$\Lambda(\vartheta) = \{\upsilon \in K : \ f(\upsilon,\eta(\vartheta,\upsilon) + J^0(\upsilon,\vartheta-\upsilon) \geq 0\}.$$

Claim i: For any $\vartheta \in K$, $\Lambda(\vartheta)$ is a nonempty set that is also closed.

According to the definition of the set Λ, it is abundantly clear that $\Lambda(\vartheta)$is not an empty set given that $\vartheta \in \Lambda(\vartheta)$ is true for each and every $\vartheta \in K$.

Let $\{\upsilon_n\}_{n\geq 1} \subset \Lambda(\vartheta)$ be a sequences that converges weakly to υ. We show that $\upsilon \in \Lambda(\vartheta)$ for each $n \geq 1$,

$$f(\upsilon_n,\eta(\vartheta,\upsilon_n) + J^0(\upsilon_n,\vartheta-\upsilon_n) \geq 0.$$

Since $f(.,\eta(\vartheta,.)$ is u.s.c, and take that account proposition (ii). Then $\upsilon \in \Lambda(\vartheta)$ for each $n \geq 1$. When we pass lim sup as $n \to \infty$, we have to

$$f(\upsilon,\eta(\vartheta,\upsilon)) + J^0(\upsilon,\vartheta - \upsilon) \geq 0.$$

Therefore, $\upsilon \in \Lambda(\vartheta)$ and $\Lambda(\vartheta)$ is weakly closed subset of K.

Depending on the equation (4.21) for all $\upsilon \in K$, there is $\vartheta \in K$ in which $\upsilon \in [\Lambda(\vartheta)]^c = X - \Lambda(\vartheta)$. Therefore, the family $\{[\Lambda(\vartheta)]^c\}$ is open covering for the compact set K, for each $\vartheta \in K$. This mean that there is a finite set $\{\vartheta_1,\vartheta_2,\dots,\vartheta_n\}$ of K in which $\{[\Lambda(\vartheta_i)]^c\}$ is a finite subcover of K for every $i = \overline{1,n}$.

Assume that $\Omega_i(\upsilon) = dis\ (\upsilon\Lambda(\vartheta_i))$ (i.e, the interval among x and the set $\Lambda(\vartheta_i)$) and define $\mathrm{Z}_i\colon \mathrm{K} \to \mathbb{R}$ by

$$\mathrm{Z}_i(\upsilon) = \frac{\Omega_i(\upsilon)}{\sum_{j=1}^{n}\Omega_i(\upsilon)}.$$

That is obvious for each of $i = \overline{1,n}$, Z_i is a lipschitz continuous Fun. for all $\upsilon \in K$ and $\sum_{i=1}^{n}\mathrm{Z}_i(\upsilon) = 1$. Let us let the operator $A\colon K \to K$ defined by :

$$A(\upsilon) = \sum_{i=1}^{n}\mathrm{Z}_i(\upsilon)\vartheta_i$$

Claim ii : The map $A(\upsilon)$ is continuous. To do this, we can acquire for any, $\upsilon_1,\upsilon_2 \in K$ the accompanying assessment:

$$\|A(\upsilon_1) - A(\upsilon_2)\| = \left\|\sum_{i=1}^{n}\big(\mathrm{Z}_i(\upsilon_1) - \mathrm{Z}_i(\upsilon_2)\big)\vartheta_i\right\|$$

$$\leq \sum_{i=1}^{n} \|\vartheta_i\| \left\| \left(\mathrm{Z}_i(v_1) - \mathrm{Z}_i(v_2) \right) \right\|$$

$$\leq M_i \sum_{i=1}^{n} \|\vartheta_i\| \|v_1 - v_2\|$$

$$\leq M \|v_1 - v_2\|.$$

This show that A is continuous mapping. By hypothesis 2.46, there is $\bar{v} \in K$ in which $A\,(\bar{v}) = \bar{v}$.

Let us define a Fun. $\phi: K \to \mathbb{R}$ in which

$$\phi(v) = f\big(v, \eta(A(v), v)\big) + J^0(v, A(v) - v).$$

Since $f\big(v, \eta(.\,, v)\big)$, is convex and proposition 2.45, on can get

$$\phi(v) = f(v\,, \eta(\textstyle\sum_{i=1}^{n} \mathrm{Z}_i(v)\vartheta_i\,, v) + J^0(v, \textstyle\sum_{i=1}^{n} \mathrm{Z}_i(v), \vartheta_i - v)$$

$$\leq \sum_{i=1}^{n} \mathrm{Z}_i(v)\big[f\big(v, \eta(\vartheta_i, v)\big) + J^0(v, \vartheta_i - v)\big].$$

Taking in to account $K \subset \bigcup_{i=1}^{n} [\Lambda(\vartheta_i)]^c$, for every $i = \overline{1, n}$, there exists at least one index $i_0 = \overline{1, n}$ in which $v \in \big[\Lambda(\vartheta_{i_0})\big]^c$. This show that $\phi(v) < 0$ for all $v \in K$. On the other hand $\phi(\bar{v}) = 0$, we have obtained a contradiction.

4.3.2 Well-Posedness

The significance of (w-p) is broadly perceived in the hypothesis of EP. presented the idea of (w-p) for a minimization issue , which is known as Tykhonov (w-p) . Because of its significance in streamlining issues.

4.3.2.1 Well-posed Of (IEP) With Metric Characterization

In this subpart, we form the (w-p) for invex equilibrium issue (for short (IEP)). At first we will give some conditions under which the equilibrium issue is Swpgs.

Definition 4.10. $\{(t_n, \upsilon_n)\} \subset T \times K$ is named to be an Appr. Seq for (IEP) if there is positive sequence $\{\lambda_n\}$ accompanied by $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$ in which

$$f(t, \upsilon_n, \left(\eta(\vartheta, \upsilon_n)\right) + \lambda_n \|\vartheta - \upsilon\| \geq 0 \ \ \forall n \in \mathbb{N}, \vartheta \in K \tag{4.22}$$

Definition 4.11. The issue (IEP) is said to be strongly (resp., weakly) w-p (resp., strongly (resp., weakly) in the generalized sense) if (IEP) has a single solution υ, and for every sequence $\{\upsilon_n\}$ accompanied by $\upsilon_n \rightarrow \upsilon$, every Appr. Seq for (IEP) converges strongly (resp., weakly) to the single solution (resp., if (IEP) has a nonempty solution set $M(t)$, and every approximate solution sequence has a subsequence which strongly (resp., weakly) converges to some point of $M(t)$.

In the following, we will establish some categorizations of (w-p) for (IEP). For any $\lambda > 0$, $\forall \vartheta \in K$ we define two sets:

$$\Omega(\lambda) = \left\{(t, \upsilon) \in T \times K : f(t, \upsilon, \left(\eta(\vartheta, \upsilon)\right) + \lambda \|\vartheta - \upsilon\| \geq 0 \ \ \forall (t, \vartheta) \in T \times K \right\}.$$

$$\Phi(\lambda) = \left\{(t, \upsilon) \in T \times K : f\left(t, \vartheta, \eta(\upsilon, \vartheta)\right) \leq \alpha\left(\eta(\vartheta, \upsilon)\right) + \beta(\vartheta, \upsilon) + \lambda \|\vartheta - \upsilon\|\right\}.$$

Lemma 4.12. Let K be a nonempty convex, closed subset of a B.S X. Assume that $f :$ T × K × K → $\mathbb{R}$, $\alpha : X \rightarrow \mathbb{R}$ and $\beta : K \times K \rightarrow R$ be three Funs. If

i. $f(t, \upsilon, \eta(\upsilon, \upsilon)) = 0 \quad \forall\, t \in T, \upsilon \in K$;
ii. $f(t, ., (\eta(\,.\,, .)\,)$ is α-β-η-Pseudo-monotone and hemi-continuous $\forall\, t \in K$;
iii. $f\left(t, \upsilon, \eta(\,.\,, \upsilon)\right)$ is convex Fun.;

iv. $\alpha(\eta(.,\upsilon))$ *and* $\beta(.,\upsilon)$ are convex Fun. $\forall\, t \in T\,, \vartheta \in K$.

Then, $\Omega(\lambda) = \Phi(\lambda)$ for all $\lambda > 0$.

Proof. Suppose that $(t,\upsilon) \in \Omega(\lambda)$. There exists $(t,\upsilon) \in T \times K$ in which

$$f(t,\upsilon,(\eta(\vartheta,\upsilon)) + \lambda\|\vartheta - \upsilon\| \geq 0 \quad \forall\vartheta \in K. \tag{4.23}$$

Since $f(t,.,(\eta(\,.,.)\,)$ is α-β-η-Pseudo-monotonicity

$$\forall\vartheta \in K \;\; f\big(t,\vartheta,\eta(\upsilon,\vartheta)\big) \leq \alpha\big(\eta(\vartheta,\upsilon)\big) + \beta(\vartheta,\upsilon) + \lambda\|\vartheta - \upsilon\| \tag{4.24}$$

Conversely, let $(t,\upsilon) \in \Phi(\lambda)$ and fix $\vartheta \in K$, since that K is convex.

$\forall\upsilon,\vartheta \in K\,, \varepsilon \in\,]0,1[\,, \upsilon_\varepsilon = \varepsilon\vartheta + (1-\varepsilon)\upsilon\,,$ then $f\big(t,\upsilon_\varepsilon,\eta(\upsilon,\upsilon_\varepsilon)\big) - \alpha\big(\eta(\upsilon_\varepsilon,\upsilon)\big) - \beta(\upsilon_\varepsilon,\upsilon) \leq \lambda\|\upsilon_\varepsilon - \upsilon\| = \varepsilon\lambda\|\vartheta - \upsilon\|.$ (4.25)

since $f(t,\upsilon_\varepsilon,\eta(.,\upsilon_\varepsilon))$ is convex, one can have

$$0 = f\big(t,\upsilon_\varepsilon,\eta(\upsilon_\varepsilon,\upsilon_\varepsilon)\big)$$

$$\leq f\big(t,\upsilon_\varepsilon,\eta(\upsilon,\upsilon_\varepsilon)\big) - \varepsilon\big[f\big(t,\upsilon_\varepsilon,\eta(\upsilon,\upsilon_\varepsilon)\big) - f\big(t,\upsilon_\varepsilon,\eta(\vartheta,\upsilon_\varepsilon)\big)\big],$$

Then,

$$\varepsilon\big[f\big(t,\upsilon_\varepsilon,\eta(\upsilon,\upsilon_\varepsilon)\big) - f\big(t,\upsilon_\varepsilon,\eta(\vartheta,\upsilon_\varepsilon)\big)\big] \leq f\big(t,\upsilon_\varepsilon,\eta(\upsilon,\upsilon_\varepsilon)\big) \tag{4.26}$$

Since $\alpha\big(\eta(.,\upsilon)\big)$ *and* $\beta(.,\upsilon)$ are convex

$$\alpha\big(\eta(\upsilon_\varepsilon,\upsilon)\big) \leq \alpha\big(\eta(\upsilon,\upsilon)\big) - \varepsilon\big[\alpha\big(\eta(\upsilon,\upsilon)\big) - \alpha\big(\eta(\vartheta,\upsilon)\big)\big] \tag{4.27}$$

$$\beta(\upsilon_\varepsilon,\upsilon) \le \beta(\upsilon,\upsilon) - \varepsilon[\beta(\upsilon,\upsilon) - \beta(\vartheta,\upsilon)] \tag{4.28}$$

From (4.25), (4.26), (4.27) and (4.28), one can get

$$\varepsilon\big[f\big(t,\upsilon_\varepsilon,\eta(\upsilon,\upsilon_\varepsilon)\big) - f\big(t,\upsilon_\varepsilon,\eta(\vartheta,\upsilon_\varepsilon)\big) + \alpha(\eta(\upsilon,\upsilon)) - \alpha\big(\eta(\vartheta,\upsilon)\big) + \beta(\upsilon,\upsilon) - \beta(\vartheta,\upsilon)\big].$$

$$\le f\big(t,\upsilon_\varepsilon,\eta(\upsilon,\upsilon_\varepsilon)\big) + \alpha\big(\eta(\upsilon,\upsilon)\big) - \alpha\big(\eta(\upsilon_\varepsilon,\upsilon)\big) + \beta(\upsilon,\upsilon) - \beta(\upsilon_\varepsilon,\upsilon)$$

$$\varepsilon\lambda\|\vartheta - \upsilon\| + \alpha\big(\eta(\upsilon_\varepsilon,\upsilon)\big) + \beta(\upsilon_\varepsilon,\upsilon) + \alpha\big(\eta(\upsilon,\upsilon)\big) - \alpha\big(\eta(\upsilon_\varepsilon,\upsilon)\big) + \beta(\upsilon,\upsilon) - \beta(\upsilon_\varepsilon,\upsilon)$$

$$\le \varepsilon\lambda\|\vartheta - \upsilon\| + \alpha\big(\eta(\upsilon,\upsilon)\big) + \beta(\upsilon,\upsilon).$$

Since $f(.,t,\eta(.,.))$ is hemicontinuous, one can have

$$\varepsilon\big[-f\big(t,\upsilon,\eta(\vartheta,\upsilon)\big) - \lambda\|\vartheta - \upsilon\|\big]$$

$$\le (1-\varepsilon)\big[\alpha\big(\eta(\upsilon,\upsilon)\big) + \beta(\upsilon,\upsilon)\big] + \varepsilon\alpha\big(\eta(\vartheta,\upsilon)\big) + \varepsilon\beta(\vartheta,\upsilon)f\big(t,\upsilon,\eta(.,\vartheta,\upsilon)\big) + \lambda\|\vartheta - \upsilon\|$$

$$\ge -\alpha\big(\eta(\vartheta,\upsilon)\big) - \beta(\vartheta,\upsilon) + \frac{t-1}{t}\big[\alpha\big(\eta(\upsilon,\upsilon)\big) + \beta(\upsilon,\upsilon)\big].$$

From remark (2.45) one can get

$$f\big(t,\upsilon,\eta(\vartheta,\upsilon)\big) + \lambda\|\vartheta - \upsilon\| \ge 0.$$

Hence, $(t,\upsilon) \in \Omega(\lambda)$. Therefore $\Omega(\lambda) = \Phi(\lambda)\ \ \forall\lambda > 0$.

Lemma 4.13. Let K be a nonempty convex-closed set of a B.S X. Suppose that $f: T \times K \times K \to \mathbb{R}$, $\alpha : X \to \mathbb{R}$ and $\beta : K \times K \to \mathbb{R}$ satisfy in the following conditions:

i. $f(.,\vartheta,(\eta(.,\vartheta))$ is l.s.c $\forall \vartheta \in K$,

ii. $\alpha(\eta(\vartheta,.)$ *and* $\beta(\vartheta,.)$ are u.s.c $\forall\vartheta \in K$.

Then, $\Phi(\lambda)$ is closed in $T \times K$ for any $\lambda > 0$.

Proof. Letting $\{(t_n,\upsilon_n)\} \subset \Phi(\lambda)$ is a sequence in which $(t_n,\upsilon_n) \to (t,\upsilon)$ in $T \times K$.

$$f\big(t_n,\vartheta,\eta(\upsilon_n,\vartheta)\big) \leq \alpha\big(\eta(\vartheta,\upsilon_n)\big) + \beta(\vartheta,\upsilon_n) + \lambda\|\vartheta - \upsilon_n\|.$$

Since $f(.,\vartheta,(\eta(.,\vartheta))$ is l.s.c and $\alpha(\eta(\vartheta,.)$ *and* $\beta(\vartheta,.)$ are u.s.c

$$f\big(t,\vartheta,\eta(\upsilon,\vartheta)\big) \leq \lim_n \inf\big[f\big(t_n,\vartheta,\eta(\upsilon_n,\vartheta)\big)\big]$$

$$\leq \lim_n \sup\big[\alpha\big(\eta(\vartheta,\upsilon_n)\big) + \beta(\vartheta,\upsilon_n) + \lambda\|\vartheta - \upsilon_n\|\big]$$

$$\leq \alpha\big(\eta(\vartheta,\upsilon)\big) + \beta(\vartheta,\upsilon) + \lambda\|\vartheta - \upsilon\|.$$

Which implies $(t,\upsilon) \in \Phi(\lambda)$. Therefore, $\Phi(\lambda)$ is closed in $T \times K$ for all $\lambda > 0$.

Theorem 4.14. Consider that K is a nonempty convex, closed subset of a B.S X. Let $f: T \times K \times K \to \mathbb{R}$, $\alpha : X \to \mathbb{R}$ and $\beta: K \times K \to \mathbb{R}$ be three Funs. If (IEP) is strongly w-p) then,

$$\Omega(\lambda) \neq \emptyset \quad \forall\lambda > 0\,, \lim_{\lambda\to 0} diam\,\big(\Omega(\lambda)\big) = 0. \tag{4.29}$$

In addition, if the following assumptions hold:

i. $f\big(t, \upsilon, \eta(v, \upsilon)\big) = 0 \quad \forall\, t \in T, v \in K,$

ii. $f\big(t, ., \eta(., .)\big)$ is α-β-η-Pseudo-monotonicity and hemicontinuous $\forall\, t \in T,$

iii. $f\big(t, v, \eta(\, ., \upsilon)\big)$ is convex $\forall (t, v) \in T \times K,$

iv. $\alpha(\eta(., v))\ and\ \ \beta(., v)$ are convex $\forall\, v \in K,$

v. $f\big(., v, \big(\eta(., v)\big)$ is l.s.c $\forall v \in K,$

vi. $\alpha\big(\eta(\vartheta, .)\big)\ and\ \beta(\vartheta, .)$ are u.s.c $\forall \vartheta \in K.$

Proof. Let (IEP) is strongly w-p. Then, (IEP) admit a unique solution $(t, v) \in T \times K$, i.e.,

$$f\big(t, v, \eta(\vartheta, \upsilon)\big) \geq 0 \quad \forall \vartheta \in K.$$

Clearly, $\Omega(\lambda) \neq \emptyset$ for any $\lambda > 0$. Then, by contradiction, we consider that $\lim\limits_{\lambda_n \to 0} diam\big(\Omega(\lambda_n)\big) > p > 0,$ for sequences $\{\lambda_n\} > 0$, one can to find two sequences $\{(t_n, v_n)\}$ and $\{(t_n, \vartheta_n)\}$ satisfying $(t_n, v_n) \in \Omega(\lambda), (t_n, \vartheta_n) \in \Omega(\lambda)$, and

$$||(t_n, v_n) - (t_n, \vartheta_n)|| > p \quad \forall n \in \mathbb{N}. \tag{4.30}$$

Since $\{(t_n, v_n)\}$ and $\{(t_n, \vartheta_n)\}$ are Appr. Seq for (IEP). By the (w-p) of (IEP), they have to converge strongly to the unique solution of (IEP) a logical inconsistency to (4.30).

Conversely, suppose that condition (4.29) holds. Let $\{(t_n, v_n)\}$ be an Appr. Seq for (IEP). Then, there exists a nonnegative sequence $\{\lambda_n\}$ with $\lambda_n \to 0$ as $n \to \infty$ in which

$$f(t, v_n, \big(\eta(\vartheta, v_n)\big) + \lambda_n ||\vartheta - \upsilon|| \geq 0 \quad \forall n \in \mathbb{N}, \vartheta \in K. \tag{4.31}$$

This yields that $\{(t_n, v_n)\} \in \Omega(\lambda_n)$. It follows from (4.29) that $\{(t_n, v_n)\}$ is a cauchy sequence and so it converges to a point $(t, v) \in T \times K$. It follows from (4.31), α-β-η-

Pseudo-monotone and since $f(t, \vartheta, (\eta(., \vartheta))$ is u.s.c, $\alpha(\eta(\vartheta, .))$ and $\beta(\vartheta, .)$ are l.s.c that

$$0 = \lim_n \inf \lambda_n \|\vartheta - \upsilon_n\|$$

$$\geq \lim_n \inf[-f(t_n, \upsilon_n, \eta(\vartheta, \upsilon_n))].$$

Then,

$$0 \geq \lim_n \inf\left[f(t_n, \vartheta(\eta(\upsilon_n, \vartheta)) - \alpha(\eta(\vartheta, \upsilon_n)) - \beta(\vartheta, \upsilon_n)\right]$$

$$\geq f(t, \vartheta(\eta(\upsilon, \vartheta)) - \alpha(\eta(\vartheta, \upsilon)) - \beta(\vartheta, \upsilon).$$

From the above and theorem (4.3), we determine (t, υ) as a solution of (IEP). The uniqueness follows immediately from (4.29).

Remark 4.15. If there is more than one solution to (IEP), then you will see that the diameter of $\Omega(\lambda)$ does not tend to zero. Instead of using the diameter, we substitute the kuratowski non compactness measure of approximating the solution set in the next result.

Theorem 4.16. Consider that T and K are nonempty, closed and convex subsets of real B.S X and E respectively. If (IEP) is Swpgs, then

$$\Omega(\lambda) \neq \emptyset \ \ \forall \lambda > 0, \lim_{\lambda \to 0} \xi(\Omega(\lambda)) = 0. \tag{4.32}$$

Moreover, if the following assumptions hold:

i. $f(t, \upsilon, \eta(\upsilon, \upsilon)) = 0 \quad \forall t \in T, \upsilon \in K,$

ii. $f(t,.,\eta(.,.))$ is α-β-η-pseudomonotonicity and hemicontinuous,

iii. $f(t,\upsilon,\eta(.,\upsilon))$ is convex $\forall(t,\upsilon) \in T \times K$,

iv. $\alpha(\eta(.,\upsilon))$ and $\beta(.,\upsilon)$ are convex $\forall \upsilon \in K$,

v. $f(.,\upsilon,\eta(.,\upsilon))$ is l.s.c $\forall \upsilon \in K$,

vi. $\alpha(\eta(\vartheta,.))$ and $\beta(\vartheta,.)$ are u.s.c $\forall \vartheta \in K$.

Then, the converse holds.

Proof. Let (IEP) be Swpgs. Then the solution set $M$of (IEP) is a nonempty. this shows that, for any $\lambda > 0, \Omega(\lambda) \neq 0$ since $M \subset \Omega(\lambda)$. Additionally, we claim that the solution set M of (IEP) is compact. indeed, for any sequence $\{(t_n, \upsilon_n)\}$ in M, $\{(t_n, \upsilon_n)\}$ is an approximating arrangement for (IEP). Consequently there is a combining aftereffect to some place of M. This suggests that M is compact.

Now, we show that $\lim_{\lambda \to 0} \xi(\Omega(\lambda)) \to 0$. It follows from $M \subset \Omega(\lambda)$ that

$$H(\Omega(\lambda), M) = \max\{(e(\Omega(\lambda), M), e(M, \Omega(\lambda))\}.$$

Because the solution set M is somewhat compact, it is possible to obtain

$$\xi(\Omega(\lambda)) \leq 2H(\Omega(\lambda), M) + \xi(M) = 2e(\Omega(\lambda), M),$$

Where $\xi(M) = 0$ since M is compact.

To show $\lim_{\lambda \to 0} \xi(\Omega(\lambda)) = 0$.

It is sufficient to display that$e(\Omega(\lambda), M) \to 0 \ as \ \lambda \to 0$.

If not, there exists a constant $c > 0 \ \ and \ \lambda_n \to 0, \ and \ \{(t_n, \upsilon_n)\} \subset \Omega(\lambda_n)$ in which for all $n \in \mathbb{N}$

$$(t_n, \upsilon_n) \notin M + B_{\frac{c}{2}}(0). \tag{4.33}$$

In which $B_{\frac{c}{2}}(0)$ represents an open ball with a center value of 0 and a radius value of $\frac{c}{2}$. Yet, $\{(t_n, \upsilon_n)\} \subset \Omega(\lambda_n)$, is an Appr. Seq for (IEP). It follows the generalized (w-p) of (IEP) that there is a subsequence converges to some point of $(t, \upsilon) \in M$, which contradicts (4.33).

On the other hand, assume that (4.31) holds. By Lemma 4.12 and Lemma 4.13, $\Omega(\lambda)$ is nonempty and closed for all $\lambda \to 0$. Using the kuratowski theorem (Kuratowski 1968), we may obtain

$$H(\Omega(\lambda), M) \to 0 \ \ as\, \lambda \to 0,$$

Where $M = \bigcap_{\lambda \to 0} \Omega(\lambda)$ is a nonempty and compact.

Let $\{(t_n, \upsilon_n)\} \subset K$ by any an approximate solution sequence for (IEP). Then there is a nonnegative sequence $\{\lambda_n\}$ accompanied by $\lambda_n \to 0 \ \ as\ n \to \infty$ in which

$$f(t_n, \upsilon_n, \eta(\vartheta, \upsilon_n)) + \lambda_n \|\vartheta - \upsilon_n\| \geq 0 \ \ \forall n \in \mathbb{N}, \vartheta \in K.$$

This means that $(t_n, \upsilon_n) \in \Omega(\lambda_n)$. This together with (4.30) indicates that

$$d[(t_n, \upsilon_n), M] \leq e(\Omega(\lambda), M) \to 0.$$

Since M is compact, it follows that there exists $(\bar{t}_n, \bar{\upsilon}_n)$ has a subsequence $\{(\bar{t}_{n_k}, \bar{\upsilon}_{n_k})\}$ converging strongly to $\{(\bar{t}_n, \bar{\upsilon}_n)\} \in M$. Therefore, the corresponding $\{(t_{n_k}, \upsilon_{n_k})\}$ subsequence of $\{(t_n, \upsilon_n)\}$ converges strongly to $\{(\bar{t}_n, \bar{\upsilon}_n)\}$. Hence (IEP) wpgs.

4.3.2.2 Well-posedness For Optimization Problem With Invex Parametric Equilibrium Constraints

In this section, let's define the formulation of the optimization problem with an invex parametric equilibrium constraint, which will be designated as (OPIPEC) from now on. The problem can be defined as follows:

$$\min h(t,a) \ \ s.t \ (t,a) \in T \times K,$$

Where $a \in M(t), T$ is a nonempty closed subset of a parametric normed space, $h : T \times K \to \mathbb{R}, f : T \times K \times K \to \mathbb{R}$, and $M(t)$ is the solution set of the parametric invex Equilibrium problem (IEP), defined by, $a \in M(t)$ if and only if

$$f\big(t,a,\eta(b,a)\big) \geq 0 \ \ \forall b \in K.$$

Definition 4.17. A sequence $\{(t_n,a_n)\} \subset T \times K$ is let to be an Appr. Seq for (OPIPEC) if

i. There is a posative sequence $\{\lambda_n\}$ with $\lambda_n \to 0$ as $n \to \infty$, then

$$f\big(t_n,a_n,\eta(b,a_n)\big) + \lambda_n\|b-a_n\| \geq 0 \ \ \forall n \in \mathbb{N}, b \in K.$$

ii. $\limsup_{n} h(t_n,a_n) \leq \inf_{e \in T, b \in M(e)} h(e,b).$

Definition 4.18. (OPIPEC) is named to be strongly (resp. weakly) w-p (resp. strongly (resp. weakly) wpgs) if (OPIPEC) has a single solution υ and for every Appr. Seq for (OPIPEC) converges strongly (resp. weakly) to the single solution (resp., if $M \neq \emptyset$ and any solution of Appr. Seq have a subsequence which strongly (resp. weakly) converges to some element of M).

The approximate set of solutions for (OPIPEC) is defined by the following:

$$\mu(\lambda,\omega)=\begin{cases}(t,a)\in T\times K: h(t,a)\le \inf_{e\in T, b\in M(e)} h(e,b)+\omega \quad and \\ \qquad f\big(t,a,\eta(b,a)\big)+\lambda_n\|b-a\| \ \ \forall b\in K.\end{cases}$$

Theorem 4.19. Let K is a non-empty convex, closed subset of a B.S X. Let $f: T\times K\times K\to\mathbb{R}, h: T\times K\to\mathbb{R}, \alpha: X\to\mathbb{R}$ and $\beta: K\times K\to\mathbb{R}$ be four Funs.

If (OPIEPC) is strongly (w-p), then

$$\mu(\lambda,\omega)\neq\emptyset \ \forall\lambda,\omega>0, diam\,\mu(\lambda,\omega)\to 0, \tag{4.34}$$

Where $(\lambda,\omega)\to(0,0)$. Moreover, if the following assumptions hold:

i. $f\big(t,\upsilon,\eta(\upsilon,\upsilon)\big)=0 \ \ \forall t\in T, \upsilon\in K,$
ii. $f\big(t,.,\eta(.,.)\big)$ is α-β-η-pseudo-monotone and hemicontiuous,$\forall t\in T,$
iii. $f\big(t,\upsilon,\eta(.,\upsilon)\big)$ is convex $\forall(t,\upsilon)\in T\times K,$
iv. $\alpha\big(\eta(.,\upsilon)\big)$ and $\beta(.,\upsilon)$ are convex $\forall\upsilon\in K,$
v. $f\big(.,\upsilon,\eta(.,\upsilon)\big)$ is l.s.c $\forall\upsilon\in K,$
vi. $\alpha\big(\eta(\vartheta,.)\big)$ and $\beta(\vartheta,.)$ are u.s.c $\forall\vartheta\in K,$
vii. h is l.s.c

Then the convers holds.

Proof. Let (OPIEPC) is strongly (w-p). Then (OPIEPC) admits a unique solution $(t,\upsilon)\in T\times K,$ i.e.,

$$\begin{cases}h(t,\upsilon)=\inf_{e\in T, y\in M(e)} h(e,y) \ \ and \\ \ f\big(t,\upsilon,\eta(\vartheta,\upsilon)\big)\ge 0 \ \ \forall\vartheta\in K.\end{cases}$$

Clearly, $\mu(\lambda, \omega) \neq \emptyset$ for any $\lambda, \omega > 0$, since $(t, \upsilon) \in \mu(\lambda, \omega)$ for any $\lambda, \omega > 0$. If $diam\, \mu(\lambda, \omega) \nrightarrow \emptyset\ as\ \lambda \to 0, \omega \to 0$. If this is the case, there must be a constant $p > 0$ and a nonnegative sequences $\{\lambda_n\}$ and $\{\omega_n\}$ with $\lambda_n \to 0, \omega_n \to 0\ as\ n \to 0$ and (t_n, υ_n), $(t_n, \vartheta_n) \in \mu(\lambda_n, \omega_n)$ in which

$$\|(t_n, \upsilon_n) - (t_n, \vartheta_n)\| > p\ \forall n \in \mathbb{N}. \tag{4.35}$$

Since $(t_n, \upsilon_n), (t_n, \vartheta_n) \in \mu(\lambda_n, \omega_n)\ \forall n \in \mathbb{N}$, so both $\{(t_n, \upsilon_n)\}\ and\ \{(t_n, \vartheta_n)\}$ are Appr. Seq for (OPIEPC). By the well-posed of (OPIEPC), they have to converge strongly to the unique solution of (OPIEPC) a contradiction to (4.35).

Conversely, assume that condition (4.34) holds. Let $\{(t_n, \upsilon_n)\}$ be an Appr. Seq for (OPIEPC). Then, there is a nonnegative sequence $\{\lambda_n\}$ with $\lambda_n \to 0\ as\ n \to \infty$ in which

$$\begin{cases} \limsup_n h(t_n, \upsilon_n) \leq \inf_{e \in T, y \in M(e)} h(e, y)\ \ and \\ f\big(t_n, \upsilon_n, \eta(\vartheta, \upsilon_n)\big) + \lambda_n \|\vartheta - \upsilon_n\|\ \ \forall n \in \mathbb{N}, \vartheta \in K. \end{cases} \tag{4.36}$$

This yields that $(t_n, \upsilon_n) \in \mu(\lambda_n, \omega_n)$. It follows from (4.34), that $\{(t_n, \upsilon_n)\}$ is a Cauchy succession thus it unites firmly to a point $(t, \upsilon) \in T \times K$. Because of (4.36) and the assumptions made in (ii), (v), and (vi), it concludes that

$$0 = \liminf_n \lambda_n \|\vartheta - \upsilon_n\|$$

$$\geq \liminf_n [-f(t_n, \upsilon_n, \eta(\vartheta, \upsilon_n)]$$

Then,

$$0 \geq \liminf_n \left[f\big(t_n, \vartheta, \eta(\upsilon_n, \vartheta)\big) - \alpha\big(\eta(\vartheta, \upsilon)\big) - \beta(\vartheta, \upsilon)\right]$$

$$\geq f\big(t, \vartheta, \eta(\upsilon, \vartheta)\big) - \alpha\big(\eta(\vartheta, \upsilon)\big) - \beta(\vartheta, \upsilon).$$

Furthermore, one can deduce from (4.36), in conjunction the condition (vii), that

$$\inf_{e \in T, y \in M(e)} h(e, y) \geq \limsup_{n} \; h(t_n, \upsilon_n)$$

$$\geq \liminf_{n} h(t_n, \upsilon_n)$$

$$\geq h(t, \upsilon).$$

So, by theorem (4.3) (t, υ) solves (OPIPEC). The uniqueness follows immediately from (4.34).

Theorem 4.20. Consider that T and K are nonempty-closed and convex set of real B.Ss X and E resp., If (POIPEC) is swpgs, then

$$\mu(\lambda, \omega) \neq 0, \; \forall \lambda, \omega > 0, \; \lim_{(\lambda,\omega) \to (0,0)} \xi\big(\mu(\lambda, \omega)\big) = 0. \tag{4.37}$$

Additionally, assuming the aforementioned premises are true:

i. $f\big(t, \upsilon, \eta(\upsilon, \upsilon)\big) = 0 \;\; \forall t \in T, \upsilon \in K,$
ii. $f\big(t, ., ., \eta(., .)\big)$ is α-β-η-pseudo-monotone and hemicontiuous, $\forall \; t \in T,$
iii. $f\big(t, \upsilon, \eta(., \upsilon)\big)$ is convex $\forall (t, \upsilon) \in T \times K,$
iv. $\alpha\big(\eta(., \upsilon)\big)$ *and* $\beta(., \upsilon)$ are convex $\forall \; \upsilon \in K,$
v. $f\big(., \upsilon, \eta(., \upsilon)\big)$ is l.s.c $\forall \upsilon \in K,$
vi. $\alpha\big(\eta(\vartheta, .)\big)$ *and* $\beta(\vartheta, .)$ are u.s.c $\forall \; \vartheta \in K,$
vii. h is l.s.c

Then, the convers holds.

Proof. By a comparable confirmation as that of hypothesis 4.16, we can get the accompanying outcome for the (w-p) of (OPIPEC).

To examine the uniqueness of evening out to (OPIPEC), we show that under reasonable circumstances, in the following outcome the (w-p) of (OPIPEC) is identical to the existence and uniqueness of soutions.

Theorem 4.21. Consider that T and K are nonempty subsets of finite dimensional B.Ss X and E resp., that are closed. Let $f: T \times K \times K \to \mathbb{R}$, and $h: T \times K \to \mathbb{R}$. Suppose that

i. $f(t, \upsilon, \eta(\upsilon, \upsilon)) = 0 \quad \forall t \in T, \upsilon \in K$,
ii. $f(t, ., \eta(., .))$ is α-β-η-pseudo-monotone and hemicontinuous, $\forall t \in T$,
iii. $f(t, \upsilon, \eta(., \upsilon))$ is convex $\forall (t, \upsilon) \in T \times K$,
iv. $f(., \upsilon, \eta(., \upsilon))$ is l.s.c $\forall \upsilon \in K$,
v. $\alpha(\eta(\vartheta, .))$ and $\beta(\vartheta, .)$ are u.s.c $\forall \vartheta \in K$,
vi. h is convex and l.s.c.

Hence, (OPIPEC) is a strongly (w-p) if and only if it has a single solution

Proof. It is not hard to see the necessity. Assume for the purposes of sufficiency that (OPIPEC) possesses just one solution (t^*, υ^*). Consequentially,

$$\begin{cases} h(t^*, \upsilon^*) = \inf\limits_{e \in T, y \in M(e)} h(e, y) \\ f(t^*, \upsilon^*, \eta(\vartheta, \upsilon^*) \geq 0 \quad \forall \vartheta \in K \end{cases} \tag{4.38}$$

Let $\{(t_n, \upsilon_n)\} \subset T \times K$ be an Appr. Seq for (OPIPEC). Then there exists $\lambda_n > 0$ with $\lambda_n \to 0$ as $n \to \infty$,

$$\begin{cases} \limsup\limits_{n \to \infty} h(t_n, \upsilon_n) \leq \inf\limits_{e \in T, \vartheta \in M(e)} h(e, \vartheta) \\ f\big(t_n, \upsilon_n, \eta(\vartheta, \upsilon_n)\big) + \lambda_n \|\vartheta - \upsilon_n\| \geq 0 \end{cases} \tag{4.39}$$

We claim that $\{(t_n, \upsilon_n)\}$ is a bounded. If not, we assume that $\|(t_n, \upsilon_n)\| \to +\infty$. Let $r_n = \frac{1}{\|(t_n,\upsilon_n)-(t^*,\upsilon^*)\|}$, and

$$(a_n, b_n) = r_n(t_n, \upsilon_n) + (1 - r_n)(t^*, \upsilon^*)$$

$$= [r_n t_n + (1 - r_n)t^*, r_n x_n + (1 - r_n)\upsilon^*].$$

Without loss of consensus, one can Consider to be that $r_n \in]0,1[$ and $(a_n, b_n) \to (a, b)$ with $(a, b) \neq (t^*, \upsilon^*)$ since $X \times E$ is finite-dimensional. If both T and K are considered in terms of their closedness and convexity, one has $(a_n, b_n) \in T \times K$. Hence, according to assumption (vi), 4.38 and 4.39, we have for any $(a, b) \in T \times K$, that:

$$h(t^*, \upsilon^*) = \lim_n \sup r_n h(t_n, \upsilon_n) + \lim_n \sup (1 - r_n) h(t^*, \upsilon^*)$$

$$\geq \lim_n \sup [r_n h(t_n, \upsilon_n) + (1 - r_n) h(t^*, \upsilon^*)]$$

$$\geq \lim_n \sup h(a_n, b_n) \tag{4.40}$$

$$\geq \lim_n \inf h(a_n, b_n)$$

$$\geq h(a, b).$$

Moreover, it follows from conditions (i),(iii),4.38 and 4.39 that

$$0 = \lim_n \inf r_n \lambda_n \|y - \upsilon_n\|$$

$$\geq \lim_n \inf -r_n\big[f\big(t_n, \upsilon_n, \eta(\vartheta, \upsilon_n)\big)\big] - (1 - r_n)[f(t^*, \upsilon^*, \eta(\vartheta, \upsilon^*))]$$

$$\geq \liminf_{n} \left[-r_n f\big(t_n, \upsilon_n, \eta(\vartheta, \upsilon_n)\big) - (1 - r_n) f(t^*, \upsilon^*, \eta(\vartheta, \upsilon^*))\right]$$

$$\geq \liminf_{n} \left[-f(a_n, b_n, \eta(\vartheta, b_n))\right] \tag{4.41}$$

$$\geq \liminf_{n} \left[f(a_n, \vartheta, \eta(b_n, \vartheta)) - \alpha(\eta(\vartheta, b_n)) - \beta(\vartheta, b_n)\right]$$

$$\geq f\big(a, \vartheta, \eta(b, \vartheta)\big) - \alpha\big(\eta(\vartheta, b)\big) - \beta(\vartheta, b).$$

Applying Theorem 4.3 implies that

$$f\big(a, b, \eta(\vartheta, b)\big) \geq 0\,, \forall \vartheta \in K.$$

Hence, from 4.40 and 4.41, (a, b) solves (OPIPEC), which is a contradiction. so $\{(t_n, \upsilon_n)\}$ is bounded. Let $\{\big(t_{n_i}, \upsilon_{n_i}\big)\}$ be any subsequence of $\{(t_n, \upsilon_n)\}$ in which $(t_{n_i}, \upsilon_{n_i}) \to (t_0, \upsilon_0)$ as $i \to \infty$. One can have

$$0 = \liminf_{n} \lambda_{n_i} \left\| \vartheta - \upsilon_{n_i} \right\|$$

$$\geq \liminf_{n} \left[-f(t_{n_i}, \upsilon_{n_i}, \eta(\vartheta, \upsilon_{n_i}))\right] \tag{4.42}$$

$$\geq \liminf_{n} \left[f(t_{n_i}, \vartheta, \eta(\upsilon_{n_i}, \vartheta)) - \alpha(\eta(\vartheta, \upsilon_{n_i})) - \beta(\vartheta, \upsilon_{n_i})\right]$$

$$\geq f(t_0, \vartheta, \eta(\upsilon_0, \vartheta) - \alpha\big(\eta(\vartheta, \upsilon_0)\big) - \beta(\vartheta, \upsilon_0).$$

Applying Theorem 4.3 implies that

$$f\big(t_0, \upsilon_0, \eta(\vartheta, \upsilon_0)\big) \geq 0, \quad \forall \vartheta \in K. \tag{4.43}$$

It follows from 4.39 and l.s.c of h that

$$\lim_{e\in T, y\in M(e)} \inf\, h(e,y) \geq \lim_{n} \sup\, h\left(t_{n_i}, a_{n_i}\right)$$

$$\geq \lim_{n} \inf\, h\left(t_{n_i}, a_{n_i}\right) \tag{4.44}$$

$$\geq h(t_0, a_0).$$

From 4.43 and 4.44, (t_0, a_0) solves (OPIPEC). Due to the uniqueness of the solution, one can get $(t_0, a_0) = (t^*, a^*)$. Hence, (t_n, a_n) converges to (t^*, a^*). Therefore, (OPIPEC) is strong (w-p).

5. CONCLUSIONS AND RECOMMENDATION

5.1 Conclusions

As stated in the introduction, the aim of the study is to present results on an EP that includes pseudomonotone bi-fun. We have given a new, more general definition of **α**-**β**-**η**-pseudomonotone bi-fun where new results have been found with the application of KKM theory. Moreover, we have found some applications related to our theorems and our main results in this paper about HVI and and well-posed of IEP with metric characterization and (w-p) for optimization issue with invex parametric.

5.2 Recommendation

1. Finding the new outcomes by new types of pseudomonotone involving the KKM mapping or fixed point theorem.

2. Appling the existence and uniqueness of the weak solution to the (w-p) in Sobolev space.

Using well-known fixed point theory, one can applying of for **α**-**β**-**η**-pseudo monotone bi-fun operator in partial differential equations.

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