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Some Results Of Trıfunctıon Equılıbrıum Problems

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SOME RESULTS OF TRIFUNCTION EQUILIBRIUM PROBLEMS

By Salih Fakhri Khalaf KHALAF

February 2023

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science

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ABSTRACT

SOME RESULTS OF TRIFUNCTION EQUILIBRIUM PROBLEM

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Master of Science in Mathematics

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The objective of this thesis is to investigate in a rigorous manner a number of equilibrium problems resulting from applied sciences. Put differently, the thesis is the study of equilibrium problems in Banach space and reflexive Banach space with the aim of obtaining results on the existence of solutions of equilibrium problems in the case of convex and bounded subsets by using a KKM mapping, compactness and convex function. To do so, the thesis is composed of five chapters. Chapter two recalls a family of essential definitions along with some Theorems. The aim of this chapter is to assist the reader to get acquainted with main results grounded thereof. Chapter three is planned to found novel results of existence and uniqueness for $GEp\,(T,\vartheta_1,\vartheta_2)$ generalizing a new class of trifunction equilibrium problems, with a novel type of monotonicity in Banach space. Chapter four is intended to prove that the issues of well-posedness for $GEp_t(T,\vartheta_1,\vartheta_2)$ which is corresponding to the existence and uniqueness of its solution. Among the important results that we obtained, under specific conditions, we were able to prove that if $GEp_t(T,\vartheta_1,\vartheta_2)$ is WP. Then, $\Xi(c) \neq \emptyset$ and $lim\ _{c\to 0} diam\,[\Xi(c)] = 0,$ where known in the same chapter. The hoped-for results in this thesis is aimed to renew and improve some corresponding solutions of several researchers.

2023, 64 pages

Keywords: Equilibrium problems, Solution existence, Well-posedness, KKM mapping, Monotone bifunction

ÖZET

ÜÇ İŞLEVLİ DENGE PROBLEMİNİN BAZI SONUÇLARI

Salih Fakhri Khalaf KHALAF
Matematik, Yüksek Lisans
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Bu tezin amacı uygulamalı bilimlerden gelen çok sayıda denge problemini incelemektir. Bir KKM dönüşümü, kompaklık ve konveks fonksiyonları kullanarak; konveks ve sınırlı alt kümelerde denge problemlerinin çözümünün varlığı uzerinde farklı bir şekilde yansıyan Banach uzaylarında denge problemlerini çalışmaktır. Bunu yapmak için tez beş bölüme ayrılmıştır. İkinci ünite bazı teoremlerle birlikte ana tanımları içerir. Bu ünitenin amacı okuyucuya gerekli alt yapıyı sağlamaktır. Üçüncü ünite, Banach uzaylarında trifunction denge problemlerinin yeni bir sınıfını genelleyen $GEp(T, \vartheta_1, \vartheta_2)$ için varlık ve teklik sonuçları için verilmiştir. Dördüncü ünite de çözümünün teklik ve varlıgı için verilen $p_t(T, \vartheta_1, \vartheta_2)$ için iyi durumluluk konusunu ispatlamak istenmiştir. Bazı özel şartlar altında elde ettiğimiz önemli sonuçlar $p_t(T, \vartheta_1, \vartheta_2)$ iyi durumluysa sağlanır. O zaman bazı ünitelerde elde ettiğimiz gibi $\Gamma(c) \neq \emptyset$ ve $diam\,[\Gamma(c)] = 0,$ sonuçları vardır. Elde edilen sonuçların araştırmacılar için yararlı olacağını düşünüyoruz.

2023, 64 sayfa

Anahtar Kelimeler: Denge problemleri, Çözüm varlığı, iyi durumluluk, KKM dönüşümü, Monoton ikilifonksiyon

PREFACE AND ACKNOWLEDGEMENTS

Praise be to God, who has bestowed upon us the gift of wisdom, and prayers and peace be upon our master Muhammad and his pure family, his chosen companions, and those who followed them in charity until the Day of Judgment.

As I put the finishing touches on this thesis, I would like to express my gratitude and appreciation to the supervising professor (Prof. Dr. Faruk POLAT) for giving me his time and suggesting the thesis topic, as well as for his valuable guidance and sound opinions, which greatly enriched the research and revealed it for what it is. Also, o express my appreciation and gratitude to my virtuous supervisor (Co-Advisor: Asst. Prof. Dr. Ayed Eleiwis HASHOOSH), since he has a prominent role, insightful opinions, and thoughtful touches on each page of my thesis. I convey my profound appreciation to the staff of the Mathematics Department, particularly the Department Head. We remember everyone who contributed to the success of the message along with me, especially my mother. In conclusion, I would want to thank everyone who assisted me in completing this research.

Salih Fakhri Khalaf KHALAF
Çankırı-2023

CONTENTS

LIST OF SYMBOLS

R	The set of real numbers
N	The set of natural numbers
Q	The set of rational numbers
W^c	The complement of W
$\in$	Belongs to
$\notin$	Does not belong to

LIST OF ABBREVIATIONS

Ap. Seq	Approximating sequence.
SS(t)	Solution set.
et al.	And others.
w.r.t	With respect to.
usc	Upper Semi Continuous
lsc	Lower Semi Continuous
H.C	Hemicontinuous
IPS	Inner product space
WP	Well-posed
WP(G)	Well-posed in the generalized sense
et al.	And others
T.V.S	Topological vector space
H.T.S	Hausdorff topological space
iff	If, and only if

1. INTRODUCTION

Equilibrium issues are a relatively contemporary mathematical field. This enables us to research and solve a broad range of problems originating in various scientific domains, including but not limited to biology, physics, economics, chemistry, engineering, mechanics, and other applied sciences. One of the foundations for its growth was the introduction of (Muu and Oettli 1992).

Moreover, Study of tri-function of equilibrium issues has been an motivating field of scientific research as it comprises and inter links different basic areas variational inequality problems and its generalizations, such as minimax problems, quasivariational, optimization problems variational-like inequalities and Game theory.

1.1 Background

The problems of equilibrium bases forward by the two famous scientists (Blum and Oettli 1994) have recently seen wide use in different sciences, whether pure or applied sciences. The equilibrium problem has many mathematical problems that are special cases of it, for example, complementary problems, mathematical programming problems, fixed point problems, variational problems see (Krause and Zorin-Kranich 2018), (Ding 1999), (Thong et al. 2022) and (Wang and Teo 2022).

Devised from the concept of the vector variational inequality problem presented by the scientist (Giannessi 1980). They are defined vector equilibrium problem as follows: given a closed convex subset K of a real topological Hausdorff vector space and a bifunction $F(x, y)$ valued in a real ordered locally convex vector space, find $x \in K$ in which $F(x, y) \geq 0$ for all $y \in K$. M. Bianchi et al. established existence of results for the unifying model, involving that F is either a quasimonotone or pseudomonotone bifunction (Bianchi et al. 1997).

Hadjisavvas and Schaible used a uniform method to get existence results for quasi-monotone vector equilibrium problems and quasimonotone (multivalued) vector variational inequality problems from an existence result for a scalar equilibrium problem of two (rather than one) quasi-monotone bifunctions (Hadjisavvas and Schaible 1998). Wang et al. proposed a new type of extra gradient technique for the variational inequality problem (Wang et al. 2001). The technique uses a new inspecting way which differs from any one in existing projection-type methods, and is of a better step size rule. It is proved to be globally convergent under new condition of generalized monotonicity.

By employing technique KKM-Fan theorem (Li et al. 2003) introduced new a class which includes a number of (scalar) implicit equilibrium problems, implicit variational inequalities, and implicit complementarity problems as special cases in Hausdorff topological vector spaces.

(Lin and Yu 2005) studied the system of generalized vector quasi-equilibrium problems and they established the existence result for its solutions by means of the Kakutani–Fan–Glicksberg fixed points theorem. Also the existence results for weakly Pareto–Nash equilibrium points for multi-objective generalized game problems and multi-objective game problems are derived.

Recent research on equilibrium problems has garnered so much interest that it has been expanded to include the trifunction class of problems, one of the most current and general cases (EP). The instance of trifunction equilibrium difficulties is crucial. Hence, in 2007, a trifunction equilibrium problem was employed for the first time as a pure mathematical object in a study by (Preda et al. 2007). Most recent trifunction articles were published in 2020 by (Chadli et al. 2020).

In the same year, Ekeland-type variational principle was extended to a vectorial form of equilibrium (Ansari 2007) demonstrated similar findings to our variational principle. He proved the existence of solutions to the vector equilibrium problem in complete quasi-metric spaces.

Sach and Lee (2009) We present sufficient conditions for the local closedness and local openness properties as well as the lower (upper) semicontinuity characteristics of the solution sets of a general model that includes as special types a large number of generalized vector quasi-equilibrium problems with set-valued maps (Sach and Lee 2009). A model application that can be viewed as a system of generalized vector quasi-equilibrium issues is also provided.

An iterative method was devised by (Zegeye and Shahzad 2011) for the purpose of finding an element in the common fixed point set of a finite family of closed relatively quasi-nonexpansive mappings, common solutions of a finite family of equilibrium issues, and common solutions of a finite family of issues of variational inequality for monotone mappings in Banach spaces. The above process was developed for the purpose of locating an element in the common fixed point set of a finite family of closed relatively. This is because our theorem has a more general scope.

In establishing logarithmic potential theory, (Hardy and Kuijlaars 2012) showed lower semi-continuity and strict convexity of the energy functional for a large class of vector equilibrium issues.

In 2014, Chen outlined the saddle point of vector-valued mapping on a cone. For the vector equilibrium issue, he proved that it exists. Then, a theorem proving the existence of the saddle point of a ϵ -cone of vector-valued mappings is shown by using this existence result (Chen 2014).

On the other hand, Han and Huang introduced a secularization outcome as well as a density theorem that was concerned with the sets of weakly efficient and efficient approximate results to a generalized vector equilibrium problem. Both of these results were concerned with the sets of approximate solutions (Han and Huang 2016).

For the Lexicographic vector equilibrium issues, (Wangkeeree and Bantaojai 2017) presented the concepts of Levitin-Polyak (LP) well-posedness (WP) and LP well-posedness in the generalized sense. Following this, some adequate requirements for the LP well-posedness of lexicographic vector equilibrium problems at the reference point are established. Additionally, the conclusions he obtained were supported by a variety of situations.

Hashoosh and Jaber presented existence results for a general type of systems of nonlinear hemi-equilibrium problems by using a fixed point theorem involving statuses of bounded and unbounded closed convex subset in real reflexive Banach space (Hashoosh and Jaber 2020).

Finally, (Hung 2022) established novel results on the error bounds for a class of vector equilibrium problems by partial order provided via a polyhedral cone generated by some matrix.

1.2 Organization of the Thesis

Equilibrium problems is a critical innovative concept and brought about general tools, such unified, rigorous and innovative tools that permitted us to study and solve a wide spectrum of problems. These are problems that have originated from different areas of sciences.

The purpose of this thesis is to present new type of trifunction in equilibrium problems. By using KKM technique and according to relevant suppositions on the study of nonlinear mappings, we will prove the existence of a results for generalized equilibrium $GEp(\vartheta_1, \vartheta_2)$ involving class of monotone operator in the setting of real reflexive Banach space.

This thesis is composed of five chapters. The first chapter is devoted to the general introduction (background and Organization) to the thesis. This chapter gave a survey of the literature on equilibrium problems. Now we introduce the following chapters:

Chapter two presents a family of essential definitions along with some Theorems. The aim of this chapter is to assist the reader to get acquainted with main results grounded thereof. The chapter starts with a section on primary definitions of normed space before it ends up with a brief introduction to equilibrium problems. We can state all of the aims of this chapter at once by saying that we are trying to ground the investigation in some concepts of convex analysis required for the upcoming chapters. This being said, the chapter appeals to some concept derived from topological spaces and monotone trifunction. Finally, we mention fact concerning of WP for optimization problems.

Chapter three is divided into three sections including the introduction. Section two recalls relevant definitions that are important to our investigations. In section three, the existence of results for a new state of the trifunctional for equilibrium problem $GEp\ (T, \vartheta_1, \vartheta_2)$ on convex and closed sets either bounded or unbounded in Banach spaces by using a generalized type of monotonicity is called $\vartheta h-$ monotone are introduced.

Chapter four is intended to be an conclusive investigation in our thesis in which we provided the necessary definitions on which to base our main findings in this field. As for the third section, it defines and extends the concept of the WP to the problem $GEp\ (T, \vartheta_1, \vartheta_2)$. In the chapter's last section, we explore a new idea of WP in optimization issues by constraints provided with a parametric form of the problem of generalized equilibrium and the suitable WP.

Finally, chapter five contains some conclusions and Future work. The solutions given in this thesis improve some corresponding results of many scholars.

2. PRELIMINARIES AND BASIC RESULTS

2.1 Introduction

We review a brief series of fundamental concepts and preliminary results in this chapter. The purpose of these findings is to provide a foundation upon which to build the primary findings of our project, which will be presented in subsequent chapters. We begin with the most fundamental definition of Banach spaces and conclude with a short introduction to the concept of WP for optimization problems with constraints. This allowed us to go into the concepts of convex analysis, equilibrium, and monotone operators.

2.2 Some Key Concepts of B$\boldsymbol{S}$

Definition 2.1 (Sutherland 2009) A function $d : X \times X \rightarrow R$ is named metric on A nonempty set X . If the next axioms are verified:

i) $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$;

ii) $d(x, y) = d(y, x)$;

iii) $d(x, y) \leq d(x, z) + d(z, y)$.

for every $x, y, z \in X$.

The set X which equipped by metric is called metric space, and abbreviated as (X, d).

Definition 2.2 (Sutherland 2009) Assume (X, d) is a metric space, then the an open and closed ball define as following:

i) $B(x_0; \varepsilon) = \{x \in X : d(x, x_0) < \varepsilon\}$ is called an open ball;

ii) $\underline{B}(x_0; \varepsilon) = \{x \in X : d(x, x_0) \leq \varepsilon\}$ is called a closed ball.

where a point $x_0 \in X$ is called the center and a real number $\varepsilon > 0$ is radius.

Definition 2.3. (Sutherland 2009) A sequence $\{x_n\}$ is bounded set if, there is a positive real number M in which $|x_n| \leq M$ for all $n \in N$.

Definition 2.4. (Kreyszig 1991) Let $\|.\| : V \to [0, \infty)$ be a real-valued function on a linear space V over the field R. Then $\|.\|$ is titled a norm if, the next axioms hold:

i) $\|u\| \geq 0$, and $\|u\| = 0$, iff $u = 0$; for all $u \in V$;

ii) $\|\alpha u\| = |\alpha| \|u\|$ for all $u \in V$ and $\alpha \in R$;

iii) $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in V$.

Example 2.5. It can easily be shown that any normed vector space is a metric space with $d(x, y) = \|x - y\|$.

Definition 2.6. (Kreyszig 1991) Let A: $U \to V$ be an operator, where U and V are normed spaces. Then, A is named continuous operator at $u_0 \in U$, if for $all\ \epsilon > 0$, there is a $\delta = \delta(\epsilon, u_0)$ and if $\|u - u_0\| < \delta$, then $\|A_U - A_{u_0}\| < \epsilon$.

Definition 2.7. (Kreyszig 1991) A sequence $\{U_n\}_{n=1}^{\infty}$ in a normed space $(X, \|.\|)$ Is said to be

i) Converges to a point $u \in X$, if and only if, for every real number $\epsilon > 0$, there is an integer N in which $\|U_n - u\| < \epsilon$, $n > N$.

ii) Cauchy sequence in X, if and only if, for every real number $\epsilon > 0$, there is an integer N in which $\|U_n - U_m\| < \epsilon,\ n, m > N$.

Definition 2.8. **(**Kreyszig 1991) A normed vector space $(X, \|.\|)$ is said to be a Banach space, if every Cauchy sequence in X is convergent in X.

Example 2.9. Let $\|.\|: X \to R$ be a function in which $\|\varkappa\| = |\varkappa\ |$ for all $\varkappa \in X$ in which $X = F$

i) Let $\varkappa \in X$ then $\| \varkappa \| = 0$ iff $|\varkappa| = 0$ iff $\varkappa = 0$;

ii) Let $\varkappa \in X$ and $\vartheta \in F$ then, $\|\vartheta\varkappa\| = |\vartheta\varkappa| = |\vartheta||\varkappa|$;

iii) Let $x, y \in X$, then

$$\|\varkappa + y\| = |\varkappa + y| \leq |\varkappa| + |y| = \|\varkappa\| + \|y\|.$$

So, X is a normed space, since R is complete space, then X is complete space. Hence, X is a Banach space.

Definition 2.10. (Kreyszig 1991; page 241) A normed space X is said to be reflexive if Range $C = X''$ in which $C: X \longrightarrow X''$ is the canonical mapping.

An essential theory concerning this space states that all normed spaces of finite dimensions are always reflexive (Kreyszig 1991; page 242).

Now, we present definition of inner product space and some relevant preliminary results from it.

Definition 2.11. (Kreyszig 1991) Consider that $\langle .,. \rangle: X \times X \to R$ is a function, where X= F is a vector space, then $\langle .,. \rangle$ is said to be an inner product on X, if the following axioms hold

i) $\langle \upsilon, \upsilon\rangle \geq 0,\ \forall \upsilon \in X$, while $\langle \upsilon, \upsilon\rangle = 0$ if, and only if $\upsilon = 0$.

ii) $\underline{\langle \upsilon, v\rangle} = \langle v, \upsilon\rangle$; where $\underline{\langle \upsilon, v\rangle}$ is implies conjugate to $\langle \upsilon, v\rangle$;

iii) $\langle \alpha \upsilon + \vartheta v, z\rangle = \alpha\langle \upsilon, z\rangle + \vartheta\langle v, z\rangle$ $, \forall \upsilon, v \in X\, ; \alpha, \vartheta \in X.$,

As a result, the set $(X, \langle .\,, .\rangle)$ is an example of an inner product space.

Proposition 2.12 (Kreyszig 1991) Let $(X, \langle .\,, .\rangle)$ be inner product space, then

i) $\langle 0, \upsilon\rangle = \langle \upsilon, 0\rangle = 0$;

ii) $\langle \upsilon, \alpha v + \vartheta z\rangle = \underline{\alpha}\langle \upsilon, v\rangle + \underline{\vartheta}\langle \upsilon, z\rangle$ $, \forall \upsilon, v, z \in X\, ; \alpha, \vartheta \in F.$

2.3 Some Key Concepts of Topological spaces and Convex Analysis

This section aim to recall the collection of the related ideas from topological spaces and Convex Analysis.

Definition 2.13. (Köthe 1983) A topological space V is titled Hausdorff topological space if for any a and ρ in V, then there is two open set, A and B in V in which $a \in A$ and $\rho \in B$, in which $A \cap B = \emptyset$.

Definition 2.14. (Köthe 1983) The topology on V induces via norm $\|.\|$ on a Banach space V is said to be norm topology or strong topology.

Definition 2.15. (Bogachev et al. 2017) Suppose that V is a B$\boldsymbol{S}$ and $\xi \subseteq V$. Then if ξ is closed in the weak topology of V, then we say ξ is weakly closed.

Definition 2.16. (Kassay and Kolumbán 1966) Let T be a real-valued function defined on convex subset K of S, where $t \in (0,1)$. Then A is called to be hemi-continuous (H.C), if

$$T(tu + (1-t)v) = T(v) \quad \forall\, u, v \in K.$$

Definition 2.17. (Kreyszig 1991) A mapping ω from a B$\boldsymbol{S}$ X into a real numbers R is called to be

i) Lower semi-continuous (or l.s.c. for short) at $x_0 \in X$ if

$$\omega(x_0) \leq \lim_n inf\, \omega(x_n).$$

ii) Upper semi-continuous (or u.s.c. for short) at $x_0 \in X$ if

$$\omega(x_0) \geq \lim_n \sup \omega(x_n).$$

For any sequence x_n of x in which $x_n \to x_0$.

Definition 2.18. (Kuratowski 1966) Assume that the nonempty set N and M of a B$\boldsymbol{S}$ X. The, a Hausdorff metric that lies in between N and M and is denoted $\hbar(\cdot\,.\cdot)$, and its definition is as shown in:

$\hbar(N, M) = max\{ð\,(N, M), ð\,(M, N)\}$,

In which,

$$ð\,(N, M) = {}^{Sup}_{a\in N} d(a, M) \;\; with\; d(a, M) = {}^{Inf}_{\rho\in M} \| a - \rho \|.$$

Note that $\{N_n\}$ of nonempty is a sequence subset of S. The sequence N_n is convergent to N by Hausdorff metric if $\hbar(N_n\,, N)$ converges to zero. Then, ð ($N_n\,, N$) → 0, if and only if $d(\,a_n\,, N\,) \to\; 0\,, \forall\, a_n \in\; N.$

Definition 2.19. (Kuratowski 1966) The measure of noncompactness Ʊ of a nonempty subset A which is defined as following

$$\mho(\mathcal{A}) = \inf\{\epsilon > 0 : \mathcal{A} \subseteq \cup_{j=1}^{n} \mathcal{A}_j, \text{diam } (\mathcal{A}_j) < \epsilon, \quad j = \overline{1,n}\}$$

in which diam A_j know that the diameter of a set A_j.

Definition 2.20. (Fenchel and Blackett 1953) A nonempty subset K of a B$\boldsymbol{S}$ X , for $\lambda \in [0,1]$. Then

i) K is called convex set if $\lambda u + (1-\lambda)v \in K$.

ii) $F: K \to X$ is named to be convex function if

$$F\big(u + \lambda(v-u)\big) \leq (1-\lambda)F(u) + \lambda F(v) \quad \forall u, v \in K, \lambda \in [0,1].$$

Remark 2.21. A nonempty subset K of a B$\boldsymbol{S}$ X , for $\lambda \in [0,1]$. Then $F: K \to X$ is said to be

i) F concave function if the function $-F$ is convex.

ii) F is affine when ever it is both convex and concave.

iii) Intersection of two convex sets is convex.

iv) Sum of two convex sets is convex.

Definition 2.22. (Fenchel and Blackett 1953) Considering A be a nonempty set of $B\boldsymbol{S}$ X , then small convex set in S contain A is said to be convex hull is denoted by conv (A) .

Theorem 2.23. (Fenchel and Blackett 1953) A nonempty subset K of a B$\boldsymbol{S}$ X. Then

i) $A \subseteq$ conv(A) ;

ii) conv(A) = intersection of all convex sets that contain A ;

iii) If A convex set $\Leftrightarrow$ A =conv(A) .

Definition 2.24. (Fenchel and Blackett 1953) Suppose that A is a nonempty and convex set if for each $u, v \in A$, $\lambda \in [0,1]$, let $(1-\lambda)u + v$ be linear combination belongs to A. Then combination is titled convex combination of u and v. In commonalty; for each $u_1, \dots, u_n \in A$, $\lambda_i \geq 0$ in which $\sum_{i=1}^{n} \lambda_i = 1$, the combination $\sum_{i=1}^{n} \lambda_i u_i \in \mathcal{A}$ is said to be convex combination.

In the following, we will recalled the topic of vector 0-diagonally convexity (abbreviated as, 0-diag), whether it is in a bifunction or trifunction.

Definition 2.25. (Chadli et al. 2002) A bifunction $\vartheta: A \times A \to R$ is namely 0-diag if, for any finite subset $\{a_1, a_2 \dots . a_n\}\, of\ A$ and $\alpha_i \geq 0, i = 1,2 \dots n$ accompanied by $a = \sum_{i=1}^{n} \alpha_i a_i$ and $\sum_{i=1}^{n} \alpha_i = 1$ one can have

$$\sum_{i=1}^{n} \alpha_i \vartheta(a, a_i) \geq 0.$$

Definition 2.26. (Chiang et al. 2004). A tri-function $\vartheta: A \times A \times A \longrightarrow R$ is namely 0-diag if for each finite sets $\{a_1, a_2, \dots, a_r\}$ and $\{\rho_1, \rho_2, \dots, \rho_r\}$ of set A, $\alpha_i \geq 0$, $i \in \{1 \dots r\}$ such that $b_0 = \sum_{i=1}^{r} \alpha_i b_i$ and $\sum_{i=1}^{r} \alpha_i = 1$, one can have

$$\sum_{i=1}^{r} \alpha_i \vartheta(a_i, b_0, b_i) \geq 0.$$

Lemma 2.27. (Murphy 2008) Suppose X and Y be two topological spaces. If $B: X \to 2^Y$ is upper semi-continuous $(u.s.c)$ of closed values. Then, B is closed.

Theorem 2.28. (Fan 1984) Let K be a nonempty set in a Housdorff vector topological space X, and the set valued mapping $F: K \to 2^X$ be a KKM map in which

i) $F(x)$ is closed for all $x \in K$;

ii) $F(x)$ is compact for some $x \in K$.

Then, $\cap \{F(x) : x \in \kappa\} \neq \emptyset$.

Definition 2.29. (Knaster et al. 1929) Consider that K be a nonempty set of a topological vector space X. Then, A mapping $T : K \to 2^X$ is titled KKM- Theorem, if for every nonempty finite subset $\{\sigma_1, \sigma_2, \cdots, \sigma_n\} \subset K$, one has $co\{\sigma_1, \sigma_2 \cdots, \sigma_n\} \subset \cup_{i=1}^{n} T(\sigma_i)$.

Lemma 2.30. (Kim et al. 2018) Suppose that X is a Hausdorff topological space, $\{B_1, B_2, \dots, B_r\}$ be nonempty closed, convex and compact sets of X. Then $conv \cup_{i=1}^{r} G(B_i)$ is compact set.

2.4 Key Concepts of Equilibrium Issues and Monotone Bifunction

In this part, we'll discuss some of the issues that have been highlighted in previous research on equilibrium issues; it's important to keep in mind that most research on equilibrium problems has been conducted on convex and closed sets in B$\boldsymbol{S}$, whether the sets are bounded or unbounded.

Consider that K be a nonempty subset of B$\boldsymbol{S}$ V in which h: $K \times K \longrightarrow R$ be real-valued bi-function and $I: K \times K \times K \longrightarrow R$ real-valued trifunction.

Find $\rho \in K$

$$h(\rho, e) - h(\rho, \rho) + I(a, \rho, e) \geq 0 \quad \forall a, e \in K. \tag{2.1}$$

This kind of issues is called mixed equilibrium problems of tri-function which was introduced by (Mahato and Nahak 2014) and we abbreviated as $Ep(h, I)$.

Below are some of the special cases of formula (2.1) that we need to remember:

i) If $h = 0$ and $\rho \equiv e$ then, one can note the formula $Ep(h, I)$ is minimizes to the trifunction equilibrium issue given via Preda et al which is to find $\rho \in K$ in which $I(a, \rho, \rho) \geq 0 \quad \forall a \in K,$ see (Noor and Oettli 1994).

ii) If $I = 0$ and $h(\rho, \rho) = 0$ then one can note the formula $Ep(h, I)$ is minimizes to the standard equilibrium issue which is to find $\rho \in K$ in which $L(\rho, e) \geq 0 \quad \forall a \in K$ (Briefly, EP) see (Blum and Oettli 1994).

iii) If $h \equiv 0$ and $I(a, \rho, e) =< Te, a - \rho >$ where $T: K \rightarrow V^*$ then one can note the formula $Ep(h, I)$ is minimizes to a problem of variational inequality as presented by find $\rho \in K$ in which $< Te, a - \rho > \forall a \in K,$ can be seen in (Noor and Oettli 1994).

iv) If $h(\rho, e) - h(\rho, \rho) = \vartheta(\rho, e)$ then, one can note the formula $Ep(h, I)$ is minimizes to the trifunction of generalized equilibrium issue given by (Abed et al. 2021) which is to find $\rho \in K$ in which $\vartheta(\rho, e) + I(a, \rho, e) \geq 0 \quad \forall a, e \in K$ and abbreviated as $Ep(\vartheta, I)$.

v) If $I(a, \rho, e) = \vartheta_2(e) - \vartheta_2(\rho)$ and $h(\rho, \rho) = 0$ then one can note the formula $Ep(L, I)$ is minimizes to the problem mixed-equilibrium (MEP), (Mahato and Nahak 2013), which is to find $\rho \in K$ in which

$$\vartheta_1(\rho, e) + I(e) - I(\rho) \geq 0 \quad \forall a \in K.$$

In this following, we recalled a novel class of monotonicity is called ϑh −monotone and defined as:

Definition 2.31. (Abed et al. 2021) Let $I: M \times M \times M \rightarrow R$ be a tri-function. Then is called ϑh −monotone if

$$I(a, \rho, e) + I(a, e, \rho) + h(\rho, e) \leq 0 \quad \forall a, \rho, e \in M.$$

Which includes many special cases under applying appropriate hypotheses.

In the following, we will recall some of the theorems and findings that related to our main results in later chapters.

Theorem 2.32. (Abed et al. 2021) Assume that $I: M \times M \times M \to R$ is ϑh–monotone trifunction H.C in the 2-term, convex in 3-term, h, $\vartheta: M \times M \to R$ be a convex in the 2-term, then the equilibrium issue $Ep(\vartheta, I)$ is equivalent to the following issue:

$$I(a, e, \rho) + h(\rho, e) \leq \vartheta(\rho, e).$$

2.5 Some Concepts of WP

The main objective of this section is to mention some results and contributions to the development of some solutions to equilibrium problems. In particular, we study the concepts of WP by parameter for a novel type of equilibrium problem $Ep(\vartheta, I)$ and for optimal problems with perturbations that include the standard equilibrium problem as a special case.

Below the formulation of optimization issues with equilibrium constraints are recalled. Assume that $F: T \times K \times K \to R$ be a trifunction and $H: T \times K \to R$ be a function in which $T \subset E$ that isn't empty. Hence, we may formulate the optimization problem subject to a generalized equilibrium constraint (OPGPEC) as follows:

(t, u) in which $(t, u) \in T \times K$ and $u \in SS(t)$,

We would like to point out that the symbol $SS(t)$ denotes the collection of solutions to the parametric generalized equilibrium issue

$Ep_t(\vartheta, I)$ defined by $u \in SS(t)$ if and only if defined by $\rho \in SS(t)$ if

$$\vartheta(t,\rho,e) + I(a,\rho,e) \geq 0 \quad \forall\, a,e \in K.$$

Definition 2.33. (Abed and Hashoosh 2021) Assume that $\{(t_n,x_n)\} \subset T \times K$ is a sequence. Then, it is said to be an Ap. seq for $Ep(\vartheta,I)$ if there is a sequence $\{\epsilon_n\} \geq 0$ where $\epsilon_n \to 0$ as $\to \infty$, $n \in N$, for any $y \in K$ one has

$$\vartheta(t,\rho_n,e) + I(a,\rho_n,e) + c_n \parallel \rho_n - e \parallel \quad \forall n \in N, a,e \in K.$$

Definition 2.34.(Abed and Hashoosh 2021) The problem $Ep(\vartheta,I)$ is called to be strongly (or weakly) WP (or strongly (or weakly) WP in the generalized-sense) if $Ep(\vartheta,I)$ has only one solution x, and any sequence $\{x_n\}$ via $x_n \to x$, any Ap. seq for $Ep(\vartheta,I)$ converges strongly (or weakly) to only one solution (resp if $Ep(\vartheta,I)$ has a nonempty solution set $SS(t)$ in addition for any approximating solution sequence has a subsequence that strongly (or weakly) converges to some point of $SS(t)$).

Here we shall discuss two of the author's identified sets (Abed and Hashoosh 2021) which were utilized in the authors' justification of their conclusions of WP for $Ep(\vartheta,I)$. For any $\epsilon > 0$ we recall two sets:

$$\Xi(\epsilon) := \{(t,x) \in T \times K : \vartheta(t,\rho_n,e) + I(a,\rho_n,e) + c \parallel \rho - e \parallel \geq 0 \text{ for all } y \in K\}.$$

And

$$\Lambda(\epsilon) := \{(t,x) \in T \times K : I(a,e,\rho) + h(\rho,e) \leq \vartheta(t,\rho,e) + c \parallel \rho - e \parallel \text{ for all } y \in K\}.$$

Lemma 2.35. (Abed and Hashoosh 2021) Consider that K is a nonempty convex, closed subset of a $B\boldsymbol{S}$ X. Consider $\vartheta: T \times K \times K \to R$, $I: K \times K \times K \to R$ and $h: K \times K \to R$ that be three function. Providing the conditions below are met

i. $\vartheta(t,\rho,\rho) = 0 \ \forall\, t \in T, \ \forall\, \rho \in K,$

ii. $I(a, \rho, e)$ is ϑh-monotone trifunction and H.C,

iii. $\vartheta(t, \rho, .)$ is convex $t \in T, \rho \in K$,

iv. $I(a, \rho, .)$ is convex $\forall a, \rho \in K$.

Consequently, $\Xi(\epsilon) = \Lambda(\epsilon)$ for all $\epsilon > 0$.

Lemma 2.36. (Abed and Hashoosh 2021) Let K be a nonempty convex closed subset of B$\boldsymbol{S}$ V. Consider that $\vartheta: T \times K \times K \to R$ and $: K \times K \times K \longrightarrow R$. Providing the conditions below are met

i. $\vartheta(.,.,e)$ is u.s.c forall $e \in K$,

ii. $I(a, e, .)$ and $h(., e)$ are $l.s.c$ forall $a, e \in K$.

Consequently, for any $\epsilon > 0$, $\Lambda(\epsilon)$ is closed in $T \times K$.

3. EXISTENCE RESULTS FOR SOME TRI-EQUILIBRIUM PROBLEMS

3.1 Introduction

The purpose of this chapter is to demonstrate the existence and uniqueness of the solution to (EP) by generalizing a new category of trifunction equilibrium problems in BS. In addition, we examine a new example of the trifunction labeled as $GEp\ (T, \vartheta_1, \vartheta_2)$ in order to discover additional conditions under which the equilibrium state holds using h- monotone subsets, namely convex and closed subsets (bounded or unbounded). We demonstrate the existence of a solution to (EP), therefore enhancing the literature's most recent properties.

To achieve the aforementioned purpose, the chapter is divided into three pieces. The second section reviews definitions that are pertinent to the investigation. Section three introduces h- monotone, a new type of monotonicity. We seek to establish the existence and uniqueness of $GEp(T, \vartheta_1, \vartheta_2)$ by the end of the chapter.

3.2 Preliminaries

Let M be a nonempty subset of real reflexive B$\boldsymbol{S}$ V.and $\vartheta_1: M \times M \longrightarrow R$ be two real-valued bifunctions $\vartheta_2: M \times\ M \times M \longrightarrow R$ is real-valued trifunction and $T: M \to V^*$ be arbitrary nonlinear mapping. By the following generalized equilibrium abbreviated as $GEp(T, \vartheta_1, \vartheta_2)$ we understand the issue of understand the issue of finding $\rho \in\ M$ in which

$$\langle T\rho, \rho - e\rangle + \vartheta_1(\rho, e) + \vartheta_2(a, \rho, e) \geq 0 \quad \forall a, e \in M. \tag{3.1}$$

It is interesting to mention below some special cases:

i. If $\vartheta_1 = 0, T = 0$ and $\rho \equiv e$ then issue $GEp(T, \vartheta_1, \vartheta_2)$ is minimizes to the trifunction equilibrium issue given by Preda et al which is to find $\rho \in M$ in which $\vartheta_2(a, \rho, \rho) \geq 0 \quad \forall a \in M$, see (Preda et al 2007).

ii. If $\vartheta_1(\rho, e) := h(\rho, e) - h(\rho, \rho)$ and $T = 0$, then issue $GEp(T, \vartheta_1, \vartheta_2)$ is minimizes to general form of mixed equilibrium issue which is to find $\rho \in M$ in which $\vartheta_2(a, \rho, e) + h(\rho, e) - h(\rho, \rho) \geq 0$, see (Chadli 2002).

iii. If $\vartheta_2 = 0$ and $T = 0$, then issue $GEp(T, \vartheta_1, \vartheta_2)$ is minimizes to the classical equilibrium issue which is to find $\rho \in M$ in which $\vartheta_1(\rho, e) \geq 0$, $\forall a \in M$; (Nikaidô and Isoda 1955).

iv. If $\vartheta_2(a, \rho, e) = \vartheta_2(e) - \vartheta_2(\rho)$ and $T = 0$, then the issue $GEp(T, \vartheta_1, \vartheta_2)$ is minimizes to the mixed equilibrium (MEP) see for instance (Moudafi (2002), which is to find $\rho \in M$ in which

$$\vartheta_1(\rho, e) + \vartheta_2(e) - \vartheta_2(\rho) \geq 0 \quad \text{for all } a \in M.$$

Here, considering the equilibrium trifunction $GEp(T, \vartheta_1, \vartheta_2)$ in which for all $a, \rho \in M$ thus, $\vartheta_2(a, \rho, \rho) = 0$.

Remark 3.1 Considering V is a B$\boldsymbol{S}$ and M be a nonempty-convex set of V. Let that $h, \beta: \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R} \cup \{+\infty\}$ be two bifunctions which include $\rho_\varepsilon = \varepsilon e + (1 - \varepsilon)\rho$, where $e, \rho \in M$, $\varepsilon \in [0.1]$. Therefore,

i. $\lim\limits_{\varepsilon \to 0} \frac{h(\rho, \rho_\varepsilon)}{\varepsilon} = 0;$

ii. $h(\rho, e) \leqslant \lim\limits_{\varepsilon \to 0} \frac{\varepsilon - 1}{\varepsilon} [\vartheta_1(\rho, \rho) + h(\rho, \rho)].$

3.3 Main Results on Tri- Equilibrium Problem

In the first part of this section, we give the valuable results of the Tri-equilibrium function in an effort to synthesize some results about the existence of an equilibrium issue $GEP(T, \vartheta_1, \vartheta_2)$ using the 0-diagonal convexity technique. Secondly from the results obtained, we can prove that there is a solution to the issue of $GEP(T, \vartheta_1, \vartheta_2)$, without assuming the boundness of a set M. Finally, we can get to the uniqueness of the solution.

The researchers describe their initial findings, which focus on equivalency issues with $GEP(T, \vartheta_1, \vartheta_2)$.

Theorem 3.2 Consider that $\vartheta_2: M \times M \times M \rightarrow R$ is ϑh–monotone trifunction H.C in the 2-term convex in 3-term, h, $\vartheta_1: M \times M \rightarrow R$ be convex in the 2-term, then the equilibrium issue $GEP(T, \vartheta_1, \vartheta_2)$ is equivalent to the issue in the following equation:

$$\vartheta_2(a, e, \rho) + h(\rho, e) \leq \langle T\rho, \rho - e\rangle + \vartheta_1(\rho, e). \tag{3.2}$$

Proof: Suppose that $GEP(T, \vartheta_1, \vartheta_2)$ has a solution $\rho \in M$ in which

$$\langle T\rho, \rho - e\rangle + \vartheta_1(\rho, e) + \vartheta_2(a, \rho, e) \geq 0 \quad \forall a, e \in M.$$

Since ϑ_2 is ϑh-monotone trifunction, then

$$\vartheta_2(a, \rho, e) + \vartheta_2(a, e, \rho) + h(\rho, e) \leq 0 \quad \forall a, e \in M.$$

So,

$$\vartheta_2(a, e, \rho) + h(\rho, e) \leq -\vartheta_2(a, \rho, e).$$

Notice that, from $GEP(T, \vartheta_1, \vartheta_2)$ it follows that

$$\vartheta_2(a, e, \rho) + h(\rho, e) \leq \vartheta_1(\rho, e) + \langle T\rho, \rho - e \rangle.$$

Therefore, $\rho \in M$ is a solution of issue (3.2).

Assume that $\rho \in A$ is a solution of problem (3.2) let $a, e \in M$, and $\rho_\varepsilon = \varepsilon e + (1 - \varepsilon)\rho$, $\varepsilon \in [0,1]$ then $\rho_\varepsilon \in M$ because M is convex set. Thus,

$$\vartheta_2(a, e, \rho) + h(\rho, e) \leq \vartheta_1(\rho, e) + \langle T\rho, \rho - e \rangle.$$

Then,

$$\vartheta_2(a, \rho_\varepsilon, \rho) + h(\rho, \rho_\varepsilon) \leq \vartheta_1(\rho, \rho_\varepsilon) + \langle T\rho, \rho - \rho_\varepsilon \rangle. \tag{3.3}$$

Since, $0 = \vartheta_2(a, \rho_r, \rho_r)$, then

$$0 = \vartheta_2(a, \rho_\varepsilon, \rho_\varepsilon)$$

$$\leq \varepsilon\vartheta_2(a, \rho_\varepsilon, e) + (1 - \varepsilon)\vartheta_2(a, \rho_\varepsilon, \rho),$$

It means that,

$$\varepsilon[\vartheta_2(a, \rho_\varepsilon, \rho) - \vartheta_2(a, \rho_\varepsilon, e)] \leq \vartheta_2(a, \rho_\varepsilon, \rho). \tag{3.4}$$

Via convexity of h, ϑ_1 in the 2-term, one can get

$$\vartheta_1(\rho, \rho_\varepsilon) \leq \varepsilon\vartheta_1(\rho, e) + (1 - \varepsilon)\vartheta_1(\rho, \rho).$$

Then,

$$\varepsilon[\vartheta_1(\rho,\rho) - \vartheta_1(\rho,e)] \leq \vartheta_1(\rho,\rho) - \vartheta_1(\rho,\rho_\varepsilon). \tag{3.5}$$

Also,

$$h(\rho,\rho_\varepsilon) \leq rh(\rho,e) + (1-\varepsilon)h(\rho,\rho).$$

So,

$$\varepsilon[h(\rho,\rho) - h(\rho,e)] \leq h(\rho,\rho) - h(\rho,\rho_\varepsilon). \tag{3.6}$$

As a result of equations (3.4), (3.5), (3.6), and (3.3), we find that

$$\varepsilon[\vartheta_2(a,\rho_\varepsilon,\rho) - \vartheta_2(a,\rho_\varepsilon,e) + \vartheta_1(\rho,\rho) - \vartheta_1(\rho,e) + h(\rho,\rho) \ - h(\rho,e)]$$

$$\leq \vartheta_2(a,\rho_\varepsilon,\rho) + \vartheta_1(\rho,\rho) - \vartheta_1(\rho,\rho_\varepsilon) \ + h(\rho,\rho) \ - h(\rho,\rho_\varepsilon) \tag{3.7}$$

$$\leq \varepsilon\langle T\rho, \rho - e\rangle - 2h(\rho,\rho_\varepsilon) + \vartheta_1(\rho,\rho) + h(\rho,\rho).$$

Since the hemicontinuouity of ϑ_2, we have

$$\varepsilon[-\vartheta_2(a,\rho,e) - \vartheta_1(\rho,e) - \langle T\rho, \rho - e\rangle]$$

$$\leq -2h(\rho,\rho_\varepsilon) + \varepsilon h(\rho,e) + (1-\varepsilon)[\vartheta_1(\rho,\rho) + h(\rho,\rho)].$$

divided by (-r), one can obtain

$$\vartheta_2(a,\rho,e) + \vartheta_1(\rho,e) + \langle T\rho, e - \rho\rangle$$
$$\geq \frac{2h(\rho,\rho_\varepsilon)}{\varepsilon} - h(\rho,e) - \frac{(1-\varepsilon)}{\varepsilon}[\vartheta_1(\rho,\rho) + h(\rho,\rho)].$$

From Remark (3.1) we observe that

$$\langle T\rho, \rho - e\rangle + \vartheta_1(\rho, e) + \vartheta_2(a, \rho, e) \geq 0 \quad \forall\, a, \rho \in M.$$

That is means b is a solution of problem $GEP(T, \vartheta_1, \vartheta_2)$.

Next, we prove the existence of the issue $GEP(T, \vartheta_1, \vartheta_2)$.

Theorem 3.3. Consider that M is a nonempty-convex set of the reflexive B$\boldsymbol{S}$ V that is closed and bounded, and also assume the following:

i. $\vartheta_1 : M \times M \to R$ is convex in 2- term, u.s.c in 1- term and 0-DC

ii. $\vartheta_2 : M \times M \times M \to R$ is ϑh-monotone, H.C in 2- term, 0-DC and (l.s.c) in 3- term.

iii. $h: M \times M \to R$ is (l.s.c) in 1- term and convex in 2- term.

iv. $\rho \longmapsto \langle T\rho, \rho - e\rangle$ u.s.c on M with respect weak*topology of V*.

Then, the problem $GEP(T, \vartheta_1, \vartheta_2)$, admits at least one solution.

Proof: Define two set -valued mappings $\Xi, S: M \to 2^V$ as follow, $\forall\, e \in M$

$$\Xi(e) = \bigcap_{a \in A} \{\rho \in \mathcal{M};\ \vartheta_1(\rho, e) + \vartheta_2(a, \rho, e) + \langle T\rho, \rho - e\rangle \geq 0 \}$$

$$S(e) = \bigcap_{a \in A} \{\rho \in \mathcal{M};\ \vartheta_2(a, e, \rho) + h(\rho, e) \leq \vartheta_1(\rho, e) + \langle T\rho, \rho - e\rangle \}.$$

In that case, the issue is solvable if and only if:

$$\bigcap_{e \in A} \Xi(e) \neq \emptyset.$$

Since $e \in \Xi(e) \Longrightarrow \Xi(e) \neq \emptyset$.

Now, we prove that $\Xi(e)$ is KKM-map. From the contrary. Let $\Xi(e)$ be not KKM, then there is a finite subset $\{\rho_1, \rho_2, \dots, \rho_n\}$ of a set M and $\alpha_i \geq 0 \quad i = 1,2, \dots, n, \ \sum_{i=1}^{n} \alpha_i = 1$, in which $\rho_0 = \sum_{i=1}^{n} \alpha_i \rho_i \notin \bigcup_{i=1}^{n} \Xi(\rho_i)$. So, $\rho_0 \notin \Xi(\rho_i) \quad \forall\, i = 1,2, \dots, n$, it means that,

$$\vartheta_1(\rho_0, \rho_i) + \vartheta_2(a_0, \rho_0, \rho_i) + \langle T\rho_0, \rho_0 - \rho_i \rangle < 0. \tag{3.8}$$

So,

$$\sum_{i=1}^{n} \alpha_i [\vartheta_1(\rho_0, \rho_i) + \vartheta_2(a_0, \rho_0, \rho_i) + \langle T\rho_0, \rho_0 - \rho_i \rangle < 0].$$

From the convexity of ϑ_1 and ϑ_2, this would contradict 0-DC. Then Ξ is KKM-mapping. Next, from Theorem (3.2) we have that $\Xi(e) \subseteq S(e)$ for all $e \in M$. Hence $S(e)$ is KKM. Since $\vartheta_2(a, \rho, .), h(., \rho)$ are (l.s.c) and $\vartheta_1(., \rho)$ and $\rho \longmapsto \langle T\rho, \rho - e \rangle$ is (u.s.c), then

$$\vartheta_2(a, e, \rho) + h(\rho, e) \leq \liminf_{n \to \infty} [\vartheta_2(a, e, \rho_n) + h(\rho_n, e)]$$

$$\leq \limsup_{n \to \infty} \vartheta_1(\rho_n, e) + \langle T\rho_n, \rho_n - e \rangle$$

$$\leq \vartheta_1(\rho, e) + \langle T\rho, \rho - e \rangle.$$

So,

$$\vartheta_2(a, e, \rho) + h(\rho, e) \leq \vartheta_1(\rho, e) + \langle T\rho, \rho - e \rangle.$$

Therefore $S(e)$ is weakly closed $\forall e \in M$, since M is a nonempty, bounded, closed and convex subset of the real reflexive B$\boldsymbol{\mathcal{S}}$ V, then A is a weakly compact and from Theorem (2.28) and Theorem (3.2), we get

$$\bigcap_{e \in A} \Xi(e) = \bigcap_{e \in A} \mathrm{S}(e) \neq \emptyset.$$

Then the problem $GEP(T, \vartheta_1, \vartheta_2)$ has a solution. And this completes the proof.

Here, we would like to point out that through the Theorem (3.3) we obtained this Corollary, but without the condition of compactness. Therefore, we are done with the proof.

Corollary 3.4. Assume that M is a nonempty subset convex of B$\boldsymbol{\mathcal{S}}$ V in which $0 \in M$. If the assumptions (i-iv) in Theorem (3.3). Moreover, the following assumptions hold

i. $\vartheta_1(\rho, 0) = 0$,

ii. $S_1 = \{\rho \in M; h(\rho, 0) \leq \langle T\rho, \rho \rangle \}$ and $S_1 = \{\rho \in M; \vartheta_2(a, 0, \rho) \leq 0\}$ are relative compact sets.

Then, $GEP(T, \vartheta_1, \vartheta_2)$ has a solution.

Proof: To demonstrate that $S(e)$ is compact for some $e \in M$ is sufficient.

$$\mathrm{S(e)} = \bigcap_{\mathrm{a} \in \mathcal{M}} \{\mathrm{b} \in \mathcal{M}; \vartheta_2(\mathrm{a, e, b}) + \mathrm{h(b, e)} \leq \vartheta_1(\mathrm{b, e}) + \langle \mathrm{Tb, b - e} \rangle \ \forall \mathrm{e} \in \mathcal{M}\}$$

$$= \bigcap_{\mathrm{a} \in \mathcal{M}} \left\{\mathrm{b} \in \mathcal{M}; \vartheta_2(\mathrm{a, e, b}) + \mathrm{h(b, e)} \leq \frac{\vartheta_1(\mathrm{b, e})}{2} + \frac{\vartheta_1(\mathrm{b, e})}{2} + \langle \mathrm{Tb, b - e} \rangle \right\}$$

Then,

$$S(e) \subseteq \bigcap_{a\in\mathcal{M}} \left\{b \in \mathcal{M}; \vartheta_2(a,e,b) \leq \frac{\vartheta_1(b,e)}{2}\right\} \cup \left\{b \in \mathcal{M}; h(b,e) \leq \frac{\vartheta_1(b,e)}{2} + \langle T\rho, \rho - e\rangle \right\}.$$

$$S(0) \subseteq \bigcap_{a\in A} \left\{b \in \mathcal{M}; \vartheta_2(a,0,b) \leq \frac{\vartheta_1(b,0)}{2}\right\} \cup \left\{b \in \mathcal{M}, h(b,0) \leq \frac{\vartheta_1(b,0)}{2} + \langle T\rho, \rho\rangle\right\}.$$

From conditions (i, ii), $\Xi(0)$ is subset of relative compact sets S_1 and S_2, then $\Xi(e)$ is a compact for some $e \in M$, closed for every $e \in M$ and by Theorem (2.28) we have

$$\bigcap_{e\in\mathcal{M}} \Xi(e) \neq \emptyset.$$

Then, $GEP(T, \vartheta_1, \vartheta_2)$ admits solution. Therefore, we are done with the proof.

We would like to point out that the next result is a development of the Theorem (3.3) after dispensing with the boundedness clause.

Theorem 3.5. Consider that the similar hypotheses that are used in Theorem (3.3) hold true even if M is not bounded. Furthermore, let us assume that the following presumptions are accurate:

i. $\vartheta_1(\rho, \rho) = 0,$

ii. For all $\rho_0 \in M$ such that $\vartheta_1(\rho, \rho_0) + \vartheta_2(a, \rho, \rho_0) + \langle T\rho, \rho - \rho_0\rangle < 0, \forall a \in M,$ where $\rho \in M$ and $\parallel \rho \parallel$ is large encugh in which $\parallel \rho \parallel \geqslant \parallel \rho_0 \parallel$.Then the issue $GEP(T, \vartheta_1, \vartheta_2)$ has a solution.

Proof. Take a set $M_n = \{e \in M;\ \parallel e \parallel \leq n, n > 0\}$ and cansider the issue is to find $\rho_n \in M_n$ in which

$$\vartheta_1(\rho_n, e) + \vartheta_2(a, \rho_n, e) + \langle T\rho_n, \rho_n - e\rangle \geq 0, \quad \forall a, e \in M_n.$$

From the boundedness of M_n and Theorem (3.3) we obtain that the issue $GEP(T, \vartheta_1, \vartheta_2)$ has at least one solution. For all $n > 0$ we will choose $\| \rho_0 \| < n_0$, then M_{n_0} has one solution ρ_{n_0} in which

$$\langle T\rho_{n_0}, \rho_{n_0} - \rho_0\rangle + \vartheta_1(\rho_{n_0}, \rho_0) + \vartheta_2(a, \rho_{n_0}, \rho_0) \geq 0 \qquad \forall a, e \in M_{n_0} \tag{3.8}$$

So, $\| \rho_{n_0} \| \leq n_0$ because $\rho_{n_0} \in M_{n_0}$ if $\| \rho_{n_0} \| = n_0$, that is ρ_{n_0} large enough, so by condition (ii), we have

$$\langle T\rho_{n_0}, \rho_{n_0} - \rho_0\rangle + \vartheta_1(\rho_{n_0}, \rho_0) + \vartheta_2(a, \rho_{n_0}, \rho_0) < 0 \quad \forall a \in M \tag{3.9}$$

Which is contradictions with (3.8). Then $\| \rho_{n_0} \| < n_0$. For each $e \in M$, one can take $\varepsilon \in (0,1)$, small enough in which $c = \varepsilon e + (1 - \varepsilon)\rho_{no} \in M_{n_0}$. Since M_{n_0} convex, then $\in M_{n_0}. From$ (3.8), for each $c \in M$, one can get

$$0 \leq \langle \mathrm{Tb_{n_0}}, \mathrm{b_{n_0}} - \mathrm{c}\rangle + \vartheta_1(\mathrm{b_{n_0}}, \mathrm{c}) + \vartheta_2(\mathrm{a}, \mathrm{b_{n_0}}, \mathrm{c})$$

$$= \langle \mathrm{Tb_{n_0}}, \mathrm{b_{n_0}} - (\mathrm{re} + (1 - \mathrm{r})\mathrm{b_{no}})\rangle + \vartheta_1(\mathrm{b_{n_0}}, \mathrm{re} + (1 - \mathrm{r})\mathrm{b_{no}})$$

$$+\vartheta_2(\mathrm{a}, \mathrm{b_{n_0}}, \mathrm{re} + (1 - \mathrm{r})\mathrm{b_{no}})$$

$$\leq \mathrm{r}\langle \mathrm{Tb_{n_0}}, \mathrm{b_{n_0}} - \mathrm{e}\rangle + \mathrm{r}\vartheta_1(\mathrm{b_{n_0}}, \mathrm{e}) + (1 - \mathrm{r})\vartheta_1(\mathrm{b_{n_0}}, \mathrm{b_{n_0}}) + \mathrm{r}\,\vartheta_2(\mathrm{a}, \mathrm{b_{n_0}}, \mathrm{e})$$

$$+(1 - \mathrm{r})\vartheta_2(\mathrm{a}, \mathrm{b_{n_0}}, \mathrm{b_{n_0}})$$

Dividing by r

$\langle \mathrm{Tb}_{\mathrm{n}_0}, \mathrm{b}_{\mathrm{n}_0} - e\rangle + \vartheta_1(\mathrm{b}_{\mathrm{n}_0}, \mathrm{e}) + \vartheta_2(\mathrm{a}, \mathrm{b}_{\mathrm{n}_0}, \mathrm{e}) \qquad \forall\, \mathrm{a}, \mathrm{e} \in \mathcal{M}.$

Then, ρ_{n_0} is a solution of $GE(\vartheta_1, \vartheta_2)$. Therefore, we are done with the proof.

Let's end this section with an introduction the uniqueness of the solution for the problem $GEP(T, \vartheta_1, \vartheta_2)$.

Theorem 3.5. If the same hypotheses in Theorem (3.3) hold. By adding:

i. ϑ_1 is a monotone bifunction,

ii. There is $\varepsilon > 0$, in which $\vartheta_2(a, \rho, e) \leq -\varepsilon\|\rho - e\|^2 \quad \forall a, \rho, e \in M$,

iii. There is $t > 0$, in which $\langle T\rho, \rho - e\rangle \leq -t\|\rho - e\|^2 \quad \forall a, \rho, e \in M$.

Therefore, there is only one solution to the problem $GEP(T, \vartheta_1, \vartheta_2)$.

Proof. The existence of solutions is implied by Theorem (3.3), so all that remains is to demonstrate the uniqueness of the solution. By assuming the contrary, that there is two solutions, say ρ_1 and ρ_2 for $GEP(T, \vartheta_1, \vartheta_2)$, so we have,

$$\vartheta_1(\rho_1, e) + \vartheta_2(a, \rho_1, e) + \langle T\rho_1, \rho_1 - e\rangle \geq 0 \qquad \forall a, e \in M$$

$$\vartheta_1(\rho_2, e) + \vartheta_2(a, \rho_2, e) + \langle T\rho_2, \rho_2 - e\rangle \geq 0 \qquad \forall a, e \in M.$$

Since $\rho_1, \rho_2 \in M$, we have,

$$\vartheta_1(\rho_1, \rho_2) + \vartheta_2(a, \rho_1, \rho_2) + \langle T\rho_1, \rho_1 - \rho_2\rangle \geq 0\,,$$

$$\vartheta_1(\rho_2, \rho_1) + \vartheta_2(a, \rho_2, \rho_1) + \langle T\rho_2, \rho_2 - \rho_1\rangle \geq 0.$$

Then,

$$\vartheta_1(\rho_1,\rho_2)+\vartheta_2(a,\rho_1,\rho_2)+\langle T\rho_1,\rho_1-\rho_2\rangle+\vartheta_1(\rho_2,\rho_1)+\vartheta_2(a,\rho_2,\rho_1)$$

$$+\langle T\rho_2,\rho_2-\rho_1\rangle\geq 0.$$

This is rewarding,

$$\vartheta_2(a,\rho_1,\rho_2)+\langle T\rho_1,\rho_1-\rho_2\rangle+\vartheta_2(a,\rho_2,\rho_1)+\langle T\rho_2,\rho_2-\rho_1\rangle$$

$$\geq\ -\vartheta_1(\rho_1,\rho_2)-\vartheta_1(\rho_2,\rho_1) \tag{3.10}$$

Since ϑ_1 is a monotone bifunction, we have

$$\vartheta_1(\rho_1,\rho_2)+\vartheta_1(\rho_2,\rho_1)\leq 0 \quad \forall\ \rho_1,\rho_2\in M,$$

$$-\vartheta_1(\rho_1,\rho_2)-\vartheta_1(\rho_2,\rho_1)\geq 0. \tag{3.11}$$

From (3.10) and (3.11) we have,

$$\vartheta_2(a,\rho_1,\rho_2)+\langle T\rho_1,\rho_1-\rho_2\rangle+\vartheta_2(a,\rho_2,\rho_1)+\langle T\rho_2,\rho_2-\rho_1\rangle\geq 0. \tag{3.12}$$

From condition (ii), one can get

$$\vartheta_2(a,\rho_1,\rho_2)\leq -\varepsilon\|\rho_1-\rho_2\|^2,$$

$$\vartheta_2(a,\rho_2,\rho_1)\leq -\varepsilon\|\rho_2-\rho_1\|^2,$$

So,

$$\vartheta_2(a, \rho_1, \rho_2) + \vartheta_2(a, \rho_2, \rho_1) \leq -2\varepsilon\|\rho_2 - \rho_1\|^2. \tag{3.13}$$

Also,

$$\langle T\rho_1, \rho_1 - \rho_2\rangle \leq -t\|\rho_1 - \rho_2\|^2.$$

And

$$\langle T\rho_2, \rho_2 - \rho_1\rangle \leq -t\|\rho_2 - \rho_1\|^2$$

$$\langle T\rho_1, \rho_1 - \rho_2\rangle + \langle T\rho_2, \rho_2 - \rho_1\rangle \leq -2t\|\rho_2 - \rho_1\|^2. \tag{3.14}$$

From (3.13), (3.14) and (3.12), one can obtain

$$\begin{aligned} 0 \leq \vartheta_2(a, \rho_1, \rho_2) + \langle T\rho_1, \rho_1 - \rho_2\rangle + \vartheta_2(a, \rho_2, \rho_1) + \langle T\rho_2, \rho_2 - \rho_1\rangle \\ \leq -2(\varepsilon + t)\|\rho_2 - \rho_1\|^2 \leq 0. \end{aligned}$$

A contradiction, then

$\|\rho_2 - \rho_1\|^2 = 0 \Longrightarrow \rho_2 = \rho_1.$

Therefore, $GEp(\vartheta_1, \vartheta_2)$ has only one solution. Therefore, we are done with the proof.

4. IMPROVEMENT OF WELL- POSEDNESS FOR EQUILIBRIUM PROLEMS WITH ϑh-MONOTONE TRIFUNCTION

4.1 Introduction

The purpose of this chapter is to demonstrate the existence and uniqueness of the WP findings for the $GEp\ (T, \vartheta_1, \vartheta_2)$ in relation to ϑh-monotone trifunction. In order to achieve this goal, we make use of a subset M of B$\boldsymbol{S}$ V. We provide a useful characterization as well as the characteristics of metric projection and sufficient requirements for different kinds of WP. We conclude this chapter with a prove that the concept of WP for $GEp\ (T, \vartheta_1, \vartheta_2)$ is corresponding to the existence and uniqueness of its solution. Related to the above, the chapter comprises of the four sections. The essential definitions are recalled in Section 2, along with references to some findings in the area. In section 3 establishes the notions for WP and extends it to equilibrium issues $GEp(T, \vartheta_1, \vartheta_2)$. In the following section (section 4), a novel idea of WP for optimization issues that involve constraints is presented. This idea is related to WP and can be described by a parameterized formula for a generalized equilibrium problem.

4.2 Explanation of optimization issues

Let us make the assumption that V and W both are B$\boldsymbol{S}$s, that M is a nonempty, closed and convex set of the B$\boldsymbol{S}$ V, and that K is a subset of W. Let us also assume that K is a subset of W. We will make this assumption until it is specifically stated differently. To make things easier to understand, let's go over some of the criteria and outcomes that have to be enforced in order to provide evidence for our most important outcomes.

Following this, we will present an introduction to the formulation of optimization problems using equilibrium constraints. Consider that $H: K \times M \longrightarrow R$ and $\vartheta_1: K \times M \times M \rightarrow R$ be two bi-functions and the optimization through the application of the generalized equilibrium constraint, abbreviated as (GOPGEC), is expressed it this way: $H(t, ℷ)\ \in K \times M$ in which $ℷ \in SS(t)$.

Where $SS(t)$ is a solution set of the generalized parametric equilibrium issue $GEp_t(T,\vartheta_1,\vartheta_2)$ defined by $\lambda \in SS(t)$ if

$$\langle T\,\lambda, \lambda - e\rangle + \vartheta_1(t,\lambda,e) + \vartheta_2(a,\lambda,e) \geq 0 \;\; \forall a,e \in M\,. \tag{4.1}$$

As an alternative to writing $\{GEp_t(t,\vartheta_1,\vartheta_2), t \in K\}$ set of generalized equilibrium issue (parametric problem).

In specific kinds, the following can be obtained:

i. If T$\equiv 0$, $\vartheta_2 \equiv 0$ and $\vartheta_1(t,\lambda,e) = -h(t,\lambda,\lambda - e)$ then the issue $GEp_t(T,\vartheta_1,\vartheta_2)$ is minimizes to parametric quasivariational inequality QVI(s) see (Zhang et al. 2009).

ii. If $\vartheta_2 \equiv 0$, T$\equiv 0$, then the issue $GEp_t(T,\vartheta_1,\vartheta_2)$ is minimizes to the Parametric equilibrium issue EP(s) see (Fang et al. $(2008)a$).

iii. If $\vartheta_2(a,\lambda,e) = \vartheta(\lambda,e), T \equiv 0$ and $\vartheta_1(t,\lambda,e) = \langle T\,\lambda, e - \lambda\rangle$. Therefore, the issue $GEp_t(T,\vartheta_1,\vartheta_2)$ is minimizes to the mixed variational inequalities (Fang et al. $(2008)\rho$).

iv. If T$\equiv 0$ then the issue $GEp_t(T,\vartheta_1,\vartheta_2)$ is minimizes to generalized equilibrium issues $Ep_t(\vartheta_1,\vartheta_2)$ see (Abed and Hashoosh 2021).

4.3 WP of $\boldsymbol{GEp_t(T,\vartheta_1,\vartheta_2)}$ with Metric Characterizations

This section provides a formalization of key concepts fundamental to the well-posed general equilibrium issues $GEp_t(T,\vartheta_1,\vartheta_2)$. Based on the results presented here, we will outline specific scenarios in which the equilibrium problem is very WP (GS).

Definition 4.1. $\{(t_n, \lambda_n)\} \subset K \times M$ is named to be approximating sequence (abbreviated as, Ap. Seq) for $GEp_t(T, \vartheta_1, \vartheta_2)$ if there is a sequence $\{c_n\}$ that is not negative by $\gamma_n \to 0$ while $n \longrightarrow \infty$ such that

$$\langle T\,\lambda_n, \lambda_n - e\rangle + \vartheta_1(t, \lambda_n, e) + \vartheta_2(a, \lambda_n, e) + \gamma_n \parallel \lambda_n - e \parallel\parallel a \parallel \geq 0,$$

$$\text{for all } n \in N, a, e \in M \tag{4.2}$$

Definition 4.2. The issue $GEp_t(T, \vartheta_1, \vartheta_2\ )$ is named to be strongly (or weakly) well-posed in the generalized if $GEp_t(T, \vartheta_1, \vartheta_2)$ has just one solution λ and for every sequence $\{\lambda_n\}$ with $\lambda_n \to \lambda$, every Ap. seq for $GEp_t(T, \vartheta_1, \vartheta_2)$ converge strongly (or weakly) of the unique solution if $GEp_t(T, \vartheta_1, \vartheta_2)$ has a non-empty solution set $SS(t)$, and any Ap. Seq solution has a subsequence that is a strongly (or weakly) converge to some point $SS(t)$.

Following this, we will proceed to develop certain criteria or characterizations for WP for $GEp_t(T, \vartheta_1, \vartheta_2\ )$. We define two sets for any $\gamma > 0$.

$$\begin{aligned} \Xi(c) = \{(t, \lambda) \in K \times A; \langle T\,\lambda, \lambda - e\rangle + \vartheta_1(t, \lambda, e) + \vartheta_2(a, \lambda, e) + \ \gamma \parallel \lambda - e \parallel \parallel a \parallel \\ \geq 0 \quad for\ all\ a, e \in M\}. \end{aligned}$$

$$N(c) = \{(s, \lambda) \in K \times M; \vartheta_2(a, e, \lambda) + h(\lambda, e)$$

$$\leq \langle T\,\lambda, \lambda - e\rangle + \vartheta_1(t, \lambda, e) + \ \gamma \parallel \lambda - e \parallel \parallel a \parallel \quad for\ all\ a, e \in M\}.$$

Lemma 4.3. Consider that M is a nonempty convex, closed subset of a B$\boldsymbol{S}$ V and let $\vartheta_1: K \times M \times M \to R$, $B_2: M \times M \times M \to R$ and $h: M \times M \to R$ be three functions, and $T: M \to V^*$ be arbitrary nonlinear mapping. Providing the conditions below are met

i. $\vartheta_1(t, \lambda, \lambda) = 0\ \ \forall\ t \in K,\ \ \forall\ \lambda \in M,$

ii. $\vartheta_2(a, ƛ, e)$ is ϑh -monotone trifunction and H.C in 2-term,

iii. $\vartheta_1(t, ƛ, .), \vartheta_2(a, ƛ, .)$ and $h\,(ƛ, .)$ are convex $\forall\, t \in K, a, ƛ \in M.$

As a result, the two sets $\Xi(c)$ and $N(c)$ are equal for all $c > 0$.

Proof. Assume that $(t, ƛ) \in \Xi(c)$, then $(t, ƛ) \in K \times M$ in which:

$$\langle T\, ƛ, ƛ - e\rangle + \vartheta_1(t, ƛ, e) + \vartheta_2(a, ƛ, e) + \gamma \parallel ƛ - e \parallel\parallel a \parallel \geq 0 \quad for\ all\ a, e \in M.$$

From, ϑ_2 is ϑh -monotone trifunction, we get

$\vartheta_2(a, ƛ, e) + \vartheta_2(a, e, ƛ) + h\,(ƛ, e) \leq 0 \quad \forall\, a,\ ƛ, e \in M.$

The equation is satisfied,

$$\vartheta_2(a, e, ƛ) + h\,(ƛ, e) \leq -\vartheta_2(a, ƛ, e)$$

$$\leq \langle T\, ƛ, ƛ - e\rangle + \vartheta_1(t, ƛ, e) + \gamma \parallel ƛ - e \parallel\parallel a \parallel.$$

Consequently, $(t, ƛ) \in N(c)$.

Conversely, let $(t, ƛ) \in N(c)$ and let $e \in A$ in which $ƛ_r = ƛ - r(ƛ - e)$, $r \in (0,1)$ then, $ƛ_r \in M$, since M convex we get

$$\vartheta_2(a, ƛ_r, ƛ) + h\,(ƛ, ƛ_r) \leq \langle T\, ƛ, ƛ - ƛ_r\rangle + \vartheta_1(t, ƛ, ƛ_r) + \gamma \parallel ƛ - ƛ_r \parallel\parallel a \parallel,$$

Since,

$$\gamma \parallel ƛ - ƛ_r \parallel\parallel a \parallel, = r\,\gamma \parallel ƛ - e \parallel\parallel a \parallel, \tag{4.3}$$

Then, the equation is satisfied,

$$\vartheta_2(a, \lambda_r, \lambda) + h(\lambda, \lambda_r) \leq \langle T\lambda, \lambda - \lambda_r\rangle + \vartheta_1(t, \lambda, \lambda_r) + r\gamma \parallel \lambda - e \parallel\parallel a \parallel.$$

As,

$$0 = \vartheta_2(a, \lambda_r, \lambda_r) \leq \vartheta_2(a, \lambda_r, \lambda) - r[\vartheta_2(a, \lambda_r, \lambda) - \vartheta_2(a, \lambda_r, e)].$$

So,

$$0 \leq \vartheta_2(a, \lambda_r, \lambda) - r[\vartheta_2(a, \lambda_r, \lambda) - \vartheta_2(a, \lambda_r, e)].$$

Then,

$$r[\vartheta_2(a, \lambda_r, \lambda) - \vartheta_2(a, \lambda_r, e)] \leqslant \vartheta_2(a, \lambda_r, \lambda). \tag{4.4}$$

Take to account that, $\vartheta_1(t, \lambda, .)$ and $h(\lambda, .)$ convex trifunction, then

$$\vartheta_1(t, \lambda, \lambda_r) \leq \vartheta_1(t, \lambda, \lambda) - r[\vartheta_1(t, \lambda, \lambda) - \vartheta_1(t, \lambda, e)]. \tag{4.5}$$

Also,

$$h(\lambda, \lambda_r) \leq rh(\lambda, e) + (1 - r)h(\lambda, \lambda) \tag{4.6}$$

$$r[h(\lambda, \lambda) - h(\lambda, e)] \leq h(\lambda, \lambda) - h(\lambda, \lambda_r)$$

From (4.4) , (4.5) and (4.6), one can obtain

$$r[[\vartheta_2(a, \lambda_r, \lambda) - \vartheta_2(a, \lambda_r, e) + \vartheta_1(t, \lambda, \lambda) - \vartheta_1(t, \lambda, e) + h(\lambda, \lambda) - h(\lambda, e)]$$

$$\leq \vartheta_2(a, \lambda_r, \lambda) + \vartheta_1(s, \lambda, \lambda) - \vartheta_1(t, \lambda, \lambda_r) + h(\lambda, \lambda) - h(\lambda, \lambda_r).$$

So, by (4.3), one can get

$$r[\vartheta_2(a, \lambda_r, \lambda) - \vartheta_2(a, \lambda_r, e) + \vartheta_1(t, \lambda, \lambda) - \vartheta_1(t, \lambda, e) + h(\lambda, \lambda) - h(\lambda, e)]$$

$$\leq r\,\gamma \parallel \lambda - e \parallel \parallel a \parallel - h(\lambda, \lambda_r) + \vartheta_1(t, \lambda, \lambda) + \langle T\,\lambda, \lambda - \lambda_r\rangle + h(\lambda, \lambda) - h(\lambda, \lambda_r)$$

So,

$$r[\vartheta_2(a, \lambda_r, \lambda) - \vartheta_2(a, \lambda_r, e) + \vartheta_1(t, \lambda, \lambda) - \vartheta_1(t, \lambda, e) + h(\lambda, \lambda) - h(\lambda, e)]$$

$$\leq r\,\gamma \parallel \lambda - e \parallel \parallel a \parallel - h(\lambda, \lambda_r) + \vartheta_1(t, \lambda, \lambda) + r\langle T\,\lambda, \lambda - e\rangle + h(\lambda, \lambda) - h(\lambda, \lambda_r)$$

Divided the above equation by $(-r)$

$$-\vartheta_2(a, \lambda_r, \lambda) + \vartheta_2(a, \lambda_r, e) - \vartheta_1(t, \lambda, \lambda) + \vartheta_1(t, \lambda, e) \geq -\gamma \parallel \lambda - e \parallel\parallel a \parallel$$

$$+\frac{2h(\lambda, \lambda_r)}{r} - \frac{\vartheta_1(t, \lambda, \lambda)}{r} - \langle T\,\lambda, \lambda - e\rangle - \frac{h(\lambda, \lambda)}{r} + h(\lambda, \lambda) - h(\lambda, e).$$

Since $\vartheta_2(a, \cdot, e)$ is H.C, and Remark (3.1). One can get $\vartheta_2(a, \lambda, e) + \vartheta_1(t, \lambda, e) \geq -\gamma \parallel \lambda - e \parallel\parallel a \parallel - \langle T\,\lambda, \lambda - e\rangle$.

Hence $(t, \lambda) \in \Xi(\gamma)$.

Consequently $\Xi(\gamma) = N(\gamma)$ for all $\gamma > 0$. Therefore, we are done with the proof.

Lemma 4.4. Consider that M is a nonempty-convex and closed set of B V, and consider that $\vartheta_1 = K \times M \times M \to R$, $\vartheta_2 : M \times M \times M \longrightarrow R$, $h : M \times M \to R$ and $T : M \to V^*$ be operator satisfy the following assumptions:

i. $\vartheta_1(.,.,e)$ is u.s.c $\forall e \in M$,

ii. $\vartheta_2(a, e, .)$ and $h(., e)$ are $l.s.c$ $\forall a, e \in M$,

iii. $\lambda \longmapsto \langle T\lambda, \lambda - e\rangle$ u.s.c on M with respect weak*topology of V^*.

Then, for every $\gamma > 0$ the set $N(\gamma)$ is closed in $K \times M$.

Proof. Consider that $\{(s_n, \lambda_n)\} \subseteq N(\gamma)$ is a sequence in which $(t_n, \lambda_n) \longrightarrow (t, \lambda)$. Then, for any $a, e \in M$

$$\vartheta_2(a, e, \lambda_n) + h(\lambda_n, e) \leq \langle T\lambda_n, \lambda_n - e\rangle + \vartheta_1(t_n, \lambda_n, e) + \gamma \parallel \lambda_n - e \parallel \parallel a \parallel$$

Take to account that, $\vartheta_2(a, \lambda, .)$ and $h(., e)$ are $l.s.c$ and $\vartheta_1(.,.,e)$ and $\lambda \longmapsto \langle T\lambda, \lambda - e\rangle$ are u.s.c, then

$$\vartheta_2(a, e, \lambda) + h(\lambda, e) \leq \lim_{n\to\infty} \inf[\vartheta_2(a, e, \lambda_n) + h(\lambda_n, e)]$$

$$\leq \lim_{n\to\infty} \sup[\langle T\lambda_n, \lambda_n - e\rangle + \vartheta_1(t_n, \lambda_n, e) + \gamma \| \lambda_n - e \| \| a \|]$$

$$\leq \langle T\lambda, \lambda - e\rangle + B_1(t, \lambda, e) + \gamma || \lambda - e \| \| a \|.$$

Then,

$$\vartheta_2(a, e, \lambda) + h(\lambda, e) \leq \langle T\lambda, \lambda - e\rangle + \vartheta_1(t, \lambda, e) + \gamma \| \lambda - e \| \| a \| \qquad \forall a, e \in \mathcal{M},$$

This argues that $(t, \lambda) \in N(\gamma)$.

Thus, $N(\gamma)$ is a closed set in $K \times A$ for all $\gamma > 0$. Therefore, we are done with the proof.

Theorem 4.5. Consider A is a nonempty-convex and closed set of B V, and let that $\vartheta_1: K \times M \times M \to R$, $\vartheta_2: M \times M \times M \to R$ and $h: M \times M \to R$ be three functions, and $T: A \to V^*$, $GEp_t(T, \vartheta_1, \vartheta_2)$ is WP then:

$$\Xi(\gamma) \neq \emptyset \, , \, lim_{\ c \to 0} diam \, [\Xi(\gamma)] = 0. \tag{4.7}$$

In addition to this, if provided the following premises are correct:

i. $\vartheta_1(.,.,e)$ is $u.s.c$ $\forall e \in M$;

ii. $\vartheta_1(t, \lambda, \lambda) = 0$ $\forall (t, \lambda) \in K \times M$;

iii. $\vartheta_2(a, \lambda, e)$ is ϑh-monotone trifunction and H.C in 2-term;

iv. $\vartheta_1(t, \lambda, .), \vartheta_2(a, \lambda, .)$ and $h(\lambda, .)$ are convex $\forall$ $t \in K, a, \lambda \in M$;

v. $\vartheta_2(a, e, .), h(., e)$ are $l.s.c$ $\forall a, e \in M$.

vi. $\lambda \mapsto \langle T\lambda, \lambda - e \rangle$ u.s.c on M with respect weak* topology of V*.

Then, the converse holds. Therefore, we are done with the proof.

Proof. Consider that $GEp_t(T, \vartheta_1, \vartheta_2)$ is WP, then $GEp_t(T, \vartheta_1, \vartheta_2)$ admit a just one solution $(t, \lambda) \in K \times M$ in which

$\langle T\,ƛ, ƛ - e\rangle + \vartheta_1(t, ƛ, e) + \vartheta_2(a, ƛ, e) \geq 0 \quad \forall a, e \in A.$

Evidently, the set $\Xi(\gamma) \neq \emptyset$, for any $\gamma > 0$, Assume, for the sake of contradiction, that the expression, $lim_{\,n\to\infty}\, diam\,[\Xi(\gamma_n)] > p > 0$ holds true for some sequence $\{\gamma_n\} > 0$. We got two sequences $\{(t_n, ƛ_n)\}$ and $\{(t_n, e_n)\} \in \Xi(\gamma_n)$ in which

$$\|\,(t_n, ƛ_n) - (t_n, e_n)\,\| > p \quad \forall n \in N. \tag{4.8}$$

Since $\{(t_n, ƛ_n)\}$ and $\{(t_n, e_n)\}$ are Ap. seq for $GEp_t(T, \vartheta_1, \vartheta_2)$. By WP of $GEp_t(T, \vartheta_1, \vartheta_2)$ they have to converge to just one solution of $GEp_t(T, \vartheta_1, \vartheta_2)$ a contradiction to (4.8).

Conversely, let's assume (4.7) is correct. Let $\{(t_n, ƛ_n)\}$ be an Ap. seq for $GEp_t(T, \vartheta_1, \vartheta_2)$. Then, there is a nonnegative sequence $\{\gamma_n\}$ by $\gamma_n \to 0$ as $n \to \infty$ which consists of the following:

$$\langle T\,ƛ_n, ƛ_n - e\rangle + \vartheta_1(t_n, ƛ_n, e) + \vartheta_2(a, ƛ_n, e) + \gamma_n \|\, ƛ_n - e\,\|\|\,a\,\| \geq 0$$

$$\forall\, n \in N, a, \in M. \tag{4.9}$$

Then, $\{(t_n, ƛ_n)\} \subseteq \Xi(\gamma_n)$. From (4.7), $\{(t_n, ƛ_n)\}$ is a cauchy sequence and, then, it converges to a point $(t, ƛ)$. The conclusions that we will obtain are as follows: (4.9), ϑ_2 is ϑh-monotone, $\vartheta_2(a, e, .), h(., e)$ are $l.s.c$ and $\vartheta_1(., ., e)\ u.s.c,$

$$0 = \lim_{n\to\infty} inf\ \gamma_n \|\,ƛ_n - e\|\ \|\,a\,\|$$

$$\geq \lim_{n\to\infty} inf[-\langle T\,ƛ_n, ƛ_n - e\rangle - \vartheta_1(t_n, ƛ_n, e) - \vartheta_2(a, ƛ_n, e)]$$

$$\geq \lim_{n\to\infty} inf[-\langle T\,ƛ_n, ƛ_n - e\rangle - \vartheta_1(t_n, ƛ_n, e) + \vartheta_2(a, e, ƛ_n) + h(ƛ_n, e)].$$

So,

$$\lim_{n\to\infty} inf[\langle T\, ƛ_n,\, ƛ_n - e\rangle + \vartheta_1(t_n,\, ƛ_n, e)] \geq \lim_{n\to\infty} inf[\vartheta_2(a, e,\, ƛ_n) + h(\, ƛ_n, e)].$$

It means that

$$\langle T\, ƛ, ƛ - e\rangle + \vartheta_1(t, ƛ, e) \geq \vartheta_2(a, e, ƛ) + h(\, ƛ, e).$$

So, by above, together by Theorem (3.3), then $(t, ƛ)$ as a solution to $GEp_t(T, \vartheta_1, \vartheta_2)$. Therefore, we are done with the proof. Because of this, the uniqueness naturally follows on from (4.7).

Remark 4.6. Remind the user that the diameter of $\Xi(\gamma)$ does not tend toward 0 if $\mathrm{GEp_t}(\mathrm{T}, \vartheta_1, \vartheta_2)$ has multiple solutions. Kuratowski's noncompactness measure of approximation solution set is used in the next result instead of the diameter.

Theorem 4.7. Considering K and M are a nonempty, closed subsets of B$\boldsymbol{S}$ W and V resp. If $GEp_t(T, \vartheta_1, \vartheta_2)$ is strongly WP(GS), then:

$$\Xi(\gamma) \neq \emptyset \text{ for all } \gamma,\ \lim_{\gamma\to 0} \mu[\Xi(\gamma)] = 0\,. \tag{4.10}$$

Furthermore, if the following conditions are met

i. $\vartheta_1(t, ƛ, ƛ) = 0\ \forall\, t \in K, ƛ \in M$,

ii. $\vartheta_1(.,.,e)$ is $u.s.c\ \ \forall\, e \in M$,

iii. $\vartheta_2(a, ƛ, e)$ is ϑh-monotone trifunction and H.C in 2- term,

iv. $\vartheta_1(t, ƛ, .), \vartheta_2(a, ƛ, .)\ and\ h\,(\, ƛ, .)$ are convex $\forall\, t \in K, a, ƛ \in M$,

v. $\vartheta_2(a, e, \cdot)$ and $h(\cdot, e)$ are $l.s.c$ $\forall a, e \in M$.

vi. $\lambda \mapsto \langle T\lambda, \lambda - e\rangle$ u.s.c on M with respect weak*topology of V*.

Then, the converse holds.

Proof. Assume that $GEp_t(T, \vartheta_1, \vartheta_2)$ be strongly WP(GS). Then the solution set SS of $GEp_t(T, \vartheta_1, \vartheta_2)$ is a nonempty. This shows that for every $\gamma > 0, \Xi(\gamma) \neq \emptyset$, because $SS \subset \Xi(\gamma)$. In addition, we assert that the solution set for $GEp_t(T, \vartheta_1, \vartheta_2)$ is compact here. Certainly, for any sequence $\{(t_n, \lambda_n)\}$ in $SS, \{(t_n, \lambda_n)\}$ is an Ap. seq for $GEp_t(T, \vartheta_1, \vartheta_2)$. Consequently, there is a converging to subsequence of some point for SS. This implies that SS is compact.

To show that $lim_{\gamma\to 0}\mu[(\Xi(\gamma)] \to 0$. Therefore, we conclude $SS \subseteq \Xi(\gamma)$ that

$L(\Xi(\gamma), SS) = max\{e(r(\gamma), SS), e(SS, \Xi(\gamma))\}$.

Due to the fact that the solution set SS is a compact, then

$$\mu\big(\Xi(\gamma)\big) \leq 2L(\Xi(\gamma), SS) + \mu(SS) = 2e(\Xi(\gamma), SS),$$

where $\mu(SS) = 0$, since SS is compact. To prove $Lim_{\gamma\to 0}\ \mu(\Xi(\gamma)) = 0$. It is sufficient to show that $e(\Xi(\gamma), SS) \to 0$ as $\gamma \to 0$. If not, there is constant $\gamma > 0$ and $\gamma_n \to 0$, and $\{(t_n, \lambda_n)\} \subset \Xi(\gamma_n)$ where

$$(t_n, \lambda_n) \notin SS + \lambda_{\frac{\gamma}{2}}(0)\ \forall n \in N\,. \tag{4.11}$$

Where $B_{\frac{\gamma}{2}}(0)$ is an open sphere by center 0 and radius $\frac{\gamma}{2}$. But $\{(t_n, \lambda_n)\} \subseteq \Xi(\gamma_n)$, is an Ap.seq for $GEp_t(T, \vartheta_1, \vartheta_2)$. It Follows the generalized WP of $GEp_t(T, \vartheta_1, \vartheta_2)$ that there is a sub-sequence converges to some point of $(t, \lambda) \in SS$ which contradicts (4.11).

On the other hand, consider that (4.10). According to Lemma (4.3) and Lemma (4.4), $\Xi(\gamma)$ is a nonempty and closed each all $\gamma > 0$, so from the Kuratowski theorem (Kuratowski 1968) we obtain,

$$L(\Xi(\gamma), SS) \to 0 \; as \; \gamma \to 0.$$

Where $SS = \bigcap_{\gamma>0} \Xi(\gamma))$ is a nonempty and compact.

Let $\{(t_n, ƛ_n)\} \subset K \times M$ be any approximate solution sequence for $GEp_t(T, \vartheta_1, \vartheta_2)$. Thus, there is a nonnegative sequence $\{\gamma_n\}$ by $\gamma_n \to 0$ as $n \to \infty$, such that

$$\langle T\, ƛ_n, ƛ_n - e\rangle + \vartheta_1(t_n, ƛ_n, e) + \vartheta_2(a, ƛ_n, e) + \gamma_n \parallel ƛ_n - e \parallel\parallel a \parallel \geq 0 \; \forall n \in N, a, ƛ \in M.$$

So, $(t_n, ƛ_n) \in \Xi(\gamma_n)$. With the addition of the account (4.10) show that

$$d[(s_n, ƛ_n), SS] \leq e(\Xi(\gamma), SS) \to 0.$$

Subsequently, SS compact, it follows that there is $(\bar{t}_n, \bar{ƛ}_n) \in SS$ in which

$$\parallel (t_n, ƛ_n) - (t^{-}{}_n, ƛ^{-}{}_n) \parallel = d[(t_n, ƛ_n), SS] \to 0.$$

Once more, depending on compactness of the solution set SS, the sequence $\left(\underline{t}_n, ƛ^{-}{}_n\right)$ have a sup-sequence $\left\{\left(\underline{t}_{n_k}, ƛ^{-}{}_{n_k}\right)\right\}$ converge strongly to $\{t^{-}{}_n, ƛ^{-}{}_n)\} \in SS$. There for the corresponding $\left\{\left(t_{n_k}, ƛ_{n_k}\right)\right\}$ sub-sequence of $\{(t_n, \rho_n)\}$ converge stronglly to $\{t^{-}{}_n, ƛ^{-}{}_n)\}$. Hence, $GEp_t(T, \vartheta_1, \vartheta_2)$ is WP(GS). Therefore, we are done with the proof.

4.4 WP VIA Optimization Problems By Generalized Parametric Equilibrium Constraints

This section, we will discuss development a new formulation for optimization problems based on equilibrium constraints. The optimization problem involving generalized equilibrium constraints, denoted by the acronym GOPGEPC, will be formulated as follows:

$H(t,ƛ)$ in which $(t,ƛ) \in K \times M$,

where $ƛ \in SS(t)$, K is a nonempty closed subset of a parametric space, $H: K \times M \longrightarrow R$, $\vartheta_1: K \times M \times M \to R$, $\vartheta_2: M \times M \times M \to R$, and $SS(t)$ is means that a solution set of the parametric generalized equilibrium $GEp_t(T, \vartheta_1, \vartheta_2)$, defined $\rho y, ƛ \in SS(t)$ if and only if:

$$\langle T\, ƛ, ƛ - e\rangle + \vartheta_1(t, ƛ, e) + \vartheta_2(a, ƛ, e) \geq 0 \quad \forall a, e \in M.$$

Definition 4.8. The set $\{(t_n, ƛ_n)\} \subset K \times M$ is called to be Ap. seq for (GOPGEPC) if:

i. there is a positive sequence $\{\gamma_n\}$ via $\gamma_n \to 0$ where $n \to \infty$ in which:

$$\langle \mathrm{T}\, ƛ_{\mathrm{n}}, ƛ_{\mathrm{n}} - \mathrm{e}\rangle + \vartheta_1\big(\mathrm{t_n}, ƛ_{\mathrm{n},}, \mathrm{e}\big) + \vartheta_2(\mathrm{a}, ƛ_{\mathrm{n}}, \mathrm{e}) + \gamma_{\mathrm{n}} \| ƛ_{\mathrm{n}} - \mathrm{e}\| \, \| \mathrm{a} \| \geqslant 0$$
$$\forall \mathrm{n} \in \mathbb{N}, \mathrm{a}, \mathrm{e} \in \mathcal{M},$$

ii. $\lim_{n\to\infty} \sup \mathrm{H}(\mathrm{t_n}, ƛ_{\mathrm{n}}) \leq \inf_{\mathrm{m}\in \mathrm{k}, \mathrm{e}\in \mathrm{I(m)}} \mathrm{H}(\mathrm{m}, \mathrm{e}).$

Definition 4.9. (GOPGEPC) is called to be strongly (or weakly) WP (or strongly (resp., weakly) WP(GS) (abbreviated as, WP(GS)) if (GOPGEPC) have just one solution and for every Ap. seq for (GOPGETC) converges strongly (or weakly) to the unique solution (or if $SS \neq \varphi$ and every solution Ap. seq have a subsequence that strongly (or weakly) converge to some point of SS).

The definition of a family for approximate solutions for (GOPGEPC):

$$\psi(c,\epsilon) = \{(s,\lambda) \in K \times M;\ H(t,\lambda) \leq H(m,e) + \epsilon,$$
$$and\ \langle T\lambda_n, \lambda_n - e\rangle + \vartheta_1(t,\lambda,e) + \vartheta_2(a,\lambda,e) + \gamma_n \| \lambda - e\| \, \| a \|$$
$$\geq 0 \qquad \forall a,e \in M.$$

The subsequent outcome supports the continuing of the expansion of previously documented theories.

Theorem 4.10. Consider A to be a non-empty, convex, and closed subset of B V. Let $\vartheta_1: K \times M \times M \to R$, $\vartheta_2: M \times M \times M \to R$, $H: K \times A \to R$, $h: M \times M \to R$ be four function and $T: M \to V^*$. If (GOPGEPC) is strongly WP. Then

$$\psi(\gamma,\epsilon) \neq \emptyset, for\ all\ \gamma,\epsilon > 0, diam\ \psi(\gamma,\epsilon) \to 0, where\ (\gamma,\epsilon) \to (0,0). \quad (4.12)$$

In addition to this, if the assumptions that are listed below are fulfilled,

i. $\vartheta_1(\cdot,\cdot,e)$ is $u.s.c$ $\forall e \in M$;

ii. $\vartheta_1(t,\lambda,.), \vartheta_2(a,\lambda,.)$ and $h(\lambda,.)$ are convex $\forall\ t \in K, a, \lambda \in M$;

iii. $\vartheta_2(a,\lambda,e)$ is ϑh -monotone trifunction and H.C in 2- term;

iv. $\vartheta_2(a,e,.), h(\cdot,e)$ are $l.s.c$ $\forall a,e \in M$;

v. H is $l.s.c.$

vi. $\lambda \longmapsto \langle T\lambda, \lambda - e\rangle$ u.s.c on M with respect weak*topology of V*.

Therefore, the opposite is true.

Proof. Assume that $(GOPGEPC)$ is strongly WP. Then (GOPGEPC) has just one solution $(t, \lambda) \in K \times M$ in which

$$\{H(t,\lambda) = H(m,e) \;\; and \; \langle T\lambda, \lambda - e\rangle + \vartheta_1(t,\lambda,e) + \vartheta_2(a,\lambda,e) \geq 0 \qquad \forall\, a, e \in M.$$

Noticeably, $\psi(\gamma,\epsilon) \neq \emptyset$ for all $\gamma, \epsilon > 0$, That's because $(t,\lambda) \in \psi(\gamma,\epsilon)$ for any $\gamma, \epsilon > 0$. If $diam\, \psi(\gamma,\epsilon) \nrightarrow 0$ while $\gamma \to 0, \epsilon \to 0$. So, there is a constant $p > 0$ and a non-negative sequences $\{\gamma_n\}$ and $\{\epsilon_n\}$ via $\gamma_n \to 0$, $\epsilon_n \to 0$ where $n \to 0$ and $(t_n, \lambda_n), (t_n, e_n) \in \psi(\gamma,\epsilon)$ such that

$$\| (t_n, \lambda_n) - (t_n, e_n) \| > p \;\; \forall n \in N. \tag{4.13}$$

Since, $(t_n, \lambda_n), (t_n, e_n) \in \psi(\gamma,\epsilon) \;\forall\, n \in N$, then $\{(t_n, \lambda_n)\}$ and $\{(t_n, e_n)\}$ are Ap. seq for (GOPGEPC). Therefore, they must strongly converge to the unique solution of (GOPGEPC) from the WP of (GOPGEPC), which is a contradiction of equation (4.13).

On the other hand, take into account the fact that the condition 4.13 is met. Let $\{(t_n, \lambda_n)\}$ be an Ap. seq for (GOPGEPC). Then there is a non-negative sequence $\{\gamma_n\}$ via $\gamma_n \to 0$ as $n \to \infty$, where

$$\begin{gathered} \lim_{n\to\infty} sup\, H(t_n, \lambda_n) \leq \inf_{m\in k, e\in I(m)} H(m,e) \;\; and \\ \langle T\rho_n, \lambda_n - e\rangle + \vartheta_1(t_n,\lambda_n,e) + \vartheta_2(a,\lambda_n,e) + \gamma_n \|\lambda_n - e\| \, \| a \| \geq 0 \\ \forall\, n \in \mathbb{N}, a, e \in \mathcal{M}. \end{gathered} \tag{4.14}$$

Thus, we conclude that $(t_n, \lambda_n) \in \psi(\gamma_n, \epsilon_n)$.

Since (4.12), we conclude that $\{(t_n, \lambda_n)\}$ is a Cauchy sequence. Subsequently, it strongly converges to the point $(t, \rho) \in K \times M$. Based on equation (4.14), and assuming statements (i), (iii), and (v)

$$0 = \lim_{n\to\infty} \inf \gamma_n \parallel ƛ_n - e \parallel\parallel a \parallel$$

$$\geq \lim_{n\to\infty} \inf[-\langle Tƛ_n, ƛ_n - e\rangle - \vartheta_1(t_n, ƛ_n, e) - \vartheta_2(a, ƛ_n, e)]$$

$$\geq \lim_{n\to\infty} \inf[-\langle Tƛ_n, ƛ_n - e\rangle - \vartheta_1(t_n, ƛ_n, e) + \vartheta_2(a, e, ƛ_n) + h(ƛ_n, e)].$$

Then,

$$\lim_{n\to\infty} \inf\ [\langle T\,ƛ_n,\ ƛ_n - e\rangle + \vartheta_1(t_n,\ ƛ_n, e)] \geq \lim_{n\to\infty} \inf[\vartheta_2(a, e,\ ƛ_n) + h(\ ƛ_n, e)].$$

It means that

$$\langle T\,ƛ, ƛ - e\rangle + \vartheta_1(t, ƛ, e) \geq \vartheta_2(a, e, ƛ) + h(\,ƛ, e).$$

Furthermore, based on equation (4.14), as well as assumption (vii), one can deduce that

$$\begin{aligned} \inf_{m\in K, e\in I(m)} H(m,e) &\geq \lim_{n\to\infty} \sup H(t_n,\ ƛ_n) \\ &\geq \lim_{n\to\infty} \inf H(t_n,\ ƛ_n) \\ &\geq H(t, ƛ). \end{aligned}$$

According to Theorem (3.3), $(t, ƛ)$ solves (GOPGEPC). The uniqueness comes directly from the fact that (4.12). As a result, we are done with the proof.

Using the same method for the demonstration of Theorem (4.7), we are able to reach the following result for the WP of (GOPGEPC).

Theorem 4.11. Consider that K and M are a nonempty, closed and convex subset of real B$\boldsymbol{S}$s W and V resp. If (GOPGEPC) is strongly WP(GS), then:

$$\psi(\gamma, \epsilon) \neq \emptyset \quad \forall \gamma, \epsilon > 0, \lim_{(\gamma,\epsilon)\to(0,0)} \mu[\psi(\gamma, \epsilon)] = 0. \tag{4.15}$$

Furthermore, if the following suppositions hold:

i. $\vartheta_1(.,.,e)$ is $u.s.c$, $\forall e \in M$,

ii. $\vartheta_1(t,\lambda,.), \vartheta_2(a,\lambda,.)$ *and* $h\ (\lambda,.)$ are convex $\forall\ t \in K, a, \lambda \in M$,

iii. $\vartheta_2(a,\lambda,e)$ is ϑh -monotone trifunction and H.C in 2- term,

iv. $\vartheta_1(t,\lambda,\lambda) = 0\ \ \forall t \in K, \lambda \in M$,

v. $\vartheta_2(a,.,e)$ and $h(.,e)$ are $l.s.c\ \forall a, e \in M$,

vi. H is $l.s.c.$

vii. $\lambda \longmapsto \langle T\lambda, \lambda - e\rangle$ u.s.c on M with respect weak*topology of V*.

Therefore, the opposite is true.

We illustration that under appropriate conditions, in the next outcome, the WP of (GOPGEPC) is equivalent to the existence and uniqueness of solution. This is done so that we can study the possibility of there being only one solution to (GOPGEPC).

Theorem 4.12. Consider that K and M be a nonempty closed and convex subsets of finite dimensional B$\mathcal{S}$s W and V resp,. Assume that $\vartheta_1 = K \times M \times M \to R, B_2: M \times M \times M \to R$, $h: M \times M \to R$ and $H: K \times M \longrightarrow R$ be four mappings, and $T: M \to V^*$ be operator. Furthermore, if the following suppositions hold:

i. $\vartheta_1(.,.,e)$ is $u.s.c$ and convex $\ \forall e \in M$,

ii. $\vartheta_1(t,\lambda,\lambda) = 0\ \ \forall\ t \in K, \lambda \in M$,

iii. $\vartheta_2(a, ג, e)$ is ϑh -monotone and H.C in 2-term,

iv. $\vartheta_2(a, e, .)$, $h(e, .)$are $l.s.c$, $\forall a, e \in M$,

v. $\vartheta_2(a, ., ג)$ *and* $h\ (ג, .)$ are convex for all $a, ג \in M$,

vi. H is convex and $l.s.c$.

vii. $ג \longmapsto \langle Tג, ג - e\rangle$ u.s.c on M with respect weak*topology of V*.

Then, (GOPGEPC) is a strongly WP if and only if it has just one solution.

Proof. The necessity is clearly. For the sufficient condition. Let's assume (GOPGEPC) has just one solution $(t^*,\ ג^*)$. It follow that:

$$\begin{cases} H(t^*,\ ג^*) = \inf\limits_{m\in K\ ,e\in I(m)} H(m, e) \\ \langle T\ ג^*,\ ג^* - e\rangle + \vartheta_1(t^*,\ ג^*, e) + \vartheta_2(a,\ ג^*, e) \geq 0 \quad \forall\ a, e \in \mathcal{M}. \end{cases} \tag{4.16}$$

Let $\{(t_n,\ ג_n)\} \subset K \times M$ be Ap. seq for $(GOPGEPC)$. Then, there is $\gamma_n > 0$ via $\gamma_n \to 0$ as $n \to \infty$ such that:

$$\begin{cases} \lim\limits_{n\to\infty} \sup H(t_n,\ ג_n) = \inf\limits_{m\in K, e\in I(m)} H(m, e) \\ \langle T\ ג_n,\ ג_n - e\rangle + \vartheta_1(t_n,\ ג_n, e) + \vartheta_2(a,\ ג_n, e) + \gamma_n \|\ ג_n - e\|\ \|\ a\ \| \geq 0\ . \end{cases} \tag{4.17}$$

Our claim that $\{(t_n,\ ג_n)\}$ is bounded. If not, we assume *that* $\|\ (t_n,\ ג_n)\ \| \to +\infty$.

$Let\ r_n = \frac{1}{\|(t_n,\ ג_n) - (t^*,\ ג^*)\|}$, and

$$\begin{aligned} (s_n, z_n) &= r_n(t_n,\ ג_n) + (1 - r_n)(t^*,\ ג^*) \\ &= [r_n t_n + (1 - r_n)t^*, r_n\ ג_n + (1 - r_n)\ ג^*]. \end{aligned}$$

One might assume that, without losing generality,

$r_n \in (0,1)$ and $(s_n, z_n) \to (s, z)$ via $(s, z) \neq (t^*,\ \lambda^*)$ since $W \times V$ is finite-dimensional, taking into consideration the closedness as well as the convexity of K and A, we have $(s_n, z_n) \in K \times M$. Therefore, according to hypothesis (vi) and $(4.16), (4.17)$ for any $(s, z) \in K \times M$, one can obtain

$$
\begin{aligned}
H(t^*,\ \lambda^*) &= \lim_{n\to\infty} sup\, r_n H(t_n,\ \lambda_n) + \lim_{n\to\infty} sup(1 - r_n) H(t^*,\ \lambda^*) \\
&\geq \lim_{n\to\infty} sup[r_n H(t_n,\ \lambda_n) + (1 - r_n) H(t^*,\ \lambda^*)] \\
&\geq \lim_{n\to\infty} sup\, H(s_n, z_n) \\
&\geq \lim_{n\to\infty} inf\ H(s_n, z_n) \\
&\geq H(s, z). \qquad (4.18)
\end{aligned}
$$

Besides, from conditions (i), (ii), (4.16) and (4.17), one can have

$$
\begin{aligned}
0 &= \lim_{n\to\infty} inf\, r_n \gamma_n \| \lambda_n - e \| \parallel a \parallel \\
&\geq \lim_{n\to\infty} inf -r_n [\langle T\, \lambda_n,\ \lambda_n - e\rangle + \vartheta_1(t_n,\ \lambda_n, e) + \vartheta_2(a,\ \lambda_n, e)] \\
&\quad - (1 - r_n)[\langle T\, \lambda^*,\ \lambda^* - e\rangle + \vartheta_1(t^*,\ \lambda^*, e) + \vartheta_2(a,\ \lambda^*, e)] \\
&\geq \lim_{n\to\infty} inf[-r_n \langle T\, \lambda_n,\ \lambda_n - e\rangle - (1 - r_n)\langle T\, \lambda^*,\ \lambda^* - e\rangle] \\
&\quad + \lim_{n\to\infty} inf[-r_n \vartheta_1(t_n,\ \lambda_n, e) - (1 - r_n)\vartheta_1(t^*,\ \lambda^*, e)]
\end{aligned}
$$

$$+\liminf_{n\to\infty}[-r_n\vartheta_2(a, \lambda_n, e) - (1-r_n)\vartheta_2(a, \lambda^*, e)]$$

$$\geq \liminf_{n\to\infty} [-\langle Tz_n, z_n - e\rangle - \vartheta_1(s_n, z_n, e) - \vartheta_2(a, z_n, e)]$$

$$\geq \liminf_{n\to\infty} [-\langle Tz_n, z_n - e\rangle - \vartheta_1(s_n, z_n, e) + \vartheta_2(a, e, z_n) + h(z_n, e)].$$

Then,

$$\liminf_{n\to\infty}[\langle Tz_n, z_n - e\rangle + \vartheta_1(s_n, z_n, e)] \geq \liminf_{n\to\infty} [\vartheta_2(a, e, z_n) + h(z_n, e)]$$

$$\langle Tz, z - e\rangle + \vartheta_1(s, z, e) \geq \vartheta_2(a, e, z) + h(z, e). \tag{4.19}$$

The conclusion reached by using Theorem (3.2) is that

$$\langle Tz, z - e\rangle + \vartheta_1(s, z, e) + \vartheta_2(a, z, e) \geq 0.$$

Hence, from (4.18), (4.19), that (s, z) solves (GOPGEPC) that is a contradiction. Then $\{(t_n, \lambda_n)\}$ is bounded. Let $\{(t_{n_i}, \lambda_{n_i})\}$ be any subsequence of $\{(t_n, \lambda_n)\}$ in which

$$(t_{n_i}, \lambda_{n_i}) \longrightarrow (t_0, \lambda_0) \; as \; i \longrightarrow \infty.$$

It follows that:

$$0 \geq \liminf_{n\to\infty} \gamma_{n_i} \| \lambda_{n_i} - e \| \, \| a \|$$

$$\geq \liminf_{n\to\infty}\left[-\langle T \lambda_{n_i}, \lambda_{n_i} - e\rangle - \vartheta_1(t_{n_i}, \lambda_{n_i}, e) - \vartheta_2(a, \lambda_{n_i}, e)\right]$$

$$\geq \liminf_{n\to\infty}\left[-\langle T \lambda_{n_i}, \lambda_{n_i} - e\rangle - \vartheta_1(t_{n_i}, \lambda_{n_i}, e) + \vartheta_2(a, e, \lambda_{n_i}) + h(\lambda_{n_i}, e)\right.$$

$$\geq -\langle T\,ƛ_0,\, ƛ_0 - e\rangle - \vartheta_1(t_0,\, ƛ_0, e) + \vartheta_2(a, e,\, ƛ_0) + h(\,ƛ_0, e). \qquad \forall a, e \in \mathcal{M}. \quad (4.20)$$

Using Theorem (3.2) indicates that

$$\langle T\,ƛ_0,\, ƛ_0 - e\rangle + \vartheta_1(t_0,\, ƛ_0, e) + \vartheta_2(a,\, ƛ_0, e) \geq 0 \qquad \forall a, e \in M. \quad (4.21)$$

It follows from (4.17) and l.s.c of H that

$$\lim_{m\in K\,,e\in SS(m)} \inf \mathrm{H}(\mathrm{m}, \mathrm{e}) \geq \lim_{i\to\infty} \sup \mathrm{H}\left(\mathrm{t}_{\mathrm{n}_\mathrm{i}},\, ƛ_{\mathrm{n}_\mathrm{i}}\right)$$

$$\geq \lim_{i\to\infty} \inf \mathrm{H}\left(\mathrm{t}_{\mathrm{n}_\mathrm{i}},\, ƛ_{\mathrm{n}_\mathrm{i}}\right)$$

$$\geq \mathrm{H}(\mathrm{t}_0,\, ƛ_0). \quad (4.22)$$

From (4.21) and (4.22) one can see $(t_0,\, ƛ_0)$ solves (GOPGEPC). With the uniqueness of a solution in consideration, we have $(t_0,\, ƛ_0) = (t^*,\, ƛ^*)$ and henceforth $(t_n,\, ƛ_n)$ converges to $(t^*,\, ƛ^*)$.So, (GOPGPEC) is strong WP. Therefore, we are done with the proof.

Theorem 4.13. Assume K and M be a nonempty, convex and closed subsets of finite dimensional B$\boldsymbol{S}$s W and V resp., and assume that $\vartheta_1: K \times M \times M \to R, \vartheta_2 = M \times M \times M \to R, h: M \times M \to R$ and $H: K \times M \to R$ be functions. Let's suppose that there is some $\epsilon > 0$ in which $\psi(\epsilon, \epsilon)$ is a not empty and bounded set, and that the following presumptions are true:

i. $\vartheta_1(.,.,e)$ is $u.s.c$ and $convex$ $\forall\, e \in M,$

ii. $\vartheta_1(t, ƛ, ƛ) = 0\ \ \forall\, t \in K, ƛ \in M,$

iii. $\vartheta_2(a, ƛ, e)$ is $\vartheta h -$ monotone and H.C in 2-term,

iv. $\vartheta_2(a,.,\lambda)$ and $h\ (\lambda,.)$ are convex $\forall\ a,\lambda \in M$,

v. $\vartheta_2(a,e,.), h(e,.)$ are l.s.c $\forall a,e \in M$,

vi. H is convex and $l.s.c.$

vii. $\lambda \longmapsto \langle T\lambda, \lambda - e\rangle$ u.s.c on M with respect weak*topology of V*.

Then (GOPGPEC) is strongly WP(GS).

Proof. Let $\{(t_n, \lambda_n)\} \subseteq K \times M$ be an Ap. seq for ($GOPGETC$).Then, there is a sequence $\{\gamma_n\} \geq 0$, via $\gamma_n \to 0$ as $n \to \infty$ in which

$$\begin{cases} \lim_{n\to\infty} sup\, H(t_n, \lambda_n) = \inf_{m\in K, e\in I(m)} H(m,e) \\ \langle T\,\lambda_n, \lambda_n - e\rangle + \vartheta_1(t_n, \lambda_n, e) + \vartheta_2(a, \lambda_n, e) + \gamma_n \| \lambda_n - e\|\ \|\ a\ \| \geq 0. \end{cases} \tag{4.23}$$

Because $\psi(\epsilon,\epsilon)$ is a nonempty and bounded. Then there is n_0 in which $(t_n,\lambda_n) \in \psi(\epsilon,\epsilon)$ for $any\ n \geq n_0$. Considering the boundedness into account of $\psi(\epsilon,\epsilon)$, there is subsequence $\{(t_{n_i}, \lambda_{n_i})\}$ for $\{(t_n, \lambda_n)\}$ where $(t_{n_i}, \lambda_{n_i}) \to (t_0, \lambda_0)$ as $i \to \infty$. As a consequence of this, as demonstrated in Theorem (4.12), (t_0, λ_0) is a solution (GOPGETC). If this is the case, then (GOPGETC) is strongly WP (GS).

5. CONCLUSION

5.1 Summing Up

Throughout our previous discussion, we pursued to study equilibrium problems in B$\boldsymbol{S}$, having focused on establishing findings on the existence of explanations of equilibrium problems in such cases as convex subsets and bounded subsets.

The above discussion has come in five main chapters. In the second chapter, we recalled a host of base definitions and theorems, with the aim of understanding the main grounded results. The third chapter, managed to establish novel results of existence and uniqueness for $GE_p(T, \vartheta_1, \vartheta_2)$ by generalizing a new class of trifunction equilibrium problems involving a new type of monotonicity in B$\boldsymbol{S}$. The final chapter concluded that the notion of WP for $GEp_t(T, \vartheta_1, \vartheta_2)$ be an equivalent to its existence and uniqueness.

5.2 Future Works

Throughout our study above, we end with the conclusion that there is much to say about equilibrium problems in B$\boldsymbol{S}$, particularly, monotonicity of trifunction. We have also observed that further work we would like to study in the future in which to establish the existence of results and Theorems of WP for a system of equilibrium problem with ϑh-Monotone Trifunction. Our research mind in the future is fixed to get this work further investigated and uncovered.

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