HARMONIC ANALYSIS IN FINITE PHASE SPACE

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MASTER OF SCIENCE

By

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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The Wigner distribution and linear canonical transforms are important tools for optics, signal processing, quantum mechanics, and mathematics. In this thesis, we study the discrete versions of Wigner distributions and linear canonical transforms. In the definition of a discrete entity we focus on two aspects: structural analogy and continuum approximation and/or limits. Based on this framework, the tradeoffs are analyzed and a compromise for a discrete Wigner distribution that meets both objectives to a high degree is presented by consolidating sampling theory and the algebraic approach. Such a compromise is necessary since it is impossible to meet the conditions to the highest possible degree. The differences between discrete and continuous time-frequency analysis are also discussed in a group theoretical perspective. In the second part of the thesis, the discrete versions of linear canonical transforms are reviewed and their connections to the continuous theory is established. As a special case the discrete fractional Fourier transform is defined and its properties are derived.

Keywords: discrete Wigner distributions, discrete time-frequency analysis, discrete linear canonical transforms, discrete fractional Fourier transform.

ÖZET

SONLU FAZ UZAYINDA HARMONİK ANALİZ

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Wigner dağılımı ve lineer kanonik dönüşümler optik, sinyal işleme, kuantum mekaniği ve matematik için önemli araçlardır. Bu tezde Wigner dağılımı ve lineer kanonik dönüşümlerin ayrık versiyonları araştırılmıştır. Herhangi bir ayrık dönüşümün tanımlanmasında temel iki amaç üzerinde durulmuştur: yapısal analoji ve sayısal yaklaşım ve/veya limitler. Bu iki koşulun kısıtları araştırılmış ve bu iki amaca optimal biçimde uyan bir ayrık Wigner dağılımı gösterilmiştir. Bu sürecte örnekleme yöntemleri ile cebirsel metodlardan aynı ayrık dağılıma ulaşılabildiği gösterilmiştir. Ne yazık ki bu iki amaca da aynı anda en yüksek düzeyde ulaşmak imkansızdır. Dolayısıyla bu iki amaca aynı anda ne düzeyde ulaşılabileceği önemli bir sorundur. Ayrık zaman-frakans analizi ve sürekli zaman-frekans analizi arasındaki farklar da grup teorisi perspektifinde incelenmiştir. Tezin ikinci kısmında ayrık lineer kanonik dönüşümler kısaca anlatılmış ve bu dönüsümlerin sürekli kanonik dönüşümlerle ilişkisi kurulmuştur. Özel durum olarak ayrık kesirli Fourier dönüşümü tanımlanmış ve özellikleri cıkarılmıştır.

Anahtar sözcükler: ayrık Wigner dağımı, ayrık zaman-frekans analizi, ayrık lineer kanonik dönuşümler, ayrık kesirli Fourier dönüşümü .

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Chapter 1

Introduction

1.1 Wigner distributions

The Wigner Distribution (WD) [1–4] is an important time-frequency representation [5–9]. It is widely used in signal analysis and processing [10], optics [11–13], and quantum mechanics [14]. A discrete-time discrete-frequency version of the WD is of great importance not only for digital signal processing but also for the other fields where the WD is utilized. In defining the discrete version of any transform or representation there are usually two distinct objectives:

- **Structural analogy**: The discrete entity should satisfy as many operational properties and relationships analogous to the continuous entity as possible.
- **Approximation and limits**: The discrete entity should approximate the samples of the continuous entity and/or the continuous entity should in some sense be the limit of the discrete entity.

The discrete Fourier transform (DFT) satisfies both of these objectives to a very high degree. Most approaches to designing the discrete WD in the literature have primarily emphasized either of the above objectives. Ideally we would desire our definition of the discrete WD also to satisfy both of these objectives. Unfortunately, a definition satisfying both objectives to the same degree as the DFT does not exist. In fact, these two objectives often seem to contradict each other and a trade-off is often necessary. Therefore it is desirable to understand to what extent these objectives can be simultaneously met and the nature of the trade-off between them.

For instance, to define the discrete WD so that it exhibits the largest possible structural analogy to the continuous case, it is possible to base its definition in group representation theory in a manner completely analogous to the definition of the continuous WD [15, 16]. Such a definition, referred to as $\mathrm{WD^m}$ in this thesis, indeed exhibits a high degree of structural analogy, but fails to approximate the continuous Wigner distribution. In fact, rather convincing arguments have been put forth that $\mathrm{WD^m}$ is the definition exhibiting the greatest possible degree of structural similarity, in the sense that this is the only definition which satisfies the discrete versions of a set of properties of the continuous WD that uniquely define it among other members of Cohen's class [17]. On the other hand, definitions obtained by sampling the continuous WD [3], such as the definition that will be referred to as $\mathrm{WD^s}$ in this thesis, while approximating the continuous WD, exhibit only a very limited degree of structural analogy and lack several of the fundamental properties that distinguish the WD from other members of Cohen's Class. In this thesis we will also study a third definition [18], referred to as $\mathrm{WD^h}$, which not only provides a good continuous approximation, but also exhibits a high degree of structural analogy, and therefore seems to be one of the most desirable definitions of the discrete WD for most purposes. In most respects, the relationship between this definition and the continuous WD, comes closest to the relationship between the DFT and the continuous FT.

In section 2.1, the derivation of the continuous WD based on the Weyl correspondence will be reviewed and this approach will be adapted to the discrete case. Based on how we choose to handle divisions by 2, this leads to the definitions of the discrete WD we refer to as $\mathrm{WD^m}$ or $\mathrm{WD^h}$. Such algebraic approaches lead to definitions exhibiting a high degree of structural analogy to the continuous case (left hand of figure 1.1.1). In section 2.2 the properties of the three discrete WDs discussed in this thesis will be compared. Section 2.3 discusses definitions of the discrete WD

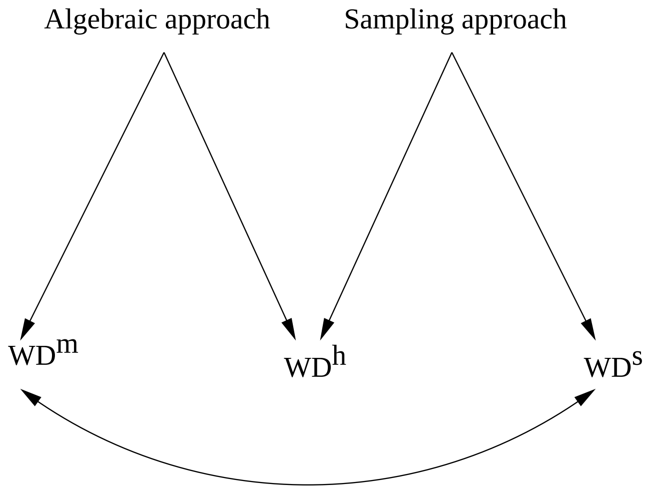

Figure 1.1.1: Approaches to defining the discrete WD and resulting definitions.

based on sampling (which achieve continuous approximation), and leads to the definitions we refer to as WD^{h} and WD^{s} (right hand of figure 1.1.1). As we can see from figure 1.1.1, WD^{h} emerges at the intersection of the algebraic approach which leads to high structural analogy, and the sampling approach which leads to continuous approximation, and therefore stands out as a definition which satisfies both of our goals to a very high degree. The relationship between WD^{m} and WD^{s} represented by the arc at the bottom of the figure is also discussed in the same section. In section 2.4 a brief discussion of some of the issues from the perspective of group theory will be presented.

The review of the literature has been postponed to section 2.5.

1.2 Linear canonical transforms

The linear canonical transforms (LCT) play an important role in optics, quantum mechanics and also found applications in signal processing [11]. The fractional Fourier transform (FrFT), as a special linear canonical transform, is widely studied in optics [13]. The FrFT and its close relationship with the WD have led to many applications in time-frequency analysis [8]. Defining the discrete versions of these transforms is also important for the fields where the continuous version is used. As with the case of WDs, in the definition of the discrete LCTs we will focus on:

- **Structural analogy**: The discrete entity should satisfy as many analogous operational properties and relationships to the continuous entity as possible.
- **Approximation and limits**: The discrete entity should approximate the samples of the continuous entity and/or the continuous entity should in some sense be the limit of the discrete entity.

Before proceeding to the discrete analogy we must note the most desirable objective. It would be good to have operators both form a matrix group[1] $SL(2,\mathbb{R})$ and act on a finite dimensional Hilbert space. However, it is stated in [13] pp. 277 that "the group $Mp(2,\mathbb{R})$ has **no** finite-dimensional matrix representation". In the literature there has been various proposals for the computation of the LCTs and as a special case the fractional Fourier transform (FrFT) [11]. However these approaches lack structural analogy and desirable properties that designate the LCTs.

It is possible to construct LCTs in a modulo sense where the matrix group becomes $SL(2,\mathbb{Z}_p)$ and the real field $\mathbb{R}$ is replaced with $\mathbb{Z}_p$. The field $\mathbb{Z}_p$ denotes the integers in $[0, p-1]$ with the group operations being additions and multiplications in $\mod p$. The representation theory of this group was studied first by Tanaka [19] in an abstract manner. An explicit construction of the metaplectic representation in the Weyl-Fourier form is given in [20–22]. The limits of the discrete metaplectic representation are studied and it is known[2] that the discrete LCTs do not have the continuous LCTs as limits [23]. We will show in section 3.3 that under certain assumptions it is possible to relate the LCTs coming from $SL(2,\mathbb{Z}_p)$ the continuous LCTs obtained from $SL(2,\mathbb{R})$.

The fractional Fourier transform is of special interest among the subgroups of $SL(2,\mathbb{R})$. In section 3.4, we will also study fractional Fourier transforms obtained from $SL(2,\mathbb{Z}_p)$ and derive their properties and connections with WD^{m}. Although

[1] Up to ± 1 sign uncertainty

[2] Private communications, Laurence Barker

the first studies on the group $SL(2, \mathbb{Z}_p)$ date back to sixties, to the best our knowledge the discrete FrFT obtained from $SO(2, \mathbb{Z}_p)$ is not studied in this detail. Authors of [16, 24] came close to defining the discrete versions of this dicrete FrFT however they do prove many of the properties given in table 3.4.1 and do not write the transform kernel in the complete form that will be presented in 3.4.

A detailed review of the literature and further discussions is postponed to section 3.3.

Chapter 2

Wigner distributions

2.1 Weyl correspondence approach to defining discrete Wigner distributions

Before defining discrete WDs, we will briefly review the development of the continuous WD based on the Weyl correspondence, an approach also known as the characteristic function operator method [6]. Our inner product convention is $\langle f, g \rangle = \int_{-\infty}^{\infty} f(u) g^*(u)\, du$. The Fourier transform is defined as

$$\mathcal{F}\{f(u)\} = F(\mu) = \int_{-\infty}^{\infty} f(u) e^{-j2\pi\mu u}\, du, \tag{2.1.1}$$

$$\mathcal{F}^{-1}\{F(\mu)\} = f(u) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu u}\, d\mu, \tag{2.1.2}$$

and the coordinate multiplication and differentiation operators are defined as

$$\mathcal{U} f(u) = u f(u), \qquad \text{in the time domain}, \tag{2.1.3}$$

$$\mathcal{D} F(\mu) = \mu F(\mu), \qquad \text{in the frequency domain}, \tag{2.1.4}$$

where $\mathcal{U}$ and $\mathcal{D}$ are related through $\mathcal{D} = \mathcal{F}^{-1}\mathcal{U}\mathcal{F}$. Exponentiation of these operators yields the time-shift and frequency-shift operators. Expressed in the time domain:

$$e^{j2\pi\bar{\mu}\mathcal{U}} f(u) = e^{j2\pi\bar{\mu}u} f(u), \tag{2.1.5}$$

$$e^{j2\pi\bar{u}\mathcal{D}} f(u) = f(u + \bar{u}). \tag{2.1.6}$$

We now apply the correspondence principle $u \to \mathcal{U}$, $\mu \to \mathcal{D}$ to the function $e^{j2\pi(\bar{\mu}u+\bar{u}\mu)}$. This is known as the Weyl correspondence[1]:

$$e^{j2\pi(\bar{\mu}u+\bar{u}\mu)} \to e^{j2\pi(\bar{\mu}\mathcal{U}+\bar{u}\mathcal{D})}. \tag{2.1.7}$$

The entity on the right-hand side will be denoted by $\rho(\bar{u},\bar{\mu})$, and can be put in the following form by employing the Baker-Campbell-Hausdorff formula [11] $e^{\mathcal{A}+\mathcal{B}} = e^{\mathcal{A}}e^{\mathcal{B}}e^{-[\mathcal{A},\mathcal{B}]/2}$ which holds when $[\mathcal{A},[\mathcal{A},\mathcal{B}]] = [\mathcal{B},[\mathcal{A},\mathcal{B}]] = 0$ (which is true in our case since $[\mathcal{U},\mathcal{D}] = \mathcal{U}\mathcal{D} - \mathcal{D}\mathcal{U} = \frac{j}{2\pi}\mathcal{I}$):

$$\begin{aligned}\rho(\bar{u},\bar{\mu}) = e^{j2\pi(\bar{\mu}\mathcal{U}+\bar{u}\mathcal{D})} &= e^{j\pi\bar{u}\bar{\mu}}e^{j2\pi\bar{\mu}\mathcal{U}}e^{j2\pi\bar{u}\mathcal{D}} \\ &= e^{-j\pi\bar{u}\bar{\mu}}e^{j2\pi\bar{u}\mathcal{D}}e^{j2\pi\bar{\mu}\mathcal{U}}.\end{aligned} \tag{2.1.8}$$

With reference to equations (2.1.5) and (2.1.6), $\rho(\bar{u},\bar{\mu})$ is an operator with the effect of combined time and frequency shifting with time-frequency shift parameters $\bar{u},\bar{\mu}$. Applying this combined shift operator to a function $f(u)$ and taking the inner product of the result with $f(u)$ yields the correlative time-frequency representation known as the ambiguity function:

$$A_f(\bar{u},-\bar{\mu}) = \langle \rho(\bar{u},\bar{\mu})f, f\rangle. \tag{2.1.9}$$

The Wigner distribution can be defined as the two-dimensional Fourier transform of the ambiguity function:

$$W_f(u,\mu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle \rho(\bar{u},\bar{\mu})f, f\rangle e^{-j2\pi(u\bar{\mu}+\mu\bar{u})}\, d\bar{u}\, d\bar{\mu}. \tag{2.1.10}$$

The above definitions can be easily put in the following forms:

$$W_f(u,\mu) = \int_{-\infty}^{\infty} f(u+\bar{u}/2)f^*(u-\bar{u}/2)e^{-j2\pi\bar{u}\mu}\, d\bar{u}, \tag{2.1.11}$$

$$= \int_{-\infty}^{\infty} F(\mu+\bar{\mu}/2)F^*(\mu-\bar{\mu}/2)e^{j2\pi\bar{\mu}u}\, d\bar{\mu}, \tag{2.1.12}$$

$$A_f(\bar{u},\bar{\mu}) = \int_{-\infty}^{\infty} f(u+\bar{u}/2)f^*(u-\bar{u}/2)e^{-j2\pi\bar{\mu}u}\, du, \tag{2.1.13}$$

$$= \int_{-\infty}^{\infty} F(\mu+\bar{\mu}/2)F^*(\mu-\bar{\mu}/2)e^{j2\pi\bar{u}\mu}\, d\mu. \tag{2.1.14}$$

[1]We note that the Weyl correspondence, the Schrödinger representation of Heisenberg group, and the Schwinger basis all refer to essentially the same thing.

After this review, two definitions of the discrete WD will be developed in a unified manner. In order to arrive at definitions of the WD which are analogous to the continuous definition in a fundamental sense, we will try to follow the same procedure for defining the continuous WD outlined in the previous section. This is in contrast to approaches based on sampling the continuous WD to arrive at a discrete definition [25–27]

We will be dealing with the discrete index sets S_1, S_2 respectively:

$$S_1 = \{0, 1, 2, 3, \ldots, N-1\}, \tag{2.1.15}$$

$$S_2 = \Big\{ -\frac{N-1}{2}, -\frac{N-3}{2}, \ldots, \frac{N-3}{2}, \frac{N-1}{2} \Big\}, \tag{2.1.16}$$

where N is odd. Our inner product convention is $\langle f, g\rangle = \sum_{n\in S} f[n]g^*[n]$. The DFT is defined as

$$\mathbf{F}\{f[n]\} = F[k] = \sum_{n\in S} f[n]e^{-j2\pi kn/N}, \tag{2.1.17}$$

$$\mathbf{F}^{-1}\{F[k]\} = f[n] = \frac{1}{N}\sum_{k\in S} F[k]e^{j2\pi kn/N}, \tag{2.1.18}$$

where S denotes S_1 or S_2. Which index set is being used will be evident from the context.

The discrete versions of the coordinate multiplication and coordinate differentiation operators can be defined easily [28–30]:

$$\mathbf{U}f[n] = nf[n], \qquad \text{in the time domain}, \tag{2.1.19}$$

$$\mathbf{D}F[k] = kF[k], \qquad \text{in the frequency domain}, \tag{2.1.20}$$

and are related through the DFT: $\mathbf{D} = \mathbf{F}^{-1}\mathbf{U}\mathbf{F}$. Analogous to the continuous case, we also have [28, 29],

$$e^{\frac{j2\pi\bar{k}\mathbf{U}}{N}} f[n] = e^{\frac{j2\pi\bar{k}n}{N}} f[n], \tag{2.1.21}$$

$$e^{\frac{j2\pi\bar{n}\mathbf{D}}{N}} f[n] = f[n+\bar{n}]. \tag{2.1.22}$$

Now, we may again analogously introduce the Weyl correspondence as [29]

$$e^{j2\pi(\bar{k}n+\bar{n}k)/N} \longrightarrow e^{j2\pi(\bar{k}\mathbf{U}+\bar{n}\mathbf{D})/N}, \tag{2.1.23}$$

with the intent of defining a discrete WD in a manner completely analogous to the definition of the continuous WD. However, it is not possible to proceed from this point onward in a completely analogous manner because unlike the continuous case, $[\mathbf{U},[\mathbf{U},\mathbf{D}]] \neq 0$, $[\mathbf{D},[\mathbf{U},\mathbf{D}]] \neq 0$, $[\mathbf{U},\mathbf{D}]] \neq \mathbf{I}$, so that we cannot apply an analogous Baker-Campbell-Hausdorff formula. For this reason it has not been possible to define a discrete WD though the Weyl correspondence [29], although the authors did not refer to this obstacle. Furthermore, this outcome is not dependent on the particular definition of $\mathbf{U}$ and $\mathbf{D}$ chosen, since the commutation relation $[\mathcal{U},\mathcal{D}] = \frac{j}{2\pi}\mathcal{I}$ does not have an analog in the discrete case for any $\mathbf{U}$ and $\mathbf{D}$ due to the following result [31]:

Proposition 1 *Two matrices* $\mathbf{A},\mathbf{B}$ *cannot have identity as the commutator:* $[\mathbf{A},\mathbf{B}] = \mathbf{AB} - \mathbf{BA} \neq \mathbf{I}$.

proof: Assume that there exists matrices $\mathbf{A},\mathbf{B}$ such that $\mathbf{AB} - \mathbf{BA} = \mathbf{I}$ and take the trace of both sides $Tr[\mathbf{AB}] - Tr[\mathbf{BA}] = Tr[\mathbf{I}]$. This leads to a contradiction since $Tr[\mathbf{AB}] - Tr[\mathbf{BA}] = 0 \neq Tr[\mathbf{I}]$. ■

This break of analogy with the continuous case, is a strong indication of the different nature of the discrete scenario. Despite this setback, we will proceed by maintaining the analogy as much as possible by employing the following discrete versions of equation (2.1.8) as two alternative definitions of $\rho[\bar{n},\bar{k}]$:

$$\rho(\bar{u},\bar{\mu}) = e^{j\pi\bar{u}\bar{\mu}} e^{j2\pi\bar{\mu}\mathcal{U}} e^{j2\pi\bar{u}\mathcal{D}} = e^{-j\pi\bar{u}\bar{\mu}} e^{j2\pi\bar{u}\mathcal{D}} e^{j2\pi\bar{\mu}\mathcal{U}}, \tag{2.1.24}$$

$$\rho^m[\bar{n},\bar{k}] = e^{\frac{j2\pi\bar{n}\bar{k}2^{-1}}{N}} e^{\frac{j2\pi\bar{k}\mathbf{U}}{N}} e^{\frac{j2\pi\bar{n}\mathbf{D}}{N}} = e^{\frac{-j2\pi\bar{n}\bar{k}2^{-1}}{N}} e^{\frac{j2\pi\bar{n}\mathbf{D}}{N}} e^{\frac{j2\pi\bar{k}\mathbf{U}}{N}}, \tag{2.1.25}$$

$$\rho^h[\bar{n},\bar{k}] = e^{\frac{j\pi\bar{n}\bar{k}}{N}} e^{\frac{j2\pi\bar{k}\mathbf{U}}{N}} e^{\frac{j2\pi\bar{n}\mathbf{D}}{N}} = e^{\frac{-j\pi\bar{n}\bar{k}}{N}} e^{\frac{j2\pi\bar{n}\mathbf{D}}{N}} e^{\frac{j2\pi\bar{k}\mathbf{U}}{N}}. \tag{2.1.26}$$

Two alternative definitions emerge from the two possible ways of handling the division by 2, both of which have their own advantages. In the first case (2.1.25), 2^{-1} denotes the $\mod N$ inverse of 2 which is given by $(N+1)/2$. In the second case (2.1.26), the division is handled in the usual sense so that 2^{-1} cancels the 2 in the numerator. Notice that the exponentials are both square roots of the same expression:

$$e^{\frac{j2\pi\bar{n}\bar{k}}{N}} = \left(e^{\frac{j\pi\bar{n}\bar{k}}{N}}\right)^2 = \left(e^{\frac{j2\pi}{N}[\bar{n}\bar{k}2^{-1}]}\right)^2. \tag{2.1.27}$$

The corresponding WDs may now be written in analogy with the continuous case as:

$$W_f(u,\mu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle \rho(\bar{u},\bar{\mu})f, f\rangle\, e^{-j2\pi(u\bar{\mu}+\mu\bar{u})}\, d\bar{u}\, d\bar{\mu}, \tag{2.1.28}$$

$$W_f^m[n,k] = \frac{1}{N}\sum_{\bar{n}\in S_1}\sum_{\bar{k}\in S_1} \langle \rho^m[\bar{n},\bar{k}]f, f\rangle\, e^{-j2\pi(n\bar{k}+k\bar{n})/N}, \tag{2.1.29}$$

$$W_f^h[n,k] = \frac{1}{N}\sum_{\bar{n}\in S_2}\sum_{\bar{k}\in S_2} \langle \rho^h[\bar{n},\bar{k}]f, f\rangle\, e^{-j2\pi(n\bar{k}+k\bar{n})/N}. \tag{2.1.30}$$

Both of these discrete definitions have been studied extensively and originate from [15] and [18] respectively.

Although it is impossible to obtain a discrete WD exactly the same way in the continuous case, we must note that the presented derivation is equivalent to starting with the discrete Rihaczek distribution [29] and then proceeding to the discrete WD. This is still legitimate in the framework of the characteristic function operator method since we can obtain any member of the Cohen's class from the other members [5].

A third definition of the discrete WD, which we will refer to as WD^{s}, will be discussed in section 2.3.

2.2 Discussion of the properties of the two WDs

In this section we will compare the two discrete WDs defined in the previous section. As already noted, WD^{m} is the definition exhibiting the greatest possible degree of structural analogy but lacks a direct connection to the continuous WD. While WD^{h} does not satisfy all the analogous properties of WD^{m}, it approximates the samples of the continuous WD.

When we speak of structural analogy, we will focus our attention on both operational time-frequency properties (look ahead to table 2.2.1) and a set of cross relationships between the WD, the ambiguity function and a set of so-called auxiliary functions (look ahead to figure 2.2.1) satisfied by the continuous WD.

2.2.1 Auxilary functions

The following auxiliary functions play an important role in the study of the continuous WD:

$$\gamma_f(u,\bar{u}) = f(u+\bar{u}/2)f^*(u-\bar{u}/2), \tag{2.2.1}$$
$$\Gamma_f(\mu,\bar{\mu}) = F(\mu+\bar{\mu}/2)F^*(\mu-\bar{\mu}/2). \tag{2.2.2}$$

To define the discrete counterparts of these entities, we must decide how to handle the division by two. For WD$^{\mathrm{m}}$ we define:

$$\gamma_f^m[n,\bar{n}] = f[n+\bar{n}2^{-1}]f^*[n-\bar{n}2^{-1}], \tag{2.2.3}$$
$$\Gamma_f^m[k,\bar{k}] = F[k+\bar{k}2^{-1}]F^*[k-\bar{k}2^{-1}], \tag{2.2.4}$$

where 2^{-1} is defined in the modulo sense and $n+2^{-1}$ is the halfway between n and $n+1$ in a circular context.

Figure 2.2.1 shows the relationships between the discrete WD, ambiguity function and auxiliary functions, which is fully analogous to a similar set of relationships satisfied by their continuous counterparts. The derivation of these relationships are elementary and fully analogous to the derivations in the continuous case and only the derivations of a subset is shown below. First note that the inner product in the definition of the WD$^{\mathrm{m}}$ in equation (2.1.29) can be further simplified by a change of variables $n \to n-\bar{n}2^{-1}$ as follows:

$$\begin{aligned}\langle \rho^m[\bar{n},\bar{k}]f,f\rangle &= \sum_{n=0}^{N-1} e^{j2\pi(2^{-1}\bar{n}\bar{k})/N} e^{j2\pi\bar{k}(n-\bar{n}2^{-1})/N} f[n-\bar{n}2^{-1}+\bar{n}]f^*[n-\bar{n}2^{-1}],\\ &= \sum_{n=0}^{N-1} f[n+\bar{n}2^{-1}]f^*[n-\bar{n}2^{-1}]e^{j2\pi\bar{k}n/N}. \end{aligned} \tag{2.2.5}$$

By combining equations (2.2.5) and (2.1.29) a simple expression for WD$^{\mathrm{m}}$ can be obtained as:

$$
\begin{aligned}
W_f^m[n,k] &= \frac{1}{N}\sum_{\bar{n},\bar{k},n'=0}^{N-1} \gamma_f^m[n',\bar{n}]e^{j2\pi\bar{k}n'/N}e^{-j2\pi(n\bar{k}+k\bar{n})/N} \\
&= \sum_{\bar{n},n'=0}^{N-1} \gamma_f^m[n',\bar{n}]e^{-j2\pi k\bar{n}/N}\delta[n-n'] \\
&= \sum_{\bar{n}=0}^{N-1} f[n+\bar{n}2^{-1}]f^*[n-\bar{n}2^{-1}]e^{-j2\pi k\bar{n}/N}, \qquad (2.2.6)
\end{aligned}
$$

Equations (2.2.5), (2.2.6) and a corresponding pair of equations for Γ_f which can be similarly derived, can be summarized in graphical form (figure 2.2.1) which is familiar from the continuous case [10, 11]. This constitutes further support for the strong structural analogy of this definition to the continuous case. To the best of our knowledge the auxiliary functions have not been defined for WD$^{\mathrm{m}}$ and the relationships depicted in this figure have not been shown.

We now turn our attention to obtaining similar results for WD$^{\mathrm{h}}$. Since division by two is actually treated as a half-integer in this case, we must more carefully examine the concept of shifting discrete functions by half an integer, since such functions are undefined for non-integer values. Since the operator $e^{\frac{j2\pi\bar{n}\mathbf{D}}{N}}$ corresponds to a shift by the integer amount $\bar{n}$, we define a half-integer shift as $e^{\frac{j\pi\bar{n}\mathbf{D}}{N}} = \mathbf{F}^{-1}e^{\frac{j\pi\bar{n}\mathbf{U}}{N}}\mathbf{F}$. Applying this operator to a periodic signal $f[n]$ defined over the set S_2 we obtain

$$
e^{\frac{j\pi\bar{n}\mathbf{D}}{N}}f[n] = \sum_{n'\in S_2} f[n']\phi(n+\frac{\bar{n}}{2}-n') \qquad (2.2.7)
$$

where

$$
\phi(u) = \frac{1}{N}\sum_{n'\in S_2} e^{j2\pi n'u/N} = \frac{\sin(\pi u)}{N\sin(\pi u/N)} \qquad (2.2.8)
$$

which is essentially an interpolation relation. Note that $\phi(u)$, the periodically replicated version of the sinc function, is the interpolation function for periodic band-limited signals [32, 33]. When the argument of $\phi(u)$ is an integer n, it reduces to $\phi(n) = \delta[n]$. Thus $\phi(u)$ is a generalization of the delta function and basis to non-integer values.

Now, we may attempt to define the discrete versions of equations (2.2.1) and (2.2.2) for $\mathrm{WD^h}$ as follows:

$$\hat{\gamma}_f[n,\bar{n}] \triangleq \left(e^{\frac{j\pi\bar{n}\mathbf{D}}{N}} f[n]\right)\left(e^{\frac{-j\pi\bar{n}\mathbf{D}}{N}} f^*[n]\right), \tag{2.2.9}$$

$$\hat{\Gamma}_f[k,\bar{k}] \triangleq \left(e^{\frac{j\pi\bar{k}\mathbf{D}}{N}} F[k]\right)\left(e^{\frac{-j\pi\bar{k}\mathbf{D}}{N}} F^*[k]\right). \tag{2.2.10}$$

Unfortunately, these two definitions are neither consistent with each other nor do they lead to a set of relationships of the form given by figure 2.2.1 [34]. The underlying reason for this is that fractional shifts as defined above are not distributive over multiplication:

$$e^{\frac{j\pi\bar{n}\mathbf{D}}{N}}\left(f[n]g[n]\right) \neq \left(e^{\frac{j\pi\bar{n}\mathbf{D}}{N}} f[n]\right)\left(e^{\frac{j\pi\bar{n}\mathbf{D}}{N}} g[n]\right). \tag{2.2.11}$$

This can be solved by defining the auxilary functions in asymmetric form [34]:

$$\gamma_f^h[n,\bar{n}] = e^{\frac{-j\pi\bar{n}\mathbf{D}}{N}}\left(f[n+\bar{n}]f^*[n]\right) \tag{2.2.12}$$

$$= e^{\frac{j\pi\bar{n}\mathbf{D}}{N}}\left(f[n]f^*[n-\bar{n}]\right), \tag{2.2.13}$$

$$\Gamma_f^h[k,\bar{k}] = e^{\frac{-j\pi\bar{k}\mathbf{D}}{N}}\left(F[k+\bar{k}]F^*[k]\right) \tag{2.2.14}$$

$$= e^{\frac{j\pi\bar{k}\mathbf{D}}{N}}\left(F[k]F^*[k-\bar{k}]\right). \tag{2.2.15}$$

We reemphasize that due to a lack of the multiplicity property above, these cannot be reduced to the form of equations (2.2.9) and (2.2.10). Nevertheless, in a certain sense these definitions are not so asymmetric since the asymmetric functions $f[n+\bar{n}]f^*[n]$ and $F[k+\bar{k}]F^*[k]$ are symmetrized by applying the operators $e^{\frac{-j\pi\bar{n}\mathbf{D}}{N}}$ and $e^{\frac{-j\pi\bar{k}\mathbf{D}}{N}}$ respectively. These definitions fully satisfy the relationships embodied in figure 2.2.1 [34].

2.2.2 Operational properties

In table 2.2.1 we list various properties which a discrete Wigner distribution may be expected to satisfy for both $\mathrm{WD^m}$ and $\mathrm{WD^h}$ which we have discussed above, and also for $\mathrm{WD^s}$ which we will discuss in a following section.

Properties involving the instantaneous frequency are not included since these do not generalize easily to the discrete-time discrete-frequency case. Also excluded

Table 2.2.1: The expressions for the discrete properties

1.	Shifts	$f[n-n_o]e^{j2\pi nk_o/N} \rightarrow W_f[n-n_o, k-k_o]$
2.	Scale	$h[n]=f[an], (a,N)=1 \rightarrow W_h[n,k]=W_f[an, ka^{-1}]$
3.	Modulation	$h[n]=f[n]g[n] \rightarrow W_h[n,k]=W_f[n,k] \star_k W_g[n,k]$
4.	Convolution	$h[n]=f[n]\star g[n] \rightarrow W_h[n,k]=W_f[n,k] \star_n W_g[n,k]$
5.	Reality	$(W_f[n,k])^* = W_f[n,k]$
6.	Time Marginal	$\sum_k W_f[n,k] = N\lvert f[n]\rvert^2$
7.	Frequency Marginal	$\sum_n W_f[n,k] = N\lvert F[k]\rvert^2$
8.	Generalized Marginals	Marginals over arbitrary discrete angles [35]
9.	Total Energy	$\sum_n \sum_k W_f[n,k] = N\times$Energy of the signal
10.	Moyal	$N\langle f_1, g_1\rangle\overline{\langle f_2, g_2\rangle} = \langle W_{f_1,f_2}, W_{g_1,g_2}\rangle$
11.	Quadratic	$W_f[n,k] = \sum_{n_1}\sum_{n_2} f[n_1]f^*[n_2]K[n,k;n_1,n_2]$
12.	Fourier Transform	$W_F[n,k] = W_f[-k,n]$
13.	Frequency Localization	$F[k]=\delta[k-k_o] \rightarrow W_f[n,k]=N\delta[k-k_o]$
14.	Time Localization	$f[n]=\delta[n-n_o] \rightarrow W_f[n,k]=N\delta[n-n_o]$
15.	Chirp Localization	$f[n]=e^{j2\pi 2^{-1}qn^2/N}, \forall q\in\mathbb{Z} \rightarrow W_f[n,k]=N\delta[k-qn]$
		$f[n]=e^{j\pi qn^2/N}, \forall q\in\mathbb{Z} \rightarrow W_f[n,k]=N\delta[k-qn]$
16.	Chirp Convolution	$F[k]e^{j2\pi 2^{-1}qk^2/N}, \forall q\in\mathbb{Z} \rightarrow W_f[n+qk,k]$
		$F[k]e^{j\pi qk^2/N}, \forall q\in\mathbb{Z} \rightarrow W_f[n+qk,k]$
17.	Chirp Multiplication	$f[n]e^{j2\pi 2^{-1}qn^2/N}, \forall q\in\mathbb{Z} \rightarrow W_f[n,k-qn]$
		$f[n]e^{j\pi qn^2/N}, \forall q\in\mathbb{Z} \rightarrow W_f[n,k-qn]$
18.	Continuous Limit	Has the continuous WD as a limit
19.	Sampling Theory	Can be utilized as a numerical tool

*All summations are carried out over one period N (N is odd).
‡All inversions are performed in mod N.
†All convolutions are circular with respect to the subscripted parameter.

Table 2.2.2: A comparison of the properties of the discrete Wigner distributions WD^m, WD^h, WD^s.

		WD^m	WD^h	WD^s
1.	Shifts	✓	✓	-
2.	Scale	✓	-	-
3.	Modulation	✓	-	-
4.	Convolution	✓	-	-
5.	Reality	✓	✓	✓
6.	Time Marginal	✓	✓	✓
7.	Frequency Marginal	✓	✓	-
8.	Generalized Marginals	✓	-	-
9.	Total Energy	✓	✓	✓
10.	Moyal	✓	✓	-
11.	Quadratic	✓	✓	✓
12.	Fourier Transform	✓	✓	-
13.	Frequency Localization	✓	✓	-
14.	Time Localization	✓	✓	✓
15.	Chirp Localization	✓	-	-
		-	✓	-
16.	Chirp Convolution	✓	-	-
		-	✓	-
17.	Chirp Multiplication	✓	-	-
		-	✓	-
18.	Continuous Limit	-	✓	-
19.	Sampling Theory	-	✓	✓

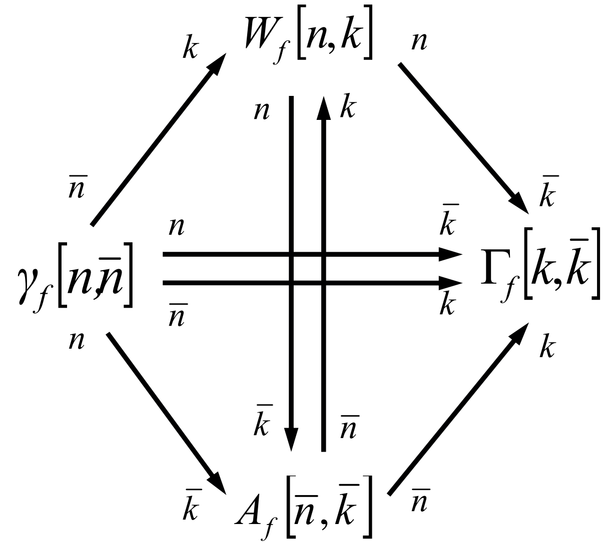

Figure 2.2.1: Graphical representation of the relationships between the two WDs, their corresponding ambiguity functions and auxiliary functions. The relationships are valid for both $\mathrm{WD^m}$ and $\mathrm{WD^h}$. Arrows denote DFTs.

are finite time support and finite frequency support properties due to difficulties in defining them in the cyclic case.

As noted in the introduction, $\mathrm{WD^m}$ satisfies all of the structural properties satisfied by the continuous WD, with the exception of those which have a meaning only in the continuous case. This is a consequence of the fact that $\mathrm{WD^m}$ and the continuous WD are two different realizations of the same underlying group-theoretical structure. Compared with $\mathrm{WD^m}$, we observe that the most important properties $\mathrm{WD^h}$ lacks are the convolution and modulation properties. On the other hand, $\mathrm{WD^m}$ lacks a direct relationship to the continuous WD in the sense of sampling or approximation. As noted in [16], $\mathrm{WD^m}$ satisfies chirp multiplication and convolution properties (properties 15, 16) for modulo chirps $e^{j2\pi(2^{-1}qn^2)/N}$. However, these chirps are equal to common chirp functions only when q is even but not when q is odd:

$$e^{j2\pi(2^{-1}qn^2)/N} \neq e^{j\pi qn^2/N}, \qquad q \text{ is odd.} \tag{2.2.16}$$

Most of the properties in table 2.2.2 are either known or can be verified easily and are omitted. Properties 14, 15, 16 are derived for $\mathrm{WD^m}$ in [24]; here we sketch the derivation for $\mathrm{WD^h}$. Although we cannot use the chirp $e^{j\pi qn^2/N}$ directly in the definition of $\mathrm{WD^h}$ since it is not periodic with N, we can still prove property 15.

Lets assume that the following equality holds for the WDs of two signals g and h.

$$W_g^h[n,k] = W_f^h[n,k-qn], \tag{2.2.17}$$

from this we can infer that

$$\gamma_g^h[n,\bar{n}] = \gamma_f^h[n,\bar{n}]e^{j2\pi qn\bar{n}/N}. \tag{2.2.18}$$

Applying the $e^{\frac{j\pi\bar{n}\mathbf{D}}{N}}$ operator to both sides gives[2]

$$g[n+\bar{n}]g^*[n] = f[n+\bar{n}]f^*[n]e^{j2\pi q(n+\bar{n}/2)\bar{n}/N}. \tag{2.2.19}$$

Now letting $n = 0$,

$$g[\bar{n}]g^*[0] = f[\bar{n}]f^*[0]e^{j\pi q\bar{n}^2/N}, \tag{2.2.20}$$

from which we conclude that the effect of chirp multiplication is shearing in the frequency direction. The other chirp properties 14, 16 can be derived in a similar manner.

Several sets of desirable properties that uniquely define the continuous WD have been proposed. First we consider the set of properties consisting of properties 1, 3-4, 5, 10, 11 in table 2.2.1. In [17] it is shown that the only discrete distribution satisfying all of these properties is WD$^{\mathrm{m}}$ and that only for odd values of N.

Within this framework we now summarize our comparison of WD$^{\mathrm{m}}$ and WD$^{\mathrm{h}}$. The use of WD$^{\mathrm{h}}$ entails a loss of a number of structural properties compared to WD$^{\mathrm{m}}$, most prominent of which are the convolution and multiplication properties. However, since WD$^{\mathrm{m}}$ cannot be directly related to the continuous WD, which severely limits its usefulness in many applications, and since WD$^{\mathrm{m}}$ is the only definition satisfying all of properties 1, 3-4, 5, 10, 11, in seeking an alternative definition it necessarily follows that we must lose at least one of these properties. Given the importance of these properties, the convolution and multiplication properties seem the most dispensable, despite their attractiveness. It is also interesting to note that the modulation and convolution properties are virtually absent from the physics literature on finite phase-space theory despite the fact that they are common in the

[2] Despite the fact that the fractional shift operator is not multiplicative in general, it is so if one of the the signals is a harmonic function with integer valued frequency.

signal processing literature. Turning our attention to the property of generalized marginals, once again in seeking an alternative definition we must abandon this property.

While we are not able to similarly argue for the inevitable loss of the scaling property 2 for $\mathrm{WD^h}$, we note that this property is satisfied by $\mathrm{WD^m}$ only for prime values of N and it does not seem to be considered among the most important properties of the WD, as also evidenced by the lack of its appearance in the above mentioned sets of essential properties. On the other hand, $\mathrm{WD^h}$ can be related to the continuous WD in the sense of sampling or approximation, and the chirps involved in the chirp multiplication and convolution properties are analogous to continuous chirps, neither of which is true for $\mathrm{WD^m}$.

It is also of interest to consider another set of properties uniquely defining the continuous WD. It is known that the only member of Cohen's class of time-frequency distribution satisfying property 14 in the continuous case is the continuous WD [36]. Although we do not give any proof in the discrete case, it is quite possible that the discrete versions of chirp localization properties uniquely define the discrete WDs in the discrete case. In this perspective the definition $\mathrm{WD^h}$ seems better than $\mathrm{WD^m}$ since it has the localization property with the common chirp signal $e^{j\pi qn^2/N}$ while $\mathrm{WD^m}$ has the localization property with the modulo chirp signal $e^{j2\pi 2^{-1}qn^2/N}$.

Therefore, all things considered, $\mathrm{WD^h}$ emerges as a definition of the discrete WD which can be used to approximate the continuous WD or related to it through sampling, and at the same time, has a group-theoretical foundation, and satisfies a large number of structural properties. The few properties it does not satisfy seem to be more or less the most dispensable ones among those which cannot be all simultaneously satisfied. As a result, this definition seems to be a very strong, if not the strongest candidate for a Wigner distribution combining the two major objectives set out in the introduction of this chapter.

2.3 Connections to the continuum and sampling

The discrete Fourier transform (DFT) satisfies both of the major objectives set out in the introduction, namely structural analogy and numerical approximation of the the continuous Fourier transform. As already stated, ideally we would like to achieve the same with our definition of the discrete Wigner distribution. The definition WD^{m} already discussed in detail, satisfies a maximal set of properties analogous to the continuous WD, but cannot be used to approximate the continuous WD, and thus completely fails to satisfy one of our objectives despite it elegance.

In this section we first consider a widely used definition for the discrete WD, which is widely used for computational purposes [8, 37]:

$$W_f^s(n,k) = \sum_{\bar{n}=0}^{M-1} f(n+\bar{n})f^*(n-\bar{n})e^{-j2\pi\bar{n}k/M}. \tag{2.3.1}$$

In the above definition, $f(n)$ denotes the samples of the signal which has duration M. The shifts inside the summation are linear and not cyclic. This definition avoids aliasing if the sampling rates are double the Nyquist rate. As shown in table 2.2.2, this definition fails to satisfy many of the desirable properties expected of a definition of the discrete WD. Therefore, despite its useful relation to the continuous WD, this definition fails to satisfy our objective regarding structural analogy to the continuous WD. Since our aim is to satisfy both of the objectives of continuous approximation and structural analogy, even if to a limited extent, the definitions WD^{m} and WD^{s} cannot be rated very highly since they do very poorly in the first and second of our objectives respectively.

Furthermore, we also note that WD^{s} is not fully analogous to the continuous WD in a formal sense either due to the absence of the 1/2 terms in the arguments of the functions.

2.3.1 Sampling and WD^{h}

In this section we will give a derivation of the definition WD^{h} based on sampling theory. We will assume that the signal is approximately confined to a finite region

$[-\Delta u/2, \Delta u/2]$ in the time domain and $[-\Delta\mu/2, \Delta\mu/2]$ in the frequency domain. Under this assumption, the WD of the signal is confined to a region $[-\Delta u/2, \Delta u/2] \times [-\Delta\mu/2, \Delta\mu/2]$, and the AF is confined to a region $[-\Delta u, \Delta u] \times [-\Delta\mu, \Delta\mu]$ due to the correlative nature [11]. Thus, the energy of the signal can be approximated as:

$$\text{Energy of the signal} \approx \int_{\frac{-\Delta u}{2}}^{\frac{\Delta u}{2}} \int_{\frac{-\Delta\mu}{2}}^{\frac{\Delta\mu}{2}} W_f(u, \mu)\, du\, d\mu. \tag{2.3.2}$$

By proper choice of Δu and $\Delta\mu$, this approximation can be made as accurate as necessary. In order to obtain a discrete WD by sampling the continuous WD, we will use the relationship between the WD and the AF and then the definition of the ambiguity function in terms of auxiliary functions. Since there are four parameters $u, \bar{u}, \mu, \bar{\mu}$ there will be four corresponding sampling rates, respectively $T_u, T_{\bar{u}}, T_\mu, T_{\bar{\mu}}$. These sampling rates must be chosen in such a way that there is no aliasing and the structural relationships inherent in figure 2.2.1 are maintained. Since the WD is the double Fourier transform of the AF, and both are confined to a finite region, we can apply the sampling theorem and use a double DFT to compute the samples of the WD from the samples of the AF. The continuous WD and AF are related as follows:

$$W_f(u, \mu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A_f(\bar{u}, -\bar{\mu}) e^{-j2\pi(u\bar{\mu}+\bar{u}\mu)}\, d\bar{u}\, d\bar{\mu}. \tag{2.3.3}$$

We will use the asymmetric form of the AF obtained by the variable substitution $u \to u + \bar{u}/2$ for sampling:

$$A_f(\bar{u}, -\bar{\mu}) = e^{j\pi\bar{u}\bar{\mu}} \int_{-\infty}^{\infty} \psi(u, \bar{u}) e^{j2\pi u\bar{\mu}}\, du. \tag{2.3.4}$$

$$= e^{-j\pi\bar{u}\bar{\mu}} \int_{-\infty}^{\infty} \Psi(\mu, \bar{\mu}) e^{j2\pi\bar{u}\mu}\, d\mu. \tag{2.3.5}$$

where $\psi(u, \bar{u}) = f(u + \bar{u}) f^*(u)$ and $\Psi(\mu, \bar{\mu}) = F(\mu - \bar{\mu}) F^*(\mu)$. The samples of the AF are

$$A_f(\bar{n}T_{\bar{u}}, -\bar{k}T_{\bar{\mu}}) = e^{j\pi\bar{n}\bar{k}T_{\bar{\mu}}T_{\bar{u}}} \int_{-\infty}^{\infty} \psi(u, \bar{n}T_{\bar{u}}) e^{j2\pi u\bar{k}T_{\bar{\mu}}}\, du \tag{2.3.6}$$

$$= e^{-j\pi\bar{n}\bar{k}T_{\bar{\mu}}T_{\bar{u}}} \int_{-\infty}^{\infty} \Psi(\mu, \bar{k}T_{\bar{\mu}}) e^{j2\pi\bar{n}T_{\bar{u}}\mu}\, d\mu. \tag{2.3.7}$$

In order to avoid aliasing in the WD domain, the following conditions must be satisfied

$$\frac{1}{T_{\bar{u}}} \geq \Delta\mu, \qquad \frac{1}{T_{\bar{\mu}}} \geq \Delta u, \qquad \text{Sampling the AF,} \tag{2.3.8}$$

$$\frac{1}{T_u} \geq 2\Delta\mu, \qquad \frac{1}{T_\mu} \geq 2\Delta u, \qquad \text{Sampling the WD.} \tag{2.3.9}$$

Returning to equation (2.3.6) and (2.3.7) which give the samples of the AF, they are still expressed in terms of continuous functions. To replace the Fourier transform of $\psi(u, \bar{n}T_{\bar{u}})$ and $\Psi(\mu, \bar{k}T_{\bar{\mu}})$ with an inverse DFT, we must observe the following relationships to avoid aliasing:

$$\frac{1}{T_u} \geq 2\Delta\mu, \qquad \frac{1}{T_{\bar{\mu}}} \geq \Delta u, \qquad \text{Constraints for eq. (2.3.6)} \tag{2.3.10}$$

$$\frac{1}{T_\mu} \geq 2\Delta u, \qquad \frac{1}{T_{\bar{u}}} \geq \Delta\mu \qquad \text{Constraints for eq. (2.3.7)} \tag{2.3.11}$$

due to the quadratic structure of $\psi(u, \bar{n}T_{\bar{u}})$ and $\Psi(\mu, \bar{k}T_{\bar{\mu}})$. Note that (2.3.10), (2.3.11) are consistent with the (2.3.8), (2.3.9) set of constraints. The samples of the AF (2.3.6), (2.3.7) will be approximated as:

$$A_f(\bar{n}T_{\bar{u}}, -\bar{k}T_{\bar{\mu}}) \approx e^{j\pi\bar{n}\bar{k}T_{\bar{\mu}}T_{\bar{u}}} \sum_n \psi(nT_u, \bar{n}T_{\bar{u}}) e^{j2\pi n\bar{k}T_u T_{\bar{\mu}}} \tag{2.3.12}$$

$$A_f(\bar{n}T_{\bar{u}}, -\bar{k}T_{\bar{\mu}}) \approx e^{j\pi\bar{n}\bar{k}T_{\bar{\mu}}T_{\bar{u}}} \sum_k \Psi(kT_\mu, \bar{k}T_{\bar{\mu}}) e^{j2\pi n\bar{k}T_{\bar{u}} T_\mu} \tag{2.3.13}$$

We must choose the sampling rates $T_u, T_{\bar{u}}, T_\mu, T_{\bar{\mu}}$ such that there is no aliasing and the definition $\mathrm{WD^h}$ is obtained. We shall choose,

$$T_{\bar{u}}T_{\bar{\mu}} = T_u T_{\bar{\mu}} \Longrightarrow T_{\bar{u}} = T_u, \tag{2.3.14}$$

$$T_{\bar{u}}T_{\bar{\mu}} = T_{\bar{u}} T_\mu \Longrightarrow T_{\bar{\mu}} = T_\mu. \tag{2.3.15}$$

This choice is necessary since the term $e^{j\pi\bar{n}\bar{k}T_{\bar{\mu}}T_{\bar{u}}}$ and $e^{j2\pi n\bar{k}T_u T_{\bar{\mu}}}$ in equation (2.3.12) must have equal periods. With these assumptions on the sampling rates, the asymmetric auxiliary functions can be written as:

$$\psi(nT_u, \bar{n}T_{\bar{u}}) = f(nT_u + \bar{n}T_{\bar{u}}) f^*(nT_u) \to f(n+\bar{n}) f^*(n), \tag{2.3.16}$$

$$\Psi(kT_\mu, \bar{k}T_{\bar{\mu}}) = F(kT_\mu - \bar{k}T_{\bar{\mu}}) F^*(kT_\mu) \to F(k-\bar{k}) F^*(k). \tag{2.3.17}$$

If we combine the constraints in equations (2.3.8), (2.3.9), (2.3.10), (2.3.11) and (2.3.14), (2.3.15) the following rates are the minimum ones:

$$T_u = T_{\bar{u}} = \frac{1}{2\triangle\mu}, \qquad T_\mu = T_{\bar{\mu}} = \frac{1}{2\triangle u}. \tag{2.3.18}$$

However, if we further assume that $\Delta u \Delta\mu$ is chosen such that $\Delta u\Delta\mu = N$ integer, then the optimal choice leads to $4N$ length DFTs. Unfortunately, the definition $\mathrm{WD^h}$ is defined only for odd length signals and as a result we can not obtain $\mathrm{WD^h}$ in this optimal sampling strategy. In order to have an odd length WD, we will choose the following sampling rates:

$$T_u = T_{\bar{u}} = \frac{1}{2\triangle\mu}, \qquad T_\mu = T_{\bar{\mu}} = \frac{2\Delta\mu}{4\Delta u\Delta\mu + 1} < \frac{1}{2\Delta u}. \tag{2.3.19}$$

Since $f(n)$ has duration $2N$, $f(n+\bar{n})f^*(n)$ also has duration $2N$. The summation in 2.3.12 can be put in to the following form:

$$A(\bar{n}T_{\bar{u}}, -\bar{k}T_{\bar{\mu}}) \approx e^{\frac{j\pi\bar{n}\bar{k}}{4N+1}} \sum_{n=-N}^{N} f(n+\bar{n})f^*(n)e^{\frac{j2\pi n\bar{k}}{4N+1}} = e^{\frac{j\pi\bar{n}\bar{k}}{4N+1}} \sum_{n=-2N}^{2N} f(n+\bar{n})f^*(n)e^{\frac{j2\pi n\bar{k}}{4N+1}} \tag{2.3.20}$$

By zero padding the $2N$ nonzero terms in $f(n)$ to $4N+1$, periodically replicating and writing as $f[n]$, the summation can now be written as:

$$A_f^h[\bar{n}, -\bar{k}] = e^{\frac{j\pi\bar{n}\bar{k}}{4N+1}} \sum_{n=-2N}^{2N} f[n+\bar{n}]f^*[n]e^{\frac{j2\pi n\bar{k}}{4N+1}}. \tag{2.3.21}$$

We replaced the linear shifts with circular shifts since the signal $f[n]$ has duration $2N$ and the correlative shifts in $f[n+\bar{n}]f^*[n]$ are in the range of $\bar{n} \in [-2N, 2N]$. Note that we obtained the ambiguity function corresponding to the $\mathrm{WD^h}$. Then the $\mathrm{WD^h}$ will be the double DFT of the $\mathrm{AF^h}$.

It is also possible to make other choices like:

$$T_u = T_{\bar{u}} = \frac{2\Delta u}{4\Delta u\Delta\mu + 1} < \frac{1}{2\Delta\mu}, \qquad T_\mu = T_{\bar{\mu}} = \frac{1}{2\Delta u}. \tag{2.3.22}$$

Since $F(k)$ has duration $2N$, $F(k-\bar{k})F^*(k)$ also has duration $2N$. The summation in 2.3.13 can be put in to the following form:

$$A(\bar{n}T_{\bar{u}}, -\bar{k}T_{\bar{\mu}}) \approx e^{\frac{j\pi\bar{n}\bar{k}}{4N+1}} \sum_{k=-N}^{N} F(k-\bar{k})F^*(k)e^{\frac{j2\pi k\bar{n}}{4N+1}} = e^{\frac{j\pi\bar{n}\bar{k}}{4N+1}} \sum_{k=-2N}^{2N} F(k-\bar{k})F^*(k)e^{\frac{j2\pi k\bar{n}}{4N+1}} \tag{2.3.23}$$

by zero padding the $2N$ nonzero terms in $F(k)$ to $4N+1$, periodically replicating and writing as $F[k]$, the summation can now be written as:

$$A_f^h[\bar{n}, -\bar{k}] = e^{\frac{j\pi\bar{n}\bar{k}}{4N+1}} \sum_{k=-2N}^{2N} F[k-\bar{k}]F^*[k]e^{\frac{j2\pi k\bar{n}}{4N+1}}. \tag{2.3.24}$$

We replaced the linear shifts with circular shifts since the signal $F[k]$ has duration $2N$ and the correlative shifts in $F[k-\bar{k}]F^*[k]$ are in the range of $\bar{k} \in [-2N, 2N]$. Note that we obtained the ambiguity function corresponding to the $\mathrm{WD^h}$. Then the $\mathrm{WD^h}$ will be the double DFT of the $\mathrm{AF^h}$.

In summary, we must sample the signal at twice the Nyquist rate and then apply zero padding such that the final length is $4N+1$ where N denotes the number of degrees of freedom. We made two operations which included redundancy. The first one is sampling at the double Nyquist rate. This is necessary and natural since the AF is quadratic and this results in frequency doubling. We further applied a zero padding which is still necessary since the AF is of correlative nature and in order to replace linear correlation with circular correlation zero padding is necessary.

2.3.2 Relationship between $\mathrm{WD^m}$ and $\mathrm{WD^s}$

In this subsection we will show a simple relationship between the definitions $\mathrm{WD^m}$ and $\mathrm{WD^s}$ defined as:

$$W_f^s(n,k) = \sum_{\bar{n}=0}^{M-1} f(n+\bar{n})f^*(n-\bar{n})e^{-j2\pi\bar{n}k/M}. \tag{2.3.25}$$

The shifts in above expression are linear. However the definition $\mathrm{WD^m}$ is defined by using cyclic shifts. If the signal $f(n)$ is zero padded to $N \geq 4M+1$ and denoted as $f[n]$, the linear shifts in the definition can be represented with cyclic shifts. $\mathrm{WD^s}$ can be written as:

$$W_f^s[n,k] = \sum_{\bar{n}=0}^{N-1} f[n+\bar{n}]f^*[n-\bar{n}]e^{-j2\pi\bar{n}k/N}. \tag{2.3.26}$$

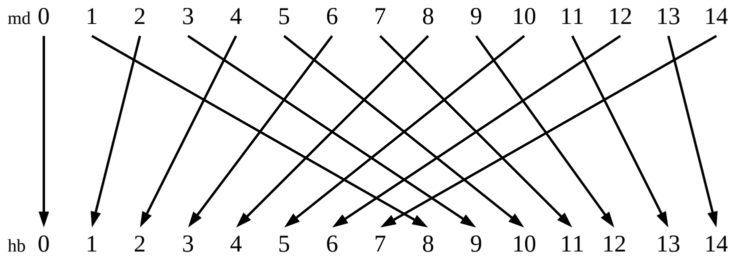

Figure 2.3.1: The permutation of the values of WD^{m} and WD^{s} along the k axis for $N = 15$. (md refers to WD^{m} and hb refers to WD^{s})

$$\begin{aligned} W_f^m[n,k] &= \sum_{\bar{n}=0}^{N-1} f[n+\bar{n}2^{-1}]f^*[n-\bar{n}2^{-1}]e^{-j2\pi k\bar{n}/N} \\ &= \sum_{\bar{n}=0}^{N-1} f[n+\bar{n}]f^*[n-\bar{n}]e^{-j2\pi(2\bar{n}k)/N} \\ &= W_f^s[n,2k], \end{aligned} \tag{2.3.27}$$

or equivalently

$$W_f^s[n,k] = W_f^m[n,2^{-1}k], \tag{2.3.28}$$

where $2^{-1}k$ is again computed modulo N. We used the substitution $\bar{n} \to 2\bar{n}$ in passing to the second line of (2.3.27). This remarkably simple relationship means that the values of either of these WDs is obtained simply by rearranging (permuting) the values of the other along the frequency axis (figure 2.3.1). It is interesting to note that the resulting permutation is in the form of a perfect shuffle and is also related to decimation in time. This relationship also means that if we know the WD according to one of these definitions, we can quickly compute WD according to the other definition by simply rearranging the values. where $f[n]$ is periodic and the shifts in the definition are cyclic.

2.4 Group theoretical discussion

In this section we will give a group theoretical discussion for the definitions of discrete WDs. We will avoid a detailed and rigorous treatment and will provide a general sketch of the theory. Before proceeding to a group theoretical foundations of the WD, we will briefly review the definitions of a group, Lie group, differentiable manifold, and Lie algebra. Rigorous definitions and detailed treatment of the subject can be found in [13, 38].

A group is a set associated with a binary operation such that the binary operation defined on this set satisfies closure, existence of identity, existence of an inverse for all members, and associativity properties. The real line with addition operation is a common example of a group. A Lie group has an extra structure in addition to the group property: a differentiable manifold. A differentiable manifold is a generalizations of a differentiable curve to higher dimensions. The real line, the sphere, and the torus are examples of differentiable manifolds. A Lie algebra is a linear space associated with a binary operation called the Lie bracket. A common example of a Lie algebra is the 3 dimensional vector space with the vector product being the Lie bracket. There exists a very important connection between Lie groups and Lie algebras. The Lie algebra is the tangent space of the Lie group near the identity element of the group. Furthermore the exponential map connects one parameter subgroups of a Lie group to the corresponding Lie algebra.

The Heisenberg group is also a Lie group and the members of the group are given as:

$$\rho(\bar{u}, \bar{\mu}, t) = e^{j2\pi(\bar{\mu}\mathcal{U} + \bar{u}\mathcal{D} + t\mathcal{I})} \tag{2.4.1}$$

$$= e^{j2\pi t\mathcal{I}} e^{j\pi\bar{u}\bar{\mu}} e^{j2\pi\bar{\mu}\mathcal{U}} e^{j2\pi\bar{u}\mathcal{D}} \tag{2.4.2}$$

where $\bar{u}, \bar{\mu}, t$ are the 3 parameters of the group. The operator $\rho(\bar{u}, \bar{\mu}, t)$ is usually called the Schrödinger representation of the Heisenberg group. The operator $\bar{\mu}\mathcal{U} + \bar{u}\mathcal{D} + t\mathcal{I}$ is called the Schrödinger representation of the Heisenberg Lie algebra where the Lie bracket is the commutator of two operators ($[\mathcal{U}, \mathcal{D}] = \mathcal{U}\mathcal{D} - \mathcal{D}\mathcal{U} = \frac{j}{2\pi}\mathcal{I}$). The derivation of the continuous WD based on the operator $\rho(\bar{u}, \bar{\mu}, t)$ is presented in section 2.1. A careful observation of this derivation reveals that the definition of the continuous WD is solely based on the Heisenberg group and the operator given in equation (2.4.2). The transition from equation (2.4.1) to (2.4.2) is not a necessary part of the definition of the continuous WD.

In order to define a discrete WD distribution in complete analogy to the continuous case, all of the ingredients in the continuous group theoretical derivation must be replaced with corresponding discrete versions. However it is not possible to extend a differentiable manifold to the discrete case. But the infinite group can be replaced with a finite group in many cases. In the case of the Heisenebrg group it is shown in proposition 1 that the Heisenberg Lie algebra can not be extended to the discrete case since it is impossible to find matrices whose commutator identity. Note that the Weyl correspondence approach written in equation (2.1.7) depends not only on the group structure but also to the Lie algebra which is connected with the differentiable manifold property of the Lie group, since the $[\mathcal{U}, \mathcal{D}] = \mathcal{U}\mathcal{D} - \mathcal{D}\mathcal{U} = \frac{j}{2\pi}\mathcal{I}$ property is used. The finite analogs of Lie groups that lack the differentiable manifold structure are usually called Lie type groups.

In the discrete case, two realizations of the finite Heisenberg group have been studied.

$$\rho^{m}[\bar{n}, \bar{k}, \tau] = e^{j2\pi\tau\mathbf{I}/N} e^{j2\pi 2^{-1}\bar{n}\bar{k}/N} e^{j2\pi\bar{k}\mathbf{U}/N} e^{j2\pi\bar{n}\mathbf{D}/N} \tag{2.4.3}$$

$$\rho^{h}[\bar{n}, \bar{k}, \tau] = e^{j2\pi\tau\mathbf{I}/N} e^{j\pi\bar{n}\bar{k}/N} e^{j2\pi\bar{k}\mathbf{U}/N} e^{j2\pi\bar{n}\mathbf{D}/N} \tag{2.4.4}$$

where the parameters $\bar{n}, \bar{k}, \tau \in S_1$ and $\bar{n}, \bar{k}, \tau \in S_2$. The operator ρ^{m} is advantageous in the sense that it leads to $\mathrm{WD^m}$, which is structurally more analogous to the continuous WD. The cost for this is the loss of the connection with continuum. On the other hand, ρ^{h} leads to $\mathrm{WD^h}$ and has a connection to continuum. The group theoretical discussions for ρ^{m} can be found in [15,21,24,39,40]. On the other hand ρ^{h} was studied in [18, 41] and has the continuous Heisenberg group as a limit [41]. We must note that the theory works best only for prime lengths and to some exceptions for odds for both of the realizations of the finite Heisenebrg group ρ^{m} and ρ^{h}.

2.5 Review of the literature and discussions

The concept of discrete phase space was first studied by J. von Neumann [42] and later further developed by A. Weil [15]. To the best of our knowledge, J. Schwinger was the first researcher to explicitly study a discrete Weyl correspondence [18].

Detailed and rigorous treatment of the phase space can be found in [14, 38, 43, 44].

As we have mentioned in the introduction, approaches to defining a discrete WD can be categorized as mainly falling under two headings: algebraic approaches and approaches based on sampling. Algebraic approaches based on the Weyl correspondence, the Schrödinger representation of the Heisenberg group, and the Schwinger basis all essentially lead to the same operators ρ^m and ρ^h defined in section 2.1. To the best of our knowledge, the operator ρ^m is first studied by A. Weil [15] and the operator ρ^h is first studied by J. Schwinger in [18]. The operator ρ^m is studied by many authors including [20, 21, 24, 39, 45] and the operator ρ^h is studied by many authors including [29, 30, 41, 46] within the context of finite phase space. Wootters [35] independently rediscovered the definition $\mathrm{WD^m}$ for prime length signals [47].

Most approaches to defining discrete WDs in the signal processing literature are based on sampling theory [25–27, 48, 49]. The earliest work we are aware of to give such a definition of a discrete WD is [3]. The work of Richman and others [24] is an exception in that it is based on group representation theory. Other works based on algebraic approaches in signal processing are [17, 29]. Sampling theory is adopted by many authors [25, 50] for the implementation and the computation of the WD. Special emphasize is given to computation of the WD without aliasing. Although these approaches lead to successful computational methods for the continuous WD, they lack structural analogy to the continuous WD. A review of sampling theory based approaches can be found in [27]. Other works in signal processing are [49, 51]. The definition $\mathrm{WD^m}$ is studied by [17, 24] in signal processing. To the best of our knowledge the definition $\mathrm{WD^h}$ is not studied in the signal processing literature. A generalization of the Shannon sampling theorem in WD domain is discussed in [52].

One very interesting and unifying approach which leads to a definition which satisfies both of our requirements of structural analogy and numerical approximation is based on the Kravchuk functions. The main motivation comes from the well-established theory in [53], and the associated Wigner distributions are developed in [54]. This approach is supported by its relation to developments involving discrete Gauss-hypergeometric functions [53]. The fundamental drawback of this approach is that it is not consistent with the conventional definition of the DFT, which must be replaced with the discrete Kravchuk-Fourier transform [55]. The Kravchuk-Fourier transform also approximate the continuous Fourier transform and is related to the DFT but is not equivalent to it. There has been many attempts to study discrete WDs along these lines [46, 54], but we are unable to judge whether it is desirable or acceptable to give up the conventional DFT.

One of our primary objectives set for the definition of a discrete entity was the continuum limits property. The limits of the finite Heisenberg group and discrete WDs are usually evaluated by comparing with a Reimann sum in the literature. A mathematically rigorous treatment of the limits for discrete operators and finite spaces is discussed in [56, 57] by using inductive resolutions. Inductive resolutions are generalizations of the limits of functions and sequences to groups, spaces and other algebraic entities.

Despite the fact that the WD is extensively used in both signal processing and quantum mechanics, and the many analogies between them, it is used in quite different contexts and forms in these two fields. Nevertheless the similarity between them point to further analogies between the phase spaces of discrete signal analysis and finite quantum mechanics. Finite phase space and discrete WDs are important concepts in the area of quantum computation [58]. However, because the discrete WDs employed in this field are chosen to be of even length, the WDs discussed here may not be of much use. It is worth noting a closely related recent work in processing quantum signals [59]. Other works in finite quantum mechanics and quantum computation literature dealing with the finite phase space are [60–66].

In this thesis we discussed discrete WDs only for the case of odd length signals. This is a consequence of the use of approaches based on structural analogies; even some approaches based on sampling theory have the same structure. The authors of [17] show that a discrete WD exists only for odd lengths, by imposing the analogs of the properties which uniquely define the WD in the continuous case. Indeed WD^{m} satisfies the generalized marginal properties only for prime length signals [35]. These all show that the parity plays an important role, especially in approaches based on structural analogies. Indeed, the definitions in [24, 45] for even length signals are very different from the ones for odd length signals. There has been proposals for overcoming these parity issues in defining the discrete WD in a unified manner for odd and even length signals [67]. A distinct definition for even length signals can be found in [68].

Chapter 3

Linear Canonical Transforms

3.1 Continuous linear canonical transforms

In this section, we will briefly review the continuous theory of LCTs and their connection with the WD from [11, 69]. We will avoid a detailed group theoretical discussion since there exists excellent sources on the subject [13, 38, 69] and such an approach may lead us too far away from our purpose. Nevertheless, we will emphasize the differences and similarities of discrete and continuous LCTs in a group theoretical perspective.

The continuous LCTs are defined as:

$$(\mathcal{C}_{\mathbf{M}}f)(u) = \int_{-\infty}^{\infty} C_{\mathbf{M}}(u,u')f(u')\,du', \tag{3.1.1}$$

$$C_{\mathbf{M}}(u,u') = A_{\mathbf{M}}e^{j\pi(\alpha u^2 - 2\beta uu' + \gamma u'^2)}, \tag{3.1.2}$$

$$A_{\mathbf{M}} = \sqrt{\beta}e^{-j\pi/4}. \tag{3.1.3}$$

There exists an ambiguity in the computation of the square roots in the above definitions. We will use the same convention in [11] which takes the root that falls in $(-\pi/2, \pi/2]$. The parameters α, β, γ are real[1], independent and are denoted with

[1] The complex case is also studied in [69]

$\mathbf{M}$ in the above definitions. LCTs can also be written in the following form:

$$C_{\mathbf{M}}(u,u') = A_{\mathbf{M}} e^{j\pi(\frac{D}{B}u^2 - \frac{2}{B}uu' + \frac{A}{B}u'^2)}, \tag{3.1.4}$$

$$A_{\mathbf{M}} = \frac{1}{\sqrt{B}} e^{-j\pi/4}, \tag{3.1.5}$$

$$\mathbf{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \qquad AD - BC = 1. \tag{3.1.6}$$

The FrFT is a special linear canonical transform corresponding to the rotation subgroup $SO(2,\mathbb{R})$.

$$\mathcal{F}^a = e^{ja\pi/4} \mathcal{C}_{\mathbb{M}} \tag{3.1.7}$$

for $-2 \leq a \leq 2$ and

$$\mathbf{M} = \begin{bmatrix} \cos(\pi a/2) & \sin(\pi a/2) \\ -\sin(\pi a/2) & \cos(\pi a/2) \end{bmatrix}. \tag{3.1.8}$$

The transform has been studied independently by many authors in different contexts and a detailed review of the literature can be found in [11].

LCTs have the following so called ABCD distortion property on the WD:

$$W_{f_{\mathbf{M}}}(Au + B\mu, Cu + D\mu) = W_f(u,\mu), \tag{3.1.9}$$

where $f_{\mathbf{M}} = (\mathcal{C}_{\mathbf{M}} f)(u)$. As a special case, fractional Fourier transform rotates the WD.

3.2 Discrete linear canonical transforms

In this section, we will define the discrete versions of LCTs and make connections to continuum. Before making the definitions we will introduce the Gauss sums since they play a major role in the construction. Indeed Gauss sums are the discrete versions of chirp integrals and Fourier transforms of chirps signals that play an important role in the continuous theory [11].

The Gauss sum G_p for an odd prime p is [70]:

$$G_p[n] = \frac{1}{\sqrt{p}} \sum_{m\in\mathbb{Z}_p} e^{\frac{j2\pi}{p}[nm^2]} = \Big(\frac{n}{p}\Big) G_p(1), \quad n \neq 0 \tag{3.2.1}$$

$G_p(1)$ is computed as:

$$G_p(1) = \begin{cases} 1, & p = 1 \mod 4; \\ j, & p = 3 \mod 4. \end{cases} \tag{3.2.2}$$

and $(\frac{n}{p})$ denotes the Legendre symbol:

$$\Big(\frac{n}{p}\Big) = \begin{cases} 1, & n = r^2 \mod p; \\ -1, & n \neq r^2 \mod p. \end{cases} \tag{3.2.3}$$

Gauss sums and the Legendre symbol play an important role in number theory [70].

The representation theory of the group $SL(2, \mathbb{Z}_p)$ was first studied by Tanaka [19] in an abstract form. An explicit construction of this representation in Weyl-Fourier form is given in [22]. A different but related treatment of the problem is [71]. The discrete LCTs are given in the following form in [23].

$$(\mathbf{C_M} f)[n] = \sum_{n'\in\mathbb{Z}_p} f[n'] C_{\mathbf{M}}[n, n'], \tag{3.2.4}$$

$$C_{\mathbf{M}}[n, n'] = G_{\mathbf{M}} e^{\frac{j2\pi}{p}[2^{-1}B^{-1}(Dn^2 - 2nn' + An'^2)]}, \tag{3.2.5}$$

$$G_{\mathbf{M}} = \frac{1}{\sqrt{p}} G_p^*[2B], \tag{3.2.6}$$

where the parameters $A, B, C, D \in \mathbb{Z}_p$.

It is stimulating to compare the definitions of continuous and discrete LCTs since they exactly share the same structure. The kernel of the discrete LCTs has the following properties which imply the unitarity of the discrete transform:

$$C_{\mathbf{M}}^{-1}[n, n'] = C_{\mathbf{M}}^*[n', n] = C_{\mathbf{M}^{-1}}[n, n']. \tag{3.2.7}$$

The effect on the discrete Wigner distribution $\mathrm{WD^m}$ can be easily proved by making the following matrix decompositions in $\mod p$.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ DB^{-1} & 1 \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ AB^{-1} & 1 \end{bmatrix} \tag{3.2.8}$$

The effect of discrete LCTs on WD$^{\mathrm{m}}$ takes a similar form to the continuous case.

$$W_f^m[n,k] = \frac{1}{\sqrt{N}} \sum_{\bar{n}=0}^{N-1} f[n+\bar{n}2^{-1}] f^*[n-\bar{n}2^{-1}] e^{-j2\pi\bar{n}k/N} \tag{3.2.9}$$

$$W_{f_{\mathbf{M}}}^m[An+Bk, Cn+Dk] = W_f^m[n,k] \tag{3.2.10}$$

where $f_{\mathbf{M}} = (\mathbf{C}_{\mathbf{M}} f)[n]$. The discrete versions of the ABCD distortion for WD$^{\mathrm{m}}$ have also been shown in [16, 24]. However authors do not give any definition for the discrete LCTs and do not derive the matrix group properties.

3.3 Continuum connections

The discrete LCTs discussed in the previous section are obtained from the theory of the group $SL(2,\mathbb{Z}_p)$. These discrete LCTs are exactly analogous to the continuous LCTs in the sense of operational properties. It has also been shown that the discrete LCTs do not have the continuous LCTs as continuous limits [23]. Nevertheless, we will show that it is possible to relate these discrete LCTs to the samples of the continuous LCTs. Thus, with regard to the two main desirable qualities set in the introduction, the discrete LCTs discussed in the previous section exhibit high structural analogy, are not related to continuous LCTs through continuous limits but nevertheless can approximate them under certain conditions.

The discrete versions of chirp signals and their connections to the continuous chirps has been discussed in [72]. It is shown that under certain assumptions it is possible to relate the continuous chirp modulated Fourier transform to the discrete versions. Various optimality and approximation considerations has been derived in [72]. In this section, we will apply the methodology in [72] to relate the continuous LCTs to the discrete LCTs.

For reference, we rewrite the continuous LCTs:

$$(\mathcal{C}_{\mathbf{M}}f)(u) = \int C_{\mathbf{M}}(u,u')f(u')\,du', \tag{3.3.1}$$

$$C_{\mathbf{M}}(u,u') = A_{\mathbf{M}}e^{j\pi(\alpha u^2 - 2\beta uu' + \gamma u'^2)}, \tag{3.3.2}$$

$$A_{\mathbf{M}} = \sqrt{\beta}e^{-j\pi/4}. \tag{3.3.3}$$

To show the connection between the continuous LCTs and the discrete LCTs, we will approximate the integral in the definition of continuous LCTs as follows:

$$\int C_{\mathbf{M}}(u,u')f(u')du' \approx \sum_{n'} C_{\mathbf{M}}(nT_u, n'T_{u'})f(n') \tag{3.3.4}$$

where

$$C_{\mathbf{M}}(nT_u, n'T_{u'}) = A_{\mathbf{M}}e^{j\pi(\alpha n^2T_u^2 - 2\beta nn'T_uT_{u'}u' + \gamma n'^2T_{u'}^2)}, \tag{3.3.5}$$

$$A_{\mathbf{M}} = \sqrt{\beta}e^{-j\pi/4}. \tag{3.3.6}$$

Note that this approximation is true up to multiplicative constants. We will choose the sampling rates as $T_u = T_{u'} = \sqrt{2/p}$, then the sampled kernel takes the following form:

$$C_{\mathbf{M}}(nT_u, n'T_{u'}) = A_{\mathbf{M}}e^{j2\pi(\alpha n^2 - 2\beta nn'u' + \gamma n'^2)/p}. \tag{3.3.7}$$

If we compare the samples of the kernel with the kernel of discrete LCTs given below:

$$C_{\mathbf{M}}[n,n'] = G_{\mathbf{M}}e^{\frac{j2\pi}{N}[2^{-1}B^{-1}(Dn^2 - 2nn' + An'^2)]}, \tag{3.3.8}$$

for the following values of the continuous LCTs parameters

$$\alpha = 2^{-1}B^{-1}D, \tag{3.3.9}$$

$$\beta = 2^{-1}B^{-1}, \tag{3.3.10}$$

$$\gamma = 2^{-1}B^{-1}A, \tag{3.3.11}$$

the two kernels given in equations (3.3.7) and (3.3.8) become equivalent up to multiplicative constants.

In summary, the approximation property shown above is valid only in a limited context. For a given discrete LCT with parameters $A, B, C, D \in \mathbb{Z}_p$, there exists a sampling strategy and corresponding continuous LCT parameters α, β, γ for which the discrete LCT approximates the continuous LCT. Note that this approximation is limited only to integer valued continuous α, β, γ parameters. Arbitrary continuous LCTs can not be approximated with the discrete ones. Furthermore, the lengths of the signals are limited to primes in the presented form of discrete LCTs. In spite of these limitations, it is possible to say that the two goals set out in the introduction have been met to a considerable degree: structural analogy and continuum approximation. Although the discrete LCTs do not have the continuous LCTs as limits, we believe that this is not very significant in a signal processing context.

3.4 The discrete fractional Fourier transform

In the continuous case, the FrFT is a special LCT that corresponds to the $SO(2,\mathbb{R})$ subgroup. In this section we will study the discrete version of the FrFT which are special discrete LCTs studied in the previous section. The discrete FrFT will correspond to the discrete rotations group $SO(2,\mathbb{Z}_p)$. The members of the group are given as:

$$R_g = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}, \qquad A^2 + B^2 = 1 \mod p. \tag{3.4.1}$$

It is known that this group has a generator. Depending on the value of p, the number of discrete rotations are given as [16]:

$$\text{Number of rotations} = \begin{cases} p-1, & p = 1 \mod 4; \\ p+1, & p = 3 \mod 4. \end{cases} \tag{3.4.2}$$

Let R_g denote the generator of the finite rotation group, then we will associate the fractional order a to the generator powers as follows:

$$(R_g)^r \longrightarrow \mathbf{F}^a, \qquad r \in \mathbb{Z} \tag{3.4.3}$$

$$\tag{3.4.4}$$

The fractional orders are:

$$a = \begin{cases} \frac{4r}{p-1}, & p = 1 \mod 4; \\ \frac{4r}{p+1}, & p = 3 \mod 4. \end{cases} \tag{3.4.5}$$

Table 3.4.1: Properties of the discrete fractional Fourier transform.

1.	Linearity	$\mathbf{F}^a\big(\sum_\ell \beta_\ell f_\ell[n]\big) = \sum_\ell \beta_\ell \mathbf{F}^a f_\ell[n]$	✓
2.	Integer Orders	$\mathbf{F}^m = \big(\mathbf{F}\big)^m$	✓
3.	Inverse	$\big(\mathbf{F}^a\big)^{-1} = \mathbf{F}^{-a}$	✓
4.	Unitarity	$\big(\mathbf{F}^a\big)^{-1} = \big(\mathbf{F}^a\big)^H$	✓
5.	Index Additivity	$\mathbf{F}^{a_2}\mathbf{F}^{a_1} = \mathbf{F}^{a_2+a_1}$	✓
6.	Commutativity	$\mathbf{F}^{a_2}\mathbf{F}^{a_1} = \mathbf{F}^{a_1}\mathbf{F}^{a_2}$	✓
7.	Associativity	$\big(\mathbf{F}^{a_1}\mathbf{F}^{a_2}\big)\mathbf{F}^{a_3} = \mathbf{F}^{a_1}\big(\mathbf{F}^{a_2}\mathbf{F}^{a_3}\big)$	✓
8.	Parseval	$\langle f, g\rangle = \langle f_a, g_a\rangle$	✓
9.	Eigenfunctions		?
10.	Wigner distribution	$W^m_{f_a} = W^m_f[An + Bk, Ak - Bn]$	✓
11.	Radon Transform-1	$\mathcal{RDN}_M W^m_{f,g}[n,k] = f_a[n]g^*_a[n]$	✓
12.	Radon Transform-2	$\mathcal{RDN}_M A^m_{f,g}[\bar{n},\bar{k}] = f_a[n2^{-1}]g^*_a[-n2^{-1}]$	✓
13.	Limits & Approximation		-
14.	Fast Algorithm $O(N \log N)$		✓

and the parameter r is in the range of $\{0, 1, 2, ..., p-2\}$ if $p = 1 \mod 4$ and in the range of $\{0, 1, 2, ..., p\}$ if $p = 3 \mod 4$.

The discrete FrFT will be defined as:

$$f_a[n] = (\mathbf{F}^a f)[n] = (G_p[2])^a \mathbf{C}_\mathbf{M}, \tag{3.4.6}$$

$$(\mathbf{F}^a f)[n] = \frac{1}{\sqrt{p}} G^*_p[2B](G_p[2])^a \sum_{n' \in \mathbb{Z}_p} f[n'] e^{\frac{j2\pi}{p}[2^{-1}B^{-1}(An^2 - 2nn' + An'^2)]}. \tag{3.4.7}$$

where $\mathbf{M}$ is the rotation matrix

$$\mathbf{M} = \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \tag{3.4.8}$$

corresponding to the fractional order a. The discrete Radon transform of $f[n, n']$ is defined as:

$$(\mathcal{RDN}_M f)[n] = \sum_{n' \in \mathbb{Z}_p} f[An - Bn', Bn + An']. \tag{3.4.9}$$

The discrete version of the projection slice theorem can be written as:

$$\sum_{n\in\mathbb{Z}_p}(\mathcal{RDN}_M f)[n]e^{\frac{-j2\pi kn}{p}} = \sum_{n,n'\in\mathbb{Z}_p} f[n,n']e^{\frac{-j2\pi k(An+Bn')}{p}}. \tag{3.4.10}$$

Table 3.4.1 summarizes the properties of the discrete fractional Fourier transform. Entries appeared in this table are adapted from [73].

3.4.1 Discrete fractional Fourier transforms and discrete rotations

Understanding discrete rotations has been pointed out as an important open problem in [11] since this might lead to a consolidation of a discrete WD and the discrete FrFT proposed in [73]. The discrete FrFT proposed in [73] is defined for all real fractional orders, hence it is necessary to find a way to define arbitrary discrete rotations on a torus if a consolidation with a discrete WD is aimed.

If we turn our attention to the discrete FrFT obtained from $SO(2,\mathbb{Z}_p)$ it is possible to define discrete rotations in a consistent manner with the discrete WD^{m}. Indeed the discrete modulo rotations $SO(2,\mathbb{Z}_p)$ are the algebraic copies of continuous rotations $SO(2,\mathbb{R})$. The structure of such discrete rotations have been discussed in [16]. Unfortunately it is difficult to relate these modulo rotations to continuous rotations. Nevertheless, it is possible to define a discrete FrFT that rotates the discrete WD^{m} in an algebraic sense.

For a graphical representation of discrete rotationally symmetric functions see Fig. 3.4.1 and Fig. 3.4.1.

3.4.2 Exponential forms

The FrFT and many subgroups of $SL(2,\mathbb{R})$ have representations in hyper differential forms [11, 69]. These indeed stem from the connection between one parameter subgroups of Lie groups and their corresponding Lie algebras. The extension of the same forms to the discrete case is rather problematic. This mainly stems from the fact that we can not find a Heisenberg Lie algebra representation in a finite linear space as proved in Proposition 1.

Nevertheless, it is still possible to express the the following subgroups in formal analogy to the continuous case.

$$e^{\frac{j2\pi}{N}[2^{-1}q\mathbf{U}^2]} \equiv \text{Chirp Multiplication} \tag{3.4.11}$$

$$e^{\frac{j2\pi}{N}[2^{-1}r\mathbf{D}^2]} \equiv \text{Chirp Convolution} \tag{3.4.12}$$

$$e^{\frac{j2\pi}{N}[n_o\mathbf{D}]} \equiv \text{Shift} \tag{3.4.13}$$

$$e^{\frac{j2\pi}{N}[k_o\mathbf{U}]} \equiv \text{Phase Shift} \tag{3.4.14}$$

The remaining of the exponential forms for fractional Fourier transforms, dilations, hyperbolic does not seem to be expressible by using the discrete operators **U** and **D** in a full analogy to the continuous case.

3.5 Discussions and review of the literature

There exists a third and important version of a discrete FrFT known as the fractional Kravchuk-Fourier transform [55]. This transform has interesting properties and fits well into the framework set for the definition of a discrete entity in the introduction. The fractional Kravchuk-Fourier transform approximates the continuous FrFT transform [55]. Furthermore it is also a fractional operator and is defined for all real fractional orders. Indeed it is closely related with the theory of discrete polynomials and discrete Gauss hypergeometric functions [53], and the Kravchuk polynomials are also related with the representation theory of $SU(2)$ [53]. In spite of these nice properties, the fractional Kravchuk-Fourier transform does not reduce to the ordinary DFT for the fractional order $a = 1$ and the operator is not defined in a cyclic form. This has been regarded as a drawback since loosing the reduction to DFT is not desirable in a signal processing framework [11, 73]. Although this may seem as a disadvantage we note that the Kravchuk polynomials are also discrete Fourier transforms defined on a symmetric space [70] and can be regarded as discrete analogs of spherical harmonics [70]. However we can not justify whether this has a significant role in the discrete theory since to the best of our knowledge this property of Kravchuk polynomials is not mentioned in [53, 55, 74]. Other works on the subject are [46, 54, 75, 76].

It seems that the best discrete FrFT that fits into the framework for defining a discrete entity is the definition obtained from Harper's matrix [73]. This definition both approximates the continuous FrFT, has as a limit [77], and has many operational properties that are analogous to the continuous case. Unfortunately, this definition does not have a closed form expression and is computed numerically. Furthermore, there does not exist a fast algorithm for the computation which limits the applicability of the transform. Some of the operational properties of this definition were tested numerically and they lack an analytical proof [11]. As a future work, we believe that the operational properties of this transform must be tested for the case of 4×4 matrices where the eigenvalue problems can be solved analytically. There has been other proposals for nearly tri-diagonal matrices that commute with the DFT matrix [78]. It may be possible to define other discrete FrFTs; however, it seems that the Harper's matrix is the simplest among various candidates that can be related with interpolations [34]. The eigenvectors of the DFT matrix has been studied by many authors [73, 79, 80, 80–86].

If we turn our attention to the discrete FrFT obtained from $SO(2, \mathbb{Z}_p)$, it can be justified that this definition has the highest possible structural analogy but fails to have a connection with the continuum. Nevertheless, since it has an explicit expression, and an $O(N \log N)$ algorithm, it is still possible to apply this transform to signal processing or to the other fields where DFT is used. Unfortunately this definition is defined only for prime length signals. This indeed stems from the structure of the group $SL(2, \mathbb{Z}_p)$ and the finite field $\mathbb{Z}_p$. In order to define a discrete FrFT for even length signals that has the highest structural analogy, it seems that major changes are necessary in the theory as noted by [20]. The structure of the group $SL(2, \mathbb{Z}_p)$ and its representations have been studied by many authors including [19–22, 39, 46, 71].

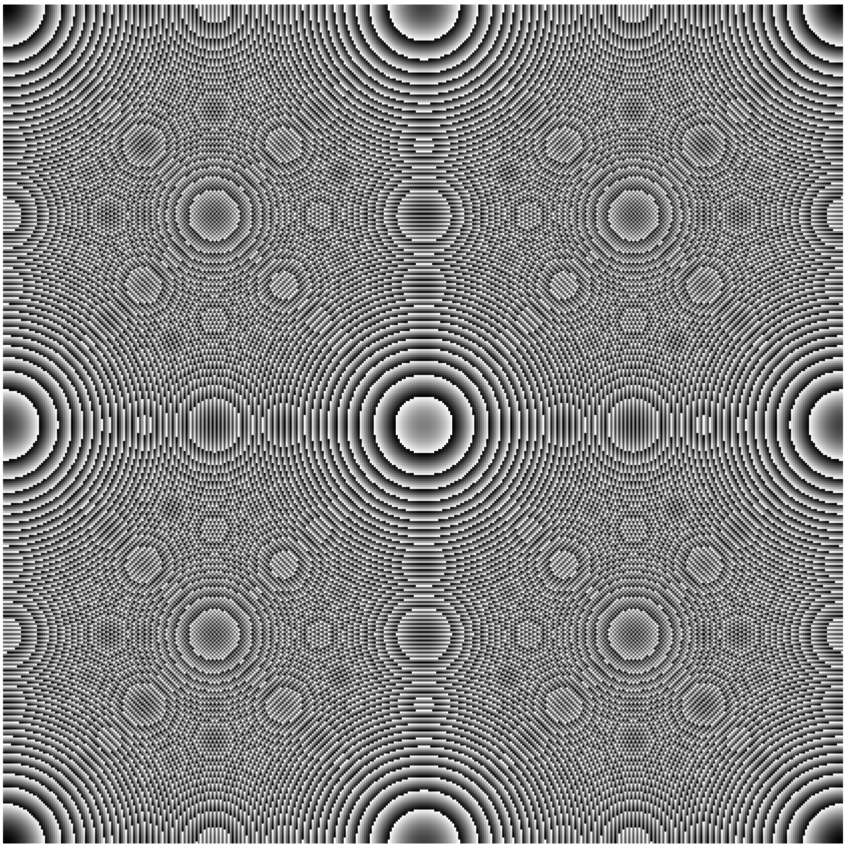

Figure 3.4.1: Gray level picture of the function $f(m, n) = m^2 + n^2 \mod 419$

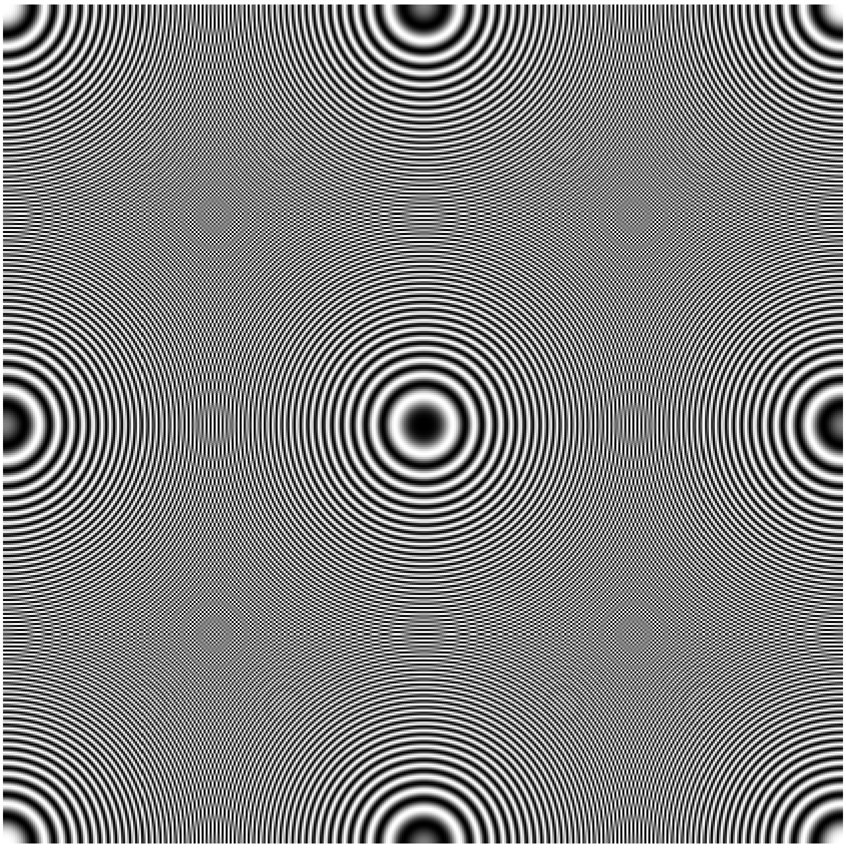

Figure 3.4.2: Gray level picture of the function $f(m,n) = \cos(2\pi(m^2+n^2)/419)$

Chapter 4

Discussions and Future Work

As stated by Leonhardt [67]: "The transcription of the continuous Wigner formalism for discrete quantum mechanics involves some interesting subtleties like the transcription of symphony for a chamber orchestra." In other words we lack some instruments in the discrete case. It is important to understand the absent instruments before proceeding to the development of a finite phase space theory. We believe that the lack of the differentiable manifold property in the discrete case sets fundamental limitations to the theory of finite phase space. Such limitations require the questioning of the structural analogy expected from a discrete candidate of a continuous entity.

In this thesis, we proposed the definition WD^{h} as a compromise for a discrete WD which is related with the representation theory of the finite Heisenberg group, can be utilized as a numerical tool, and has the continuous WD as a limit. This framework is analogous to the framework that makes DFT analogous to the continuous Fourier transform. Due to the trade-offs between the structural analogy and continuum approximation properties, the proposed WD^{h} is not the best in sense of structural analogy. However, we believe that it fits well in to the needs of signal analysis. As a future work, the same framework must be applied to a WD for even length signals. Although there exist some proposals for a discrete WD for even length signals [24, 45, 47] it is not certain at the moment what is the best one fitting to the framework set for the definition of a discrete entity. Indeed the definitions in [24, 45] satisfy similar sets of properties.

There exists a similarity between algebraic coding theory and digital signal processing [70]. Since the DFT plays an important role in algebraic coding theory it is possible for a discrete FrFT to have applications in coding theory. The WD in $GF(2^n)$ and the associated discrete LCTs and FrFTs will be discussed elsewhere.

The proposed sampling methodology for WD^{h} can also be extended to defining the discrete analogous of other members of the Cohen's class. This may be used for defining structurally analogous discrete versions of Cohen's class that have continuum connections.

Due to the lack of a Heisenberg algebra in the discrete case, it is necessary to modify the discrete version of an uncertainty principle. In the continuous case, there is also a close relationship with the WD and the Heisenberg uncertainty principle. The discrete version of such a relationship for WD^{m} and WD^{h} will be discussed elsewhere by using the discrete uncertainty principles in [70].

Appendix A

Representation of $SL(2, \mathbb{Z}_p)$

In this appendix, a modified proof of the group structure of discrete LCTs will be given. The derivation is modified from [23].

Let

$$\mathbf{M}_1 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}, \quad \mathbf{M}_2 = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}, \quad \mathbf{M}_3 = \mathbf{M}_2\mathbf{M}_1 = \begin{bmatrix} A_3 & B_3 \\ C_3 & D_3 \end{bmatrix}$$

The elements of $\mathbf{M}_3$ are explicitly given as:

$$\begin{aligned} A_3 &= A_2A_1 + B_2C_1 \\ B_3 &= A_2B_1 + B_2D_1 \\ C_3 &= C_2A_1 + D_1C_1 \\ D_3 &= C_2B_1 + D_2D_1 \end{aligned}$$

The kernel of the discrete LCT after a concatenation $\mathbf{C}_{\mathbf{M}_2}\mathbf{C}_{\mathbf{M}_1}$ will be:

$$\begin{aligned} C_{\mathbf{M}_3}[n, n'] &= G_{\mathbf{M}_2}G_{\mathbf{M}_1} \sum_{n'' \in \mathbb{Z}_p} C_{\mathbf{M}_2}[n, n'']C_{\mathbf{M}_1}[n'', n'] \\ &= G_{\mathbf{M}_2}G_{\mathbf{M}_1} \sum_{n'' \in \mathbb{Z}_p} e^{\frac{j2\pi}{N}[2^{-1}B_2^{-1}(D_2n^2 - 2nn'' + A_2n''^2) + 2^{-1}B_1^{-1}(D_1n''^2 - 2n''n' + A_1n'^2)]} \\ &= G_{\mathbf{M}_2}G_{\mathbf{M}_1} e^{\frac{j2\pi}{N}[2^{-1}B_2^{-1}D_2n^2 + 2^{-1}B_1^{-1}A_1n'^2]} \sum_{n'' \in \mathbb{Z}_p} e^{\frac{j2\pi}{N}[2^{-1}(B_2^{-1}A_2 + B_1^{-1}D_1)n''^2 - (B_2^{-1}n + B_1^{-1}n')n'']} \end{aligned}$$

In order the compute a summation of the following form:

$$S = \frac{1}{\sqrt{p}} \sum_{n \in \mathbb{Z}_p} e^{\frac{j2\pi}{p}[an^2 - bn]}$$

we will make a change of variables as: $n \longrightarrow n + r$. The summation can now be written as:

$$\begin{aligned} S &= \frac{1}{\sqrt{p}} \sum_{n \in \mathbb{Z}_p} e^{\frac{j2\pi}{p}[a(n+r)^2 - b(n+r)]} \\ &= \frac{1}{\sqrt{p}} \sum_{n \in \mathbb{Z}_p} e^{\frac{j2\pi}{p}[an^2 + ar^2 + 2anr - bn - br]} \\ &= \frac{1}{\sqrt{p}} e^{\frac{j2\pi}{p}[ar^2 - br]} \sum_{n \in \mathbb{Z}_p} e^{\frac{j2\pi}{p}[an^2 + 2anr - bn]} \end{aligned}$$

if we choose $r = 2^{-1}a^{-1}b$, the summation takes the following form:

$$\begin{aligned} S &= \frac{1}{\sqrt{p}} e^{\frac{j2\pi}{p}[ar^2 - br]} \sum_{n \in \mathbb{Z}_p} e^{\frac{j2\pi}{p}[an^2]} \\ &= e^{\frac{j2\pi}{p}[ar^2 - br]} \left(\frac{a}{p}\right) G_p[1] \\ &= e^{\frac{j2\pi}{p}[ar^2 - 2ar^2]} \left(\frac{a}{p}\right) G_p[1] \\ &= e^{\frac{j2\pi}{p}[-ar^2]} \left(\frac{a}{p}\right) G_p[1] \end{aligned}$$

This result will be applied to the summation:

$$\sum_{n'' \in \mathbb{Z}_p} e^{\frac{j2\pi}{N}[2^{-1}(B_2^{-1}A_2 + B_1^{-1}D_1)n''^2 - (B_2^{-1}n + B_1^{-1}n')n'']}$$

where

$$\begin{aligned} a &= 2^{-1}(B_2^{-1}A_2 + B_1^{-1}D_1) \\ b &= (B_2^{-1}n + B_1^{-1}n') \\ r &= 2^{-1}a^{-1}b = (B_2^{-1}A_2 + B_1^{-1}D_1)^{-1}(B_2^{-1}n + B_1^{-1}n') \end{aligned}$$

After application of this result the kernel $C_{\mathbf{M}_3}[n, n']$ take sthe following form:

$$C_{\mathbf{M}_3}[n, n'] = G_{\mathbf{M}_2} G_{\mathbf{M}_1} e^{\frac{j2\pi}{N}[2^{-1}B_2^{-1}D_2n^2 + 2^{-1}B_1^{-1}A_1n'^2 - 2^{-1}(B_2^{-1}A_2 + B_1^{-1}D_1)r^2]} G_p[2^{-1}(B_2^{-1}A_2 + B_1^{-1}D_1)]$$

In this case, the kernel $C_{\mathbf{M}_3}[n,n']$ consists of two terms. A constant in front and terms consisting of discrete chirps. We will first show that the constant in the combined kernel $\mathbf{C}_{\mathbf{M}_2}\mathbf{C}_{\mathbf{M}_1}$ is consistent with $\mathbf{C}_{\mathbf{M}_3}$.

$$\begin{aligned}\Big(\frac{2B_1}{p}\Big)G_p^*[1]\Big(\frac{2B_2}{p}\Big)G_p^*[1]\Big(\frac{2^{-1}(B_2^{-1}A_2+B_1^{-1}D_1)}{p}\Big)G_p[1] &= \Big(\frac{2B_1A_2+2B_2D_1}{p}\Big)G_p^*[1]\\ &= G_p^*[2B_3]\end{aligned}$$

As a second step, the followings prove that the chirped terms in kernel of the combined operator $\mathbf{C}_{\mathbf{M}_2}\mathbf{C}_{\mathbf{M}_1}$ is consistent with $\mathbf{C}_{\mathbf{M}_3}$

$$\begin{aligned}2^{-1}B_3^{-1}D_3 &= 2^{-1}B_2^{-1}D_2 - 2^{-1}B_3^{-2}(B_1^2)(B_2^{-1}A_2+B_1^{-1}D_1)\\ D_3 &= B_3B_2^{-1}D_2 - 2^{-1}B_3^{-1}(B_1^2)(B_2^{-1}A_2+B_1^{-1}D_1)\\ B_1B_2D_3 &= B_3B_1D_2 - B_3^{-1}(B_1^2)(B_1A_2+B_2D_1)\\ B_2((A_2D_2-1)B_2^{-1}B_1+D_2D_1) &= B_3D_2 - B_1\\ (A_2D_2-1)B_1+B_2D_2D_1 &= (A_2B_1+B_2D_1)D_2 - B_1\end{aligned}$$

in a similar way

$$\begin{aligned}2^{-1}B_3^{-1}A_3 &= 2^{-1}B_1^{-1}A_1 - 2^{-1}(B_2^{-1}A_2+B_1^{-1}D_1)B_3^{-2}B_2^2\\ B_1B_2A_3 &= B_2B_3A_1 - (B_1A_2+B_2D_1)B_3^{-1}B_2^2\\ B_1(A_1A_2+B_2(A_1D_1-1)B_1^{-1}) &= B_3A_1 - B_2\\ B_1A_1A_2+B_2(A_1D_1-1) &= (B_1A_2+B_2D_1)A_1 - B_2\end{aligned}$$

and finally:

$$\begin{aligned}B_3^{-1} &= 2^{-1}(B_2^{-1}A_2+B_1^{-1}D_1)2B_3^{-2}B_1B_2\\ B_3 &= (B_2^{-1}A_2+B_1^{-1}D_1)B_1B_2\end{aligned}$$

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