REMARKS ON THE TRACE FORMULA FOR FINITE GROUPS

by

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To my mother

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ABSTRACT

REMARKS ON THE TRACE FORMULA FOR FINITE GROUPS

Let G be a finite group and consider a subgroup Γ of G along with a representation $\rho : \Gamma \to \mathrm{GL}(V_\rho)$ of Γ on a finite dimensional $\mathbb{C}$-linear vector space V_ρ. We get the trace formula by computing the trace of the operator $\mathrm{Ind}_\Gamma^G(\rho)(f) : \mathrm{Ind}_\Gamma^G(V_\rho) \to \mathrm{Ind}_\Gamma^G(V_\rho)$ for a test function $f : G \to \mathbb{C}$. Our aim is to present this computation in two different ways, that is, by expressing the spectral side $J(\rho, f)$ and the geometric side $I(\rho, f)$ of the formula. The expression $J(\rho, f)$ is a sum over non-isomorphic irreducible representations of G having the multiplicities of the irreducible representations emerging in the decomposition of the induced representation $\mathrm{Ind}_\Gamma^G(\rho)$ within and the geometric side $I(\rho, f)$ of the trace formula is obtained by calculating the conjugacy classes of the groups Γ and G. We call the generalized formula $J(\rho, f) = \mathrm{Tr}(\mathrm{Ind}_\Gamma^G(\rho)(f)) = I(\rho, f)$ the trace formula for the finite group G with respect to the subgroup Γ and $\rho : \Gamma \to \mathrm{GL}(V_\rho)$.

ÖZET

SONLU GRUPLAR İÇİN İZ FORMÜLÜ ÜZERİNE NOTLAR

G sonlu bir grup olsun ve bir Γ altgrubunu ele alalım, ayrıca $\rho : \Gamma \to \mathrm{GL}(V_\rho)$ altgrubun sonlu $\mathbb{C}$-lineer uzayı V_ρ üzerine bir temsil olsun. İz formülünü, $\mathrm{Ind}_\Gamma^G(\rho)(f) : \mathrm{Ind}_\Gamma^G(V_\rho) \to \mathrm{Ind}_\Gamma^G(V_\rho)$ operatörünün bir f fonksiyonu için izini hesaplayarak yazmaktayız. Amacımız bu hesabın iki farklı yolunu sunmaktır: formülün spektral kısmı olan $J(\rho, f)$ ve geometrik kısmı olan $I(\rho, f)$'yi yazmak. Formülün spektral kısmı, G'nin izomorfik olmayan indirgenemez temsilleri üzerinden bir toplamdır öyle ki spektral kısım $\mathrm{Ind}_\Gamma^G(\rho)$ temsilinin dekompozisyonunda beliren temsillerin katsayılarını içerir ve formülün geometrik kısmı Γ ve G gruplarının eşlenik sınıflarının hesaplanmasıyla ifade edilir. Genel formül olan $J(\rho, f) = \mathrm{Tr}(\mathrm{Ind}_\Gamma^G(\rho)(f)) = I(\rho, f)$ eşitliği G sonlu grubu için Γ altgrubu ve onun $\rho : \Gamma \to \mathrm{GL}(V_\rho)$ temsiline göre iz formülü olarak adlandırılır.

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$[a_{ij}]$	matrix with the entries a_{ij}
B	the Borel subgroup of $\mathrm{GL}(2, \mathbb{F}_p)$
$\mathcal{B}_V$	basis of the vector space V
$\mathbb{C}$	the complex numbers
$\mathbb{C}[G]$	the group ring G over $\mathbb{C}$
$\mathrm{End}_{\mathbb{C}}(V)$	the set of all endomorphisms of the $\mathbb{C}$-linear vector space V
e	the identity element in G
$\mathbb{F}_p$	finite field of order $p = q^k$
$\mathbb{F}_q$	finite field of order q
f	test function
G, H	groups
G_γ	the centralizer of the set $\{\mathit{\Gamma}\}$ of the group G
$\mid G \mid$	order of the group G
$[G : H]$	the index of $\mathit{\Gamma}$ in G
$\mathrm{GL}(2, \mathbb{F}_p)$	the general linear group of degree 2 over the finite field $\mathbb{F}_p$
$\mathrm{GL}(V)$	the general linear group over vector space V
g^{-1}	the inverse element of g in G
$I(\rho, f)$	the spectral side of the trace formula
$\mathrm{Ind}_{\mathit{\Gamma}}^G(V_\rho)$	the induced space from V_ρ
$\mathrm{Ind}_{\mathit{\Gamma}}^G(\rho)$	induced representation on G on the vector space $\mathrm{Ind}_{\mathit{\Gamma}}^G(V_\rho)$
$\mathrm{Ind}_{\mathit{\Gamma}}^G(\rho)(f)$	operator
$J(\rho, f)$	the geometric side of the trace formula
K_f	kernel of the operator $\mathrm{Ind}_{\mathit{\Gamma}}^G(\rho)(f)$
$m(\pi_0, \mathrm{Ind}_{\mathit{\Gamma}}^G(\rho))$	multiplicities of π_0 in the decomposition of $\mathrm{Ind}_{\mathit{\Gamma}}^G(\rho)$
$N(\alpha)$	the (field) norm of α
$\mathcal{N}_{\mathrm{Ind}_{\mathit{\Gamma}}^G(V_\rho)}$	orthonormal basis of $\mathrm{Ind}_{\mathit{\Gamma}}^G(V_\rho)$
p, q	prime numbers
$\mathcal{R}_{\mathit{\Gamma}/G}$	the complete set of representatives of the coset space $\mathit{\Gamma}/G$

$\mathrm{Res}_H^G E$	restriction from G to H on E
$\mathrm{Tr}(r(f))$	the trace of the operator $r(f)$
$\mathrm{Tr}(\alpha)$	the (field) trace of α
V, W	vector spaces
$V \oplus W$	the direct sum of V and W
$V \otimes W$	the tensor product of V and W
$V \times W$	the direct product of V and W
V_ρ	the vector space on which the representation ρ is defined
$\mathbb{Z}$	the set of integers
$\mathbb{Z}/n\mathbb{Z}$	the group of integers modulo n
$\mathit{\Gamma}$	subgroup of G
$\{\mathit{\Gamma}\}$	the set of representatives of the conjugacy classes in $\mathit{\Gamma}$
$\mathit{\Gamma}_\gamma$	the centralizer of the set $\{\mathit{\Gamma}\}$ of the group G
$\mathit{\Gamma} g \sqcup \mathit{\Gamma} h$	disjoint union of the cosets $\mathit{\Gamma} g$ and $\mathit{\Gamma} h$
δ_{ij}	Kronecker delta
$\mathit{\Pi}(G)$	the set of all isomorphism classes $[\pi]$
π_0, ρ	representations
$[\pi]$	isomorphism class of irreducible representations π of G over $\mathbb{C}$
$[\varphi(g)]_{\mathcal{B}}$	the coordinate vector of $\varphi(g)$ relative to the basis $\mathcal{B}$
$\langle \varphi \mid \psi \rangle$	the inner product of φ and ψ
$\overline{\varphi(x)}$	the conjugate of $\varphi(x)$
$\chi_\rho(\gamma)$	the trace of $\rho(\gamma)$

1. INTRODUCTION

The trace formula for finite groups is a formula where the trace of the operator $\mathrm{Ind}_{\Gamma}^{G}(\rho)(f) : \mathrm{Ind}_{\Gamma}^{G}(V_\rho) \to \mathrm{Ind}_{\Gamma}^{G}(V_\rho)$ is computed spectrally and geometrically. The formula represents the trivial case of the Arthur trace formula.

In the preliminaries, we explain what induced spaces and induced representations are. Furthermore, we investigate the relationship between the spaces V_ρ and $\mathrm{Ind}_{\Gamma}^{G}(V_\rho)$. In this section, we see particularly that the induced space $\mathrm{Ind}_{\Gamma}^{G}(V_\rho)$ is a $\mathbb{C}[G]$- module and that this space is isomorphic to the direct sum $\bigoplus\limits_{r\in R} rV_\rho$ which is a sum over representatives of G of the left cosets G/Γ.

In the next chapter, Trace Formula for Finite Groups, we mainly consult the article [1]. Here, we state the main theorem of this thesis, the trace formula for finite groups, and present the proof. In the proof, we see how one can approach the trace formula, namely, spectrally and geometrically. For the spectral side, the multiplicities $m(\pi_0, \mathrm{Ind}_{\Gamma}^{G}(\rho))$ are to be calculated which are exactly the multiplicities appearing in the decomposition of the induced representation $\mathrm{Ind}_{\Gamma}^{G}(\rho)$ in distinct irreducible representations π_0 of G. On the geometric side, for the calculations, we view two partitions of the group G, namely $\Gamma_\gamma \leq \Gamma \leq G$ and $\Gamma_\gamma \leq G_\gamma \leq G$ where Γ_γ is the centralizer of the set $\{\Gamma\}$ of the representatives of the conjugacy classes of Γ of the group Γ, and G_γ is the centralizer of the set $\{\Gamma\}$ of the group G.

In the following chapter, The Trace Formula for $\mathbb{Z}/n\mathbb{Z}$, we firstly write the trace formula for the group $\mathbb{Z}/6\mathbb{Z}$ with respect to its non-trivial subgroups and the trivial representation of the subgroups. Then we examine the general case, namely for the group $\mathbb{Z}/n\mathbb{Z}$ where n is a natural number greater than 1, with respect to the subgroup $\mathbb{Z}/m\mathbb{Z}$ where $m \mid n$ and the trivial representation of $\mathbb{Z}/m\mathbb{Z}$.

In the last chapter, The Trace Formula for $\mathrm{GL}(2,\mathbb{F}_3)$, we write the trace formula for the group $\mathrm{GL}(2,\mathbb{F}_3)$ with respect to its Borel subgroup and the trivial representation of the Borel group where previously we give the character table of the group $\mathrm{GL}(2,\mathbb{F}_p)$ refering to the books [2] and [3].

Finally, throughout the study of the groups $\mathbb{Z}/n\mathbb{Z}$ and $\mathrm{GL}(2,\mathbb{F}_3)$, we choose the test function f to be defined as $f(g)=1$ for all $g\in G$. There are other various options to choose the test function, of course.

2. PRELIMINARIES

In this chapter, we explain what induced spaces and induced representations are. The trace formula for finite groups gives us the trace of the operator $\mathrm{Ind}_{\Gamma}^{G}(\rho)(f)$. This is an operator defined on the induced space $\mathrm{Ind}_{\Gamma}^{G}(V_\rho)$ which we study in this chapter. For the definitions in this chapter one can consult [2] and for the propositions and the theorem in the chapter one can refer to the sources [2] and [3].

2.1. Induced Spaces and Induced Representations

Definition 2.1. *Consider a finite group G. Let Γ be a subgroup of G and let $\rho : \Gamma \to GL(V)$ be a representation of Γ on a finite dimensional vector space V over $\mathbb{C}$ where we set symbolically $V = V_\rho$. We call the space*

$$Ind_{\Gamma}^{G}(V_\rho) = \{\varphi : G \to V_\rho \mid \varphi(\gamma x) = \rho(\gamma)(\varphi(x))\ \forall \gamma \in \Gamma,\ \forall x \in G\} \tag{2.1}$$

the induced vector space from V_ρ.

Proposition 2.1. *The induced space $Ind_{\Gamma}^{G}(V_\rho)$ is a $\mathbb{C}[G]$-module.*

Proof. Define a $\mathrm{Ind}_{\Gamma}^{G}(V_\rho)$-valued map on $G \times \mathrm{Ind}_{\Gamma}^{G}(V_\rho)$, namely $\alpha : G \times \mathrm{Ind}_{\Gamma}^{G}(V_\rho) \to \mathrm{Ind}_{\Gamma}^{G}(V_\rho)$ by

$$\alpha(x, \varphi(y)) = x\varphi(y) = \varphi(xy). \tag{2.2}$$

We show that this is an action of G on $\mathrm{Ind}_{\Gamma}^{G}(V_\rho)$.

Let $x, y, z \in G$. We get

(i) $\alpha(e, \varphi(y)) = e\varphi(y) = \varphi(ey) = \varphi(y) \implies \alpha(e, \varphi(y)) = \varphi(y)$, where e is the identity element in G.

(ii) $\alpha(x, \alpha(y, \varphi(z))) = x\alpha(y, \varphi(z)) = x(y\varphi(z)) = xy\varphi(z) = \alpha(xy, \varphi(z))$.

Together with this action $\mathrm{Ind}_{\Gamma}^{G}(V_\rho)$ turns into a $\mathbb{C}[G]$-module. Indeed, $\mathrm{Ind}_{\Gamma}^{G}(V_\rho)$ is an additive abelian group. Define an addition operation on $\mathrm{Ind}_{\Gamma}^{G}(V_\rho)$ as

$$(\varphi+\psi)(x) := \rho(\gamma)[\varphi(x)+\psi(x)]. \tag{2.3}$$

(i) Let $\varphi, \psi \in \mathrm{Ind}_{\Gamma}^{G}(V_\rho)$. Then for all $x \in G$ and $\gamma \in \Gamma$

$$\begin{aligned}(\varphi+\psi)(\gamma x) &= \rho(\gamma)((\varphi+\psi)(x))\\ &= \rho(\gamma)(\varphi(x)+\psi(x))\\ &= \rho(\gamma)[\varphi(x)+\psi(x)]\\ &= \rho(\gamma)(\phi(x))+\rho(\gamma)(\psi(x))\\ &= \varphi(\gamma x)+\psi(\gamma x)\end{aligned} \tag{2.4}$$

$\implies \varphi+\psi \in \mathrm{Ind}_{\Gamma}^{G}(V_\rho)$.

(ii) Now, the addition is associative. Let $\phi, \psi, \eta \in \mathrm{Ind}_{\Gamma}^{G}(V_\rho)$. Then for all $x \in G$ and $\gamma \in \Gamma$

$$\begin{aligned}[(\varphi+\psi)+\eta](\gamma x) &= \rho(\gamma)[(\varphi+\psi)+\eta](x)\\ &= \rho(\gamma)\{[\varphi(x)+\psi(x)]+\eta(x)\}\\ &= [\rho(\gamma)(\varphi(x))+\rho(\gamma)(\psi(x))]+\rho(\gamma)(\eta(x))\\ &= [\varphi(\gamma x)+\psi(\gamma x)]+\eta(\gamma x)\\ &= \varphi(\gamma x)+[\psi(\gamma x)+\eta(\gamma x)]\\ &= \rho(\gamma)(\varphi(x))+[\rho(\gamma)(\psi(x))+\rho(\gamma)(\eta(x))]\\ &= \rho(\gamma)\{\varphi(x)+[\psi(x)+\eta(x)]\}\\ &= [\varphi+(\psi+\eta)](\gamma x)\end{aligned} \tag{2.5}$$

$\implies (\varphi+\psi)+\eta = \varphi+(\psi+\eta)$.

(iii) Next, the additive identity element of $\mathrm{Ind}_{\Gamma}^{G}(V_\rho)$ is the function $\mathbf{0} : G \to V_\rho$ such that $\mathbf{0}(x) := \mathbf{0}_{V_\rho}$ where $\mathbf{0}_{V_\rho}$ is the additive identity of the vector space V_ρ.

(iv) Let $\varphi \in \mathrm{Ind}_{\Gamma}^{G}(V_\rho)$. The additive inverse of φ is $-\varphi$. If $\varphi+\psi = 0$, i.e. $(\varphi+\psi)(x) = 0(x)$, then $(\varphi+\psi)(\gamma x) = \varphi(\gamma x)+\psi(\gamma x) = 0(\gamma x) = 0_{V_\rho}$. We have for all $x \in G$ and $\gamma \in \Gamma$

$$\varphi(\gamma x) + \psi(\gamma x) = \rho(\gamma)[\varphi(x) + \psi(x)] = \rho(\gamma)(\varphi(x)) + \rho(\gamma)(\psi(x)) = 0_V \qquad (2.6)$$

$$\implies \rho(\gamma)(\psi(x)) = -\rho(\gamma)(\varphi(x))$$
$$\implies \psi(x) = -\varphi(x)$$

(v) For the commutativity of the addition we have for all $x \in G$

$$(\varphi + \psi)(x) = \varphi(x) + \psi(x) = \psi(x) + \varphi(x) = (\psi + \varphi)(x) \qquad (2.7)$$

since $(V_\rho, +)$ is an abelian group.

(vi) Now, the scalar multiplication is defined as

$$(a\varphi)(\gamma x) := a\varphi(\gamma x). \qquad (2.8)$$

For the associativity of the scalar multiplication, we have for all $x \in G$ and $\gamma \in \mathit{\Gamma}$

$$(ab)\varphi(\gamma x) = (ab)\varphi(\gamma x) = a(b\varphi(\gamma x)) = a(b\varphi)(\gamma x) \qquad (2.9)$$

where $a, b \in F$ over which V_ρ is a vector space.

(vii) Furthermore, the scalar multiplication is distributive over field addition, that is, $x \in G$, $\gamma \in \mathit{\Gamma}$ and $a, b \in F$,

$$[(a+b)\varphi](\gamma x) = (a+b)\varphi(\gamma x) = a\varphi(\gamma x) + b\varphi(\gamma x) = (a\varphi)(\gamma x) + (b\varphi)(\gamma x); \quad (2.10)$$

and the scalar multiplication is distributive over the vector addition, i.e. $x \in G$, $\gamma \in \mathit{\Gamma}$ and $a, b \in F$,

$$[a(\varphi+\psi)](\gamma x) = a(\varphi+\psi)(\gamma x) = a\varphi(\gamma x) + a\psi(\gamma x) = (a\varphi)(\gamma x) + (a\psi)(\gamma x). \quad (2.11)$$

(viii) Finally, the identity of the scalar multiplication is the multiplicative identity of the vector space V_ρ. □

Definition 2.2. *The space $Ind_{\mathit{\Gamma}}^G(V_\rho)$ is called the induced module via the representation ρ. We denote the corresponding induced representation as $Ind_{\mathit{\Gamma}}^G(\rho)$; the representation of the group G on the induced vector space $Ind_{\mathit{\Gamma}}^G(V_\rho)$. This representation is defined by*

$$[Ind_{\mathit{\Gamma}}^G(\rho)(g)](\varphi)(x) = \varphi(xg) \qquad (2.12)$$

for all $g \in G$, for all $\varphi \in Ind_{\mathit{\Gamma}}^G(V_\rho)$ and for all $x \in G$, that is, the right-regular *representation of G on the vector space $Ind_{\mathit{\Gamma}}^{G}(V_\rho)$.*

Previously, we set V_ρ as a vector space over the field F. In the following, we put $F = \mathbb{C}$.

Proposition 2.2. *$Ind_{\mathit{\Gamma}}^{G}(\rho)$ is indeed a representation.*

Proof.

Claim 1. *We show that*

(i) $[Ind_{\mathit{\Gamma}}^{G}(\rho)(g)](\varphi) \in Ind_{\mathit{\Gamma}}^{G}(V_\rho)$,
(ii) $[Ind_{\mathit{\Gamma}}^{G}(\rho)(g)](\varphi_1 + \varphi_2) = [Ind_{\mathit{\Gamma}}^{G}(\rho)(g)](\varphi_1) + [Ind_{\mathit{\Gamma}}^{G}(\rho)(g)](\varphi_2)$,
(iii) $[Ind_{\mathit{\Gamma}}^{G}(\rho)(g)](\lambda\varphi) = \lambda[Ind_{\mathit{\Gamma}}^{G}(\rho)(g)](\varphi)$,
(iv) $[Ind_{\mathit{\Gamma}}^{G}(\rho)(g)]^{-1} = [Ind_{\mathit{\Gamma}}^{G}(\rho)(g^{-1})]$

for all $g \in G$; $\varphi, \varphi_1, \varphi_2 \in Ind_{\mathit{\Gamma}}^{G}(V_\rho)$; $\lambda \in \mathbb{C}$; where $\rho : \mathit{\Gamma} \to V_\rho$ is a representation of $\mathit{\Gamma}$ over $\mathbb{C}$.

(i) We have for all $g, x \in G$, $\gamma \in \mathit{\Gamma}$ and $\varphi \in \mathrm{Ind}_{\mathit{\Gamma}}^{G}(V_\rho)$

$$[\mathrm{Ind}_{\mathit{\Gamma}}^{G}(\rho)(g)](\varphi)(\gamma x) = \varphi(\gamma x g) = \rho(\gamma)(\varphi(xg)). \tag{2.13}$$

That is, $[\mathrm{Ind}_{\mathit{\Gamma}}^{G}(\rho)(g)](\varphi) \in \mathrm{Ind}_{\mathit{\Gamma}}^{G}(V_\rho)$.

(ii) Next, we have for arbitrary $\varphi_1, \varphi_2 \in \mathrm{Ind}_{\mathit{\Gamma}}^{G}(V_\rho)$, for all $g, x \in G$ and for all $\lambda \in \mathbb{C}$,

$$\begin{aligned}[\mathrm{Ind}_{\mathit{\Gamma}}^{G}(\rho)(g)](\varphi_1 + \varphi_2)(x) &= (\varphi_1 + \varphi_2)(x + g) \\ &= \varphi_1(x + g) + \varphi_2(x + g) \\ &= [\mathrm{Ind}_{\mathit{\Gamma}}^{G}(\rho)(g)](\varphi_1)(x) + [\mathrm{Ind}_{\mathit{\Gamma}}^{G}(\rho)(g)](\varphi_2)(x).\end{aligned} \tag{2.14}$$

(iii) Furthermore, for all $\varphi \in \mathrm{Ind}_{\mathit{\Gamma}}^{G}(V_\rho)$, for all $g, x \in G$ and for all $\lambda \in \mathbb{C}$

$$
\begin{aligned}
\operatorname{Ind}_{\Gamma}^{G}(\rho)(g)(\lambda\varphi)(x) &= (\lambda\varphi)(x+g) \\
&= \lambda\varphi(x+g) \\
&= \lambda[\operatorname{Ind}_{\Gamma}^{G}(\rho)(g)](\varphi)(x).
\end{aligned} \tag{2.15}
$$

(iv) Finally, for all $\varphi \in \operatorname{Ind}_{\Gamma}^{G}(V_\rho)$ and $g, x \in G$

$$
\begin{aligned}
[\operatorname{Ind}_{\Gamma}^{G}(\rho)(g^{-1})](\varphi)(x) &= \varphi(x+(-g)) \\
&= \varphi(x-g) \\
&= \{[\operatorname{Ind}_{\Gamma}^{G}(\rho)(g)](\varphi)(x)\}^{-1}
\end{aligned} \tag{2.16}
$$

because, $\varphi(x+g)\varphi(x-g) = 1 \implies \varphi(x-g) = [\varphi(x+g)]^{-1}$. □

2.2. The Relation Between The Spaces V_ρ and $\operatorname{Ind}_{\Gamma}^{G}(V_\rho)$

Proposition 2.3. *The induced space $\operatorname{Ind}_{\Gamma}^{G}(V_\rho)$ is isomorphic to the direct sum $\bigoplus_{r\in R} rV_\rho$, where R is the complete set of representatives in G of the left cosets G/Γ, that is, $\operatorname{Ind}_{\Gamma}^{G}(V_\rho) \simeq \bigoplus_{r\in R} rV_\rho$.*

Proof. We embed V_ρ in $\operatorname{Ind}_{\Gamma}^{G}(V_\rho)$ by sending each $v \in V_\rho$ onto the function $\varphi_v : G \to V_\rho$ which is defined by

$$
\varphi_v(g) = \begin{cases} \rho(g)v, & g \in \Gamma \\ 0_{V_\rho}, & g \in G - \Gamma. \end{cases} \tag{2.17}
$$

Consider the decomposition of G into left classes modulo Γ, i.e. $G = \bigsqcup_{r\in R} r\Gamma$. For every $\varphi \in \operatorname{Ind}_{\Gamma}^{G}(V_\rho)$ and for $r \in R$ define a function $\varphi_r \in \operatorname{Ind}_{\Gamma}^{G}(V_\rho)$ by

$$
\varphi_r(g) = \begin{cases} \varphi(g), & g \in \Gamma r^{-1} \\ 0_{V_\rho}, & \text{otherwise.} \end{cases} \tag{2.18}
$$

We have $r^{-1}\varphi_r(g) \in V_\rho$ for all g so that $\varphi = \sum_{r\in R} r(r^{-1}\varphi_r)$, that is $\operatorname{Ind}_{\Gamma}^{G}(V_\rho) \simeq \bigoplus_{r\in R} rV_\rho$. □

Corollary 2.1. *The dimension of the induced space $\operatorname{Ind}_{\Gamma}^{G}(V_\rho)$ is $[G : H] \dim V_\rho$.*

Proposition 2.4. *The diagram*

$$\begin{tikzcd} \mathbb{C}[G] \times V_\rho \arrow[r, "canonical"] \arrow[dr, "B"'] & \bigoplus_{r \in R} rV_\rho \arrow[d, dashed, "\exists! B_0"] \\ & W \end{tikzcd}$$

commutes.

Proof. For each $g \in \mathbb{C}[G]$ and $v \in V_\rho$ we write

$$f(g, v) = \bigoplus_{r \in R} f_r(g, v) \in \bigoplus_{r \in R} rV_\rho. \tag{2.19}$$

Since B is bilinear and f is linear in each component, B_0, defined by $B_0(f(g, v)) = B(g, v)$, is well-defined and linear. Suppose there is another map B_0' such that $B_0' \circ f = B$. Then

$$B_0(f(g, v)) = B(g, v) = B_0'(f(g, v)), \tag{2.20}$$

i.e. B_0 is unique. □

Corollary 2.2. $Ind_\Gamma^G(V_\rho) \simeq \mathbb{C}[G] \otimes_{\mathbb{C}[\Gamma]} V_\rho$.

Proposition 2.5. *We have the transitivity that*

$$Ind_H^G(V) = Ind_\Gamma^G(Ind_H^\Gamma(V)). \tag{2.21}$$

Proof. Indeed, $\mathbb{C}[G] \otimes_{\mathbb{C}[\Gamma]} \mathrm{Ind}_H^\Gamma V = \mathbb{C}[G] \otimes_{\mathbb{C}[\Gamma]} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}[H]} V = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$. □

Definition 2.3. *We denote by $Res_H^G E$ the restriction from G to H on E.*

Theorem 2.1 (Frobenius Reciprocity)**.** *Using the isomorphism in Corollary 2.0.2., we get the canonical isomorphism*

$$Hom_{\mathbb{C}[G]}(Ind_H^G W, E) \simeq Hom_{\mathbb{C}[H]}(W, Res_H^G E). \tag{2.22}$$

Remark 2.1. *We have the equality*

$$\dim Hom_{\mathbb{C}[G]}(Ind_H^G W, E) = \dim Hom_{\mathbb{C}[H]}(W, Res_H^G E), \tag{2.23}$$

also written in the following notation,

$$\langle Ind_H^G \tau, \sigma\rangle_G = \langle \tau, Res_H^G \sigma\rangle_H. \tag{2.24}$$

Remark 2.2. *If τ and σ both are irreducible, then the multiplicity of σ in $Ind_H^G\tau$ equals the multiplicity of τ in $Res_H^G\sigma$.*

Definition 2.4. *The space $Ind_\Gamma^G(V_\rho)$ is an inner product space together with the inner product $\langle \bullet \mid \bullet \rangle : Ind_\Gamma^G(V_\rho) \times Ind_\Gamma^G(V_\rho) \to \mathbb{C}$ on $Ind_\Gamma^G(V_\rho)$ defined by*

$$\langle \varphi_1 \mid \varphi_2 \rangle := \sum_{i=1}^{\iota} {}^H[\varphi_2(g_i)]_{\mathcal{B}_{V_\rho}} [\varphi_1(g_i)]_{\mathcal{B}_{V_\rho}} \quad \forall \varphi_1, \varphi_2 \in Ind_\Gamma^G(V_\rho). \tag{2.25}$$

Remark 2.3. *The map defined in Definition 4 is indeed an inner product. The conjugate symmetry and linearity is clear where the map is also positive-definite as shown in what follows*

$$\begin{aligned} \langle \varphi, \varphi \rangle &= \sum_{i=1}^{\iota} {}^H[\varphi(g_i)]_{\mathcal{B}_{V_\rho}} [\varphi(g_i)]_{\mathcal{B}_{V_\rho}} \\ &= \sum_{i=1}^{\iota} \mid [\varphi(g_i)]_{\mathcal{B}_{V_\rho}} \mid^2 > 0 \end{aligned} \tag{2.26}$$

where $\mid [\varphi(g_i)]_{\mathcal{B}_{V_\rho}} \mid^2 > 0$ for all $1 \le i \le \iota$ provided that $\varphi \neq \boldsymbol{0}_{V_\rho}$.

Remark 2.4. *The scalar product neither depends on a choice of a complete set of representatives $\mathcal{R}_{\Gamma\backslash G} = \{g_1, ..., g_\iota\} \subset G$ of the coset space $\Gamma\backslash G$ nor a choice of an ordered basis $\mathcal{B}_{V_\rho} = \{v_1, ..., v_d\}$ of V_ρ over $\mathbb{C}$.*

3. TRACE FORMULA FOR FINITE GROUPS

This chapter takes its heading from the theorem "Trace formula for finite groups". Examining induced spaces and induced representations in the previous chapter, we now delve into the trace formula for finite groups, giving the main theorem in this master thesis study for which one can refer to the article [1]. In the chapter, essentially, we present the theorem and its proof which can be found in the article [1].

Definition 3.1. *Let $r : G \to GL(V_r)$ be a representation of the finite group G on a d-dimensional vector space V_r over complex numbers. For any function $f : G \to \mathbb{C}$, there is a $\mathbb{C}$-linear operator $r(f) : V_r \to V_r$ defined on V_r as follows*

$$(r(f))(v) = \sum_{g\in G} f(g)r(g)(v) \quad \forall v \in V_r. \tag{3.1}$$

Here, we are interested in computing the trace of the linear operator $\mathrm{Ind}_{\Gamma}^{G}(\rho)(f) : \mathrm{Ind}_{\Gamma}^{G}(V_\rho) \to \mathrm{Ind}_{\Gamma}^{G}(V_\rho)$.

Corollary 3.1. *The operator $Ind_{\Gamma}^{G}(\rho)(f)$ is defined for all $\varphi \in Ind_{\Gamma}^{G}(V_\rho)$ and for all $x \in G$ by*

$$(Ind_{\Gamma}^{G}(\rho)(f)\varphi)(x) = \sum_{g\in G} f(g)[Ind_{\Gamma}^{G}(\rho)(g)](\varphi)(x) = \sum_{g\in G} f(g)\varphi(xg). \tag{3.2}$$

Remark 3.1. *Substituting $xg \rightsquigarrow y$ and partitioning G as $G = \bigsqcup_{i=1}^{\iota} \Gamma g_i$ we have for all $\varphi \in Ind_{\Gamma}^{G}(V_\rho)$ and for all $x \in G$*

$$\begin{aligned}(Ind_{\Gamma}^{G}(\rho)(f)\varphi)(x) &= \sum_{y\in G} f(x^{-1}y)\varphi(y)\\ &= \sum_{i=1}^{\iota}\sum_{\gamma\in\Gamma} f(x^{-1}\gamma g_i)\varphi(\gamma g_i)\\ &= \sum_{i=1}^{\iota}\sum_{\gamma\in\Gamma} f(x^{-1}\gamma g_i)\rho(\gamma)(\varphi(g_i)).\end{aligned} \tag{3.3}$$

Moreover, with the counting measure, $(Ind_{\Gamma}^{G}(\rho)(f)\varphi)(x)$ *is an integral operator with the kernel* $K_f : G \times G \to End_{\mathbb{C}}(V_\rho)$ *given by*

$$K_f(x,y) = \sum_{\gamma \in \Gamma} f(x^{-1}\gamma g_i)\rho(\gamma) \quad \forall x, y \in G. \tag{3.4}$$

Theorem 3.1 (Trace formula for finite groups)**.** *For an arbitrary test function* $f : G \to \mathbb{C}$*, the trace* $\mathrm{Tr}(\mathrm{Ind}_{\Gamma}^{G}(\rho)(f))$ *of the operator*

$$\mathrm{Ind}_{\Gamma}^{G}(\rho)(f) : \mathrm{Ind}_{\Gamma}^{G}(V_\rho) \to \mathrm{Ind}_{\Gamma}^{G}(V_\rho) \tag{3.5}$$

on the $\mathbb{C}$*-linear space* $\mathrm{Ind}_{\Gamma}^{G}(V_\rho)$ *satisfies the identity*

$$J(\rho, f) = \mathrm{Tr}(\mathrm{Ind}_{\Gamma}^{G}(\rho)(f)) = I(\rho, f), \tag{3.6}$$

where

$$J(\rho, f) = \sum_{\pi_0 \in [\pi] \in \Pi(G)} m(\pi_0, \mathrm{Ind}_{\Gamma}^{G}(\rho))\mathrm{Tr}(\pi_0(f)) \tag{3.7}$$

and

$$I(\rho, f) = \sum_{\gamma \in \{\Gamma\}} \chi_\rho(\gamma) \frac{\mid G_\gamma \mid}{\mid \Gamma_\gamma \mid} \sum_{t \in G_\gamma \backslash G} f(t^{-1}\gamma t) \tag{3.8}$$

where $\{\Gamma\}$*: a set consisting of all representatives for the conjugacy classes in* Γ*;* $\Gamma_\gamma = \{\delta \in \Gamma \mid \delta^{-1}\gamma\delta = \gamma\}$ *and* $G_\gamma = \{g \in G \mid g^{-1}\gamma g = \gamma\}$ *for* $\gamma \in \{\Gamma\}$*.*

Here, $J(\rho, f)$ *and* $I(\rho, f)$ *are called the spectral side and the geometric side of the trace formula for the finite group* G *with respect to the subgroup* Γ *and* $\rho : \Gamma \to GL(V_\rho)$*, respectively.*

Proof. To give a proof for the spectral side $J(\rho, f)$ of the equation of the trace formula, observe that the representation $\mathrm{Ind}_{\Gamma}^{G}(\rho)$ decomposes as follows

$$\mathrm{Ind}_{\Gamma}^{G}(\rho) = \bigoplus_{\pi_0 \in [\pi] \in \Pi(G)} m(\pi_0, \mathrm{Ind}_{\Gamma}^{G}(\rho))\pi_0 \tag{3.9}$$

where $[\pi]$ is the set of non-isomorphic irreducible representations of G over $\mathbb{C}$ with the corresponding positive integer coefficients together with zero and $\Pi(G)$ denotes the set of all isomorphism classes $[\pi]$ of irreducible representations of G over $\mathbb{C}$. Note that for the coefficients $m(\pi_0, \mathrm{Ind}_\Gamma^G(\rho))$ (multiplicities) we have the formula (cf. Remark 4.1.)

$$m(\pi_0, \mathrm{Ind}_\Gamma^G(\rho)) = \frac{1}{\mid G \mid} \sum_{g \in G} \pi_0(g) \overline{\mathrm{Ind}_\Gamma^G(\rho)(g)}. \tag{3.10}$$

Also, for any function $f : G \to \mathbb{C}$, the linear operator $\mathrm{Ind}_\Gamma^G(\rho)(f)$ associated with the induced representation $\mathrm{Ind}_\Gamma^G(\rho)$ decomposes as follows

$$\mathrm{Ind}_\Gamma^G(\rho)(f) = \bigoplus_{\pi_0 \in [\pi] \in \Pi(G)} m(\pi_0, \mathrm{Ind}_\Gamma^G(\rho)) \pi_0(f) \tag{3.11}$$

Then the trace $\mathrm{Tr}(\mathrm{Ind}_\Gamma^G(\rho)(f))$ of the integral operator $\mathrm{Ind}_\Gamma^G(\rho)(f) : \mathrm{Ind}_\Gamma^G(V_\rho) \to \mathrm{Ind}_\Gamma^G(V_\rho)$ is given by

$$\begin{aligned} \mathrm{Tr}(\mathrm{Ind}_\Gamma^G(\rho)(f)) &= \mathrm{Tr} \left(\bigoplus_{\pi_0 \in [\pi] \in \Pi(G)} m(\pi_0, \mathrm{Ind}_\Gamma^G(\rho)) \pi_0(f) \right) \\ &= \sum_{\pi_0 \in [\pi] \in \Pi(G)} m(\pi_0, \mathrm{Ind}_\Gamma^G(\rho)) \mathrm{Tr}(\pi_0(f)) \\ &= J(\rho, f). \end{aligned} \tag{3.12}$$

Now we compute the geometric side $I(\rho, f)$ of the trace formula. Firstly, note that we have, for the entry a_{ij} of the matrix $A = [a_{ij}]$, the following equalities

$$a_{ij} = \sum_{i=1}^{n} a_{ij} \delta_{ij} = \sum_{i=1}^{n} a_{ij} \langle e_i, e_j \rangle = \left\langle \sum_{i=1}^{n} a_{ij} e_i, e_j \right\rangle = \langle T(e_j), e_j \rangle, \tag{3.13}$$

where δ_{ij} is the Kronecker delta and $\{e_1, ..., e_n\}$ is an orthonormal basis of the vector space on which an operator T is defined. Now, with respect to the ordered orthonormal basis $\mathcal{N}_{\mathrm{Ind}_\Gamma^G V_\rho} = \{\varphi_{ij}\}_{\substack{i=1,\dots,\iota \\ j=1,\dots,d}}$ of the vector space $\mathrm{Ind}_\Gamma^G(V_\rho)$, the matrix associated with the operator $\mathrm{Ind}_\Gamma^G(\rho)(f)$ is given by

$$\begin{bmatrix} \langle \mathrm{Ind}_{\Gamma}^{G}(\rho)(\varphi_{11}) \mid \varphi_{11}\rangle & \langle \mathrm{Ind}_{\Gamma}^{G}(\rho)(\varphi_{12}) \mid \varphi_{11}\rangle & \cdots & \langle \mathrm{Ind}_{\Gamma}^{G}(\rho)(\varphi_{\iota d}) \mid \varphi_{11}\rangle \\ \langle \mathrm{Ind}_{\Gamma}^{G}(\rho)(\varphi_{11}) \mid \varphi_{12}\rangle & \langle \mathrm{Ind}_{\Gamma}^{G}(\rho)(\varphi_{12}) \mid \varphi_{11}\rangle & \cdots & \langle \mathrm{Ind}_{\Gamma}^{G}(\rho)(\varphi_{\iota d}) \mid \varphi_{12}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathrm{Ind}_{\Gamma}^{G}(\rho)(\varphi_{11}) \mid \varphi_{\iota d}\rangle & \langle \mathrm{Ind}_{\Gamma}^{G}(\rho)(\varphi_{12}) \mid \varphi_{\iota d}\rangle & \cdots & \langle \mathrm{Ind}_{\Gamma}^{G}(\rho)(\varphi_{\iota d}) \mid \varphi_{\iota d}\rangle \end{bmatrix}. \tag{3.14}$$

Therefore, the trace $\mathrm{Tr}(\mathrm{Ind}_{\Gamma}^{G}(\rho)(f))$ of the operator $\mathrm{Ind}_{\Gamma}^{G}(\rho)(f)$ is given by

$$\mathrm{Tr}(\mathrm{Ind}_{\Gamma}^{G}(\rho)(f)) = \sum_{(s,t)=(1,1)}^{(\iota,d)} \langle \mathrm{Ind}_{\Gamma}^{G}(\rho)(f)(\varphi_{st}) \mid \varphi_{st}\rangle. \tag{3.15}$$

We calculate as follows

$$\begin{aligned} \mathrm{Tr}(\mathrm{Ind}_{\Gamma}^{G}(\rho)(f)) &= \sum_{(s,t)=(1,1)}^{(\iota,d)} \langle \mathrm{Ind}_{\Gamma}^{G}(\rho)(f)(\varphi_{st}) \mid \varphi_{st}\rangle \\ &= \sum_{(s,t)=(1,1)}^{(\iota,d)} \sum_{i_0=1}^{\iota} {}^{H}[\varphi_{st}(g_{i_0})]_{\mathcal{N}_{V_\rho}} [\mathrm{Ind}_{\Gamma}^{G}(\rho)(f)(\varphi_{st}(g_{i_0}))]_{\mathcal{N}_{V_\rho}} \\ &= \sum_{(s,t)=(1,1)}^{(\iota,d)} \sum_{i_0=1}^{\iota} {}^{H}[\varphi_{st}(g_{i_0})]_{\mathcal{N}_{V_\rho}} \left[\sum_{i=1}^{\iota} K_f(g_{i_0}, g_i)\varphi_{st}(g_i) \right]_{\mathcal{N}_{V_\rho}} \\ &= \sum_{(s,t)=(1,1)}^{(\iota,d)} \sum_{i_0=1}^{\iota} {}^{H}[\varphi_{st}(g_{i_0})]_{\mathcal{N}_{V_\rho}} \left[\sum_{i=1}^{\iota} \sum_{\gamma\in\Gamma} f(g_{i_0}^{-1}\gamma g_i)\rho(\gamma)\varphi_{st}(g_i) \right]_{\mathcal{N}_{V_\rho}} \\ &= \sum_{(s,t)=(1,1)}^{(\iota,d)} \sum_{i_0=1}^{\iota} \sum_{j_0=1}^{d} \overline{\langle \varphi_{st}(g_{i_0}) \mid e_{j_0}\rangle} \sum_{i=1}^{\iota} \sum_{\gamma\in\Gamma} f(g_{i_0}^{-1}\gamma g_i)\langle \rho(\gamma)\varphi_{st}(g_i) \mid e_{j_0}\rangle. \end{aligned} \tag{3.16}$$

where $i_0 = 1, ..., \iota$; $j_0 = 1, ..., d$.

We have

$$\varphi_{st}(g_{i_0}) = \begin{cases} \rho(g_{i_0} g_s^{-1})e_t = e_t, & i_0 = s \\ 0_{V_\rho}, & \text{otherwise.} \end{cases} \tag{3.17}$$

Also, for $\gamma \in \Gamma$; $s, i, i_0 = 1, ..., \iota; t, j_0 = 1, ..., d$,

$$\rho(\gamma)\varphi_{st}(g_i) = \varphi_{st}(\gamma g_i) = \tilde{\rho}(\gamma g_i g_s^{-1})e_t = \begin{cases} \rho(\gamma g_i g_s^{-1})e_t = \rho(\gamma)e_t, & i = s \\ 0, & i \neq s. \end{cases} \tag{3.18}$$

Therefore, for a fixed $s = 1, ..., \iota$ and a fixed $t = 1, ..., d$,

$$\begin{aligned} \langle \mathrm{Ind}_{\Gamma}^{G}(\rho)(f)(\varphi_{st}) \mid \varphi_{st}\rangle &= \sum_{i_0=1}^{\iota}\sum_{j_0=1}^{d} \overline{\langle \varphi_{st}(g_{i_0}) \mid e_{j_0}\rangle} \sum_{i=1}^{\iota}\sum_{\gamma\in\Gamma} f(g_{i_0}^{-1}\gamma g_i)\langle \rho(\gamma)\varphi_{st}(g_i) \mid e_{j_0}\rangle \\ &= \sum_{i=1}^{\iota}\sum_{\gamma\in\Gamma} f(g_s^{-1}\gamma g_i)\langle \rho(\gamma)\varphi_{st}(g_i) \mid e_t\rangle \\ &= \sum_{\gamma\in\Gamma} f(g_s^{-1}\gamma g_s)\langle \rho(\gamma)\varphi_{st}(g_s) \mid e_t\rangle \\ &= \sum_{\gamma\in\Gamma} f(g_s^{-1}\gamma g_s)\langle \rho(\gamma)e_t \mid e_t\rangle, \end{aligned} \tag{3.19}$$

which leads to the identity

$$\mathrm{Tr}(\mathrm{Ind}_{\Gamma}^{G}(\rho)(f)) = \sum_{(s,t)=(1,1)}^{(\iota,d)} \langle \mathrm{Ind}_{\Gamma}^{G}(\rho)(f)(\varphi_{st}) \mid \varphi_{st}\rangle = \sum_{(s,t)=(1,1)}^{(\iota,d)} \sum_{\gamma\in\Gamma} f(g_s^{-1}\gamma g_s)\langle \rho(\gamma)e_t \mid e_t\rangle. \tag{3.20}$$

Introduce two subgroups $\Gamma_\gamma \leq \Gamma$ and $G_\gamma \leq G$ for $\gamma \in \{\Gamma\}$ as follows

$$\begin{aligned} \Gamma_\gamma &= \{\delta \in \Gamma \mid \delta^{-1}\gamma\delta = \gamma\} \\ G_\gamma &= \{g \in G \mid g^{-1}\gamma g = \gamma\} \end{aligned} \tag{3.21}$$

where $\{\Gamma\}$ is a set consisting of all representatives γ for the conjugacy classes C_γ^{Γ} in Γ.

The subgroup Γ decomposes as

$$\Gamma = \bigsqcup_{\gamma\in\{\Gamma\}} C_\gamma^{\Gamma} = \bigsqcup_{\gamma\in\{\Gamma\}} C_\gamma = \bigsqcup_{\gamma\in\{\Gamma\}} \{\delta^{-1}\gamma\delta \mid \delta \in \mathcal{R}_{\Gamma_\gamma\backslash\Gamma}\} \tag{3.22}$$

where $\mathcal{R}_{\Gamma_\gamma\backslash\Gamma}$ is a fixed complete set of representatives of $\Gamma_\gamma\backslash\Gamma$. $\square$

We write the trace formula as follows

$$
\begin{aligned}
\mathrm{Ind}_{\Gamma}^{G}(\rho)(f)(\varphi_{st}) \mid \varphi_{st}\rangle &= \sum_{\substack{1\le s\le\iota\\1\le t\le d}}\sum_{\gamma\in\Gamma} f(g_s^{-1}\gamma g_s)\langle\rho(\gamma)e_t \mid e_t\rangle\\
&= \sum_{\substack{1\le s\le\iota\\1\le t\le d}}\sum_{\gamma\in\{\Gamma\}}\sum_{\omega\in C_\gamma} f(g_s^{-1}\omega g_s)\langle\rho(\omega)e_t \mid e_t\rangle\\
&= \sum_{\substack{1\le s\le\iota\\1\le t\le d}}\sum_{\gamma\in\{\Gamma\}}\sum_{\delta\in\mathcal{R}_{\Gamma_\gamma\backslash\Gamma}} f(g_s^{-1}\delta_{-1}\gamma\delta g_s)\langle\rho(\delta_{-1}\gamma\delta)e_t \mid e_t\rangle\\
&= \sum_{\gamma\in\{\Gamma\}}\sum_{\delta\in\mathcal{R}_{\Gamma_\gamma\backslash\Gamma}}\sum_{\substack{1\le s\le\iota\\1\le t\le d}} f(g_s^{-1}\delta_{-1}\gamma\delta g_s)\langle\rho(\delta_{-1}\gamma\delta)e_t \mid e_t\rangle\\
&= \sum_{\gamma\in\{\Gamma\}}\sum_{\delta\in\mathcal{R}_{\Gamma_\gamma\backslash\Gamma}}\left(\sum_{1\le s\le\iota} f(g_s^{-1}\delta_{-1}\gamma\delta g_s)\right)\left(\sum_{1\le t\le d}\langle\rho(\delta_{-1}\gamma\delta)e_t \mid e_t\rangle\right)\\
&= \sum_{\gamma\in\{\Gamma\}}\sum_{\delta\in\mathcal{R}_{\Gamma_\gamma\backslash\Gamma}}\left(\sum_{1\le s\le\iota} f(g_s^{-1}\delta_{-1}\gamma\delta g_s)\right)\mathrm{Tr}(\delta^{-1}\gamma\delta)\\
&= \sum_{\gamma\in\{\Gamma\}}\sum_{\delta\in\mathcal{R}_{\Gamma_\gamma\backslash\Gamma}}\left(\sum_{1\le s\le\iota} f(g_s^{-1}\delta_{-1}\gamma\delta g_s)\right)\mathrm{Tr}(\rho(\gamma))\\
&= \sum_{\gamma\in\{\Gamma\}}\sum_{\delta\in\mathcal{R}_{\Gamma_\gamma\backslash\Gamma}}\left(\sum_{1\le s\le\iota} f(g_s^{-1}\delta_{-1}\gamma\delta g_s)\right)\chi_\rho(\gamma)\\
&= \sum_{\gamma\in\{\Gamma\}}\chi_\rho(\gamma)\sum_{\delta\in\mathcal{R}_{\Gamma_\gamma\backslash\Gamma}}\sum_{1\le s\le\iota} f(g_s^{-1}\delta_{-1}\gamma\delta g_s).
\end{aligned}
\tag{3.23}
$$

Now, we view the partitions $\Gamma_\gamma \le \Gamma \le G$ and $\Gamma_\gamma \le G_\gamma \le G$ of G and decompose the group G accordingly in two different ways as follows

$$
G = \bigsqcup_{g_i\in\mathcal{R}_{\Gamma\backslash G}} \Gamma g_i = \bigsqcup_{g_i\in\mathcal{R}_{\Gamma\backslash G}}\left(\bigsqcup_{\delta\in\mathcal{R}_{\Gamma_\gamma\backslash\Gamma}}\Gamma_\gamma\delta\right) g_i = \bigsqcup_{s=1}^{\iota}\bigsqcup_{\delta\in\mathcal{R}_{\gamma_\gamma\backslash\Gamma}}\Gamma_\gamma\delta g_s
$$

$$
G = \bigsqcup_{t\in\mathcal{R}_{G_\gamma\backslash G}} G_\gamma t = \bigsqcup_{t\in\mathcal{R}_{G_\gamma\backslash G}}\left(\bigsqcup_{y\in\mathcal{R}_{\Gamma_\gamma\backslash G_\gamma}}\Gamma_\gamma y\right) t = \bigsqcup_{t\in\mathcal{R}_{G_\gamma\backslash G}}\bigsqcup_{t\in\mathcal{R}_{\Gamma_\gamma\backslash G_\gamma}}\Gamma_\gamma y t
$$

We have

$$
\begin{aligned}
\mathrm{Tr}(\mathrm{Ind}_{\Gamma}^{G}(\rho)(f)) &= \sum_{\gamma\in\{\Gamma\}}\chi_\rho(\gamma)\sum_{\delta\in\mathcal{R}_{\Gamma_\gamma\backslash\Gamma}}\sum_{1\le s\le t} f(g_s^{-1}\delta^{-1}\gamma\delta g_s)\\
&= \sum_{\gamma\in\{\Gamma\}}\chi_\rho(\gamma)\sum_{\delta\in\mathcal{R}_{\Gamma_\gamma\backslash\Gamma}} f(x^{-1}\gamma x)
\end{aligned}
\tag{3.24}
$$

and

$$\begin{aligned}
\mathrm{Tr}(\mathrm{Ind}_{\Gamma}^{G}(\rho)(f)) &= \sum_{\gamma\in\{\Gamma\}} \chi_\rho(\gamma) \sum_{\delta\in\mathcal{R}_{\Gamma_\gamma\backslash\Gamma}} f(x^{-1}\gamma x) \\
&= \sum_{\gamma\in\{\Gamma\}} \chi_\rho(\gamma) \sum_{\delta\in\mathcal{R}_{\Gamma_\gamma\backslash\Gamma}} \sum_{y\in\mathcal{R}_{\Gamma_\gamma\backslash G_\gamma}} f(t^{-1}y^{-1}\gamma y t) \\
&= \sum_{\gamma\in\{\Gamma\}} \chi_\rho(\gamma) \frac{\mid G_\gamma \mid}{\mid \Gamma_\gamma \mid} \sum_{t\in\mathcal{R}_{G_\gamma\backslash G}} f(t^{-1}\gamma t) \\
&= I(\rho, f).
\end{aligned} \tag{3.25}$$

4. THE TRACE FORMULA FOR $\mathbb{Z}/n\mathbb{Z}$

In order to investigate the trace formula for some specific abelian group, we first write the spectral side of the formula for that group. We here study the group $\mathbb{Z}/6\mathbb{Z}$ and then the general case $\mathbb{Z}/n\mathbb{Z}$. For that reason, in what follows we compute the multiplicities from the formula for induced characters (see Remark 6). Then we find the set $\{\Gamma\}$ and the orders $\mid G_{\Gamma} \mid$ and $\mid \Gamma_{\gamma} \mid$ so that we write the geometric side of the trace formula. The group $\mathbb{Z}/6\mathbb{Z}$ has two non-trivial subgroups. In this chapter we write the trace formula with respect to these subgroups and their trivial representations. Moreover, we write the trace formula for $\mathbb{Z}/n\mathbb{Z}$ with respect to the subgroup $\mathbb{Z}/m\mathbb{Z}$ and the trivial representation. In an introductory part of the chapter, we give some definitions concerning class functions. The Definitions 6 and 7 are from [4].

4.1. A Short Note On Class Functions

Definition 4.1. *A function ϕ on G satisfying*

$$\phi(tst^{-1}) = \phi(s) \tag{4.1}$$

for $s, t \in G$ is called a class function.

Definition 4.2. *If ϕ is a class function on Γ, consider the function ϕ' on G defined by the formula*

$$\phi'(\gamma) = \frac{1}{\mid \Gamma \mid} \sum_{\gamma \in \Gamma} \phi(\gamma). \tag{4.2}$$

We denote ϕ' by $Ind_{\Gamma}^{G}(\phi)$ and say that ϕ' is induced by ϕ. That is,

$$Ind_{\Gamma}^{G}(\phi)(\gamma) = \frac{1}{\mid \Gamma \mid} \sum_{\gamma \in \Gamma} \phi(\gamma). \tag{4.3}$$

Remark 4.1. *In particular, characters are class functions. In other words, characters are constant on conjugacy classes.*

We have the following identities for the multiplicities $m(\pi_0, Ind_{\Gamma}^{G}(\boldsymbol{1}_{\Gamma}))$

$$\begin{aligned} m(\pi_0, Ind_{\Gamma}^{G}(\boldsymbol{1}_{\Gamma})) &= \langle \pi_0, Ind_{\Gamma}^{G}(\boldsymbol{1}_{\Gamma}) \rangle_G \\ &= \frac{1}{\mid G \mid} \sum_{g \in G} \pi_0(g) \overline{Ind_{\Gamma}^{G}(\boldsymbol{1}_{\Gamma})(g)} \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} m(\chi_{\pi_0}, \chi_{Ind_{\Gamma}^{G}(\boldsymbol{1}_{\Gamma})}) &= \frac{1}{\mid G \mid} \sum_{g \in G} \chi_{\pi_0}(g) \overline{\chi_{Ind_{\Gamma}^{G}(\boldsymbol{1}_{\Gamma})}(g)} \\ &= \frac{1}{\mid \Gamma \mid} \sum_{\gamma \in \Gamma} \chi_{\pi_0}(\gamma) \overline{\chi_{Ind_{\Gamma}^{G}(\boldsymbol{1}_{\Gamma})}(\gamma)} \\ &= \frac{1}{\mid \Gamma \mid} \sum_{\gamma \in \Gamma} \chi_{\pi_0}(\gamma) \end{aligned} \tag{4.5}$$

where $\pi_0 \in [\pi] \in \Pi(G)$ *are non-isomorphic irreducible representations of* G*.*

4.2. The Trace Formula For $G = \mathbb{Z}/6\mathbb{Z}$ With Respect to The Subgroup $\Gamma = \langle \overline{2} \rangle$ and The Trivial Representation $\mathbf{1}_{\Gamma} : \Gamma \to \mathbf{GL}(\mathbb{C})$

Let $G = \mathbb{Z}/6\mathbb{Z}$ and let $\Gamma = \langle \overline{2} \rangle \leq G$ be a subgroup of order 3. Consider the trivial representation $\mathbf{1}_{\Gamma}$ of Γ. In order to write the spectral side of the trace formula for $G = \mathbb{Z}/6\mathbb{Z}$, we first find the multiplicities in the decomposition of the induced representation $\mathrm{Ind}_{\Gamma}^{G}(\mathbf{1}_{\Gamma})$. The induced representation is of dimension $[G : \Gamma] \dim(\mathbb{C}) = 2$.

Now, we note the character table of the group of integers modulo n, $\mathbb{Z}/6\mathbb{Z}$, as follows

Table 4.1. Character Table of $G = \mathbb{Z}/6\mathbb{Z}$.

g	0	1	2	3	4	5
χ_0	1	1	1	1	1	1
χ_1	1	ω	ω^2	ω^3	ω^4	ω^5
χ_2	1	ω^2	ω^4	1	ω^2	ω^4
χ_3	1	ω^3	1	ω^3	1	ω^3
χ_4	1	ω^4	ω^2	1	ω^4	ω^2
χ_5	1	ω^5	ω^4	ω^3	ω^2	ω

where set $\omega = e^{\frac{2\pi i}{6}}$.

We know that, since G is abelian, all the irreducible representations are 1-dimensional. These are the following characters

$$\chi_k(m) = e^{\frac{2\pi ikm}{6}} = \pi_k(m) \tag{4.6}$$

for $k = 0, 1, 2, 3, 4, 5$. By Remark 6, we have

$$\begin{aligned} m_k(\pi_k, \mathrm{Ind}_{\Gamma}^{G}(\mathbf{1}_{\Gamma})) &= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\chi_k(\gamma) \\ &= \frac{1}{3}(\chi_k(0) + \chi_k(2) + \chi_k(4)) \\ &= \frac{1}{3}(1 + e^{\frac{2\pi ik}{3}} + e^{\frac{4\pi ik}{3}}). \end{aligned} \tag{4.7}$$

If $k \equiv 0 \mod 3$, then $m_k = 1$, otherwise $m_k = 0$. Thus,

$$\mathrm{Ind}_{\Gamma}^{G}(\mathbf{1}_{\Gamma}) = \chi_0 \oplus \chi_3. \tag{4.8}$$

For the spectral side we get

$$\begin{aligned} J(\mathbf{1}_{\Gamma}, f) &= \sum_{k=0}^{5} m(\pi_k, \mathrm{Ind}_{\Gamma}^{G}(\mathbf{1}_{\Gamma}))\mathrm{Tr}(\pi_k(f)) \\ &= \mathrm{Tr}(\pi_0(f)) + \mathrm{Tr}(\pi_3(f)) \\ &= 2f(0) + 2f(2) + 2f(4). \end{aligned} \tag{4.9}$$

Now, we calculate as $\{\Gamma\} = \{\bar{0}, \bar{2}, \bar{4}\}$, and $G_\gamma = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\} = G$ and $\Gamma_\gamma = \{\bar{0}, \bar{2}, \bar{4}\} = \Gamma$ for all $\gamma \in \{\Gamma\}$. Then, combining, the calculation of the geometric side is immediate. It is

$$I(\mathbf{1}_\Gamma, f) = 2f(0) + 2f(2) + 2f(4). \tag{4.10}$$

Note that we can also compute the multiplicities $m(\pi_k, \mathrm{Ind}_\Gamma^G(\mathbf{1}_\Gamma))$ by a direct calculation in the following manner.

We find a basis for the induced space $\mathrm{Ind}_\Gamma^G(\mathbb{C})$. The dimension of the induced space is calculated to be 2. Let $\{\varphi_1, \varphi_2\}$ be a basis of $\mathrm{Ind}_\Gamma^G(\mathbb{C})$. Set φ_1 to be the identity map. Now, let φ be an arbitrary element of the induced space. Then we can write $a\varphi_2 \circ b\varphi_1 = \varphi$ where $a, b \in \mathbb{C}$. We have $a\varphi_2(g) \circ b\varphi_1(g) = \varphi(g)$ for all $g \in G$. Put $g = \gamma + x$. Set $\rho = \mathbf{1}_\Gamma$ for the convenience. Then

$$a\varphi_2(\gamma + x) \circ b\varphi_1(\gamma + x) = \varphi(g) = \varphi(\gamma + x) = \rho(\gamma)(\varphi(x)). \tag{4.11}$$

We have

$$ab\varphi_2(2\gamma + x) = \varphi(\gamma + x) \tag{4.12}$$

so that

$$\varphi_2(2\gamma + x) = \rho(\gamma)(\varphi(x)), \tag{4.13}$$

for all $x \in G$ and $\gamma \in \Gamma$. By the direct sum decomposition of the induced representation $\mathrm{Ind}_\Gamma^G(\rho)$ of the trivial representation of Γ, the representation $\mathrm{Ind}_\Gamma^G(\rho)$ has two components, i.e. it can be written as a 2×2 matrix. Now, we have

$$[\mathrm{Ind}_\Gamma^G(\rho)(1)](\varphi_1)(x) = \varphi_1(x + 1) = x + 1 = \varphi_1(x) + 1 \tag{4.14}$$

and

$$[\mathrm{Ind}_\Gamma^G(\rho)(1)](\varphi_2)(x) = \varphi_2(x + 1) = \rho(2\gamma)(\varphi_2(x)) = \varphi_2(x) \tag{4.15}$$

so that

$$\mathrm{Ind}_\Gamma^G(\mathbf{1}_\Gamma) = \pi_1 \oplus \pi_2 = \iota_G\pi_1 \oplus \iota_G\pi_2 \tag{4.16}$$

where we get that

$$m(\pi_1, \mathrm{Ind}_{\Gamma}^{G}(\mathbf{1}_{\Gamma})) = m(\pi_2, \mathrm{Ind}_{\Gamma}^{G}(\mathbf{1}_{\Gamma})) = 1. \tag{4.17}$$

One may compare (4.4) and (4.5) with (4.3).

4.3. The Trace Formula For $G = \mathbb{Z}/6\mathbb{Z}$ With Respect to The Subgroup $\Gamma = \langle\bar{3}\rangle$ and The Trivial Representation $\mathbf{1}_{\Gamma} : \Gamma \to \mathbf{GL}(\mathbb{C})$

Let $G = \mathbb{Z}/6\mathbb{Z}$ and let $\Gamma = \langle\bar{3}\rangle \leq G$ be a subgroup of order 2. Consider the trivial representation 1_{Γ} of Γ. Firstly, we find the multiplicities in the decomposition of the induced representation $\mathrm{Ind}_{\Gamma}^{G}(\mathbf{1}_{\Gamma})$. The dimension of the induced representation is 3. We have

$$\begin{aligned} m_k(\pi_k, \mathrm{Ind}_{\Gamma}^{G}(\mathbf{1}_{\Gamma})) &= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\chi_k(\gamma) \\ &= \frac{1}{2}(\chi_k(0) + \chi_k(3)) \\ &= \frac{1}{2}(1 + e^{\frac{\pi i k}{3}}). \end{aligned} \tag{4.18}$$

For $k = 0, 2, 4$ $m_k = 1$ and otherwise $m_k = 0$. Thus,

$$\mathrm{Ind}_{\Gamma}^{G}(\mathbf{1}_{\Gamma}) = \chi_0 \oplus \chi_2 \oplus \chi_4. \tag{4.19}$$

For the spectral side we get

$$\begin{aligned} J(\mathbf{1}_{\Gamma}, f) &= \sum_{k=0}^{5} m(\pi_k, \mathrm{Ind}_{\Gamma}^{G}(\mathbf{1}_{\Gamma}))\mathrm{Tr}(\pi_k(f)) \\ &= \mathrm{Tr}(\pi_0(f)) + \mathrm{Tr}(\pi_2(f)) + \mathrm{Tr}(\pi_4(f)) \\ &= 3f(0) + 3f(3). \end{aligned} \tag{4.20}$$

Now, we calculate as $\{\Gamma\} = \{\bar{0}, \bar{3}\}$, and $G_{\gamma} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\} = G$ and $\Gamma_{\gamma} = \{\bar{0}, \bar{3}\} = \Gamma$

for all $\gamma \in \{\Gamma\}$. Thus, combining, the geometric side is easy to compute as follows

$$I(\mathbf{1}_{\Gamma}, f) = 3f(0) + 3f(3). \tag{4.21}$$

4.4. The Trace Formula For $G = \mathbb{Z}/n\mathbb{Z}$ With Respect to The Subgroup $\mathbb{Z}/m\mathbb{Z}$ and The Trivial Representation $\mathbf{1}_\Gamma : \Gamma \to \mathbf{GL}(\mathbb{C})$

Let $G = \mathbb{Z}/n\mathbb{Z}$ and $\Gamma = \mathbb{Z}/m\mathbb{Z}$ where $m \mid n$. Choose the trivial representation $\mathbf{1}_\Gamma$ of Γ and choose the test function f such that $f(g) = 1$ for all $g \in \mathbb{Z}/n\mathbb{Z}$. In the following, ζ stands for the nth root of unity, i.e. $\zeta = e^{2\pi ij/n}$.

The character table of the group $\mathbb{Z}/n\mathbb{Z}$ is given below.

Table 4.2. Character Table of $G = \mathbb{Z}/n\mathbb{Z}$.

g	0	1	2	$\cdots$	$n-1$
χ_0	1	1	1	$\cdots$	1
χ_1	1	ζ	ζ^2	$\cdots$	ζ^{n-1}
χ_2	1	ζ^2	ζ^4	$\cdots$	$\zeta^{2(n-1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_{n-1}	1	ζ^{n-1}	$\zeta^{2(n-1)}$	$\cdots$	$\zeta^{(n-1)^2}$

For the multiplicities, we have

$$\begin{aligned} m_k(\pi_k, \mathrm{Ind}_\Gamma^G(\mathbf{1}_\Gamma)) &= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\chi_k(\gamma) \\ &= \frac{1}{m}(\chi_k(0) + \cdots + \chi_k(m-1)) \\ &= \frac{1}{m}\sum_{j=0}^{m-1} e^{2\pi kj/n}. \end{aligned} \tag{4.22}$$

If $k \equiv 0 \mod n$,

$$\frac{1}{m}\sum_{j=0}^{m-1} e^{2\pi kj/n} = 1. \tag{4.23}$$

If $k \not\equiv 0 \mod n$,

$$\frac{1}{m}\sum_{j=0}^{m-1} e^{2\pi kj/n} = \frac{1}{m}\frac{1-e^{2\pi ik(m+1)/n}}{1-e^{2\pi ik/n}}. \tag{4.24}$$

We compute as follows

$$\begin{aligned} J(\mathbf{1}_\mathit{\Gamma}, f) &= \sum_{k=0}^{n-1} m(\pi_k, \mathrm{Ind}_\mathit{\Gamma}^G(\mathbf{1}_\mathit{\Gamma}))\mathrm{Tr}(\pi_k(f)) \\ &= n + \sum_{k=1}^{n-1} \frac{1-e^{2\pi ik(m+1)/n}}{1-e^{2\pi ik/n}}\mathrm{Tr}(\pi_k(f)) \\ &= n + \frac{1-e^{2\pi ik(m+1)/n}}{1-e^{2\pi ik/n}}\left(\sum_{k=1}^{n-1}\sum_{j=0}^{n-1}\pi_k(j)\right) \\ &= n+0 \\ &= n \\ &\equiv 0 \mod n. \end{aligned} \tag{4.25}$$

We compute as

$$\{\mathit{\Gamma}\} = \mathbb{Z}/m\mathbb{Z}, \tag{4.26}$$

$$G_\gamma = G = \mathbb{Z}/n\mathbb{Z} \tag{4.27}$$

and

$$\mathit{\Gamma}_\gamma = \mathit{\Gamma} = \mathbb{Z}/m\mathbb{Z} \tag{4.28}$$

for $\gamma \in \{\mathit{\Gamma}\}$. We get

$$I(\mathbf{1}_\mathit{\Gamma}, f) = n \equiv 0 \mod n. \tag{4.29}$$

5. THE TRACE FORMULA FOR GL$(2, \mathbb{F}_3)$

In this chapter, we investigate the group $\mathrm{GL}(2, \mathbb{F}_3)$. Firstly, we find the conjugacy classes of the general linear group of degree 2 over the finite field $\mathbb{F}_p$ with p elements. Then we write the character table of the group from James' and Liebeck's book [3] where also the indexing of the characters are studied from that source. In the second section, we write the spectral and the geometric side of the trace formula for $\mathrm{GL}(2, \mathbb{F}_3)$. For computing the conjugacy classes of the group $\mathrm{GL}(2, \mathbb{F}_p)$ we refer to the source [2].

5.1. The Character Table Of GL$(2, \mathbb{F}_p)$

In order to express the spectral and the geometric sides of the trace formula for $G = \mathrm{GL}(2, \mathbb{F}_3)$ we firstly write the character table of the group $\mathrm{GL}(2, \mathbb{F}_p)$. For that purpose, the conjugacy classes of $\mathrm{GL}(2, \mathbb{F}_p)$ are to be found.

Let $g \in G$. We examine the eigenvalues of g. The element g has two eigenvalues. There are two cases: the eigenvalues of g is contained in $\mathbb{F}_p$ or not.

Let us look at the first case. Suppose both eigenvalues are equal to the same element $\alpha \in \mathbb{F}_p$. Then we have

$$c_1(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \qquad \text{or} \qquad c_2(\alpha) = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}. \tag{5.1}$$

If the eigenvalues are distinct, say α, β; then

$$c_3(\alpha, \beta) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \tag{5.2}$$

There are $p + 1$ matrices of the form $c_1(\alpha)$, $p - 1$ matrices of the form $c_2(\alpha)$ and $\frac{1}{2}(p-1)(p-2)$ of the form $c_3(\alpha, \beta)$.

In the latter case, where the eigenvalues of g do not belong to $\mathbb{F}_p$, the eigenvalues are contained in the unique quadratic extension of $\mathbb{F}_p$, say L. Denoting by $p(X)$ the characteristic polynomial of g, we know that $p(X)$ is irreducible over $\mathbb{F}_p$ and its roots, say $\alpha, \overline{\alpha}$, are conjugate over $\mathbb{F}_p$. Noting $\mathrm{Tr}(\alpha) = \alpha + \overline{\alpha}$ and $\mathrm{N}(\alpha) = \alpha\overline{\alpha}$, we have $p(X) = X^2 - \mathrm{Tr}(\alpha)\mathbb{F}_p + \mathrm{N}(\alpha)$.

Let $v \in \mathbb{F}_p$ where $v \neq 0$. Then $\{v, gv\}$ form a basis for $\mathbb{F}_p^2$ over $\mathbb{F}_p$, because there would be an element $\lambda \in \mathbb{F}_p$ otherwise such that $gv = \lambda v$. By the Cayley-Hamilton Theorem, we get

$$p(g) = 0 \implies c_4(\alpha) = \begin{pmatrix} 0 & -\mathrm{N}(\alpha) \\ 1 & \mathrm{Tr}(\alpha) \end{pmatrix}. \tag{5.3}$$

There are $p^2 - p$ elements in $L - \mathbb{F}_p$, therefore there are $\frac{1}{2}(p^2 - p)$ matrices of the form $c_4(\alpha)$.

The character table of $\mathrm{GL}(2, \mathbb{F}_p)$ is from [3], as follows

Table 5.1. Character Table of $G = \mathrm{GL}(2, \mathbb{F}_p)$.

g	$c_1(\alpha)$	$c_2(\alpha)$	$c_3(\alpha)$	$c_4(r)$
λ_i	$\overline{\alpha}^{2i}$	$\overline{\alpha}^{2i}$	$(\overline{\alpha\beta})^i$	$\overline{r}^{i(1+p)}$
ψ_i	$p\overline{\alpha}^{2i}$	0	$(\overline{\alpha\beta})^i$	$-r^{-i(1+p)}$
$\psi_{i,j}$	$(p+1)\overline{\alpha}^{i+j}$	$\overline{\alpha}^{i+j}$	$\overline{\alpha}^i\overline{\beta}^j + \overline{\alpha}^j\overline{\beta}^i$	0
χ_i	$(p-1)\overline{\alpha}^i$	$-\overline{\alpha}^i$	0	$-(\overline{r}^i + \overline{r}^{ip})$

where $\mathrm{Tr}(\alpha) = r + r^p$ and $\mathrm{N}(\alpha) = r^{1+p}$. The matrix $c_4(\alpha)$ has eigenvalues r and r^p.

For λ_i, we have $0 \leq i \leq p-2$. Therefore, there are $p-1$ characters λ_i, each of degree 1. For ψ_i, we have $0 \leq i \leq p-2$, there are $p-1$ characters ψ_i, each of degree

p. For $\psi_{i,j}$, we have $0 \leq i < j \leq p-2$. Therefore there are $\frac{(p-1)(p-2)}{2}$ characters $\psi_{i,j}$, each of degree $p+1$. Lastly, for χ_i, we consider the numbers j with $0 \leq j_k \leq p^2-1$ and $(p+1) \nmid j_k$. Precisely, one of the numbers j_1 and j_2 with $j_1 \equiv j_2 p \mod (p^2-1)$ is chosen to belong to the indexing set for the characters χ_i. Therefore, there are $\frac{p^2-p}{2}$ characters χ_i, each of degree $p-1$. Note also that there are p^2-1 conjugacy classes of $\mathrm{GL}(2, \mathbb{F}_p)$.

5.2. The Trace Formula For $G = \mathbf{GL}(2, \mathbb{F}_3)$ With Respect to The Borel Subgroup B and The Trivial Representation $\mathbf{1}_B : B \to \mathbf{GL}(\mathbb{C})$

Let B be the subgroup of $G = \mathrm{GL}(2, \mathbb{F}_p)$ consisting of all upper triangular matrices $[a_{ij}]$ with $a_{11}, a_{22} \in \mathbb{F}_p^\times$, $a_{12} \in \mathbb{F}_p$ and $a_{21} = 0$; setting

$$B = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \Big| a_{11}, a_{22} \in \mathbb{F}_p^\times;\ a_{12} \in \mathbb{F}_p \right\}, \tag{5.4}$$

the group B is called the Borel subgroup of G. The order of the Borel subgroup is $(p-1)^2 p$.

Consider now $p = 3$. We have $\mid B \mid = 12$. We list the elements M_i of B, $0 \leq i \leq 12$, as follows

$$\begin{aligned} B = \Bigg\{ M_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, M_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, M_4 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \\ M_5 &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, M_6 = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, M_7 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, M_8 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \\ M_9 &= \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, M_{10} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, M_{11} = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}, M_{12} = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} \Bigg\} \end{aligned} \tag{5.5}$$

Moreover, we have the character table for $\mathrm{GL}(2, \mathbb{F}_3)$ as follows.

Table 5.2. Character Table of $G = \mathrm{GL}(2, \mathbb{F}_3)$.

$\mathbf{g}$	$c_1(\alpha)$	$c_2(\alpha)$	$c_3(\alpha)$	$c_4(r)$
χ_1	1	1	1	1
χ_2	α^2	α^2	$(\alpha\beta)^2$	$\overline{r}^4$
χ_3	3	0	1	-1
χ_4	$3\alpha^2$	0	$\alpha\beta$	$-\overline{r}^4$
χ_5	4α	α	$\alpha+\beta$	0
χ_6	2α	$-\alpha$	0	$-(\overline{r}+\overline{r}^3)$
χ_7	$2\alpha^2$	$-\alpha^2$	0	$-(\overline{r}^2+1)$
χ_8	$2\alpha^2$	$-\alpha^2$	0	$-(\overline{r}^2+1)$

We calculate the multiplicities $m_k(\pi_k, \mathrm{Ind}_B^G(\mathbf{1}_B)) = \frac{1}{|B|}\sum_{i=1}^{12} \chi_k(M_i)$ all to be 1. Next, the numbers $\mathrm{Tr}(\pi_k(f))$ are to be computed. We define the test function f by the rule $f(g) = 1$ for all $g \in G$. We get for all $0 \leq k \leq 8$

$$\mathrm{Tr}(\pi_k(f)) \equiv 0 \mod 3, \tag{5.6}$$

where the number of conjugacy classes of $\mathrm{GL}(2, \mathbb{F}_3)$ is 8 and there are 2 elements of the matrix type $c_1(\alpha)$, 16 elements of the matrix type $c_2(\alpha)$, 12 elements of the matrix type $c_3(\alpha, \beta)$ and 18 elements of the matrix type $c_4(r)$. Therefore,

$$J(\mathbf{1}_\Gamma, f) = 0 \mod 3. \tag{5.7}$$

Next, we have

$$\{B\} = \left\{ S_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, S_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, S_3 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \right.$$
$$\left. S_4 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, S_5 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, S_6 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \right\}.$$

We compute and write the order of the sets B_{S_i} and G_{S_i} for each $1 \leq i \leq 6$ as follows $|B_{S_1}| = |B_{S_2}| = 12$, $|B_{S_3}| = |B_{S_5}| = 4$, $|B_{S_4}| = |B_{S_6}| = 6$ and $|G_{S_1}| = |G_{S_2}| = 48$, $|G_{S_3}| = |G_{S_5}| = 4$, $|G_{S_4}| = |G_{S_6}| = 6$, where we compute the orders with SageMath[1] codes which can be found in the appendix.

Furthermore, the indices are $[G : G_{S_1}] = [G : G_{S_2}] = 1$, $[G : G_{S_3}] = [G : G_{S_5}] = 12$ and $[G : G_{S_4}] = [G : G_{S_6}] = 8$. Therefore, the geometric side of the trace formula for $\mathrm{GL}(2, \mathbb{F}_3)$ is calculated as

$$I(\mathbf{1}_B, f) = 48 \equiv 0 \mod 3. \tag{5.8}$$

[1]SageMath, 2025

6. CONCLUSION

In the master thesis, we study the trace formula for finite groups which in the further studies of this subject can be considered as a part of an area of both algebra and geometry. This subject may present perspectives of group-theoretic studies as we apply the trace formula for the specific group about which we want to gain information. In this manner, indeed, one can view the structure of groups spectrally and geometrically where one gets (complex) numbers from applying the trace formula for the groups related.

In this representation-theoretic study, we investigate relatively less complicated groups like $\mathbb{Z}/n\mathbb{Z}$ and $\mathrm{GL}(2, \mathbb{F}_q)$ (we had chosen $q = 3$ as a basic example). One can also work on a much more complicated group, namely the monster group [2] whose order is approximately 8×10^{53}. This would give us valuable insights about a very complicated group structure, which provides us with the equipment of a better understanding of what a group is. In particular, further studies of this direction may contribute to the knowledge about geometric presence of groups immensely where number-theoretic outputs of the study plus thereby provided intuition surely exhibit how significant the subject is.

To sum up, our study on the trace formula for finite groups is to be considered as a study of group theory where representation theory comes into play; in other words, in the future we would like to deepen the study regarding it as a group-theoretic investigation. In this respect, one may examine characteristics of a very large group, that is to say, of the monster group. Since the significance of the subject of the trace formula for finite groups is huge, considering its group-theoretic and both geometric and number-theoretic sides, the related investigations would be fruitful.

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[2] Hereby, we thank Prof. İkeda very much for the idea and discussions about the topic where upon his advice we read the article "A short introduction to Monstrous Moonshine" of Valdo Tatitscheff. In our discussions we outlined that the first step of the application of the trace formula for the monster group is to find conjugacy classes of the group.

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APPENDIX A: SAGEMATH CODES

Here are the exemplary SageMath codes [3], i.e. Code 1 and Code 2, for finding the order of the centralizer of the element $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ in $\mathrm{GL}(2, \mathbb{F}_3)$ and for finding the order of the centralizer of the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in the Borel subgroup $B \leq \mathrm{GL}(2, \mathbb{F}_3)$, respectively, where the codes also list the elements of the related centralizer and the order of the group $\mathrm{GL}(2, \mathbb{F}_3)$ and the order of the Borel subgroup B for the ease.

Code 1

```
# Define the finite field (change 3 to any prime power, e.g., 5, 7, etc.)
F = GF(3)

# Define the general linear group GL(2, F)
G = GL(2, F)

# Define the matrix g = [[2, 1], [0, 2]]
g = matrix(F, [[2, 1], [0, 2]])

# Compute the centralizer of g in G
centralizer = G.centralizer(g)

# Output
print("Matrix g:")
print(g)
print("Centralizer of g in GL(2, F):")
for g in centralizer:
    print(g)
print(f"\nNumber of elements in the centralizer: {len(centralizer)}")
print("Order of the group GL(2, F_3):", G.order())
```

[3]We used the artificial intelligence ChatGPT (OpenAI, 2023) to write the codes.

Code 2

```
# Step 1: Define the finite field F_3
F = GF(3)

# Step 2: Define GL(2, F_3)
G = GL(2, F)

# Step 3: Manually build the Borel subgroup
B = []

for a in F:
    for d in F:
        if a != 0 and d != 0:  # a, d must be units for invertibility
            for b in F:
                M = Matrix(F, [[a, b], [0, d]])
                B.append(G(M))  # Wrap as GL(2, F) element

# Step 4: Define the matrix A = [[1, 1], [0, 1]]
A = G(Matrix(F, [[1, 1], [0, 1]]))

# Step 5: Compute centralizer of A in Borel subgroup
centralizer = [g for g in B if g * A == A * g]

# Step 6: Output the results
print("Matrix A:")
print(A)
print("Centralizer of A in the Borel subgroup of GL(2, F_3):")
for g in centralizer:
    print(g)
print(f"\nNumber of elements in the centralizer: {len(centralizer)}")
print(f"Order of the Borel subgroup of GL(2, F_3): {len(B)}")
print("Order of the group GL(2, F_3):", G.order())
```