Feedback Control of Nonlinear Parabolic Equations by Finite Determining Parameters

by

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This is to certify that I have examined this copy of a master's thesis by

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and have found that it is complete and satisfactory in all respects,

and that any and all revisions required by the final

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ABSTRACT

In this thesis, we study on stabilization of the solutions of second order nonlinear parabolic equations by using finite parameters feedback control systems. In particular, the zero steady state solution of initial boundary value problems, for nonlinear parabolic equations (with Neumann and Dirichlet boundary conditions) is examined. Finite parameters, such as, spatial averages (finite volume elements), finitely many Fourier modes and finitely many nodal observables and controllers are used. Moreover, continuous dependence of the solutions on the initial data is shown.

ÖZETÇE

Bu tezde, sonlu sayıda parametreli geribildirim denetim sistemlerini kullanarak, ikinci dereceden doğrusal olmayan parabolik denklemlerin çözümlerinin kararlılığını çalışıyoruz. Özellikle, doğrusal olmayan parabolik denklemler için başlangıç sınır değer problemlerinin (Neumann ve Dirichlet sınır koşulları ile) sıfır-durgun-durum (zero steady state) çözümünün kararlılığı incelenmiştir. Sonlu parametreler olarak, konumsal ortalamalar (sonlu hacim elemanları), sonlu Fourier modları ve sonlu düğümsel gözlemlenebilirler (observables) ve geribildirim denetleyicileri kullanılmıştır. Buna ek olarak, çözümlerin başlangıç değerine sürekli bağımlı olduğu gösterilmiştir.

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TABLE OF CONTENTS

Chapter 1

PRELIMINARIES

1.1 Introduction

The determining parameters are crucial in the feedback control theory. C. Foias and G. Prodi (1967), in [3], present the first rigorous proof about the description of the long term behaviours of the solutions of the particular dissipative equations, by a finite number of degrees of freedom. Specifically, it is shown that the asymptotic behavior of solutions of initial boundary value problem for two dimensional Navier-Stokes equations can be determined by first N Fourier modes.

A. Azouani and E.S. Titi ([1]) studied the problem of stabilization of the problem

$$u_t(x,t) - \nu u_{xx}(x,t) - \alpha u(x,t) + u^3(x,t) = 0, \tag{1.1}$$

$$u_x(0,t) = u_x(L,t) = 0, \quad \forall t \geq 0, \tag{1.2}$$

where $\nu, \alpha > 0$, by using controllers depending on finite number of parameters. They studied the following feedback control system.

$$u_t(x,t) - \nu u_{xx}(x,t) - \alpha u(x,t) + u^3(x,t) = -\mu I_h(u), \tag{1.3}$$

$$u_x(0,t) = u_x(L,t) = 0, \qquad \forall t \geq 0, \tag{1.4}$$

$$u(x,0) = u_0, . \tag{1.5}$$

As an extended application of this study, in the article [5], E. Lunasin and E. S. Titi develop some algorithms for the finite-dimensional feedback control of infinite-dimensional dissipative evolution equations. Specifically, a feedback control system is developed for the stabilization of the solution of those equations. The computational studies are done on one-dimensional Chafee-Infante and one-dimensional Kuramoto-Sivashinsky equation:

$$u_t(t,x) + u_{xxxx}(t,x) + u_{xx}(t,x) + u(t,x)u_x(t,x) = 0 \quad \text{in} \quad (0,\infty) \times \mathbb{R}. \tag{1.6}$$

A recent study about the feedback control by using finite number of determining parameters is done by V. Kalantarov and E. S. Titi, in [4]. Different from the previous studies, the feedback control is used to stabilize the solutions of nonlinear damped wave equations. In particular, the nonlinear damped and strong damped wave equations

$$\partial_t^2 u - \Delta u + b\partial_t u - \alpha + f(u) = -\mu w, \quad x \in \Omega, \quad t > 0, \tag{1.7}$$

and

$$\partial_t^2 u - \Delta u + b\Delta\partial_t u - \lambda u + f(u) = -\mu w, \quad x \in \Omega, \quad t > 0, \tag{1.8}$$

are studied. Various feedback controllers are employed to stabilize the solutions of:

(i) equation (1.7) with Neuman boundary conditions by using finite volume elements,

(ii) equation (1.7) with Dirichlet boundary conditions by using one feedback controller based on a proper subdomain $\Omega \subset \mathbb{R}^n$,

(iii) equations (1.7) and (1.8) with Dirichlet boundary conditions by using finite number of Fourier modes,

(iv) equation (1.8) by using finite number of nodal values.

In this thesis, following [1], we propose a feedback control for nonlinear parabolic equations by using finite number of determining parameters, such as spatial averages, nodal values, Fourier modes and nodal observables. We will consider the following initial-boundary value problem for a one dimensional second order nonlinear reaction-diffusion equation of the form

$$u_t(x,t) - \nu u_{xx}(x,t) - \alpha u(x,t) + |u(x,t)|^{p-2}u(x,t) = 0, \tag{1.9}$$

$$u_x(0,t) = u_x(L,t) = 0, \qquad \forall t \geq 0, \tag{1.10}$$

$$u(x,0) = u_0, \quad u_0 \in L^2(0,L). \tag{1.11}$$

where $p \geq 2$ and ν and α are positive constants.

Let us explain the structure of the thesis briefly:

In Chapter 1, we give some preliminary information about function spaces, useful identities and inequalities, and some auxiliary lemmas and theorems.

In Chapter 2, we present three examples of the interpolant operators. These are interpolant operators based on spatial averages, nodal values and finite number of Fourier modes.

In Chapter 3, we demonstrate how an interpolant operator can be used as a feed-back controller. Specifically, in our first subsection 3.1, we use an interpolant operator based on the spatial averages to stabilize the zero solution of equation (1.9) with Neuman boundary conditions (1.10). In the second subsection 3.2, we use another interpolant operator which is based on the Fourier modes to stabilize the zero solution of equation (1.9) with Dirichlet boundary conditions

$$u(x,0) = u(x,L) = 0.$$

In Chapter 4, we study the stabilization of the zero solution of zero steady state solution of the problem (1.9)-(1.11) in L^2 and H^1 norm.

Finally, in Chapter 5, we introduce a new feedback control system

$$u_t(x,t) - \nu u_{xx}(x,t) - \alpha u(x,t) + |u(x,t)|^{p-2}u(x,t) = -\mu h \sum_{k=1}^{N} u(\overline{x}_k,t)\delta(x-x_k),$$

with the periodic boundary conditions

$$u(x,t) = u(x+L,t).$$

Remark 1.1.1. *Throughout this thesis, the parameters ν, α and μ are positive numbers and $p \geq 2$.*

1.2 Function Spaces

Lebesgue Spaces

For $1 \leq p \leq \infty$, $L^p(\Omega)$ is the space of all Lebesque measurable functions defined on Ω with the following norm:

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$
$$\|f\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_{x\in\Omega}|f(x)|, \quad p = \infty.$$

Note that the space $L^p(\Omega)$ is a Banach space. Moreover, $L^2(\Omega)$ is a Hilbert space. The inner product in $L^2(\Omega)$ is defined as follows:

$$(f,g)_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx.$$

In the following chapters, the norm and the inner product in $L^2(0,L)$ will be denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively.

The Space of Test Functions

The space of test functions, $C_c^\infty(\Omega)$, is defined as follows:

$$C_c^\infty(\Omega) = \{\eta \in C^\infty : \ \mathrm{supp}(\eta) \text{ is compact in } \Omega\},$$

where

$$\mathrm{supp}(\eta) = \overline{\{\mathrm{x} \in \Omega : \ \eta(\mathrm{x}) \neq 0\}}.$$

Note that the space of test functions $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for $1 \leq p < \infty$.

The Space of Locally Integrable Functions

The space of the locally integrable functions is denoted by $L_{1,\mathrm{loc}}(\Omega)$ and defined as follows:

$$L_{1,\mathrm{loc}}(\Omega) = \{f : \Omega \to \mathbb{R} \text{ measurable} : f|_K \in L^1(\Omega), \ \forall K \subset \Omega, K \text{ compact}\}.$$

Sobolev Space $H^1(\Omega)$

Definition 1.2.1 (Weak Derivative)**.** *Assume that $\alpha = (\alpha_1, .., \alpha_n)$ is a given multi-index, u and v are locally integrable functions over Ω. Assume that $\forall \eta \in C_c^\infty(\Omega)$ the following equation is satisfied:*

$$\int_\Omega u(x) D^\alpha \eta(x) dx = (-1)^{|\alpha|} \int_\Omega v(x)\eta(x)dx,$$

where

$$D^\alpha \eta(x) = \frac{\partial^{|\alpha|}\eta(x)}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}.$$

Then the function v is called α-th weak derivative of u on the domain Ω.

The Sobolev space $H^1(\Omega)$ is an inner product space of all functions $u \in L^2(\Omega)$ that have all first order weak derivatives u_{x_i}'s are from $L^2(\Omega)$ for $i = 1, 2, ..., n$. The inner product in $H^1(\Omega)$ is defined as folllows:

$$(u, v)_{H^1(\Omega)} = \int_\Omega [u(x)v(x) + \sum_{j=1}^n u_{x_j}(x) v_{x_j}(x)] dx.$$

The $H^1(\Omega)$ norm is defined as follows:

$$\|u\|_{H^1(\Omega)} = \left(\int_\Omega (u^2(x) + |\nabla u(x)|^2) dx \right)^{\frac{1}{2}}. \tag{*}$$

In the following chapters, the H^1-norm on the interval $[0, L]$ will be defined as follows:

$$\|u\|^2_{H^1(0,L)} := \frac{1}{L^2} \int_0^L u^2(x)dx + \int_0^L u_x^2(x)dx.$$

This norm clearly is equivalent to the norm $(*)$. The closure of the space $C_c^\infty(\Omega)$ in the space $H^1(\Omega)$ is defined as the Hilbert space $H_0^1(\Omega)$. The inner product in $H_0^1(\Omega)$ is defined as follows:

$$(u, v)_{H_0^1(\Omega)} = \int_\Omega \nabla u(x) \cdot \nabla v(x) dx.$$

The Dual Space of $H_0^1(\Omega)$:$H^{-1}(\Omega)$

(See [2]) The dual space of $H_0^1(\Omega)$, $(H_0^1(\Omega))^*$, is denoted by $H^{-1}(\Omega)$ and defined as follows:

$$H^{-1}(\Omega) = \{u : H_0^1(\Omega) \to \mathbb{R} |\text{ u is bounded and linear}\}.$$

The pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ is denoted by $\langle\cdot,\cdot\rangle$.

Definition 1.2.2. *The H^{-1} norm is defined as follows:*

$$\|u\|_{H^{-1}(\Omega)} = \sup\{\langle u, w\rangle| \quad w \in H_0^1(\Omega), \|w\|_{H_0^1} \leq 1\}.$$

1.3 Useful Identities and Inequalities

Lemma 1.3.1 (Parseval Identity). *Suppose that $f \in L^2(0, L)$, with period L. Let a_k, b_k be real Fourier coefficients of f. Then we have the following equality:*

$$\frac{2}{L}\int_0^L |f(x)|^2 dx = \frac{|a_0|^2}{2} + \sum_{k=1}^{\infty}(|a_k|^2 + |b_k|^2),$$

where

$$a_0 = \frac{2}{L}\int_0^L f(x)dx,$$

$$a_k = \frac{2}{L}\int_0^L f(x)\cos(\frac{2k\pi x}{L})dx, \qquad k \in \mathbb{Z}^+,$$

$$b_k = \frac{2}{L}\int_0^L f(x)\sin(\frac{2k\pi x}{L})dx, \qquad k \in \mathbb{Z}^+.$$

Lemma 1.3.2 (Gronwall's Inequality). *Let $u(t)$ be a real-valued function which satisfies the following differential inequality:*

$$\frac{d}{dt_+}u(t) \leq v(t)u(t) + h(t), \qquad \forall t \in [0, T].$$

Then

$$u(t) \leq u(0)exp\left(\int_0^t v(r)dr\right) + \int_0^t exp\left(\int_s^t v(\tau)d\tau\right) h(s)ds, \ \forall t \in [0, T].$$

Gronwall's inequality for integral:

Let $u(t)$ and $v(t)$ be nonnegative continuous functions for $t \geq 0$, and satisfy the following inequality:

$$u(t) \leq C + \int_0^t u(r)v(r)dr, \quad \forall t \geq 0,$$

where $C \geq 0$. Then

$$u(t) \leq C \exp\left(\int_0^t v(r)dr\right), \qquad \forall t \geq 0.$$

Lemma 1.3.3 (**Cauchy-Schwarz Inequality**)**.** *Let H be an inner product space with . Then for any $u, v \in H$, the following inequality holds:*

$$|(u,v)_H| \leq \|u\|_H \|v\|_H.$$

Lemma 1.3.4 (**Hölder Inequality**)**.** *Assume that $p \in [1, \infty]$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^1(\Omega)$ and*

$$\|uv\|_{L^1(\Omega)} \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.$$

Lemma 1.3.5 (**Young's Inequality**)**.** *Let p and q be positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for any nonnegative real numbers a and b the following inequality holds:*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{1.12}$$

Young's Inequality with ϵ:

For some $\epsilon > 0$, the following inequality holds:

$$ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}. \tag{1.13}$$

Lemma 1.3.6 (**Poincaré Inequality**)**.** *For all $\varphi \in H_0^1(0, L)$, the following inequality holds true:*

$$\|\varphi\|^2 \leq (\lambda_1)^{-1} \|\varphi'\|^2, \tag{1.14}$$

where λ_1 is the smallest eigenvalue of the operator $-\frac{d^2}{dx^2}$ with Dirichlet boundary conditions.

Proof. Let us write Fourier expansion of φ.

$$\begin{aligned} \varphi(x) &= \sum_{k=1}^{\infty} \varpi_k w_k(x) \\ \varphi''(x) &= \sum_{k=1}^{\infty} \varphi_k w_k''(x) \\ &= -\sum_{k=1}^{\infty} \varphi_k \lambda_k w_k(x). \end{aligned}$$

Consider the following

$$
\begin{aligned}
\int_0^L (\varphi'(x))^2 dx &= -\int_0^L \varphi''(x)\varphi(x)dx \\
&= \int_0^L \left(\sum_{k=1}^{\infty} \varphi_k \lambda_k w_k(x)\right)\left(\sum_{m=1}^{\infty} \varphi_m w_m(x)\right) dx \\
&= \int_0^L \sum_{k=1}^{\infty} \lambda_k \varphi_k^2 w_k^2(x) dx \\
&= \sum_{k=1}^{\infty} \lambda_k \varphi_k^2 \\
&\geq \lambda_1 \sum_{k=1}^{\infty} \varphi_k^2 \\
&= \lambda_1 \|\varphi\|^2
\end{aligned}
$$

□

Lemma 1.3.7 (**Poincaré-type Inequality,**(e.g. [4])). *For all $u \in H_0^1(0,L)$, the following inequality holds true:*

$$
\|u(t) - \sum_{k=1}^{N} (u(x,t), w_k) w_k\|^2 \leq (\lambda_{N+1})^{-1} \|u_x(t)\|^2, \tag{1.15}
$$

where w_k's are orthonormal basis for $L^2(0,L)$ and λ_1 is the smallest eigenvalue of the operator $-\frac{d^2}{dx^2}$ with Dirichlet boundary conditions.

Lemma 1.3.8 (**Jensen Inequality**). *Assume that f is a convex function in $\mathbb{R}$. Let $\psi \in L^1(\Omega)$ is a positive function and $u \in C(\overline{\Omega})$. Then the following inequality holds true:*

$$
\frac{\int_\Omega f(u(x))\psi(x)dx}{\int_\Omega \psi(x)dx} \geq f\left(\frac{\int_\Omega u(x)\psi(x)dx}{\int_\Omega \psi(x)dx}\right).
$$

Proof. Let us consider the following:

$$
k_0 := \left(\frac{\int_\Omega u(x)\psi(x)dx}{\int_\Omega \psi(x)dx}\right).
$$

Since f is a convex function, there exists a number m such that

$$
f(u) \geq f(k_0) + m(u - k_0). \tag{1.16}
$$

Let us multiply both sides of inequality (1.16) by ψ and integrate over the region Ω. Thus, we get

$$\int_\Omega f(u(x))\psi(x)dx \geq f(k_0)\int_\Omega \psi(x)dx + m\int_\Omega u(x)\psi(x)dx - mk_0\int_\Omega \psi(x)dx. \quad (1.17)$$

Let us write the value of k_0 in the last inequality (1.17). Hence, we obtain

$$\int_\Omega f(u(x))\psi(x)dx \geq f\left(\frac{\int_\Omega u(x)\psi(x)dx}{\int_\Omega \psi(x)dx}\right)\int_\Omega \psi(x)dx.$$

$\square$

Lemma 1.3.9 (**Monotonicity Inequality**). *If $p \geq 2$, then there exists a number $d_0(p,n)$ such that for each $a, b \in \mathbb{R}^n$ the following inequality holds true*

$$(|a|^{p-2}a - |b|^{p-2}b, a-b) \geq d_0|a-b|^p.$$

Proof. Let $J : \mathbb{R} \to \mathbb{R}$ be a function defined as

$$J(p) := (|a|^{p-2}a - |b|^{p-2}b, a-b), \quad \forall p \geq 2.$$

$$\begin{aligned} J(p) &= (|a|^{p-2}a - |b|^{p-2}b, a-b) \\ &= (\int_0^1 \frac{d}{ds}[|sa+(1-s)b|^{p-2}(sa+(1-s)b)]ds, a-b) \\ &= \int_0^1 |sa+(1-s)b|^{p-2}|a-b|^2 ds \\ &+ (p-2)\int_0^1 |sa+(1-s)b|^{p-4}(sa+(1-s)b, a-b)^2 ds. \end{aligned}$$

Since $p \geq 2$ we get

$$J(p) \geq \int_0^1 |sa+(1-s)b|^{p-2}|a-b|^2 ds. \quad (1.18)$$

If $|a| \geq |b-a|$ then we have

$$\begin{aligned} |sa+(1-s)b| &= |a+(1-s)(b-a)| \\ &\geq ||a|-(1-s)|a-b|| \geq s|a-b|. \end{aligned}$$

Thus,

$$\begin{aligned} J(p) &\geq |a-b|^2 \int_0^1 s^{p-2}|a-b|^{p-2} ds \\ &= \frac{1}{p-1}|a-b|^p. \end{aligned}$$

If $|a| < |b-a|$ then we have

$$\begin{aligned} |sa+(1-s)b| &= |a+(1-s)(b-a)| \\ &\leq |a| + (1-s)|a-b| \leq (2-s)|a-b|. \end{aligned}$$

Therefore,

$$|sa+(1-s)b|^{p-2} \geq \frac{[|sa+(1-s)b|^2]^{\frac{p}{2}}}{(2-s)^2|a-b|^2}.$$

By using Jensen's Inequality (1.3.8), we obtain from (1.18) the following inequality

$$\begin{aligned} J(p) &\geq |a-b|^2 \int_0^1 \frac{[|sa+(1-s)b|^2]^{\frac{p}{2}}}{(2-s)^2|a-b|^2} ds \\ &\geq \frac{1}{4}\left(\int_0^1 |sa+(1-s)b|^2 ds\right)^{\frac{p}{2}} \\ &= \frac{1}{4}\frac{1}{3^{\frac{p}{2}}}(|a|^2+(a,b)+|b|^2)^{\frac{p}{2}} \\ &= \frac{1}{4}\frac{1}{3^{\frac{p}{2}}}\left(\frac{3|a+b|^2}{4}+\frac{|a-b|^2}{4}\right)^{\frac{p}{2}} \\ &\geq \frac{1}{4}\frac{1}{12^{\frac{p}{2}}}|a-b|^p. \end{aligned}$$

This completes the proof. □

1.4 Useful Lemmas and Theorems

Lemma 1.4.1 (See [6]). *If $f \in \dot{H}^1_p(0,L)$, where $\dot{H}^1_p(0,L)$ is the completion of $C^\infty_p(0,L) = \{f|_{[0,L]} : f \in C^\infty$ and f is periodic with period $L\}$ with $\int_0^L f(x)dx = 0$, then*

$$\int_0^L |f(x)|^2 dx \leq \left(\frac{L^2}{4\pi^2}\right) \int_0^L |f'(x)|^2 dx.$$

Proof. Let us consider the Fourier series expansion of f.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2\pi i n x}{L}\right).$$

Since

$$\int_0^L f(x)dx = 0,$$

we have

$$c_0 = \frac{1}{L}\int_0^L f(x)dx = 0.$$

Thus we can write the Fourier series expansion of f as follows:

$$f(x) = \sum_{n=-\infty, n\neq 0}^{\infty} c_n \exp\left(\frac{2\pi i n x}{L}\right).$$

Let us take the derivative of $f(x)$ with respect to x.

$$f'(x) = \sum_{n=-\infty, n\neq 0}^{\infty} \left(\frac{2\pi i n}{L}\right) c_n \exp\left(\frac{2\pi i n x}{L}\right).$$

By using Parseval Identity 1.3.1 we have the following equality:

$$\int_0^L |f'(x)|^2 dx = L \sum_{n=-\infty, n\neq 0}^{\infty} \left| \left(\frac{2\pi n}{L}\right) c_n \right|^2.$$

Thus, we get

$$\begin{aligned} \int_0^L |f'(x)|^2 dx &= \sum_{n=-\infty, n\neq 0}^{\infty} L \left(\frac{4\pi^2 n^2}{L^2}\right) |c_n|^2 \\ &\geq \left(\frac{4\pi^2}{L^2}\right) \sum_{n=-\infty, n\neq 0}^{\infty} L|c_n|^2. \end{aligned}$$

Thanks to Parseval Identity 1.3.1 in the right-hand side of the last inequality, we obtain

$$\int_0^L |f'(x)|^2 dx \geq \left(\frac{4\pi^2}{L^2}\right) \int_0^L |f(x)|^2 dx. \tag{1.19}$$

Hence, this completes the proof. $\square$

Lemma 1.4.2 (See [7]). *Let V, H, V^* be three Hilbert spaces such that $V \subset H \equiv H^* \subset V^*$, where V^* and H^* are dual spaces of V and H, respectively. If $u \in L^2(0,T;V)$ and $u_t \in L^2(0,T;V^*)$, then the following equality holds true:*

$$\frac{d}{dt}\|u(t)\|^2 = 2\langle u_t, u\rangle_H.$$

Theorem 1.4.3. *Assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz function, i.e. f satisfies the following inequality*

$$\|f(x) - f(y)\|_{\mathbb{R}^n} \leq L(B)\|x - y\|_{\mathbb{R}^n}$$

for every $x, y \in B$, where $B \subset \mathbb{R}^n$ is any bounded set.
Then there exists a $T = T(x_0)$ such that the initial value problem

$$\frac{dx}{dt} = f(x), \qquad x(0) = x_0 \tag{1.20}$$

has a unique solution on $[0, T]$.

Theorem 1.4.4. *Let $x(t)$ be a solution of the initial value problem (1.20). Then $x(t)$ has a maximal interval of existence $[0, T^*)$ if and only if $\|x(t)\|_{\mathbb{R}^n} \to \infty$ as $t \to T^*$.*

Definition 1.4.5 (Weak* Convergence). *Let X be a Banach space. A sequence $\{f_n\} \in X^*$ is said to be weak* (weak star) convergent to f if $f_n(x) \to f(x)$ for all $x \in X$.*

Theorem 1.4.6 (Alaoglu weak* compactness, see [6]). *Let H be a Banach space and $\{u_n\}$ be a bounded sequence in the dual space H^*. Then $\{u_n\}$ has a weak* convergent subsequence.*

Theorem 1.4.7 (See [6]). *Assume that $\{u_n\}$ is a bounded sequence in a reflexive Banach space, U. Then there exists a subsequence of $\{u_n\}$ which is weakly convergent in U.*

Theorem 1.4.8 (See [6]). *Assume that $u \in L^2(0,T;H^1(0,L))$ and $u_t \in L^2(0,T;H^{-1})$. Then (i) $u : [0,T] \to L^2(0,L)$ is continuous*

$$\sup_{t\in[0,T]} |u(t)| \leq C(\|u(t)\|_{L^2(0,T;H^1(0,L))} + \|u_t(t)\|_{L^2(0,T;H^{-1}(0,L))}),$$

and (ii)

$$\frac{d}{dt}\|u(t)\|^2 = 2\langle u_t, u\rangle,$$

for almost every $t \in [0, T]$.

Theorem 1.4.9. *Let* $X \subset\subset H \subset Y$ *be Banach spaces and* X *be reflexive. Assume that* u_n *is a uniformly bounded sequence in* $L^2(0, T; X)$, *and* $\frac{du_n}{dt}$ *is uniformly bounded in* $L^p(0, T; Y)$, *for some* $p > 1$. *Then there exists a subsequence that converges strongly in* $L^2(0, T; H)$.

Lemma 1.4.10 (See [8])**.** *Let* u_n *be a bounded sequence of functions in* $L^p(0, L)$ $(1 \le p \le \infty)$. *If* $u \in L^p(0, L)$ *and* $u_n \to u$ *almost everywhere then* $u_n \to u$ *weakly in* $L^p(0, L)$.

Lemma 1.4.11 (See [8])**.** *Let* X *be a Banach space and* X^* *be the dual space of* X. *If for* $1 < p \le \infty$,

$$\begin{cases} u_n \to u \quad \textit{weak* in} \quad L^p(0, T; X), \\ u_n' \to u' \quad \textit{weak* in} \quad L^p(0, T; X), \end{cases} \tag{1.21}$$

then

$$u_n(0) \to u(0) \quad \textit{weak* in} \quad X. \tag{1.22}$$

Remark 1.4.12. *In Lemma* 1.4.11, *for a reflexive Banach space* X, *weak* convergence is the same as the weak convergence.*

Chapter 2

EXAMPLES OF INTERPOLANT OPERATORS

In this section, we define a linear map $I_h : H^1(0,L) \to L^2(0,L)$ which satisfies for every $\varphi \in H^1(0,L)$, the following inequality

$$\|\varphi - I_h(\varphi)\| \le ch\|\varphi\|_{H^1(0,L)}, \tag{2.1}$$

where $c > 0$ is a constant. We use this map, I_h, as an interpolant operator. We introduce three types of interpolant operators and show that they satisfy inequality (2.1), for every $\varphi \in H^1(0,L)$.

2.1 Interpolant operator based on spatial averages (finite volume elements).

The interpolant operator based on spacial averages is defined as follows:

$$I_h(\varphi) = \sum_{k=1}^{N} \overline{\varphi}_k \chi_{J_k}(x),$$

where $J_k = \left[(k-1)\frac{L}{N}, k\frac{L}{N}\right)$, for $k = 1, ..., N-1$, and $J_N = \left[(N-1)\frac{L}{N}, L\right]$. $\chi_{J_k}(x)$ is the characteristic function on the interval J_k and

$$\overline{\varphi}_k = \frac{1}{|J_k|}\int_{J_k} \varphi(x)dx = \frac{N}{L}\int_{J_k} \varphi(x)dx$$

are the local averages of the solution for $k = 1, ..., N$. In the following proposition, we will show that this interpolant operator satisfy inequality (2.1).

Proposition 2.1.1. *Let $\varphi \in H^1(0,L)$ and $h = \frac{L}{N}$. Then the following inequalities hold true:*

$$\|\varphi - \sum_{k=1}^{N} \overline{\varphi}_k \chi_{J_k}\| \le h\|\varphi'\| \le h\|\varphi\|_{H^1(0,L)}, \tag{2.2}$$

$$\|\varphi\|^2 \leq h \sum_{k=1}^{N} \overline{\varphi}_k^2 + \left(\frac{h}{2\pi}\right)^2 \|\varphi'\|^2. \tag{2.3}$$

Proof.

$$\|\varphi - \sum_{k=1}^{N} \overline{\varphi}_k \chi_{J_k}\|^2 = \int_0^L \left(\varphi(x) - \sum_{k=1}^{N} \overline{\varphi}_k \chi_{J_k}(x)\right)^2 dx.$$

Since

$$\sum_{k=1}^{N} \chi_{J_k}(x) = 1,$$

we can write

$$\varphi(x) = \varphi(x) \sum_{k=1}^{N} \chi_{J_k}(x).$$

$$\begin{aligned}
\|\varphi - \sum_{k=1}^{N} \overline{\varphi}_k \chi_{J_k}\|^2 &= \int_0^L \left(\varphi(x) \sum_{k=1}^{N} \chi_{J_k}(x) - \sum_{k=1}^{N} \overline{\varphi}_k \chi_{J_k}(x)\right)^2 dx \\
&= \int_0^L \left(\sum_{k=1}^{N} (\varphi(x) - \overline{\varphi}_k) \chi_{J_k}(x)\right)^2 dx \\
&= \int_0^L \sum_{k,l=1}^{N} (\varphi(x) - \overline{\varphi}_k)(\varphi(x) - \overline{\varphi}_l) \chi_{J_k}(x) \chi_{J_l}(x) dx.
\end{aligned}$$

We know that $\chi_{J_k}(x)\chi_{J_l}(x) = \chi_{J_k}(x)\delta_{kl}$. Thus, we have:

$$\begin{aligned}
\|\varphi - \sum_{k=1}^{N} \overline{\varphi}_k \chi_{J_k}\|^2 &= \int_0^L \sum_{k=1}^{N} (\varphi(x) - \overline{\varphi}_k)^2 \chi_{J_k}(x) dx \\
&= \sum_{k=1}^{N} \int_{J_k} (\varphi(x) - \overline{\varphi}_k)^2 dx
\end{aligned} \tag{2.4}$$

Since $\int_{J_k} (\varphi(x) - \overline{\varphi}_k) dx = 0$ and $\varphi(x) \in H^1(0, L)$, we can apply Lemma 1.4.1 in equality (2.4). We obtain:

$$\int_{J_k} (\varphi(x) - \overline{\varphi}_k)^2 dx \leq \left(\frac{h}{2\pi}\right)^2 \int_{J_k} (\varphi'(x))^2 dx. \tag{2.5}$$

By summing inequality (2.5) from $k = 1$ to $k = N$, we get:

$$\begin{aligned}
\|\varphi - \sum_{k=1}^{N} \overline{\varphi}_k \chi_{J_k}\|^2 &\leq \left(\frac{h}{2\pi}\right)^2 \sum_{k=1}^{N} \int_{J_k} (\varphi'(x))^2 dx \\
&= \left(\frac{h}{2\pi}\right)^2 \int_0^L (\varphi'(x))^2 dx \\
&= \left(\frac{h}{2\pi}\right)^2 \|\varphi'\|^2 \\
&= h^2 \|\varphi\|_{H^1}^2.
\end{aligned}$$

Hence,

$$\|\varphi - \sum_{k=1}^{N} \overline{\varphi}_k \chi_{J_k}\| \leq h\|\varphi'\| \leq h\|\varphi\|_{H^1}.$$

Now, we will show that

$$\|\varphi\|^2 \leq h \sum_{k=1}^{N} \overline{\varphi}_k^2 + \left(\frac{h}{2\pi}\right)^2 \|\varphi'\|^2.$$

Let us consider the inequality

$$\int_{J_k} (\varphi(x) - \overline{\varphi}_k)^2 dx = \int_{J_k} \varphi^2(x) dx - 2\overline{\varphi}_k \int_{J_k} \varphi(x) dx + \int_{J_k} \overline{\varphi}_k{}^2 dx.$$

By definition of $\overline{\varphi}_k$,

$$\int_{J_k} (\varphi(x) - \overline{\varphi}_k)^2 dx = \int_{J_k} \varphi^2(x) dx - \overline{\varphi}_k{}^2 h.$$

By using Lemma 1.4.1, we get:

$$\int_{J_k} \varphi^2(x) dx - \overline{\varphi}_k{}^2 h \leq \left(\frac{h}{2\pi}\right)^2 \int_{J_k} (\varphi'(x))^2 dx. \tag{2.6}$$

By summing inequality (2.6) with respect to k from $k = 1$ to $k = N$, we obtain

$$\sum_{k=1}^{N} \int_{J_k} \varphi^2(x) dx - h \sum_{k=1}^{N} \overline{\varphi}_k{}^2 \leq \left(\frac{h}{2\pi}\right)^2 \sum_{k=1}^{N} \int_{J_k} (\varphi'(x))^2 dx.$$

$$\int_0^L \varphi^2(x) dx - h \sum_{k=1}^{N} \overline{\varphi}_k{}^2 \leq \left(\frac{h}{2\pi}\right)^2 \int_0^L (\varphi'(x))^2 dx$$

$$\|\varphi\|^2 - h \sum_{k=1}^{N} \overline{\varphi}_k{}^2 \leq \left(\frac{h}{2\pi}\right)^2 \|\varphi'\|^2.$$

Therefore,

$$\|\varphi\|^2 \leq h \sum_{k=1}^{N} \overline{\varphi}_k{}^2 + \left(\frac{h}{2\pi}\right)^2 \|\varphi'\|^2.$$

This completes the proof of the proposition. $\square$

2.2 Interpolant operator based on nodal values.

We define this type of interpolant operator as follows

$$I_h(\varphi) = \sum_{k=1}^{N} \varphi(x_k)\chi_{J_k}(x),$$

where x_k are the points in J_k for $k = 1, ..., N$ and χ_{J_k} are the same as in the previous example. In the next proposition, it will be shown that this operator satisfy inequality (2.1).

Proposition 2.2.1. *For every $\varphi \in H^1(0, L)$, the following inequality holds*

$$\|\varphi - \sum_{k=1}^{N} \varphi(x_k)\chi_{J_k}\| \leq h\|\varphi'\|.$$

Proof.

$$\begin{aligned}
\|\varphi - \sum_{k=1}^{N} \varphi(x_k)\chi_{J_k}\|^2 &= \int_0^L \left(\varphi(x) - \sum_{k=1}^{N} \varphi(x_k)\chi_{J_k}\right)^2 dx \\
&= \int_0^L \left(\sum_{k=1}^{N} \varphi(x)\chi_{J_k}(x) - \sum_{k=1}^{N} \varphi(x_k)\chi_{J_k}(x)\right)^2 dx \\
&= \int_0^L \sum_{k=1}^{N} (\varphi(x) - \varphi(x_k))^2 \chi_{J_k}(x) dx \\
&= \sum_{k=1}^{N} \int_{J_k} (\varphi(x) - \varphi(x_k))^2 dx \\
&= \sum_{k=1}^{N} \int_{J_k} \left(\int_{x_k}^{x} \varphi'(y) dy\right)^2 dx \\
&\leq \sum_{k=1}^{N} \int_{J_k} \left(\int_{J_k} |\varphi'(y)| dy\right)^2 dx \\
&\leq h \sum_{k=1}^{N} \left(\int_{J_k} |\varphi'(y)| dy\right)^2. \qquad (2.7)
\end{aligned}$$

Let us apply Cauchy-Schwarz Inequality (1.3.3) in the right-hand side of inequality (2.7), we obtain:

$$\begin{aligned}
\|\varphi - \sum_{k=1}^{N} \varphi(x_k)\chi_{J_k}\|^2 &\leq h \sum_{k=1}^{N} \int_{J_k} 1dy \int_{J_k} |\varphi'(y)|^2 dy \\
&= h^2 \sum_{k=1}^{N} \int_{J_k} |\varphi'(y)|^2 dy \\
&\leq h^2 \|\varphi'\|^2.
\end{aligned}$$

Hence, we conclude:

$$\|\varphi - \sum_{k=1}^{N} \varphi(x_k)\chi_{J_k}\| \leq h\|\varphi'\|.$$

□

2.3 Projection onto Fourier modes as an interpolant operator.

In this example, the interpolant operator will be constructed by using the projection onto Fourier modes. We will present two examples of interpolant operator by using Fourier modes.

Example 1.

In this example, we will define I_h as follows:

$$I_h(\varphi) = \sum_{k=1}^{N} \hat{\varphi}_k \cos(\frac{kx\pi}{L}),$$

where

$$\hat{\varphi}_k = \frac{2}{L} \int_0^L \varphi(x) \cos(\frac{\pi k x}{L}) dx. \tag{2.8}$$

Next proposition implies that this interpolant operator satisfies inequality (2.1).

Proposition 2.3.1. *Let $\varphi \in H^1(-L, L)$ be an even function and $\hat{\varphi}_k$ be defined as in* (2.8). *Suppose that*

$$\int_0^L \varphi(x)dx = 0.$$

Then the following inequality holds:

$$\|\varphi - \sum_{k=1}^{N} \hat{\varphi}_k \cos(\frac{kx\pi}{L})\| \leq ch\|\varphi'\|,$$

where $h = \frac{L}{N}$.

Proof. Since φ is an even function from $H^1(-L, L)$, we can write its Fourier series extension with period $2L$ as follows:

$$\varphi(x) = \frac{\hat{\varphi}_0}{2} + \sum_{k=1}^{\infty} \hat{\varphi}_k \cos(\frac{kx\pi}{L}),$$

where

$$\hat{\varphi}_k = \frac{2}{L}\int_0^L \varphi(x)\cos(\frac{kx\pi}{L})dx, \quad k = 0, 1, 2, ...$$

First, let us take the derivative of φ with respect to x.

$$\varphi'(x) = -\sum_{k=1}^{\infty} \hat{\varphi}_k(\frac{k\pi}{L})\sin(\frac{k\pi x}{L}).$$

Note that by using Parseval Identity 1.3.1 and Poincaré-type Inequality 1.3.7, we get:

$$\begin{aligned}
\int_0^L |\varphi'(x)|^2 dx &= \frac{L}{2}\sum_{k=1}^{\infty}(\frac{k\pi}{L})^2|\hat{\varphi}_k|^2 \\
&\geq \frac{L}{2}\sum_{k=N+1}^{\infty}(\frac{k\pi}{L})^2|\hat{\varphi}_k|^2 \\
&\geq \frac{(N+1)^2\pi^2}{2L}\sum_{k=N+1}^{\infty}|\hat{\varphi}_k|^2 \\
&= \frac{(N+1)^2\pi^2}{2L}\|\varphi - \sum_{k=1}^{N}\hat{\varphi}_k\cos(\frac{kx\pi}{L})\|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\varphi - \sum_{k=1}^{N} \hat{\varphi}_k \cos(\frac{kx\pi}{L})\| &\leq \frac{\sqrt{2L}}{(N+1)\pi}\|\varphi'\| \\
&\leq ch\|\varphi'\|.
\end{aligned}$$

where

$$c = \frac{\sqrt{2L}}{\pi} \quad \text{and} \quad h = \frac{1}{N}.$$

□

Example 2

Let us define the interpolant operator as

$$I_h(\varphi) = \sum_{k=1}^{N} \hat{\varphi}_k \sin(\frac{k\pi x}{L}),$$

where

$$\hat{\varphi}_k = \frac{2}{L}\int_0^L \varphi(x)\sin(\frac{k\pi x}{L})dx.$$

As in the following propositions, we will show that this interpolant operator also satisfies inequality (2.1).

Proposition 2.3.2. *Let $\varphi \in H^1(0, L)$. Then the following inequality holds:*

$$\|\varphi - \sum_{k=1}^{N} \hat{\varphi}_k \sin(\frac{k\pi x}{L})\| \leq ch\|\varphi'\|,$$

where $h = \frac{L}{N}$.

Proof. Let us write Fourier expansion of φ as follows:

$$\varphi(x) = \sum_{k=1}^{\infty} \hat{\varphi}_k \sin(\frac{k\pi x}{L}),$$

where

$$\hat{\varphi}_k = \frac{2}{L}\int_0^L \varphi(x)\sin(\frac{kx\pi}{L})dx, \quad k = 0, 1, 2, ...$$

Let us take the derive of φ with respect to x.

$$\varphi'(x) = \sum_{k=1}^{\infty} \hat{\varphi}_k \frac{k\pi}{L}\cos(\frac{k\pi x}{L}).$$

Thanks to Parseval Identity 1.3.1, we obtain:

$$\begin{aligned}
\int_0^L |\varphi'(x)|^2 dx &= \frac{L}{2}\sum_{k=1}^{\infty}(\frac{k\pi}{L})^2|\hat{\varphi}_k|^2 \\
&\geq \frac{L}{2}\sum_{k=N+1}^{\infty}(\frac{k\pi}{L})^2|\hat{\varphi}_k|^2 \\
&\geq \frac{(N+1)^2\pi^2}{2L}\sum_{k=N+1}^{\infty}|\hat{\varphi}_k|^2 \\
&= \frac{(N+1)^2\pi^2}{2L}\|\varphi - \sum_{k=1}^{N}\hat{\varphi}_k \mathrm{sin}(\frac{kx\pi}{L})\|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}\|\varphi - \sum_{k=1}^{N} \hat{\varphi}_k \sin(\frac{kx\pi}{L})\| &\leq \frac{\sqrt{2L}}{(N+1)\pi}\|\varphi'\| \\ &\leq ch\|\varphi'\|.\end{aligned}$$

where

$$c = \frac{\sqrt{2L}}{\pi} \quad \text{and} \quad h = \frac{1}{N}.$$

□

Chapter 3

INTERPOLANT OPERATOR AS FEEDBACK CONTROLLER

In this chapter, there will be two problems with different stabilizers and distinct boundary conditions. We will present the stabilization of initial boundary value problem for nonlinear reaction-diffusion equation by using controller involving interpolants based on the spatial averages and Fourier modes.

3.1 Spatial Averages as Stabilizer

In this subsection, we will consider the following feedback control system:

$$u_t(x,t) - \nu u_{xx}(x,t) - \alpha u(x,t) + |u(x,t)|^{p-2}u(x,t) = -\mu \sum_{k=1}^{N} \overline{u}_k \chi_{J_k}(x) \tag{3.1}$$

$$u_x(0,t) = u_x(L,t) = 0, \quad \forall t \geq 0, \tag{3.2}$$

$$u(x,0) = u_0. \tag{3.3}$$

The feedback controllers are the interpolant operators based on the spatial averages of which is mentioned in Chapter 2.
Next, we will show that the stabilization of the zero solution of the initial- boundary value problem (1.9)-(1.11) under the following condition

$$\mu \geq \nu(\frac{2\pi}{h})^2 > \alpha \tag{3.4}$$

where $h = \frac{L}{N}$.

Theorem 3.1.1. *Assume that* (3.4) *holds. Then for each solution* $u(t)$ *of the problem* (3.1)-(3.3), *we have* $\|u(t)\| \to 0$, *as* $t \to \infty$.

Proof. Let us take the L^2 inner product of equation (3.1) with u.

$$\int_0^L u_t(x,t)u(x,t)dx - \nu \int_0^L u_{xx}(x,t)u(x,t)dx - \alpha \int_0^L u^2(x,t)dx + \int_0^L |u(x,t)|^p dx$$
$$= -\mu \sum_{k=1}^{N} \int_{J_k} \overline{u}_k u(x,t)dx. \tag{3.5}$$

Note that

$$\int_0^L u_t(x,t)u(x,t)dx = \frac{1}{2}\frac{d}{dt}\|u(t)\|^2$$

and by using integration by parts and the boundary conditions (3.2), we have

$$\int_0^L u_{xx}(x,t)u(x,t)dx = -\int_0^L {u_x}^2(x,t)dx.$$

Thus, equality (3.5) implies:

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \nu \int_0^L {u_x}^2(x,t)dx - \alpha\|u(t)\|^2 + \|u(t)\|^p_{L^p(0,L)} = -\mu \sum_{k=1}^{N} \overline{u}_k \int_{J_k} u(x,t)dx$$

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \nu\|u_x(t)\|^2 - \alpha\|u(t)\|^2 + \|u(t)\|^p_{L^p(0,L)} = -\mu\frac{L}{N}\sum_{k=1}^{N} {\overline{u}_k}^2.$$

This implies

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \nu\|u_x(t)\|^2 + \mu h\sum_{k=1}^{N} {\overline{u}_k}^2 - \alpha\|u(t)\|^2 = -\|u(t)\|^p_{L^p(0,L)}. \tag{3.6}$$

It is clear that

$$\nu\|u_x(t)\|^2 + \mu h\sum_{k=1}^{N} {\overline{u}_k}^2 = \nu\frac{4\pi^2}{h^2}\left(\frac{h^2}{4\pi^2}\|u_x(t)\|^2 + h\sum_{k=1}^{N} {\overline{u}_k}^2\right) + \left(\mu h - \nu\frac{4\pi^2}{h}\right)\sum_{k=1}^{N} {\overline{u}_k}^2.$$

By using the condition (2.3) and the condition (3.4), we obtain:

$$\nu\|u_x(t)\|^2 + \mu h\sum_{k=1}^{N} {\overline{u}_k}^2 \geq \nu\frac{4\pi^2}{h}\|u(t)\|^2. \tag{3.7}$$

Employing (3.7) in (3.6), we obtain the following inequality:

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \left(\frac{4\pi^2\nu}{h} - \alpha\right)\|u(t)\|^2 \leq 0.$$

By using Gronwall's Inequality (1.3.2), we get:

$$\|u(t)\|^2 \leq e^{-2\left(\nu\frac{4\pi^2 N^2}{L^2} - \alpha\right)t}\|u_0\|^2.$$

Hence, $\|u(t)\|$ tends to 0 as $t \to \infty$, with an exponential rate. $\square$

3.2 Fourier Modes as Stabilizer

Next, we will consider a new feedback control system for the following initial-boundary value problem

$$u_t(x,t) - \nu u_{xx}(x,t) - \alpha u(x,t) + |u(x,t)|^{p-2}u(x,t) = 0, \tag{3.8}$$

$$u(0,t) = u(L,t) = 0, \quad \forall t \geq 0, \tag{3.9}$$

$$u(x,0) = u_0. \tag{3.10}$$

We will construct a feedback control system for this new problem (3.8)-(3.10) by using the interpolant operator based on the Fourier modes:

$$u_t(x,t) - \nu u_{xx}(x,t) - \alpha u(x,t) + |u(x,t)|^{p-2}u(x,t) = -\mu \sum_{k=1}^{N} u_k(t)\sin(\frac{k\pi x}{L}) \tag{3.11}$$

$$u(0,t) = u(L,t) = 0 \tag{3.12}$$

$$u(x,0) = u_0, \tag{3.13}$$

where

$$u_k(t) = \frac{2}{L}\int_0^L u(x,t)\sin(\frac{k\pi x}{L})dx,$$

under the following conditions

$$\mu \geq 2\alpha, \tag{3.14}$$

and

$$(\lambda_{N+1})^{-1} \leq \frac{\nu}{\alpha^2 L^2 + 2\alpha L}, \tag{3.15}$$

where λ_{N+1} is the $(N+1)$-th eigenvalue of the operator $-\frac{d^2}{dx^2}$ under the homogeneous Dirichlet boundary conditions. In the following theorem, we will show that all solutions of the problem (3.8)-(3.10) tend to zero, as t tends to infinity.

Theorem 3.2.1. *Assume that the conditions* (3.14) *and* (3.15) *hold. Then we have*

$$\|u(t)\|_{H^1(0,L)} \to 0$$

as $t \to \infty$, *for every solution* $u(t)$ *of the problem* (3.11)- (3.13).

Proof. In order to show that $\|u(t)\|_{H^1(0,L)} \to 0$ as $t \to \infty$, first we will show that

$$\|u(t)\| \to 0, \text{ as } \ t \to \infty.$$

Let us write the Fourier series expansion of u as follows:

$$u(x,t) = \sum_{k=1}^{N} u_k(t) \sin(\frac{k\pi x}{L}) + \sum_{k=N+1}^{\infty} u_k(t) \sin(\frac{k\pi x}{L}),$$

and take L^2 inner product of equation (3.11) with u.

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \nu\|u_x(t)\|^2 + \int_0^L |u(x,t)|^p dx - \alpha\|u(t)\|^2 \\ = -\mu \int_0^L \left(\sum_{k=1}^{N} u_k(t)\sin(\frac{k\pi x}{L})\right)\left(\sum_{l=1}^{\infty} u_l(t)\sin(\frac{l\pi x}{L})\right) dx.\end{aligned}$$

Since $\sum\limits_{k=1}^{N} u_k(t)\sin(\frac{k\pi x}{L})$ and $\sum\limits_{k=N+1}^{\infty} u_k(t)\sin(\frac{k\pi x}{L})$ are orthogonal to each other, we get the following equality:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \nu\|u_x(t)\|^2 + \int_0^L |u(x,t)|^p dx - \alpha\|u(t)\|^2 \\ = -\mu \sum_{k=1}^{N} u_k^2(t) \int_0^L \sin^2(\frac{k\pi x}{L}) dx \\ = -\frac{\mu L}{2}\sum_{k=1}^{N} u_k^2(t).\end{aligned}$$

By using Parseval Identity 1.3.1, we obtain:

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \nu\|u_x(t)\|^2 + \int_0^L |u(x,t)|^p dx + \left(\frac{\mu L}{2} - \alpha L\right)\sum_{k=1}^{N} u_k^2(t) = \alpha L \sum_{k=N+1}^{\infty} u_k^2(t).$$

Thanks to the Poincaré-type Inequality 1.3.7, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \nu\|u_x(t)\|^2 + \int_0^L |u(x,t)|^p dx + \left(\frac{\mu L}{2} - \alpha L\right)\sum_{k=1}^{N} u_k^2(t) \\ \le \alpha L (\lambda_{N+1})^{-1}\|u_x(t)\|^2.\end{aligned}$$

By using the assumption (3.14) we get:

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + (\nu - \alpha L(\lambda_{N+1})^{-1})\|u_x(t)\|^2 \le 0. \tag{3.16}$$

Thanks to the assumption (3.15) and Poincaré Inequality 1.3.6, we obtain followings:

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \frac{\nu}{2}\|u_x(t)\|^2 \leq 0$$
$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \frac{\nu}{2}\lambda_1\|u(t)\|^2 \leq 0.$$

By taking integral from 0 to t, we get:

$$\|u(t)\|^2 \leq \|u_0\|^2 e^{-\nu\lambda_1 t}.$$

Therefore,

$$\lim_{t\to\infty} \|u(t)\|^2 = 0. \tag{3.17}$$

Now, let us show that

$$\|u_x(t)\|^2 \to 0, \text{as} \quad t \to \infty.$$

Let us rewrite (3.11) in the following form:

$$u_t(x,t) - \nu u_{xx}(x,t) + (\mu-\alpha)\sum_{k=1}^{N} u_k(t) sin(\frac{k\pi x}{L}) + |u(x,t)|^{p-2}u(x,t)$$
$$= \alpha \sum_{k=N+1}^{\infty} u_k(t) sin(\frac{k\pi x}{L}) \tag{3.18}$$

Let us multiply (3.18) by u_t and integrate with respect to x from 0 to L.

$$\|u_t(t)\|^2 + \frac{d}{dt}\left[\frac{\nu}{2}\|u_x(t)\|^2 + \frac{(\mu-\alpha)L}{4}\sum_{k=1}^{N} u_k^2(t) + \frac{1}{p}\int_0^L |u(x,t)|^p dx\right]$$
$$= \frac{\alpha L}{2}\sum_{k=N+1}^{\infty} u_k(t)u_k'(t).$$

By using Cauchy-Schwarz (1.3.3) and Young's inequality 1.3.5, we obtain

$$\|u_t(t)\|^2 + \frac{d}{dt}\left[\frac{\nu}{2}\|u_x(t)\|^2 + \frac{(\mu-\alpha)L}{4}\sum_{k=1}^{N} u_k^2(t) + \frac{1}{p}\int_0^L |u(x,t)|^p dx\right]$$
$$\leq \frac{\alpha L}{2}\left(\sum_{k=N+1}^{\infty} u_k^2(t)\right)^{\frac{1}{2}}\left(\sum_{k=N+1}^{\infty} (u_k'(t))^2\right)^{\frac{1}{2}}$$
$$\leq \frac{\alpha L}{2}\left(\sum_{k=N+1}^{\infty} u_k^2(t)\right)^{\frac{1}{2}}\|u_t(t)\|$$
$$\leq \|u_t(t)\|^2 + \frac{\alpha^2 L^2}{8}\sum_{k=N+1}^{\infty} u_k^2(t).$$

This implies

$$\frac{d}{dt}\left[\frac{\nu}{2}\|u_x(t)\|^2 + \frac{(\mu-\alpha)L}{4}\sum_{k=1}^{N} u_k^2(t) + \frac{1}{p}\int_0^L |u(x,t)|^p dx\right]$$
$$\leq \frac{\alpha^2 L^2}{8}\sum_{k=N+1}^{\infty} u_k^2(t). \tag{3.19}$$

Now, let us multiply equality (3.18) by u and integrate over x from 0 to L. We obtain

$$\frac{d}{dt}\|u(t)\|^2 + 2\nu\|u_x(t)\|^2 + (\mu-\alpha)L\sum_{k=1}^{N} u_k^2(t) + 2\int_0^L |u(x,t)|^p dx$$
$$= \alpha L \sum_{k=N+1}^{\infty} u_k^2(t). \tag{3.20}$$

Let us multiply equality (3.20) by some $\epsilon > 0$, and add (3.19). Thanks to Poincaré-type Inequality 1.3.7, we get

$$\frac{d}{dt}\left[\frac{\nu}{2}\|u_x(t)\|^2 + \frac{(\mu-\alpha)L}{4}\sum_{k=1}^{N} u_k^2(t) + \frac{1}{p}\int_0^L |u(x,t)|^p dx + \epsilon\|u(t)\|^2\right]$$
$$+2\epsilon\nu\|u_x(t)\|^2 + (\mu-\alpha)\epsilon L\sum_{k=1}^{N} u_k^2(t) + 2\epsilon\int_0^L |u(x,t)|^p dx$$
$$\leq (\frac{\alpha^2 L^2}{8} + \epsilon\alpha L)\sum_{k=N+1}^{\infty} u_k^2(t)$$
$$\leq (\frac{\alpha^2 L^2}{8} + \epsilon\alpha L)(\lambda_{N+1})^{-1}\|u_x(t)\|^2. \tag{3.21}$$

Note that by using Poincaré Inequality 1.3.6, we can write the following inequality

$$\|u_x(t)\|^2 = \frac{1}{2}\|u_x(t)\|^2 + \frac{1}{2}\|u_x(t)\|^2$$
$$\geq \frac{1}{2}\|u_x(t)\|^2 + \lambda_1\frac{1}{2}\|u(t)\|^2. \tag{3.22}$$

By using (3.22), we obtain from (3.21)the following inequality

$$\frac{d}{dt}E(t) + \epsilon\nu\|u_x(t)\|^2 + \epsilon\nu\lambda_1\|u(t)\|^2 + (\mu-\alpha)\epsilon L\sum_{k=1}^{N} u_k^2(t) + 2\epsilon\int_0^L |u(x,t)|^p dx$$
$$\leq (\frac{\alpha^2 L^2}{8} + \epsilon\alpha L)(\lambda_{N+1})^{-1}\|u_x(t)\|^2,$$

where

$$E(t) := \frac{\nu}{2}\|u_x(t)\|^2 + \frac{(\mu-\alpha)L}{4}\sum_{k=1}^{N} u_k^2(t) + \frac{1}{p}\int_0^L |u(x,t)|^p dx + \epsilon\|u(t)\|^2.$$

This implies that

$$\frac{d}{dt}E(t) + \left(\epsilon\nu - (\frac{\alpha^2L^2}{8} + \epsilon\alpha L)(\lambda_{N+1})^{-1}\right)\|u_x(t)\|^2 + \epsilon(\mu-\alpha)L\sum_{k=1}^{N} u_k^2(t)$$
$$+2\epsilon\int_0^L |u(x,t)|^p dx + \epsilon\nu\lambda_1\|u(t)\|^2 \leq 0.$$

Let us choose $\epsilon = \frac{1}{4}$, and

$$\frac{\nu}{4} - (\frac{\alpha^2L^2}{8} + \frac{\alpha L}{4})(\lambda_{N+1})^{-1} \geq \frac{\nu}{8},$$

i.e.

$$(\lambda_{N+1})^{-1} \leq \frac{\nu}{\alpha}^2 L^2 + 2\alpha L.$$

We can write that

$$\frac{d}{dt}E(t) + \frac{\nu}{8}\|u_x(t)\|^2 + \frac{\nu\lambda_1}{8}\|u(t)\|^2 + \frac{(\mu-\alpha)L}{8}\sum_{k=1}^{N} u_k^2(t)$$
$$+\frac{1}{8}\int_0^L |u(x,t)|^p dx \leq 0.$$

Therefore, we obtain

$$\frac{d}{dt}E(t) + \delta E(t) \leq 0, \tag{3.23}$$

for some $\delta > 0$ small enough. By integrating (3.23) over $(0,t)$, we obtain

$$E(t) \leq E(0)e^{-\delta t}.$$

This implies that

$$\lim_{t\to\infty} E(t) = \lim_{t\to\infty}\left[\frac{\nu}{2}\|u_x(t)\|^2 + \frac{(\mu-\alpha)L}{4}\sum_{k=1}^{N} u_k^2(t) + \frac{1}{p}\int_0^L |u(x,t)|^p dx + \epsilon\|u(t)\|^2\right] = 0. \tag{3.24}$$

Since all terms in $E(t)$ are nonnegative, we obtain from (3.24) that

$$\lim_{t\to\infty} \|u_x(t)\|^2 = 0. \tag{3.25}$$

By combining the equalities (3.17) and (3.25), we conclude that, for all solutions of the problem (3.11)-(3.13) we have

$$\lim_{t\to\infty} \|u(t)\|_{H^1} = 0.$$

$\square$

Chapter 4

STABILITY

In this chapter, we will show that a general feedback control interpolant operator I_h which satisfies the condition (2.1), stabilizes the problem (1.9)- (1.11).

4.1 Stabilization in L^2 norm

In the following theorem, we will show the stabilization of zero solution of zero steady state solution of (1.9)- (1.11). So we consider the problem:

$$u_t(x,t) - \nu u_{xx}(x,t) - \alpha u(x,t) + |u(x,t)|^{p-2}u(x,t) = -\mu I_h(u), \tag{4.1}$$

$$u_x(0,t) = u_x(L,t) = 0, \qquad \forall t \geq 0, \tag{4.2}$$

$$u(x,0) = u_0, \tag{4.3}$$

with parameters such that

$$\mu \geq (2\alpha + 3\nu L^{-2}), \qquad \nu \geq \mu c^2 h^2. \tag{4.4}$$

Theorem 4.1.1. *Suppose that $T > 0$, and the conditions* (2.1) *and* (4.4) *are satisfied. Then the initial-boundary value problem* (4.1)-(4.3) *has a unique solution $u \in C(0,T;L^2(0,L)) \cap L^2(0,T;H^1(0,L))$. Also this solution u depends continuously on the initial data. Moreover, it satisfies the following equalities:*

$$\lim_{t\to\infty} \|u(t)\|^2 = 0;$$

and for every $\tau \geq 0$

$$\lim_{t\to\infty} \int_t^{t+\tau} \|u_x(s)\|^2 ds = 0.$$

Proof.

Remark 4.1.2. *Existence and uniqueness of the weak solutions of the problem will be show in the Appendix* 7*, in Theorem* 7.0.5*.*

Let us show that

$$\lim_{t\to\infty} \|u(t)\|^2 = 0.$$

Let us rewrite (4.1) as follows:

$$\begin{aligned} u_t(x,t) - \nu u_{xx}(x,t) + \frac{\nu}{L^2} u(x,t) - \left(\alpha + \frac{\nu}{L^2}\right) u(x,t) \\ = -|u(x,t)|^{p-2} u(x,t) - \mu I_h(u). \end{aligned} \tag{4.5}$$

Let us take L^2 inner product of (4.1) with $u(x,t)$. Then, we obtain:

$$\begin{aligned} \int_0^L u_t(x,t)u(x,t) - \nu u_{xx}(x,t)u(x,t) + \frac{\nu}{L^2} u^2(x,t) - \left(\alpha + \frac{\nu}{L^2}\right) u^2(x,t) dx \\ = - \int_0^L |u(x,t)|^p dx - \mu \int_0^L I_h(u) u(x,t) dx. \end{aligned}$$

This implies

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \nu \|u_x(t)\|^2 + \frac{\nu}{L^2}\|u(t)\|^2 = \left(\alpha + \frac{\nu}{2}\right) \|u(t)\|^2 \\ -\|u(t)\|^p_{L^p(0,L)} - \mu \int_0^L I_h(u) u(x,t) dx. \end{aligned}$$

Let us write

$$I_h(u)u(x,t) = (I_h(u) - u(x,t))u(x,t) + u^2(x,t).$$

By Cauchy-Schwarz Inequality (1.3.3), we have:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \nu \|u_x(t)\|^2 + \frac{\nu}{L^2}\|u(t)\|^2 \le \left(\alpha + \frac{\nu}{L^2}\right) \|u(t)\|^2 \\ -\|u(t)\|^p_{L^p(0,L)} - \mu \|u(t)\|^2 + \mu \|u(t)\| \|u(t) - I_h(u)\|. \end{aligned}$$

By applying Young's Inequality 1.3.5, we obtain the following inequality:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \nu \|u_x(t)\|^2 + \frac{\nu}{L^2}\|u(t)\|^2 \le \left(\alpha + \frac{\nu}{L^2}\right) \|u(t)\|^2 \\ - \|u(t)\|^p_{L^p(0,L)} - \frac{\mu}{2}\|u(t)\|^2 + \frac{\mu}{2}\|u(t) - I_h(u)\|^2. \end{aligned} \tag{4.6}$$

From inequality (2.1) and definition of $H^1(0,L)$ norm, we obtain:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \nu\|u_x(t)\|^2 + \frac{\nu}{L^2}\|u(t)\|^2 &\leq \left(\alpha + \frac{\nu}{L^2}\right)\|u(t)\|^2 - \|u(t)\|^p_{L^p(0,L)} \\ &- \frac{\mu}{2}\|u(t)\|^2 + \frac{\mu}{2}c^2h^2\|u(t)\|^2_{H^1(0,L)} \\ &= \left(\alpha + \frac{\nu}{L^2}\right)\|u(t)\|^2 - \|u(t)\|^p_{L^p(0,L)} - \frac{\mu}{2}\|u(t)\|^2 \\ &+ \frac{c^2h^2\mu}{2}\left(\frac{1}{L^2}\|u(t)\|^2 + \|u_x(t)\|^2\right). \end{aligned} \tag{4.7}$$

By using our assumption (4.4),

$$\mu \geq (2\alpha + \frac{3\nu}{L^2}), \qquad \nu \geq \mu c^2 h^2,$$

in inequality (4.7), we obtain the following inequality:

$$\begin{aligned}\frac{d}{dt}\|u(t)\|^2 + 2\nu\|u_x(t)\|^2 + \frac{2\nu}{L^2}\|u(t)\|^2 &\leq \left(2\alpha + \frac{2\nu}{L^2}\right)\|u(t)\|^2 - 2\|u(t)\|^p_{L^p(0,L)} \\ &- \mu\|u(t)\|^2 + \frac{\nu}{L^2}\|u(t)\|^2 + \nu\|u_x(t)\|^2. \end{aligned} \tag{4.8}$$

Let us rewrite inequality (4.8). Thus, we have following inequality:

$$\begin{aligned}\frac{d}{dt}\|u(t)\|^2 + \nu\|u_x(t)\|^2 + \frac{2\nu}{L^2}\|u(t)\|^2 &\leq \left(2\alpha + \frac{3\nu}{L^2}\right)\|u(t)\|^2 \\ &- \mu\|u(t)\|^2 - 2\int_0^L |u(x,t)|^p dx. \end{aligned}$$

Since $\mu \geq (2\alpha + \frac{3\nu}{L^2})$, we have:

$$\frac{d}{dt}\|u(t)\|^2 + \nu\|u_x(t)\|^2 + \frac{2\nu}{L^2}\|u(t)\|^2 \leq -2\|u(t)\|^p_{L^p(0,L)}.$$

Therefore, we have:

$$\frac{d}{dt}\|u(t)\|^2 + \nu\|u_x(t)\|^2 + \frac{2\nu}{L^2}\|u(t)\|^2 \leq 0. \tag{4.9}$$

We can eliminate $\|u_x(t)\|^2$ from the left-hand side of inequality (4.9). Thus, we obtain:

$$\frac{d}{dt}\|u(t)\|^2 \leq -\frac{2\nu}{L^2}\|u(t)\|^2.$$

By integrating both sides over $(0,t)$, we get:

$$\|u(t)\|^2 \leq e^{\frac{-2\nu t}{L^2}}\|u_0\|^2. \tag{4.10}$$

Therefore,

$$\lim_{t\to\infty} \|u(t)\|^2 \leq \lim_{t\to\infty} e^{\frac{-2\nu t}{L^2}} \|u_0\|^2 = 0. \tag{4.11}$$

Hence, since $\|u(t)\|^2 \geq 0$, for every $t \geq 0$, inequality (4.11) implies:

$$\lim_{t\to\infty} \|u(t)\|^2 \equiv 0. \tag{4.12}$$

Now, let us show that for every $\tau \geq 0$, the following equality holds

$$\lim_{t\to\infty} \int_t^{t+\tau} \|u_x(s)\|^2 ds = 0.$$

Let us take integral of both sides of inequality (4.9) from t to $t+\tau$. Observe that $\forall \tau > 0$ we have:

$$\begin{aligned}
\|u(t+\tau)\|^2 + \nu \int_t^{t+\tau} \left(\|u_x(s)\|^2 + \frac{2}{L^2}\|u(s)\|^2 \right) ds &\leq \|u(t)\|^2 \\
\nu \int_t^{t+\tau} \left(\|u_x(s)\|^2 + \frac{2}{L^2}\|u(s)\|^2 \right) ds &\leq \|u(t)\|^2 \\
\nu \int_t^{t+\tau} \|u_x(s)\|^2 ds &\leq \|u(t)\|^2.
\end{aligned}$$

Since $\nu \geq 0$ and $lim_{t\to\infty}\|u(t)\|^2 \equiv 0$, we obtain:

$$\lim_{t\to\infty} \int_t^{t+\tau} \|u_x(s)\|^2 ds = 0, \quad \forall \tau > 0.$$

□

4.2 Stabilization in H^1 norm

In this section, we will show that the feedback control interpolant operator I_h stabilizes the zero solution of the problem (1.9) and (1.11) in H^1 norm. In other words, we will show:

$$\lim_{t\to\infty} \|u(t)\|^2_{H^1} = 0.$$

Lemma 4.2.1. *The solution u of the initial boundary problem* (4.1)-(4.2) *with the initial value $u_0 \in H^1$, satisfies the following inequality*

$$\lim_{t\to\infty} \|u(t)\|^2_{H^1} = 0.$$

Proof. Note that in Theorem 4.1.1, we have shown that the solution, u, of the problem (4.1)- (4.3) satisfies

$$\lim_{t\to\infty} \|u(t)\|^2 = 0.$$

It is enough to show that

$$\lim_{t\to\infty} \|u_x(t)\|^2 = 0.$$

Let us rewrite equation (4.1) as follows:

$$\begin{aligned} u_t(x,t) + \frac{\nu}{L^2}u(x,t) - \nu u_{xx}(x,t) - (\alpha + \frac{\nu}{L^2})u(x,t) + |u(x,t)|^{p-2}u(x,t) \\ = -\mu I_h(u). \end{aligned} \tag{4.13}$$

Let us take L^2 inner product of (4.13) with $-u_{xx}(x,t)$. Thus, we get:

$$\begin{aligned} \int_0^L -u_t(x,t)u_{xx}(x,t)dx - \frac{\nu}{L^2}\int_0^L u(x,t)u_{xx}(x,t)dx + \nu\int_0^L u_{xx}^2(x,t)dx \\ +(\alpha + \frac{\nu}{L^2})\int_0^L u(x,t)u_{xx}(x,t)dx \\ = \int_0^L |u(x,t)|^{p-2}u(x,t)u_{xx}(x,t)dx + \mu\int_0^L I_h(u)u_{xx}(x,t)dx. \end{aligned}$$

By using integration by parts and the boundary conditions (4.2), we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_x(t)\|^2 + \frac{\nu}{L^2}\|u_x(t)\|^2 + \nu\|u_{xx}(t)\|^2 - (\alpha + \frac{\nu}{L^2})\|u_x(t)\|^2 \\ = \int_0^L |u(x,t)|^{p-2}u(x,t)u_{xx}(x,t)dx + \mu\int_0^L I_h(u)u_{xx}(x,t)dx. \end{aligned}$$

The first integral in the right-hand side of the last inequality is

$$\int_0^L |u(x,t)|^{p-2}u(x,t)u_{xx}(x,t)dx = -(p-1)\int_0^L u_x^2(x,t)|u(x,t)|^{p-2}dx.$$

Let us write the following:

$$I_h(u)u_{xx}(x,t) = (I_h(u) - u(x,t))u_{xx}(x,t) + u(x,t)u_{xx}(x,t).$$

Therefore,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_x(t)\|^2 + \nu\|u_{xx}(t)\|^2 + \frac{\nu}{L^2}\|u_x(t)\|^2 - (\alpha + \frac{\nu}{L^2})\|u_x(t)\|^2 \\ = -(p-1)\int_0^L u_x^2(x,t)|u(x,t)|^{p-2}dx + \mu\int_0^L (I_h(u) - u(x,t))u_{xx}(x,t)dx \\ - \mu\|u_x(t)\|^2. \end{aligned} \tag{4.14}$$

This implies

$$\frac{1}{2}\frac{d}{dt}\|u_x(t)\|^2 + \nu\|u_{xx}(t)\|^2 + \frac{\nu}{L^2}\|u_x(t)\|^2 - (\alpha + \frac{\nu}{L^2})\|u_x(t)\|^2$$
$$\leq \mu \int_0^L (I_h(u) - u(x,t))u_{xx}(x,t)dx - \mu\|u_x(t)\|^2$$

Thanks to Cauchy-Schwarz Inequality (1.3.3), we get:

$$\frac{1}{2}\frac{d}{dt}\|u_x(t)\|^2 + \nu\|u_{xx}(t)\|^2 + \frac{\nu}{L^2}\|u_x(t)\|^2 \leq (\alpha + \frac{\nu}{L^2})\|u_x(t)\|^2 - \mu\|u_x(t)\|^2$$
$$+ \mu\|I_h(u) - u(t)\|\|u_{xx}(t)\|. \tag{4.15}$$

Let us apply Young's Inequality 1.3.5, in the last term of inequality (4.15), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_x(t)\|^2 + \nu\|u_{xx}(t)\|^2 + \frac{\nu}{L^2}\|u_x(t)\|^2 \leq (\alpha + \frac{\nu}{L^2} - \mu)\|u_x(t)\|^2$$
$$+ \frac{\nu}{2}\|u_{xx}(t)\|^2 + \frac{\mu^2}{2\nu}\|I_h(u) - u(t)\|^2.$$

By the assumption (2.1), we get

$$\frac{1}{2}\frac{d}{dt}\|u_x(t)\|^2 + \frac{\nu}{2}\|u_{xx}(t)\|^2 + \frac{\nu}{L^2}\|u_x(t)\|^2 \leq (\alpha + \frac{\nu}{L^2} - \mu)\|u_x(t)\|^2 + \frac{h^2\mu^2c^2}{2\nu}\|u_x(t)\|^2_{H^1}$$
$$= (\alpha + \frac{\nu}{L^2} - \mu)\|u_x(t)\|^2 + \frac{h^2\mu^2c^2}{2\nu}\left(\|u_x(t)\|^2 + \frac{1}{L^2}\|u(t)\|^2\right)$$
$$= (\alpha + \frac{\nu}{L^2} + \frac{h^2\mu^2c^2}{2\nu} - \mu)\|u_x(t)\|^2 + \frac{h^2\mu^2c^2}{2\nu L^2}\|u(t)\|^2.$$

Thanks to the assumption (4.4), $\nu \geq \mu c^2 h^2$, and $\mu \geq (2\alpha + \frac{3\nu}{L^2})$, we have:

$$\frac{1}{2}\frac{d}{dt}\|u_x(t)\|^2 + \frac{\nu}{2}\|u_{xx}(t)\|^2 + \frac{\nu}{L^2}\|u_x(t)\|^2 \leq -\frac{\nu}{2L^2}\|u_x(t)\|^2 + \frac{\mu}{2}\|u(t)\|^2. \tag{4.16}$$

This implies:

$$\frac{d}{dt}\|u_x(t)\|^2 \leq -\frac{3\nu}{L^2}\|u_x(t)\|^2 + \mu\|u(t)\|^2. \tag{4.17}$$

By applying Gronwall's Inequality (1.3.2) to (4.17) we obtain:

$$\|u_x(t)\|^2 \leq \|u_x(0)\|^2 \exp(-\frac{3\nu t}{L^2}) + \mu \int_0^t \exp\left(-\frac{3\nu t}{L^2} + \frac{3\nu s}{L^2}\right)\|u(s)\|^2 ds. \tag{4.18}$$

We will show that the right-hand side of inequality (4.18) tends to 0 as $t \to \infty$.

In the first term we have:

$$\lim_{t\to\infty} \|u_x(0)\|^2 \exp(-\frac{3\nu t}{L^2}) = 0. \tag{4.19}$$

Let us consider the second term:

$$\begin{aligned}\lim_{t\to\infty}\mu\frac{\int_0^t \exp(\frac{3\nu s}{L^2})\|u(s)\|^2 ds}{\exp(\frac{3\nu t}{L^2})} &= \mu\lim_{t\to\infty}\frac{\exp(\frac{3\nu t}{L^2})\|u(t)\|^2}{\frac{3\nu}{L^2}\exp(\frac{3\nu t}{L^2})}\\ &= \mu\frac{L^2}{3\nu}\lim_{t\to\infty}\|u(t)\|^2.\end{aligned}$$

Since we have from Theorem 4.1.1 that $\lim_{t\to\infty}\|u(t)\|^2 = 0$, we obtain

$$\lim_{t\to\infty}\mu\frac{\int_0^t \exp(\frac{3\nu s}{L^2})\|u(s)\|^2 ds}{\exp(\frac{3\nu t}{L^2})} = 0. \tag{4.20}$$

Thanks to the equalities (4.19) and (4.20), we get:

$$\lim_{t\to\infty}\|u_x(t)\|^2 = 0. \tag{4.21}$$

Therefore, the equalities (4.12) and (4.21) imply:

$$\lim_{t\to\infty}\|u(t)\|_{H^1}^2 = 0.$$

Hence, this completes the proof. □

Chapter 5

NODAL OBSERVABLES AND FEEDBACK CONTROLLERS

In this chapter, we will investigate a different feedback control system for our problem (1.9)-(1.11). The observables will be chosen as the values of solutions $u(\overline{x}_k)$ at the points $\overline{x}_k \in J_k$, where $J_k = [(k-1)\frac{L}{N}, k\frac{L}{N}]$ and feedback is at some points $x_k \in J_k$, for $k = 1, ..., N$. We will present a new feedback control system based on nodal observables and feedback controllers. Let us consider the periodic boundary value problem

$$\begin{aligned} u_t(x,t) - \nu u_{xx}(x,t) - \alpha u(x,t) + |u(x,t)|^{p-2}u(x,t) \\ = -\mu h \sum_{k=1}^{N} u(\overline{x}_k, t)\delta(x - x_k) \end{aligned} \tag{5.1}$$

$$u(x,t) = u(x+L,t), \tag{5.2}$$

where $h = \frac{L}{N}$ for $a \in [0, L]$, $\delta(x-a) \in H^{-1}_{per}(0, L)$ and for every $\varphi \in H^1_{per}(0, L)$, $\delta(x-a)$ is extended periodically such that

$$\langle \delta(\cdot - a), \varphi \rangle = \varphi(a). \tag{5.3}$$

Lemma 5.0.2. *Let $x_k, \overline{x}_k \in J_k$, where $J_k = [(k-1)\frac{L}{N}, k\frac{L}{N}]$ and $h = \frac{L}{N}$. Then for every $\varphi \in H^1(0, L)$ the following inequalities hold true:*

$$\sum_{k=1}^{N} |\varphi(x_k) - \varphi(\overline{x}_k)|^2 \le h\|\varphi'\|^2 \tag{5.4}$$

and

$$\|\varphi\|^2 \le 2\left[h\sum_{k=1}^{N} |\varphi(x_k)|^2 + h^2\|\varphi'\|^2\right]. \tag{5.5}$$

Proof. Since C^1 is dense in H^1, it is enough to prove inequality (5.4) for every $\varphi \in C^1$. Let $\varphi \in C^1$ be arbitrary.

$$\begin{aligned} |\varphi(x_k) - \varphi(\overline{x}_k)|^2 &\leq \left| \int_{\overline{x}_k}^{x_k} \varphi'(s)ds \right|^2 \\ &\leq \left(\int_{J_k} |\varphi'(s)|ds \right)^2. \end{aligned}$$

Then by Hölder inequality, we have

$$\begin{aligned} |\varphi(x_k) - \varphi(\overline{x}_k)|^2 &\leq |J_k| \int_{J_k} |\varphi'(s)|^2 ds \\ &= h \int_{J_k} |\varphi'(s)|^2 ds. \end{aligned}$$

Let us sum the last inequality over k from $k = 1$ to $k = N$. Then, we get

$$\begin{aligned} \sum_{k=1}^{N} |\varphi(x_k) - \varphi(\overline{x}_k)|^2 &\leq h \sum_{k=1}^{N} \int_{J_k} |\varphi'(s)|^2 ds \\ &= h \int_0^L |\varphi'(s)|^2 ds \\ &= h\|\varphi'\|^2. \end{aligned}$$

This completes the proof of inequality (5.4).

Next, let us show inequality (5.5). Let us observe that

$$|\varphi(x)| \leq |\varphi(x_k)| + \int_{J_k} |\varphi'(s)|ds.$$

This implies that

$$|\varphi(x)|^2 \leq 2 \left[|\varphi(x_k)|^2 + \left(\int_{J_k} |\varphi'(s)|^2 ds \right)^2 \right].$$

Let us integrate the last inequality with respect to x over J_k.

$$\begin{aligned} \int_{J_k} |\varphi(x)|^2 dx &\leq 2 \int_{J_k} |\varphi(x_k)|^2 dx + 2 \int_{J_k} \left(\int_{J_k} |\varphi'(s)|^2 ds \right)^2 dx \\ &= 2h|\varphi(x_k)|^2 + 2h \left(\int_{J_k} |\varphi'(s)|^2 ds \right)^2. \end{aligned}$$

Thanks to Cauchy-Schwarz Inequality (1.3.3) and by summing over k from $k=1$ to $k=N$, we have

$$\int_{J_k} |\varphi(x)|^2 dx \leq 2h\left[|\varphi(x_k)|^2 + h\int_{J_k} |\varphi'(s)|^2 ds\right]$$
$$\sum_{k=1}^{N}\int_{J_k} |\varphi(x)|^2 dx \leq 2h\left[\sum_{k=1}^{N}|\varphi(x_k)|^2 + h\|\varphi'\|^2\right].$$

Hence, this completes this completes the proof. ☐

Theorem 5.0.3. *Let $\mu > 4\alpha$ and h be small enough such that $\nu \geq 2\mu h^2$. Then for every $T>0$, and every $u_0 \in L^2_{per}(0,L)$ the system (5.1) has a unique solution u such that*

$$u \in C(0,T; L^2_{per}(0,L)) \cap L^2(0,T; H^1_{per}(0,L)) \cap L^p(0,T; L^p_{per}(0,L)),$$

where $p \geq 2$ and

$$u_t \in L^2(0,T; H^{-1}_{per}),$$

Moreover,

$$\lim_{t\to\infty} \|u(t)\| = 0, \quad \text{and for every,} \quad \tau > 0 \quad \lim_{t\to\infty}\int_t^{t+\tau} \|u_x(t)\|^2 ds = 0.$$

Proof. Let $u \in H^1(0,L)$ and let us take $H^{-1}(0,L)$ action of equation (5.1) with u. Thanks to Lemma 1.4.2, we get

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 - \nu\int_0^L u_{xx}(x,t)u(x,t)dx - \alpha\int_0^L u^2(x,t)dx + \int_0^L |u(x,t)|^p dx$$
$$= -\mu h\sum_{k=1}^{N} u(\overline{x}_k,t)\int_0^L \delta(x-x_k)u(x,t)dx.$$

By using integration by parts and the property of δ-function, (5.3), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \nu\|u_x(t)\|^2 &= \alpha\|u(t)\|^2 - \int_0^L |u(x,t)|^p dx - \mu h\sum_{k=1}^{N} u(\overline{x}_k,t)u(x_k,t)\\ &= \alpha\|u(t)\|^2 - \int_0^L |u(x,t)|^p dx\\ &\quad - \mu h\sum_{k=1}^{N}|u(x_k,t)|^2 + \mu h\sum_{k=1}^{N}(u(x_k,t)-u(\overline{x}_k,t))u(x_k,t).\end{aligned}$$

Thanks to Young's Inequality 1.3.5, we have

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \nu\|u_x(t)\|^2 \le \alpha\|u(t)\|^2 - \int_0^L |u(x,t)|^p dx \\ - \frac{\mu}{2}h\sum_{k=1}^N |u(x_k,t)|^2 + \frac{\mu}{2}h\sum_{k=1}^N |u(x_k,t) - u(\overline{x}_k,t)|^2$$

Let us use the inequalities (5.4) and (5.5) from Lemma 5.0.2. Then, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \nu\|u_x(t)\|^2 \le \alpha\|u(t)\|^2 - \int_0^L |u(x,t)|^p dx \\
&- \frac{\mu}{2}h\sum_{k=1}^N |u(x_k,t)|^2 + \frac{\mu}{2}h^2\|u_x(t)\|^2 \\
&\le \alpha\|u(t)\|^2 - \int_0^L |u(x,t)|^p dx - \frac{\mu}{4}\|u(t)\|^2 + \mu h^2\|u_x(t)\|^2.
\end{aligned}$$

Therefore, we have

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + (\nu - \mu h^2)\|u_x(t)\|^2 \le (\alpha - \frac{\mu}{4})\|u(t)\|^2 - \int_0^L |u(x,t)|^p dx.$$

By using our assumptions

$$\nu \ge 2\mu h^2 \quad \text{and} \quad \mu > 4\alpha,$$

and omitting the term $-\int_0^L |u(x,t)|^p dx$ from the right-hand side of the last inequality, we obtain

$$\frac{d}{dt}\|u(t)\|^2 + \nu\|u_x(t)\|^2 + (\frac{\mu}{2} - 2\alpha)\|u(t)\|^2 \le 0. \tag{5.6}$$

Then we have

$$\frac{d}{dt}\|u(t)\|^2 + (\frac{\mu}{2} - 2\alpha)\|u(t)\|^2 \le 0. \tag{5.7}$$

Let us denote $z(t) = \|u(t)\|^2$ and apply Gronwall's Inequality (1.3.2) to (5.7). We obtain

$$z(t) \le z(0)e^{-(\frac{\mu}{2}-2\alpha)t}.$$

In other words, we get

$$\|u(t)\|^2 \le \|u_0\|^2 e^{-(\frac{\mu}{2}-2\alpha)t}.$$

Since

$$\|u(t)\|^2 \geq 0 \quad \text{and} \quad \lim_{t\to\infty} \|u(t)\|^2 \leq 0,$$

we have

$$\lim_{t\to\infty} \|u(t)\| = 0.$$

We will show that

$$\forall \tau > 0 \quad \lim_{t\to\infty} \int_t^{t+\tau} \|u_x(t)\|^2 ds = 0.$$

Let us integrate inequality (5.6) from t to $t+\tau$, for an arbitrary $\tau > 0$. We get

$$\|u(t+\tau)\|^2 + \nu \int_t^{t+\tau} \|u_x(s)\|^2 ds + (\frac{\mu}{2} - 2\alpha) \int_t^{t+\tau} \|u(s)\|^2 ds \leq \|u(t)\|^2. \tag{5.8}$$

This implies

$$\nu \int_t^{t+\tau} \|u_x(s)\|^2 ds \leq \|u(t)\|^2.$$

Since

$$\nu > 0, \quad \|u_x(s)\|^2 \geq 0 \quad \text{and} \quad \lim_{t\to\infty} \|u(t)\| = 0$$

we obtain that

$$\int_t^{t+\tau} \|u_x(s)\|^2 ds \leq 0.$$

Next, we will show the uniqueness of the solution. Let $u(x,t)$ and $v(x,t)$ be two solutions of the problem (5.1)-(5.1) and denote $w(x,t) = u(x,t) - v(x,t)$. We have the following equation

$$\begin{aligned} w_t(x,t) - \nu w_{xx}(x,t) - \alpha w(x,t) + |u(x,t)|^{p-2}|u(x,t)| - |v(x,t)|^{p-2}|v(x,t)| \\ = -\mu h \sum_{k=1}^{N} w(\overline{x}_k, t)\delta(x - x_k). \end{aligned} \tag{5.9}$$

Let us take H^{-1} action of (5.9) on w and apply Lemma 1.4.2. By following similar procedure in the previous steps, we get

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2 + \nu\|w_x(t)\|^2 - \alpha\|w(t)\|^2$$

$$= -\int_0^L (|u(x,t)|^{p-2}|u(x,t)| - |v(x,t)|^{p-2}|v(x,t)|)w(x,t)dx - \mu h \sum_{k=1}^{N} w(\overline{x}_k, t)w(x_k, t).$$

Thanks to Monotonicity Inequality 1.3.9, we obtain

$$\int_0^L (|u(x,t)|^{p-2}|u(x,t)| - |v(x,t)|^{p-2}|v(x,t)|)w(x,t)dx \geq d\int_0^L |w(x,t)|^p dx,$$

where $d > 0$ is a constant. This implies

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w(t)\|^2 + \nu\|w_x(t)\|^2 - \alpha\|w(t)\|^2 &\leq -d\int_0^L |w(x,t)|^p dx - \mu h\sum_{k=1}^N w(\overline{x}_k,t)w(x_k,t)\\
&\leq -\mu h\sum_{k=1}^N w(\overline{x}_k,t)w(x_k,t)\\
&\leq -\mu h\sum_{k=1}^N |w(x_k,t)|^2 + \mu h\sum_{k=1}^N (w(x_k,t) - w(\overline{x}_k,t))w(x_k,t).
\end{aligned}$$

By using Young's inequality 1.3.5 and apply Lemma 5.0.2, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w(t)\|^2 + \nu\|w_x(t)\|^2 - \alpha\|w(t)\|^2 &\leq -\frac{\mu}{2}h\sum_{k=1}^N |w(x_k,t)|^2\\
&\quad + \frac{\mu}{2}h\sum_{k=1}^N |w(x_k,t) - w(\overline{x}_k,t)|^2\\
&\leq -\frac{\mu}{2}h\sum_{k=1}^N |w(x_k,t)|^2 + \frac{\mu}{2}h^2\|w_x(t)\|^2\\
&\leq -\frac{\mu}{4}\|w(t)\|^2 + \mu h^2\|w_x(t)\|^2.
\end{aligned}$$

Therefore, we obtain

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2 + (\nu - \mu h^2)\|w_x(t)\|^2 \leq (\alpha - \frac{\mu}{4})\|w(t)\|^2.$$

By using our assumptions $\nu \geq 2\mu h^2$ and $\mu > 4\alpha$, we have

$$\frac{d}{dt}\|w(t)\|^2 + \nu\|w_x(t)\|^2 + (\frac{\mu}{2} - 2\alpha)\|w(t)\|^2 \leq 0.$$

Since $\nu > 0$, we get

$$\frac{d}{dt}\|w(t)\|^2 + (\frac{\mu}{2} - 2\alpha)\|w(t)\|^2 \leq 0.$$

Thanks to Gronwall's Inequality (1.3.2), we obtain

$$\|w(t)\|^2 \leq e^{(2\alpha - \frac{\mu}{2})t}\|w(0)\|^2. \tag{5.10}$$

Therefore, since $\|w(0)\| = \|u(0) - v(0)\| = 0$, from inequality (5.10) we have that

$$\|w(t)\| = 0 \quad \forall t \in [0, T].$$

i.e. the solution is unique. Furthermore, inequality (5.10) implies continuous dependence of the solutions on the initial data.

Hence, this completes the proof. $\square$

Chapter 6

CONCLUSION

In this study, we have shown that the stability solutions of a second order nonlinear parabolic equation by using finite parameters feedback control. Our motivation is the study of A. Azouani and E.S. Titi in [1] and V. Kalantarov and E. S. Titi in [4]. We are inspired by the articles [1] and [4] to show our technical results.

We aimed to adapt the results of [1] to the nonlinear parabolic equation of the following form:

$$u_t(x,t) - \nu u_{xx}(x,t) - \alpha u(x,t) + |u(x,t)|^{p-2}u(x,t) = -\mu I_h, \tag{6.1}$$

with the initial value

$$u(x,0) = u_0, \qquad u_0 \in L^2(0,L).$$

The feedback control of this initial value problem under Neumann and Dirichlet boundary conditions is done with various feedback controllers, such as, finite volume elements, finitely many Fourier modes and nodal observables and controllers.

We can describe our results as follows:

(i) The existence and uniqueness of the solutions of equation (6.1) with the Neumann boundary conditions:

$$u_x(0,t) = u_x(L,t) = 0,$$

and the initial value

$$u(x,0) = u_0, \qquad u_0 \in L^2(0,L).$$

(ii) The stabilization of (6.1) in L^2 norm with the Neumann boundary conditions, by using *spatial averages* as interpolant operator.

(iii) The stabilization of (6.1) in H^1 norm with the homogeneous Dirichlet boundary conditions:

$$u(0,t) = u(L,t) = 0,$$

by using *finite Fourier modes.*

(iv) The stabilization of (6.1) in H^1 norm with the following periodic boundary conditions:

$$u(x,t) = u(x+L,t) = 0,$$

by using *finite nodal values and controllers.*

Chapter 7

APPENDIX

Weak Solutions

Consider the following one dimensional nonlinear reaction-diffusion equation

$$u_t(x,t) - \nu u_{xx}(x,t) - \alpha u(x,t) + |u(x,t)|^{p-2}u(x,t) = h(x,t), \tag{7.1}$$

with the Neumann boundary conditions

$$u_x(0,t) = u_x(L,t) = 0, \qquad \forall t \geq 0, \tag{7.2}$$

and the initial value

$$u(x,0) = u_0. \tag{7.3}$$

We will prove that the initial-boundary value problem (7.1)-(7.3) has a unique solution which is defined as follows:

Definition 7.0.4 (Weak Solution)**.** *Suppose that* $u_0 \in L^2(0,L)$.
Then the function $u : [0,\infty] \to L^2(0,L)$ *is called a weak solution of the initial-boundary value problem* (7.1)-(7.3) *if*
(i) for each $T > 0$,

$$u \in L^2(0,T;H_0^1(0,L)) \cap C(0,T;L^2(0,L)),$$
$$u_t \in L^2(0,T;H^{-1}(0,L)),$$

(ii) for every $w \in H_0^1(0,L)$*, the following equality holds for all* $t \in \mathbb{R}^+$:

$$\langle u_t(t), w\rangle - \nu \int_0^L u_{xx}(x,t)w(x)dx - \alpha \int_0^L u(x,t)w(x)dx$$
$$= -\int_0^L |u(x,t)|^{p-2}u(x,t)w(x)dx + \int_0^L h(x,t)w(x)dx,$$

where $\langle \cdot, \cdot \rangle$ *denotes the duality pairing between* $H_0^1(0, L)$ *and* $H^{-1}(0, L)$,

(iii) $u(x, 0) = u_0$.

Theorem 7.0.5. *If* $u_0 \in H_0^1 \cap L^p(0, L)$, *and* $h \in L^2(0, T; L^2(0, L))$, *then the initial-boundary value problem* (7.1)-(7.3) *has a unique weak solution* u.

Proof. We will prove this theorem by Galerkin method. Let us choose base functions $\{w_j, \ \ j \in \mathbb{N}\}$ in H_0^1 such that w_j's are eigenfunctions of the Sturm-Liouville problem

$$-w_j''(x) = \lambda_j w_j(x), \quad x \in [0, L],$$
$$w_j(0) = w_j(L) = 0,$$

where

$$w_j(x) = \sqrt{\frac{2}{\pi}} \sin(\frac{k\pi}{L} x), \quad k = 1, 2, ...$$

We look for the approximate weak solution of the form

$$u_m(x, t) = \sum_{j=1}^{m} u_{mj}(t) w_j(x),$$

which satisfies the system of equations:

$$(u_m'(x,t), w_j(x)) - \nu(u_{mxx}(x,t), w_j(x)) - \alpha(u_m(x,t), w_j(x))$$
$$= -(|u_m(x,t)|^{p-2} u_m(x,t), w_j(x)) + (h(x,t), w_j(x)), \quad j = 1, ..., m. \tag{7.4}$$
$$(u_m(0), w_j(x)) = \xi_{mj}, \quad j = 1, ..., m, \tag{7.5}$$

where

$$u_m'(x,t) = \frac{\partial}{\partial t} u_m(x,t), \quad u_{mx}(x,t) = \frac{\partial}{\partial x} u_m(x,t),$$

and ξ_{mj} are constants such that

$$\sum_{j=1}^{m} \xi_{mj} w_j \to u_0 \quad \text{in} \quad H_0^1 \cap L^p(0, L),$$

as $m \to \infty$. We can rewrite the system (7.4)-(7.5) as a system of ordinary differential equations:

$$u_{mj}'(t) + \nu\lambda_j u_{mj}(t) - \alpha u_{mj}(t) = -G_j(u_{mj}(t)) + \int_0^L h(x,t) w_j(x) dx, \quad j = 1, ..., m, \tag{7.6}$$

$$u_{mj}(0) = \xi_{mj}, \quad j = 1, ..., m, \tag{7.7}$$

where

$$G_j(u_{mj}(t)) = \int_0^L |\sum_{j=1}^m u_{mj}(t)w_j(x)|^{p-2}u_{mj}(t)dx$$

for $j = 1, ..., m$.

Note that G_j is a polynomial of u_{mj} of order $p-1$. Therefore, G_j's are locally Lipschitz for $j = 1, 2, ..., m$. From equality (7.6), the functions

$$F_j(t) := \int_0^L h(x,t)w_j(x)dx$$

are continuously differentiable, for $j = 1, 2, ..., m$.

This implies that F_j's are locally Lipschitz for $j = 1, 2, ..., m$. Thanks to Theorem 1.4.3, the initial value problem (7.4)-(7.5) has a unique solution on some interval $[0, T^*]$. By using Theorem 1.4.4, if u_m remains bounded on every interval $[0, T]$, then we can extend solution to the interval $[0, \infty)$.

Now, we will prove that the sequence $\{u_m(x,t)\}$, is uniformly bounded. Let us multiply equation (7.4) by $u_{mj}(t)$ and sum with respect to $j = 1, 2, ..., m$. We obtain

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|^2 + \nu\|u_{mx}(t)\|^2 - \alpha\|u_m(t)\|^2 + \|u_m(t)\|^p_{L^p(0,L)} = (h, u_m). \tag{7.8}$$

By applying Cauchy-Schwarz (1.3.3) and Young Inequality (1.3.5) to the right-hand side of (7.8), we obtain:

$$\frac{d}{dt}\|u_m(t)\|^2 + 2\nu\|u_{mx}(t)\|^2 - (2\alpha+1)\|u_m(t)\|^2 + 2\|u_m(t)\|^p_{L^p(0,L)} \leq \|h(t)\|^2. \tag{7.9}$$

By integrating inequality (7.9) over the interval $(0, t)$, we get:

$$\begin{aligned} &\|u_m(t)\|^2 + 2\nu\int_0^t \|u_{mx}(\tau)\|^2 d\tau + 2\int_0^t \|u_m(\tau)\|^p_{L^p(0,L)} d\tau \\ &\leq \sum_{j=1}^m \xi_{mj}^2 + |2\alpha+1|\int_0^t \|u_m(\tau)\|^2 d\tau + \int_0^t \|h(\tau)\|^2 d\tau \end{aligned}$$

Since $\sum_{j=1}^m \xi_{mj}^2$ is uniformly bounded, we deduce that

$$\|u_m(t)\|^2 + 2\nu\int_0^t \|u_{mx}(\tau)\|^2 d\tau + 2\int_0^t \|u_m(\tau)\|^p_{L^p(0,L)} d\tau \leq H_T, \quad \forall t \in [0, T^*], \tag{7.10}$$

where H_T depends on $\|u_0\|$, $\int_0^T \|h(\tau)\|^2 d\tau$ and T. From the inequality (7.10), we obtain

$$\|u_m(t)\|^2 = \sum_{j=1}^{m} u_{mj}^2(t) \leq H_T, \quad \forall t \in [0,T].$$

Hence, we have a uniform *a priori* estimates:

- u_m is uniformly bounded in $L^\infty(0,T; L^2(0,L))$,
- u_m is uniformly bounded in $L^2(0,T; H_0^1(0,L))$,
- u_m is uniformly bounded in $L^p(0,T; L^p(0,L))$.

Now, let us multiply the equation (7.4) by $\lambda_1 u_{mj}$ and u'_{mj} respectively and sum over j. We obtain the following inequalities

$$\begin{aligned}(u'_m(x,t), -u_{mxx}(x,t)) + \nu\|u_{mxx}(t)\|^2 - \alpha(u_m(x,t), -u_{mxx}(x,t))\\ +(|u_m(x,t)|^{p-2}u_m(x,t), -u_{mxx}(x,t)) = (h(x,t), -u_{mxx}(x,t))\\ \leq \|h(t)\|\|u_{mxx}(t)\|,\end{aligned} \tag{7.11}$$

and

$$\begin{aligned}\|u'_m(t)\|^2 + \nu(u_{mx}(x,t), u'_{mx}(x,t)) - \alpha(u_m(x,t), u'_m(x,t))\\ +(|u_m(x,t)|^{p-2}u_m(x,t), u'_m(x,t)) = (h(x,t), u'_m(x,t))\\ \leq \|h(t)\|\|u'_m(t)\|.\end{aligned} \tag{7.12}$$

We integrate both inequalities (7.11) and (7.12) over $(0,t)$. Then we obtain

$$\begin{aligned}\frac{1}{2}\|u_{mx}(t)\|^2 + \nu\int_0^t \|u_{mxx}(\tau)\|^2 d\tau - \alpha\int_0^t \|u_{mx}(\tau)\|^2 d\tau\\ - (p-1)\int_0^t \||u_m(\tau)|^{\frac{p-2}{2}}u_{mx}(\tau)\|^2 d\tau\\ \leq \frac{1}{2}\|u_{mx}(0)\|^2 + \int_0^t \|h(\tau)\|\|u_{mxx}(\tau)\| d\tau,\end{aligned} \tag{7.13}$$

and

$$\int_0^t \|u'_m(\tau)\|^2 d\tau + \frac{\nu}{2}\|u_{mx}(\tau)\|^2 - \frac{\alpha}{2}\|u_m(t)\|^2 + \frac{1}{p}\|u_m(t)\|^p_{L^p(0,L)}$$
$$\leq \frac{\alpha}{2}\|u_m(0)\|^2 + \frac{\nu}{2}\|u_{mx}(0)\|^2 + \frac{1}{p}\|u_m(0)\|^p_{L^p(0,L)} + \int_0^t \|h(\tau)\|\|u'_m(\tau)\|d\tau. \tag{7.14}$$

Hence,we obtain the following *a priori* estimates:

$$\begin{cases} u_m \text{ is uniformly bounded in } \ L^\infty(0,T;H_0^1 \cap L^p(0,L)), \\ u_m \text{ is uniformly bounded in } \ L^2(0,T;H_0^1(0,L)), \\ u'_m \text{ is uniformly bounded in } \ L^2(0,T;L^2(0,L)). \end{cases} \tag{7.15}$$

Therefore, by Theorem 1.4.7 and Theorem 1.4.6, there is a subsequence u_m, such that

- $u_m \to u$ weak* in $L^\infty(0,T;H_0^1 \cap L^p(0,L))$,
- $u_m \to u$ weakly in $L^2(0,T;H_0^1(0,L))$,
- $u'_m \to u'$ weakly in $L^2(0,T;L^2(0,L))$.

Thanks to Theorem 1.4.9, we can extract a subsequence of u_m, denoted by u_m such that

$$u_m \to u \ \ \textit{strongly in} \ \ \ L^2(0,T;H_0^1(0,L)).$$

By Lemma 1.4.10, we can extract a subsequence, denoted by u_m, such that u_m almost everywhere converges to u in domain $Q_T = [0,L] \times [0,T]$. Therefore, $|u_m|^{p-2}u_m \to |u|^{p-2}u$ almost everywhere in Q_T. Note that (7.15) implies that, for $p \geq 2$,

$$u_m \ \textit{is uniformly bounded in} \ L^{\frac{p}{p-1}}(0,L).$$

Thanks to Lemma 1.4.10, we obtain that

$$|u_m|^{p-2}u_m \to |u|^{p-2}u \ \ \textit{weakly in} \ \ L^{\frac{p}{p-1}}(0,T;L^{\frac{p}{p-1}}(0,L)).$$

From Lemma 1.4.11 and Remark 1.4.12 we get

$$u_m(0) = \sum_{j=1}^m \xi_{mj} w_j \to u(0) \ \ \textit{weakly in} \ \ L^2(0,L). \tag{7.16}$$

We know that $u_m(0) \to u(0)$ strongly in $H_0^1(0,L)$. Therefore, $u_m(0) \to u_0$ weakly in $L^2(0,L)$. Since the limit is unique, we obtain

$$u(0) = u_0(x).$$

Now, we pass to the limit in (7.4). Observe that each term in the left hand side of the equation (7.4) is weakly convergent in $L^{\frac{p}{p-1}}(0,T)$, for $p \geq 2$. Therefore, the following holds in $L^{\frac{p}{p-1}}(0,T)$.

$$\begin{aligned}(u_m'(x,t), w_j(x)) + \nu(u_{mx}(x,t), w_{jx}(x)) - \alpha(u_m(x,t), w_j(x))\\ = -(|u_m(x,t)|^{p-2}u_m(x,t), w_j(x)) + (h(x,t), w_j(x)), \quad \forall j = 1,2,....\end{aligned} \tag{7.17}$$

Since u',u_{xx} and f and belong to $L^2(0,T;L^2(0,L))$, $|u|^{p-2}u \in L^2(0,T;L^2(0,L))$ and (7) holds in $L^2(0,T;L^2(0,L))$. □

Uniqueness

Now, we will show the uniqueness of the solution of the problem (7.1)-(7.3).

Proof. Suppose that $u(x,t)$ and $v(x,t)$ are two solutions of the problem. Let $w(x,t) := u(x,t) - v(x,t)$. Thus, we have:

$$w_t(x,t) - \nu w_{xx}(x,t) - \alpha w(x,t) + |u(x,t)|^{p-2}u(x,t) - |v(x,t)|^{p-2}v(x,t) = 0, \tag{7.18}$$

$$w_x(0,t) = w_x(L,t) = 0, \qquad \forall t \geq 0, \tag{7.19}$$

$$w(x,0) = 0. \tag{7.20}$$

Let us multiply equation (7.18) by $w(x,t)$ and integrate with respect to x over $[0,L]$. Thus, we have:

$$\int_0^L [w_t(x,t)w(x,t) - \nu w_{xx}(x,t)w(x,t) - \alpha w^2(x,t)]dx + \int_0^L (|u(x,t)|^{p-2}u(x,t) - |v(x,t)|^{p-2}v(x,t))w(x,t)dx = 0.$$

Thanks to the integration by parts, boundary conditions and Monotonicity Inequality 1.3.9, we have:

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2 + \nu\|w_x(t)\|^2 + d_0\|w(t)\|^p_{L^p(0,L)} \leq \alpha\|w(t)\|^2,$$

where d_0 is a number. This implies

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2 \leq \alpha\|w(t)\|^2. \tag{7.21}$$

By applying Gronwall's Inequality (1.3.2) in inequality (7.21) we obtain:

$$\|w(t)\|^2 \leq e^{-2\alpha t}\|w(0)\|^2 = 0, \quad \forall t \in [0,T]. \tag{7.22}$$

Hence, this implies the uniqueness of the solution. Furthermore, note that inequality (7.22) implies the continuous dependence of the solutions of the problem on the initial data. $\square$

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