# OSCILLATION PROPERTIES OF STOPPED RANDOM WALKS ON INFINITE TREES

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By

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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# ABSTRACT

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We investigate a random walk on a branching tree with vertices labeled $1, 2, \ldots, r$ which terminates if two consecutive vertices of a walk are labeled with close numbers. It is proved that expected absorption times and absorption probabilities have surprising oscillating dependencies on starting states.

*Keywords:* branching random walk, absorption probabilities, expected absorption times.

# ÖZET

# SONSUZ AĞAÇLAR ÜZERİNDEKİ DURDURULAN RASSAL YÜRÜYÜŞLERİN SALINIM ÖZELLİKLERİ

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Düğümleri $1, 2, \ldots, r$ ile numaralandırılmış bir dallanma ağacı üzerinde ardışık iki düğümünün numaraları biribirine yakın olduğunda duran rassal yürüyüşleri inceliyoruz. Beklenen yutma zamanlarının ve yutma olasılıklarının başlangıç konumunlarına bağlılıklarının özellikleri araştırılıyor.

*Anahtar sözcükler*: dallanma rassal yürüyüşü, yutma olasılıkları, beklenen yutma zamanları.

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# Contents

# Chapter 1

# Introduction

In this thesis we investigate expected absorption times and absorption probabilities in branching random walk.

In Chapter 2, we introduce discrete time Markov chains and provide basic definitions and properties as the Markov property, $n$-step transition probabilities, expected absorption times and absorption probabilities, recurrence and transience.

In Chapter 3, we define a branching random walk starting at one of $r$ labeled vertices of a regular rooted tree where at each step of the walk we randomly jump to one of these labeled vertices of the next generation. The walk terminates if the difference of labels of two successive vertices is at most one. The aim of this chapter is to investigate the behaviour of expected absorption times. We prove that the expected absorption times have surprising oscillation properties.

In Chapter 4, we investigate absorption probabilities of the random walk defined in the previous chapter. We prove that the absorption probabilities have surprising oscillation properties.

In Chapter 5, we obtain the exact expressions of the expected absorption times. Then we define a random walk which terminates if the difference of labels of two consecutive vertices is at most $s$ and formulate hypothesises related to oscillating behaviour of expected absorption times and absorption probabilities.

Numerical results for both even and odd cases are given to support our hypothesises. In the end, we define a random walk by forbidding jumps stopping the walk. We observe that limiting probabilities have no oscillation behaviour in contrast to absorption probabilities.

# Chapter 2

# Fundamental Concepts

In this chapter we formulate basic properties of discrete time Markov chains [1], [2], [3].

**1.** Let $(\Omega, \Sigma, P)$ be a probability space and $(X_n)_{n\geq 0}$ be a sequence of random variables whose ranges are in a finite or countable set $X$. We start with the following definition:

**Definition 2.1.** The sequence $(X_n)_{n\geq 0}$ is called a *Markov chain* if for $n \geq 0$ and $i_0, \ldots, i_{n-1}, i, j$ in $X$

$$P\{X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\} = P\{X_{n+1} = j | X_n = i\} \quad (0.1)$$

under the assumption that $P\{X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\} > 0$.

By this definition we see that given the present and past values of $(X_n)_{n\geq 0}$, the probability of future behaviour is independent of the history of the chain. The property in (0.1) is referred as the *Markov property* of $(X_n)_{n\geq 0}$ and shows its memorylessness. But since the random variables $X_0, \ldots, X_n$ are dependent, the Markov property gives a model of dependence among random variables.

The set $X$ is called the *state space* or *phase space* of the chain. Probabilities

$$P\{X_{n+1} = j | X_n = i\} = p_{ij}^{n,n+1} \quad (0.2)$$

are called the *transition probabilities* from state $i$ to state $j$ and $p_{i_0} = P\{X_0 = i_0\}$ is the *initial probability distribution.*

If the transition probabilities do not change over time, i.e. $p_{ij}^{n,n+1} = p_{ij}$ then $(X_n)_{n\geq 0}$ is called a *homogeneous Markov chain.* If we arrange the numbers $p_{ij}$ in a matrix $\mathbb{P}$=$||p_{ij}||$ with $p_{ij} \geq 0$ and $\sum_j p_{ij} = 1$ for each $i$ in $X$, the corresponding matrix is called the *transition probability matrix* of the chain.

**Theorem 2.2.** *([1]) A discrete time stochastic process $(X_n)_{0\leq n\leq N}$ is called a Markov chain if and only if for all $i_0, \dots, i_N$ in X*

$$P\{X_N = i_N, \dots, X_1 = i_1, X_0 = i_0\} = p_{i_{N-1}i_N} \dots p_{i_1 i_2} p_{i_0 i_1} p_{i_0}. \tag{0.3}$$

By means of Theorem 2.2 we observe that the transition probabilities and the initial probability distribution completely specify the joint distribution of $X_0, \dots, X_n$ for $n = 0, \dots, N$.

**Example 2.3.** *Random walk without barriers.* Let $X = \mathbb{Z}$ and let $\zeta_1, \dots, \zeta_n$ be a sequence of independent Bernoulli random variables with $P\{\zeta_i = +1\} = p$ and $P\{\zeta_i = -1\} = q = 1 - p$. We define $S_n = \sum_{i=1}^n \zeta_i$ for $n \geq 1$ and $S_0 = 0$. Then the random walk $(S_n)_{n\geq 0}$ starting at $i = 0$ with transition probabilities $p_{ij} = p$ for $j = i + 1$ and $p_{ij} = q$ for $j = i - 1$ is a Markov chain since

$$\begin{aligned} P\{S_{n+1} = i_{n+1} | S_n = i_n, \dots, S_0 = i_0\} =& P\{S_n + \zeta_{n+1} = i_{n+1} | S_n = i_n, \dots, S_0 = i_0\} \\ =& P\{S_n + \zeta_{n+1} = i_{n+1} | S_n = i_n\} \\ =& P\{\zeta_{n+1} = i_{n+1} - i_n\}. \end{aligned}$$

**Example 2.4.** *Branching process.* Suppose that an organism produces a random number $\zeta$ of offspring and dies immediately. Assume that $P\{\zeta = i\} = p_i$ for $i = 0, 1, \dots$ with $\sum_i p_i = 1$. Let $X_n$ denote the size of the population at the $n - th$ generation and at the initial time there be just one organism i.e. $X_0 = 1$. We also suppose that all offspring are acting independently. If $\zeta_i^{(n-1)}$ denote the number of offspring produced by the $i - th$ organism at the $n - 1 - st$ generation then the number of offspring produced for the $n - th$ generation is

$$X_n = \zeta_1^{(n-1)} + \zeta_2^{(n-1)} + \dots + \zeta_{X_{n-1}}^{(n-1)}. \tag{0.4}$$

The process $(X_n)_{n\geq 1}$ defined by (0.4) is called a *branching process* and is actually a Markov chain since

$$\begin{aligned}P\{X_{n+1}=i_{n+1}|X_n=i_n,\dots,X_1=i_1\}=&P\{X_{n+1}=i_{n+1}|X_n=i_n\}\\=&P\{\zeta_1^{(n)}+\zeta_2^{(n)}+\cdots+\zeta_{i_n}^{(n)}=i_{n+1}\}.\end{aligned}$$

**2.** Let $(X_n)_{n\geq 0}$ be a homogeneous Markov chain with initial vector $\Pi$=$(p_i)$ and transition matrix $\mathbb{P}$=$||p_{ij}||$. Then for any $t\geq 0$ the probability that the process moves from state $i$ to state $j$ in $n$ transitions (or in $n$ steps) is

$$P\{X_{n+t}=j|X_t=i\}=p_{ij}^{(n)} \tag{0.5}$$

and the probability that finding process at point $j$ at time $n$ is

$$P\{X_n=j\}=p_j^{(n)}. \tag{0.6}$$

One can arrange the probabilities in (0.5) and (0.6) in matrices $\mathbb{P}^{(n)}$=$||p_{ij}^{(n)}||$ and $\Pi^{(n)}$=$||p_j^{(n)}||$, respectively.

It is a crucial fact that $p_{ij}^{(n)}$ satisfy the *Kolmogorov-Chapman equation*

$$p_{ij}^{(n+m)}=\sum_k p_{ik}^{(n)}p_{kj}^{(m)}. \tag{0.7}$$

Indeed, using the total probability law we get

$$\begin{aligned}p_{ij}^{(n+m)}=&P\{X_{n+m}=j|X_0=i\}=\sum_k P\{X_{n+m}=j,X_n=k|X_0=i\}\\=&\sum_k P\{X_{n+m}=j|X_n=k\}P\{X_n=k|X_0=i\}=\sum_k p_{ik}^{(n)}p_{kj}^{(m)}.\end{aligned}$$

By (0.7) we obtain the equations

$$p_{ij}^{(m+1)}=\sum_k p_{ik}p_{kj}^{(m)} \tag{0.8}$$

and

$$p_{ij}^{(n+1)}=\sum_k p_{ik}^{(n)}p_{kj} \tag{0.9}$$

called *backward* and *forward* equations, respectively.

Let us consider the unconditional probabilities in (0.6). By means of the method used to prove (0.7), we see that the probabilities $p_j^{(n)}$ satisfy the equation

$$p_j^{(n+m)} = \sum_k p_k^{(n)} p_{kj}^{(m)}. \tag{0.10}$$

Now we investigate the $n$-step transition probabilities by matrix analysis. For this purpose, we write (0.7) in matrix form

$$\mathbb{P}^{(n+m)} = \mathbb{P}^{(n)}\mathbb{P}^{(m)}. \tag{0.11}$$

Clearly, (0.8) and (0.9) have the matrix forms

$$\mathbb{P}^{(n+1)} = \mathbb{P}\mathbb{P}^{(n)} \quad and \quad \mathbb{P}^{(n+1)} = \mathbb{P}^{(n)}\mathbb{P},$$

respectively. Since $\mathbb{P}^{(1)} = \mathbb{P}$ it immediately follows that $\mathbb{P}^{(n)} = \mathbb{P}^n$. Thus we conclude that the $n$-step transition probabilities are the entries of the $n-th$ power of the probability matrix $\mathbb{P}$ for homogeneous Markov chains.

We now consider the law of $X_0$ i.e. $P\{X_0 = i\} = p_i$. The probability of finding the process in state $j$ at time $n$ in (0.6) actually equals the $j-th$ entry of the $n$-vector $\Pi\mathbb{P}^n$. In fact, by (0.10) we obtain

$$p_j^{(n)} = P\{X_n = j\} = \sum_k p_k p_{kj}^{(n)} = (\Pi\mathbb{P}^n)_j.$$

**3.** We are now passing to discuss the notion of expected absorption times and absorption probabilities. For this purpose, we start with introducing the class structure of Markov chains.

**Definition 2.5.** We say that $i$ *leads to* $j$ which is written $i \to j$ if for some $n \geq 0$

$$P\{X_n = j | X_0 = i\} > 0$$

and say that $i$ *communicates with* $j$ which is written $i \leftrightarrow j$ if $i \to j$ and $j \to i$.

It can be readily seen that $\leftrightarrow$ is an equivalence relation on $X$ and thus we can partition the states of the chain into *communicating classes*.

A communicating class $C$ is known to be a *closed class* if $i \in C$ and $i \to j$ then $j \in C$. It means that if you enter this class, you stay there forever. If $\{i\}$ is a closed class then $i$ is called an *absorbing state*. If the state space $X$ itself is a communicating class then the chain $(X_n)_{n\geq 0}$ is called *irreducible*.

**Definition 2.6.** Let $(X_n)_{n\geq 0}$ be a Markov chain with transition probability matrix $\mathbb{P}$ and $K$ be a closed communicating class for $X$. The *absorption time* of $K \subset X$ is the random variable $\tau^K : \Omega \to \{0, 1, \dots\} \cup \{\infty\}$ given by

$$\tau^K(\omega) = \inf\{n \geq 0 : X_n(\omega) \in K\} \tag{0.12}$$

where we agree that the infimum of empty set is infinity.

If $(X_n)_{n\geq 0}$ is initially in $i$, the probability that the process is absorbed in $K$ is called *absorption probability* and can be stated as

$$u_i^K = P\{\tau^K < \infty | X_0 = i\} = P_i\{\tau^K < \infty\}, \tag{0.13}$$

and the mean time until the process is absorbed in $K$ is called *expected absorption time* and can be stated as

$$v_i^K = E\{\tau^K | X_0 = i\} = E_i\{\tau^K\}. \tag{0.14}$$

By (0.13) we observe that if $X_0 = i \in K$, $\tau^K = 0$ and thus $u_i^K = 1$. But if we consider the case $X_0 = i \notin K$, we see that $\tau^K$ is at least one. Therefore, using the Markov property we get

$$P_i\{\tau^K < \infty | X_1 = j\} = P_j\{\tau^K < \infty\} = u_j^K.$$

Then the total probability law by summing (0.13) over $j$ with $X_1 = j$ simply gives

$$u_i^K = \sum_{j \in X} p_{ij} u_j^K. \tag{0.15}$$

We see that absorption probabilities satisfy above linear system of equations, but we have a more comprehensive result as follows:

**Theorem 2.7.** *([1]) Let $X$ be a countable set. Then the vector of absorption probabilities $u^K = (u_i^K : i \in X)$ is the minimal non-negative solution to the system of linear equations*

$$\begin{cases} u_i^K = 1, & i \in K \\ u_i^K = \sum_{j \in X} p_{ij} u_j^K, & i \notin K. \end{cases}$$

Similarly, by (0.14) we observe that if $X_0 = i \in K$, $\tau^K = 0$ and thus $v_i^K = 0$. But if we consider the case $X_0 = i \notin K$, we see that $\tau^K$ is at least one. Therefore, using the Markov property we get

$$E_i\{\tau^K | X_1 = j\} = 1 + E_j\{\tau^K\}.$$

Then summing (0.14) over $j$ with $X_1 = j$ simply gives

$$v_i^K = 1 + \sum_{j \notin K} p_{ij} v_j^K.$$

Now we have the following theorem that gives a general result for expected absorption times.

**Theorem 2.8.** *([1]) Let $X$ be a countable set. Then the vector of expected absorption times $v^K = (v_i^K : i \in X)$ is the minimal non-negative solution to the system of linear equations*

$$\begin{cases} v_i^K = 0, & i \in K \\ v_i^K = 1 + \sum_{j \notin K} p_{ij} v_j^K, & i \notin K. \end{cases}$$

**Remark 2.9.** In Theorem 2.7 and Theorem 2.8, we used the phrases absorption probabilities and expected absorption times instead of hitting probabilities and mean hitting times whom the author used in [1] since we assume that $K \subset X$ is a closed communicating class.

**Remark 2.10.** In Theorem 2.7 and Theorem 2.8, we have minimal non-negative solutions to the systems of linear equations. But if the state space $X$ is finite, the solutions of these linear systems are unique.

**4.** Now we discuss the classification of states of a Markov chain in terms of asymptotic properties of the probabilities $p_{ii}^{(n)}$ and $p_{ij}^{(n)}$. Let $(X_n)_{n\geq 0}$ be a Markov chain with the probability matrix $\mathbb{P}$. The *probability of first return to state $i$ at time $n$* is

$$f_{ii}^{(n)} = P\{X_n = i, X_k \neq i, 1 \leq k \leq n-1 | X_0 = i\}$$

and for $i \neq j$ the *probability of first arrival at state $j$ at time $n$* is

$$f_{ij}^{(n)} = P\{X_n = j, X_k \neq j, 1 \leq k \leq n-1 | X_0 = i\}.$$

Let $X_0 = i$ be given. The probability that we arrive at $j$ in $n$ transitions is

$$p_{ij}^{(n)} = \sum_{k=1}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)}. \tag{0.16}$$

Indeed, $T_j$ be the time of first arrival at $j$. Using the total probability law and the Markov property we get

$$\begin{aligned} p_{ij}^{(n)} &= \sum_{k=1}^{n} P\{X_n = j | T_j = k, X_0 = i\} P\{T_j = k | X_0 = i\} \\ &= \sum_{k=1}^{n} P\{X_n = j | X_k = j\} P\{T_j = k | X_0 = i\} \\ &= \sum_{k=1}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)}. \end{aligned}$$

If $j = i$ (0.16) becomes

$$p_{ii}^{(n)} = \sum_{k=1}^{n} f_{ii}^{(k)} p_{ii}^{(n-k)}$$

and represents the probability of return to state $i$ in $n$ transitions.

Now we define the *probability of a particle that leaves state $i$ will sooner or later return to the same state* as

$$f_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)}.$$

If $f_{ii} = 1$, the state $i$ is said to be *recurrent*, and if $f_{ii} < 1$, it is *transient*.

Let $\mu_i$ be the *average time of return* defined by

$$\mu_i = \sum_{n=1}^{\infty} n f_{ii}^{(n)}.$$

Then a recurrent state $i$ is *positive* if $\mu_i < \infty$, and it is a *null* state if $\mu_i = \infty$.

We give alternative characterizations for the recurrence property of states since calculating the functions $f_{ii}^{(n)}$ is very sophisticated.

**Theorem 2.11.** *([2]) (a) The state $i$ is recurrent if and only if*

$$\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty.$$

*(b) If state $j$ is recurrent and $i \leftrightarrow j$ then state $i$ is also recurrent.*

**Theorem 2.12.** *([2]) If state $j$ is transient then*

$$\sum_{n=1}^{\infty} p_{ij}^{(n)} < \infty$$

*for every $i$, and therefore $p_{ij}^{(n)} \to 0, \quad n \to \infty$.*

We say that state $j$ has *period* $d = d(j)$ if $p_{jj}^{(n)} > 0$ only for the values of $n$ of the form $dm$ where $d$ is the largest number satisfying this property. It means that $d$ is the greatest common divisor of the numbers $n$ such that $p_{jj}^{(n)} > 0$. If $d(j) = 1$ then the state $j$ is known as *aperiodic*.

Now we consider recurrent states as follows:

**Theorem 2.13.** *([2])Let $j$ be a recurrent state with $d(j) = 1$.*

*(a) If $i$ communicates with $j$, then*

$$p_{ij}^{(n)} \to \frac{1}{\mu_j}, \quad n \to \infty.$$

*If in addition $j$ is a positive state then*

$$p_{ij}^{(n)} \to \frac{1}{\mu_j} > 0, \quad n \to \infty.$$

*If, however, $j$ is a null state, then*

$$p_{ij}^{(n)} \to 0, \quad n \to \infty.$$

*(b) If $i$ and $j$ belong to different communicating classes, then*

$$p_{ij}^{(n)} \to \frac{f_{ij}}{\mu_j}, \quad n \to \infty.$$

Let $d \geq 1$ be the period of an irreducible Markov chain $(X_n)_{n\geq 0}$. The chain may have a special structure concerning transitions from one group of states to another. Let $i_0$ be a state in $X$, the class $X$ is divided into subclasses in the following sense:

$$\begin{aligned} C_0 &= \{j \in X : p_{i_0 j}^{(n)} > 0 \Rightarrow n \equiv 0 \pmod{d}\}, \\ C_1 &= \{j \in X : p_{i_0 j}^{(n)} > 0 \Rightarrow n \equiv 1 \pmod{d}\}, \\ &\vdots \\ C_{d-1} &= \{j \in X : p_{i_0 j}^{(n)} > 0 \Rightarrow n \equiv d-1 \pmod{d}\}. \end{aligned} \tag{0.17}$$

Clearly $X = C_0 + C_1 + \cdots + C_{d-1}$. We observe that after one step from $C_0$ the particle enters $C_1$, it enters $C_2$ in the next transition and so on. It returns to a given set of states after $d$ transitions. This is called *cyclic property* associated with the states. Indeed, let $i \in C_p$ and $p_{ij} > 0$. Let $n$ be such that $p_{i_0 i}^{(n)} > 0$. Since we have $n = md + p$ i.e. $n \equiv p \pmod{d}$, $n + 1 \equiv p + 1 \pmod{d}$. Therefore $p_{i_0 j}^{(n+1)} > 0$ and $j \in C_{p+1}$.

**Theorem 2.14.** *([2]) Let $j$ be a recurrent state and let $d(j) > 1$.*

*(a) If $i$ and $j$ belong to the same class, and if $i$ belongs to cyclic subclass $C_r$ and $j$ to $C_{r+a}$, then*

$$p_{ij}^{(nd+a)} \to \frac{d}{\mu_j}.$$

*(b) With an arbitrary $i$,*

$$p_{ij}^{(nd+a)} \to \left(\sum_{r=0}^{\infty} f_{ij}^{(rd+a)}\right) \frac{d}{\mu_j}, \quad a = 0, 1, \ldots, d-1.$$

**5.** Let $(X_n)_{n\geq 0}$ be a Markov chain with a finite state space $X = \{1, \ldots, N\}$ and probability transition matrix $\mathbb{P}$. If we suppose $\mathbb{P}^n > 0$ for some $n$ then the corresponding matrix is called *regular*. The most important fact about a regular Markov chain is existence of the limiting distribution $\pi = (\pi_1, \ldots, \pi_N)$, where $\pi_j > 0$ for each $j \in X$ and $\sum_j \pi_j = 1$, and is independent of the initial state $i$, such a distribution is called *ergodic*.

Formally, for $j = 1, \ldots, N$ we have

$$\lim_{n\to\infty} P\{X_n = j|X_0 = i\} = \lim_{n\to\infty} p_{ij}^{(n)} = \pi_j.$$

It means that in the long run the probability that we find the process in state $j$ is approximately $\pi_j$ no matter which value the chain had initially.

Now we have the following theorem, called Ergodic Theorem, explains the long run behaviour of Markov chains.

**Theorem 2.15.** *([2]) Let $\mathbb{P}$=$||p_{ij}||$ be the transition matrix of a chain with a finite state space $X = \{1, \ldots, N\}$.*

*(a) There is an $n_0$ such that*

$$\min_{i,j} p_{ij}^{(n_0)} > 0$$

*if and only if there are numbers $\pi_1, \ldots, \pi_N$ such that*

$$\pi_j > 0, \quad \sum_j \pi_j = 1$$

*and*

$$p_{ij}^{(n)} \to \pi_j, \quad n \to \infty$$

*for every $i \in X$.*

*(b) The numbers $(\pi_1, \ldots, \pi_N)$ satisfy the equations*

$$\pi_j = \sum_k \pi_k p_{kj}, \quad j = 1, \ldots, N.$$

# Chapter 3

# Oscillation Properties of Expected Absorption Times

In this chapter we give a definition of a branching random walk and explore expected times until the walk is stopped. Our aim is to prove the oscillation behaviour of expected absorption times.

## 3.1 Formulation of Results

We consider a random walk on $I(r) = \{1, 2, \ldots, r\}$ where the probability to jump from any point to any point is $\frac{1}{r}$. Let $(S_n^i)_{n \in Z_{\geq 0}}$ be the random walk starting at point $i \in I(r) : S_0^i = i$. The walk terminates at $l - th$ step if its $l - th$ move length is 0 or 1: $S_{l-1}^i = j$ and $S_l^i \in \{j-1, j, j+1\}$ (we adopt $1 - 1 \equiv 1$ and $r + 1 \equiv r$). Actually $(S_n^i)_{n \in Z_{\geq 0}}$ can be treated as a branching random walk on a regular rooted tree, where at each step we randomly choose one offspring out of $r$ labeled offsprings and the walk terminates if the difference between labels of two consecutively chosen offsprings is at most one. Branching random walks are widely explored [4], the walk when the number of unlabeled offsprings at each step is random is a well-known Galton-Watson tree [5], [6], [7].

Let the random variable $X(i)$ be the time when the random walk $(S_n^i)_{n\in Z_{\geq 0}}$ terminates and $E_i$ be the expectation of $X(i)$. We will investigate the behaviour of expected absorption times $E_i$ as a function of $i$. Since by symmetry $E_i = E_{r-i+1}$ for $i = 1, \dots, \lfloor \frac{r}{2} \rfloor$, we will investigate $E_1, \dots, E_k$ for $r = 2k$ and $E_1, \dots, E_{k+1}$ for $r = 2k+1$. It can be readily shown that $E_1 > E_2$, one could expect similar monotonicity relationships $E_i > E_{i+1}$ for $i = 1, 2, \dots \lceil \frac{r}{2} \rceil$ between other expectations. On the contrary, it turns out that $E_2$ is the smallest one among all expectations and there is the following oscillating hierarchy between expected absorption times:

**Theorem 3.1.** *Expected absorption times $E_i$, $i = 1, 2, \dots, r$ satisfy the following inequalities:*

- $r = 2k = 4l$

$$E_1 > E_3 > \cdots > E_{k-3} > E_{k-1} > E_k > E_{k-2} > \cdots > E_4 > E_2. \tag{1.1}$$

- $r = 2k = 4l + 2$

$$E_1 > E_3 > \cdots > E_{k-4} > E_{k-2} > E_k > E_{k-1} > E_{k-3} > \cdots > E_4 > E_2. \tag{1.2}$$

- $r = 2k + 1 = 4l + 1$

$$E_1 > E_3 > \cdots > E_{k-1} > E_{k+1} > E_k > E_{k-2} > \cdots > E_4 > E_2. \tag{1.3}$$

- $r = 2k + 1 = 4l + 3$

$$E_1 > E_3 > \cdots > E_{k-4} > E_{k-2} > E_k > E_{k+1} > E_{k-1} > \cdots > E_4 > E_2. \tag{1.4}$$

## 3.2 Proof of Oscillation Properties

It can be readily shown that the expectations $E_i, 1 \leq i \leq r$ satisfy the following system of $r$ linear equations with $r$ unknowns:

$$\begin{aligned}
E_1 &= \frac{1}{r} + \frac{1}{r} + \sum_{j=3}^{r} \frac{1}{r}(1 + E_j) \\
E_2 &= \frac{1}{r} + \frac{1}{r} + \frac{1}{r} + \sum_{j=4}^{r} \frac{1}{r}(1 + E_j) \\
E_i &= \sum_{j=1}^{i-2} \frac{1}{r}(1 + E_j) + \frac{1}{r} + \frac{1}{r} + \frac{1}{r} + \sum_{j=i+2}^{r} \frac{1}{r}(1 + E_j) \quad if \quad 3 \leq i \leq r-2 \qquad (2.1) \\
E_{r-1} &= \sum_{j=1}^{r-3} \frac{1}{r}(1 + E_j) + \frac{1}{r} + \frac{1}{r} + \frac{1}{r} \\
E_r &= \sum_{j=1}^{r-2} \frac{1}{r}(1 + E_j) + \frac{1}{r} + \frac{1}{r}.
\end{aligned}$$

If $r$ is even then by symmetry $E_i = E_{r-i+1}$ for $i = 1, \ldots, \frac{r}{2}$ and if $r$ is odd then by symmetry $E_i = E_{r-i+1}$ for $i = 1, \ldots, \frac{r-1}{2}$. Therefore, we can reduce the number of equations and unknowns to $\frac{r}{2}$ and $\frac{r+1}{2}$ in even and odd cases, respectively. Let us start with the even case $r = 2k$. After reducing the number of unknowns to $\frac{r}{2} = k$, we get the following linear system of $k$ equations:

$$\begin{aligned}
&(r-1)E_1 - E_2 - 2\sum_{j=3}^{k} E_j = r \\
&-E_1 + (r-1)E_2 - E_3 - 2\sum_{j=4}^{k} E_j = r \\
&-2\sum_{j=1}^{i-2} E_j - E_{i-1} + (r-1)E_i - E_{i+1} - 2\sum_{j=i+2}^{k} E_j = r \quad if \quad 3 \leq i \leq k-2 \qquad (2.2) \\
&-2\sum_{j=1}^{k-3} E_j - E_{k-2} + (r-1)E_{k-1} - E_k = r \\
&-2\sum_{j=1}^{k-2} E_j - E_{k-1} + rE_k = r.
\end{aligned}$$

Let us take the difference of each equation in the system (2.2) with the subsequent one and get the system of $k-1$ equations:

$$\begin{aligned} & r(E_1 - E_2) = E_3 \\ & r(E_i - E_{i+1}) = E_{i+2} - E_{i-1} \quad if \quad 2 \leq i \leq k-2 \\ & (r+1)(E_{k-1} - E_k) = E_{k-1} - E_{k-2}. \end{aligned} \tag{2.3}$$

The system (2.2) also yields the following system of $k-2$ equations:

$$\begin{aligned} & (r+1)(E_{k-2} - E_k) = E_k - E_{k-3} \\ & (r+1)(E_{k-3} - E_k) = E_k - E_{k-2} + E_{k-1} - E_{k-4} \\ & (r+1)(E_i - E_{i+3}) = E_{i+4} - E_{i+1} + E_{i+2} - E_{i-1} \quad if \quad 2 \leq i \leq k-4 \\ & (r+1)(E_1 - E_4) = E_5 - E_2 + E_3. \end{aligned} \tag{2.4}$$

where the first equation is obtained by considering the difference of $k-2-nd$ and $k-th$ equations of (2.2), the second equation is obtained by considering the difference of $k-3-rd$ and $k-th$ equations of (2.2), and for $j = 3, \ldots, k-2$ the $j-th$ equation is obtained by considering the difference of $k-j-1-st$ and $k-j+2-nd$ equations of (2.2).

Let us define numbers $a_i$, $i = 0, 1, \ldots, k-3$ by $a_0 = \frac{1}{r+1}$ and $a_i = (r+1-a_{i-1})^{-1}$ for each integer $1 \leq i \leq k-3$. Then the system (2.4) yields the following system of $k-2$ equations:

$$\begin{aligned} & E_{k-2} - E_k = a_0(E_k - E_{k-3}) \\ & E_{k-2-i} - E_{k+1-i} = a_i(E_{k-i} - E_{k-3-i}) \quad if \quad 1 \leq i \leq k-4 \\ & E_1 - E_4 = a_{k-3}E_3. \end{aligned} \tag{2.5}$$

The first equations of (2.4) and (2.5) coincide and for $j = 2, \ldots, k-2$ the $j-th$ equation of (2.5) is obtained from $j-th$ and $j-1-th$ equations of (2.4).

The systems (2.3) and (2.5) readily yield:

$$E_1 - E_2 = \frac{E_3}{r} \quad and$$

$$
\begin{aligned}
E_2 - E_3 &= \frac{E_4 - E_1}{r} = -\frac{a_{k-3}E_3}{r} \\
E_3 - E_4 &= \frac{E_5 - E_2}{r} = -\frac{a_{k-4}(E_4 - E_1)}{r} = \frac{a_{k-4}a_{k-3}E_3}{r} \\
&\vdots \\
E_{k-2} - E_{k-1} &= \frac{E_k - E_{k-3}}{r} = (-1)^{k-1}\frac{E_3 \prod_{n=1}^{k-3} a_n}{r} \\
E_{k-1} - E_k &= \frac{E_{k-1} - E_{k-2}}{r+1} = a_0(E_{k-1} - E_{k-2}) = (-1)^k \frac{E_3 \prod_{n=0}^{k-3} a_n}{r}.
\end{aligned} \tag{2.6}
$$

Thus, for $2 \le i \le k-1$ we have

$$
E_i - E_{i+1} = (-1)^{i+1}\frac{E_3 \prod_{n=k-i-1}^{k-3} a_n}{r}. \tag{2.7}
$$

Now since $0 < a_n < 1$ and $E_1 - E_2 = E_3/r$ the relationships (1.1) and (1.2) readily follow from (2.7).

Let us consider the odd case $r = 2k+1$. Reducing the number of unknowns $r$ to $\frac{r+1}{2} = k+1$, one can get the following linear system of $k+1$ equations:

$$
\begin{aligned}
&(r-1)E_1 - E_2 - 2\sum_{j=3}^{k} E_j - E_{k+1} = r \\
&-E_1 + (r-1)E_2 - E_3 - 2\sum_{j=4}^{k} E_j - E_{k+1} = r \\
&if \quad 3 \le i \le k-2 \\
&-2\sum_{j=1}^{i-2} E_j - E_{i-1} + (r-1)E_i - E_{i+1} - 2\sum_{j=i+2}^{k} E_j - E_{k+1} = r \\
&-2\sum_{j=1}^{k-3} E_j - E_{k-2} + (r-1)E_{k-1} - E_k - E_{k+1} = r \\
&-2\sum_{j=1}^{k-2} E_j - E_{k-1} + (r-1)E_k = r \\
&-2\sum_{j=1}^{k-1} E_j + rE_{k+1} = r.
\end{aligned} \tag{2.8}
$$

Let us take the difference of each equation in the system (2.8) with the subsequent one and get the following system of $k$ equations:

$$\begin{aligned}
&r(E_1 - E_2) = E_3 \\
&r(E_i - E_{i+1}) = E_{i+2} - E_{i-1} \quad if \quad 2 \le i \le k-1 \\
&r(E_k - E_{k+1}) = E_k - E_{k-1}.
\end{aligned} \tag{2.9}$$

The system (2.8) also yields the following system of $k-1$ equations:

$$\begin{aligned}
&r(E_{k-1} - E_k) = E_{k+1} - E_{k-2} \\
&(r+1)(E_{k-2} - E_{k+1}) = E_k - E_{k-1} + E_k - E_{k-3} \\
&(r+1)(E_i - E_{i+3}) = E_{i+4} - E_{i+1} + E_{i+2} - E_{i-1} \quad if \quad 2 \le i \le k-3 \\
&(r+1)(E_1 - E_4) = E_5 - E_2 + E_3.
\end{aligned} \tag{2.10}$$

where the first equation is obtained by considering the difference of $k-1-st$ and $k-th$ equations of (2.8), the second equation is obtained by considering the difference of $k-2-nd$ and $k+1-st$ equations of (2.8), and for $j = 3, \ldots, k-1$ the $j-th$ equation is obtained by considering the difference of $k-j-th$ and $k-j+3-rd$ equations of (2.8).

Let us define numbers $b_i$, $i = 0, 1, \ldots, k-2$ by $b_0 = \frac{1}{r}$ and $b_i = (r+1-b_{i-1})^{-1}$ for each integer $1 \le i \le k-2$. Then the system (2.10) yields the following system of $k-1$ equations:

$$\begin{aligned}
&E_{k-1} - E_k = b_0(E_{k+1} - E_{k-2}) \\
&E_{k-1-i} - E_{k+2-i} = b_i(E_{k+1-i} - E_{k-2-i}) \quad if \quad 1 \le i \le k-3 \\
&E_1 - E_4 = b_{k-2}E_3.
\end{aligned} \tag{2.11}$$

The first equations of (2.10) and (2.11) coincide and for $j = 2, \ldots, k-1$ the $j-th$ equation of (2.11) is obtained from $j-th$ and $j-1-th$ equations of (2.10).

The systems (2.9) and (2.11) readily yield:

$$E_1 - E_2 = \frac{E_3}{r} \qquad and$$

$$
\begin{aligned}
E_2 - E_3 &= \frac{E_4 - E_1}{r} = -\frac{b_{k-2}E_3}{r} \\
E_3 - E_4 &= \frac{E_5 - E_2}{r} = -\frac{b_{k-3}(E_4 - E_1)}{r} = \frac{b_{k-3}b_{k-2}E_3}{r} \\
&\ \ \vdots \\
E_{k-1} - E_k &= \frac{E_{k+1} - E_{k-2}}{r} = (-1)^k \frac{E_3 \prod_{n=1}^{k-2} b_n}{r} \\
E_k - E_{k+1} &= \frac{E_k - E_{k-1}}{r} = b_0(E_k - E_{k-1}) = (-1)^{k+1} \frac{E_3 \prod_{n=0}^{k-2} b_n}{r}.
\end{aligned} \tag{2.12}
$$

Thus, for $2 \leq i \leq k$ we have

$$
E_i - E_{i+1} = (-1)^{i+1} \frac{E_3 \prod_{n=k-i}^{k-2} b_n}{r}. \tag{2.13}
$$

Now since $0 < b_n < 1$ and $E_1 - E_2 = E_3/r$ the relationships (1.3) and (1.4) readily follow from (2.13). The proof of Theorem 3.1 is completed.

# Chapter 4

# Oscillation Properties of Absorption Probabilities

In this chapter we formulate and prove the oscillation properties of absorption probabilities.

## 4.1 Formulation of Results

We consider a random walk defined in Section 3.1. Let the random variable $X(i)$ be the time when the random walk $(S_n^i)_{n\in Z_{\geq 0}}$ terminates and let $p_{i,j}$ be the probability that the random walk $(S_n^i)_{n\in Z_{\geq 0}}$ starting at point $i$ terminates at point $j$. Then absorption probabilities $P_j = \frac{1}{r}\sum_{i=1}^{r} p_{i,j}$ for $1 \leq j \leq r$ under natural assumption that the starting state $i \in I(r)$ of the walk is chosen with probability $\frac{1}{r}$.

We will investigate the behaviour of absorption probabilities $P_j$ as a function of $j$. Since by symmetry $P_j = P_{r-j+1}$ for $j = 1, \dots, \lfloor \frac{r}{2} \rfloor$ we will investigate $P_1, \dots, P_k$ if $r = 2k$ and $P_1, \dots, P_{k+1}$ if $r = 2k+1$. It can be readily shown that $P_1 < P_2$, one could expect similar monotonicity relationship $P_j < P_{j+1}$ for $j = 1, 2, \dots \lceil \frac{r}{2} \rceil$.

On the contrary, it turns out that $P_2$ is the greatest one among all probabilities and there is the following oscillating hierarchy between absorption probabilities:

**Theorem 4.1.** *Absorption probabilities satisfy the following inequalities:*

- $r = 2k = 4l$

$$P_1 < P_3 < \cdots < P_{k-3} < P_{k-1} < P_k < P_{k-2} < \cdots < P_4 < P_2. \tag{1.1}$$

- $r = 2k = 4l + 2$

$$P_1 < P_3 < \cdots < P_{k-4} < P_{k-2} < P_k < P_{k-1} < P_{k-3} < \cdots < P_4 < P_2. \tag{1.2}$$

- $r = 2k + 1 = 4l + 1$

$$P_1 < P_3 < \cdots < P_{k-1} < P_{k+1} < P_k < P_{k-2} < \cdots < P_4 < P_2. \tag{1.3}$$

- $r = 2k + 1 = 4l + 3$

$$P_1 < P_3 < \cdots < P_{k-4} < P_{k-2} < P_k < P_{k+1} < P_{k-1} < \cdots < P_4 < P_2. \tag{1.4}$$

## 4.2 Proof of Oscillation Properties

We start with the even case $r = 2k$. For each fixed $j = 1, \ldots, k$, the absorption probability $P_j = \frac{1}{r}\sum_{i=1}^{r} p_{i,j}$. In order to evaluate $p_{i,j}$ we fix $j$ and write the system of equations $p_{i,j}$, $i = 1, \ldots, r$. Since these systems have different forms for $j = 1, 2, 3, 4$ and $j \geq 5$ there are five different equation systems.

The system for evaluation related to the probability that the walk terminates at $j = 1$ is:

$$
\begin{aligned}
p_{i,1} &= \frac{1}{r} + \sum_{n=i+2}^{r} \frac{1}{r} p_{n,1} \quad if \quad 1 \leq i \leq 2 \\
p_{i,1} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,1} + \sum_{n=i+2}^{r} \frac{1}{r} p_{n,1} \quad if \quad 3 \leq i \leq r-2 \\
p_{i,1} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,1} \quad if \quad r-1 \leq i \leq r.
\end{aligned}
\tag{2.1}
$$

The system for evaluation related to the probability that the walk terminates at $j = 2$ is:

$$
\begin{aligned}
p_{i,2} &= \frac{1}{r} + \sum_{n=i+2}^{r} \frac{1}{r} p_{n,2} \quad if \quad 1 \leq i \leq 2 \\
p_{3,2} &= \frac{1}{r} + \frac{1}{r} p_{1,2} + \sum_{n=5}^{r} \frac{1}{r} p_{n,2} \\
p_{i,2} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,2} + \sum_{n=i+2}^{r} \frac{1}{r} p_{n,2} \quad if \quad 4 \leq i \leq r-2 \\
p_{i,2} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,2} \quad if \quad r-1 \leq i \leq r.
\end{aligned}
\tag{2.2}
$$

The system for evaluation related to the probability that the walk terminates at $j = 3$ is:

$$
\begin{aligned}
p_{1,3} &= \sum_{n=3}^{r} \frac{1}{r} p_{n,3} \\
p_{2,3} &= \frac{1}{r} + \sum_{n=4}^{r} \frac{1}{r} p_{n,3} \\
p_{i,3} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,3} + \frac{1}{r} + \sum_{n=i+2}^{r} \frac{1}{r} p_{n,3} \quad if \quad 3 \leq i \leq 4 \\
p_{i,3} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,3} + \sum_{n=i+2}^{r} \frac{1}{r} p_{n,3} \quad if \quad 5 \leq i \leq r-2 \\
p_{i,3} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,3} \quad if \quad r-1 \leq i \leq r.
\end{aligned}
\tag{2.3}
$$

The system for evaluation related to the probability that the walk terminates at $j = 4$ is:

$$
\begin{aligned}
p_{i,4} &= \sum_{n=i+2}^{r} \frac{1}{r} p_{n,4} \quad if \quad 1 \le i \le 2 \\
p_{i,4} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,4} + \frac{1}{r} + \sum_{n=i+2}^{r} \frac{1}{r} p_{n,4} \quad if \quad 3 \le i \le 5 \\
p_{i,4} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,4} + \sum_{n=i+2}^{r} \frac{1}{r} p_{n,4} \quad if \quad 6 \le i \le r-2 \\
p_{i,4} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,4} \quad if \quad r-1 \le i \le r.
\end{aligned}
\tag{2.4}
$$

The system for evaluation related to the probability that the walk terminates at $j = m$ for $5 \le m \le k$ is:

$$
\begin{aligned}
p_{i,m} &= \sum_{n=i+2}^{r} \frac{1}{r} p_{n,m} \quad if \quad 1 \le i \le 2 \\
p_{i,m} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,m} + \sum_{n=i+2}^{r} \frac{1}{r} p_{n,m} \quad if \quad 3 \le i \le m-2 \\
p_{i,m} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,m} + \frac{1}{r} + \sum_{n=i+2}^{r} \frac{1}{r} p_{n,m} \quad if \quad m-1 \le i \le m+1 \\
p_{i,m} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,m} + \sum_{n=i+2}^{r} \frac{1}{r} p_{n,m} \quad if \quad m+2 \le i \le r-2 \\
p_{i,m} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,m} \quad if \quad r-1 \le i \le r.
\end{aligned}
\tag{2.5}
$$

Let $q_j = \sum_{i=1}^{r} p_{i,j}$, where $1 \le j \le k$.

The side by side summation of the system (2.1) yields

$$
2(p_{1,1} + p_{r,1}) + 3 \sum_{i=2}^{r-1} p_{i,1} = 2. \ \ Thus, \quad q_1 = \frac{2 + p_{1,1} + p_{r,1}}{3}.
$$

Similarly, by side by side summation of the systems (2.2)-(2.5) we get

$$
q_j = \frac{3 + p_{1,j} + p_{r,j}}{3}
$$

for $2 \le j \le k$.

The system (2.1) readily yields:

$$\begin{aligned} p_{1,1} &= \frac{1}{r+1} + \frac{1}{r+1}(q_1 - p_{2,1}) \\ p_{2,1} &= \frac{1}{r+1} + \frac{1}{r+1}(q_1 - p_{1,1} - p_{3,1}) \\ p_{i,1} &= \frac{1}{r+1}(q_1 - p_{i-1,1} - p_{i+1,1}) \;\; if \;\; 3 \leq i \leq r-1 \\ p_{r,1} &= \frac{1}{r+1}(q_1 - p_{r-1,1}). \end{aligned} \tag{2.6}$$

The system (2.2) readily yields:

$$\begin{aligned} p_{1,2} &= \frac{1}{r+1} + \frac{1}{r+1}(q_2 - p_{2,2}) \\ p_{i,2} &= \frac{1}{r+1} + \frac{1}{r+1}(q_2 - p_{i-1,2} - p_{i+1,2}) \;\; if \;\; 2 \leq i \leq 3 \\ p_{i,2} &= \frac{1}{r+1}(q_2 - p_{i-1,2} - p_{i+1,2}) \;\; if \;\; 4 \leq i \leq r-1 \\ p_{r,2} &= \frac{1}{r+1}(q_2 - p_{r-1,2}). \end{aligned} \tag{2.7}$$

Similarly, the systems (2.3)-(2.5) readily yield:

$$\begin{aligned} p_{1,m} &= \frac{1}{r+1}(q_m - p_{2,m}) \\ p_{i,m} &= \frac{1}{r+1}(q_m - p_{i-1,m} - p_{i+1,m}) \;\; if \;\; 2 \leq i \leq m-2 \\ p_{i,m} &= \frac{1}{r+1} + \frac{1}{r+1}(q_m - p_{i-1,m} - p_{i+1,m}) \;\; if \;\; m-1 \leq i \leq m+1 \\ p_{i,m} &= \frac{1}{r+1}(q_m - p_{i-1,m} - p_{i+1,m}) \;\; if \;\; m+2 \leq i \leq r-1 \\ p_{r,m} &= \frac{1}{r+1}(q_m - p_{r-1,m}). \end{aligned} \tag{2.8}$$

Now we wish to express $p_{1,j}$ and $p_{r,j}$ in terms of $q_j$ for $j = 1, \ldots, k$:

Let us define numbers $c_i$, $i = 0, 1, \ldots, r-1$ by

$$c_1 = 1 - \frac{1}{(r+1)^2} \;\; and \;\; c_i = 1 - \frac{1}{c_{i-1}(r+1)^2}$$

for each integer $2 \leq i \leq r-1$.

Then the system (2.6) yields the following two relationships:

$$p_{1,1} = \left( \sum_{l=1}^{r-1} \frac{(-1)^{l+1}}{(r+1)^{r-l} \prod_{m=l}^{r-1} c_m} - \frac{1}{(r+1)^r \prod_{m=1}^{r-1} c_m} \right) q_1 + \frac{1}{(r+1)c_{r-1}}$$
$$- \frac{1}{(r+1)^2 c_{r-2} c_{r-1}},$$
$$p_{r,1} = \left( \sum_{l=1}^{r-1} \frac{(-1)^{l+1}}{(r+1)^{r-l} \prod_{m=l}^{r-1} c_m} - \frac{1}{(r+1)^r \prod_{m=1}^{r-1} c_m} \right) q_1 + \frac{1}{(r+1)^{r-1} \prod_{m=1}^{r-1} c_m}$$
$$- \frac{1}{(r+1)^r \prod_{m=1}^{r-1} c_m}.$$

Similarly, the system (2.7) yields the following two relationships:

$$p_{1,2} = \left( \sum_{l=1}^{r-1} \frac{(-1)^{l+1}}{(r+1)^{r-l} \prod_{m=l}^{r-1} c_m} - \frac{1}{(r+1)^r \prod_{m=1}^{r-1} c_m} \right) q_2 + \frac{1}{(r+1)c_{r-1}}$$
$$- \frac{1}{(r+1)^2 c_{r-2} c_{r-1}} + \frac{1}{(r+1)^3 \prod_{m=r-3}^{r-1} c_m},$$
$$p_{r,2} = \left( \sum_{l=1}^{r-1} \frac{(-1)^{l+1}}{(r+1)^{r-l} \prod_{m=l}^{r-1} c_m} - \frac{1}{(r+1)^r \prod_{m=1}^{r-1} c_m} \right) q_2 - \frac{1}{(r+1)^{r-2} \prod_{m=2}^{r-1} c_m}$$
$$+ \frac{1}{(r+1)^{r-1} \prod_{m=1}^{r-1} c_m} - \frac{1}{(r+1)^r \prod_{m=1}^{r-1} c_m}.$$

Finally, for $3 \leq j \leq k$, the system (2.8) yields the following two relationships:

$$p_{1,j} = \left( \sum_{l=1}^{r-1} \frac{(-1)^{l+1}}{(r+1)^{r-l} \prod_{m=l}^{r-1} c_m} - \frac{1}{(r+1)^r \prod_{m=1}^{r-1} c_m} \right) q_j + \frac{(-1)^j}{(r+1)^{j-1} \prod_{m=r-j+1}^{r-1} c_m}$$
$$- \frac{(-1)^j}{(r+1)^j \prod_{m=r-j}^{r-1} c_m} + \frac{(-1)^j}{(r+1)^{j+1} \prod_{m=r-j-1}^{r-1} c_m},$$
$$p_{r,j} = \left( \sum_{l=1}^{r-1} \frac{(-1)^{l+1}}{(r+1)^{r-l} \prod_{m=l}^{r-1} c_m} - \frac{1}{(r+1)^r \prod_{m=1}^{r-1} c_m} \right) q_j + \frac{(-1)^{j+1}}{(r+1)^{r-j} \prod_{m=j}^{r-1} c_m}$$
$$- \frac{(-1)^{j+1}}{(r+1)^{r-j+1} \prod_{m=j-1}^{r-1} c_m} + \frac{(-1)^{j+1}}{(r+1)^{r-j+2} \prod_{m=j-2}^{r-1} c_m}.$$

Summarizing last six relationships we get that for $j = 1, \ldots, k$, $p_{1,j}$ and $p_{r,j}$ have the following form:

$$p_{1,j} = f(r)q_j + g(r,j),$$
$$p_{r,j} = f(r)q_j + h(r,j)$$

where

$$f(r) = \sum_{l=1}^{r-1} \frac{(-1)^{l+1}}{(r+1)^{r-l}\prod_{m=l}^{r-1} c_m} - \frac{1}{(r+1)^r \prod_{m=1}^{r-1} c_m},$$
$$g(r,1) = \frac{1}{(r+1)c_{r-1}} - \frac{1}{(r+1)^2 c_{r-2}c_{r-1}},$$
$$if \quad 2 \le j \le k$$
$$g(r,j) = \frac{(-1)^j}{(r+1)^{j-1}\prod_{m=r-j+1}^{r-1} c_m} - \frac{(-1)^j}{(r+1)^j \prod_{m=r-j}^{r-1} c_m} + \frac{(-1)^j}{(r+1)^{j+1}\prod_{m=r-j-1}^{r-1} c_m},$$
$$h(r,1) = \frac{1}{(r+1)^{r-1}\prod_{m=1}^{r-1} c_m} - \frac{1}{(r+1)^r \prod_{m=1}^{r-1} c_m},$$
$$h(r,2) = -\frac{1}{(r+1)^{r-2}\prod_{m=2}^{r-1} c_m} + \frac{1}{(r+1)^{r-1}\prod_{m=1}^{r-1} c_m} - \frac{1}{(r+1)^r \prod_{m=1}^{r-1} c_m},$$
$$if \quad 3 \le j \le k$$
$$h(r,j) = \frac{(-1)^{j+1}}{(r+1)^{r-j}\prod_{m=j}^{r-1} c_m} - \frac{(-1)^{j+1}}{(r+1)^{r-j+1}\prod_{m=j-1}^{r-1} c_m} + \frac{(-1)^{j+1}}{(r+1)^{r-j+2}\prod_{m=j-2}^{r-1} c_m}.$$

By the definitions of $q_j$, $p_{1,j}$ and $p_{r,j}$ for $j = 1, \ldots, k$, we have:

$$q_1 = \frac{2 + g(r,1) + h(r,1)}{3 - 2f(r)},$$
$$q_j = \frac{3 + g(r,j) + h(r,j)}{3 - 2f(r)} \quad if \quad 2 \le j \le k.$$

Then $q_j$ for $j = 1, \ldots, k$ can be expressed as follows:

$$q_1 = \frac{2 + \frac{1}{(r+1)c_{r-1}} - \frac{1}{(r+1)^2 c_{r-2}c_{r-1}} + \frac{1}{(r+1)^{r-1}\prod_{m=1}^{r-1} c_m} - \frac{1}{(r+1)^r \prod_{m=1}^{r-1} c_m}}{3 - 2f(r)},$$
$$q_2 = \frac{3 + \frac{1}{(r+1)c_{r-1}} - \frac{1}{(r+1)^2 c_{r-2}c_{r-1}} + \frac{1}{(r+1)^3 \prod_{m=r-3}^{r-1} c_m} - \frac{1}{(r+1)^{r-2}\prod_{m=2}^{r-1} c_m}}{3 - 2f(r)}$$
$$+ \frac{\frac{1}{(r+1)^{r-1}\prod_{m=1}^{r-1} c_m} - \frac{1}{(r+1)^r \prod_{m=1}^{r-1} c_m}}{3 - 2f(r)},$$

$$q_j = \frac{3 + \frac{(-1)^j}{(r+1)^{j-1}\prod_{m=r-j+1}^{r-1} c_m} - \frac{(-1)^j}{(r+1)^j \prod_{m=r-j}^{r-1} c_m} + \frac{(-1)^j}{(r+1)^{j+1}\prod_{m=r-j-1}^{r-1} c_m}}{3-2f(r)}$$
$$+ \frac{\frac{(-1)^{j+1}}{(r+1)^{r-j}\prod_{m=j}^{r-1} c_m} - \frac{(-1)^{j+1}}{(r+1)^{r-j+1}\prod_{m=j-1}^{r-1} c_m} + \frac{(-1)^{j+1}}{(r+1)^{r-j+2}\prod_{m=j-2}^{r-1} c_m}}{3-2f(r)} \quad if \quad 3 \le j \le k.$$

Since $P_j = \frac{q_j}{r}$ for $j = 1, \dots, k$ we get:

$$P_1 - P_2 = \frac{-1 - \frac{1}{(r+1)^3 \prod_{m=r-3}^{r-1} c_m} + \frac{1}{(r+1)^{r-2}\prod_{m=2}^{r-1} c_m}}{r(3-2f(r))}.$$
$$P_2 - P_3 = \frac{\frac{1}{(r+1)c_{r-1}} + \frac{1}{(r+1)^4 \prod_{m=r-4}^{r-1} c_m} - \left(\frac{1}{(r+1)^r \prod_{m=1}^{r-1} c_m} + \frac{1}{(r+1)^{r-3}\prod_{m=3}^{r-1} c_m}\right)}{r(3-2f(r))}.$$

$P_j - P_{j+1}$ for $j = 3, \dots, k-1$ is written in the form

$$\frac{\frac{(-1)^j}{(r+1)^{j-1}\prod_{m=r-j+1}^{r-1} c_m} + \frac{(-1)^j}{(r+1)^{j+2}\prod_{m=r-j-2}^{r-1} c_m} + \frac{(-1)^{j+1}}{(r+1)^{r-j+2}\prod_{m=j-2}^{r-1} c_m} + \frac{(-1)^{j+1}}{(r+1)^{r-j-1}\prod_{m=j+1}^{r-1} c_m}}{r(3-2f(r))}.$$

Now let us show that $3 - 2f(r)$ is positive. Straightforward calculation yields that $c_i > \frac{1}{r+1}$ for $i = 1, \dots, r-1$. Thus, for any $l$ and indices $i_1, i_2, \dots, i_l$ we have $(r+1)^l c_{i_1} c_{i_2} \cdots c_{i_l} > 1$. Determine $t_i = (r+1)c_i > 1$ for $i = 1, \dots, r-1$. Then $r + 1 > t_i > r$ since $1 > c_i > \frac{r}{r+1}$. Thus, $t_i > t_{i+1}$ for $1 \le i \le r-2$. Now

$$\begin{aligned}
f(r) &= \sum_{l=1}^{r-1} \frac{(-1)^{l+1}}{(r+1)^{r-l}\prod_{m=l}^{r-1} c_m} - \frac{1}{(r+1)^r \prod_{m=1}^{r-1} c_m} \\
&= \sum_{l=1}^{r-1} \frac{(-1)^{l+1}}{\prod_{m=l}^{r-1} t_m} - \frac{1}{(r+1)\prod_{m=1}^{r-1} t_m} \\
&= \sum_{l=1}^{\frac{r}{2}-1} \frac{1}{\prod_{m=2l+1}^{r-1} t_m}\left(1 - \frac{1}{t_{2l}}\right) + \frac{1}{\prod_{m=1}^{r-1} t_m}\left(1 - \frac{1}{r+1}\right) < \left(1 - \frac{1}{r+1}\right)\sum_{l=0}^{\frac{r}{2}-1} \frac{1}{r^{2l+1}} \\
&= \left(1 - \frac{1}{r+1}\right)\sum_{l=0}^{\frac{r}{2}-1} \frac{r^{2l}}{r^{r-1}} = \frac{r}{(r+1)r^{r-1}}\frac{r^r - 1}{r^2-1} < \frac{r^r}{r^{r-1}(r^2-1)} = \frac{r}{r^2-1} < \frac{3}{2}.
\end{aligned}$$

Therefore, $3 - 2f(r) > 0$.

Now we investigate signs of $P_j - P_{j+1}$ for $j = 1, \ldots, k-1$:

$$P_1 - P_2 = \frac{-1}{r(3-2f(r))} + \frac{-1 + \frac{1}{(r+1)^{r-5}\prod_{m=2}^{r-4} c_m}}{r(3-2f(r))\frac{1}{(r+1)^3\prod_{m=r-3}^{r-1} c_m}} < 0$$

since both terms are negative.

Similarly,

$$P_2 - P_3 = \frac{1 - \frac{1}{(r+1)^{r-1}\prod_{m=1}^{r-2} c_m}}{r(3-2f(r))\frac{1}{(r+1)c_{r-1}}} + \frac{1 - \frac{1}{(r+1)^{r-7}\prod_{m=3}^{r-5} c_m}}{r(3-2f(r))\frac{1}{(r+1)^4\prod_{m=r-4}^{r-1} c_m}} > 0$$

since both terms are positive.

$$\vdots$$

Similarly, since

$$P_{k-2} - P_{k-1} = \frac{(-1)^k \left(1 - \frac{1}{(r+1)^7\prod_{m=k-4}^{k+2} c_m}\right)}{r(3-2f(r))(r+1)^{k-3}\prod_{m=k+3}^{r-1} c_m} + \frac{(-1)^k \left(1 - \frac{1}{(r+1)c_{k-1}}\right)}{r(3-2f(r))(r+1)^k\prod_{m=k}^{r-1} c_m}$$

the sign of $P_{k-2} - P_{k-1}$ is positive (negative) for even (odd) values of $k$ and since

$$P_{k-1} - P_k = \frac{(-1)^{k+1} \left(1 - \frac{1}{(r+1)^2 c_k c_{k+1}}\right)}{r(3-2f(r))(r+1)^{k-2}\prod_{m=k+2}^{r-1} c_m} + \frac{(-1)^{k+1} \left(1 - \frac{1}{(r+1)^2 c_{k-3} c_{k-2}}\right)}{r(3-2f(r))(r+1)^{k+1}\prod_{m=k-1}^{r-1} c_m}$$

the sign of $P_{k-1} - P_k$ is positive (negative) for odd (even) values of $k$.

Now we investigate signs of $P_j - P_{j+2}$ for $j = 1, \ldots, k-2$:

$$P_1 - P_3 = \frac{\frac{-1+\frac{1}{(r+1)c_{r-4}}}{(r+1)^3\prod_{m=r-3}^{r-1} c_m} - \frac{1-\frac{1}{(r+1)c_2}}{(r+1)^{r-3}\prod_{m=3}^{r-1} c_m} - 1 + \frac{1}{(r+1)^4\prod_{m=r-4}^{r-1} c_m} - \frac{1}{(r+1)^r\prod_{m=1}^{r-1} c_m}}{r(3-2f(r))} < 0.$$

Thus, $P_1 < P_3$.

$$P_2 - P_4 = \frac{\frac{1-\frac{1}{(r+1)c_{r-2}}}{(r+1)c_{r-1}} + \frac{1-\frac{1}{(r+1)c_{r-5}}}{(r+1)^4\prod_{m=r-4}^{r-1} c_m} + \frac{1-\frac{1}{r+1}}{(r+1)^{r-1}\prod_{m=1}^{r-1} c_m} + \frac{1-\frac{1}{(r+1)c_3}}{(r+1)^{r-4}\prod_{m=4}^{r-1} c_m}}{r(3-2f(r))} > 0.$$

Thus, $P_2 > P_4$.

$$P_3 - P_5 = \frac{\frac{-1+\frac{1}{(r+1)c_{r-3}}}{(r+1)^2 c_{r-2}c_{r-1}} + \frac{-1+\frac{1}{(r+1)c_{r-6}}}{(r+1)^5\prod_{m=r-5}^{r-1} c_m} + \frac{-1+\frac{1}{(r+1)c_1}}{(r+1)^{r-2}\prod_{m=2}^{r-1} c_m} + \frac{-1+\frac{1}{(r+1)c_4}}{(r+1)^{r-5}\prod_{m=5}^{r-1} c_m}}{r(3-2f(r))} < 0.$$

Thus, $P_3 < P_5$.

$$\vdots$$

$P_{k-2} - P_k$

$$= \frac{\frac{(-1)^k\left(1-\frac{1}{(r+1)c_{k+2}}\right)}{(r+1)^{k-3}\prod_{m=k+3}^{r-1} c_m} + \frac{(-1)^k\left(1-\frac{1}{(r+1)c_{k-1}}\right)}{(r+1)^{k}\prod_{m=k}^{r-1} c_m} + \frac{(-1)^k\left(1-\frac{1}{(r+1)c_{k-4}}\right)}{(r+1)^{k+3}\prod_{m=k-3}^{r-1} c_m} + \frac{(-1)^k\left(1-\frac{1}{(r+1)c_{k-1}}\right)}{(r+1)^{k}\prod_{m=k}^{r-1} c_m}}{r(3-2f(r))}.$$

Thus, $P_{k-2} > P_k$ if $k$ is even, $P_{k-2} < P_k$ if $k$ is odd.

These two sequences of inequalities readily imply the relationships (1.1) and (1.2).

Let us consider the odd case $r = 2k+1$. For each fixed $j = 1, \ldots, k+1$, the absorption probability $P_j = \frac{1}{r}\sum_{i=1}^{r} p_{i,j}$. In order to evaluate $p_{i,j}$ we fix $j$ and write the system of equations $p_{i,j}$, $i = 1, \ldots, r$. Since these systems have different forms for $j = 1, 2, 3, 4$ and $j \geq 5$ there are five different equation systems.

The system for evaluation related to the probability that the walk terminates at $j = 1$ is:

$$\begin{aligned} p_{i,1} &= \frac{1}{r} + \sum_{n=i+2}^{r} \frac{1}{r} p_{n,1} \quad if \quad 1 \leq i \leq 2 \\ p_{i,1} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,1} + \sum_{n=i+2}^{r} \frac{1}{r} p_{n,1} \quad if \quad 3 \leq i \leq r-2 \\ p_{i,1} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,1} \quad if \quad r-1 \leq i \leq r. \end{aligned} \tag{2.9}$$

The system for evaluation related to the probability that the walk terminates at $j = 2$ is:

$$
\begin{aligned}
p_{i,2} &= \frac{1}{r} + \sum_{n=i+2}^{r} \frac{1}{r} p_{n,2} \quad if \quad 1 \leq i \leq 2 \\
p_{3,2} &= \frac{1}{r} + \frac{1}{r} p_{1,2} + \sum_{n=5}^{r} \frac{1}{r} p_{n,2} \\
p_{i,2} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,2} + \sum_{n=i+2}^{r} \frac{1}{r} p_{n,2} \quad if \quad 4 \leq i \leq r-2 \\
p_{i,2} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,2} \quad if \quad r-1 \leq i \leq r.
\end{aligned} \tag{2.10}
$$

The system for evaluation related to the probability that the walk terminates at $j = 3$ is:

$$
\begin{aligned}
p_{1,3} &= \sum_{n=3}^{r} \frac{1}{r} p_{n,3} \\
p_{2,3} &= \frac{1}{r} + \sum_{n=4}^{r} \frac{1}{r} p_{n,3} \\
p_{i,3} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,3} + \frac{1}{r} + \sum_{n=i+2}^{r} \frac{1}{r} p_{n,3} \quad if \quad 3 \leq i \leq 4 \\
p_{i,3} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,3} + \sum_{n=i+2}^{r} \frac{1}{r} p_{n,3} \quad if \quad 5 \leq i \leq r-2 \\
p_{i,3} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,3} \quad if \quad r-1 \leq i \leq r.
\end{aligned} \tag{2.11}
$$

The system for evaluation related to the probability that the walk terminates at $j = 4$ is:

$$
\begin{aligned}
p_{i,4} &= \sum_{n=i+2}^{r} \frac{1}{r} p_{n,4} \quad if \quad 1 \leq i \leq 2 \\
p_{i,4} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,4} + \frac{1}{r} + \sum_{n=i+2}^{r} \frac{1}{r} p_{n,4} \quad if \quad 3 \leq i \leq 5 \\
p_{i,4} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,4} + \sum_{n=i+2}^{r} \frac{1}{r} p_{n,4} \quad if \quad 6 \leq i \leq r-2 \\
p_{i,4} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,4} \quad if \quad r-1 \leq i \leq r.
\end{aligned} \tag{2.12}
$$

The system for evaluation related to the probability that the walk terminates at $j = m$ for $5 \leq m \leq k+1$ is:

$$\begin{aligned}
p_{i,m} &= \sum_{n=i+2}^{r} \frac{1}{r} p_{n,m} \quad if \quad 1 \leq i \leq 2 \\
p_{i,m} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,m} + \sum_{n=i+2}^{r} \frac{1}{r} p_{n,m} \quad if \quad 3 \leq i \leq m-2 \\
p_{i,m} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,m} + \frac{1}{r} + \sum_{n=i+2}^{r} \frac{1}{r} p_{n,m} \quad if \quad m-1 \leq i \leq m+1 \\
p_{i,m} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,m} + \sum_{n=i+2}^{r} \frac{1}{r} p_{n,m} \quad if \quad m+2 \leq i \leq r-2 \\
p_{i,m} &= \sum_{n=1}^{i-2} \frac{1}{r} p_{n,m} \quad if \quad r-1 \leq i \leq r.
\end{aligned} \tag{2.13}$$

Let $\rho_j = \sum_{i=1}^{r} p_{i,j}$, where $1 \leq j \leq k+1$.

The side by side summation of the system (2.9) yields

$$2(p_{1,1} + p_{r,1}) + 3\sum_{i=2}^{r-1} p_{i,1} = 2. \quad Thus, \quad \rho_1 = \frac{2 + p_{1,1} + p_{r,1}}{3}.$$

Similarly, by side by side summation of the systems (2.10)-(2.13) we get

$$\rho_j = \frac{3 + p_{1,j} + p_{r,j}}{3} \quad for \quad 2 \leq j \leq k+1.$$

The system (2.9) readily yields:

$$\begin{aligned}
p_{1,1} &= \frac{1}{r+1} + \frac{1}{r+1}(\rho_1 - p_{2,1}) \\
p_{2,1} &= \frac{1}{r+1} + \frac{1}{r+1}(\rho_1 - p_{1,1} - p_{3,1}) \\
p_{i,1} &= \frac{1}{r+1}(\rho_1 - p_{i-1,1} - p_{i+1,1}) \quad if \quad 3 \leq i \leq r-1 \\
p_{r,1} &= \frac{1}{r+1}(\rho_1 - p_{r-1,1}).
\end{aligned} \tag{2.14}$$

The system (2.10) readily yields:

$$
\begin{aligned}
p_{1,2} &= \frac{1}{r+1} + \frac{1}{r+1}(\rho_2 - p_{2,2}) \\
p_{i,2} &= \frac{1}{r+1} + \frac{1}{r+1}(\rho_2 - p_{i-1,2} - p_{i+1,2}) \ \ if \ \ 2 \le i \le 3 \\
p_{i,2} &= \frac{1}{r+1}(\rho_2 - p_{i-1,2} - p_{i+1,2}) \ \ if \ \ 4 \le i \le r-1 \\
p_{r,2} &= \frac{1}{r+1}(\rho_2 - p_{r-1,2}).
\end{aligned}
\tag{2.15}
$$

Similarly, the systems (2.11)-(2.13) readily yield:

$$
\begin{aligned}
p_{1,m} &= \frac{1}{r+1}(\rho_m - p_{2,m}) \\
p_{i,m} &= \frac{1}{r+1}(\rho_m - p_{i-1,m} - p_{i+1,m}) \ \ if \ \ 2 \le i \le m-2 \\
p_{i,m} &= \frac{1}{r+1} + \frac{1}{r+1}(\rho_m - p_{i-1,m} - p_{i+1,m}) \ \ if \ \ m-1 \le i \le m+1 \\
p_{i,m} &= \frac{1}{r+1}(\rho_m - p_{i-1,m} - p_{i+1,m}) \ \ if \ \ m+2 \le i \le r-1 \\
p_{r,m} &= \frac{1}{r+1}(\rho_m - p_{r-1,m}).
\end{aligned}
\tag{2.16}
$$

Now we wish to express $p_{1,j}$ and $p_{r,j}$ in terms of $\rho_j$ for $j = 1, \dots, k+1$:

Let us define numbers $d_i$, $i = 0, 1, \dots, r-1$ by

$$d_1 = 1 - \frac{1}{(r+1)^2} \ \ and \ \ d_i = 1 - \frac{1}{d_{i-1}(r+1)^2}$$

for each integer $2 \le i \le r-1$.

Then the system (2.14) yields the following two relationships:

$$
\begin{aligned}
p_{1,1} = &\left( \sum_{l=1}^{r-1} \frac{(-1)^l}{(r+1)^{r-l} \prod_{m=l}^{r-1} d_m} + \frac{1}{(r+1)^r \prod_{m=1}^{r-1} d_m} \right) \rho_1 + \frac{1}{(r+1)d_{r-1}} \\
&- \frac{1}{(r+1)^2 d_{r-2} d_{r-1}}, \\
p_{r,1} = &\left( \sum_{l=1}^{r-1} \frac{(-1)^l}{(r+1)^{r-l} \prod_{m=l}^{r-1} d_m} + \frac{1}{(r+1)^r \prod_{m=1}^{r-1} d_m} \right) \rho_1 - \frac{1}{(r+1)^{r-1} \prod_{m=1}^{r-1} d_m} \\
&+ \frac{1}{(r+1)^r \prod_{m=1}^{r-1} d_m}.
\end{aligned}
$$

Similarly, the system (2.15) yields the following two relationships:

$$p_{1,2} = \left( \sum_{l=1}^{r-1} \frac{(-1)^l}{(r+1)^{r-l} \prod_{m=l}^{r-1} d_m} + \frac{1}{(r+1)^r \prod_{m=1}^{r-1} d_m} \right) \rho_2 + \frac{1}{(r+1)d_{r-1}}$$
$$- \frac{1}{(r+1)^2 d_{r-2} d_{r-1}} + \frac{1}{(r+1)^3 \prod_{m=r-3}^{r-1} d_m},$$
$$p_{r,2} = \left( \sum_{l=1}^{r-1} \frac{(-1)^l}{(r+1)^{r-l} \prod_{m=l}^{r-1} d_m} + \frac{1}{(r+1)^r \prod_{m=1}^{r-1} d_m} \right) \rho_2 + \frac{1}{(r+1)^{r-2} \prod_{m=2}^{r-1} d_m}$$
$$- \frac{1}{(r+1)^{r-1} \prod_{m=1}^{r-1} d_m} + \frac{1}{(r+1)^r \prod_{m=1}^{r-1} d_m}.$$

For $3 \leq j \leq k+1$, the system (2.16) yields the following two relationships:

$$p_{1,j} = \left( \sum_{l=1}^{r-1} \frac{(-1)^l}{(r+1)^{r-l} \prod_{m=l}^{r-1} d_m} + \frac{1}{(r+1)^r \prod_{m=1}^{r-1} d_m} \right) \rho_j + \frac{(-1)^j}{(r+1)^{j-1} \prod_{m=r-j+1}^{r-1} d_m}$$
$$- \frac{(-1)^j}{(r+1)^j \prod_{m=r-j}^{r-1} d_m} + \frac{(-1)^j}{(r+1)^{j+1} \prod_{m=r-j-1}^{r-1} d_m},$$
$$p_{r,j} = \left( \sum_{l=1}^{r-1} \frac{(-1)^l}{(r+1)^{r-l} \prod_{m=l}^{r-1} d_m} + \frac{1}{(r+1)^r \prod_{m=1}^{r-1} d_m} \right) \rho_j + \frac{(-1)^j}{(r+1)^{r-j} \prod_{m=j}^{r-1} d_m}$$
$$- \frac{(-1)^j}{(r+1)^{r-j+1} \prod_{m=j-1}^{r-1} d_m} + \frac{(-1)^j}{(r+1)^{r-j+2} \prod_{m=j-2}^{r-1} d_m}.$$

Summarizing last six relationships we get that for $j = 1, \ldots, k+1$, $p_{1,j}$ and $p_{r,j}$ have the following form:

$$p_{1,j} = u(r)\rho_j + v(r,j),$$
$$p_{r,j} = u(r)\rho_j + z(r,j)$$

where

$$u(r) = \sum_{l=1}^{r-1} \frac{(-1)^l}{(r+1)^{r-l} \prod_{m=l}^{r-1} d_m} + \frac{1}{(r+1)^r \prod_{m=1}^{r-1} d_m},$$
$$v(r,1) = \frac{1}{(r+1)d_{r-1}} - \frac{1}{(r+1)^2 d_{r-2} d_{r-1}},$$
$$if \quad 2 \leq j \leq k+1$$
$$v(r,j) = \frac{(-1)^j}{(r+1)^{j-1} \prod_{m=r-j+1}^{r-1} d_m} - \frac{(-1)^j}{(r+1)^j \prod_{m=r-j}^{r-1} d_m} + \frac{(-1)^j}{(r+1)^{j+1} \prod_{m=r-j-1}^{r-1} d_m},$$

$$z(r,1) = -\frac{1}{(r+1)^{r-1}\prod_{m=1}^{r-1} d_m} + \frac{1}{(r+1)^{r}\prod_{m=1}^{r-1} d_m},$$
$$z(r,2) = \frac{1}{(r+1)^{r-2}\prod_{m=2}^{r-1} d_m} - \frac{1}{(r+1)^{r-1}\prod_{m=1}^{r-1} d_m} + \frac{1}{(r+1)^{r}\prod_{m=1}^{r-1} d_m},$$
$$if \quad 3 \le j \le k+1$$
$$z(r,j) = \frac{(-1)^j}{(r+1)^{r-j}\prod_{m=j}^{r-1} d_m} - \frac{(-1)^j}{(r+1)^{r-j+1}\prod_{m=j-1}^{r-1} d_m} + \frac{(-1)^j}{(r+1)^{r-j+2}\prod_{m=j-2}^{r-1} d_m}.$$

Regarding the definitions of $\rho_j$, $p_{1,j}$ and $p_{r,j}$ for $j = 1, \ldots, k+1$, we have:

$$\rho_1 = \frac{2 + v(r,1) + z(r,1)}{3 - 2u(r)},$$
$$\rho_j = \frac{3 + v(r,j) + z(r,j)}{3 - 2u(r)} \quad if \quad 2 \le j \le k+1.$$

Then $\rho_j$ for $j = 1, \ldots, k+1$ can be expressed as follows:

$$\rho_1 = \frac{2 + \frac{1}{(r+1)d_{r-1}} - \frac{1}{(r+1)^2 d_{r-2} d_{r-1}} - \frac{1}{(r+1)^{r-1}\prod_{m=1}^{r-1} d_m} + \frac{1}{(r+1)^{r}\prod_{m=1}^{r-1} d_m}}{3 - 2u(r)},$$
$$\rho_2 = \frac{3 + \frac{1}{(r+1)d_{r-1}} - \frac{1}{(r+1)^2 d_{r-2} d_{r-1}} + \frac{1}{(r+1)^3\prod_{m=r-3}^{r-1} d_m} + \frac{1}{(r+1)^{r-2}\prod_{m=2}^{r-1} d_m}}{3 - 2u(r)}$$
$$+ \frac{\frac{-1}{(r+1)^{r-1}\prod_{m=1}^{r-1} d_m} + \frac{1}{(r+1)^{r}\prod_{m=1}^{r-1} d_m}}{3 - 2u(r)},$$
$$\rho_j = \frac{3 + \frac{(-1)^j}{(r+1)^{j-1}\prod_{m=r-j+1}^{r-1} d_m} - \frac{(-1)^j}{(r+1)^{j}\prod_{m=r-j}^{r-1} d_m} + \frac{(-1)^j}{(r+1)^{j+1}\prod_{m=r-j-1}^{r-1} d_m}}{3 - 2u(r)}$$
$$+ \frac{\frac{(-1)^j}{(r+1)^{r-j}\prod_{m=j}^{r-1} d_m} - \frac{(-1)^j}{(r+1)^{r-j+1}\prod_{m=j-1}^{r-1} d_m} + \frac{(-1)^j}{(r+1)^{r-j+2}\prod_{m=j-2}^{r-1} d_m}}{3 - 2u(r)} \quad if \quad 3 \le j \le k+1.$$

Now let us show that $3 - 2u(r)$ is positive. Straightforward calculation yields that $d_i > \frac{1}{r+1}$ for $i = 1, \ldots, r-1$. Thus, for any $l$ and indices $i_1, i_2, \ldots, i_l$ we have $(r+1)^l d_{i_1} d_{i_2} \cdots d_{i_l} > 1$. Determine $e_i = (r+1)d_i > 1$ for $i = 1, \ldots, r-1$. Then $r+1 > e_i > r$ since $1 > d_i > \frac{r}{r+1}$. Thus, $e_i > e_{i+1}$ for $1 \le i \le r-2$. Now

$$
\begin{aligned}
u(r) &= \sum_{l=1}^{r-1} \frac{(-1)^l}{(r+1)^{r-l}\prod_{m=l}^{r-1} d_m} + \frac{1}{(r+1)^r \prod_{m=1}^{r-1} d_m} \\
&= \sum_{l=1}^{r-1} \frac{(-1)^l}{\prod_{m=l}^{r-1} e_m} + \frac{1}{(r+1)\prod_{m=1}^{r-1} e_m} \\
&= \sum_{l=1}^{\frac{r-1}{2}} \frac{1}{\prod_{m=2l}^{r-1} e_m}\left(1-\frac{1}{e_{2l-1}}\right) + \frac{1}{(r+1)\prod_{m=1}^{r-1} e_m} < \left(1-\frac{1}{r+1}\right)\sum_{l=0}^{\frac{r-3}{2}} \frac{1}{r^{2l+1}} + \frac{1}{r+1} \\
&= \left(1-\frac{1}{r+1}\right)\sum_{l=0}^{\frac{r-3}{2}} \frac{r^{2l}}{r^{r-2}} + \frac{1}{r+1} = \frac{r}{(r+1)r^{r-2}}\frac{r^{r-1}-1}{r^2-1} + \frac{1}{r+1} \\
&< \frac{r^{r-1}}{r^{r-2}(r^2-1)} + \frac{1}{r+1} = \frac{2r-1}{r^2-1} < \frac{3}{2}.
\end{aligned}
$$

Therefore, $3-2u(r)>0$.

Now we investigate signs of $P_j - P_{j+1}$ for $j=1,\ldots,k$:

$$
\begin{aligned}
P_1 - P_2 &= \frac{-1 - \frac{1}{(r+1)^3\prod_{m=r-3}^{r-1} d_m} - \frac{1}{(r+1)^{r-2}\prod_{m=2}^{r-1} d_m}}{r(3-2u(r))} < 0. \\
P_2 - P_3 &= \frac{\frac{1}{(r+1)d_{r-1}} + \frac{1}{(r+1)^4\prod_{m=r-4}^{r-1} d_m} + \frac{1}{(r+1)^r\prod_{m=1}^{r-1} d_m} + \frac{1}{(r+1)^{r-3}\prod_{m=3}^{r-1} d_m}}{r(3-2u(r))} > 0.
\end{aligned}
$$

$P_j - P_{j+1}$ for $j=3,\ldots,k$ is written in the form

$$
\frac{\frac{(-1)^j}{(r+1)^{j-1}\prod_{m=r-j+1}^{r-1} d_m} + \frac{(-1)^j}{(r+1)^{j+2}\prod_{m=r-j-2}^{r-1} d_m} + \frac{(-1)^j}{(r+1)^{r-j+2}\prod_{m=j-2}^{r-1} d_m} + \frac{(-1)^j}{(r+1)^{r-j-1}\prod_{m=j+1}^{r-1} d_m}}{r(3-2u(r))}.
$$

It can be readily seen that the sign of $P_j - P_{j+1}$ is negative (positive) for odd (even) values of $j=3,\ldots,k$.

Summing two consecutive equations of the above system, we investigate signs of $P_j - P_{j+2}$ for $j=1,\ldots,k-1$:

$$
\begin{aligned}
P_1 - P_3 &= \frac{\left(-1+\frac{1}{(r+1)d_{r-1}}\right) + \frac{-1+\frac{1}{(r+1)d_{r-4}}}{(r+1)^3\prod_{m=r-3}^{r-1} d_m} + \frac{1}{(r+1)^r\prod_{m=1}^{r-1} d_m} + \frac{1-\frac{1}{(r+1)d_2}}{(r+1)^{r-3}\prod_{m=3}^{r-1} d_m}}{r(3-2f(r))} \\
&= \frac{A+B+C+D}{r(3-2f(r))}.
\end{aligned}
$$

Now

$$\left|\frac{A}{D}\right| = \frac{1-\frac{1}{(r+1)d_{r-1}}}{\frac{1-\frac{1}{(r+1)d_2}}{(r+1)^{r-3}\prod_{m=3}^{r-1}d_m}} = \frac{(r+1)d_1-1}{(r+1)d_2-1}(r+1)^{r-3}\frac{1}{d_1}\prod_{m=2}^{r-1}d_m = \frac{\prod_{m=2}^{r-1}e_m}{e_1}\frac{e_1-1}{e_2-1}$$

$$> \frac{\prod_{m=3}^{r-1}e_m}{e_1}(e_1-1) = \frac{(r+1)\prod_{m=3}^{r-1}e_m}{(r+1)^2-1}(e_1-1) > \frac{(r+1)e_{r-1}^{r-3}}{(r+1)^2-1}(e_{r-1}-1)$$

$$> \frac{r^2(r+1)e_{r-1}^{r-5}}{(r+1)^2-1}(e_{r-1}-1) > e_{r-1}(e_{r-1}-1) > r(r-1) > 1.$$

Similarly,

$$\left|\frac{B}{C}\right| = \frac{\frac{1-\frac{1}{(r+1)d_{r-4}}}{(r+1)^3\prod_{m=r-3}^{r-1}d_m}}{\frac{1}{(r+1)^r\prod_{m=1}^{r-1}d_m}} = \frac{(r+1)d_{r-4}-1}{r+1}(r+1)^{r-3}\prod_{m=1}^{r-3}d_m$$

$$> \frac{e_{r-1}^{r-3}}{r+1}(e_{r-4}-1) > \frac{r^2e_{r-1}^{r-5}}{r+1}(e_{r-1}-1) > e_{r-1}(e_{r-1}-1) > 1.$$

Thus, $P_1 < P_3$.

$$P_2 - P_4 = \frac{\frac{1-\frac{1}{(r+1)d_{r-2}}}{(r+1)d_{r-1}} + \frac{1-\frac{1}{(r+1)d_{r-5}}}{(r+1)^4\prod_{m=r-4}^{r-1}d_m} + \frac{-1+\frac{1}{r+1}}{(r+1)^{r-1}\prod_{m=1}^{r-1}d_m} + \frac{-1+\frac{1}{(r+1)d_3}}{(r+1)^{r-4}\prod_{m=4}^{r-1}d_m}}{r(3-2f(r))}$$

$$= \frac{E+F+G+H}{r(3-2f(r))}.$$

$$\left|\frac{E}{H}\right| = \frac{\frac{1-\frac{1}{(r+1)d_{r-2}}}{(r+1)d_{r-1}}}{\frac{1-\frac{1}{(r+1)d_3}}{(r+1)^{r-4}\prod_{m=4}^{r-1}d_m}} = \frac{(r+1)d_{r-2}-1}{(r+1)d_{r-3}-1}(r+1)^{r-5}\prod_{m=3}^{r-3}d_m > \frac{e_{r-2}-1}{e_{r-3}-1}\prod_{m=3}^{r-3}e_m$$

$$> e_{r-1}^{r-4}(e_{r-2}-1) > e_{r-1}(e_{r-1}-1) > 1.$$

Similarly,

$$\left|\frac{F}{G}\right| = \frac{\frac{1-\frac{1}{(r+1)d_{r-5}}}{(r+1)^4\prod_{m=r-4}^{r-1}d_m}}{\frac{1-\frac{1}{r+1}}{(r+1)^{r-1}\prod_{m=1}^{r-1}d_m}} = \frac{(r+1)d_{r-5}-1}{r}(r+1)^{r-5}\prod_{m=1}^{r-4}d_m$$

$$> ((r+1)d_{r-5}-1)(r+1)^{r-6}\prod_{m=1}^{r-4} d_m = \frac{\prod_{m=1}^{r-4} e_m}{(r+1)^2}(e_{r-5}-1) > \frac{e_{r-1}^{r-4}}{(r+1)^2}(e_{r-1}-1)$$

$$> \frac{r^3 e_{r-1}^{r-7}}{(r+1)^2}(e_{r-1}-1) > e_{r-1}(e_{r-1}-1) > 1.$$

Thus, $P_2 > P_4$.

$$P_3 - P_5 = \frac{\frac{-1+\frac{1}{(r+1)d_{r-3}}}{(r+1)^2 d_{r-2}d_{r-1}} + \frac{-1+\frac{1}{(r+1)d_{r-6}}}{(r+1)^5\prod_{m=r-5}^{r-1} d_m} + \frac{1-\frac{1}{(r+1)d_1}}{(r+1)^{r-2}\prod_{m=2}^{r-1} d_m} + \frac{1-\frac{1}{(r+1)d_4}}{(r+1)^{r-5}\prod_{m=5}^{r-1} d_m}}{r(3-2f(r))}$$

$$= \frac{I+J+K+L}{r(3-2f(r))}.$$

$$\left|\frac{I}{L}\right| = \frac{\frac{1-\frac{1}{(r+1)d_{r-3}}}{(r+1)^2 d_{r-2}d_{r-1}}}{\frac{1-\frac{1}{(r+1)d_4}}{(r+1)^{r-5}\prod_{m=5}^{r-1} d_m}} = \frac{(r+1)d_{r-3}-1}{(r+1)d_4-1}(r+1)^{r-7}\prod_{m=4}^{r-4} d_m = \frac{e_{r-3}-1}{e_4-1}\prod_{m=4}^{r-4} e_m$$

$$> (e_{r-3}-1)\prod_{m=5}^{r-4} e_m > e_{r-1}(e_{r-1}-1) > 1.$$

Similarly,

$$\left|\frac{J}{K}\right| = \frac{\frac{1-\frac{1}{(r+1)d_{r-6}}}{(r+1)^5\prod_{m=r-5}^{r-1} d_m}}{\frac{1-\frac{1}{(r+1)d_1}}{(r+1)^{r-2}\prod_{m=2}^{r-1} d_m}} = \frac{(r+1)d_{r-6}-1}{(r+1)d_1-1}(r+1)^{r-7}\prod_{m=1}^{r-7} d_m = \frac{e_{r-6}-1}{e_1-1}\prod_{m=1}^{r-7} e_m$$

$$> (e_{r-6}-1)\prod_{m=2}^{r-7} e_m > e_{r-1}(e_{r-1}-1) > 1.$$

Thus, $P_3 < P_5$.

$$\vdots$$

$$P_{k-1} - P_{k+1}$$

$$= \frac{\frac{(-1)^{k-1}\left(1-\frac{1}{(r+1)d_{k+2}}\right)}{(r+1)^{k-2}\prod_{m=k+3}^{r-1} d_m} + \frac{(-1)^{k-1}\left(1-\frac{1}{(r+1)d_{k-1}}\right)}{(r+1)^{k+1}\prod_{m=k}^{r-1} d_m} + \frac{(-1)^{k-1}\left(-1+\frac{1}{(r+1)d_{k-3}}\right)}{(r+1)^{k+3}\prod_{m=k-2}^{r-1} d_m} + \frac{(-1)^{k-1}\left(-1+\frac{1}{(r+1)d_k}\right)}{(r+1)^{k}\prod_{m=k+1}^{r-1} d_m}}{r(3-2f(r))}$$

$$= \frac{M+N+O+P}{r(3-2f(r))}.$$

$$\left|\frac{M}{P}\right| = \frac{\frac{1-\frac{1}{(r+1)d_{k+2}}}{(r+1)^{k-2}\prod_{m=k+3}^{r-1} d_m}}{\frac{1-\frac{1}{(r+1)d_k}}{(r+1)^{k}\prod_{m=k+1}^{r-1} d_m}} = (r+1)^2 d_k d_{k+1}\frac{(r+1)d_{k+2}-1}{(r+1)d_k-1} = e_k e_{k+1}\frac{e_{k+2}-1}{e_k-1}$$

$$> e_{k+1}(e_{k+2}-1) > e_{r-1}(e_{r-1}-1) > 1.$$

Similarly,

$$\left|\frac{N}{O}\right| = \frac{\frac{1-\frac{1}{(r+1)d_{k-1}}}{(r+1)^{k+1}\prod_{m=k}^{r-1} d_m}}{\frac{1-\frac{1}{(r+1)d_{k-3}}}{(r+1)^{k+3}\prod_{m=k-2}^{r-1} d_m}} = (r+1)^2 d_{k-3} d_{k-2}\frac{(r+1)d_{k-1}-1}{(r+1)d_{k-3}-1} = e_{k-3} e_{k-2}\frac{e_{k-1}-1}{e_{k-3}-1}$$

$$> e_{k-2}(e_{k-1}-1) > e_{r-1}(e_{r-1}-1) > 1.$$

Thus, $P_{k-1} < P_{k+1}$ if $k$ is even, $P_{k-1} > P_{k+1}$ if $k$ is odd.

These two sequences of inequalities readily imply the relationships (1.3) and (1.4). The proof of Theorem 4.1 is completed.

# Chapter 5

# Concluding Remarks

In this chapter we give important remarks related to future work and compare the results of Theorem 4.1 and the limiting behaviour of a branching random walk defined in this chapter.

In the proof of Theorem 3.1, all expected absorption times are expressed in terms of $E_3$. One can obtain the exact expressions of these expectations by solving the systems (2.7) and (2.13). Thus,

$$E_3 = \frac{r}{r-1-A}$$

where

$$A = 2k - 4 + \frac{2}{r} - 3\frac{a_{k-3}}{r} - (2k-7)\frac{a_{k-4}a_{k-3}}{r} - \frac{1}{r}\sum_{j=0}^{k-5}(2j+2)(-1)^{k-j}\prod_{n=j}^{k-3} a_n$$

and

$$E_3 = \frac{r}{r-1-B}$$

where

$$B = 2k - 3 + \frac{2}{r} - 3\frac{b_{k-2}}{r} - (2k-6)\frac{b_{k-3}b_{k-2}}{r} - \frac{1}{r}\sum_{j=0}^{k-4}(2j+1)(-1)^{k-j+1}\prod_{n=j}^{k-2} b_n$$

in cases $r = 2k$ and $r = 2k+1$, respectively.

Let $(S_n^{i,\leq s})_{n\in Z_{\geq 0}}$ be the random walk starting at point $i \in I(r) : S_0^{i,\leq s} = i$ and terminating at $l-th$ step if its $l-th$ move length is less than or equal to s: $S_{l-1}^{i,\leq s} = j$ and $S_l^{i,\leq s} \in [j-s, j+s]$ ( by definitions $j-s \equiv 1$ if $j-s<1$ and $j+s \equiv r$ if $j+s>r$). Let the random variable $X^{\leq s}(i)$ be the time when the random walk $(S_n^{i,\leq s})_{n\in Z_{\geq 0}}$ terminates and $E_i^{\leq s}$ be the expectation of $X^{\leq s}(i)$.

The linear system of expectations $E_i^{\leq s}$, $1 \leq i \leq r$ of $s$ neighbourhood can be written as:

$$E_1^{\leq s} = \frac{1}{r} + \frac{s}{r} + \sum_{j=s+2}^{r} \frac{1}{r}(1+E_j^{\leq s})$$

$$E_i^{\leq s} = \frac{i-1}{r} + \frac{1}{r} + \frac{s}{r} + \sum_{j=i+s+1}^{r} \frac{1}{r}(1+E_j^{\leq s}) \;\; if \;\; 2 \leq i \leq s+1$$

$$if \;\; s+2 \leq i \leq r-s-1$$

$$E_i^{\leq s} = \sum_{j=1}^{i-s-1} \frac{1}{r}(1+E_j^{\leq s}) + \frac{s}{r} + \frac{1}{r} + \frac{s}{r} + \sum_{j=i+s+1}^{r} \frac{1}{r}(1+E_j^{\leq s})$$

$$E_i^{\leq s} = \sum_{j=1}^{i-s-1} \frac{1}{r}(1+E_j^{\leq s}) + \frac{s}{r} + \frac{1}{r} + \frac{r-i}{r} \;\; if \;\; r-s \leq i \leq r-1$$

$$E_r^{\leq s} = \sum_{j=1}^{r-s-1} \frac{1}{r}(1+E_j^{\leq s}) + \frac{s}{r} + \frac{1}{r}.$$

We do expect that the behaviour of expected absorption times in this case is also oscillating but in the following more sophisticated sense. For example, consider the odd case $r = 2k+1$, then by symmetry $E_i^{\leq s} = E_{r-i+1}^{\leq s}$ for $i = 1, \ldots, \frac{r+1}{2}$. Let $k+1 = sp+q$. We distribute $k+1$ expectations into $p+1$ groups: $G_1 = \{E_1^{\leq s}, \ldots, E_s^{\leq s}\}, G_2 = \{E_{s+1}^{\leq s}, \ldots, E_{2s}^{\leq s}\}, \ldots, G_p = \{E_{(p-1)s+1}^{\leq s} \ldots, E_{ps}^{\leq s}\}, G_{p+1} = \{E_{ps+1}^{\leq s}, \ldots, E_{k+1}^{\leq s}\}$. The inequality $G_i < G_j$ will mean that $E' < E''$ for any two expectations $E' \in G_i$ and $E'' \in G_j$. We expect that $G_i, i = 1, \ldots, p+1$ satisfy inequalities (1.1)-(1.4) and if for some $i<j$ $E_i^{\leq s}$ and $E_j^{\leq s}$ belong to $G_k$, then for odd $k$ $E_i^{\leq s} > E_j^{\leq s}$ and for even $k$ $E_i^{\leq s} < E_j^{\leq s}$. The numerical results support the hypothesis formulated above:

If $r = 13$ and $s = 2$ then:

$E_1^{\leq 2} = 3.200051$, $E_2^{\leq 2} = 2.984779$, $E_3^{\leq 2} = 2.768460$, $E_4^{\leq 2} = 2.798540$,

$E_5^{\leq 2} = 2.812140$, $E_6^{\leq 2} = 2.809020$, $E_7^{\leq 2} = 2.807974$

and $E_1^{\leq 2} > E_2^{\leq 2} > E_5^{\leq 2} > E_6^{\leq 2} > E_7^{\leq 2} > E_4^{\leq 2} > E_3^{\leq 2}$.

If $r = 13$ and $s = 3$ then:

$E_1^{\leq 3} = 2.480078$, $E_2^{\leq 3} = 2.323323$, $E_3^{\leq 3} = 2.164878$, $E_4^{\leq 3} = 2.005490$,

$E_5^{\leq 3} = 2.037821$, $E_6^{\leq 3} = 2.059782$, $E_7^{\leq 3} = 2.072043$

and $E_1^{\leq 3} > E_2^{\leq 3} > E_3^{\leq 3} > E_7^{\leq 3} > E_6^{\leq 3} > E_5^{\leq 3} > E_4^{\leq 3}$.

Similarly, we can formulate a hypothesis for the even case $r = 2k$. Numerical results are as follows:

If $r = 12$ and $s = 2$ then:

$E_1^{\leq 2} = 2.997369$, $E_2^{\leq 2} = 2.781032$, $E_3^{\leq 2} = 2.563484$,

$E_4^{\leq 2} = 2.596044$, $E_5^{\leq 2} = 2.610575$, $E_6^{\leq 2} = 2.606651$

and $E_1^{\leq 2} > E_2^{\leq 2} > E_5^{\leq 2} > E_6^{\leq 2} > E_4^{\leq 2} > E_3^{\leq 2}$.

If $r = 12$ and $s = 3$ then:

$E_1^{\leq 3} = 2.334060$, $E_2^{\leq 3} = 2.176302$, $E_3^{\leq 3} = 2.016323$,

$E_4^{\leq 3} = 1.856343$, $E_5^{\leq 3} = 1.893091$, $E_6^{\leq 3} = 1.919754$

and $E_1^{\leq 3} > E_2^{\leq 3} > E_3^{\leq 3} > E_6^{\leq 3} > E_5^{\leq 3} > E_4^{\leq 3}$.

We also expect oscillation properties of absorption probabilities of the random walk $(S_n^{i,\leq s})_{n \in Z_{\geq 0}}$. Let $p_{i,j}^{\leq s}$ be the probability that the random walk $(S_n^{i,\leq s})_{n \in Z_{\geq 0}}$ starting at point $i$ terminates at point $j$. For fixed $j \in \{1, \dots, r\}$, the linear system of probabilities $p_{i,j}^{\leq s}$ of $s$ neighbourhood can be written as:

$$
\begin{aligned}
&rp_{i,j}^{\leq s} - \sum_{k=i+s+1}^{r} p_{k,j}^{\leq s} = \gamma_i \quad if \quad 1 \leq i \leq s+1 \\
&-\sum_{k=1}^{i-s-1} p_{k,j}^{\leq s} + rp_{i,j}^{\leq s} - \sum_{k=i+s+1}^{r} p_{k,j}^{\leq s} = \gamma_i \quad if \quad s+2 \leq i \leq r-s-1 \\
&-\sum_{k=1}^{i-s-1} p_{k,j}^{\leq s} + rp_{i,j}^{\leq s} = \gamma_i \quad if \quad r-s \leq i \leq r
\end{aligned}
\tag{0.1}
$$

where

$$
\gamma_i = \begin{cases} 1, & \text{for all } i \in \{1, \dots, j+s\} \text{ if } j \in \{1, \dots, s+1\} \\ 1, & \text{for all } i \in \{j-s, \dots, j+s\} \text{ if } j \in \{s+2, \dots, r-s-1\} \\ 1, & \text{for all } i \in \{j-s, \dots, r\} \text{ if } j \in \{r-s, \dots, r\} \\ 0, & \text{otherwise.} \end{cases}
$$

Regarding the system (0.1) we note that absorption probabilities

$$
P_j^{\leq s} = \frac{1}{r} \sum_{i=1}^{r} p_{i,j}^{\leq s} \quad and \quad \sum_{j=1}^{r} P_j^{\leq s} = 1.
$$

Similar to expected absorption times, we do expect that the behaviour of absorption probabilities in this case is also oscillating but in the following more sophisticated sense. For example, consider the odd case $r = 2k+1$, then by symmetry $P_j^{\leq s} = P_{r-j+1}^{\leq s}$ for $j = 1, \dots, \frac{r+1}{2}$. Let $k+1 = sp+q$. We distribute $k+1$ probabilities into $p+1$ groups: $H_1 = \{P_1^{\leq s}, \dots, P_s^{\leq s}\}, H_2 = \{P_{s+1}^{\leq s}, \dots, P_{2s}^{\leq s}\}, \dots, H_p = \{P_{(p-1)s+1}^{\leq s} \dots, P_{ps}^{\leq s}\}, H_{p+1} = \{P_{ps+1}^{\leq s}, \dots, P_{k+1}^{\leq s}\}$. The inequality $H_i < H_j$ will mean that $P' < P''$ for any two probabilities $P' \in H_i$ and $P'' \in H_j$. We expect that $H_i, i = 1, \dots, p+1$ satisfy inequalities (1.1)-(1.4) and if for some $i < j$ $P_i^{\leq s}$ and $P_j^{\leq s}$ belong to $H_k$, then for odd $k$ $P_i^{\leq s} < P_j^{\leq s}$ and for even $k$ $P_i^{\leq s} > P_j^{\leq s}$. The numerical results support the hypothesis formulated above:

If $r = 13$ and $s = 2$, using $P_j^{\leq 2} = \frac{1}{13} \sum_{i=1}^{13} p_{i,j}^{\leq 2}$ for $j = 1, \dots, 7$ we have:

$P_1^{\leq 2} = \frac{744038784529}{14044285015605}$, $P_2^{\leq 2} = \frac{976603795454}{14044285015605}$, $P_3^{\leq 2} = \frac{1210299028862}{14044285015605}$,

$P_4^{\leq 2} = \frac{1177803460549}{14044285015605}$, $P_5^{\leq 2} = \frac{1163110568270}{14044285015605}$, $P_6^{\leq 2} = \frac{1166481172598}{14044285015605}$,

$P_7^{\leq 2} = \frac{1167611395081}{14044285015605}$

and $P_3^{\leq 2} > P_4^{\leq 2} > P_7^{\leq 2} > P_6^{\leq 2} > P_5^{\leq 2} > P_2^{\leq 2} > P_1^{\leq 2}$.

If $r = 13$ and $s = 3$, using $P_j^{\leq 3} = \frac{1}{13}\sum_{i=1}^{13} p_{i,j}^{\leq 3}$ for $j = 1, \ldots, 7$ we have:

$P_1^{\leq 3} = \frac{11421331946332}{215094122259016}$, $P_2^{\leq 3} = \frac{14014960627976}{215094122259016}$, $P_3^{\leq 3} = \frac{16636540997384}{215094122259016}$,

$P_4^{\leq 3} = \frac{19273726011848}{215094122259016}$,$P_5^{\leq 3} = \frac{18738795860092}{215094122259016}$, $P_6^{\leq 3} = \frac{18375423919160}{215094122259016}$,

$P_7^{\leq 3} = \frac{18172563533432}{215094122259016}$

and $P_4^{\leq 3} > P_5^{\leq 3} > P_6^{\leq 3} > P_7^{\leq 3} > P_3^{\leq 3} > P_2^{\leq 3} > P_1^{\leq 3}$.

Similarly, we can formulate a hypothesis for the even case $r = 2k$. Numerical results are as follows:

If $r = 12$ and $s = 2$, using $P_j^{\leq 2} = \frac{1}{12}\sum_{i=1}^{12} p_{i,j}^{\leq 2}$ for $j = 1, \ldots, 6$ we have:

$P_1^{\leq 2} = \frac{324912449742}{5608732106592}$, $P_2^{\leq 2} = \frac{426027121422}{5608732106592}$, $P_3^{\leq 2} = \frac{527707796946}{5608732106592}$,

$P_4^{\leq 2} = \frac{512489528178}{5608732106592}$, $P_5^{\leq 2} = \frac{505697482050}{5608732106592}$, $P_6^{\leq 2} = \frac{507531674958}{5608732106592}$

and $P_3^{\leq 2} > P_4^{\leq 2} > P_6^{\leq 2} > P_5^{\leq 2} > P_2^{\leq 2} > P_1^{\leq 2}$.

If $r = 12$ and $s = 3$, using $P_j^{\leq 3} = \frac{1}{12}\sum_{i=1}^{12} p_{i,j}^{\leq 3}$ for $j = 1, \ldots, 6$ we have:

$P_1^{\leq 3} = \frac{15892503222}{272994497208}$, $P_2^{\leq 3} = \frac{19481415198}{272994497208}$, $P_3^{\leq 3} = \frac{23120875230}{272994497208}$,

$P_4^{\leq 3} = \frac{26760335262}{272994497208}$, $P_5^{\leq 3} = \frac{25924348182}{272994497208}$, $P_6^{\leq 3} = \frac{25317771510}{272994497208}$

and $P_4^{\leq 3} > P_5^{\leq 3} > P_6^{\leq 3} > P_3^{\leq 3} > P_2^{\leq 3} > P_1^{\leq 3}$.

Let us define a random walk which never stops by prohibiting absorbing states: the walk $(\tilde{S}_n^{i,\leq s})_{n \in Z_{\geq 0}}$ starting at $i$ has a finite state space $I(r) = \{1, 2, \ldots, r\}$, the probability that its $l-th$ move length is less than or equal to s (:= as above) is 0 and the probabilities of all other states are equal. Then the transition probability matrix $\tilde{P}^{\leq s} = ||\tilde{p}_{ij}^{\leq s}||$ of the walk has the following form:

$$\tilde{p}_{ij}^{\leq s} = \begin{cases} \frac{1}{r-s-i}, & \text{for all } j \in I(r) - \{1, \ldots, i+s\} \text{ if } i \in \{1, \ldots, s\} \\ \frac{1}{r-2s-1}, & \text{for all } j \in I(r) - \{i-s, \ldots, i+s\} \text{ if } i \in \{s+1, \ldots, r-s\} \\ \frac{1}{i-s-1}, & \text{for all } j \in I(r) - \{i-s, \ldots, r\} \text{ if } i \in \{r-s+1, \ldots, r\} \\ 0, & \text{otherwise.} \end{cases}$$

It can be readily seen that the transition matrix is ergodic (there exists a number $n_0(s)$ such that $\min_{i,j}(\tilde{p}_{ij}^{\leq s})^{(n_0(s))} > 0$) and there is a unique stationary distribution $\pi = (\pi_1, \ldots, \pi_r)$ satisfying $\sum_{j=1}^{r} \pi_j = 1$ (by symmetry $\pi_j = \pi_{r-j+1}$ for $j = 1, \ldots, \lfloor \frac{r}{2} \rfloor$). Straightforward calculations show that for $r = 2k$ and $r = 2k+1$ we get the following stationary distributions respectively:

$$\pi_j = \begin{cases} \frac{r-s-j}{(r-s-1)(r-s)}, & j \in \{1, \ldots, s\} \\ \frac{r-2s-1}{(r-s-1)(r-s)}, & j \in \{s+1, \ldots, k\}, \end{cases}$$

$$\pi_j = \begin{cases} \frac{r-s-j}{(r-s-1)(r-s)}, & j \in \{1, \ldots, s\} \\ \frac{r-2s-1}{(r-s-1)(r-s)}, & j \in \{s+1, \ldots, k+1\}. \end{cases}$$

Thus, there is no any oscillation behaviour of limiting probabilities in contrast to Theorem 4.1:

if $r = 2k$: $\pi_1 > \cdots > \pi_s > \pi_{s+1} = \cdots = \pi_k$,

if $r = 2k+1$: $\pi_1 > \cdots > \pi_s > \pi_{s+1} = \cdots = \pi_{k+1}$.

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