DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

SEMIGROUP METHODS IN THE REPRESENTATION THEORY OF THE INFINITE SYMMETRIC GROUP

by
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İZMİR

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We have read the thesis entitled "**SEMIGROUP METHODS IN THE REPRESENTATION THEORY OF THE INFINITE SYMMETRIC GROUP** " completed by **SEDEF TAŞKIN** under supervision of **PROF. DR. SELÇUK DEMİR** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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Sedef TAŞKIN

SEMIGROUP METHODS IN THE REPRESENTATION THEORY OF THE INFINITE SYMMETRIC GROUP

ABSTRACT

This thesis is devoted to giving a classification of all the unitary representations of the infinite symmetric group, which consists of bijections on natural numbers, that are continuous with respect to a suitable topology in the infinite symmetric group. For the first time in 1972 this classification was done by A. Lieberman without the use of semigroups . We work out the semigroup approach to this problem developed by G. I. Olshansky. In particular, we associate a sequence of representations of finite inverse symmetric semigroups to any unitary representation of the infinite symmetric group.

Keywords: Infinite symmetric group, finite inverse symmetric semigroups, continuous unitary respresentations, partial permutations, good representations, A. Lieberman Theorem

SONSUZ SİMETRİK GRUBUN TEMSİL TEORİSİNDE YARIGRUP METODLARI

ÖZ

Bu tezde, doğal sayılar üzerindeki permütasyonlardan oluşan sonsuz simetrik grubun uygun bir topolojisine göre sürekli olan üniter temsillerinin sınıflandırılması çalışılmıştır. Bu sınıflandırma ilk olarak 1972'de A. Lieberman tarafından yarıgruplar kullanılmadan yapıldı. Biz bu problemi, G. I. Olshansky tarafından geliştirilen yarıgrup methoduyla çalıştık. Özel olarak, sonsuz simetrik grubun üniter temsillerinin her birini sonlu ters simetrik yarıgrupların bir dizi temsilleriyle ilişkilendirdik.

Anahtar kelimeler: Sonsuz simetrik grup, sonlu ters simetrik yarıgruplar, sürekli üniter temsiller, kısmi permütasyonlar, iyi temsiller, A. Lieberman Teorem

CONTENTS

CHAPTER ONE
INTRODUCTION

Representation theory was initiated in 1896 with the studies of the German mathematician F. G. Frobenius. The idea motivating the representation theory is to make an abstract algebraic object more concrete by describing its elements by linear transformation of vector spaces and the algebraic operations in terms of addition and multiplication of linear transformations. The algebraic objects which are suitable for such a description include semigroups and groups. In other words, representation theory reduces problems in abstract algebra to problems in linear algebra, which is a well understood subject. Moreover, the vector space on which an abstract object is represented can be infinite-dimensional, and by this allowance, for instance, a Hilbert space, methods of analysis can be applicable.

In mathematics, there are many applications of the representation theory. So far, a vast number of connections to other branches of mathematics have been explored, in particular to combinatorics (especially representations of the finite symmetric groups $S(n)$), number theory, Lie algebras, group theory. Moreover, there are applications of representation theory in algebraic and geometric computational complexity, cyro-electron imaging, digital signal processing, holographic algorithms and quantum computing, machine learning and pattern recognation. Peculiarly, the infinite symmetric group and its representations have an important place in the field of modern theoretical physics such as gauge theories, gravitation and integrable systems.

Roughly speaking, representation theory is the study of the ways in which a given group may act on vector spaces. The description of representation theory which we give is very simple, but it is splitted substantially when we would like to be more specific. In particular, the common objects of representation theory of groups are locally compact groups such as finite groups, Lie groups, etc. Nevertheless, the representation theory of nonlocally compact groups such as groups of diffeomorphisms, certain groups of intinite matrices, is very rich. It is a remarkable fact that to any some representation of such a group, a sequence of representations of certain finite dimensional semigroups is associated naturally. By using this technique, one may get a classification of all the unitary representations. The aim of this thesis is to make a present of an extensive introduction to this topic on a special example of the infinite symmetric group, $S(\infty)$ which we will define soon.

In this thesis, we study Lieberman Theorem which gives a classification of all the unitary representations of $S(\infty)$ that are continuous relative to pointwise convergence topology in $S(\infty)$. For the first time it was proved by A. Lieberman without using the semigroups, to find his proof see Lieberman 1972. However, semigroup approach takes advantage of handy facts from the representation theory of finite semigroups. There are many reasons to study with semigroups from the representation theoretic point of view. First of all, obviously finite semigroups have less complicated structure than infinite groups. However, by using representations of finite semigroups, we can obtain a classification of certain representations of some infinite groups. The aim of this thesis is to introduce this subject by an example: the infinite symmetric group. On the other hand, the objects of a representation theory in a complex Hilbert space are sets with a kind of topological and algebraic structure that does not go beyond that of some subsets of bounded linear operators on Hilbert space. In the light of this fact, another reason appearing immediately is that the bounded linear operators on Hilbert space forms a semigroup under composition, which occupies a significant place in functional analysis. A difficult problem in representation theory is for a given representation whether there is a decomposition into irreducible representations or not. We give the answer of the question in Chapter 2 for finite semigroups: representations of finite semigroups break up into a direct sum of irreducible representations. In addition, any irreducible representation of a finite semigroup is finite dimensional.

We haven't described our main object of study yet, the infinite symmetric group, $S(\infty)$. There are different meaningful realizations of the infinite symmetric group. Mostly, $S(\infty)$ is defined as the group of all bijections of the set of natural numbers $\mathbb{N} = \{1, 2, 3, ...\}$ to itself. In this thesis, alternatively, we will consider $S(\infty)$ as the group of all infinite permutation matrices, i.e. infinite matrices whose number of 1's in any row or column is exactly one. We are acquainted with the notion of considering the elements of finite symmetric group $S(n)$ as $n \times n$ permutation matrices: to any $\sigma \in S(n)$ we associate the corresponding matrix by

$$\sigma_{ij} = \begin{cases} 1 & i\sigma = j \\ 0 & otherwise. \end{cases}$$

In exactly the same way, the group of bijections of $\mathbb{N}$ turns into the group of all infinite permutation matrices. Seeing elements of $S(\infty)$ as infinite permutation matrices will help us understand more easily the topology that will be constructed on the infinite symmetric group in Chapter 3. $S(n)$ can be imbedded into $S(\infty)$: any $n \times n$ permutation matrix σ is identified with the infinite permutation matrix

$$\begin{pmatrix} \sigma & 0 \\ 0 & I \end{pmatrix}$$

where I denotes the infinite identity matrix. In Chapter 3, we give a detailed information about the realizations of the infinite symmetric group.

Let us now describe the content of the thesis in a little more detail.

The second chapter of the thesis deals with the representation theory of semigroups. It contains necessary definitions and results from semigroup representation theory. A representation of a semigroup Γ is a couple (H, τ) where H is a complex Hilbert space, called the representation space, and τ is a morphism which associates a bounded linear operator $H \to H$ to each element of Γ. Our main interest is in finding the building blocks of the representations which are called the irreducible representations. At this point, some properties of bounded linear operators on a Hilbert space are used in order to obtain a characterization of irreducible representations. It is natural to ask whether we have any relation between two representations of a semigroup. We construct an equivalence relation between representations and we will not distinguish between representations and equivalence classes.

In Chapter 3, we introduce the infinite symmetric group $S(\infty)$. More clearly, it begins with the definition of the infinite symmetric group, then develops its basic properties. In this chapter, some of its subgroups are defined, such as $S^o(\infty)$, $S_n(\infty)$, $\tilde{S}_n(\infty)$. The first subgroup $S^o(\infty)$ is the union of the infinite chain

$$S(1) \subset S(2) \subset S(3) \subset ... \tag{1.1}$$

of growing finite symmetric groups. $S^o(\infty)$ is countable and it is reasonable to consider it as a natural infinite analog of the finite symmetric groups $S(n)$. The elements of the second subgroup $S_n(\infty)$ are infinite permutation matrices whose upper left $n \times n$ submatrix is the identity matrix of size n. $S(\infty)$ has a topological group structure for which $\{S_n(\infty)\}_{n\geq 1}$ is a fundamental system of neighborhoods of the identity element.

In Chapter 4, we describe some representations of the infinite symmetric group. In this chapter we use the first realization of the infinite symmetric group, i.e. we consider the elements of $S(\infty)$ as bijections from $\mathbb{N}$ to itself. First we fix a natural number n and an irreducible representation π of $S(n)$. For this pair (n, π) we construct a representation of $S(\infty)$, which will be denoted by $T(n, \pi)$. These representations are proven to be irreducible and continuous with respect to the group topology of $S(\infty)$ which is constructed in Chapter 3.

In Chapter 5, we introduced the partial permutations. A partial permutation is simply a bijection defined on a subset of $\mathbb{N}$. The set of all partial permutations is denoted by $\Gamma(\infty)$. By the same way as we did for $S(\infty)$, we may consider $\Gamma(\infty)$ as the set of all infinite 0-1 matrices, i.e. matrices whose number of 1's in any row or column is at most one. There is a detailed information about the two realizations of $\Gamma(\infty)$ in the begining of the chapter. The infinite symmetric group is a subgroup of $\Gamma(\infty)$. Similarly we define the semigroup $\Gamma(n)$ of all the square 0-1 matrices of order n. The semigroups $\Gamma(n)$ are called the finite inverse symmetric semigroups. $\Gamma(\infty)$ has a topological structure and its topology is compatible with the group topology of $S(\infty)$ defined in Chapter 3. Simply, any neighborhood of an element γ of $\Gamma(\infty)$ consists of the elements whose left upper $n \times n$ submatrices are the same as γ. One of the striking fact is that with respect to this topology $S^o(\infty)$ is dense in $\Gamma(\infty)$.

In Chapter 6, we relate the unitary representations of $S^o(\infty)$, continuous unitary representations of $S(\infty)$ and representations of $\Gamma(\infty)$. For a unitary representation T^o of $S^o(\infty)$, we define $H_n(T^o)$ as the subspace of $S_n^o(\infty) = S_n(\infty) \bigcap S^o(\infty)$-invariant vectors in $H(T^o)$. These subspaces constitutes a chain

$$H_1(T^o) \subseteq H_2(T^o) \subseteq H_3(T^o) \subseteq \ldots \tag{1.2}$$

We denote their union by $H_\infty(T^o) = \bigcup_{n \in \mathbb{N}} H_n(T^o)$. We say that a unitary representation of $S^o(\infty)$ is a good representation if $H_\infty(T^o)$ is dense in $H(T^o)$. We showed that any good representation of $S^o(\infty)$ can be extended to a continuous unitary representation of $S(\infty)$ and $\Gamma(\infty)$ which acts in the same representation space $H(T^o)$. By the density of $S^o(\infty)$, these extensions are unique. To extend a unitary representation T^o of $S^o(\infty)$ to a representation of $\Gamma(\infty)$, we first construct the representations τ_n of the semigroups $\Gamma(n)$, $\forall n \in \mathbb{N}$. The extension representation τ of $\Gamma(\infty)$ is is obtained from the representations τ_n.

In Chapter 7, we introduce the finite inverse symmetric semigroups $\Gamma(m)$, $m \in \mathbb{N}$ and their representations. $\Gamma(m)$ is the semigroup of all $m \times m$ 0-1 matrices satisfying the following property: number of 1's in any row or column is at most one. Let $k \leq m$ and $\varepsilon_{k,m} \in \Gamma(m)$ be the element

$$\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}_{m \times m} \tag{1.3}$$

Let τ be an irreducible representation of the semigroup $\Gamma(m)$. We denote the image of $\tau(\varepsilon_{k,m})$ by $H_k(\tau)$. Let $n = n(\tau)$ be the smallest natural number such that $H_{n(\tau)}(\tau) \neq 0$. We proved that $H_n(\tau)$ is an irreducible representation of $S(n)$ and any representation of $\Gamma(m)$ is uniquely determined by the number $n(\tau)$ and the representation of $S(n)$ in $H_n(\tau)$. After giving necessary results about representations of the semigroups $\Gamma(m)$, we prove the A. Lieberman Theorem.

Lieberman Theorem is the main theorem of this thesis. This theorem classifies the representations of the infinite symmetric group. In particular, it has two major statements: the first one says that the representations $T(n, \pi)$, where $n = 1, 2, ...$, and π is a unitary irreducible representation of $S(n)$, and the trivial representation are all continuous irreducible representations of $S(\infty)$; the second one says that any continuous unitary representation of $S(\infty)$ is a direct sum of irreducible representations. To prove Lieberman Theorem, in essence, we use the Olshansky semigroup approach. This approach first allows us to obtain representations of the finite semigroups $\Gamma(n)$, $n \in \mathbb{N}$ from any representation of the infinite symmetric group (uniquely). Moreover, it helps us classify all of the irreducible representations of the finite semigroups $\Gamma(n)$, $n \in \mathbb{N}$. Thanks to this classification, we classify the representations of $S(\infty)$, which is the purpose of this thesis.

CHAPTER TWO
PRELIMINARIES

Let Γ be a semigroup with unity. An involution $* : \Gamma \longrightarrow \Gamma$ is a bijection which satisfies the following:

$$\gamma^{**} = \gamma, \;\; (\gamma_1\gamma_2)^* = \gamma_2^*\gamma_1^* \tag{2.1}$$

for every $\gamma, \gamma_1, \gamma_2 \in \Gamma$.

From now on, the semigroup Γ will be assumed to possess an involution.

Proposition 1. *Let Γ be a semigroup with unity. Then $1^* = 1$ where 1 denotes the unity of the semigroup.*

Proof. For all $\gamma \in \Gamma$, we have $\gamma = \gamma^{**} = (\gamma^*)^* = (\gamma^*1)^* = 1^*\gamma$.

Taking $\gamma = 1$, since 1 is the identity, $1 = 1^*1 = 1^*$. □

Let H be a complex Hilbert space. A linear operator $u : H \to H$ is bounded if there is a constant C such that

$$\|u(x)\| \leq C\|x\| \tag{2.2}$$

for all $x \in H$. Using the linearity of u

$$\left\| u\left(\frac{x}{\|x\|}\right) \right\| = \left\| \frac{u(x)}{\|x\|} \right\| = \frac{\|u(x)\|}{\|x\|} \tag{2.3}$$

we see that u is bounded, satisfying equation 2.2 if and only if

$$sup_{\|x\|=1}\|u(x)\| \leq C. \tag{2.4}$$

We denote by $B(H)$ the set of all bounded linear operators $u : H \to H$. $B(H)$ form a vector space. For all $u, v \in B(H)$, $u + v$ is the transformation with $(u + v)(x) = u(x) + v(x)$ for all $x \in H$ and for every constant C, Cu is the operator $x \mapsto Cu(x)$ for all $x \in H$. On $B(H)$ we define the operator norm by

$$\|u\| = sup_{\|x\|=1}\|u(x)\| \tag{2.5}$$

for all $u \in B(H)$.

Let $\Gamma(H) = \{u \in B(H) : \|u\| \leq 1\}$. Then we have:

Proposition 2. *$\Gamma(H)$ is a semigroup with involution (the usual conjugation of operators).*

Proof. By definition of norm, we already have

$$\|st\| \leq \|s\| \cdot \|t\| \tag{2.6}$$

for all $s, t \in B(H)$. So the composition of any two operators from $\Gamma(H)$ has norm less than or equal to 1. This makes $\Gamma(H)$ a semigroup. Thus it only remains to show that $\Gamma(H)$ is closed under involution. To do this, we show that $\|s^*\| = \|s\|, \forall s \in B(H)$. Since

$$\|s(x)\|^2 = \langle s(x), s(x)\rangle = |\langle (s^*s)(x), x\rangle| \leq \|(s^*s)(x)\| \cdot \|x\| \leq \|s^*s\| \cdot \|x\|^2 \tag{2.7}$$

From this we see that $\|s\| \leq \sqrt{\|s^*s\|} \leq \sqrt{\|s^*\|}\sqrt{\|s\|}$. Squaring both sides we get $\|s\| \leq \|s^*\|$. Similarly, $\|s^*\| \leq \|s^{**}\| = \|s\|$.

Hence $\Gamma(H)$ is closed under involution. □

The semigroup $\Gamma(H)$ is equipped with the weak operator topology. (There is an explanation about the weak operator topology in Appendix A.) From now on, when we say that a subspace (a net) of $\Gamma(H)$ is weakly closed (weakly convergent), we mean that it is a closed subspace (convergent net) with respect to the weak operator topology of $\Gamma(H)$.

Definition 1. *A representation of a semigroup Γ is a complex Hilbert space H along* *with the morphism $\tau : \Gamma \longrightarrow \Gamma(H)$ which commutes with involution and maps 1 into 1. If Γ is equipped with a topology then τ is supposed to be continuous with respect to weak operator topology on $\Gamma(H)$. Thus for every $\gamma, \gamma_1, \gamma_2 \in \Gamma$*

$$\tau(\gamma_1\gamma_2) = \tau(\gamma_1)\tau(\gamma_2), \quad \tau(\gamma^*) = \tau(\gamma)^*, \quad \tau(1) = 1_{B(H)} \tag{2.8}$$

The Hilbert space of a representation τ will be denoted by $H(\tau)$. We denote such a representation by $(H(\tau), \tau)$ or simply τ.

We say that $(H(\tau), \tau)$ is a unitary representation of the semigroup Γ if for all invertible elements $\gamma \in \Gamma$, the associated linear operator $\tau(\gamma)$ is a unitary linear operator.

Proposition 3. *Let $S = \{\gamma \in \Gamma \mid \gamma^*\gamma = \gamma\gamma^* = 1\}$. S is a subgroup of Γ. If τ is a representation of Γ, then $\tau|_S$ is a unitary representation of S.*

Proof. By Proposition 1, we know that $1^* = 1$ and so $1 \in S$, so that S is unital. If $\alpha, \beta \in S$, then

$$(\alpha\beta)(\alpha\beta)^* = \alpha\beta\beta^*\alpha^* = 1 \tag{2.9}$$

and

$$(\alpha\beta)^*(\alpha\beta) = \beta^*\alpha^*\alpha\beta = 1 \tag{2.10}$$

which shows that S is closed under multiplication. By construction of S each element has an inverse in S. Hence S is a subgroup of Γ.

Next, if $\sigma \in S$ then

$$\tau(\sigma)^* = \tau(\sigma^*) = \tau(\sigma^{-1}) = \tau(\sigma)^{-1} \tag{2.11}$$

So $\tau(\sigma)$ is unitary. □

Definition 2. *Let τ be a representation of the semigroup Γ. A closed subspace S of $H(\tau)$ is said to be invariant under Γ if for all $\gamma \in \Gamma$ and for all $s \in S$, $\tau(\gamma)(s) \in S$.*

Proposition 4. *Let τ be a representation of Γ. If $H \subset H(\tau)$ is invariant under Γ, then the orthogonal complement $H^{\perp}$ is invariant, too.*

Proof. Let $h \in H, h' \in H^{\perp}, \gamma \in \Gamma$. Since H is invariant under τ,

$$\langle \tau(\gamma)(h'), h \rangle = \langle h', \tau(\gamma)^*(h) \rangle = \langle h', \tau(\gamma^*)(h) \rangle = 0 \tag{2.12}$$

This shows that $\tau(\gamma)(h') \in H^{\perp}$. Hence $H^{\perp}$ is also an invariant subspace. □

Definition 3. *A representation τ of the semigroup Γ is said to be irreducible if there is no proper nontrivial closed invariant subspace in $H(\tau)$.*

A closed subspace M of a Hilbert space H is stable under a set of operators $\Lambda \subset B(H)$ if $T(M) \subset M$ for every $T \in \Lambda$. A set $\Lambda \subset B(H)$ is said to be irreducible, if there is no such a $M \subset H$.

Proposition 5. *A subspace M of a Hilbert space H is stable under a set of operators $\Lambda \subset B(H)$ if and only if it is stable under the weakly closed algebra A generated by Λ.*

Proof. If M is stable under Λ then it is stable under the algebra $\tilde{A}$ generated by Λ. Now given $X \in A$, let N be a directed set and let $\{X_n\}_{n \in N}$ be a net with $X_n \in \tilde{A}$ such that $X_n \to X$ weakly in $B(H)$. For any $x \in M$ and $y \in M^{\perp}$ we have $X_n(x) \in M$ and so $0 = \langle X_n(x), y \rangle \to \langle X(x), y \rangle$. This proves that $X(x) \perp M^{\perp}$,i.e. $X(M) \subset M$ and so M is stable under A. Since $\Lambda \subset A$ the stability of M under A clearly implies that M is stable under Λ. □

We say that an operator algebra A is self adjoint if $A = A^*$, that is for all $T \in A$, $T^* \in A$. Let A' denote the set of continuous operators commuting with all operators in A.

$$A' = \{M \in B(H) : MN = NM, \forall N \in A\} \tag{2.13}$$

Theorem 1. *Let A be a self adjoint algebra of continuous operators in complex Hilbert space . Then A is irreducible if and only if $A' = \mathbb{C}I$.*

Proof. If A is reducible then by $A = A^*$ we have $H = M + M^{\perp}$ where M and $M^{\perp}$ are nonzero stable subspaces. Then we claim that A' contains the projection $P : H \to M$ and so $A \neq \mathbb{C}I$. We first show that a subspace M is stable under Λ if and only if the associated projection belongs to Λ'. In view of Proposition 5 stability with respect to Λ and A mean the same thing. Since $A = A^*$ and M is stable under A, then so is $M^{\perp}$. We need to show that $PT = TP$ for all $T \in A$ if $M, M^{\perp}$ are stable under Λ.

Since M is stable, $TP(M) \subset T(M) \subset M$. Thus if x is any element of H then $TP(x) = P(x')$ for some $x' \in H$ and so by $P^2 = P$ we obtain $PTP = TP$. Similarly since $M^{\perp}$ is also stable we have $(I - P)T(I - P) = T(I - P)$, that is to say $PTP = PT$. Hence if $M, M^{\perp}$ are stable then $PT = PTP = TP$. Conversely if $PT = TP$ then $T(M) = TP(H) = PT(H) = P(H) = M$ and in addition $(I - P)T = T(I - P)$. By the same reasoning we also get $T(M^{\perp}) = T(I - P)(H) = (I - P)T(H) \subset (I - P)(H) = M^{\perp}$.

Conversely, if A is irreducible then the only projections in A' are 0 and I . Every weakly closed self adjoint algebra larger than $\mathbb{C}I$ contains a proper projection (It is a consequence of a sequence of theorems and a detailed information can be found in Gaal (1973) see the References page.). Thus A is irreducible if and only if the commutant A' is equal to $\mathbb{C}I$. □

Theorem 2. *A representation $\tau : \Gamma \to \Gamma(H)$ is irreducible if and only if the algebra generated the operators $\tau(\gamma)(\gamma \in \Gamma)$ is dense in $B(H)$ in the weak operator topology.*

Proof. If A is irreducible, then by Theorem 1, $A' = \mathbb{C}I$. Since all operators commutes with scalar operators we have $A = B(H)$. Now suppose $A = B(H)$ so that a $T \in A'$ commutes with every projection $P : H \to H$. We fix $x \neq 0$ in H and let P denote the projection $P : H \to \mathbb{C}x$. Then $T(x) = \lambda(x)x = PT(x) = \lambda(x)x$ for some $\lambda(x) \in \mathbb{C}$ implying $T(x) = \lambda(x)x, \forall x \in H$. Take $x, y \in H$ as linearly independent vectors and consider

$$T(x + y) = \lambda(x + y)(x + y) = T(x) + T(y) = \lambda(x)x + \lambda(y)y \tag{2.14}$$

this implies that we have $(\lambda(x)-\lambda(x+y))x=(\lambda(y)-\lambda(x+y))y$. But x, y are chosen as linearly independent vectors so we have $\lambda(x)=\lambda(x+y)=\lambda(y)$. Thus $T(x)=\lambda x$ where λ is a fixed element of $\mathbb{C}$ and so $T=\lambda I$. Hence $A'=\mathbb{C}I$ and so irreducible by Theorem 1. $\square$

Definition 4. *Let H be a Hilbert space and $\{z_\alpha\}$ a family of vectors in H which are labelled by a set A. The set $\{z_\alpha\}$ is said to be total in H if its linear span is dense in H.*

Definition 5. *Let τ be a representation of the semigroup Γ. A vector $z \in H(\tau)$ is said to be cyclic if the family $\{\tau(\gamma)(z)|\gamma \in \Gamma\}$ is total in $H(\tau)$. A subspace $H_o \subset H(\tau)$ is said to be cyclic if the family $\{\tau(\gamma)(z)|\gamma \in \Gamma, z \in H_o\}$ is total in $H(\tau)$.*

If τ is irreducible then any nonzero vector (or subspace) is cyclic. A cyclic span of a subspace $H \subset H(\tau)$ (or of a vector z) is the smallest invariant closed subspace containing it.

Definition 6. *Let $(\tau, H(\tau))$ and $(\tau', H(\tau'))$ be two representations of Γ. We say that τ and τ' are equivalent if there exists a linear isometry $I : H(\tau) \to H(\tau')$ such that $I\tau(\gamma)=\tau^{'}(\gamma)I$ for every $\gamma \in \Gamma$.*

Lemma 1. *Let $\{z_\alpha\}$ and $\{z^{'}_\alpha\}$, $\alpha \in A$, be two total families in the spaces H and $H^{'}$, respectively. Suppose $\langle z_\alpha, z_\beta\rangle = \langle z^{'}_\alpha, z^{'}_\beta\rangle$ for any $\alpha, \beta \in A$. Then there exists a linear isometry $H \longrightarrow H'$ which maps z_α into $z^{'}_\alpha$.*

Proof. Let us define $I : H \to H'$ by $I(z_\alpha)=z'_\alpha$ for all $\alpha \in A$ and extend linearly on $span\{z_\alpha|\alpha \in A\}$. Then for any $x, y \in span\{z_\alpha|\alpha \in A\}$, we clearly have $\langle I(x), I(y)\rangle = \langle x, y\rangle$. Now suppose x is a limit of a sequence from $span\{z_\alpha|\alpha \in A\}$, that is, $x=\lim_{n\to\infty} x_n$ where $x_n \in span\{z_\alpha|\alpha \in A\}$ for every $n \in \mathbb{N}$. Consider the image sequence $\{I(x_n)\}_{n\in\mathbb{N}}$. It is a Cauchy sequence. Indeed, let $\varepsilon > 0$

$$\|I(x_n)-I(x_m)\| = \|I(x_n-x_m)\| = \|x_n-x_m\| \tag{2.15}$$

For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|x_n - x_m\| < \varepsilon$ for every $n \geq N$ since

$\{x_n\}_{n\in\mathbb{N}}$ is a convergent sequence. So that $\{I(x_n)\}_{n\in\mathbb{N}}$ is Cauchy, which means it is convergent. Define $I(x) = \lim_{n\to\infty} I(x_n)$.

Next, we claim that I is an isometry. Suppose that x, y are limits of sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}$, respectively from $span\{z_\alpha | \alpha \in A\}$. Then

$$\langle x, y\rangle = \langle \lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\rangle = \lim_{n\to\infty}\langle x_n, y_n\rangle = \lim_{n\to\infty}\langle I(x_n), I(y_n)\rangle = \langle I(x), I(y)\rangle \tag{2.16}$$

So that $I : H \to H'$ is an isometry. □

Corollary 1. *Let τ and τ' be two representations of Γ. Suppose that there exist cyclic vectors $z \in H(\tau)$ and $z' \in H(\tau')$ such that*

$$\langle \tau(\gamma)(z), z\rangle = \langle \tau'(\gamma)(z'), z'\rangle \tag{2.17}$$

for every $\gamma \in \Gamma$. Then τ and τ' are equivalent.

Proof. z and z' are cyclic vectors means that $\overline{span\{\tau(\gamma)(z) : \gamma \in \Gamma\}} = H(\tau)$ and $\overline{span\{\tau'(\gamma)(z') : \gamma \in \Gamma\}} = H(\tau')$, that is, the families $\{\tau(\gamma)(z) : \gamma \in \Gamma\}$ and $\{\tau'(\gamma)(z') : \gamma \in \Gamma\}$ are total families in $H(\tau)$ and $H(\tau')$, respectively.

For any $\gamma_1, \gamma_2 \in \Gamma$ we have

$$\langle \tau(\gamma_1)(z), \tau(\gamma_2)(z)\rangle = \langle \tau(\gamma_2^*\gamma_1)(z), z\rangle = \langle \tau^{'}(\gamma_2^*\gamma_1)(z^{'}), z^{'}\rangle = \langle \tau^{'}(\gamma_1)(z^{'}), \tau^{'}(\gamma_2)(z^{'})\rangle \tag{2.18}$$

So by Lemma 1, there exists a linear isometry $I : H(\tau) \to H(\tau^{'})$ which maps $\tau(\gamma)z$ into $\tau^{'}(\gamma)z^{'}$ for every $\gamma \in \Gamma$. We claim that I gives an equivalence between τ and $\tau^{'}$, that is, $I\tau(\gamma) = \tau'(\gamma)I$ for every $\gamma \in \Gamma$.

If $x \in span\{\tau(\gamma)(z) : \gamma \in \Gamma\}$, it can be easily seen that $I\tau(\gamma)(x) = \tau'(\gamma)I(x)$. Suppose x is a limit of a sequence from $span\{\tau(\gamma)(z)|\gamma \in \Gamma\}$, that is, $x = \lim_{n\to\infty} x_n$

where $x_n \in span\{\tau(\gamma)(z)|\gamma \in \Gamma\}$ for every $n \in \mathbb{N}$. Then

$$I\tau(\gamma)(x) = I\tau(\gamma)(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} I\tau(\gamma)(x_n) = \lim_{n\to\infty} \tau'(\gamma)I(x_n) = \tau'(\gamma)I(x) \tag{2.19}$$

This proves our claim. □

A zero in Γ is an element 0 such that $\gamma 0 = 0\gamma = 0$ for every $\gamma \in \Gamma$.

Lemma 2. *Suppose that Γ contains 0 and let τ be an irreducible representation of Γ. Then either $\tau(0) = 0$ or $dim(H(\tau)) = 1$ and $\tau(\gamma) = 1$ for any $\gamma \in \Gamma$ (i.e. the trivial representation).*

Proof. Suppose that Γ contains 0 and let τ be an irreducible representation of Γ. Consider the subspace $H = \overline{span\{\tau(0)(\xi) : \xi \in H(\tau)\}}$. We claim that H is a closed and Γ-invariant subspace of $H(\tau)$.

For any $\gamma \in \Gamma$,

$$\tau(\gamma)\tau(0)(\xi) = \tau(\gamma 0)(\xi) = \tau(0)(\xi) \in H \tag{2.20}$$

so that H is Γ-invariant. By irreducibility either $H = H(\tau)$ or $H = 0$.

If $H = 0$, then $\tau(0)(\xi) = 0 \;\; \forall \xi \in H(\tau)$, which implies that $\tau(0) = 0$.

Now suppose $H = H(\tau)$. If $x \in H(\tau) = H$ is such that $\sum_{i=1}^{k} \tau(0)\xi_i = x$ for some $\xi_i \in H(\tau)$, then

$$\tau(\gamma)(x) = \tau(\gamma)\sum_{i=1}^{k} \tau(0)\xi_i = \sum_{i=1}^{k} \tau(\gamma)\tau(0)\xi_i = \sum_{i=1}^{k} \tau(\gamma 0)\xi_i = x \tag{2.21}$$

This implies that $\tau(\gamma) = 1$ for any $\gamma \in \Gamma$.

Finally, let us see that $dim(H(\tau)) = 1$. For any nonzero $\xi \in H(\tau)$, the subspace $H' = span\{\tau(0)\xi\}$ is a 1-dimensional closed invariant subspace of $H(\tau)$ and so $H' = H = H(\tau)$. □

Lemma 3. *Let* Γ *be finite. Then any of its representations breaks up into a direct sum of irreducible representations, and any irreducible representation is finite dimensional.*

Proof. A finite dimensional representation of Γ is a direct sum of irreducible representations. Because, if P_1 is an invariant subspace of positive, minimal dimension, then it is irreducible. (We can find such a subspace because the representation is finite dimensional, so its invariant subspaces have finite dimension. By minimality of dimension, P_1 is irreducible. Because otherwise it would contain a proper invariant subspace of dimension less than P_1.) By Proposition 4, $P_1^{\perp}$ is an invariant subspace and so by the same reasoning $\tau|_{P_1^{\perp}}$ has an irreducible subspace P_2. In finitely many steps we obtain a decomposition $H(\tau) = P_1 \oplus ... \oplus P_k$ where each P_i is an irreducible subspace of $H(\tau)$.

Now we consider any representation $(\tau, H(\tau))$ of Γ. Take a nonzero vector x from its representation space $H(\tau)$. Then the cyclic span of this nonzero vector $\overline{span\{\tau(\gamma)(x) : \gamma \in \Gamma\}}$ is an invariant finite dimensional subspace because Γ is a finite semigroup. Hence it contains a nonzero irreducible subspace say H. We have $H(\tau) = H \oplus H^{\perp}$. Now take a nonzero vector from $H^{\perp}$ and consider its cyclic span again. Continuing in this manner, we can write $H(\tau)$ as a direct sum of irreducible representations. So any of its irreducible representation must be finite dimensional. Otherwise it would contain a finite dimensional invariant subspace. $\square$

CHAPTER THREE
THE INFINITE SYMMETRIC GROUP

In introduction, we have mentioned that the infinite symmetric group has fundamentally two meaningful realizations. Generally, $S(\infty)$ is defined as the set of all bijections from $\mathbb{N} = \{1, 2, ...\}$ to itself. The set $S(\infty)$ acts on $\mathbb{N}$ on the right, the image of $i \in \mathbb{N}$ under $\sigma \in S(\infty)$ being written as $i\sigma$. Let $\sigma, \beta \in S(\infty)$. Their product, which we call composition of σ and β, $\alpha = \sigma\beta$ is defined as follows $i\alpha = (i\sigma)\beta$ for all $i \in \mathbb{N}$. This binary operation is associative and has identity I which sends every natural number to itself. Every element $\sigma \in S(\infty)$ has an inverse defined by $i\sigma^{-1} = j$ if $j\sigma = i$. Thus $S(\infty)$ is a group under compositon of bijections.

We may consider $S(\infty)$ as the set of all infinite permutation matrices, i.e matrices whose number of 1's in any row or column is exactly 1. To see this we assign to any $\sigma \in S(\infty)$ an infinite permutation matrix (σ_{ij})

$$\sigma_{ij} = \begin{cases} 1 & i\sigma = j \\ 0 & otherwise. \end{cases}$$

It is easy to see that any permutation matrix is of the form (σ_{ij}) for some $\sigma \in S(\infty)$ and composition of the bijections in $S(\infty)$ turns into the product of matrices. The set of infinite permutation matrices becomes a group under the multiplication of matrices. The identity of the group is I whose all diagonal entries are 1 and the inverse of each element is its transpose. In this thesis, we have used matrix description of $S(\infty)$. Because at the end of this Chapter, we will construct a group topolgy on $S(\infty)$ and the matrix description will be very useful to describe this topology more clearly. Moreover, in Chapter 5, the reason why we use the matrix description will make sense more obviously. The first description of $S(\infty)$ will only be used in Chapter 4.

Let us describe some important subgroups of $S(\infty)$. An infinite permutation matrix $\sigma \in S(\infty)$ is said to be finite if the number of 0's on the diagonal is finite. The set of all finite elements of $S(\infty)$ will be denoted by $S^o(\infty)$.

We will consider $S(n)$, the group of all permutation matrices of order n. We can identify each $\gamma \in S(n)$ with the element of $S(\infty)$ of the form

$$\begin{pmatrix} \gamma & 0 \\ 0 & I \end{pmatrix} \in S(\infty)$$

Hence for any $\gamma \in S(n)$, we have finitely many 0's on the diagonal (Indeed, there can be at most n zeros on the diagonal.), which implies that $S(n) \subset S^o(\infty)$. So we have $\bigcup_{n\in\mathbb{N}} S(n) \subset S^o(\infty)$. On the other hand, any $\sigma \in S^o(\infty)$ can be considered as an element of $S(m)$ for a sufficiently large m. Thus we have

$$\bigcup_{n\in\mathbb{N}} S(n) = S^o(\infty). \tag{3.1}$$

Proposition 6. *$S^o(\infty)$ is a countable normal subgroup in $S(\infty)$.*

Proof. For every natural number n, the group $S(n)$ is countable (finite) and by the equation 3.1 $S^o(\infty)$ is a countable union of countable sets. Since countable union of countable sets is countable, $S^o(\infty)$ is countable. So we only need to show the normality of $S^o(\infty)$. Let $\sigma \in S(\infty)$, $\tau \in S^o(\infty)$. Since $S^o(\infty)$ is the union of $S(n)$, $\forall n \in \mathbb{N}$, τ can be seen as an element of $S(m)$ for some $m \in N$. Then, one can also see that $\sigma\tau\sigma^{-1}$ has at most m zeros on the diagonal. Indeed, if $k > m$ then $\tau_{kk} = 1$. An ij th term is obtained by the multiplication $(\sigma\tau\sigma^t)_{ij} = \sigma_{ik}\tau_{kp}\sigma_{jp}$. If $k = p$, then $\sigma_{ik}\tau_{kk}\sigma_{jk} = \sigma_{ik}\sigma_{jk} = \sigma_{ik}(\sigma^t)_{kj} = I_{ij}$. Thus $(\sigma\tau\sigma^t)_{ij} = 1$ if $i = j$ and $k > m$ we obtain that $\sigma\tau\sigma$ could have at most m zeros on the diagonal. $\square$

Given a number $n = 1, 2, 3, ...$, we define the set $S_n(\infty)$ consisting of the elements whose the upper left $n \times n$ submatrix of the associated infinite permutation matrices are the identity matrix of size n

$$\begin{pmatrix} I_n & 0 \\ 0 & K \end{pmatrix} \in S(\infty)$$

where K is an infinite permutation submatrix and the set $\tilde{S}_n(\infty)$ whose elements has 0-1 matrix of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in S(\infty)$$

where A is an nxn matrix and where B is an infinite permutation submatrix.

$S_n(\infty)$ and $\tilde{S}_n(\infty)$ are subgroups of $S(\infty)$. For every $n \in \mathbb{N}$ we have that $S_{n+1}(\infty) \subset S_n(\infty)$ and $\bigcap_{n\in\mathbb{N}} S_n(\infty) = \{I\}$

Proposition 7. *$\tilde{S}_n(\infty)$ is the semidirect product of its normal subgroup $S_n(\infty)$ and $S(n)$.*

Proof. First we can show that $S_n(\infty)$ is normal in $\tilde{S}_n(\infty)$ by using a similar argument as in the proof of Proposition 6. It is quite clear that any $\sigma \in S_n(\infty) \cap S(n)$ must be identity, I. Thus we only need to check $\tilde{S}_n(\infty) = S_n(\infty)S(n)$. To do this we take a $\sigma \in \tilde{S}_n(\infty)$ and define τ and γ as $\tau_{ij} = \sigma_{ij}$ if $i \leq n$ and $\tau_{ij} = I_{ij}$ if $i > n$, $\gamma_{ij} = \sigma_{ij}$ if $i > n$ and $\gamma_{ij} = I_{ij}$ if $i \leq n$. Then $\tau \in S(n)$, $\gamma \in S_n(\infty)$ and $\sigma = \gamma\tau$. Then by Appendix C, we conclude that $\tilde{S}_n(\infty)$ is the semidirect product of its normal subgroup $S_n(\infty)$ and $S(n)$. □

By Theorem 12 in Appendix A2, in $S(\infty)$ there exists a (unique) structure of topological group for which $\{S_n(\infty)\}_{n\geq 1}$ is a fundamental system of neighborhoods of the unity. By Corollary 4 in Appendix A2, for any element $\sigma \in S(\infty)$ the classes $\{\sigma S_n(\infty)\}_{n\geq 1}$ are fundamental systems of neighborhoods of σ. This means that for any $\sigma \in S(\infty)$ a neighborhood V of σ contains the elements $\beta \in S(\infty)$ such that for a sufficiently large n, the upper left $n \times n$ submatrices of σ and β are the same. By Corollary 5, $S_n(\infty)$ and $\tilde{S}_n(\infty)$ are open in $S(\infty)$. Indeed,

$$\tilde{S}_n(\infty) = \bigcup_{\sigma \in S(n)} \sigma S_n(\infty). \tag{3.2}$$

Remark 1. *This topology on the infinite symmetric group is not compact. In particular for $S^o(\infty)$, the family $\{S_n(\infty)\}_{n\geq 1}$ is an open cover for and has no finite subcover.*

CHAPTER FOUR
CONSTRUCTING A FAMILY OF UNITARY REPRESENTATIONS

Let π be an irreducible unitary representation of the group $S(n)$. We can consider π as a representation of $\tilde{S}_n(\infty)$ by the following way. Let $\sigma \in \tilde{S}_n(\infty)$. By Proposition 7 in Chapter 3, we may write $\sigma = \sigma_1\sigma_2$ where $\sigma_1 \in S(n)$, $\sigma_2 \in S_n(\infty)$. Then we set $\pi(\sigma) = \pi(\sigma_1)$. Obviously, π is a unitary representation of $\tilde{S}_n(\infty)$.

π induces a unitary representation $T = T(n, \pi)$ of $S(\infty)$. By definition $H(T)$ is the space of all functions $f : S(\infty) \to H(\pi)$ such that

$$f(\sigma_1\sigma) = \pi(\sigma_1)f(\sigma), \ \ \forall\sigma_1 \in \tilde{S}_n(\infty), \forall\sigma \in S(\infty). \tag{4.1}$$

Now we fix coset representatives $\{\sigma_k\}_{k\in I}$ so that $S(\infty) = \sqcup_{k\in I}\sigma_{\dot{k}}$ where $\sigma_{\dot{k}}$ denotes $\tilde{S}_n(\infty)\sigma_k$. Indeed, by definition,

$$f(\beta\sigma_k) = \pi(\beta)f(\sigma_k) \tag{4.2}$$

for all $\beta \in \tilde{S}_n(\infty)$. Since any element $\sigma \in S(\infty)$ can be written uniquely in the form $\sigma = \beta\sigma_k$ for some $k \in I$ and $\beta \in \tilde{S}_n(\infty)$, the set of values $\{f(\sigma_k)\}_{k\in I}$ determines f.

$H(T)$ becomes an inner product space defined by for all $f, g \in H(T)$

$$\langle f, g\rangle = \sum_{k\in I}\langle f(\sigma_{\dot{k}}), g(\sigma_{\dot{k}})\rangle_{H(\pi)} \tag{4.3}$$

Since π is the unitary representation of $\tilde{S}_n(\infty)$, for every $\beta \in \tilde{S}_n(\infty)$

$$\langle f(\beta\sigma_k), g(\beta\sigma_k)\rangle_{H(\pi)} = \langle\pi(\beta)f(\sigma_k), \pi(\beta)g(\sigma_k)\rangle_{H(\pi)} = \langle f(\sigma_k), g(\sigma_k)\rangle_{H(\pi)} \tag{4.4}$$

Thus the notation $\langle f(\sigma_{\dot{k}}), g(\sigma_{\dot{k}})\rangle_{H(\pi)}$ in equation 4.3 is correct.

This inner product induces a norm on $H(T)$:

$$\|f\|^2 = \sum_{k\in I} \|f(\sigma_k^{\cdot})\|_{H(\pi)}^2 \tag{4.5}$$

and by definition $||f|| < \infty$ for all $f \in H(T)$. The action of $S(\infty)$ in $H(T)$ is given as follows:

$$T(\sigma)f(\sigma') = f(\sigma'\sigma), \ \ \forall \sigma, \sigma' \in S(\infty). \tag{4.6}$$

Proposition 8. *$(T, H(T))$ is a unitary representation of $S(\infty)$.*

Proof. We claim that $\langle T(\sigma)(f), T(\sigma)(g)\rangle = \langle f, g\rangle$ for all $f, g \in H(T)$.

$$\begin{array}{rcl} \langle T(\sigma)f, T(\sigma)g\rangle & = & \sum_{k\in I}\langle T(\sigma)f(\sigma_k^{\cdot}), T(\sigma)g(\sigma_k^{\cdot})\rangle_{H(\pi)} \\ & = & \sum_{k\in I}\langle f(\sigma_k^{\cdot}\sigma), g(\sigma_k^{\cdot}\sigma)\rangle_{H(\pi)} \\ & = & \sum_{m\in I}\langle f(\sigma_m^{\cdot}), g(\sigma_m^{\cdot})\rangle_{H(\pi)} \\ & = & \langle f, g\rangle \end{array} \tag{4.7}$$

□

We have an another description of $T(n, \pi)$. First we define this representation and then we will show that they are actually equivalent representations. From now on, for this chapter, we use the usual realization of $S(\infty)$, that is, we consider the elements of $S(\infty)$ as bijections from $\mathbb{N}$ to itself.

Let $\Omega = \Omega(n)$ denote the set of all injective maps $\omega : \{1, 2, ..., n\} \to \mathbb{N}$. If we denote $k\omega = i_k$, then any ω can be seen as an ordered sequence of distinct numbers $\omega = (i_1, i_2, ..., i_n)$. One may also consider $\omega = (i_1, i_2, ..., i_n)$ as an infinite 0-1 matrix whose entries are

$$\omega_{kl} = \begin{cases} 1 & l = i_k = k\omega \\ 0 & otherwise. \end{cases}$$

$S(\infty)$ has an action on $\Omega(n)$ given by for $\omega = (i_1, i_2, ..., i_n) \in \Omega(n)$ and $\sigma \in S(\infty)$

$$\omega\sigma = (i_1\sigma, ..., i_n\sigma), \tag{4.8}$$

and the action of $S(\infty)$ on $\Omega(n)$ corresponds to the multiplication of elements of $\Omega(n)$ by the elements of $S(\infty)$ from right; $\omega\sigma$. Also, $\tilde{S}_n(\infty)$ has an action on $\Omega(n)$ given by for $\sigma' \in \tilde{S}_n(\infty)$ and $\omega = (i_1, i_2, ..., i_n) \in \Omega(n)$

$$\sigma'\omega = (i_{1\sigma'}, ..., i_{n\sigma'}), \tag{4.9}$$

and the action of $\tilde{S}_n(\infty)$ on $\Omega(n)$ corresponds to the multiplication of elements of $\Omega(n)$ by the elements of $\tilde{S}_n(\infty)$ from left; $\sigma'\omega$.

We define $H'(T')$ as the space of all the functions $f : \Omega(n) \to H(\pi)$ such that

$$f(\sigma'\omega) = \pi(\sigma')f(\omega) \;\; \forall\omega \in \Omega \;\; \forall\sigma' \in \tilde{S}_n(\infty) \tag{4.10}$$

$$\|f\|^2 = \sum_{\omega\in\Omega} \|f(\omega)\|^2_{H(\pi)} < \infty \tag{4.11}$$

Then $S(\infty)$ acts on $H'(T')$ as follows:

$$T'(\sigma)f(\omega) = f(\omega\sigma) \tag{4.12}$$

for every $\sigma \in S(\infty)$ and $\omega \in \Omega$.

Now let's show that $(T, H(T))$ and $(T', H'(T'))$ are equivalent representations.

Let $\Delta : S(\infty) \to \Omega(n)$ be defined for any $\sigma \in S(\infty)$

$$\Delta(\sigma) = (1\sigma, 2\sigma, ..., n\sigma) \tag{4.13}$$

We have the following observation.

Proposition 9. *1. Δ is onto.*

2. $\Delta(\beta\sigma) = \beta\Delta(\sigma)$ for all $\beta \in \tilde{S}_n(\infty)$

Proof. 1. Clear.

2. Let $\beta \in \tilde{S}_n(\infty)$ and $\sigma \in S(\infty)$. Denote $k\sigma = i_k$ for all $k \in \mathbb{N}$. Then $k(\beta\sigma) = (k\beta)\sigma = i_{k\beta}$. So that $\Delta(\beta\sigma) = (i_{1\beta}, ..., i_{n\beta})$ which is equivalent to $\beta(i_1, ..., i_n) = \beta\Delta(\sigma)$.

□

Recall that we fixed the coset representatives of $\tilde{S}_n(\infty)$, which are $\{\sigma_k\}_{k\in I}$, so that $S(\infty) = \sqcup_{k\in I}\tilde{S}_n(\infty)\sigma_k$. Applying Δ to both sides of the last equation, since the action of $\tilde{S}_n(\infty)$ and $S(n)$ on $\Omega(n)$ are the same and by Proposition 9, we have $\Omega(n) = \sqcup_{k\in I}S(n)\omega_k$ where $\omega_k = \Delta(\sigma_k)$. (The union is still disjoint because $\Delta^{-1}(S(n)\omega_k) \subseteq \tilde{S}_n(\infty)\sigma_k$.)

Let us take any $f \in H(T)$. We define a function $\delta_f : \Omega(n) \to H(\pi)$ by

$$\delta_f(\omega_k) = f(\sigma_k) \tag{4.14}$$

for all $k \in I$ and for all $\beta \in \tilde{S}_n(\infty)$

$$\delta_f(\beta\omega_k) = f(\beta\sigma_k) \tag{4.15}$$

So that $\delta_f \circ \Delta = f$. The induced function δ_f is well-defined. In order to show this suppose that for $\sigma_1, \sigma_2 \in S(\infty)$, $\Delta(\sigma_1) = \Delta(\sigma_2)$. So there exists $\gamma \in S_n(\infty)$ such that $\sigma_1 = \gamma\sigma_2$. Since π acts trivially on $S_n(\infty)$, we have

$$f(\sigma_1) = f(\gamma\sigma_2) = \pi(\gamma)f(\sigma_2) = f(\sigma_2) \tag{4.16}$$

so that δ_f is well-defined:

$$\delta_f \circ \Delta(\sigma_1) = f(\sigma_1) = f(\sigma_2) = \delta_f \circ \Delta(\sigma_2) \tag{4.17}$$

Moreover δ_f has the following properties. For all $\beta \in \tilde{S}_n(\infty)$, $k \in I$

$$\delta_f(\beta\omega_k) = f(\beta\sigma_k) = \pi(\beta)f(\sigma_k) = \pi(\beta)\delta_f(\omega_k) \tag{4.18}$$

and

$$\|\delta_f\|^2 = \sum_{\omega\in\Omega(n)} \|\delta_f(\omega)\|^2_{H(\pi)} = \sum_{k\in I} \|\delta_f(\beta\omega_k)\|^2_{H(\pi)} = \sum_{k\in I} \|f(\beta\sigma_k)\|^2_{H(\pi)} = \|f\|^2_{H(T)} < \infty \tag{4.19}$$

so that $\delta_f \in H'(T')$. On the other hand, for any $f \in H(T)$, there is only one induced function $\delta_f \in H'(T')$ by its construction. Furthermore, for any $g \in H'(T')$ one can find an $f \in H(T)$ such that $\delta_f = g$ (just take f as $g \circ \Delta$). Thus this construction induces a bijection between $H(T)$ and $H'(T')$; $\delta : H(T) \to H'(T')$ given by $\delta(f) = \delta_f$ for any $f \in H(T)$. This induced function δ is an isometry by equation 4.19 and sets up an equivalence between $H(T)$ and $H'(T')$, that is

$$T'(\sigma)\delta = \delta T(\sigma) \tag{4.20}$$

for all $\sigma \in S(\infty)$.

From now on, we will use the second description $(T', H'(T'))$ and we denote it by $(T, H(T))$ instead of $(T', H'(T'))$.

For any $m = 1, 2, ...$, let $H_m(T)$ be the subspace of all $S_m(\infty)$-invariant vectors in $H(T)$ and P_m be the projection in $H(T) \to H_m(T)$.

$$H_m(T) = \{f \in H(T) | T(\sigma)f = f, \ \forall\sigma \in S_m(\infty)\} \tag{4.21}$$

Now we will describe a subset $\Omega_m = \Omega_m(n)$ of $\Omega(n)$. We call Ω_m the set of partial permutations $\gamma \in \Omega$ which satisfy $i_1, i_2, ..., i_n \leq m$, where $m \geq n$.

We have one more special subspace $H_m^{'}(T)$ of $H(T)$ which consists of functions whose support is a subset of Ω_m.

$$H'_m(T) = \{f \in H(T)|\text{supp}f \subseteq \Omega_m\} \tag{4.22}$$

where

$$\text{supp}f = \overline{\{\omega \in \Omega|f(\omega) \neq 0\}}. \tag{4.23}$$

Let $S(n)$^ denote the set of all irreducible unitary representations of $S(n)$.

Lemma 4. *Let $T = T(n, \pi)$, where $n = 1, 2, ...,$and $\pi \in S(n)$^*

1. *If $m < n$ then $H_m(T) = \{0\}$;*
2. *If $m \geq n$ then $H_m(T) = H'_m(T)$;*
3. *If $m \geq n$, $\forall f \in H(T)$, $P_m f = \chi_m f$ where χ_m is the characteristic function of Ω_m.*

Proof. Let $f \in H'_m(T)$ be arbitrary. Then supp$f \subseteq \Omega_m$. We claim first that f is $S_m(\infty)$-invariant. Let $\sigma \in S_m(\infty)$. Then for any $\omega \in \Omega_m$,

$$T(\sigma)f(\omega) = f(\sigma\omega) = f(\omega), \ \ \forall \sigma \in S_m(\infty). \tag{4.24}$$

Hence $f \in H_m(T)$.

Now take $f \in H_m(T)$. Then $f(\omega) = T(\sigma)f(\omega) = f(\omega\sigma)$ for every $\sigma \in S_m(\infty)$ is constant along the $S_m(\infty)$-orbits in $\Omega(n)$. We have two cases: either the orbit containing ω is finite or infinite. If $\omega \nsubseteq \{1, ..., m\}$, then the $S_m(\infty)$-orbit of ω can not be finite. But the sum of $\|f(\omega)\|$ is finite. So that if $\omega \nsubseteq \{1, ..., m\}$, then $\|f(\omega)\|$ must be 0. On the other hand, if $\omega \subseteq \{1, ..., m\}$, then the $S_m(\infty)$-orbit of ω contains only

ω. Thus the support of f must be contained in $\Omega_m(n)$, which means that $f \in H'_m(T)$. So (2) is proved.

(1) follows from the fact that $H'_m(T) = 0\ \forall m < n$.

Since $H_m(T) = H'_m(T)$, for all $f \in H(T)$ the equality $P_m f = \chi_m f$ follows. This shows (3). $\square$

Lemma 5. *Let $T = T(n, \pi)$, where $n = 1, 2, ...,$and $\pi \in S(n)^\wedge$. Then $H'_n(T)$ is isomorphic to $H(\pi)$ as the representations of $S(n)$.*

Proof. By definition

$$H'_n(T) = \{f : \Omega_n(n) \to H(\pi) : f(\sigma) = \pi(\sigma)f(I_n), \forall \sigma \in \Omega_n(n)\} \tag{4.25}$$

where I_n denotes the partial permutation $(1, 2, ..., n)$. Thus $f \in H'_n(T)$ is determined by $f(I_n) \in H(T)$ and any choice of $f(I_n) \in H(\pi)$ determines an element of $H'_n(T)$. Therefore we have an isomorphism of vector spaces $H'_n(T) \to H(\pi)$ which is given by $f \mapsto f(I_n)$. Furthermore

$$T(\sigma)f(I_n) = f(I_n\sigma) = f(\sigma I_n) = \pi(\sigma)f(I_n) \tag{4.26}$$

for all $\sigma \in S(n)$, so that the map is $S(n)$-linear. $\square$

Lemma 6. *Let $T = T(n, \pi)$, where $n = 1, 2, ...$ and $\pi \in S(n)^\wedge$. For any $\sigma \in S(\infty)$, $T(\sigma) : H(T) \to H(T)$ carries $H'_n(T)$ to isomorphically $K_\sigma = \{f \in H(T) | \mathrm{supp} f \subseteq \Omega_n(n)\sigma^{-1}\}$ with the inverse $T(\sigma^{-1})$.*

Proof. Let $f \in H'_n(T)$. Then $T(\sigma)f(\omega) = f(\omega\sigma)$ which is 0 unless $\omega\sigma \in \Omega_n(n)$, or equivalently unless $\omega \in \Omega_n(n)\sigma^{-1}$. Thus $T(\sigma)f \in K_\sigma$.

On the other hand, if $f \in K_\sigma$, then $T(\sigma^{-1})f(\omega) = f(\omega\sigma^{-1}) \neq 0$ if $\omega\sigma^{-1} \in \Omega_n(n)\sigma^{-1}$, i.e. $\omega \in \Omega_n(n)$. So that $T(\sigma^{-1})f \in H'_n(T)$. $\square$

Corollary 2. *Let $T = T(n,\pi)$, where $n = 1,2,...$ and $\pi \in S(n)^\wedge$. Then $H'_n(T)$ is cyclic in $H(T)$ as a subspace. That is,*

$$\overline{span\{T(\sigma)f | \sigma \in S(\infty), f \in H'_n(T)\}} = H(T) \tag{4.27}$$

Proof. Let $f \in H(T)$. For any $\sigma \in S(\infty)$ we denote $\chi_\sigma f$ by f_σ where χ_σ is the characteristic function of $K_\sigma = \{f \in H(T) | \text{supp} f \subseteq \Omega_n(n)\sigma^{-1}\}$. Let $\varepsilon > 0$. Then there exist some $\sigma^1, ..., \sigma^m \in S(\infty)$ and $f_{\sigma^i} \in K_{\sigma^i}$ such that

$$||f - \sum_{i=1}^{m} f_{\sigma^i}|| < \varepsilon. \tag{4.28}$$

By the isomorphism in Lemma 6, there exists $g_i \in H'_n(T)$ such that $f_{\sigma^i} = T(\sigma^i)g_i$ and so

$$||f - \sum_{i=1}^{m} f_{\sigma^i}|| = ||f - \sum_{i=1}^{m} T(\sigma^i)g_i|| < \varepsilon. \tag{4.29}$$

Since $\sum_{i=1}^{m} T(\sigma^i)g_i \in span\{T(\sigma)f | \sigma \in S(\infty), f \in H'_n(T)\}$, the result follows. $\square$

Lemma 7. *$T = T(n,\pi)$ is an irreducible representation for any $n = 1,2,...$ and $\pi \in S(n)^\wedge$.*

Proof. By Lemma 6 and Corollary 2, $H'_n(T)$ is $S(n)$-irreducible and cyclic in $H(T)$. By Lemma 4, $H'_n(T) = H_n(T)$, so $H_n(T)$ carries the same properties.

Suppose that there is a decomposition $H(T) = H \oplus H^\perp$ into closed invariant subspaces $H, H^\perp$. For any $f \in H_n(T)$, its projections on H and $H^\perp$ are $S_n(\infty)$-invariant, too. Indeed, if $f = f_1 + f_2$, where $f_1 \in H$ and $f_2 \in H^\perp$, then for any $\sigma \in S_n(\infty)$

$$f = T(\sigma)f = T(\sigma)(f_1 + f_2) = T(\sigma)f_1 + T(\sigma)f_2 \tag{4.30}$$

Since $H, H^\perp$ are $S(\infty)$-invariant, $T(\sigma)f_1 \in H$ and $T(\sigma)f_2 \in H^\perp$. But f has the unique expression $f = f_1 + f_2$, so we must have $T(\sigma)f_1 = f_1$ and $T(\sigma)f_2 = f_2$. Thus both $f_1, f_2 \in H_n(T)$. Then

$$H_n(T) = (H_n(T) \cap H) \oplus (H_n(T) \cap H^{\perp}) \qquad (4.31)$$

Since $H_n(T)$ is $S_(n)$-irreducible, we have either $H_n(T) \subseteq H$ or $H_n(T) \subseteq H^{\perp}$. But $H_n(T)$ is cyclic, so that either $H = H(T)$ or $H^{\perp} = H(T)$ and so $T(n, \pi)$ is irreducible. □

Lemma 8. *The representations $T = T(n, \pi)$, where $n = 1, 2, ...$ and $\pi \in S(n)^\wedge$ are pairwise inequivalent.*

Proof. Suppose for $n_1, n_2 \in \mathbb{N}$ ($n_1 \leq n_2$) and $\pi_1, \pi_2 \in S(n)^\wedge$, the representations $T_1 = T(n_1, \pi_1)$ and $T_2 = T(n_2, \pi_2)$ are equivalent representations. Then there exists a linear isometry $I : H(T_1) \to H(T_2)$ such that for every $\sigma \in S(\infty)$, $IT_1(\sigma) = T_2(\sigma)I$. Now let $f \in H_{n_1}(T_1)$, that is, for every $\sigma \in S_{n_1}(\infty)$, $T_1(\sigma)f = f$. Then $IT_1(\sigma)f = I(f) = T_2(\sigma)I(f)$ for every $\sigma \in S_{n_1}(\infty)$. Thus $I(f) \in H_{n_1}(T_2)$. Since I is an isomorphism of vector spaces and by definition of n_1 and the subspace $H_{n_1}(T_1)$ is nonzero, there exists some $0 \neq f \in H_{n_1}(T_1)$ such that $0 \neq I(f) \in H_{n_1}(T_2)$. However then this means that $n_1 = n_2$. Because n_2 is the smallest number k making $H_k(T_2)$ nonzero. By Lemma 5, π_1 and π_2 are representations of $S(n)$ in $H_{n_1}(T_1)$ and $H_{n_1}(T_2)$, respectively. Then for any $\sigma \in S(n)$ and $f \in H_{n_1}(T_1)$, we have

$$IT_1(\sigma)f = I\pi_1 f = T_2(\sigma)I(f) = \pi_2(\sigma)If \qquad (4.32)$$

so that I sets up an equivalence between π_1 and π_2. Thus the representations $T = T(n, \pi)$, where $n = 1, 2, ...$ and $\pi \in S(n)^\wedge$ are pairwise inequivalent. □

Lemma 9. *The representations $T(n, \pi)$ constructed as above are continuous representations with respect to the group topology of $S(\infty)$ defined in Chapter 3.*

Proof. Let $\sigma \in S(\infty)$ be close to I, that is, σ is an element of a sufficiently close neighborhood of the identity I. Our claim is now reduced to show that $T(\sigma)f$ is close to f for any fixed $f \in H(T)$. For any $\varepsilon > 0$, we can find a natural number $m \in \mathbb{N}$

such that $\|f - f_m\| \leq \frac{\varepsilon}{2}$ where $f_m = P_m f = \chi_m f$. Then for all $\sigma \in S_m(\infty)$, we have $T(\sigma) f_m = f_m$. Hence

$$\begin{aligned} \|T(\sigma)f - f\| &= \|T(\sigma)(f - f_m) - (f - f_m)\| \\ &\leq \|T(\sigma)(f - f_m)\| + \|f - f_m\| \\ &= 2\|f - f_m\| \leq \varepsilon. \end{aligned} \tag{4.33}$$

$\square$

CHAPTER FIVE
PARTIAL PERMUTATIONS

A partial permutation is a triple $\{D, \gamma, I\}$ where $D, I \subset \mathbb{N}$ and γ is a bijection from D to I. We denote the triplet shortly by γ. The set of all the partial permutations is denoted by $\Gamma(\infty)$. We say that $i\gamma$ is defined if $i \in D$; then $i\gamma$ denotes the image of i under γ. The product $\gamma = \gamma_1\gamma_2$ of elements $\gamma_1, \gamma_2 \in \Gamma(\infty)$ is constructed as follows: for $i \in \mathbb{N}$ $i\gamma$ is defined if and only if $i\gamma_1$ and $(i\gamma_1)\gamma_2$ are defined and then $i\gamma = (i\gamma_1)\gamma_2$. This product is associative so $\Gamma(\infty)$ is a semigroup. The semigroup has the unity 1and the zero 0; $i1$ is defined and equals to i for all $i \in \mathbb{N}$ but $i0$ is never defined.

The semigroup $\Gamma(\infty)$ may be considered as the semigroup of all the infinite 0-1 matrices (i.e. matrix whose values are 0 and 1) with the following property:

($\star$) In its any row or column the number of 1's is at most one.

To see this we assign to any $\gamma \in \Gamma(\infty)$ a matrix, (γ_{ij}) as follows: $\gamma_{ij} = 1$ if $i \in Dom(\gamma), i\gamma = j$ and $\gamma_{ij} = 0$ otherwise. Since γ is a bijection, in any row or column of (γ_{ij}) , there is at most one 1, and the other terms are 0. That is, (γ_{ij}) satisfies the property ($\star$). Moreover any 0-1 matrix with the property ($\star$) can be obtained by this way. The product in $\Gamma(\infty)$ turns into the product of matrices. In this thesis, we use matrix description of $\Gamma(\infty)$ mostly. Unless otherwise stated, the elements of $\Gamma(\infty)$ are regarded as matrices.

For given $\gamma \in \Gamma(\infty)$ let's denote the transpose of γ by γ^*. Then $* : \gamma \to \gamma^*$ is an involution of $\Gamma(\infty)$. Clearly $1^* = 1$ and $0^* = 0$.

The group $S(\infty)$ is a subgroup of $\Gamma(\infty)$. More precisely, $S(\infty)$ is the set of elements $\gamma \in \Gamma(\infty)$ which satisfy $\gamma^*\gamma = \gamma\gamma^* = 1$.

In a similar way, we may define the semigroup $\Gamma(n)$ of all the square $n \times n$ 0-1 matrices whose number of 1's in any row or column is at most one.

For any $n = 1, 2, ...$, let $\theta_n : \Gamma(\infty) \to \Gamma(n)$ be the map sending every $\gamma \in \Gamma(\infty)$ to its upper left $n \times n$ submatrix.

Lemma 10. *θ_n is onto and sends 1 to 1 and 0 to 0. Moreover θ_n commutes with involution.*

Proof. Let $\gamma \in \Gamma(n)$. One can imbed $\Gamma(n)$ into $\Gamma(\infty)$ by the following way:

$$\gamma\hat{} = \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix} \in \Gamma(\infty).$$

Then $\theta_n(\gamma\hat{}) = \gamma$. So θ_n is surjective.

Since the identity of $\Gamma(n)$ is a 0-1 matrix whose all diagonal entries are 1, and 0 of $\Gamma(n)$ is the zero matrix of size n, one can see that $\theta_n(1) = 1$ and $\theta_n(0) = 0$. Finally, let $\gamma \in \Gamma$. Then since the image of transpose of γ under θ_n is just the transpose of the upper left $n \times n$ submatrix of γ^*, we have $\theta_n(\gamma^*) = \theta_n(\gamma)^*$. □

The map θ_n gives a topological structure to $\Gamma(\infty)$ by the following way. For every $\gamma \in \Gamma(\infty)$ and $n \in \mathbb{N}$, we define $U(\gamma, n)$ as

$$U(\gamma, n) = \{\beta \in \Gamma(\infty) : \theta_n(\gamma) = \theta_n(\beta)\} \tag{5.1}$$

Let ζ consists of the sets $U(\gamma, n)$ for every $\gamma \in \Gamma(\infty)$ and $n \in \mathbb{N}$. ζ is a base for a topology on $\Gamma(\infty)$. Indeed, for any $\gamma \in \Gamma(\infty)$ the sets $U(\gamma, n)$ contains γ. On the other hand, for two sets $U(\gamma, n)$,$U(\alpha, m)$ from ζ where $\gamma, \alpha \in \Gamma(\infty)$ and $n, m \in \mathbb{N}$ (without loss of generality suppose $n \leq m$) if there exists an element $\beta \in U(\gamma, n) \cap U(\alpha, m)$, then $\theta_n(\alpha) = \theta_n(\beta) = \theta_n(\gamma)$. So that $U(\beta, n) \subset U(\gamma, n) \cap U(\alpha, m)$ for $U(\beta, n) \in \zeta$. Since the topology generated by ζ consists of arbitrary unions of sets in ζ, any open subset of $\Gamma(\infty)$ with respect to this topology is of the form

$$U(\gamma_i, n_i; i \in I) = \{\beta \in \Gamma(\infty) : \theta_{n_i}(\gamma_i) = \theta_{n_i}(\beta), \text{for some i } \in I\} \tag{5.2}$$

where I is an arbitrary index set.

A net $\{\gamma_\lambda\}_{\lambda\in\Lambda}$, where λ is a directed set, is said to be convergent to $\gamma_o \in \Gamma(\infty)$ if for every $n \in \mathbb{N}$, there exists a $\alpha_n \in \Lambda$ such that

$$\theta_n(\gamma_\lambda) = \theta_n(\gamma_o) \tag{5.3}$$

for every $\lambda \geq \alpha_n$. In this topology, any neighborhood of a given $\gamma \in \Gamma(\infty)$ consists of elements whose upper left $n \times n$ submatrices are the same as that of γ for a sufficiently large n. For instance, neighborhoods of the identity are the sets whose elements contain I_n for some $n \in \mathbb{N}$ in the upper left $n \times n$ submatrix. Restricting this neighborhoods to $S(\infty)$, we get the family of sets $\{S_n(\infty)\}_{n\in\mathbb{N}}$. So that we have the following result:

Lemma 11. *(i) Restricted topology of $\Gamma(\infty)$ to $S(\infty)$ is the group topology of $S(\infty)$ defined in Chapter 3.*
(ii) The involution $* : \gamma \to \gamma^*$ is continuous relative to the topology of $\Gamma(\infty)$.

Proof. (i) In the topology of $\Gamma(\infty)$, the neighborhood of $\sigma \in S(\infty)$ consists of elements whose upper left $n \times n$ submatrices are the same as that of σ. Since the neighborhoods of σ in the group topology of $S(\infty)$ are $\{\sigma S_n(\infty)\}_{n\in\mathbb{N}}$, these topologies coincide.

(ii) Let $\{\gamma_\lambda\}_{\lambda\in\Lambda}$, where Λ is a directed set, converge to γ in $\Gamma(\infty)$. Then for a sufficiently large n there exists $\alpha_n \in \Lambda$, such that $\theta_n(\gamma_\lambda) = \theta_n(\gamma)$ for all $\lambda \geq \alpha_n$.Then by Lemma 10,

$$\theta_n(\gamma_\lambda)^* = \theta_n(\gamma_\lambda^*) = \theta_n(\gamma^*) = \theta_n(\gamma)^* \tag{5.4}$$

for all $\lambda \geq \alpha_n$. Hence the involution $* : \gamma \to \gamma^*$ is continuous. □

Recall that in the begining of the Chapter 3, we mentioned that the motivation of seeing elements of $S(\infty)$ comes from the topological structure which we will put on $S(\infty)$. So far, we have constructed the topology on $\Gamma(\infty)$ and proved that the restricted topology of $\Gamma(\infty)$ to $S(\infty)$ is the group topology of $S(\infty)$ defined in Chapter 3. The topology on $\Gamma(\infty)$ is given by the equality of submatrices which are the images of the maps θ_n, $n \in \mathbb{N}$. Hence the matrix description aids us in comprehending the topological structure of $S(\infty)$.

Remark 2. *By Lemma 11, (i) part, we can say that this topology on* $\Gamma(\infty)$ *is not compact. Because by Remark 1 in Chapter 3 we know that the restricetd topology to* $S(\infty)$ *is not compact.*

Let

$$\ell^2 = \left\{ x = (x_n)_{n\in\mathbb{N}} : \|x\| = \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^2 < \infty \right\} \tag{5.5}$$

The semigroup $\Gamma(\infty)$ has a natural action in the Hilbert space ℓ^2. If we write the vectors $x \in \ell^2$ as column vectors, then the action of $\gamma \in \Gamma(\infty)$ is γx. In fact, since there are only 0 and 1's in the matrix γ, we have for all $x \in \ell^2$

$$||\gamma x|| \leq ||x|| < \infty \tag{5.6}$$

which means $\gamma x \in \ell^2$. Hence the elements of $\Gamma(\infty)$ are operators in ℓ^2. Then $\Gamma(\infty)$ becomes a subsemigroup of $\Gamma(\ell^2) = \{\gamma \in B(\ell^2) : \|\gamma\| \leq 1\}$.

Lemma 12. *The topology of* $\Gamma(\infty)$ *coincides with the weak operator topology of* $\Gamma(l^2)$.

Proof. Let $\{\gamma_\lambda\}_{\lambda\in\Lambda}$, where Λ is a directed set, converge to γ in the topology of $\Gamma(\infty)$. That means for every $n \in \mathbb{N}$ there exists $\alpha_n \in \Lambda$ such that $\theta_n(\gamma_\lambda) = \theta_n(\gamma)$ for all $\lambda \geq \alpha_n$. Then by definition of the inner product of l^2, we have for all $x, y \in l^2$

$$\langle(\gamma_\lambda - \gamma)x, y\rangle = \sqrt{\sum_{i=1}^{\infty}((\gamma_\lambda - \gamma)x)_i\overline{y_i}} \to 0. \tag{5.7}$$

Next we need to show that the convergence in the weak operator topology of $\Gamma(l^2)$ implies the convergence in the topology of $\Gamma(\infty)$. Let $\{\gamma_\lambda\}_{\lambda\in\Lambda}$, where Λ is a directed set, converge weakly to γ in $\Gamma(\infty)$. That is $\forall x, y \in l^2$ the sequence $\langle\gamma_\lambda(x), y\rangle$ converges to $\langle\gamma(x), y\rangle$. Equivalently, $\forall x, y \in l^2$ the sequence $\langle(\gamma_\lambda - \gamma)(x), y\rangle$ converges to 0. For sufficiently n's, we have in $\Gamma(l^2)$

$$\langle(\theta_n(\gamma_\lambda) - \theta_n(\gamma))(x), y\rangle \leq \langle(\gamma_\lambda - \gamma)x, y\rangle. \tag{5.8}$$

(Here we consider the matrices $\theta_n(\gamma_\lambda)$ as elements of $\Gamma(\infty)$ by the imbedding in Lemma 10.) So that $\langle(\theta_n(\gamma_\lambda) - \theta_n(\gamma))(x), y\rangle$ converges to 0 for all $x, y \in l^2$. In particular letting $y = (\theta_n(\gamma_\lambda) - \theta_n(\gamma))(x)$, we see that $||(\theta_n(\gamma_\lambda) - \theta_n(\gamma))(x)||$ converges to 0 for all $x \in l^2$. Hence $\theta_n(\gamma_\lambda) = \theta_n(\gamma)$, where n is sufficiently large. So that $\{\gamma_\lambda\}$ tends to γ in the topology of $\Gamma(\infty)$. $\square$

Let us denote by $S_n^o(\infty)$ the intersection $S_n(\infty) \cap S^o(\infty)$. Then any $\sigma \in S_n^o(\infty)$ is also an element of $S_n(\infty)$ and $S^o(\infty)$, which implies that $\sigma \in S(m)$ for some $m \in \mathbb{N}$. If $m \leq n$, then σ is the identity of $S(\infty)$. If $m > n$, then σ is a finite element whose number of zeros on the diagonal is at most $m - n$.

Since $S^o(\infty) \subseteq S(\infty)$, the map θ_n is defined on $S^o(\infty)$.

Lemma 13. *i. The restricted map $\theta_n : S^o(\infty) \to \Gamma(n)$ is onto.*

ii. There is a bijection between $\Gamma(n)$ and $S_n^o(\infty) \setminus S^o(\infty)/S_n^o(\infty)$ which is induced by θ_n. More precisely, for given $\sigma_1, \sigma_2 \in S^o(\infty)$, we have $\theta_n(\sigma_1) = \theta_n(\sigma_2)$ if and only if σ_1 and σ_2 belong to the same double class modulo $S_n^o(\infty)$.

Proof. i. Let $\gamma \in \Gamma(n)$ be arbitrary. We need to find a $\sigma \in S^o(\infty)$ whose upper left $n \times n$ submatrix is γ. Let k denote the number of 1's in γ. We need to find a $\sigma \in S^o(\infty)$ such that $\theta_n(\sigma) = \gamma$. First there exist $\sigma', \sigma'' \in S(n) \subset S^o(\infty)$ such that $\gamma = \sigma' \varepsilon_k \sigma''$ where

$$\varepsilon_k = \begin{pmatrix} 1_k & 0 \\ 0 & 0_{n-k} \end{pmatrix}$$

But then we may set $\sigma = \sigma' \sigma_{(k)} \sigma''$ where

$$\sigma_{(k)} = \begin{pmatrix} 1_k & 0 & 0 & 0 \\ 0 & 0_{n-k} & 1_{n-k} & 0 \\ 0 & 1_{n-k} & 0_{n-k} & 0 \\ 0 & 0 & 0 & 1_k \end{pmatrix}$$

ii. Let $\psi : S_n^o(\infty) \setminus S^o(\infty)/S_n^o(\infty) \to \Gamma(n)$ be defined for any $\sigma \in S^o(\infty)$

$$\psi(S_n^o(\infty)\sigma S_n^o(\infty)) = \theta_n(\sigma) \in \Gamma(n) \tag{5.9}$$

ψ is well defined and 1-1. Because any of two double cosets $S_n^o(\infty)\sigma_1 S_n^o(\infty)$ and $S_n^o(\infty)\sigma_2 S_n^o(\infty)$ are the same cosets if and only if the upper left $n \times n$ submatrices of σ_1 and σ_2 are the same matrices, that is $\theta_n(\sigma_1) = \theta_n(\sigma_2)$.

Let $\gamma \in \Gamma(n)$. By (i) , the map $\theta_n : S^o(\infty) \to \Gamma(n)$ is onto. So there exists a $\sigma \in S^o(\infty)$ such that $\theta_n(\sigma) = \gamma$. Hence $\psi(S_n^o(\infty)\sigma S_n^o(\infty)) = \theta_n(\sigma) = \gamma$, which proves that ψ is onto. Hence ψ is a bijection between $\Gamma(n)$ and $S_n^o(\infty) \setminus S^o(\infty)/S_n^o(\infty)$ which is induced by θ_n.

□

Now, by using Lemma 13, we will prove an important result which says that the subgroup $S^o(\infty)$ is dense in $S(\infty)$. By using the density, we will construct some representations of the finite semigroups $\Gamma(n)$, $n \in \mathbb{N}$ in the next chapter.

Corollary 3. $S^o(\infty)$ *(and hence* $S(\infty)$*) is dense in* $\Gamma(\infty)$.

Proof. Let $n \in \mathbb{N}$ and $\gamma_o \in \Gamma(\infty)$. Then $\theta_n(\gamma_o) \in \Gamma(n)$. By lemma 13, for every $n \in \mathbb{N}$, there exists $\gamma_n \in S^o(\infty)$ such that $\theta_n(\gamma_o) = \theta_n(\gamma_n)$. Then $(\gamma_n)_{n\in\mathbb{N}}$ is a net in $S^o(\infty)$ converging to γ_o. So $S^o(\infty)$ is dense in $\Gamma(\infty)$. Since $S^o(\infty) \subset S(\infty)$, the group $S(\infty)$ is dense in $\Gamma(\infty)$. □

CHAPTER SIX
GOOD REPRESENTATIONS

Assume that T is a continuous unitary representation of $S(\infty)$. Let $H_n(T)$ be the subspace of all $S_n(\infty)$-invariant vectors in $H(T)$ where $n \in \mathbb{N}$. For all $n \in \mathbb{N}$, we have $H_n(T) \subset H_{n+1}(T)$. Indeed if f is $S_n(\infty)$-invariant vector in $H(T)$, since $S_{n+1}(\infty) \subset S_n(\infty)$, we have that f is $S_{n+1}(\infty)$-invariant. So

$$H_1(T) \subset H_2(T) \subset ... \subset H_n(T) \subset H_{n+1}(T) \subset ... \tag{6.1}$$

Let $H_\infty(T) = \bigcup_{n\in\mathbb{N}} H_n(T)$.

In a similar way, let T^o be a unitary representation of $S^o(\infty)$ and $H_n(T^o)$ be the subspace of all $S_n^o(\infty)$-invariant vectors in $H(T^o)$. We denote the union of these subspaces $H_n(T^o)$ by $H_\infty(T^o)$.

Proposition 10. *Let T be a unitary representation of $S(\infty)$ and $T^o = T|_{S^o(\infty)}$. Then we have $H_n(T^o) = H_n(T)$ for all $n \in \mathbb{N}$ and $H_\infty(T) = H_\infty(T^o)$.*

Proof. For any representation T of $S(\infty)$, the restricted morphism to $S^o(\infty)$, $T^o = T|_{S^o(\infty)}$ gives a representation of $S^o(\infty)$. In this case $H_n(T) = H_n(T^o)$ for all $n \in \mathbb{N}$. Indeed, any $f \in H_n(T)$ is $S_n^o(\infty)$-invariant since $S_n^o(\infty) \subset S_n(\infty)$. So $H_n(T) \subseteq H_n(T^o)$. Now take any $f \in H_n(T^o)$ and $\sigma \in S_n(\infty)$.The density of $S^o(\infty)$ in $S(\infty)$ implies the density of $S_n^o(\infty)$ in $S_n(\infty)$. So that there exists a net $\{\sigma_\lambda\}_{\lambda\in\Lambda}$, where Λ is a directed set, in $S_n^o(\infty)$, converging to σ. Then $T(\sigma_\lambda)f = f$ for all $\lambda \in \Lambda$. Since T is a continuous representation, the image net $\{T(\sigma_\lambda)\}_{\lambda\in\Lambda}$ converges $T(\sigma)$. Let $\varepsilon > 0$. There exists an $\alpha \in \Lambda$ such that $||T(\sigma)f - T(\sigma_\lambda)f|| < \varepsilon$ for all $\lambda \geqslant \alpha$. Hence $||T(\sigma)f - f|| < \varepsilon$ for all $\varepsilon > 0$. Thus $T(\sigma)f = f$. So $H_n(T^o) \subseteq H_n(T)$. From this we conclude that $H_\infty(T) = H_\infty(T^o)$. $\square$

Lemma 14. *i. $H_m(T^o)$ is $S(n)$-invariant for any $m \geq n$.*

ii. The subspace $H_\infty(T^o)$ is $S^o(\infty)$-invariant.

Proof. i. It is enough to show that $H_m(T^o)$ is $S(m)$-invariant, since $S(n) \subseteq S(m)$ for all $n \leq m$. Let $f \in H_m(T^o)$ and $\sigma \in S(m)$. We will show that $T^o(\sigma)f$ is $S_m(\infty)$-invariant. Let $\beta \in S_m(\infty)$. Since $\beta\sigma = \sigma\beta$ and f is $S_m(\infty)$-invariant, we have

$$T^o(\beta)T^o(\sigma)f = T^o(\sigma)T^o(\beta)f = T^o(\sigma)f \tag{6.2}$$

ii. Let $\sigma \in S^o(\infty)$. Then $\sigma \in S(n)$ for some $n \in \mathbb{N}$. Let $f \in H_\infty(T^o)$. Then $f \in H_m(T^o)$ for some $m \in \mathbb{N}$. If $m \geq n$, then by part (i), $T(\sigma)f \in H_m(T^o) \subset H_\infty(T^o)$. If $m < n$, since $f \in H_m(T^o) \subset H_n(T^o)$, we have $T(\sigma)f \in H_n(T^o) \subset H_\infty(T^o)$. So that $H_\infty(T^o)$ is $S^o(\infty)$-invariant.

□

Now, we introduce the notion of good representations of the subgroup $S^o(\infty)$. Good representations are really crucial. In fact, good representations will help us reduce the problem of working representations of the infinite symmetric group to the problem of working that of finite semigroups $\Gamma(n)$, $n \in \mathbb{N}$.

Definition 7. *We say that a unitary representation T^o of $S^o(\infty)$ is good if $H_\infty(T^o)$ is dense in $H(T^o)$.*

If T^o is a unitary irreducible representation, since $H_\infty(T^o)$ is an invariant subspace (Lemma 14), it is enough to check that $H_\infty(T^o) \neq \{0\}$.

Lemma 15. *Suppose that T^o is a unitary good representation of $S^o(\infty)$. Then T^o is continuous with respect to the group topology on $S^o(\infty)$ described in Chapter 3.*

Proof. It is enough to prove the continuity at $I \in S^o(\infty)$. Let $f \in H(T^o)$ be fixed. We need to show that $T^o(\sigma)(f)$ is close to f when $\sigma \in S^o(\infty)$ is sufficiently close to I. Given $\varepsilon > 0$, by density of $H_\infty(T^o)$ in $H(T^o)$ we can find an m in such a way that

$$||f - P_m(f)|| \leq \frac{\varepsilon}{2} \tag{6.3}$$

Now suppose $\sigma \in S_m(\infty)$. Then $T(\sigma)(P_m f) = P_m f$, thus

$$\begin{aligned} \|T(\sigma)f - f\| &= \|T(\sigma)(f - P_m f) - (f - P_m f)\| \\ &\leq \|T(\sigma)(f - P_m f)\| + \|f - P_m f\| \\ &= 2\|f - P_m f\| \leq \varepsilon. \end{aligned} \tag{6.4}$$

□

It is time to prove one of the most important theorem of this thesis; Olshansky Theorem. By this theorem, we will show that any good representation of $S^o(\infty)$ gives a continuous unitary representation of $S(\infty)$ and a representation of $\Gamma(\infty)$ and conversely. In the proof, we will use the density of the subgroup $S^o(\infty)$ in $\Gamma(\infty)$. The construction in the proof will demonstrate the power of the semigroup approach in the following way: First we will write a representation for $\Gamma(n)$ for each $n \in \mathbb{N}$. Then by using all these representations we get a representation of $\Gamma(\infty)$ (so a representation of $S(\infty)$).

Theorem 3. *Assume that T^o is a unitary representation of $S^o(\infty)$. Then the following are equivalent:*
(i) T^o is a good representation of $S^o(\infty)$.
(ii) T^o is extended to a continuous unitary representation T of $S(\infty)$ which acts in $H(T^o)$.
(iii) T^o is extended to a representation τ of the semigroup $\Gamma(\infty)$ which acts in $H(T^o)$.

Proof. $(iii) \Rightarrow (ii)$: Assume that τ is the extended representation of $\Gamma(\infty)$. We know that $S(\infty)$ consists of elements $\sigma \in \Gamma(\infty)$ which satisfy $\sigma^*\sigma = \sigma\sigma^*\sigma = 1$. Then by Proposition 3 in Chapter 2, the restricted representation $T = \tau|_{S(\infty)}$ is unitary. On the other hand, τ is continuous. Since the restricted topology of $\Gamma(\infty)$ to $S(\infty)$ is the group topology of $S(\infty)$, $T = \tau|_{S(\infty)}$ is a continuous unitary representation of $S(\infty)$ which is extended from the good representation T^o of $S^o(\infty)$.

$(ii) \Rightarrow (i)$: Let $x \in H(T), \varepsilon > 0$ and $K_m(x)$ be the closed convex hull of the transforms $T(\sigma)(x)$ where $\sigma \in S_m(\infty)$. By continuity of T there exists an $n \in \mathbb{N}$ such that for all $\sigma \in S_n(\infty)$, $||T(\sigma)(x) - x|| < \varepsilon$. This implies that $K_n(x) \subset C(x, \varepsilon)$ where $C(x, \varepsilon)$ is the closed ball centred at x with radius ε. By definition of T, each $T(\sigma)$, $\sigma \in S(\infty)$, has norm bounded by 1 and by Definition 12 every Hilbert space is uniformly convex Banach space, then by Appendix D, Theorem 13, there exists $x_o \in K_n(x)$ such that $T(\sigma)(x_o) = x_o$ for all $\sigma \in S_n(\infty)$ which implies $x_o \in C(x, \varepsilon)$. Hence $||x - x_o|| < \varepsilon$, proving the density of $H_\infty(T)$ in $H(T)$. By Proposition 10, $H_\infty(T^o)$ is dense in in $H(T^o)$.

$(i) \Rightarrow (iii)$: Assume that T^o is a good representation of $S^o(\infty)$, i.e. $\overline{H_\infty(T^o)} = H(T^o)$. By P_n we will denote the orthogonal projection $H(T^o) \to H_n(T^o)$. Assume also that n is so large that $H_n(T^o) \neq \{0\}$. By Lemma 13 (i), $\theta_n(S^o(\infty)) = \Gamma(n)$. So we have a map $\tau_n : \Gamma(n) \to \Gamma(H_n(T^o))$ such that $\tau_n(\theta_n(\sigma)) = P_n T^o(\sigma)|_{H_n(T^o)}$ for every $\sigma \in S^o(\infty)$. In order to show that τ_n is well defined, we need to prove that if $\theta_n(\sigma_1) = \theta_n(\sigma_2)$, then we have shown above $P_n T^o(\sigma_1) P_n = P_n T^o(\sigma_2) P_n$.

By Lemma 13 (ii) we have for any $\sigma_1, \sigma_2 \in S^o(\infty)$

$$\theta_n(\sigma_1) = \theta_n(\sigma_2) \Leftrightarrow S_n^o(\infty)\sigma_1 S_n^o(\infty) = S_n^o(\infty)\sigma_2 S_n^o(\infty). \tag{6.5}$$

Thus $\theta_n(\sigma_1) = \theta_n(\sigma_2)$ implies that $\sigma_1 = \beta\sigma_2\beta'$ for some $\beta, \beta' \in S_{n^o}(\infty)$. Then for any $f \in H_n(T^o)$

$$\begin{aligned} P_n T^o(\sigma_1) P_n(f) &= P_n T^o(\beta\sigma_2\beta') P_n(f) \\ &= P_n T^o(\beta) T^o(\sigma_2) T^o(\beta')(f') \\ &= P_n T^o(\beta) T^o(\sigma_2)(f') \end{aligned}$$

where f' denotes the image $P_n(f)$, and the last row follows from the fact that f' is $S_n^o(\infty)$-invariant. Now since $H_\infty(T^o)$ is $S^o(\infty)$-invariant, $T^o(\sigma_2)f' \in H_k(T^o)$ for some $k \in \mathbb{N}$. Since $\beta \in S^o(\infty)$, then $\beta \in S(m)$ for some $m \in \mathbb{N}$. We have two cases to consider:

I. Case: $k \geq m$: By lemma 14, $H_k(T^o)$ is $S(m)$-invariant which means

$$P_n T^o(\sigma_1) P_n(f) = P_n T^o(\beta) T^o(\sigma_2)(f') = P_n T^o(\sigma_2) P_n(f). \tag{6.6}$$

II. Case: $k < m$: We know $H_k(T^o) \subset H_m(T^o)$. Since $H_m(T^o)$ is $S(m)$-invariant, we have

$$P_n T^o(\sigma_1) P_n(f) = P_n T^o(\beta) T^o(\sigma_2)(f') = P_n T^o(\sigma_2) P_n(f). \tag{6.7}$$

In each case we have

$$P_n T^o(\sigma_1) P_n = P_n T^o(\sigma_2) P_n. \tag{6.8}$$

Hence τ_n is well-defined.

Let $m > n$. Define $\theta_{n,m} : \Gamma(m) \to \Gamma(n)$ as follows: for any $\gamma \in \Gamma(m)$, $\theta_{n,m}(\gamma)$ is the upper left $n \times n$ submatrix of γ. So $\theta_{n,m}$ satisfies $\theta_{n,m} \circ \theta_m = \theta_n$. Let $\gamma \in \Gamma(m)$. If $\sigma \in S^o(\infty)$ is such that $\gamma = \theta_m(\sigma)$, then $P_n \tau_m(\gamma) = P_n \tau_m(\theta_m(\sigma))$. So by definition of τ_m,

$$P_n \tau_m(\theta_m(\sigma)) P_n = P_n P_m T^o(\sigma) P_n = P_n T^o(\sigma) P_n = \tau_n(\theta_n(\sigma)) = \tau_n(\theta_{n,m}(\gamma)). \tag{6.9}$$

Hence we have $\tau_n(\theta_{n,m}(\gamma)) = P_n \tau_m(\gamma) P_n$.

Let $\gamma \in \Gamma(\infty)$ and $\sigma_n \in S^o(\infty)$ be such that $\theta_n(\gamma) = \theta_n(\sigma_n)$ for every $n \in \mathbb{N}$. Then we have a sequence $(\sigma_n)_{n \in \mathbb{N}}$ in $S^o(\infty)$ which converges to $\gamma \in \Gamma(\infty)$. This sequence is Cauchy. In order to show this, we need to prove that for any open neighborhood $S_n^o(\infty)$ of I, there exists an $N \in \mathbb{N}$ such that $\sigma_n^{-1}\sigma_m \in S_n^o(\infty)$ and $\sigma_m^{-1}\sigma_n \in S_n^o(\infty)$ for every $n, m \geq N$. We have chosen this sequence in such a way that for any $m \geq n$, $\theta_n(\sigma_n) = \theta_n(\sigma_m)$. So we have $\theta_n(\sigma_n^{-1}\sigma_m) = \theta_n(I)$ for every $m \geq n$. That is, $\sigma_k^{-1}\sigma_m \in S_n^o(\infty)$ for every $k, m \geq n$. (Similarly $\sigma_m^{-1}\sigma_k \in S_n^o(\infty)$ for every $k, m \geq n$) This shows that $(\sigma_n)_{n \in \mathbb{N}}$ is Cauchy.

Now consider the operator sequence $(P_n T^o(\sigma_n) P_n)_{n \in \mathbb{N}}$. Let us show that this sequence is strongly convergent (See Appendix A), that is convergent with respect to strong operator topology. To show this, first take an $f \in H_\infty(T^o)$ and let $\varepsilon > 0$.

$$\begin{aligned}
&\|P_nT^o(\sigma_n)P_n(f) - P_mT^o(\sigma_m)P_m(f)\| \\
&= \|P_nT^o(\sigma_n)P_n(f) - P_nT^o(\sigma_n)(f) + P_nT^o(\sigma_n)(f) - P_mT^o(\sigma_m)(f)+ \\
&P_mT^o(\sigma_m)(f) - P_mT^o(\sigma_m)P_m(f)\| \\
&\leq \|P_nT^o(\sigma_n)P_n(f) - P_nT^o(\sigma_n)(f)\| + \|P_nT^o(\sigma_n)(f) - P_mT^o(\sigma_m)(f)\| \\
&+\|P_mT^o(\sigma_m)(f) - P_mT^o(\sigma_m)P_m(f)\| \\
&\leq \|P_nf - f\| + \|P_mf - f\| + \|P_nT^o(\sigma_n)(f) - P_mT^o(\sigma_m)(f)\|.
\end{aligned} \tag{6.10}$$

Since $f \in H_\infty(T^o)$, there exists an $m_1 \in \mathbb{N}$ such that $P_{m_1}f = f$. On the other hand, $H_\infty(T^o)$ is $S^o(\infty)$-invariant. So that $T(\sigma_n)f$ and $T(\sigma_m)f$ lie in $H_\infty(T^o)$. We know that $(\sigma_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $S^o(\infty)$ and T^o is a continuous unitary representation of $S^o(\infty)$. By Lemma 22 the image sequence $(T(\sigma_n)(f))_{n\in\mathbb{N}}$ is Cauchy, so that there exists an $m_2 \in \mathbb{N}$ such that for all $m, n > m_2$

$$\|P_nT^o(\sigma_n)(f) - P_mT^o(\sigma_m)(f)\| \leq \varepsilon. \tag{6.11}$$

Now choose $N = max\{m_1, m_2\}$, then

$$\|P_nT^o(\sigma_n)P_n(f) - P_mT^o(\sigma_m)P_m(f)\| \leq \varepsilon. \tag{6.12}$$

Let us take any $f \in H(T^o)$. By density of $H_\infty(T^o)$ in $H(T^o)$, for any $\varepsilon > 0$, there exists a $K \in \mathbb{N}$ such that $\|f - P_Kf\| \leq \frac{\varepsilon}{3}$. Furthermore the image net $(T(\sigma_n)(f))_{n\in\mathbb{N}}$ is Cauchy, so that there exists an $M \in \mathbb{N}$ such that for all $m, n > M$

$$\|P_nT^o(\sigma_n)(f) - P_mT^o(\sigma_m)(f)\| \leq \frac{\varepsilon}{3}. \tag{6.13}$$

Now choose $N = max\{K, M\}$ the equation 6.10 becomes

$$\|P_nT^o(\sigma_n)P_n(f) - P_mT^o(\sigma_m)P_m(f)\| \leq \varepsilon. \tag{6.14}$$

This proves that for any $f \in H(T^o)$, the sequence $(P_nT^o(\sigma_n)P_n(f))_{n\in\mathbb{N}}$ is Cauchy. Hence we conclude that $(P_nT^o(\sigma_n)P_n)_{n\in\mathbb{N}}$ is a strongly convergent sequence in $\Gamma(H(T^o))$. Let us denote the strong limit by $\tau(\gamma)$. We have for every $f \in H(T^o)$

$$\tau(\gamma)f = \lim_{n\to\infty} P_nT^o(\sigma_n)P_n(f) \tag{6.15}$$

Since $sup\{\|P_nT^o(\sigma_n)P_n(f)\| : n \in \mathbb{N}\} \leq 1$ for all $f \in H(T^o)$ by Banach-Steinhaus Theorem, the strong limit $\tau(\gamma)$ is a bounded linear operator whose norm is at most 1. For every $\gamma \in \Gamma(\infty)$, we can associate a bounded linear operator $\tau(\gamma) \in \Gamma(H(T^o))$ in this way, that is, we have a map $\tau : \Gamma(\infty) \to \Gamma(H(T^o))$; $\gamma \mapsto \tau(\gamma)$ for every $\gamma \in \Gamma(\infty)$.

For every $\gamma \in \Gamma(\infty)$, the operator $\tau(\gamma)$ has the following property: for any $k \in \mathbb{N}$ and $f \in H(T^o)$, by equation 6.9 and the definition of τ_k

$$P_k\tau(\gamma)P_k(f) = \lim_{n\to\infty} P_kP_nT^o(\sigma_n)P_nP_k(f) = P_kT^o(\sigma_k)P_kf = \tau_k(\theta_k(\sigma_k)) \tag{6.16}$$

τ is a representation of the semigroup $\Gamma(\infty)$. First let us show that τ is continuous. Let $\varepsilon > 0$, $f \in H(T^o)$, $\alpha, \beta \in \Gamma(\infty)$. We need to show that $\tau(\alpha)(f)$ is close to $\tau(\beta)(f)$ when α is sufficiently close to β. By construction of τ

$$\tau(\alpha)(f) = \lim_{n\to\infty} P_nT^o(\alpha_n)P_n(f) \tag{6.17}$$

where $\alpha_n \in S^o(\infty)$ and $\theta_n(\alpha_n) = \theta_n(\alpha)$ for every $n \in \mathbb{N}$ and

$$\tau(\beta)(f) = \lim_{n\to\infty} P_nT^o(\beta_n)P_n(f) \tag{6.18}$$

where $\beta_n \in S^o(\infty)$ and $\theta_n(\beta_n) = \theta_n(\beta)$ for every $n \in \mathbb{N}$. Then

$$\begin{aligned} \|\tau(\alpha)(f) - \tau(\beta)(f)\| &= \|\lim_{n\to\infty} P_n(T^o(\alpha_n) - T^o(\beta_n))P_n(f)\| \\ &= \lim_{n\to\infty} \|P_n(T^o(\alpha_n) - T^o(\beta_n))P_n(f)\| \\ &= \lim_{n\to\infty} \|(T^o(\alpha_n) - T^o(\beta_n))P_n(f)\| \end{aligned} \tag{6.19}$$

Since $H_\infty(T^o)$ is dense in $H(T^o)$, there exists $M \in \mathbb{N}$ such that $\|P_M f - f\| < \frac{\varepsilon}{3}$. Then

$$\begin{aligned}
&\|(T^o(\alpha_n) - T^o(\beta_n))P_n(f)\| \\
&= \|T^o(\alpha_n)P_n(f) - T^o(\alpha_n)(f) + T^o(\alpha_n)(f) - T^o(\beta_n)(f) + T^o(\beta_n)(f) - T^o(\beta_n)P_n(f)\| \\
&\leq \|T^o(\alpha_n)(P_n(f) - f)\| + \|T^o(\beta_n)(P_n(f) - f)\| + \|T^o(\alpha_n)(f) - T^o(\beta_n)(f)\| \\
&= 2\|P_n(f) - f\| + \|T^o(\alpha_n)(f) - T^o(\beta_n)(f)\|
\end{aligned} \tag{6.20}$$

Since T^o is continuous, $\|T^o(\alpha_n)(f) - T^o(\beta_n)(f)\| \leq \frac{\varepsilon}{3}$ when $\theta_K(\alpha_n) = \theta_K(\beta_n)$ for a sufficiently large $K \in \mathbb{N}$. Then choose $N = max\{M, K\}$, we have for all $n \geq N$

$$\|(T^o(\alpha_n) - T^o(\beta_n))P_n(f)\| \leq \varepsilon \tag{6.21}$$

and $\theta_N(\alpha) = \theta_N(\alpha_n) = \theta_N(\beta_n) = \theta_N(\beta)$. So we conclude that $\tau(\alpha)(f)$ is close to $\tau(\beta)(f)$ when $\theta_N(\alpha) = \theta_N(\beta)$.

τ commutes with involution. Let $\gamma \in \Gamma(\infty)$

$$\tau(\gamma)^*(f) = (\lim_{n\to\infty} P_n T^o(\sigma_n)P_n(f))^* = \lim_{n\to\infty} P_n T^o(\sigma_n^*)P_n(f) = \tau(\gamma^*)(f) \tag{6.22}$$

since the involution is continuous by Lemma 11 and when $\theta_n(\sigma_n) = \theta_n(\gamma)$ we have $\theta_n(\sigma_n^*) = \theta_n(\gamma^*)$ by Lemma 10.

τ sends 1 to 1. Since $\theta_n(I) = \theta_n(1)$ for every $n \in \mathbb{N}$

$$\tau(1)(f) = \lim_{n\to\infty} P_n T^o(I)P_n(f) = \lim_{n\to\infty} P_n(f) = f \tag{6.23}$$

So $\tau(1) = 1$.

Finally we need to show that $\forall \gamma_1, \gamma_2 \in \Gamma(\infty)$, τ satisfies

$$\tau(\gamma_1\gamma_2) = \tau(\gamma_1)\tau(\gamma_2) \tag{6.24}$$

First let $\gamma_1 \in S^o(\infty)$ be fixed. For any $\gamma_2 \in \Gamma(\infty)$ since $S^o(\infty)$ is dense in $\Gamma(\infty)$, there exists a net $\{\sigma_\lambda\}_{\lambda\in\Lambda}$, where Λ is a directed set, in $S^o(\infty)$ converging to γ_2. By continuity of τ we know that the net $\{T^o(\sigma_\lambda)\}_{\lambda\in\Lambda} = \{\tau(\sigma_\lambda)\}_{\lambda\in\Lambda}$ converges to $\tau(\gamma_2)$. By Appendix A, Theorem 8, the multiplication is seperately continuous in $\Gamma(\infty)$ with respect to weak operator topology, so that $\tau(\gamma_1)\tau(\sigma_\lambda)$ converges to $\tau(\gamma_1)\tau(\gamma_2)$. Again using continuity of T^o, the net $\{T^o(\gamma_1)T^o(\sigma_\lambda)\}_{\lambda\in\Lambda} = \{T^o(\gamma_1\sigma_\lambda)\}_{\lambda\in\Lambda}$ converges to $\tau(\gamma_1\gamma_2)$ for any $\gamma_2 \in \Gamma(\infty)$. Hence $\tau(\gamma_1\gamma_2) = \tau(\gamma_1)\tau(\gamma_2)$ for all $\gamma_2 \in \Gamma(\infty)$. Now, repeating the same process for γ_1, we conclude the result. $\square$

Remark 3. *By density of $S^o(\infty)$ in $S(\infty)$ and $\Gamma(\infty)$, the extension representations T and τ in Theorem 3 are unique.*

Remark 4. *By Theorem 3, we see that good representations of $S^o(\infty)$, continuous unitary representations of $S(\infty)$ and representations of $\Gamma(\infty)$ are the same representations.*

In Chapter 4 we constructed the irreducible unitary representations $T(n, \pi)$, where $n \in \mathbb{N}$ and π is an irreducible unitary representation of $S(n)$. Now we will extend them to $\Gamma(\infty)$. The semigroup $\Gamma(\infty)$ has a natural action in $\Omega(n)$ given by matrix multiplication $\omega\gamma$ for all $\omega \in \Omega$ and $\gamma \in \Gamma(\infty)$. So the action of $\Gamma(\infty)$ in $H(T(n, \pi))$ is

$$\tau(\gamma)f(\omega) = f(\omega\gamma) \tag{6.25}$$

if $\omega\gamma \neq 0$ and if $\omega\gamma = 0$, then we define

$$\tau(\gamma)f(\omega) = 0. \tag{6.26}$$

τ is a representation of $\Gamma(\infty)$ which is an extension of $T(n, \pi)$.

Theorem 4. *If T is a good representation of $S^o(\infty)$ and n is so large that $H_n(T) \neq \{0\}$, then the map $\tau_n : \Gamma(n) \to \Gamma(H_n(T))$ which is constructed in the proof of Theorem 3 is a representation of $\Gamma(n)$ in $H_n(T)$.*

Proof. For any $\gamma \in \Gamma(n)$ and $\sigma \in S^o(\infty)$ such that $\theta_n(\sigma) = \gamma$, we have defined

$\tau_n : \Gamma(n) \to \Gamma(H_n(T))$ as $\tau_n(\gamma) = P_n T(\sigma) P_n$. We need to show that

$$\tau_n(\gamma_1\gamma_2) = \tau_n(\gamma_1)\tau_n(\gamma_2) \tag{6.27}$$

for all $\gamma_1, \gamma_2 \in \Gamma(n)$, τ_n commutes with involution and $\tau_n(1) = 1$. By τ we denote the representation of $\Gamma(\infty)$ which exists by Theorem 3. Let ε_n be the infinite matrix

$$\varepsilon_n = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \in \Gamma(\infty)$$

and P_n be the orthogonal projection $H(T)$ onto $H_n(T)$. Then we first prove that $\tau(\varepsilon_n) = P_n$. Since $\varepsilon_n = \varepsilon_n^2 = \varepsilon_n^*$, the map $\tau(\varepsilon_n)$ is a projection.

Let us denote the image of $\tau(\varepsilon_n)$ by H'_n, that is $H'_n = \tau(\varepsilon_n)(H(T))$. We claim that $H'_n = H_n(T)$. Indeed, if $f \in H'_n$, then $f = \tau(\varepsilon_n)g$ for some $g \in H(T)$. Then for any $\sigma \in S^o_n(\infty)$ we have

$$\tau(\sigma)f = \tau(\sigma)\tau(\varepsilon_n)(g) = \tau(\sigma\varepsilon_n)(g) = \tau(\varepsilon_n)(g) = f \tag{6.28}$$

since $\sigma\varepsilon_n = \varepsilon_n$. Hence $H'_n \subseteq H_n(T)$.

By construction of τ and τ_n, we have

$$P_n\tau(\varepsilon_n)P_n = \tau_n(\theta_n(\varepsilon_n)) = \tau_n(\theta_n(1)) = P_n T(1) P_n = 1|_{H_n(T)}. \tag{6.29}$$

Hence $H_n(T) \subseteq H'_n$ and $H_n(T) = H'_n$ so $P_n = \tau(\varepsilon_n)$.

Now we use this fact to show that τ_n is multiplicative. We may imbed $B(H_n(T))$ into $B(H(T))$ by the map $A \to P_n A P_n$. Furthermore we can also imbed $\Gamma(n)$ into $\Gamma(\infty)$ by the map $\Gamma(n) \to \Gamma(\infty)$ given by $\gamma \mapsto \gamma^\wedge$ where

$$\gamma^\wedge = \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix} \in \Gamma(\infty)$$

for every $\gamma \in \Gamma(n)$.

By this imbedding we have

$$\tau_n(\gamma) = P_n\tau_n(\gamma)P_n = P_n\tau(\gamma^\wedge)P_n = \tau(\varepsilon_n)\tau(\gamma^\wedge)\tau(\varepsilon_n) = \tau(\varepsilon_n\gamma^\wedge\varepsilon_n) = \tau(\gamma^\wedge) \quad (6.30)$$

Then

$$\tau_n(\gamma_1\gamma_2) = \tau(\gamma_1{}^\wedge\gamma_2{}^\wedge) = \tau(\gamma_1{}^\wedge)\tau(\gamma_2{}^\wedge) = \tau_n(\gamma_1)\tau_n(\gamma_2). \quad (6.31)$$

Moreover

$$\tau_n(\gamma^*) = \tau(\gamma^{\wedge *}) = \tau(\gamma^\wedge)^* = \tau_n(\gamma)^*, \quad (6.32)$$

and

$$\tau_n(1) = \tau_n(\theta_n(1)) = P_nT^o(1)|_{H_n(T)} = 1_{H_n(T)}. \quad (6.33)$$

Thus τ_n is a representation. □

For any $k > n$ we let

$$\varepsilon_{n,k} = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \in \Gamma(k)$$

Lemma 16. *If T is a good representation of $S^o(\infty)$, then $\tau_k(\varepsilon_{n,k})$ is the orthogonal projection from $H(\tau_k) = H_k(T)$ to $H_n(T)$.*

Proof. By Theorem 3, the good representation T of $S^o(\infty)$ can be extended to a representation τ of $\Gamma(\infty)$. By Theorem 4, we have $P_n = \tau(\varepsilon_n)$. Then

$$\tau_k(\varepsilon_{n,k}) = \tau_k(\theta_k(\varepsilon_n)) = P_k\tau(\varepsilon_n)|_{H_k(T)} = P_kP_n|_{H_k(T)} = P_n|_{H_k(T)}. \quad (6.34)$$

□

CHAPTER SEVEN
LIEBERMAN THEOREM

By Theorem 3, having a good representation of $S^o(\infty)$ is equivalent to having a continuous unitary representation of $S(\infty)$ and a representation of $\Gamma(\infty)$. Hence studying continuous unitary representations of $S(\infty)$ is reduced to studying representations of the semigroup $\Gamma(\infty)$. In the proof of Theorem 3, in the $(i) \to (iii)$ part, first we introduced the representations τ_n of the $\Gamma(n)$, $n \in \mathbb{N}$ (the fact that they are representations of $\Gamma(n)$ was proved in Theorem 4). Then we constructed a representation τ of $\Gamma(\infty)$ as a strong limit of the operator sequence $(P_n \tau_n P_n)_{n\in\mathbb{N}}$, where P_n denotes the orthogonal projection from representation space of $S^o(\infty)$ to the subspace of $S^o_n(\infty)$-invariant vectors. So our main aim turns into studying the representations of the finite semigroups $\Gamma(m)$, $m \in \mathbb{N}$.

An element $\varepsilon \in \Gamma(m)$ is said to be idempotent if $\varepsilon = \varepsilon^* = \varepsilon^2$. Any idempotent element of $\Gamma(m)$ has the only non-zero entries on the diagonal. We denote an idempotent element by ε_X where X is the set of all indices i of the non-zero diagonal entries of ε_X. Note that $\varepsilon_\emptyset = 0$ and $\varepsilon_{\{1,\dots,m\}} = 1_{\Gamma(m)}$.

There is a natural partial ordering in the set of idempotents: $\varepsilon_X \leq \varepsilon_Y$ if $X \subseteq Y$.

For any $n = 1, \dots, m$ we set $\varepsilon_n = \varepsilon_{\{1,\dots,n\}}$ if $n \neq 0$ and $\varepsilon_n = 0$ if $n = 0$. Then any idempotent element $\varepsilon_X \in \Gamma(m)$ can be written as $\sigma \varepsilon_n \sigma^{-1}$ where $n = Card(X)$ and $\sigma \in S(m)$.

Let τ be a representation of $\Gamma(m)$. Then $\tau(\varepsilon_X) = \tau(\varepsilon_X)^* = \tau(\varepsilon_X)^2$. Hence $\tau(\varepsilon_X)$ is a projection for any $X \subseteq \{1, \dots, m\}$. Let $H_X(\tau)$ be its range.

Lemma 17. *Let τ be a representation of $\Gamma(m)$. If $X \subseteq Y \subseteq \{1, \dots, m\}$, then $H_X(\tau) \subseteq H_Y(\tau)$.*

Proof. Take $f \in H_X(\tau)$. Then $\tau(\varepsilon_X)(g) = f$ for some $g \in H(\tau)$.

$$\tau(\varepsilon_Y)(f) = \tau(\varepsilon_Y)\tau(\varepsilon_X)(g) = \tau(\varepsilon_X\varepsilon_Y)(g) = \tau(\varepsilon_X)(g) = f \tag{7.1}$$

Hence $f \in H_Y(\tau)$. □

Denote the image of $\tau(\varepsilon_n)$ by $H_n(\tau)$ when $n \neq 0$ and $H_\emptyset(\tau)$ when $n = 0$. Let $n(\tau)$ be the least number n such that $H_n(\tau) \neq \{0\}$.

Lemma 18. *Let τ be a representation of $\Gamma(m)$ and $n = n(\tau) \neq 0$. $\tau(\varepsilon_X) \neq 0$ when $Card(X) \geq n(\tau)$ and $\tau(\varepsilon_X) = 0$ when $Card(X) < n(\tau)$.*

Proof. Let $t = Card(X)$. Then $\varepsilon_X = \sigma\varepsilon_t\sigma^{-1}$ for some $\sigma \in S(m)$, or $\varepsilon_t = \sigma^{-1}\varepsilon_X\sigma$. Suppose $t \geq n(\tau)$. Since, by Lemma 17, $H_n(\tau) \subseteq H_t(\tau)$ we have $H_t(\tau) \neq \emptyset$, which means $\tau(\varepsilon_t) \neq 0$. Then $\tau(\sigma^{-1}\varepsilon_X\sigma) \neq 0$. Hence $\tau(\varepsilon_X) \neq 0$ when $t = Card(X) \geq n(\tau)$.

Now suppose $t < n(\tau)$. By definition of $n(\tau)$, $H_t(\tau) = \{0\}$, that is, $\tau(\varepsilon_t) = 0$. Equivalently $\tau(\sigma)\tau(\varepsilon_t)\tau(\sigma^{-1}) = \tau(\sigma\varepsilon_t\sigma^{-1}) = \tau(\varepsilon_X) = 0$. □

Lemma 19. *Let τ be a representation of $\Gamma(m)$ and $n = n(\tau) \neq 0$, $\gamma \in \Gamma(m)$ and $|\gamma|$ be the number of non-zero entries of γ. Then $\tau(\gamma) \neq 0$ when $|\gamma| \geq n(\tau)$ and $\tau(\gamma) = 0$ when $|\gamma| < n(\tau)$.*

Proof. Let X be the set of indices i of the entries such that $\gamma_{ij} \neq 0$ for some $j \in \mathbb{N}$. Then $\gamma^*\gamma = \varepsilon_X$. So we have

$$\tau(\gamma) \neq 0 \Leftrightarrow \tau(\gamma)^*\tau(\gamma) \neq 0 \Leftrightarrow \tau(\gamma^*\gamma) \neq 0 \Leftrightarrow \tau(\varepsilon_X) \neq 0 \tag{7.2}$$

Then by Lemma 18, the result follows. □

Lemma 20. *Let τ be an irreducible representation of $\Gamma(m)$ and $n = n(\tau) \neq 0$. Then $H_n(\tau)$ is invariant and irreducible under $S(n)$.*

Proof. Let $f \in H_n(T)$ and $\sigma \in S(n)$, we claim that $\tau(\sigma)(f) \in H_n(T)$. Then

$$\tau(\varepsilon_n)\tau(\sigma)(f) = \tau(\varepsilon_n\sigma)(f) = \tau(\sigma)(f) \tag{7.3}$$

so that $\tau(\sigma)(f) \in H_n(\tau)$. Hence $H_n(\tau)$ is $S(n)$-invariant.

By Lemma 3, τ is finite-dimensional. By Theorem 2, the operators $\tau(\gamma)$, $\gamma \in \Gamma(m)$, span $B(H(\tau))$. So to check its irreducibility it is sufficient to prove that the operators $\tau(\sigma)|_{H_n(\tau)}$, $\sigma \in S(n)$, span $B(H_n(\tau))$. Notice that the operators $\tau(\varepsilon_n)\tau(\gamma)\tau(\varepsilon_n)|_{H_n(\tau)}, \gamma \in \Gamma(m)$ span $B(H_n(\tau))$. Set $\gamma' = \varepsilon_n\gamma\varepsilon_n$. By definition of n and Lemma 18, we have $\tau(\gamma') = 0$ unless $|\gamma'| = n$ (Since there can be at most n nonzero terms in γ'). But $|\gamma'| = n$ if and only if $\theta_n(\gamma') \in \theta_n(S(n))$. So the operators $\tau(\sigma)|_{H_n(\tau)}, \sigma \in S(n)$ span $B(H_n(\tau))$. □

Lemma 21. *Let τ be an irreducible representation of $\Gamma(m)$ and $n = n(\tau) \neq 0$. Let $\pi \in S(n)$^ be the representation of $S(n)$ in $H_n(\tau)$. Then τ is uniquely determined by (n, π).*

Proof. For any $\xi \in H(\pi) = H_n(\tau)$ and $\gamma \in \Gamma(m)$ we have

$$\langle\tau(\gamma)\xi, \xi\rangle = \langle\tau(\gamma)\tau(\varepsilon_n)\xi, \tau(\varepsilon_n)\xi\rangle = \langle\tau(\varepsilon_n\gamma\varepsilon_n)\xi, \xi\rangle \tag{7.4}$$

$$= \begin{cases} \langle\pi(\sigma)\xi, \xi\rangle & if \theta_n(\varepsilon_n\gamma\varepsilon_n) = \theta_n(\sigma), \sigma \in S(n) \\ 0 & otherwise. \end{cases}$$

By definition of $n(\tau)$ and by Lemma 19, $\tau(\varepsilon_n\gamma\varepsilon_n) = 0$ if $|\varepsilon_n\gamma\varepsilon_n| < n$. Since

$$\varepsilon_n\gamma\varepsilon_n = \begin{pmatrix} \theta_n(\gamma) & 0 \\ 0 & 0 \end{pmatrix}_{m\times m} \in \Gamma(m) \tag{7.5}$$

then $|\varepsilon_n\gamma\varepsilon_n|$ can be at most n. So if $|\varepsilon_n\gamma\varepsilon_n| = n$, which means $\theta_n(\varepsilon_n\gamma\varepsilon_n) = \theta_n(\sigma)$ for some $\sigma \in S(n)$, then $\tau(\varepsilon_n\gamma\varepsilon_n)|_{H_n(\tau)} = \tau(\sigma)|_{H_n(\tau)} = \pi(\sigma)$.

If τ' is any other representation of $\Gamma(m)$ and π is the representation of $S(n)$ in

$H_n(\tau') = H(\pi)$, then we have by above argument for any $\xi \in H(\pi)$ and $\gamma \in \Gamma(m)$

$$\langle \tau'(\gamma)\xi, \xi \rangle = \langle \tau(\gamma)\xi, \xi \rangle \tag{7.6}$$

Any nonzero $\xi \in H_n(\tau)$ is cyclic in $H(\tau)$. Because the subspace $\overline{span\{\tau(\gamma)\xi : \gamma \in \Gamma(m)\}}$ is invariant nonzero subspace of $H(\tau)$ but τ is irreducible (Similar for τ'). So it is enough to apply Corollary 1. $\square$

Now for any pair (n, π) where $n = 1, ..., m$ and $\pi \in S(n)$^ we construct an irreducible representation $\tau = \tau(n, m; \pi)$ such that $n = n(\tau)$ and π is the representation of $S(n)$ in $H_n(\tau)$.

Let $\Omega_m(n)$ denote the set of all infinite 0-1 matrices γ for which there are some distinct natural numbers $i_1, ..., i_n \leq m$ such that $\gamma_{1i_1} = ...\gamma_{ni_n} = 1$ and the other entries are 0. Any $\gamma \in \Omega_m(n)$ may be regarded as an ordered sequence of distinct numbers $\gamma = (i_1, ..., i_n)$. The space $H(\tau)$ is the space of all the functions $f : \Omega_m(n) \to H(\pi)$ such that

$$f(\sigma'\omega) = \pi(\sigma')f(\omega) \tag{7.7}$$

for every $\omega \in \Omega_m(n)$ and $\sigma' \in S(n)$. The norm in $H(\tau)$ is defined by

$$||f||^2 = \sum_{\omega} ||f(\omega)||^2_{H(\pi)} < \infty \tag{7.8}$$

and $\Gamma(m)$ acts by the rule

$$\tau(\gamma)f(\omega) = f(\omega\gamma) \tag{7.9}$$

for every $\gamma \in \Gamma(m)$ and $\omega \in \Omega_m(n)$.

τ is a representation of $\Gamma(m)$. For any $X \subseteq \{1, ..., m\}$, $H_X(\tau)$ consists of those f which are concentrated on $\{\omega | \omega \subseteq X\}$. Because

$$H_X(\tau) = \tau_X(H(\tau)) = \{\tau(\varepsilon_X)(f) | f \in H(\tau)\} \tag{7.10}$$

and $\tau(\varepsilon_X)(f)(\omega) = f(\omega\varepsilon_X) \neq 0$ if and only if $\omega \subseteq X$. Thus for all $k \leq m$

$$H_k(\tau) = \{f \in H(\tau) : \text{supp} f \subseteq \Omega_k(n)\}. \tag{7.11}$$

Since $\Omega_k(n) = \emptyset$ for all $1 \leq k < n$ and the only function whose support is $\emptyset$ is zero of $H(\tau)$. So $H_k(\tau) \neq \{0\}$ if $k \geq n$. It follows that $n(\tau) = n$ and that the representation of $S(n)$ in $H_n(\tau)$ is π. Finally $H_n(\tau)$ being cyclic under $\Gamma(m)$ (even $S(m)$), the irreducibility of τ can be proved in exactly the same way as Corollary 2 and Lemma 7 in Chapter 4, respectively.

Theorem 5. *The representations $\tau(n, m; \pi)$ where $n = 1, ..., m$ and $\pi \in S(n)^\wedge$ together with the trivial representation of $\Gamma(m)$ exhaust all the irreducible representations of $\Gamma(m)$.*

Proof. Let τ be an irreducible representation of $\Gamma(m)$. If $n(\tau) = n \neq 0$ then τ is equivalent to some $\tau(n, m; \pi)$ by Lemma 21. But if $n(\tau) = 0$ then by definition, since $\varepsilon_\emptyset = 0$

$$H_\emptyset(\tau) = \{\tau(\varepsilon_\emptyset)f = \tau(0)f : f \in H(\tau)\} \neq 0 \tag{7.12}$$

so that there exists a nonzero element $\tau(0)f \in H_\emptyset(\tau)$. Since $\tau(0)$ is nonzero, by Lemma 2, τ is the trivial representation. $\square$

Remark 5. *Let $T = T(n, \pi)$ be the representation of $S(\infty)$ as constructed in Chapter 4. Let $m \geq n$. Consider the representation τ_m of $\Gamma(m)$ in the space $H_m(T)$ which is obtained by Theorem 3. Then τ_m is equivalent to $\tau(n, m; \pi) = \tau$. By Lemma 21, τ is determined by (n, π). Since π is the representation of $S(n)$ both in $H_n(\tau)$ and $H_n(\tau_m)$, it is enough to show that $n(\tau) = n(\tau_m) = n$. We know by the above argument (equation 7.11 and the following part) that $n(\tau) = n$. On the other hand $H_k(\tau_m)$ is the image of $\tau_m(\varepsilon_k)$ and by Lemma 16, $\tau_m(\varepsilon_k)$ is the orthogonal projection from $H_m(T)$ to $H_k(T)$. Hence $H_k(T) = H_k(\tau_m) \neq 0$ if and only if $k \geq n$ (We have shown in Lemma 4, (3) part, the least number k making $H_k(T)$ nonzero is n). So that $n(\tau_m) = n$. Hence τ_m is equivalent to $\tau(n, m; \pi) = \tau$. By Theorem 5, the representations τ_m of $\Gamma(m)$*

in the space $H_m(T(n,\pi))$ together with the trivial representation are all irreducible representations of $\Gamma(m)$.

Theorem 6. *(Lieberman)*

i. The representations

$$\{T(n,\pi)|n = 1,2,...,\pi \in S(n)\hat{}\} \tag{7.13}$$

together with the trivial representation, exhaust all the continuous irreducible unitary representations of $S(\infty)$.

ii. Any continuous unitary representation of $S(\infty)$ can be decomposed into a direct sum of irreducible representations.

Proof. i. Let T be an irreducible continuous unitary representation of $S(\infty)$. By Theorem 3, $T|_{S^o(\infty)}$ is a good representation of $S^o(\infty)$ which acts in the same space $H(T)$. $T|_{S^o(\infty)}$ is also irreducible by the density of $S^o(\infty)$ in $S(\infty)$. Because if there were a closed $S^o(\infty)$-invariant subspace K of $H(T)$, then it would also be $S(\infty)$-invriant. Indeed, let $x \in K$ and $\sigma \in S(\infty)$. Then there exists a net $\{\sigma_\lambda\}_{\lambda\in\Lambda}$, where Λ is a directed set, in $S^o(\infty)$ converging to σ. Then $T(\sigma_\lambda)x = x$ for all $\lambda \in \Lambda$ and by continuity of T, the net $\{T(\sigma_\lambda)x\}_{\lambda\in\Lambda} = \{x\}_{\lambda\in\Lambda}$ converges to $T(\sigma)x$ so that $x = T(\sigma)x$, which shows that x is $S(\infty)$-invariant. Hence K can not be a proper subspace since T is irreducible.

Let $H_n(T)$ denote the subspace of all $S_n(\infty)$-invariant vectors of $H(T)$ and $n(T)$ denote the smallest natural number n such that $H_n(T) \neq 0$.

We define $S_o(\infty) = S(\infty)$. So if $n(T) = 0$, then $H_0(T) = \{f \in H(T) : T(\sigma)f = f, \forall\sigma \in S(\infty)\}$ is a nonzero invariant subspace of $H(T)$. By irreducibility of T , $H_0(T)$ must be the whole space $H(T)$. This means that every vector in $H(T)$ is $S(\infty)$-invariant, so that T is the trivial representation.

Suppose $n(T) = n \neq 0$. Let us show that T is equivalent to some $T(n, \pi)$ where $n \in \mathbb{N}$ and $\pi \in S(n)^\wedge$ as defined in the Chapter 4. We will denote the induced representations from $S(\infty)$ to $\Gamma(\infty)$ by τ and the representations of $\Gamma(m)$ by τ_m for $m \geq n$ (by Theorem 3).

Remember that in the proof of Theorem 4, we use the imbedding $B(H(T)) \to B(H_m(T))$ given by $A \mapsto P_m A P_m$. Since T is an irreducible representation of $S^o(\infty)$, by Theorem 2 for any $m \geq n$, the linear span of the set of operators $P_m T(\sigma) P_m$ is weakly dense in $B(H_m(T))$. By the definition of τ_m, we know $\tau_m(\theta_m(\sigma))) = P_m T(\sigma) P_m$ for every $\sigma \in S^o(\infty)$. So the span of the set of operators $\{\tau_m(\gamma) : \gamma \in \Gamma(m)\}$ is weakly dense in $B(H_m(T))$. This means that τ_m is an irreducible representation of $\Gamma(m)$.

By Lemma 16, the operator $\tau_m(\varepsilon_{n,m})$ is the orthogonal projection from $H_m(T)$ to $H_n(\tau_m) = H_n(T) \neq 0$. So $n(\tau_m) = n$. By Lemma 21 and Theorem 5, $\tau_m = \tau(n, m; \pi)$, where π is the representation of $S(n)$ in $H_n(T)$. (By Lemma 20, $H_n(\tau_m)$ is invariant and irreducible under $S(n)$.)

Now consider the case $m = n$. Let $\gamma \in \Gamma(n)$. By Lemma 19, if $|\gamma| < n$, then $\tau_n(\gamma) = 0$. So $\tau_n(\gamma) \neq 0$ if $\gamma \in S(n)$ and $\tau_n(\gamma) = 0$ if $\gamma \notin S(n)$.

Take any $x \in H_n(T) = H(\pi)$, and $\sigma \in S^o(\infty)$.

$$\langle T(\sigma)x, x\rangle = \langle T(\sigma)P_n x, P_n x\rangle = \langle P_n T(\sigma) P_n x, x\rangle = \langle \tau_n(\theta_n(\sigma))x, x\rangle \quad (7.14)$$

$$= \begin{cases} \langle \pi(\theta_n(\sigma))x, x\rangle & if \theta_n(\sigma) \in S(n) \\ 0 & otherwise. \end{cases}$$

The last equality shows that if $\sigma \in S(n)$, then $\langle T(\sigma)x, x\rangle = \langle \pi(\sigma)x, x\rangle$ for every $x \in H_n(T)$. $H_n(T)$ is cyclic in $H(T)$. Because the subspace

$$\overline{span\{T(\sigma)x : \sigma \in S^o(\infty), x \in H_n(T)\}} \quad (7.15)$$

is an invariant nonzero subspace of $H(T)$ but T is irreducible. On the other hand, by Lemma 5 in Chapter 4 we have the same equality 7.14 for $T(n, \pi)$. By Corollary 1, T is uniquely determined by (n, π). So T is equivalent to $T(n, \pi)$.

ii. Suppose that T is a continuous unitary representation of $S(\infty)$. Equivalently, T is a good representation of $S^o(\infty)$. We will show that T contains an irreducible representation of $S^o(\infty)$. Hence T can be written as a direct sum of irreducible representations.

Let n be so big that $H_n(T) \neq \{0\}$. By the proof of the Theorem 3, we have an induced representation τ_n of the semigroup $\Gamma(n)$ in $H_n(T)$. (In the proof of Theorem 3 it is constructed and in Theorem 4 we have proved that τ_n is a representation.) Since $\Gamma(n)$ is finite, by Lemma 3 its representation $H_n(T) = H(\tau_n)$ contains an invariant and irreducible subspace H. Let

$$H' = \overline{span\{T(\sigma)x : \sigma \in S^o(\infty), x \in H\}} \tag{7.16}$$

and T' be the corresponding representation of $S^o(\infty)$ in H'. Then H is both $\Gamma(n)$-irreducible and $S^o(\infty)$-cyclic in $H(T')$.

For any $\sigma \in S^o(\infty)$, we have

$$P_nT(\sigma)H = P_nT(\sigma)P_nH = \tau_n(\theta_n(\sigma))(H) \subseteq H \tag{7.17}$$

Since $H_n(T') = P_n(H')$, it follows that $H_n(T') = H$.

Now suppose that H' has a decomposition $H' = K \oplus K^\perp$ into invariant subspaces $K, K^\perp$. Then $H_n(T') = (H_n(T') \cap K) \oplus (H_n(T') \cap K^\perp)$. (Indeed, if a vector is invariant relative to $S_n(\infty)$, then its projections on K and $K^\perp$ are invariant, too. (See the proof of Lemma 7.)) The subspaces $H_n(T') \cap K$ and $H_n(T') \cap K^\perp$ are invariant under the action of τ_n in H. Since $H_n(T') = H$ is $\Gamma(n)$-irreducible, we have either $H \subseteq K$ or $H \subseteq K^\perp$. But H is $S^o(\infty)$-cyclic, so either $K = H(T')$ or $K^\perp = H(T')$. Hence T' is irreducible.

$\square$

CHAPTER EIGHT
CONCLUSION

In this thesis, we aim to give an elegant proof to Lieberman theorem by using the Olshansky semigroup approach. First we have taken a natural number n and an irreducible unitary representation π of $S(n)$. From this representation, we have constructed a continuous irreducible unitary representation $T(n,\pi)$ of $S(\infty)$. Then we have obtained a different description of these representations. Next we have introduced the notion of a good representation for the subgroup $S^o(\infty)$. By Theorem 3, we have showed that a good representation of $S^o(\infty)$ can be extended to a continuous unitary representation of $S(\infty)$ and a representation of $\Gamma(\infty)$. To extend a good representation to a representation of $\Gamma(\infty)$, first we have written the representations of finite semigroups $\Gamma(n)$, then taken a strong limit to get a representation of $\Gamma(\infty)$. Thus, by Theorem 3, studying continuous unitary representations of $S(\infty)$ is reduced to studying representations of the semigroup $\Gamma(n)$. Then we have introduced some properties of the representations of the finite inverse symmetric semigroups. By using the results about the representations of the finite inverse symmetric semigroups that we obtained, we have proven the Lieberman Theorem.

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APPENDICES

A1: Topologies on $B(H)$, The Space of Bounded Linear Operaators on a Hilbert Space H

Let H be a Hilbert space. By $B(H)$ we denote the algebra of all bounded linear operators on H. We have a norm on $B(H)$

$$||u|| = sup_{||x||=1}||u(x)|| \tag{A1}$$

for all $u \in H$. This norm induces a metric, so $B(H)$ is a metric space. So the norm topology is just defined to be the metric topology. For any $\epsilon > 0$, an open set containing $A \in B(H)$ in the norm topology is of the form $O(A, \epsilon) = \{B \in B(H) : ||A-B|| < \epsilon\}$.

Let us construct the strong operator topology on $B(H)$.

We fix a vector $x_o \in H$ and consider the map $\phi_{x_o} : B(H) \to H$ defined by $\phi_{x_o} : u \mapsto u(x_o)$, for all $u \in B(H)$. We topologize H by its norm and we take the weakest topology on $B(H)$ which makes the map ϕ_{x_o} continuous.

Let $y \in H$, $\epsilon > 0$. In the norm topology of H, any open centred at y with radius ϵ will be denoted by $B(y, \epsilon) = \{x \in H : ||y - x|| < \epsilon\}$. Then $\phi_{x_o}^{-1}(B(y, \epsilon))$ must be open in this new topology:

$$\phi_{x_o}^{-1}(B(y, \epsilon)) = \{u \in B(H) : \phi_{x_o}(u) = u(x_o) \in B(y, \epsilon)\} \tag{A2}$$

This means $||u(x_o) - y|| < \epsilon$ for all $u \in \phi_{x_o}^{-1}(B(y, \epsilon))$. If $y = u_o(x_o)$ for some $u_o \in B(H)$, then we get

$$\phi_{x_o}^{-1}(B(y, \epsilon)) = \{u \in B(H) : ||(u - u_o)(x)|| < \epsilon\} \tag{A3}$$

Then a subbase for this topology is the collection of all sets of the form

$$O(u_o; x, \epsilon) = \{u \in B(H) : ||(u - u_o)(x)|| < \epsilon\} \tag{A4}$$

The strong operator topology is defined as the weakest topology on $B(H)$ that makes the all maps ϕ_{x_o}, $x_o \in H$, continuous. Then a base will be all finite intersections of the sets A4 It follows that a base is the collection of all sets of the form

$$O(u_o; x_1, ..., x_k, \epsilon) = \{u \in B(H) : ||(u - u_o)(x_i)|| < \epsilon, i = 1, ..., k\} \tag{A5}$$

The weak operator topology of $B(H)$ is defined as follows:

We fix a continuous linear functional y' on H and a vector $x \in H$ and consider the linear map $\alpha : B(H) \to \mathbb{C}$ given by $A \mapsto y'(A(x))$. The weakest topology on $B(H)$ which makes all those functions continuous is the weak operator topology on $B(H)$.

By Riesz Representation Theorem, there exists $y \in H$ such that $y'(x) = \langle x, y\rangle$ for all $x \in H$. Let $z \in \mathbb{C}$ be such that $z = y'(A_o(x))$, for some $A_o \in B(H)$, i.e. $z = \langle A_o(x), y\rangle$. We will consider the inverse images of opens in $\mathbb{C}$ under α, which are $\alpha^{-1}(B(z, \epsilon))$, where $B(z, \epsilon)$ denotes the open ball centred at $z \in \mathbb{C}$ with radius ϵ in the usual topology of $\mathbb{C}$. Then

$$\begin{aligned} \alpha^{-1}(B(z, \epsilon)) &= \{A \in B(H) : \alpha(A) = y'(A(x)) \in B(z, \epsilon)\} \\ &= \{A \in B(H) : \langle A(x), y\rangle \in B(z, \epsilon)\} \\ &= \{A \in B(H) : |\langle A(x), y\rangle - \langle A_o(x), y\rangle| < \epsilon\} \end{aligned} \tag{A6}$$

So a subbase for the weak operator topology is the collection of all sets of the form

$$O(A_o; x, y, \epsilon) = \{A \in B(H) : |\langle (A - A_o)(x), y\rangle| < \epsilon\} \tag{A7}$$

As always, a base is the collection of all finite intersections of such sets.

Proposition 11. *Let $A_\lambda, \lambda \in I$, where I is a directed set, be a net in $B(H)$ and $A \in B(H)$. Then*

(1) *A_λ converges to A in the norm topology if and only if, $||A_\lambda - A||$ converges to 0, where the norm is that of $B(H)$.*

(2) *A_λ converges to A in the strong topology if and only if for each $x \in H$, $||A_\lambda x - Ax||$ converges to 0.*

(3) *A_λ converges to A in the weak topology if and only if for each $x \in H$, $\langle A_\lambda x, y\rangle$ converges to $\langle Ax, y\rangle$.*

Proof. **(1)** A_λ converges to A in the norm topology if and only if for every open $O(A, \epsilon) = \{B \in B(H) : ||A - B|| < \epsilon\}$ containing A, there exists an α such that $A_\lambda \in O(A, \epsilon)$ for all $\lambda > \alpha$. That is $||A_\lambda - A|| < \epsilon$ for all $\lambda > \alpha$, which is equivalent to say that $||A_\lambda - A||$ converges to 0.

(2) A_λ converges to A in the strong topology if and only if for every open $O(A; x, \epsilon) = \{B \in B(H) : ||(B - A)(x)|| < \epsilon\}$ there exists an α such that $A_\lambda \in O(A; x, \epsilon)$ for all $\lambda > \alpha$. Equivalently, $||A_\lambda(x) - A(x)|| < \epsilon$ for all $\lambda > \alpha$,that is, $||A_\lambda(x) - A(x)||$ converges to 0.

(3) Similarly to (1) and (2). □

Proposition 12. *Norm convergence implies strong convergence and strong convergence implies weak convergence.*

Proof. Let $A_\lambda, \lambda \in I$, where I is a directed set, be a net in $B(H)$ converging to $A \in B(H)$ in the norm topology. That means $||A_\lambda - A||$ converges to 0. The definition of norm in $B(H)$ (supremum property) implies that for each $x \in H$, $||A_\lambda x - Ax||$ converges to 0. This gives us strong convergence.

Now, let $A_\lambda, \lambda \in I$, where I is a directed set, be a net in $B(H)$ converging to $A \in B(H)$ in the strong topology. Then for each $x \in H$, $||A_\lambda x - Ax||$ converges to 0. For any $x, y \in H$, we have

$$|\langle A_\lambda x - Ax, y\rangle| \leq ||A_\lambda x - Ax||||y|| \tag{A8}$$

Since the right hand side converges to 0, $\langle A_\lambda x - Ax, y\rangle$ converges to 0, for any $x, y \in H$, showing the weak convergence. $\square$

As a quick result of this proposition we have:

Proposition 13. *The weak topology is weaker than the strong topology and the strong topology is weaker than the norm topology.*

Proof. Let E be a closed set in the strong topology. Denote by $\overline{E}$ the closure of this set in the norm topology. Let $A \in \overline{E}$, then there exists a net $A_\lambda, \lambda \in I$, where I is a directed set, in E such that $||A_\lambda - A||$ converges to 0. But by Proposition 10, this net A_λ converges strongly in E to A. Thus $A \in E$. Hence E is closed in also norm topology. Any closed set in the strong topology is closed in the norm topology. So the strong topology is weaker than the norm topology. By the same way, it can be shown that any weakly closed set is strongly closed. Thus the weak topology is weaker than the strong topology. $\square$

But the converses of the statements in Proposition 11 are not true in general. Now we give an example of a sequence strongly convergent but not norm convergent.

Example 1. *We consider a sequence $(T_n : l^2 \to l^2)_{n\in\mathbb{N}}$ which is defined by*

$$T_n(x) = (0, 0, ..., 0, x_{n+1}, x_{n+2}, ...) \tag{A9}$$

here $x = (x_1, x_2, ...) \in l^2$ and each of the first n components is 0. This operator T_n is linear and bounded for each $n \in \mathbb{N}$. Clearly $(T_n)_{n\in\mathbb{N}}$ is convergent in the strong topology. Because $T_n x \to 0x = 0$. But, $(T_n)_{n\in\mathbb{N}}$ does not converge in the norm topology because $||T_n - 0|| = ||T_n|| = 1$.

Second, we consider an example of a sequence weakly convergent but not strong convergent.

Example 2. *Let* $(T_n : l^2 \to l^2)_{n\in\mathbb{N}}$ *be defined by*

$$T_n(x) = (0, 0, ..., 0, x_1, x_2, ...) \tag{A10}$$

here $x = (x_1, x_2, ...) \in l^2$ *and each of the first* n *components is 0. For each* $n \in \mathbb{N}$, T_n *is linear and bounded. Let* f *be a linear functional on* l^2*. By Riesz representaion theorem, there is a* $z = (z_1, z_2, ...) \in l^2$ *such that*

$$f(x) = \langle x, z\rangle = \sum_{j=1}^{\infty} x_j \overline{z_j} \tag{A11}$$

Then, setting $j = n + k$ *we have*

$$f(T_n x) = \langle T_n x, z\rangle = \sum_{j=n+1}^{\infty} x_{j-n}\overline{z_j} = \sum_{k=1}^{\infty} x_k \overline{z_{n+k}} \tag{A12}$$

Using the Cauchy-Shwarz inequality

$$|f(T_n x)|^2 = |\langle T_n x, z\rangle|^2 \leq \sum_{k=1}^{\infty} |x_k|^2 \sum_{m=n+1}^{\infty} |z_m|^2 \tag{A13}$$

The right hand-side approaches to 0 as $n \to \infty$*. Hence* $f(T_n x) \to 0 = f(0x)$*. This means that* $(T_n)_{n\mathbb{N}}$ *is convergent to 0 in the weak topology. But,* $(T_n)_{n\mathbb{N}}$ *is not convergent strongly. Since*

$$||T_m x - T_n x|| = \sqrt{1^2 + 1^2} = \sqrt{2} \;\; (m \neq n) \tag{A14}$$

We have a multiplication map on $B(H)$, given by the usual composition of operators in $B(H)$. Closedness of multiplication can be seen easily: If $A, B \in B(H)$, then their multiplication AB is bounded, since

$$||AB|| = sup_{||x||=1}||A(B(x))|| \leq ||A||||B|| < \infty \tag{A15}$$

so that $AB \in B(H)$.

Theorem 7. *Multiplication is continuous with respect to the norm topology and discontinuous with respect to strong and weak topologies.*

Proof. First we will show the continuity of multiplication in the norm topology. Let $\epsilon > 0$. Suppose that the nets $\{A_\lambda\}_{\lambda\in\Lambda}$,$\{B_\alpha\}_{\alpha\in\Lambda}$, where Λ is a directed set, converge A, B in the norm topology, respectively. Wlog suppose $||B|| \geq ||A||$. We have

$$\begin{aligned} ||A_\lambda B_\alpha - AB|| &\leq ||A_\lambda B_\alpha - A_\lambda B + A_\lambda B - AB|| \\ &\leq ||A_\lambda B_\alpha - A_\lambda B|| + ||A_\lambda B - AB|| \\ &\leq ||A_\lambda|| ||B - B_\alpha|| + ||A - A_\lambda|| ||B|| \\ &\leq (||A - A_\lambda|| + ||A||)||B - B_\alpha|| + ||A - A_\lambda|| ||B|| \end{aligned} \tag{A16}$$

Then there exists $\lambda_o \in \Lambda$ such that $||A - A_\lambda|| \leq ||B|| - ||A||$ for all $\lambda \geq \lambda_o$, there exists $\lambda_1 \in \Lambda$ such that $||A - A_\lambda|| \leq \frac{\epsilon}{2||B||}$ for all $\lambda \geq \lambda_1$ and there exists $\alpha_o \in \Lambda$ such that $||B - B_\alpha|| \leq \frac{\epsilon}{2||B||}$ for all $\alpha \geq \alpha_o$. Now setting $\gamma = max\{\alpha_o, \lambda_o, \lambda_1\}$, we then have

$$||A_\lambda B_\alpha - AB|| \leq \epsilon \tag{A17}$$

for all $\lambda, \alpha \geq \gamma$. This proves our first claim. $\square$

Secondly, we will show that the multiplication is discontinuous in the strong topology.

Step 1: Let $N = \{A \in B(H) : A^2 = 0\}$ be the set of nilpotent operators of nilpotency index 2. We claim that N is dense in $B(H)$ with respect to strong topology.

Step 2: If we show this, then for any $A \in B(H)$ there exists a net $A_\lambda \subset N$ such that $A_\lambda \to A$, so $(A_\lambda, A_\lambda) \to (A, A)$, by the definition of convergence in the product topology. If the multiplication were net continuous, then $A_\lambda^2 \to A^2$. But $A_\lambda \subset N$, so for each λ we have $A_\lambda^2 = 0$. Then $0 \to A^2$, so that $A^2 = 0$. But A was arbitrary, in particular we could have taken A to be the identity map on H, getting a contradiction.

Proof of the claim in step 1: Suppose that

$$O(A_o; x_1, ..., x_k, \epsilon) = \{A \in B(H) : ||(A - A_o)(x_i)|| < \epsilon, i = 1, ..., k\} \qquad \text{(A18)}$$

is an arbitrary basic strong neighborhood. Wlog suppose that the x_i's are linearly independent. For each i find a vector g_i such that $||A_o x_i - g_i|| < \epsilon$ and such that the span of g_i's has only 0 in common with the span of the x_i's; so long as the underlying Hilbert space is infinite dimensional, this is possible. Let A be the operator such that $Ax_i = g_i$ and $Ag_i = 0$ and $Ah = 0$ whenever $h \perp x_i$ and $h \perp g_i$.

Clearly A is nilpotent of index 2, and A belongs to the prescribed neighborhood.

Finally to show the discontinuity of multiplication in the weak topology, we use the same argument as we did in the strong operator topology.

Step 1: Since the strong operator topology is larger than the weak topology, a strongly dense set is necessarily weakly dense.

Step 2: Just replace everywhere strong by weak, it works, too.

Theorem 8. *Multiplication map on $B(H)$ is seperately continuous with respect to weak operator topology, that is continuous when one of the arguments is fixed.*

Proof. Suppose that the net $\{A_\lambda\}_{\lambda \in \Lambda}$, where Λ is a directed set converges weakly to A, i.e. $\langle A_\lambda(x), y\rangle \to \langle A(x), y\rangle$ for all $x, y \in H$. Then in particular $\langle A_\lambda(Bx), y\rangle \to \langle A(B(x)), y\rangle$ for all $x, y \in H$. Hence the multiplication is continuous with respect to weak operator topology. (Similar for the other argument) □

A2: Topological Groups

Let G be a group. A topology on G is said to be compatible with the group structure if the mapping $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous. A topological group is a pair (G, τ), where G is a group and τ is a topology on G compatible with the group structure.

Theorem 9. *If a and b are fixed elements of a topological group G, then each of the following mappings is a homeomorphism of G:*

1. $x \mapsto x^{-1}$
2. $x \mapsto ax$
3. $x \mapsto xb$
4. $x \mapsto axb$
5. $x \mapsto axa^{-1}$
6. *For a fixed $a \in G$, the mapping $x \mapsto xax^{-1}$ is continuous.*

Proof.

1. The mapping $x \mapsto x^{-1}$ is continuous and self-inverse.
2. $x \mapsto (a, x) \mapsto ax$ is the composition of two continuous mappings; the inverse mapping is $x \mapsto a^{-1}x$.
3. Similar to (2).
4. $x \mapsto ax \mapsto (ax)b$ is the composition of two homeomorphism.
5. Similar to (4).
6. $x \mapsto (xa, x^{-1}) \mapsto (xa)x^{-1}$ is the composition of two homeomorphism mappings, the first mapping being continuous because its compositions with the coordinate projections (namely, $x \mapsto xa$ and $x \mapsto x^{-1}$) are continuous.

$\square$

Corollary 4. *If G is a topological group and β is a fundamental system of the identity element e, then, for each $a \in G$, the classes $\{aV : V \in \beta\}$ are fundamental systems of* *neighborhoods of* a*. Also,* $\{V^{-1} : V \in \beta\}$ *is a fundamental system of neighborhoods of* e*.*

Proof. The mapping $m : G \to G; (x \mapsto ax)$ is a homeomorphism of G, sending e to a, and therefore transforming β into a fundamental system of neighborhoods of a. Because for any neighborhood V of G, we have

$$m^{-1}(V) = \{x \in G : ax \in V\} = \{x \in G : x \in a^{-1}V\} = a^{-1}V \tag{A19}$$

Since $a \in V$, $e \in a^{-1}V$. Then there exists an open $U \in \beta$ such that $U \in a^{-1}V$. Then $aU \subset V$. Hence $\{aU : U \in \beta\}$ is a fundamental system of neighborhoods of the element $a \in G$. (Similarly for $\{Ua : U \in \beta\}$)

Finally, since $i : G \to G(x \mapsto x^{-1})$ is a homeomorphism, for any neighborhood U of e, $i^{-1}(U) = \{x \in G : x^{-1} \in U\} = U^{-1}$ contains e. Then there exists $V \in \beta$ such that $V \subseteq U^{-1}$ implying $V^{-1} \subseteq U$, proving that $\{V^{-1} : V \in \beta\}$ is a fundamental system of neighborhoods of identity. □

Corollary 5. *If* A *is a subset of a topological group* G *and if* U *is an open subset of* G*, then the following subsets are also open:* U^{-1}, AU, UA*.*

Proof. U^{-1} is open if U is open since i is a homeomorphism. Since $x \mapsto ax$ and $a \mapsto xa$ are homeomorphisms of G,

$$AU = \bigcup_{a \in A} aU; \;\; UA = \bigcup_{a \in A} Ua \tag{A20}$$

are unions of open sets. Hence they are open. □

Theorem 10. *If* G *is a topological group, then the class* C *of all neighborhoods of the identity* e *has the following properties:*

1. $e \in V$ *for all* $V \in C$*;*

2. *If* $V, W \in C$*, then* $V \cap W \in C$*;*

3. *If $V \in C$ then there exists $W \in C$ such that $WW \subset V$;*

4. *If $V \in C$, then $V^{-1} \in C$;*

5. *If $V \in C$ and $a \in G$ then $aVa \in C$;*

6. *If $V \in C$ and $V \subset W$ then $W \in C$.*

Proof. (1), (2), (6) are general properties of the filter of neighborhoods of a point of a topological space.(Note that the term 'neighborhood' means any superset of an open set.)

3. The mapping $f(x, y) = xy$ is continuous at (e, e), and $f(e, e) = e$. Given any neighborhood V of e, choose a neighborhood A of (e, e) in $G \times G$ such that $f(A) \subset V$. One can suppose that $A = W \times W$ for some $W \in C$ (Such neighborhoods of (e, e) are basic.) Thus $WW = f(W \times W) = f(A) \subset V$.

4,5. $x \mapsto x^{-1}$ and $x \mapsto axa^{-1}$ are homeomorphisms of G mapping e to e.

□

The properties listed in Theorem 10, are characteristic of compability of a topology with the group structure:

Theorem 11. *If G is a group with identity e, and C is a nonempty class of subsets of G satifying (1)-(6) of Theorem 10, then there exists a unique topology on G such that*

1. *The topology is compatible with the group structure,*

2. *C is the system of all neighborhoods of e.*

Proof. <u>Uniqueness:</u> follows by Corollary 4.

Existence: By (1), (2) and (6), C is a filter of sets containing e. For each $a \in G$, define $C_a = \{aV : V \in C\}$; since $x \mapsto ax$ is bijective and maps e to a, C_a is a filter of sets containing a. In particular $C_e = C$.

We assert that the family $(C_a)_{a \in G}$ is the family of neighborhood filters for a topology on G. Given $N \in C_x$, it is to be shown that there exists $M \in C_x$ such that $y \in M$ implies $N \in C_y$. Say $N = xV$, $V \in C$. By (3) choose $W \in C$ such that $WW \subset V$, and set $M = xW$. Then $y \in M$ implies that $yW \subset xWW \subset xV = N$, thus $N \supset yW \in C_y$ and therefore $N \in C_y$. It remains to show that the topology on G defined in the preceding paragraph is compatible with the group structure.

Fix $a, b \in G$. To see that $(x, y) \mapsto xy$ is continuous at (a, b), a neighborhood D of ab given, say $D = abV$, $V \in C$; it will suffice to find $U \in C$ such that $(aU)(bU) \subset abV$, i.e. $(b^{-1}Ub)U \subset V$. By (3) choose $W \in C$ with $WW \subset V$, then $b^{-1}Wb \in C$ by (5), and, setting $U = (bWb^{-1}) \cap W$ one has $U \in C$ and $(b^{-1}Ub)U \subset [b^{-1}(bWb^{-1})b]W = WW \subset V$ implying $(aU)(bU) \subset abV$.

Finally, fixing $a \in G$, it is to be shown that $x \mapsto x^{-1}$ is continuous at a. Given any $S \in C_{a^{-1}}$, say $S = a^{-1}V$, $V \in C$ we seek $W \in C$ such that $(aW)^{-1} \subset a^{-1}V$, i.e. $W^{-1} \subset a^{-1}Va$, ,i.e. $W \subset a^{-1}Va$. Indeed, $a^{-1}Va \in C$ by (4) and (5), thus $W = a^{-1}Va$ is suitable. □

Theorem 12. *Let G be a group with identity e, and let β be a nonempty class of subsets of G satisfying the following properties:*

1. *$e \in B$ for all $B \in \beta$.*

2. *If $B_1, B_2 \in \beta$ then there exists $B_3 \in \beta$ such that $B_3 \subset B_1 \cap B_2$.*

3. *If $B \in \beta$ then there exists $C \in \beta$ such that $CC \subset B$.*

4. *If $B \in \beta$ then there exists $C \in \beta$ such that $C^{-1} \subset B$.*

5. *If $B \in \beta$ and $a \in G$ then there exists $C \in \beta$ such that $C \subset aBa^{-1}$.*

Then there exists a unique topology on G such that

1. *the topology is compatible with the group structure,*

2. *β is a fundamental systen of neighborhoods of e.*

Proof. β is a base for precisely one filter C, namely, $C = \{V : V \supset B \textit{ for some } B \in \beta\}$.

Evidently C satisfies the hypothesis (1)-(6) of the Theorem 11. □

Definition 8. *A sequence $(g_n)_{n\in\mathbb{N}}$ in a topological group G is said to be Cauchy if for every open neighborhood U of $e \in G$ there exists an $n \in \mathbb{N}$ such that $\forall n, m \geq N$, $g_n^{-1} g_m \in U$ and $g_m^{-1} g_n \in U$.*

Lemma 22. *Suppose that G is a Hausdorff topological space and $T : G \to \Gamma(H)$ is a continuous unitary representation of G and $(g_n)_{n\in\mathbb{N}}$ is a Cauhcy sequence in G. Then for every $x \in H$, the image sequence $(T(g_n)(x))_{n\in\mathbb{N}}$ is Cauhcy.*

Proof. Suppose that $(g_n)_{n\in\mathbb{N}}$ is a Cauhcy sequence in G. That is, if for every open neighborhood U of $e \in G$ there exists an $N \in \mathbb{N}$ such that $\forall n, m \geq N$, $g_n^{-1} g_m \in U$ and $g_m^{-1} g_n \in U$. Let $x \in H$ and $\epsilon > 0$. Since T is continuous $\|x - T(u)x\| < \epsilon$ when u is sufficiently close to e, that is $u \in U$ where U is some open neighborhood of e. Now, by assumption, there exists an $N \in \mathbb{N}$ such that $\forall n, m \geq N$, $g_n^{-1} g_m \in U$. So $g_m \in g_n U$. Then for $m, n \geq N$, $g_m = g_n u_{m,n}$ for some $u_{m,n} \in U$ and

$$\begin{aligned} \|T(g_n)x - T(g_m)x\| &= \|T(g_n)x - T(g_n u_{m,n})x\| \\ &= \|T(g_n)(x - T(u_{m,n})x)\| \\ &= \|x - T(u_{m,n})x\| \leq \epsilon. \end{aligned} \tag{A21}$$

Hence $(T(g_n)x)_{n\in\mathbb{N}}$ is Cauhcy. □

For more detailed information you can look Berberian 1974.

A3: Semidirect Products

Suppose that M and N are two groups and $\psi : T \rightarrow Aut(M)$ is a group homomorphism. Let's construct a group denoted by $M \rtimes_{\psi} N$, or just by $M \rtimes N$. The set on which the binary operation is defined on is $M \times N$, and the operation is as follows:

$$(m, n)(m', n') = (m\psi(n)(m'), nn') \tag{A22}$$

It is a group and its identity is $(1, 1)$. The inverse of an element $(m, n) \in M \rtimes N$ is $(\psi(n^{-1})(m^{-1}), n^{-1})$. Let's denote this group by G. G is called the semidirect product of M and N. We can identify M with $M \times \{1\}$ and hence consider as a normal subgroup of G. We can identify N with $\{1\} \times N$ and consider as a subgroup of G. These subgroups M and N have the following properties: $M \trianglelefteq G$, $N \leq G$, $M \cap N = 1, G = UN$.

Conversely, if a group G has subgroups M and N satisfying these properties, then G is isomorphic to a semidirect product $M \rtimes_{\psi} N$ where $\psi : N \rightarrow Aut(M)$ is given by $\psi(n)(m) = nmn^{-1}$ for every $m \in M, n \in N$.

A4: Ergodic Theory

Definition 9. *A generalized sequence is any set $\{y_\alpha\}$ of terms y_α between which a transitive relation $y_\alpha < y_\beta$ (read y_β is a successor of y_α) is defined. We allow the same element to occur repeatedly in the sequence.*

Definition 10. *A generalized sequence $\{y_\alpha\}$ of points of a given topological space converges to a limit a if and only if given any neighborhood $U(a)$ of a, each term of the sequence has a successor all of whose successors lie in $U(a)$.*

Let E be a Banach space. By a fix-point of E we mean an element a such that $N(a) = a$ for all $N \in B(E)$.

Elements not fix-points will have in general many transforms; the convex hull of the set of transforms $N(x)$ of any element x, consisting of the weighted means or averages

$$\sum \lambda N, \ \ (\lambda \geqslant 0, \sum \lambda = 1) \tag{A23}$$

of its transforms, will play a significant role below.

By the successors of a mean of transforms of y, we signify the means of its transforms. Since

$$\sum \mu_j(\sum \lambda_i N_i(y))M_j = \sum \mu_j\lambda_i(N_iM_j)(y), \ \ (\mu_j\lambda_i \geq 0, \sum \mu_j\lambda_i = 1) \tag{A24}$$

these are means of transforms of y; for the same reason, the predecessor-successor relation is transitive.

Definition 11. *An element y of E is ergodic if and only if the means of its transforms converge in the sense of Definition 10 to a fix-point.*

Definition 12. *A uniformly convex Banach space is one in which, given $\epsilon > 0$, there exists $\delta > 0$ so small that $||y|| \leq ||x|| \leq 1$ and $||x-y|| \geq \epsilon$ implies $||\frac{1}{2}(x+y)|| \leq ||x|| - \delta$*

That Hilbert space is uniformly convex follows from

$$||x+y||^2 + ||x-y||^2 = 2||x||^2 + 2||y||^2 \tag{A25}$$

whence $||\frac{1}{2}(x+y)|| \leq (||x|| - \frac{1}{4}||x-y||^2)^2$.

Lemma 23. *Let F be any closed convex subset of any uniformly convex Banach space. Then there is one and only one point of F of least norm.*

Proof. Let T be the infimum of the norms of the elements of F. Evidently $||a|| = ||b|| = T$ would imply $||\frac{1}{2}(a+b)|| < T$ if $a \neq b$, which is impossible; hence there is only one such a point. To prove there is such a point, choose $\{y_n\}$ from F so that $||y_{n+1}|| \leq ||y_n|| < T+\frac{1}{n}$. Then $||y_{n+1}-y_n|| > \epsilon$ implies by definition $||\frac{1}{2}(y_{n+1}+y_n)|| \leq T+\frac{1}{n}-\delta$ and hence $n \leq \delta$ since $\frac{1}{2}(y_{n+1}+y_n) \in F$. Therefore $\{y_n\}$ is convergent; its limit a will evidently be a point in F, of norm T, completing the proof. $\square$

It is a corollary that if F is the closed convex hull $F(y)$ of the set of transforms of some y, then this point is carried by every S into a point of F of no greater norm, and so into itself. Consequently

Theorem 13. *If for all $S \in B(E)$ has moduli bounded by one, and E is uniformly convex, then the closed convex hull $F(y)$ of the set of transforms of any element y contains at least one fix-point.*

For more detailed information you can look Alaoglu & Birkhoff 1940.