EXTENSION PROBLEM AND BASES FOR SPACES OF INFINITELY DIFFERENTIABLE FUNCTIONS

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Extension problem and bases for spaces of infinitely differentiable functions
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April 2017

We certify that we have read this dissertation and that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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ABSTRACT

EXTENSION PROBLEM AND BASES FOR SPACES OF INFINITELY DIFFERENTIABLE FUNCTIONS

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We examine the Mityagin problem: how to characterize the extension property in geometric terms. We start with three methods of extension for the spaces of Whitney functions. One of the methods was suggested by B. S. Mityagin: to extend individually the elements of a topological basis. For the spaces of Whitney functions on Cantor sets $K(\gamma)$, which were introduced by A. Goncharov, we construct topological bases. When the set $K(\gamma)$ has the extension property, we construct a linear continuous extension operator by means of suitable individual extensions of basis elements. Moreover, we use local Newton interpolations to contruct an extension operator. In the end, we show that for the spaces of Whitney functions, there is no complete characterization of the extension property in terms of Hausdorff measures or growth of Markov's factors.

Keywords: Whitney functions, Extension operator, Topological bases, Hausdorff measures, Markov factors.

ÖZET

SONSUZ TÜREVLENEBİLİR FONKSİYON UZAYLARI İÇİN GENİŞLETME PROBLEMİ VE TABANLAR

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Genişletme özelliğini geometrik terimler aracılığıyla nasıl tanımlanabileceği üzerine olan Mityagin problemini inceledik. Whitney fonksiyon uzayları için bilinen üç genişletme metodu ile başladık. Bunlardan birisi B. S. Mityagin tarafından önerilen metod: topolojik bazın elemanlarını ayrı ayrı genişletmektir. A. Goncharov tarafından tanıtılan Cantor kümeleri $K(\gamma)$ için tanımlanmış olan Whitney fonksiyon uzaylarında bir topolojik taban oluşturduk. Eğer $K(\gamma)$ kümesinin genişletme özelliği varsa, tabanın elemanlarının birbirinden ayrı uygun genişlemeleri ile bir doğrusal sürekli genişletme operatörü oluşturduk. Ayrıca, yerel Newton interpolasyonlarını kullanarak da genişletme operatörü oluşturduk. Sonunda, Whitney fonksiyon uzaylarının genişletme özelliğinin Hausdorff ölçümleri veya Markov faktörlerinin büyümesi cinsinden eksiksiz tanımlanamayacağını gösterdik.

Anahtar sözcükler: Whitney fonksiyonları, Genişletme operatörü, Topolojik taban, Hausdorff ölçümleri, Markov faktörleri.

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Contents

List of Figures

List of Notations

Chapter 1

Introduction

Let $K \subset \mathbb{R}^d$ be a compact set, $p \in \mathbb{N}$ and B be a closed cube containing K. The Whitney space $\mathcal{E}^p(K)$ of extendable jets contains all $f \in \prod_{|\alpha| \le p} C(K)$ such that there exists $F \in C^p(B)$ with $F^{(\alpha)}|_K = f^{(\alpha)}, \ \ |\alpha| \le p$. Thus the space $\mathcal{E}^p(K)$ consists of traces of C^p-functions defined on B, so it is a factor space and it should be equipped with the quotient norm $|||\, f|||_{\,p} = \inf |F|_{p,B}$ where infimum is taken over all possible extensions of f to $F \in C^p(B)$. Here $x = (x_1, \ldots, x_d), (\alpha)$ denotes a multiindex $(\alpha) = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d, |\alpha| = \alpha_1 + \ldots + \alpha_d$,

$$f^{(\alpha)} := \frac{\partial^{|\alpha|} f}{\partial {x_1}^{\alpha_1} \ldots \partial {x_d}^{\alpha_d}} \tag{1.1}$$

and

$$|F|_{p,B} := \sup\{|F^{(\alpha)}(x)| : x \in B, |\alpha| \le p\}. \tag{1.2}$$

For given K, the spaces $C^\infty(K)$ and $\mathcal{E}(K) := \mathcal{E}^\infty(K) = \cap_{p=0}^\infty \mathcal{E}^p(K)$ are defined as the corresponding projective limits, τ_Q is the quotient topology of $\mathcal{E}(K)$. Then $\mathcal{E}(K)$ is the space of jets $f : K \to \mathbb{R}$ extendable to $\mathcal{C}^\infty$-functions on $\mathbb{R}^d$. Let $R_y^p f(x) = f(x) - T_y^p f(x)$ be the Taylor remainder centered at y and

$$r_p(f,t) := \sup\{|(R_y^p f)^{(\alpha)}(x)|.|x-y|^{|\alpha|-p} : x, y \in K, 0 < |x-y| \le t, |\alpha| \le p\}. \tag{1.3}$$

We consider the space $E^p(K)$ of jets f on K such that $r_p(f,t) \to 0$ as $t \to 0$. Its topology τ_W is given by the seminorms

$$\|f\|_p = |f|_p + \sup r_p(f,t) \tag{1.4}$$

$p = 0, 1, \ldots$, where

$$|f|_p = \sup\{|f^{(\alpha)}(x)| : x \in K, |\alpha| \leq p\}. \tag{1.5}$$

We say $f \in E(K)$ if $f \in E^p(K), \forall p$. Then $E(K)$ is a Fréchet space; its topology is defined by the sequence of seminorms $(\|.\|_p)_0^\infty$.

Theorem 1.0.1. *(Whitney's Extension Theorem [31]) For all $p < \infty$, there exists a linear continuous extension operator $W_p : E^p(K) \to C^p(B)$. And the restriction operator $R : C^\infty(B) \to E(K)$ is surjective.*

For a sketch of the proof see Section 2.3.

Therefore, $E^p(K) = \mathcal{E}^p(K)$ for $p \leq \infty$. Since both spaces are complete and τ_W is not stronger than τ_Q, by the open mapping theorem, the topologies are equivalent, that is, any non-empty open set of one topology contains a non-empty open set of the other topology. For details about the spaces of Whitney functions see [5].

For each compact set $K \subset \mathbb{R}^d$, Whitney functions can be extended from jets of finite order $\mathcal{E}^p(K)$ to the functions in $C^p(B)$. By Whitney Extension Theorem, this extension can be done by means of a linear continuous operator provided $p < \infty$. In the case $p = \infty$, the possibility of such an extension by means of a linear continuous operator depends on a structure of the set K. The existence of an extension operator in the C^∞ case was first proved by B.S. Mityagin [18] and R.T. Seeley [24].

Definition 1.0.2. Let K be a compact subset of $\mathbb{R}^d$. Then K is called C^∞-determining if for each $f \in C^\infty(\mathbb{R}^d)$, the condition $f|_K = 0$ implies $f^{(k)}|_K = 0$ for all $k \in \mathbb{N}^d$.

Clearly, $K \subset \mathbb{R}$ is C^∞-determining if and only if K is perfect. Thus, if we have C^∞-determining subsets of $\mathbb{R}^d$, or perfect subsets of $\mathbb{R}$, the function $f \in \mathcal{E}(K)$ will describe all derivatives of f and the jet $(f^{(\alpha)})$ will be completely defined by f.

Definition 1.0.3. ([27]) A compact set K has the *extension property (EP)* if there exists a linear continuous extension operator $W : \mathcal{E}(K) \longrightarrow C^\infty(\mathbb{R}^d)$.

Example 1.0.4. Let us consider the simplest example of a set without (EP): $K = \{0\} \subset \mathbb{R}$ does not have the extension property. Assume to the contrary that an extension operator exists. Then we have

$$\forall p \ \ \exists q, C \quad : |Wf|_{p,[-1,1]} \leq C\|f\|_q, \quad \forall f \in \mathcal{E}(K). \tag{1.6}$$

For $p = 0$, choose a corresponding $q = q(p)$. Define a jet $f^{(q+1)} = 1$ and $f^{(i)} = 0$ for all $i \neq q+1$. Clearly, this is a Whitney jet since it is a restriction of the polynomial $F(x) = \frac{1}{(q+1)!}x^{q+1}$ on the set K. Then $\|f\|_q = 0$, whereas $Wf \neq 0$, so the left hand side of the equation (1.6) is positive.

Generalizing this, any set K with an isolated point does not have EP.

B. S. Mityagin posed in 1961 ([18], p.124) the following problem (in our terms): *What is a geometric characterization of the extension property?*

1.1 Notations and Auxiliary Results

For each set A, let $\#(A)$ be the cardinality of A, $|A|$ be the diameter of A. Also, $[a]$ is the greatest integer less than or equal to a, $\sum_{k=m}^{n}(\cdots) = 0$ and $\prod_{k=m}^{n}(\cdots) = 1$ if $m > n$. The symbol $\sim$ denotes the strong equivalence: $a_n \sim b_n$ means that $a_n = b_n(1 + o(1))$ for $n \to \infty$.
Let K be a perfect compact subset of $I = [0,1], y \in K$ and $f \in \mathcal{E}(X)$. Taylor polynomial (of order q) of f at y is the polynomial

$$T_y^q f(x) = \sum_{k=0}^{q} \frac{f^{(k)}(y)}{k!}(x-y)^k. \tag{1.7}$$

Then the Fréchet topology τ in the space $\mathcal{E}(K)$ can be given by the norms

$$\| f \|_q = |f|_{q,K} + \sup \left\{ \frac{|(R_y^q f)^{(k)}(x)|}{|x-y|^{q-k}} : x, y \in K, x \neq y, k = 0, 1, \dots, q \right\}$$

for $q \in \mathbb{Z}_+$, where $|f|_{q,K} = \sup\{|f^{(k)}(x)| : x \in K, k \leq q\}$ and $R_y^q f(x) = f(x) - T_y^q f(x)$ is the Taylor remainder.

Theorem 1.1.1. *If $F^{(q+1)}$ is continuous on an open interval I that contains y, and $x \in I$, then there exists a number c between y and x such that*

$$R_y^q F(x) = \frac{F^{(q+1)}(c)}{(q+1)!}(x-y)^{q+1}. \tag{1.8}$$

Given $f \in \mathcal{E}(K)$, let $|||\, f \,|||_q = \inf \ |\, F \,|_{q,I}$, where the infimum is taken over all possible extensions of f to $F \in C^\infty(I)$. From the definition of Taylor remainder, we have $R_y^q f(x) = R_y^q F(x)$ for any extension F. By the Lagrange form of the Taylor remainder given in the above theorem, we have $||\, f \,||_q \leq 3\,|\, F \,|_{q,I}$. The quotient topology τ_Q, given by the norms $(|||\cdot|||_{q=0}^\infty)$, is complete and, by the open mapping theorem, is equivalent to τ. Hence for any q there exist $r \in \mathbb{N}$, $C > 0$ such that

$$|||\, f \,|||_q \leq C \,||\, f \,||_r \tag{1.9}$$

for any $f \in \mathcal{E}(K)$. In general, extensions F that realize $|||\, f \,|||_q$ for a given function f essentially depend on q. Of course, the extension property of K means the existence of a simultaneous extension which is suitable for all norms.

1.2 Tidten-Vogt Linear Topological Characterization of EP

A sequence $(f_i)_{i\in\mathbb{Z}}$ (finite or infinite) of linear maps between linear spaces $(M_i)_{i\in\mathbb{Z}}$

$$\dots \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_n} M_n \xrightarrow{f_{n+1}} M_{n+1} \longrightarrow \dots \tag{1.10}$$

is said to be exact if $\operatorname{im} f_i = \ker f_{i+1}, \forall i$.

Remark 1.2.1. Let F be a Fréchet space and E be a closed subspace of F. Then E and F/E are likewise Fréchet spaces (see [17]). If $J : E \longrightarrow F$ is the inclusion and $R : F \longrightarrow F/E$ is the quotient map, then

$$0 \longrightarrow E \xrightarrow{J} F \xrightarrow{R} F/E \longrightarrow 0 \tag{1.11}$$

is a short exact sequence of Fréchet spaces.

Let I be a closed cube containing K and $\mathcal{F}(K, I) = \{F \in C^\infty(I) : F^{(\alpha)}|_K = 0,\ \forall \alpha\}$ be the ideal of flat on K functions. The Whitney space $\mathcal{E}(K)$ of extendable jets consists of traces on K of C^∞-functions defined on I, so it is a factor space of $C^\infty(I)$ and the restriction operator $R : C^\infty(I) \longrightarrow \mathcal{E}(K)$ is surjective. This means that the sequence

$$0 \longrightarrow \mathcal{F}(K, I) \xrightarrow{J} C^\infty(I) \xrightarrow{R} \mathcal{E}(K) \longrightarrow 0 \tag{1.12}$$

is exact. The sequence is called split exact if $C^\infty(I) \cong \mathcal{F}(K, I) \oplus \mathcal{E}(K)$. If it splits then a right inverse to R exists. Then R^{-1} is the desired linear continuous extension operator W and K has EP.

Definition 1.2.2. Let X be a Fréchet space with an increasing system of seminorms $(||\cdot||_k)_{k=0}^\infty$. X is said to have a *dominating norm* $||\cdot||_p$ if

$$(DN)\ \ \forall q \in \mathbb{N}\ \ \exists r \in \mathbb{N}, C \geq 1\ \ : ||\cdot||_q^2 \leq C\,||\cdot||_p\,||\cdot||_r. \tag{1.13}$$

Moreover, the following is also an important invariant in structure theory of Fréchet spaces. For all $x' \in X'$,

$$(\Omega)\ \ \forall p\ \ \exists q\ \ \forall r\ \ \exists \epsilon, C\ \ : ||\cdot||_q^* \leq C\,(||\cdot||_p^*)^\epsilon\,(||\cdot||_r^*)^{1-\epsilon}. \tag{1.14}$$

Theorem 1.2.3. *(Splitting theorem, [30]) Let E, F and G be Fréchet-Hilbert spaces and let*

$$0 \longrightarrow F \xrightarrow{j} G \xrightarrow{q} E \longrightarrow 0 \tag{1.15}$$

be a short exact sequence with continuous linear maps. If E has property (DN) and F has the property (Ω), then the sequence splits.

In [27] M. Tidten applied D. Vogt's theory of splitting of short exact sequences of Fréchet spaces (see e.g. [17], Chapter 30) and presented the following important linear topological characterization of EP:

Corollary 1.2.4. *[27] A compact set K has the extension property if and only if the space $\mathcal{E}(K)$ has a dominating norm (satisfies the condition (DN)).*

1.3 Polynomial Interpolation and Divided Differences

There are several ways to approximate a continuous function f on a closed interval $[a, b]$ by polynomials. We consider here interpolating polynomials.

Definition 1.3.1. A polynomial p is called an interpolating polynomial of f at the set of distinct points $(x_i)_{i=0}^n$ if $p(x_i) = f(x_i), \forall i \leq n$.

Theorem 1.3.2. *[22] Let $(x_i)_{i=0}^n$ be any set of distinct points in $[a, b]$ and let $f \in C[a, b]$. Then there is exactly one interpolating polynomial of degree n.*

Proof. It is easy to see that there cannot be two different interpolating polynomials of order n. Assume to the contrary that p_1 and p_2 are two interpolating polynomials of order n. Then $p_1 - p_2$ will be a polynomial of order n with $n+1$ zeros. Hence, p_1 must be equal to p_2. To show existence, we use the Lagrange interpolation formula. Let $w(x) = (x - x_0)(x - x_1)\dots(x - x_n)$, Lagrange fundamental polynomials of order n are defined as

$$l_k(x) = \frac{w(x)}{(x - x_k)w'(x_k)}, \quad k = 0, 1, \dots, n. \tag{1.16}$$

If $k = i$ then $l_k(x_i) = 1$, otherwise $l_k(s_i) = 0$. Lagrange interpolating polynomial is given by the formula

$$L_n f(x) = \sum_{k=0}^{n} f(x_k) l_k(x). \tag{1.17}$$

□

Another important polynomial interpolation is the Newton interpolation. Let

$$n_0(x) = 1, \quad n_j(x) = \prod_{k=0}^{j-1}(x - x_k), \quad j = 1, \ldots, n. \tag{1.18}$$

Newton interpolating polynomial is defined as

$$N_n(x) = \sum_{j=0}^{n} c_j n_j \tag{1.19}$$

where the coefficients c_j of the interpolating polynomial are said to be divided differences of order j for the function f and denoted by $[x_0, \ldots, x_j]f$. The divided difference of order zero is the value of function itself, i.e $[x_0]f = f(x_0)$ and

$$[x_i, \ldots, x_{i+k}]f = \frac{[x_{i+1}, \ldots, x_{i+k}]f - [x_i, \ldots, x_{i+k-1}]f}{x_{i+k} - x_i}. \tag{1.20}$$

By Theorem (1.3.2) interpolating polynomials and the coefficients c_n are equal. Using the equation (1.17), for all $n \geq 1$, we get

$$[x_0, \ldots, x_n]f = \sum_{k=0}^{n} \frac{f(x_k)}{w'_k(x)}. \tag{1.21}$$

Theorem 1.3.3. *[22]Let f be a continuous function on the interval I and $(x_i)_{i=0}^n$ be a set of distint points from I. Then, there exists $c \in I$ such that*

$$[x_0, \ldots, x_n]f = \frac{f^{(n)}(c)}{n!} \tag{1.22}$$

By the above theorem, one can say that the coefficients in Newton interpolating polynomial have similarities with the Taylor polynomial. For more information, see [22].

1.4 Weakly Equilibrium Cantor-Type Sets $K(\gamma)$

Weakly equilibrium Cantor-type set $K(\gamma)$ was introduced by A. Goncharov in [13]. Given sequence $\gamma = (\gamma_s)_{s=1}^{\infty}$ with $0 < \gamma_s < 1/4$, let $r_0 = 1$ and $r_s = \gamma_s r_{s-1}^2$

for $s \in \mathbb{N}$. Define $P_2(x) = x(x-1)$, $P_{2^{s+1}} = P_{2^s}(P_{2^s} + r_s)$ and $E_s = \{x \in \mathbb{R} : P_{2^{s+1}}(x) \leq 0\}$ for $s \in \mathbb{N}$. Then $E_s = \cup_{j=1}^{2^s} I_{j,s}$, where the s-th level *basic intervals* $I_{j,s}$ are disjoint and $\max_{1 \leq j \leq 2^s} |I_{j,s}| \to 0$ as $s \to \infty$. Here, $(P_{2^s} + r_s/2)(E_s) = [-r_s/2, r_s/2]$, so the sets E_s are polynomial inverse images of intervals. Since $E_{s+1} \subset E_s$, we have a Cantor-type set

$$K(\gamma) := \cap_{s=0}^{\infty} E_s. \tag{1.23}$$

Varying the parameters allows us to construct sets with various properties. The set $K(\gamma)$ can be considered as a generalization of the classical quadratic Julia set, but as opposed to Julia sets, it is more flexible with respect to its features. For more information, see [2].

In what follows we will consider only γ satisfying the assumptions

$$\gamma_k \leq 1/32 \quad \text{for} \quad k \in \mathbb{N} \quad \text{and} \quad \sum_{k=1}^{\infty} \gamma_k < \infty. \tag{1.24}$$

The lengths $l_{j,s}$ of the intervals $I_{j,s}$ of the $s-$th level are not the same, but, provided (1.24), we can estimate them in terms of the parameter $\delta_s = \gamma_1 \gamma_2 \cdots \gamma_s$ ([13], Lemma 6):

$$\delta_s < l_{j,s} < C_0\, \delta_s \quad \text{for} \quad 1 \leq j \leq 2^s, \tag{1.25}$$

where $C_0 = \exp(16 \sum_{k=1}^{\infty} \gamma_k)$. Each $I_{j,s}$ contains two *adjacent* basic subintervals $I_{2j-1,s+1}$ and $I_{2j,s+1}$. Let $h_{j,s} = l_{j,s} - l_{2j-1,s+1} - l_{2j,s+1}$ be the distance between them. By Lemma 4 in [13], $h_{j,s} > (1 - 4\gamma_{s+1}) l_{j,s}$. Therefore,

$$h_{j,s} \geq 7/8 \cdot l_{j,s} > \frac{7}{8} \cdot \delta_s \quad \text{for all } j. \tag{1.26}$$

In addition, by Theorem 1 in [13], the level domains

$$D_s = \{z \in \mathbb{C} : |P_{2^s}(z) + r_s/2| < r_s/2\} \tag{1.27}$$

form a nested family and $K(\gamma) = \cap_{s=0}^{\infty} \overline{D}_s$.

We decompose all zeros of P_{2^s} into s groups. Let $X_0 = \{x_1, x_2\} = \{0, 1\}, X_1 = \{x_3, x_4\} = \{l_{1,1}, 1 - l_{2,1}\}, \cdots, X_k = \{l_{1,k}, l_{1,k-1} - l_{2,k}, \cdots, 1 - l_{2^k,k}\}$ for $k \leq s-1$. Thus, $X_k = \{x : P_{2^k}(x) + r_k = 0\}$ contains all zeros of $P_{2^{k+1}}$ that are not zeros of P_{2^k}. Set $Y_s = \cup_{k=0}^{s} X_k$. Then $P_{2^s}(x) = \prod_{x_k \in Y_{s-1}} (x - x_k)$. Clearly, $\#(X_s) = 2^s$ for $s \in \mathbb{N}$ and $\#(Y_s) = 2^{s+1}$ for $s \in \mathbb{Z}_+$. We refer $s-$*th type* points to the elements of X_s.

The points of Y_s can be ordered using, as in [11], the *rule of increase of the type*. First we take points from X_0 and X_1 in the ordering given above. The set $X_2 = \{x_5, \cdots, x_8\}$ consists of the points of the second type. We take x_{j+4} as the point which is the closest to x_j for $1 \leq j \leq 4$. Here, $x_5 = x_1 + l_{1,2}$, $x_6 = x_2 - l_{4,2}$, etc. Similarly, $X_k = \{x_{2^k+1}, \cdots, x_{2^{k+1}}\}$ can be defined by the previous points. For $1 \leq j \leq 2^k$, the point x_j is an endpoint of a certain basic interval of k-th level. Let us take x_{j+2^k} as its another endpoint. Thus, $x_{j+2^k} = x_j \pm l_{i,k}$, where the sign and i are uniquely defined by j. In the same way, any N points can be chosen on each basic interval.

For example, suppose $2^n \leq N < 2^{n+1}$ and the points $(z_k)_{k=1}^N$ are chosen on $I_{j,s}$ by this rule. Then the set includes all 2^n zeros of $P_{2^{s+n}}$ on $I_{j,s}$ (points of the type $\leq s+n-1$) and some $N - 2^n$ points of the type $s+n$.

1.5 Potential Theory

In [15], our main concern was the extension property of the set $K(\gamma)$ and characterization of EP in terms of Hausdorff measures and Markov's factors. In Potential Theory, R. Nevanlinna [19] and H. Ursell [29] proved that there is no complete characterization of polarity of compact sets in terms of Hausdorff measures. We presented two Cantor-type sets for which the smaller set (with respect to Hausdorff measure) has extension property whereas the larger set does not have EP. Let us recall the necessary notions from Potential Theory.

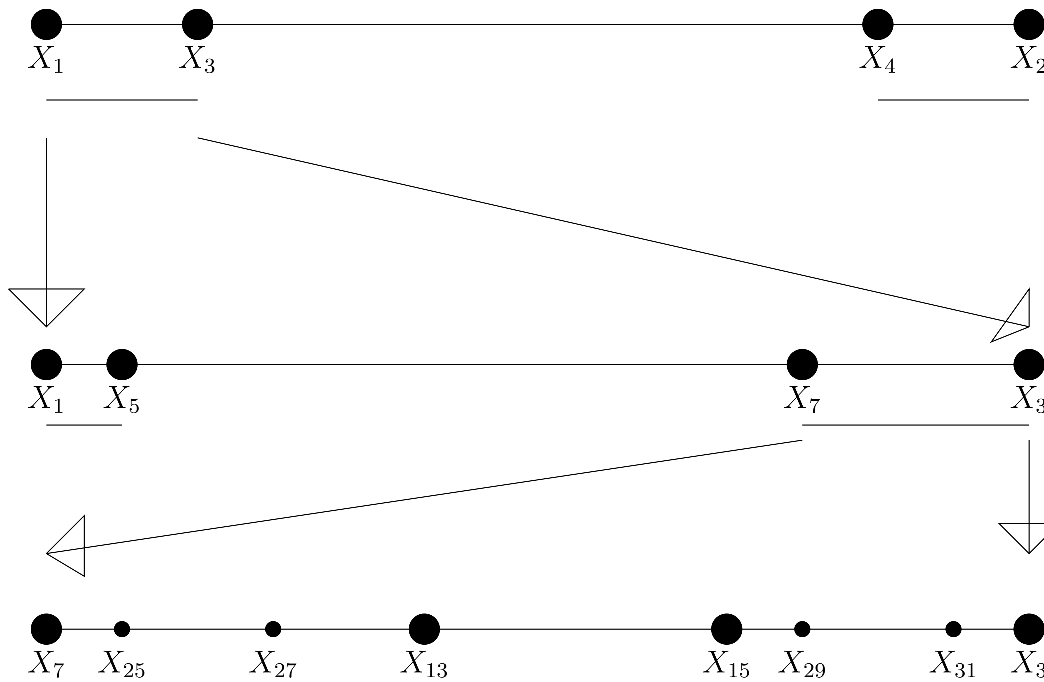

Figure 1.1: Ordering of Points

Definition 1.5.1. If μ is a Borel measure with support in K, then its logarithmic energy is defined as

$$I(\mu) := \int\int \log \frac{1}{|z-t|} d\mu(t)d\mu(z). \tag{1.28}$$

The logarithmic capacity $Cap(K)$ of K is defined by the formula

$$\log \frac{1}{Cap(K)} := \inf\{I(\mu) : \mu \geq 0, supp(\mu) \subseteq K\} \tag{1.29}$$

The equilibrium measure ω_K of a set K of positive capacity is the unique probabity measure ω_K minimizing the energy integrals in (1.29). Logarithmic potential of equilibrium measure defined as

$$U^{\omega_K}(z) := \int \log \frac{1}{|z-t|} d\omega_K(t). \tag{1.30}$$

$U^{\omega_K}(z)$ has the following properties

$$U^{\omega_K}(z) \leq \log(Cap\,(K)^{-1}) \quad for \quad z \in \mathbb{C}, \tag{1.31}$$

$$U^{\omega_K}(z) := \log(Cap\,(K)^{-1}) \quad q.e \quad z \in K, \tag{1.32}$$

where "q.e" means quasieverywhere, except on a set of zero capacity. The zero capacity sets are also called polar sets. If K is not polar then ω_K is its equilibrium measure.

We are interested in the minimal value of the logarithmic energy $\log(Cap\,(K)^{-1})$, which is also called the Robin constant of K and is denoted by

$$Rob(K) = \log(1/Cap(K)) \leq \infty. \tag{1.33}$$

For the level domains D_s defined for $K(\gamma)$ by (1.27), we have $R_s = Rob(\bar{D}_s) = 2^{-s} \cdot \log \frac{2}{r_s} \uparrow R \leq \infty$.

If $R < \infty$ then $\rho_s = R - R_s \downarrow 0$, $\rho_0 = R - \log 2$, $R_0 = \log 2$, as $r_0 = 1$. $\delta_0 := 1$,

$$\delta_s = \gamma_1 \ldots \gamma_s, \quad r_s = \gamma_s \cdot r_{s-1}^2 = \gamma_s \cdot \gamma_{s-1}^2 \ldots \gamma_1^{2^{s-1}} = \delta_s \cdot r_1 \cdot r_2 \ldots r_{s-1} \Rightarrow r_s < \delta_s < \gamma_s. \tag{1.34}$$

If $R < \infty$ then $2^{-s} \cdot \log \delta_s \to 0$ (by equation 10 in [13]).

$$R_s = 2^{-s} \cdot \log 2 + \sum_{k=1}^{s} 2^{-k} \cdot \log \frac{1}{\gamma_k} = 2^{-s} \cdot \log \frac{2}{\delta_s} + \sum_{k=1}^{s-1} 2^{-k-1} \cdot \log \frac{1}{\delta_k} \tag{1.35}$$

Therefore, the set $K(\gamma)$ is non-polar if and only if $Rob(K(\gamma)) = \sum_{k=1}^{\infty} 2^{-k} \cdot \log \frac{1}{\gamma_k} = \sum_{k=1}^{\infty} 2^{-k-1} \cdot \log \frac{1}{\delta_k} < \infty$.

1.6 The Equivalent Conditions and Negation of Conditions for Extension Property

We will characterize EP of $K(\gamma)$ in terms of the values $B_k = 2^{-k-1} \cdot \log \frac{1}{\delta_k}$. Thus, $Rob(K(\gamma)) = \sum_{k=1}^{\infty} B_k$. The main condition for EP is (compare with (3) in [12]):

$$(Y) \quad \frac{B_{n+s}}{\sum_{k=s}^{n+s} B_k} \to 0 \quad \text{as} \quad n \to \infty \quad \text{uniformly with respect to} \quad s. \tag{1.36}$$

We see that this condition allows polar sets.

Example 1. Let $\gamma_1 = \exp(-4B)$ and $\gamma_k = \exp(-2^k B)$ for $k \geq 2$, where $B \geq \frac{1}{4}\log 32$, so (1.24) is valid. Here, $B_k = B$ for all k. Hence (1.36) is satisfied and the set $K(\gamma)$ is polar.

The condition (1.36) means that

$$\forall\varepsilon\,\exists s_0,\,\exists n_0:\ B_{s+n} < \varepsilon(B_s + \cdots + B_{s+n}) \text{ for } n \geq n_0,\ s \geq s_0. \tag{1.37}$$

Clearly, instead of $\exists s_0$ one can take above $\forall s_0$. Let us show that (1.37) is equivalent to

$$\forall\varepsilon_1\ \forall m \in \mathbb{Z}_+ \exists N:\ B_{s+n-m} + \cdots + B_{s+n} < \varepsilon_1(B_s + \cdots + B_{s+n-m-1}),\ n \geq N, s \geq 1. \tag{1.38}$$

Indeed, the value $m = 0$ in (1.38) gives (1.37) at once. For the converse, remark that in (1.38) we can take on the right side $\varepsilon_1(B_s + \cdots + B_{s+n})$, so here we consider (1.38) in this new form. Suppose (1.37) is valid, given ε_1 and m, take $\varepsilon = \varepsilon_1/(m+1)$ and the corresponding value n_0 from (1.37). Take $N = n_0 + m$. Then for $n \geq N$ and $0 \leq k \leq m$ we have $n - k \geq n_0$, so

$$B_{s+n-k} < \varepsilon(B_s + \cdots + B_{s+n-k}) < \varepsilon(B_s + \cdots + B_{s+n}). \tag{1.39}$$

Summing these inequalities, we obtain a new form of (1.38).

It follows that the negation of the main condition can be written as

$$\exists\varepsilon\ \exists m:\ \forall N\,\exists n > N:\ \sum_{s+n-m}^{s+n} B_k > \varepsilon \sum_{s}^{s+n-m-1} B_k \text{ for } s = s_j \uparrow \infty. \tag{1.40}$$

Also, (1.37) is equivalent to

$$\forall\varepsilon\ \ \exists m, n_0, s_0:\ B_{s+n} < \varepsilon(B_{s+n-m} + \cdots + B_{s+n-1}) \text{ for } n \geq n_0,\ s \geq s_0. \tag{1.41}$$

Indeed, comparison of right sides of inequalities shows that (1.41) implies (1.37). Conversely, given ε, take n_0 such that (1.37) is valid with $\varepsilon/(1+\varepsilon)$ instead of ε.

Take $m = n_0$. Then for $n \geq n_0, s \geq s_0$ we have $\tilde{s} = s+n-m \geq s_0$ and, by (1.37), $B_{s+n} = B_{\tilde{s}+m} < \frac{\varepsilon}{1+\varepsilon}(B_{\tilde{s}} + \cdots B_{\tilde{s}+m})$, which is (1.41).

We will also use a "geometric" version of (1.41) in terms of (δ_k)

$$\forall M \;\; \exists m, n_0, s_0 : \;\; \delta_{s+n-1}\, \delta^2_{s+n-2} \cdots \delta^{2^{m-1}}_{s+n-m} < \delta^M_{s+n} \;\text{ for } n \geq n_0,\; s \geq s_0. \tag{1.42}$$

Chapter 2

Three Methods of Extension

There are three main methods to construct an extension operator for compact sets if it exists. The method of extending the elements of topological basis of $\mathcal{E}(K)$ introduced by B. S. Mityagin [18]. In [20] Pawłucki and Pleśniak constructed the extension operator by using Lagrange interpolation polynomials for compact sets preserving Markov's inequality. A. Goncharov [7] presented a compact set K without Markov's property, such that the corresponding Whitney space has the extension operator.

2.1 Extending the Elements of Basis

To follow the method introduced by Mityagin, the space must have topological bases. The basis is found for the space $C^\infty[-1,1]$ in (Th.14, [18]), for the case of compact sets with nonempty interior in (Th.2.4, [30]), for Cantor sets in (Th.1, [12]) , see also [9].

Let E be a locally convex topological vector space with topology τ defined by a family of seminorms. A complete, metric, locally convex space is called a Fréchet space, and has a topology defined by a countable family of seminorms.

Definition 2.1.1. A sequence $(e_n)_{n=1}^{\infty}$ in a locally convex space E is called a Schauder basis of E, if $\forall x \in E$, there exists uniquely determined sequence of constants $(\xi_n(x))_{n=1}^{\infty}$, such that $x = \sum_{n=1}^{\infty} \xi_n(x) e_n$.

Definition 2.1.2. A Schauder basis $(e_n)_{n=1}^{\infty}$ of E is called an absolute basis, if for each continuous seminorm p on E, there is a continuous seminorm q on E and there exists $C > 0$ such that

$$\sum_{n=1}^{\infty} |\xi_n(x)| p(e_n) \leq C q(x) \quad \forall x \in E. \tag{2.1}$$

Definition 2.1.3. Let $A = (a_{ip})_{i \in I, n \in \mathbb{N}}$ (the set I is supposed to be countable, in general $I \neq \mathbb{N}$) be a matrix of real numbers such that $0 \leq a_{ip} \leq a_{i\bar{p}}$ for $p < \bar{p}$. Köthe space, defined by the matrix A, is the locally convex space $K(A)$ of all sequences $s = (s_i)$ such that

$$|s|_p := \sum_{i \in I} a_{ip} |s_i| < \infty \quad \forall p \in \mathbb{N} \tag{2.2}$$

with the topology generated by the system of seminorms $|\cdot|_p, p \in \mathbb{N}$.

By the Dynin-Mityagin theorem [17] a Fréchet space E with an absolute basis $(e_n)_{n=1}^{\infty}$ and a topology generated by increasing system of seminorms $\{\|\cdot\|_p\}_{p \in \mathbb{N}}$, is isomorphic to the Köthe space defined by the matrix $A = (a_{np})$ where $a_{np} = \|e_n\|_p$.

By Corollary (1.2.4), a compact set K has the extension property if and only if the space $\mathcal{E}(K)$ has the property (DN). Let the compact K has the extension property and $\mathcal{E}(K)$ has a topological basis $(e_n)_{n=1}^{\infty}$. Then any function in $\mathcal{E}(K)$ has representation

$$f = \sum_{n=1}^{\infty} \xi_n \cdot e_n \tag{2.3}$$

and can be extended to the function defined on the whole space by

$$W(f) = \sum_{n=1}^{\infty} \xi_n \cdot W(e_n). \tag{2.4}$$

Here $\{W(e_n)\}_{n \geq 1}$ are suitable extensions of basis elements. See Theorem 2.4 in [30] about possibility of $W(e_n)$ in the case when K has nonempty interior.

However, we do not know whether each space $\mathcal{E}(K)$ has a topological basis, even though $\mathcal{E}(K)$ is complemented in $C^\infty(I)$. This is a particular case of the Mityagin-Pełczyński problem: Suppose X is a nuclear Fréchet space with basis and E is a complemented subspace of X. Does E possess a basis? The space $X = s$ of rapidly decreasing sequences, which is isomorphic to $C^\infty(I)$, presents the most important unsolved case.

2.2 The Pawłucki - Pleśniak Extension Operator

Definition 2.2.1. A compact set K is said to have the Markov property if there exists positive constants M and m such that

$$\sup_{x\in K}|\nabla p(x)| \leq M(\deg p)^m \sup_{x\in K}|p(x)| \tag{2.5}$$

for al polynomialsl p.

Given a compact set K with $Cap(K) > 0$, Green's function of K with a pole at ∞ is the unique function $g_K(z,\infty)$ defined in the outer domain $G = \mathbb{C}_\infty - K$ with the properties:

- g_K is nonnegative and harmonic in G,
- $g_K(z,\infty) - \log|z|$ is harmonic in a neighborhood of $z = \infty$, and
- $\lim_{z\to z_0, z\in G} g_K(z,\infty) = 0$ for q.e. $z_0 \in \partial G$.

Remark 2.2.2. Green's function is also related with the equilibrium measure;

$$\lim_{z\to\infty}(g(z,\infty) - \log|z|) = \log(Cap(K)^{-1}). \tag{2.6}$$

To examine the relation between Markov's property and the Green's function, let us give the definition of the $g_K(z)$ concerning polynomials.

Lemma 2.2.3. *Green's function with a pole at ∞ for $G \subset \mathbb{C}_\infty$ and $K = \mathbb{C}_\infty - G$ is also defined as*

$$g_K(z) := \sup\left\{\frac{\log|(p(z))|}{\deg p} : p \in \mathcal{P}, \quad |p|_K \leq 1\right\} \tag{2.7}$$

where $\mathcal{P}$ denotes the set of all polynomials.

Proof. From the Bernstein theorem

$$g_K(z) \geq \sup\left\{\frac{\log|(p(z))|}{\deg p} : p \in \mathcal{P}, \quad |p|_K \leq 1\right\}. \tag{2.8}$$

Let us choose for all n a monic polynomial $p_n(z)$ of degree n such that $K \subset \{z \in \mathbb{C} : |p_n(z)| \leq 1\}$. Green's function for the set K is $g_n(z) = \log|p(z)|/n$. By a suitable choice of the polynomials p_n, the intersection of corresponding level domains give the set K (see e.g. Proposition 9.8 in [4]). $\square$

Definition 2.2.4. Green's function of the set $\mathbb{C}_\infty - K$ is defined to be Hölder continuous if there exists $C, \nu > 0$ satisfying

$$g_K(z) \leq C(dist(z, K))^\nu, \quad \forall z \in \mathbb{C}. \tag{2.9}$$

It is easy to prove (HCP) of Green's function implies Markov property of the compact set K by using Cauchy's integral formula. In some cases the Markov property is equivalent to the Hölder continuity of the Green's function. Totik [28] showed that this is true for Cantor-type sets. For any compact set, the problem remains open.

Definition 2.2.5. Let $U \subset \mathbb{R}^d$. U is uniformly polynomially cuspidal (UPC) if there exists $M, m \in \mathbb{R}^+$ and $n \in \mathbb{N}$ such that for each point $x \in \bar{U}$, there is a polynomial map $h_x : \mathbb{R} \to \mathbb{R}^d$ of degree at most n satisfying;

- $h_x((0,1]) \subset U$ and $h_x(0) = x$,
- $dist(h_x(t), \mathbb{R}^d - U) \geq Mt^m, \quad \forall x \in U$ and $t \in (0,1]$.

Let $U \subset \mathbb{R}^d$ be a uniformly polynomially cuspidal compact set. Siciak's extremal function of U, introduced in [25] by

$$\Phi_U(x) = \sup_{k\geq 1}\{\sup\{|p(x)|^{1/k} : p \in \mathcal{P}_k, \|p\|_U \leq 1\}\}, \tag{2.10}$$

for $x \in \mathbb{C}^d$, has Hölder continuity property (HCP)

$$\Phi_U(x) \leq 1 + C_1\delta^{\mu} \quad if \quad dist(x, U) \leq \delta \leq 1, \tag{2.11}$$

with some positive constants C_1 and μ independent of δ. Consequently, Siciak's extremal function is the generalized Green's function and has Hölder continuity property. Thus, (UPC) compact sets have Markov property as well.

Now equip $C^\infty(U)$ with another topology. Following Zerner [32], we set $d_{-1}(f) := \|f\|_U$, $d_0(f) := dist_U(f, \mathcal{P}_0)$ and for $k = 1, 2, \ldots,$

$$d_k(f) := \sup_{n\geq 1} n^k dist_U(f, \mathcal{P}_n). \tag{2.12}$$

The notation $dist_U(f, \mathcal{P}_n)$ stands for best approximation to f on U by polynomials of degree at most n. Then by Jackson's theorem, each d_k is a seminorm on $C^\infty(U)$. Denote by τ_J the topology in $C^\infty(U)$ defined by the system of seminorms $d_k (k = -1, 0, \ldots)$.

W. Pawłucki -W. Pleśniak extension operator is constructed for the family of sets with polynomial cusps [20]. Authors used telescoping series that contain Lagrange interpolation polynomials with Fekete nodes. Pleśniak generalized the result to any Markov set in [21]. Let us explain the extension operator.

Let K be a compact subset of $\mathbb{R}^d$ and $(e_i)_{i=1}^{m_k}$ be a basis for the space of polynomials of order k where $m_k = comb(n + k, k)$. Let $\{t^k\} = \{t_1, \ldots, t_k\}$ be a system of k points of $\mathbb{R}^d$. Define the Vandermonde determinant for $i, j \in \{1, \ldots, k\}$ as

$$V(\{t^k\}) = \det[e_j(t_i)]. \tag{2.13}$$

If the determinant is nonzero, then we have Lagrange fundamental polynomials defined as

$$l_j(x, \{t^k\}) = \frac{V(t_1, \ldots, t_{j-1}, x, t_{j+1}, \ldots, t_k)}{V(\{t^k\})}. \tag{2.14}$$

Following [25], if p is a polynomial with degree $\leq k$ and $\{t^{m_k}\}$ is a system of m_k points of $\mathbb{R}^d$ such that the Lagrange fundamental polynomials defined, then for $x \in \mathbb{R}^d$

$$p(x) = \sum_{j=1}^{m_k} p(t_j) l_j(x, \{t^{m_k}\}). \tag{2.15}$$

Definition 2.2.6. A system $\{t^k\}$ of k points of K is called $Fekete-Leja$ system of extremal points of K with order k if $V(\{t^k\}) \geq V(\{s^k\})$ for all systems $\{s^k\} \subset K$.

Let $K \subset \mathbb{R}^d$ be a (UPC) compact set. For $k = 1, 2, \ldots,$ define u_k on $\mathbb{R}^d$ to be smooth functions (i.e. C^∞) such that $u_k(x) = 1$ for $x \in K$, and $u_k(x) = 0$ for $dist(x, K) \geq \epsilon_k > 0$. ϵ_k are chosen so that the Siciak extremal function has Hölder continuity. Also, let the derivatives of the smoothing function u_k satisfy

$$|u_k^{(\alpha)}(x)| \leq \frac{C_\alpha}{\epsilon^{|\alpha|}}, \quad \forall x \in \mathbb{R}^d \;\; and \;\; \alpha \in \mathbb{Z}_+^d, \tag{2.16}$$

where the constants C_α depends only on the order of derivation α. Let $f \in C^\infty(K)$, the Pawłucki - Pleśniak extension operator is defined as:

$$Lf = u_1 L_1 f + \sum_{k=1}^{\infty} u_k (L_{k+1} f - L_k f) \tag{2.17}$$

where

$$L_k f(x) = \sum_{j=1}^{m_k} f(t_j) l_j(x, \{t^{m_k}\}). \tag{2.18}$$

Lf is a smooth function on $\mathbb{R}^d$ and $Lf(x) = f(x)$ for all $x \in K$. Linearity of the operator is direct from the definition. Using Hölder continuity and Markov propery of the set we get the upper bound for the derivatives. Continuity of the operator follows since the Jackson topology and the quotient topology are equivalent for Markov sets.

For compact sets with EP without the Markov property ([7], [3]), the Pawłucki-Pleśniak extension operator is not continuous in τ_J, but it may be bounded in τ. M. Altun and A. Goncharov showed in [3] that the local version of this operator is bounded in τ.

2.3 Whitney Extension Operator and Other Methods

Let X be a closed subset of $\mathbb{R}^d$. The classical Whitney extension operator for the space $\mathcal{E}^p(X)$ jets of finite order, is constructed by using the Whitney covering and partition of unity [31]. The Whitney covering of the set is briefly finding disjoint cubes Q_i whose union contains the whole set $\mathbb{R}^d - X$ and $diam(Q_i), \forall i \in \mathbb{N}$ are comparable with the distance of the cubes Q_i to the set X.

The partition of unity ϕ_i is defined on the Whitney covering Q_i having $\phi_i|_{Q_i} = 1$ and $supp(\phi_i) \subset Q_i(x_i, r_i + \epsilon)$ (where $Q(x, r)$ means the ball centered at x with radius r). Then define the functions $\Phi_i : \mathbb{R}^d \to [0, 1]$,

$$\Phi_i(x) = \frac{\phi_i(x)}{\sum_i \phi_i}, \tag{2.19}$$

for which $\sum_i \Phi_i(x) = 1$ for all $x \in \mathbb{R}^d - X$. For each Q_i in Whitney covering, let $y_i \in K$ be the point at the shortest distance to X, i.e., $d(Q_i, y_i) = d(Q_i, X)$. Since X is closed such points exist for all i. Define the function $Wf : \mathbb{R}^d \to \mathbb{R}$ with $Wf(x) = f(x), \forall x \in X$ and if $x \in \mathbb{R}^d - X$

$$Wf(x) = \sum_i T^p_{y_i} f(x) \Phi_i(x). \tag{2.20}$$

This simple extension operator W, $f \to Wf$, is linear. If $f : X \to \mathbb{R}$ is a continuous function, then $Wf : \mathbb{R}^d \to \mathbb{R}$ defined above is continuous in $\mathbb{R}^d$. For a proof see [26].

In [6] L. Frerick, E. Jordá, and J. Wengenroth showed that, provided some conditions, W can be generalized to the case $\mathcal{E}(K)$, where K is a compact subset of $\mathbb{R}^d$. Instead of Taylor polynomials, the authors used a kind of interpolation by means of certain local measures. A linear tame extension operator was presented for $\mathcal{E}(K)$, provided K satisfies a local form of Markov's inequality.

There are some other methods to construct W for closed sets. Seeley's extension [24] is constructed from a half space. Stein's extension (Chapter VI.2.2 [26]) uses sets with Lipschitz boundary $Lip(\alpha, F)$, where α-Hölder continuous functions are mapped continuously to $Lip(\alpha, \mathbb{R}^d)$. However, these methods, in order to define $W(f, x)$ at some point x, essentially require existence of a line through x with a ray where f is defined, so these methods cannot be applied for compact sets.

Chapter 3

Bases In Spaces of Whitney Functions

With the concept of basis, one can provide some methods of approximation of vectors and operators in the space. Bases of a Banach space was introduced by Schauder and continued with the theory of bases in topological vector spaces by Banach. The basis problem Banach asked in his book on the theory of linear operators is whether every separable Banach space posseses a basis or not.

Let the set $K(\gamma) \subset \mathbb{R}$ be defined by means of a sequence of parameters $\gamma = (\gamma_s)_{s=0}^{\infty}$. In this chapter, we examine the basis problem for the set $K(\gamma)$. Given $x \in \mathbb{R}$, by $d_k(x, A)$ we denote distances from x to the points of A arranged, in nondecreasing order, so $d_k(x, A) = |x - a_{m_k}| \nearrow$.

3.1 Uniform Distribution of Points

Consider the sequences $\gamma = (\gamma_s)_{s=1}^{\infty}$ with $0 < \gamma_s \leq 1/32$ and $\sum_{s=1}^{\infty} \gamma_s < \infty$. Let $r_0 = 1, P_2(x) = x(x-1)$ and $r_s = \gamma_s r_{s-1}^2, P_{2^{s+1}} = P_{2^s}(P_{2^s} + r_s)$ for $s \in \mathbb{N}$. Then

$E_s := \{x \in \mathbb{R} : \ P_{2^{s+1}}(x) \leq 0\} = \cup_{j=1}^{2^s} I_{j,s}$, where the $s-$th level *basic intervals* $I_{j,s}$ are disjoint. Since $E_{s+1} \subset E_s$, we have a Cantor-type set $K(\gamma) := \cap_{s=0}^{\infty} E_s$.

For the length $\ell_{j,s}$ of the interval $I_{j,s}$, by Lemma 6 in [13], we have:

$$\delta_s < \ell_{j,s} < C_0\, \delta_s \quad \text{for} \quad 1 \leq j \leq 2^s, \tag{3.1}$$

where $\delta_0 := 1, \delta_s := \gamma_1 \gamma_2 \cdots \gamma_s$ for $s \in \mathbb{N}$ and $C_0 = \exp(16 \sum_{k=1}^{\infty} \gamma_k)$. Clearly,

$$r_k = \delta_k\, \delta_{k-1}\, \delta_{k-2}^2\, \delta_{k-3}^4 \cdots \delta_0^{2^{k-1}}. \tag{3.2}$$

Each $I_{j,s}$ contains two *adjacent* basic subintervals $I_{2j-1,s+1}$ and $I_{2j,s+1}$. Let $h_{j,s} = \ell_{j,s} - \ell_{2j-1,s+1} - \ell_{2j,s+1}$ be the distance between them. As in (Lemma 4, [13]),

$$h_{j,s} \geq \frac{7}{8} \cdot \ell_{j,s} > \frac{7}{8} \cdot \delta_s \tag{3.3}$$

and

$$\ell_{2j-1,s+1} + \ell_{2j,s+1} < 4\, \ell_{j,s} \tag{3.4}$$

for all s and $1 \leq j \leq 2^s$.

We decompose zeros of P_{2^s} into s groups: $X_0 = \{x_1, x_2\} = \{0, 1\}, X_1 = \{x_3, x_4\} = \{\ell_{1,1}, 1 - \ell_{2,1}\}, \cdots, X_k = \{\ell_{1,k}, \ell_{1,k-1} - \ell_{2,k}, \cdots, 1 - \ell_{2^k,k}\}$ for $k \leq s-1$, so X_k contains all zeros of $P_{2^{k+1}}$ that are not zeros of P_{2^k}. If $Y_s = \cup_{k=0}^{s} X_k$, then $P_{2^s}(x) = \prod_{x_k \in Y_{s-1}} (x - x_k)$. Clearly, $\#(X_s) = 2^s$ for $s \in \mathbb{N}$ and $\#(Y_s) = 2^{s+1}$ for $s \in \mathbb{Z}_+$. We refer *$s-$th type* points to the elements of X_s.

We put all points $(x_k)_{k=1}^{\infty}$ in $\cup_{k=0}^{\infty} X_k$ in order by means of *the rule of increase of type*. The order of $(x_k)_{k=1}^{4}$ is given above. To put the points from X_2 in order we increasingly arrange the points from Y_1, so $Y_1 = \{x_1, x_3, x_4, x_2\}$. After this we increase the index of each point by 4. This gives the ordering $X_2 = \{x_5, x_7, x_8, x_6\}$. Similarly, indices of increasingly arranged points from $Y_{k-1} = \{x_{i_1}, x_{i_2}, \cdots, x_{i_{2^k}}\}$ define the ordering $X_k = \{x_{i_1+2^k}, x_{i_2+2^k}, \cdots, x_{i_{2^k}+2^k}\}$. We see that $x_{j+2^k} = x_j \pm \ell_{m,k}$, where the sign and m are uniquely defined by j.

A useful feature of this order is that for each N, the points $Z := (x_k)_{k=1}^{N}$ are distributed uniformly on $K(\gamma)$ in the following sense.

Suppose $2^n \leq N < 2^{n+1}$. Then the binary representation

$$N = 2^n + 2^m + \cdots + 2^i + \cdots + 2^w \text{ with } 0 \leq w < \cdots < i < \cdots < m < n \quad (3.5)$$

generates the decomposition $Z = Z_n \cup Z_m \cup \cdots \cup Z_w$ with $Z_n = Y_{n-1}$ and $Z_m \cup \cdots \cup Z_w \subset X_n$. Here, for fixed $i \in \mathcal{N} := \{w, \cdots, m, n\}$, each basic interval of $i-$th level contains exactly one point of Z_i, so $\#(Z_i) = 2^i$ and the points of Z_i are uniformly distributed on $K(\gamma)$. Also, for each N, s and for each $i, j \leq 2^s$

$$|\#(Z \cap I_{i,s}) - \#(Z \cap I_{j,s})| \leq 1, \quad (3.6)$$

so any two intervals of the same level contain the same number of points of Z or, perhaps, one of the intervals contains one extra point x_k, compared to another interval.

Remark 3.1.1. If we choose, by this rule, the set Z on any $I_{j,s}$, then the points pf $Z \cap I_{2j-1,s+1}$ are also uniformly distributed, whereas the distribution of points of $Z \cap I_{2j,s+1}$ on the right subinterval is either uniform or bilaterally symmetric.

In what follows we will associate with a number N not only the sets Z and $\mathcal{N}$, but also the product $\prod_{i \in \mathcal{N}} r_i$, where r_i is defined in (3.2). We combine together all δ_k that constitute this product and arrange them in nondecreasing order:

$$\prod_{i \in \mathcal{N}} r_i = \prod_{m=1}^{N} \rho_m = \prod_{k=0}^{n} \delta_k^{s_k(N)} \quad \textit{with} \quad \sum_{k=0}^{n} s_k(N) = N. \quad (3.7)$$

For example, $N = 2^n$ gives $\prod_{m=1}^{N} \rho_m = r_n$, whereas $N = 21$ generates $\mathcal{N} = \{0, 2, 4\}$ and $\prod_{m=1}^{21} \rho_m = \delta_4\, \delta_3\, \delta_2^3\, \delta_1^5\, \delta_0^{11}$.

For each $N \geq 1$ and $k \geq 0$, the corresponding degrees are given by the formula

$$s_k(N) = [2^{-k-1}(N + 2^k)].$$

From here it follows that, for $N+1 = 2^m(2p+1)$, the values $s_k(N)$ and $s_k(N+1)$ coincide for all k except for $k = m$, where $s_m(N+1) = s_m(N) + 1$.

In the same way, we can choose the set $Z = (x_{k,j,s})_{k=1}^N$ on any $I_{j,s}$. If $2^n \le N < 2^{n+1}$, then Z includes all 2^n zeros of $P_{2^{s+n}}$ on $I_{j,s}$ and some $N - 2^n$ points of the type $s+n$. Here, $r_{i,s} = \delta_{s+i}\,\delta_{s+i-1}\,\delta_{s+i-2}^2 \cdots \delta_s^{2^{i-1}}$ and $\prod_{i\in\mathcal{N}} r_{i,s} = \prod_{m=1}^N \rho_{m,s}$. In terms of the last product we can estimate the sup-norm of the function $f_N(x) = \prod_{k=1}^N (x - x_{k,j,s})$ on the set $K(\gamma) \cap I_{j,s}$.

Lemma 3.1.2. *With the above notations*

$$(7/8)^N \prod_{k=1}^N \rho_{k,s} \le |f_N|_{0,K(\gamma)\cap I_{j,s}} \le C_0^N \prod_{k=1}^N \rho_{k,s}.$$

Proof. For brevity, let us consider the interval $I_{j,s} = [0,1]$, since the proof for the general case is the same. Thus we drop the subscripts j and s. For a fixed $x \in K(\gamma)$ we have

$$|f_N(x)| = \prod_{k=1}^N |x - x_k| = \prod_{i\in\mathcal{N}} \prod_{x_k\in Z_i} |x - x_k|. \tag{3.8}$$

For each $i \in \mathcal{N}$ we consider the chain of basic intervals containing x: $x \in I_{j_0,i} \subset I_{j_1,i-1} \subset \cdots \subset I_{j_i,0} = [0,1]$. Since the points from Z_i are uniformly distributed, the interval $I_{j_0,i}$ contains just one point from Z_i, as well as $I_{j_1,i-1} \setminus I_{j_0,i}$. Also, $\#(Z \cap (I_{j_k,i-k} \setminus I_{j_{k-1},i-k+1})) = 2^{k-1}$ for $1 \le k \le i$. Therefore,

$$\prod_{x_k\in Z_i} |x - x_k| \le \ell_{j_0,i}\,\ell_{j_1,i-1}\,\ell_{j_2,i-2}^2 \cdots \ell_{j_i,0}^{2^{i-1}} < C_0^{2^i} r_i, \tag{3.9}$$

by (3.1) and (3.2). From this, $|f_N(x)| \le C_0^N \prod_{i\in\mathcal{N}} r_i$ and

$$|f_N|_{0,K(\gamma)} \le C_0^N \prod_{k=1}^N \rho_k. \tag{3.10}$$

The bound (3.10) is sharp with respect to the product $\prod_{k=1}^N \rho_k$. Indeed, let us consider $|f_N(x_{N+1})|$. As above, $x_{N+1} \in I_{j_0,n} \subset I_{j_1,n-1} \subset \cdots \subset I_{j_i,0}$. Hence by (3.3),

$$\prod_{x_k\in Z_n} |x_{N+1} - x_k| \ge \ell_{j_0,n}\,h_{j_1,n-1}\,h_{j_2,n-2}^2 \cdots h_{j_i,0}^{2^{n-1}} > (7/8)^{2^n}\, r_n. \tag{3.11}$$

As for $\prod_{x_k\in Z_m}|x_{N+1}-x_k|$, we observe that 2^{m+1} basic intervals of the level $m+1$ contain $2^m+\cdots+2^w$ points from $A := Z_m\cup\cdots\cup Z_w$. Since the number of points is less than 2^{m+1}, the point x_{N+1} must be on some $I_{j,m+1}$ which is free of points from A. Otherwise, $x_{N+1}\in I_{i,m+1}$ with $\#(\tilde Z\cap I_{i,m+1})\ge 2^{n-m-1}+2$, where $\tilde Z=(x_k)_{k=1}^{N+1}$. Recall that each interval of the level $m+1$ contains 2^{n-m-1} points from Z_n. In particular, $\#(\tilde Z\cap I_{j,m+1})=2^{n-m-1}$, contrary to (3.6). It follows that x_{N+1} and the nearest point to it from Z_m are located on different intervals of the $m+1-$st level. Arguing as above, we see that

$$\prod_{x_k\in Z_m}|x_{N+1}-x_k|\ge h_{j_0,m}\,h_{j_1,m-1}\,h_{j_2,m-2}^2\cdots h_{j_i,0}^{2^{m-1}}>(7/8)^{2^m}\,r_m. \tag{3.12}$$

In a similar fashion, $\prod_{x_k\in Z_i}|x_{N+1}-x_k|>(7/8)^{2^i}\,r_i$ for each $i\in\mathcal N$. Therefore,

$$|f_N|_{0,K(\gamma)}\ge|f_N(x_{N+1})|\ge(7/8)^N\prod_{k=1}^N\rho_k. \tag{3.13}$$

□

Remark 3.1.3. Given $x\in K(\gamma)$, the value $|f_N(x)|$ has also the representation in terms of distances, $|f_N(x)|=\prod_{k=1}^N d_k(x,Z)$. The lengths of basic intervals of the same level may be rather different (we can say only that $\ell_{j,s}<C_0\,\ell_{i,s}$, by (3.1)). For this reason, as k increases and x,y belong to different parts of the set, the values $d_k(x,Z)$ and $d_k(y,Z)$ may increase in quite different fashions. Nevertheless, the product $\prod_{k=1}^N\rho_k$ is defined by N only, so it does not depend on the choice of x. The procedure above indicates that, for each x, there is a correspondence between $(\rho_k)_{k=1}^N$ and $(d_k(x,Z))_{k=1}^N$.

Let $2^n\le N<2^{n+1}$ and a basic interval $I=I_{j,s}$ be given. Suppose $Z=(x_k)_{k=1}^N$ and $\tilde Z=(x_k)_{k=1}^{N+1}$ are chosen on I by the rule of increase of type. Let $C_1=(8/7)\cdot(C_0+1)$.

Lemma 3.1.4. *For each $x\in\mathbb R$ with* $\operatorname{dist}(x,K(\gamma)\cap I_{j,s}))\le\delta_{s+n}$ *and $z\in\tilde Z$ we have*

$$\delta_{s+n}\prod_{k=2}^N d_k(x,Z)\le C_1^N\prod_{k=2}^{N+1}d_k(z,\tilde Z). \tag{3.14}$$

Proof. As above, we take $s = 0, j = 1$. Let $\tilde{x} \in K(\gamma)$ realize the distance above with $\tilde{x} \in I_{j_0,n} \subset I_{j_1,n-1} \subset \cdots \subset I_{j_n,0} = I$. Also, let $x_p \in Z_p \subset Z$ be such that $d_1(x, Z) = |x - x_p|$. Of course, x_p may coincide with $\tilde{x}$. We observe that x_p also belongs to $I_{j_0,n}$, since otherwise the point from $Z \cap I_{j_0,n}$ will realize $d_1(x, Z)$. Therefore,

$$d_1(x, Z) = d_1(x, Z_p) = |x - x_p| \le |x - \tilde{x}| + |\tilde{x} - x_p| \le \delta_n + \ell_{j_0,n} \le (C_0 + 1)\delta_n. \quad (3.15)$$

For each $i \in \mathcal{N}$ with $i \neq p$ we take q with $n - i \le q \le n$. Then the interval $I_{j_q,n-q}$ contains 2^{q+i-n} points from Z_i. If $x_k \in Z_i \cap I_{j_q,n-q}$ then

$$|x - x_k| \le |x - \tilde{x}| + |\tilde{x} - x_k| \le \delta_{s+n} + \ell_{j_q,n-q} < (C_0 + 1)\delta_{n-q}. \quad (3.16)$$

We combine these inequalities for all admissible q. This gives $\prod_{x_k \in Z_i} |x - x_k| \le (C_0 + 1)^{2^i} r_i$.

The terms $d_k(x, Z_p)$ for $2 \le k \le 2^p$ can be handled in much the same way. On combining all these, we get the bound

$$\delta_{s+n} \prod_{k=2}^{N} d_k(x, Z) \le (C_0 + 1)^N \prod_{i \in \mathcal{N}} r_i. \quad (3.17)$$

We proceed to show that

$$\prod_{k=2}^{N+1} d_k(z, \tilde{Z}) \ge (7/8)^N \prod_{i \in \mathcal{N}} r_i. \quad (3.18)$$

Lemma 3.1.2 gives this inequality for $N + 1 = 2^{n+1}$ or $z = z_{N+1}$ with any N, so let $N + 1 < 2^{n+1}$.

First consider the case $w = 0$ in (3.5). Then $N = 2^n + 2^m + \cdots + 2^u + 2^{v-1} + 2^{v-2} + \cdots + 2 + 1$ with some $1 \le v < u$ and, correspondingly, $N + 1 = 2^n + 2^m + \cdots + 2^u + 2^v$. Fix $z \in \tilde{Z}$ and the chain $z \in I_{j_0,n} \subset I_{j_1,n-1} \subset \cdots \subset I_{j_n,0}$. Here, z is an endpoint of $I_{j_0,n}$.

Suppose that $z \in Z_n$ and another endpoint of $I_{j_0,n}$ is $z_p \in Z_p \subset \tilde{Z}$. We will estimate separately $\prod_{x_k \in Z_q} |z - x_k|$ for q that participate in the binary decomposition of $N + 1$.

For $q = n$ we have, as in Lemma 3.1.2, $d_1(z, Z_n) = \ell_{j_1,n-1}, d_2(z, Z_n) \geq h_{j_2,n-2}, \cdots$ and $\prod_{x_k \in Z_n} |z - x_k| > (7/8)^{2^n-2}\, r_n/\delta_n$. For each q with $p < q \leq m$ we have the bound $\prod_{x_k \in Z_q} |z - x_k| \geq (7/8)^{2^q}\, r_q$. Indeed, z_p belongs to $I_{j_{n-q-1},q+1}$ and each interval of the $q+1-$st level may contain at most one point from the set $Z_q \cup \cdots \cup Z_p \cup \cdots Z_v$. Hence, the nearest to z point x_k from Z_q is on the adjacent interval of the $q+1-$st level and $d_1(z, Z_q) \geq h_{j_{n-q},q}$. Continuing in this fashion, by (3.3), we get the bound for given q.

We now handle the case $q = p$. Here, $d_1(z, Z_p) = |z - z_p| = \ell_{j_0,n} > \delta_n$. Since $z_p \in I_{j_{n-p},p}$, this interval cannot contain another point from Z_p. Therefore, $d_2(z, Z_p) \geq h_{j_{n-p+1},p-1} \geq 7/8\, \delta_{p-1}$ and $\prod_{x_k \in Z_p} |z - x_k| \geq (7/8)^{2^p}\, r_p\, \delta_n/\delta_p$.

To deal with indices $q < p$, let us take the nearest to z point z_{p_1} from the set $\tilde{Z} \setminus (Z_n \cup Z_m \cup \cdots \cup Z_p) = Z_t \cup \cdots Z_{p_1} \cdots \cup Z_v$. For each q with $p_1 < q \leq t$ we have, as above, $\prod_{x_k \in Z_q} |z - x_k| \geq (7/8)^{2^q}\, r_q$. If $q = p_1$ then $\prod_{x_k \in Z_{p_1}} |z - x_k| \geq (7/8)^{2^{p_1}}\, r_{p_1}\, \delta_p/\delta_{p_1}$. Indeed, the interval $I_{j_{n-p-1},p+1}$ contains z_p, so it cannot contain another point from $Z_p \cup \cdots Z_{p_1} \cdots \cup Z_v$. Therefore, z and z_{p_1} must be on different intervals of the $p+1-$st level. Then $d_1(z, Z_{p_1}) \geq h_{j_{n-p},p}$. Now $I_{j_{n-p_1},p_1}$ contains z_{p_1}, so $d_2(z, Z_{p_1}) \geq h_{j_{n-p_1+1},p_1-1}$ and the rest of the proof for this case runs as before.

Continuing this line of reasoning and combining all bounds together, we see that

$$(8/7)^N \cdot \prod_{k=2}^{N+1} d_k(z, \tilde{Z}) \geq \frac{r_n}{\delta_n}\, r_m \cdots r_p \frac{\delta_n}{\delta_p} r_t \cdots\, r_{p_1} \frac{\delta_p}{\delta_{p_1}} \cdots\, r_{p_2} \frac{\delta_{p_1}}{\delta_{p_2}} \cdots\, r_{p_k} \frac{\delta_{p_{k-1}}}{\delta_{p_k}}$$

with $p_k = v$. The right hand side here is $r_n \cdots r_u\, r_v/\delta_v$, which exceeds $\prod_{i \in \mathcal{N}} r_i = r_n \cdots r_u\, r_{v-1} \cdots r_1\, r_0$, since $r_{v-1}\, r_{v-2} \cdots r_1\, r_0 = r_v^2/\delta_v$, as is easy to check. This yields (3.18).

The cases $z \in Z_p \subset \tilde{Z}$ and $w > 0$ are very similar. Now (3.17) and (3.18) together give the result. $\square$

Now we consider the same N and $\tilde{Z}$, as above, but arrange their points in increasing order. Thus, $\tilde{Z} = (z_k)_{k=1}^{N+1} \subset I$ with increasing z_k. For $q = 2^m - 1$ with

$m < n$ and $1 \le j \le N+1-q$, let $J = \{z_j, \cdots, z_{j+q}\}$ be 2^m consecutive points from $\tilde{Z}$. Given j, we consider all possible chains of strict inclusions of segments of natural numbers:

$$[j, j+q] = [a_0, b_0] \subset [a_1, b_1] \subset \cdots \subset [a_{N-q}, b_{N-q}] = [1, N+1], \tag{3.19}$$

where $a_k = a_{k-1}$, $b_k = b_{k-1}+1$ or $a_k = a_{k-1}-1$, $b_k = b_{k-1}$ for $1 \le k \le N-q$.

Definition 3.1.5. For fixed J, let $\Pi(J)$ denote the minimum of the products $\prod_{k=1}^{N-q}(z_{b_k} - z_{a_k})$ for all possible chains.

Lemma 3.1.6. *For each $J \subset \tilde{Z}$ there exists $\tilde{z} \in J$ such that $\prod_{k=q+2}^{N+1} d_k(\tilde{z}, \tilde{Z}) \le \Pi(J)$.*

Proof. We take again $I = [0,1]$, as the same proof with the corresponding change of indices is valid in general case. Fix $J \subset \tilde{Z}$. Let $J_k = [z_{a_k}, z_{b_k}]$. Then (3.19), which defines $\Pi(J)$, generates strict inclusions $J \subset J_1 \subset \cdots \subset J_{N-q} = I$ with $\Pi(J) = \prod_{k=1}^{N-q} |J_k|$.

Clearly, $\#(\tilde{Z} \setminus J) = N - q$ and each $z \in \tilde{Z} \setminus J$ appears as an endpoint of some J_k. Let w_k be the endpoint of J_k in its first appearance, so w_k is not an endpoint of $J_1, \cdots, J_{k-1}$. This gives an enumeration of $\tilde{Z} \setminus J$. Our aim is to find $\tilde{z} \in J$ and a permutation $(w_{i_k})_{k=1}^{N-q}$ such that for $1 \le k \le N-q$ we have $d_{q+1+k}(\tilde{z}, \tilde{Z}) \le |J_{i_k}|$. Multiplying these inequalities immediately yields the result. Given $\tilde{z}$, let d_k be shorthand for $d_k(\tilde{z}, \tilde{Z})$.

Recall that $2^n + 1 \le \#(\tilde{Z}) \le 2^{n+1}$, so $Y_{n-1} \subset \tilde{Z} \subset Y_n$, and $\#(J) = 2^m$. The points from $\tilde{Z}$ are distributed uniformly on I. Hence, for each basic interval of $n-m+1-$st level we have

$$2^{m-1} \le \#(\tilde{Z} \cap I_{j,n-m+1}) \le 2^m. \tag{3.20}$$

We observe that J may be located on v consecutive intervals of this level with $1 \le v \le 3$. Indeed, if $J \subset I_1 \cup I_2 \cup I_3 \cup I_4$ with $J \cap I_k \ne \emptyset$, then all points from $\tilde{Z}$ on $I_2 \cup I_3$ are included in J and, by (3.20), $\#(J) \ge 2^m + 2$, a contradiction. Let us consider all possible values of v.

1) $J \subset I_1 := I_{j,n-m+1}$ for some j. Here, $J = \tilde{Z} \cap I_1$ and any point $z \in J$ may serve as $\tilde{z}$, since distances d_k for $1 \leq k \leq 2^m$ are realized on points from I_1. If $q+2 \leq k \leq N+1$ then d_k is $|\tilde{z} - w_{i_k}|$ for some $w_{i_k} \in \tilde{Z} \setminus J$ and $d_k < |J_{i_k}|$, since $\tilde{z} \in J_{i_k}$ and w_{i_k} is an endpoint of this interval.

2) $J \subset I_1 \cup I_2$. Suppose first that these intervals are adjacent, that is $I_1 = I_{2j-1,n-m+1}$ and $I_2 = I_{2j,n-m+1}$. Let $p := \#(J \cap I_1)$, that is $z_j, \cdots, z_{j+p-1}$ belong to I_1, whereas $z_{j+p}, \cdots, z_{j+q} \in I_2$. Suppose, for definiteness, that $p \leq 2^{m-1}$, that is at least a half of J is on the right interval. The right endpoint of I_2 belongs to Y_{n-1}, so it is z_{j+r} for some r with $r \geq q$. Thus, $\#(\tilde{Z} \cap I_2) = r - p + 1$, where $r - q$ of these points are from $\tilde{Z} \setminus J$. We take $\tilde{z} = z_{j+p}$, the left endpoint of I_2. Then $d_k = z_{j+p+k-1} - \tilde{z}$ for $1 \leq k \leq r-p+1$, since lengths of basic intervals are smaller than gaps between them. In particular, $d_{r-p+1} = z_{j+r} - \tilde{z} = |I_2|$. The next distances will be realized on points from $I_1: \, d_{r-p+2} = \tilde{z} - z_{j+p-1}, \cdots, d_{r+1} = \tilde{z} - z_j$. Since $\#(\tilde{Z} \cap I_2) \leq 2^m \leq r+1$, the value d_{2^m} is $\tilde{z} - z_{j+i}$ for some i with $j \leq i \leq j+p-1$.

Now, for d_k with $q+1 \leq k \leq r+1$ we take $w_{i_k} = z_{j+k-1}$ on I_2. Then $|J_{i_k}| > z_{j+k-1} - z_j > d_k = \tilde{z} - z_{j+r-k+1}$, as $z_{j+k-1} > \tilde{z}$ and $z_{j+r-k+1} \geq z_j$. Notice that the next values of d_k (for $r+2 \leq k \leq N+1$) will be realized on some points $w_{i_k} \in \tilde{Z} \setminus J$. As in the first case, $d_k < |J_{i_k}|$.

The same reasoning applies to the case of nonadjacent intervals. Let $I_1 = I_{2j-2,n-m+1}$ and $I_2 = I_{2j-1,n-m+1} \subset I_{j,n-m}$. Then we take $\tilde{z}$ as the endpoint of interval containing at least a half of J, let be again I_2. Here, p points from $I_1 \cap J$ realize some $d_{M+1}, \cdots, d_{M+p}$ and we put into correspondence to them the first p points from $I_{j,n-m} \setminus J$. All other d_k are realized on some w_{i_k} with $d_k < |J_{i_k}|$.

3) Let $J \subset I_1 \cup I_2 \cup I_3$. Here, I_2 is completely filled with points of J. One of endpoints of I_2 belongs to Y_{n-m-1}. We take this point as $\tilde{z}$. The rest of the proof runs as before. $\square$

Lemma 3.1.7. *Let $2^n \leq N < 2^{n+1}$ and $Z = (x_{k,j,s})_{k=1}^N$ be chosen on $I_{j,s}$ by the rule of increase of type. Suppose $x, y \in I_{i,s+n-q} \subset I_{j,s}$ for some $q < n$ (in general,*

$x, y \notin K(\gamma)$). If $m = \#(I_{i,s+n-q} \cap Z)$, then

$$\prod_{k=m+1}^{N} \frac{d_k(y,Z)}{d_k(x,Z)} \le \exp(2^q). \tag{3.21}$$

Proof. For brevity, let $I_{j,s} = [0,1], Z = (x_k)_{k=1}^N$. Fix $q < n$, any $I_{i,n-q}$ and x, y on this interval. Recall that $I_{i,n-q}$ may contain from 2^q to 2^{q+1} points of Z. Clearly, the distances $d_k(y,Z)$ and $d_k(x,Z)$ for $1 \le k \le m$ are realized on some points from $Z \cap I_{i,n-q}$, so we can consider the ratios $|y - x_p|/|x - x_p|$ for $x_p \in Z \setminus I_{i,n-q}$ only. Let $I_{i,n-q} \subset I_{i_1,n-q-1} \subset \cdots \subset I_{i_{n-q},0}$. If $x_p \in I_{i_1,n-q-1} \setminus I_{i,n-q}$ then

$$|y - x_p| \le \ell_{i_1,n-q-1}, \ |x - x_p| \ge h_{i_1,n-q-1} \tag{3.22}$$

and $|y - x_p|/|x - x_p| \le 8/7$, by (3.3). There are at most 2^{q+1} such points x_p. They contribute $(8/7)^{2^{q+1}}$ into the common product. For the next step, when $x_p \in I_{i_2,n-q-2} \setminus I_{i+1,n-q-1}$, we have $|y - x_p| \le |x - x_p| + \ell_{i,n-q}$ with $|x - x_p| \ge h_{i_2,n-q-2} \ge 7/8\, l_{i_2,n-q-2}$. Here, $|y - x_p|/|x - x_p| \le 1 + 8/7 \cdot 4^{-2}$, by (3.4). Continuing in this way , we get at most 2^{q+k} terms in the general product, which of the terms is bounded above by $1 + 8/7 \cdot 4^{-k}$. Here, $2 \le k \le n - q$. Therefore,

$$\prod_{k=m+1}^{N} \frac{d_k(y,Z)}{d_k(x,Z)} \le (8/7)^{2^{q+1}} \prod_{k=2}^{n-q} (1 + 8/7 \cdot 4^{-k})^{2^{q+k}}, \tag{3.23}$$

which does not exceed $\exp(2^q)$, that is easy to check. $\square$

Recall the condition (Y) (1.36) for EP which is given in terms of $B_k = 2^{-k-1} \cdot \log \frac{1}{\delta_k}$.

$$\frac{B_{n+s}}{\sum_{k=s}^{n+s} B_k} \to 0 \ \text{ as } \ n \to \infty \tag{3.24}$$

uniformly with respect to s.

This condition can be written as

$$\forall M \ \exists m, k_0 : \quad M \cdot B_k \le B_{k-m} + \cdots + B_k \ \text{ for } \ k \ge k_0. \tag{3.25}$$

Here we will use a stronger one

$$\forall M, Q\ \exists m, k_0 : \quad Q + M \cdot B_k \le B_{k-m} + \cdots + B_k \ \text{ for } \ k \ge k_0, \tag{3.26}$$

which will provide continuity of the extension operator constructed by interpolation of functions on the whole set.

In constructing of a Faber basis in the space $\mathcal{E}(K(\gamma))$ we also use the clause

$$\forall Q\ \exists m, k_0 : \quad Q \le B_{k-m} + \cdots + B_k \ \text{ for } \ k \ge k_0. \tag{3.27}$$

All these conditions have "geometric" forms in terms of (δ_k). For example (3.26) is

$$\forall M, Q\ \exists m, k_0 : \quad Q^{2^k} \cdot \delta_{k-m}^{2^m} \cdots \delta_{k-1}^2 \cdot \delta_k \le \delta_k^M \ \text{ for } \ k \ge k_0.$$

Let us illustrate the difference between conditions (3.25)-(3.27) for the case of monotone sequence $(B_k)_{k=1}^{\infty}$. Since $\gamma_s \le 1/32$ are only allowed here, we have $B_k \ge (\log 32) \cdot k\, 2^{-k-1}$. On the other hand, the values of B_k may be as large as we wish for small sets $K(\gamma)$. The condition (3.25) is valid if $B_k \searrow, B_k \nearrow B < \infty$ or $B_k \nearrow \infty$, but slowly, with subexponential growth (that is $k^{-1} \log B_k \to 0$), by Theorem 7.1 in [15]. In turn, we have (3.26) for $B_k \searrow B > 0$, $B_k \nearrow B < \infty$ or $B_k \nearrow \infty$ of subexponential growth, whereas (3.27) is satisfied with $B_k \searrow B > 0$ and any $B_k \nearrow$. Thus, the sequence of constant B_k satisfies all three conditions.

3.2 Faber Basis in the Space $\mathcal{E}(K(\gamma))$

Consider the Whitney space $\mathcal{E}(K(\gamma))$ with the norms

$$\| f \|_q = |f|_{q,K(\gamma)} + \sup \left\{ |(R_y^q f)^{(k)}(x)| \cdot |x-y|^{k-q} : x, y \in K(\gamma), x \ne y, k = 0, 1, \ldots, q \right\}$$

for $q \in \mathbb{Z}_+$, where $|f|_{q,K(\gamma)} = \sup\{|f^{(k)}(x)| : x \in K(\gamma), k \le q\}$ and $R_y^q f(x) = f(x) - T_y^q f(x)$ is the Taylor remainder. By the open mapping theorem, for any q there exist $r \in \mathbb{N}$, $C > 0$ such that

$$|||\, f \,|||_q \le C \,||\, f \,||_r \tag{3.28}$$

for any $f \in \mathcal{E}(K)$. Here, $|||\, f\,|||_q = \inf\ |\,F\,|_{q,[0,1]}$, where the infimum is taken over all possible extensions of f to $F \in C^\infty[0,1]$.

Here we use an adjustment of the construction in [11], where A. Goncharov considered the case of geometrically symmetric Cantor sets. Let $e_0 \equiv 1$ and $e_N(x) = \prod_1^N (x - x_k)$ for $N \in \mathbb{N}$, where the points $(x_k)_1^\infty$ are chosen on $K(\gamma)$ by the rule of increase of type. The divided differences define linear continuous functionals $\xi_N(f) = [x_1, x_2, \cdots, x_{N+1}]f$ on $\mathcal{E}(K(\gamma))$. By the properties of divided differences, the system $(e_N, \xi_N)_{N=0}^\infty$ is biorthogonal and functionals $(\xi_N)_{N=0}^\infty$ are total on $\mathcal{E}(K(\gamma))$, that is, whenever $\xi_N(f) = 0$ for all N, then $f = 0$. We show that $(e_N)_{N=0}^\infty$ is a topological basis in the space $\mathcal{E}(K(\gamma))$ provided the set $K(\gamma)$ is sufficiently small. Thus, for small sets $K(\gamma)$, the space $\mathcal{E}(K(\gamma))$ possesses a strict polynomial basis. Recall that a polynomial topological basis $(P_n)_{n=0}^\infty$ in a functional space is called a Faber (or strict polynomial) basis if $\deg P_n = n$ for all n.

Lemma 3.2.1. *For each N and $p = 2^u < N/2$ we have $||e_N||_p \leq C \cdot C_0^N\, N^p \cdot \prod_{k=2p+1}^N \rho_k$, where C does not depend on N, $\prod_{k=1}^N \rho_k$ is the product generated by N.*

Proof. By Lemma 3.1.2 for $I_{j,s} = [0,1]$, we have $|e_N|_{q,K(\gamma)} \leq C_0^N \prod_{k=1}^N \rho_k$. Our first goal is to generalize it:

$$|e_N|_{q,K(\gamma)} \leq C_0^{N-q}\, N^q \prod_{k=q+1}^{N} \rho_k \tag{3.29}$$

is valid for $q < N$. For a fixed x, we use the representation $|e_N(x)| = \prod_{k=1}^N d_k(x,Z)$. The $q-$th derivative of e_N at the point x is the sum of $\frac{N!}{(N-q)!}$ products, where each product contains $N-q$ terms of the type $(x - x_k)$. Hence $|e_N^{(q)}(x)| \leq N^q \prod_{k=q+1}^N d_k(x,Z)$. Here, for each k, we take the smallest $m = m(k)$ with $d_k(x,Z) \leq \ell_{j_m,i-m} < C_0\, \delta_{i-m}$. By the Remark (3.1.3), $\delta_{i-m} \leq \rho_k$. The last inequality may be strict if we take x on a part of the set with high density of points x_k, for example near the origin.

To deal with $\| e_N \|_p$, let us fix $i \leq p$ and $x \neq y$ in $K(\gamma)$. For brevity, let $R := (R_x^p e_N)^{(i)}(y)$. We consider two cases: x, y belong to the same interval or two different intervals of the level $n-u$.

In the first case, let $x, y \in I_{j,n-u}$, by the Lagrange form for the Taylor remainder, we have $|R| \cdot |x-y|^{i-p} \leq |e_N^{(p)}(\theta)| + |e_N^{(p)}(x)|$ for some $\theta \in I_{j,n-u}$.

Let $m := \#(I_{j,n-u} \cap Z)$. Since the points from Z are distributed uniformly on $K(\gamma)$, we have $p \leq m \leq 2p$. Hence,

$$|e_N^{(p)}(\theta)| \leq N^p \cdot \prod_{k=p+1}^{N} d_k(\theta, Z) \leq N^p \cdot \prod_{k=m+1}^{N} d_k(\theta, Z), \tag{3.30}$$

as all distances here do not exceed 1. By Lemma 3.1.7 and the argument in the proof of (3.29), $|e_N^{(p)}(\theta)| \leq e^p N^p \cdot \prod_{k=m+1}^N d_k(x, Z) \leq e^p N^p C_0^{N-m} \cdot \prod_{k=m+1}^N \rho_k$. On the other hand, $|e_N^{(p)}(x)| \leq N^p C_0^{N-p} \cdot \prod_{k=p+1}^N \rho_k$. Thus, in the first case, $|R| \cdot |x-y|^{i-p} \leq 2e^p N^p C_0^N \cdot \prod_{k=2p+1}^N \rho_k$, since the last product dominates the products of ρ_k involved in the estimation of both terms.

In the second case, let $y \notin I_{j,n-u}$, so $|x-y| \geq h_{j_1,n-u-1}$, where $x \in I_{j,n-u} \subset I_{j_1,n-u-1}$. By (3.3), $|x-y| > 7/8 \cdot \delta_{n-u-1}$. Now,

$$|R| \cdot |x-y|^{i-p} \leq |e_N^{(i)}(y)| \cdot |x-y|^{i-p} + \sum_{k=i}^{p} |e_N^{(k)}(x)| \cdot \frac{|x-y|^{k-p}}{(k-i)!}.$$

By (3.29), $|e_N^{(k)}(x)| \cdot |x-y|^{k-p} \leq C_0^{N-k} N^k \prod_{m=k+1}^N \rho_m \cdot (8/7)^{p-k} \delta_{n-u-1}^{k-p}$. Recall that, for given N, the interval $I_{j,n-u}$ contains at least p points from Z. Therefore, $\rho_{k+1} \cdots \rho_p \leq \delta_{n-u-1}^{p-k}$ and

$$|e_N^{(k)}(x)| \cdot |x-y|^{k-p} \leq (8/7)^p C_0^N N^k \prod_{m=p+1}^{N} \rho_m. \tag{3.31}$$

Clearly, $N^i + \sum_{k=i}^p \frac{N^k}{(k-i)!} < N^p e$. Combining these yields $|R| \cdot |x-y|^{i-p} \leq (8/7)^p C_0^N N^p \prod_{m=p+1}^N \rho_m$. We compare estimations for both cases. The result follows. $\square$

We proceed to estimate the divided difference $|\xi_N(f)|$ for $f \in \mathcal{E}(K(\gamma))$, $\xi_N(f) = [z_1, \cdots, z_{N+1}]f$. As in Lemma 3.1.6, $\tilde{Z} = (z_k)_{k=1}^{N+1}$ is the set $(x_k)_{k=1}^{N+1}$ arranged in increasing order. Recall that N generates the product $\prod_{k=1}^{N} \rho_k = \prod_{k=0}^{n} \delta_k^{s_k(N)}$, whereas for $N+1$ we have $\prod_{k=1}^{N+1} \tilde{\rho}_k = \prod_{k=0} \delta_k^{s_k(N+1)}$ with $s_k(N) = s_k(N+1)$ for all k except one value, which is not larger than $n+1$. Therefore,

$$\prod_{k=1}^{N+1} \tilde{\rho}_k \geq \prod_{k=1}^{N} \rho_k \cdot \delta_{n+1}. \tag{3.32}$$

Lemma 3.2.2. *For each N and $q = 2^m - 1 < N$ we have*

$$|\xi_N(f)| \leq (16/7)^N \, |||f|||_q \prod_{k=q+1}^{N} \rho_k^{-1}. \tag{3.33}$$

Proof. As in (17) from [15],

$$|\, \xi_N(f) \,| \leq 2^{N-q} \, |||f|||_q \, (\Pi(J_0))^{-1}, \tag{3.34}$$

where $\Pi(J_0) = min_{1 \leq j \leq N+1-q} \Pi(J)$ for $\Pi(J)$ defined in Lemma 3.1.6, so it is enough to estimate from below $\prod_{k=q+2}^{N+1} d_k(z, \tilde{Z})$ uniformly for $z \in \tilde{Z}$.

Arguing as in the proof of (3.13), we see that for each $x \in K(\gamma)$

$$|e_{N+1}(x)| = \prod_{k=1}^{N+1} d_k(x, \tilde{Z}) \geq d_1(x, \tilde{Z}) \cdot (7/8)^N \prod_{k=2}^{N+1} \tilde{\rho}_k.$$

Similarly, $\prod_{k=q+2}^{N+1} d_k(x, \tilde{Z}) \geq (7/8)^{N-q-1} \prod_{k=q+2}^{N+1} \tilde{\rho}_k$, because of the correspondence between $d_k(x, \tilde{Z})$ and $\tilde{\rho}_k$ for $1 \leq k \leq N+1$. We remove $q+1$ smallest terms from both parts of (3.32). This gives $\prod_{k=q+2}^{N+1} \tilde{\rho}_k \geq \prod_{k=q+1}^{N} \rho_k$, and the lemma follows. $\square$

Theorem 3.2.3. *Dynin-Mityagin criterion ([18],Theorem 9), let E be a nuclear space and $\{x'_k \in E', x_k \in E, k = 1, 2, \dots\}$ be a biorthogonal system satisfying the conditions:*

- *the set of functionals x'_k is total over E,*

- *for any seminorm p in E there exists a seminorm $q \geq p$ such that*

$$\sup_{p(x_k)>0} q(x'_k)p(x_k) < \infty. \tag{3.35}$$

Then the system $\{x'_k, x_k\}$ is an absolute basis in E.

Next is Theorem 1 from [11] adapted to our case.

Theorem 3.2.4. *Suppose $(B_k)_{k=1}^{\infty}$ satisfies (3.27). Then the sequence $(e_N)_{N=0}^{\infty}$ is a Schauder basis in the space $\mathcal{E}(K(\gamma))$.*

Proof. By Theorem (3.2.3), it is enough to show that for every p there exist r such that the sequence $(\| e_N\|_p \cdot |\, \xi_N|_{-r})_{N=0}^{\infty}$ is bounded. Here, $|\cdot|_{-r}$ denotes the dual norm: for $\xi \in \mathcal{E}'(K(\gamma))$, let $|\,\xi|_{-r} = \sup\{|\,\xi(f)|,\ \|f\|_r \leq 1\}$.

We can consider only p of the form $p = 2^u$. In order to apply Lemma 3.1.6 we have to take only q of the type $q = 2^k - 1$. For this reason, given arbitrary u, let $q = 2^{v+1} - 1$ where $v = v(u)$ will be specified later. Then $r = r(q)$ will be defined by (3.28).

Fix N with $2^n \leq N < 2^{n+1}$. We take N so large that Lemmas 3.2.1 and 3.2.2 can be applied. Then $|\,\xi_N|_{-r} \leq C\,(16/7)^N \prod_{k=q+1}^{N} \rho_k^{-1}$ for C defined by (3.28), and

$$\|\, e_N\|_p \cdot |\,\xi_N|_{-r} \leq \tilde{C}\,(3\,C_0)^N\, N^p \prod_{k=2p+1}^{q} \rho_k \leq \tilde{C}\,(3\,C_0)^N\, N^p \prod_{k=2p+1}^{2^v} \rho_k, \tag{3.36}$$

where $\tilde{C}$ does not depend on N. We can decrease the upper index of the product above as all ρ_k do not exceed 1.

We see that the product $\prod_{k=2p+1}^{2^v} \rho_k$ takes its maximal possible value in the case of minimal density of points of Z, when each basic interval of the level $n-u$ contains p points from Z. At worst, $\rho_1, \cdots, \rho_p$ do not exceed δ_{n-u}, whereas $\rho_{p+1}, \cdots, \rho_{2p} = \delta_{n-u-1},\ \ \rho_{2p+1}, \cdots, \rho_{4p} = \delta_{n-u-2}$, etc. On the other hand, values

N closed to 2^{n+1} will give maximal density of Z on $K(\gamma)$ with $\rho_1, \cdots, \rho_p \leq \delta_{n-u+1},\ \rho_{p+1}, \cdots, \rho_{2p} = \delta_{n-u}$, etc. Thus, in any case,

$$\prod_{k=2p+1}^{2^v} \rho_k \leq \delta_{n-u-2}^{2p}\, \delta_{n-u-3}^{4p} \cdots \delta_{n-v}^{2^{v-1}} = \exp[-2^n\, (B_{n-u-2} + \cdots + B_{n-v})].$$

We claim that RHS of (3.36) is bounded for a suitable choice of v. It suffices to prove that $N \log(3\, C_0) + p \log N \leq 2^n\, (B_{n-u-2} + \cdots + B_{n-v})$. Since $N < 2^{n+1}$, it is reduced to $2 \log(3\, C_0) + (n+1)2^{-n}\, p \log 2 \leq B_{n-u-2} + \cdots + B_{n-v}$, which is valid for large n if we take $v = m + u + 2$, where m is chosen in (3.27) for $Q = 2 \log(3\, C_0) + 1$. $\square$

Remarks. 1. With the assumption of the stronger condition (3.26), the set $K(\gamma)$ has, in addition, the extension property. The second part of the proof of Theorem 5.3 from [15] (for $j = 1, s = 0$) actually shows that the sequence $(\|\tilde{e}_N\|_p \cdot |\xi_N|_{-r})_{N=0}^{\infty}$ is bounded. Here, $\tilde{e}_N$ is a suitable extension of e_N. Since, for each extension F of $f \in \mathcal{E}(K(\gamma))$, we have $||f||_p \leq 3|F|_p$, this proof implies also that $(e_N)_{N=0}^{\infty}$ is a basis provided (3.26).

2. We guess that, using analytic properties of polynomials P_{2^s}, it is possible to replace the coefficient C_0^N in Lemma 3.2.1 with N^Q for some Q. It is interesting to analyze the possibility to improve the bound (3.34), by changing the exponential growth of the coefficient with a polynomial growth.

3.3 Local Polynomial Bases

In general, the system $(e_N, \xi_N)_{N=0}^{\infty}$ does not have the basis property. Following [11], we use local interpolations to construct bases for any considered case. Suppose we are given a nondecreasing sequence of natural numbers $(n_s)_{s=0}^{\infty}$. Let $N_s = 2^{n_s}$, $M_s^{(l)} = N_{s-1}/2 + 1$, $M_s^{(r)} = N_{s-1}/2$ for $s \geq 1$ and $M_0 = 0$. Here, (l) and (r) mean *left* and *right* respectively. We choose, by the rule of increase of type, N_s points $(x_{k,j,s})_{k=1}^{N_s}$ on each s-th level basic interval $I_{j,s}$. Set

$$e_{N,1,0}(x) = \prod_{k=1}^{N}(x - x_{k,1,0}) = \prod_{k=1}^{N}(x - x_k) \tag{3.37}$$

for $x \in K(\gamma)$ and $N = 0, 1, \cdots, N_0$. Given $s \geq 1$ and j with $1 \leq j \leq 2^s$, for $M_s^{(a)} \leq N \leq N_s$ we take $e_{N,j,s}(x) = \prod_{k=1}^{N}(x - x_{k,j,s})$ if $x \in K(\gamma) \cap I_{j,s}$ and $e_{N,j,s} = 0$ on $K(\gamma)$ otherwise. Here, the superscript a in M_s is l for odd j and $a = r$ if j is even. Thus, we interpolate a function f on the interval $I_{j,s}$ up to degree N_s, whereupon we continue the process on subintervals, preserving the previous nodes of interpolation. All nodes $x_{k,j,s}$ are taken from the sequence $(x_k)_1^{\infty}$.

By Z we denote the points $(x_{k,j,s})_{k=1}^{N}$. Let $(z_{k,j,s})_{k=1}^{N}$ be the same set, but arranged in increasing order. The functionals $\xi_{N,j,s}(f) = [z_{1,j,s}, \cdots, z_{N+1,j,s}]f$ are biorthogonal to functions $e_{k,i,s}$ for $N, k \in [M_s^{(a)}, N_s]$ and $i, j \in [1, 2^s]$. But, in general, $\xi_{N,j,s}$ are not biorthogonal to $e_{k,i,q}$. For example, as is easy to check, $\xi_{N,1,s+1}(e_{N_s,1,s}) \neq 0$ for $M_{s+1}^{(l)} \leq N \leq N_s$. For this reason, as in [11], we take the functionals

$$\eta_{N,j,s}(f) = \xi_{N,j,s}(f) - \sum_{k=N}^{N_{s-1}} \xi_{N,j,s}(e_{k,i,s-1})\, \xi_{k,i,s-1}(f) \tag{3.38}$$

for $I_{j,s} \subset I_{i,s-1}$ and $N = M_s^{(a)}, M_s^{(a)} + 1, \cdots, N_s$. We see that the subtrahend in $\eta_{N,j,s}$ is a kind of biorthogonal projection of $\xi_{N,j,s}$ in the dual space on the subspace spanned on $(\xi_{k,i,s-1})_{k=N}^{N_{s-1}}$. Also, let $\eta_{N,1,0} = \xi_{N,1,0}$ for $0 \leq N \leq N_0$. By Lemma 2 in [11], the system $(e, \eta) := (e_{N,j,s}, \eta_{N,j,s})_{s=0,\, j=1,\, N=M_s}^{\infty,\ 2^s,\ N_s}$ is biorthogonal. Increasing N by one means an inclusion of one more point into the interpolation set, so, if $\eta_{N,j,s}(f) = 0$ for all functionals, then $f(x_k) = 0$ for all k. Since the set under consideration is perfect, the functionals η are total on $\mathcal{E}(K(\gamma))$. Provided a suitable choice of the sequence $(n_s)_0^{\infty}$, the system (e, η) has the basis property.

Here, we take $n_0 = n_1 = 2$ and $n_s = [\log_2 \log \frac{1}{\delta_s}]$ for $s \geq 2$. Then $n_s \leq n_{s+1}$ and

$$\frac{1}{2}\log\frac{1}{\delta_s} < N_s \leq \log\frac{1}{\delta_s} \ \text{ for } s \geq 2. \tag{3.39}$$

In the next lemma we consider any $s,\ I_{j,s} \subset I_{i,s-1}$ and N, k, as in (3.38), that is $M_s^{(a)} \le N \le k \le N_{s-1}$. Also, $\prod_{m=1}^{N} \rho_{m,s}$ corresponds (in the sense of the proof of (3.13)) to the product $\prod_{m=1}^{N} d_m(x_{N+1,j,s}, Z)$, whereas $\prod_{m=1}^{k} \rho_{m,s-1}$ does for $\prod_{m=1}^{k} d_m(x_{k+1,i,s-1}, W)$, where $W = (x_{m,i,s-1})_{m=1}^{k}$ are chosen by the same rule, as Z, but on the interval $I_{i,s-1}$.

Lemma 3.3.1. *In the above notation, $\delta_{s-1}^{k-N} \prod_{m=q+1}^{N} \rho_{m,s} \le \prod_{m=q+1}^{k} \rho_{m,s-1}$ for $q < N$.*

Proof. We have $N_{s-1}/2 \le M_s \le N \le N_{s-1}$, so $2^n \le N \le 2^{n+1}$ with $n := n_{s-1} - 1$. We can omit the case $N = 2^{n+1}$ with $\prod_{m=1}^{N} \rho_{m,s} = \delta_{n+s+1}\delta_{n+s}\delta_{n+s-1}^2 \cdots \delta_s^{2^n}$, since then $k = 2^{n+1}$ as well with $\prod_{m=1}^{k} \rho_{m,s-1} = \delta_{n+s}\delta_{n+s-1}\delta_{n+s-2}^2 \cdots \delta_{s-1}^{2^n}$ and the desired inequality is evident. Therefore, $2^n \le N < 2^{n+1}$, $\rho_{m,s} \in \{\delta_{n+s}, \cdots, \delta_s\}$, whereas $\rho_{m,s-1} \in \{\delta_{n+s}, \cdots, \delta_{s-1}\}$ for all considered m.

Let $U = W \cap I_{2i-1,s}$ and $V = W \cap I_{2i,s}$. Let us show that $\#(U) \le \#(Z)$ and $\#(V) \le \#(Z)$. By that, the densities of points from W on each $s-$th level subinterval of $I_{i,s-1}$ do not exceed that of Z. Thereby, $\rho_{m,s} \le \rho_{m,s-1}$ for $1 \le m \le N$. Also $k - N$ does not exceed the number of points on the adjacent to $I_{j,s}$ interval. This implies that $\rho_{m,s-1} = \delta_{s-1}$ for $N+1 \le m \le k$, which gives the desired conclusion.

Suppose that $I_{j,s}$ is the right subinterval of $I_{i,s-1}$, that is $j = 2i$. If $k = 2q+1$ then $N \ge q+1$. Here, $\#(U) = q+1, \#(V) = q$, so both do not exceed $\#(Z)$. If $k = 2q$ then $N \ge q = \#(U) = \#(V)$.

Similarly, for $j = 2i - 1$ and $k = 2q + 1$ we have $N \ge q + 1 = \#(U)$ with $\#(V) = q$, whereas $k = 2q$ gives $\#(U) = \#(V) = q \le N$.

Since $2^n \le N < 2^{n+1}$, the set Z contains all points of the $n + s - 1-$st level on $I_{j,s}$ and at least one point of the $n + s-$st level is not included in the set. Therefore, $\rho_{1,s} = \delta_{n+s}, \rho_{2,s} = \delta_{n+s-1}$ and $\rho_{3,s} \in \{\delta_{n+s-1}, \delta_{n+s-2}\}$ depending on N. As in Section 2, we have $\prod_{m=1}^{N} \rho_{m,s} = \prod_{r=s}^{s+n} \delta_r^{s_r(N)}$. But $\#(W \cap I) \le \#(Z)$, where I is the subinterval of $s-$th level containing $x_{k+1,i,s-1}$. As was mentioned in Section 2, the distribution of points from $W \cap I$ is uniform or bilateral symmetric to uniform. This mean that $\rho_{m,s-1} \geq \rho_{m,s}$ for $1 \leq m \leq N$ and the degrees $s_r(k)$ in the representation $\prod_{m=1}^{N} \rho_{m,s-1} = \prod_{r=s-1}^{s+n} \delta_r^{s_r(k)}$ do not exceed the corresponding $s_r(N)$, except the value $r = s-1$ if $\#(W \cap I_{j,s}) < \#(Z)$. In addition, we have $\rho_{N+1,s-1} = \cdots = \rho_{k,s-1} = \delta_{s-1}$. This completes the proof. □

We are able to give an analog of Theorem 2 from [11].

Theorem 3.3.2. *Let $(n_s)_{s=0}^{\infty}$ be chosen as above. Then the system $(e_{N,\,j,\,s}, \eta_{N,\,j,\,s})_{s=0,\,j=1,\,N=M_s}^{\infty,\;\;2^s,\;\;N_s}$ is a Schauder basis in the space $\mathcal{E}(K(\gamma))$.*

Proof. Since in the proof of Lemmas 3.2.1 and 3.2.2 we use only (3.1), (3.3) and the properties of uniformly distributed points, the same reasoning applies to the local case. For $2^n \leq N < 2^{n+1}$ with $N > 2p$ we have

$$||e_{N,\,j,\,s}||_p \leq C \cdot C_0^N \, N^p \cdot \prod_{m=2p+1}^{N} \rho_{m,s},$$

where the values $\rho_{m,s}$ belong to the set $\{\delta_{n+s}, \cdots, \delta_s\}$ and correspond to the points $Z = (x_{m,j,s})_{m=1}^{N} \subset I_{j,s}$. In the proof we use the notation C for any coefficient that does not depend on N, s and j.

As in Lemma 3.2.2, $|\xi_{N,\,j,\,s}(f)| \leq (16/7)^N \, |||f|||_q \; \prod_{m=q+1}^{N} \rho_{m,s}^{-1}$ for $N > q+1$ with q of the form $2^m - 1$.

In order to estimate the subtrahend in (3.38), we use a rough bound (8) from [11]:

$$|\,\xi_{N,\,j,\,s}(e_{k,\,i,\,s-1})\,| = \frac{|\,e_{k,\,i,\,s-1}^{(N)}(\theta)|}{N!} \leq \binom{k}{N} \ell_{i,s-1}^{k-N} \leq (2\,C_0)^k \, \delta_{s-1}^{k-N},$$

since for each $\theta \in I_{j,s}$ we have $|e_{k,\,i,\,s-1}^{(N)}(\theta)| \leq \frac{k!}{(k-N)!} \prod_{m=N+1}^{k} d_m(\theta, Z)$ and distances d_m do not exceed $\ell_{i,s-1}$. As above, $|\xi_{k,\,i,\,s-1}(f)| \leq (16/7)^k \, |||f|||_q \; \prod_{m=q+1}^{k} \rho_{m,s-1}^{-1}$.

By Lemma 3.3.1, $|\,\xi_{N,\,j,\,s}(e_{k,\,i,\,s-1})\,| \cdot |\xi_{k,\,i,\,s-1}(f)| \leq (32\,C_0/7)^k \, |||f|||_q \; \prod_{m=q+1}^{N} \rho_{m,s}^{-1}$.

There are at most N_{s-1} terms of this type in the sum in (3.38). Hence, by

(3.28),

$$|\eta_{N,j,s}|_{-r} \le C \prod_{m=q+1}^{N} \rho_{m,s}^{-1} \, [(16/7)^N + N_{s-1}(32\, C_0/7)^{N_{s-1}}]. \tag{3.40}$$

The expression in brackets is less than $2\, N_s (5\, C_0)^{N_s}$. Therefore,

$$||e_{N,j,s}||_p \, |\eta_{N,j,s}|_{-r} \le C \, N_s^{p+1} C_2^{N_s} \prod_{m=2p+1}^{q} \rho_{m,s}$$

with $C_2 = 5\, C_0^2$. Here we can use a rough bound $\rho_{m,s} \le \delta_s$ that implies

$$||e_{N,j,s}||_p \, |\eta_{N,j,s}|_{-r} \le C \, \exp[(p+1) \log N_s + N_s \, \log C_2 + (q - 2p) \log \delta_s],$$

which is bounded, by (4.3), if we take $q > 2p + \log C_2$ and r that corresponds to this q in the sense of the bound (3.28). $\square$

Remark 3.3.3. The argument of the theorem can be applied as well to any sequence $(n_s)_{s=0}^{\infty}$ with $n_s \uparrow \infty$ (non-strictly) and $2^{n_s} \le \log \frac{1}{\delta_s}$. This gives a variety of bases in the space $\mathcal{E}(K(\gamma))$. The question about quasi-equivalence of these bases (see [11], p.359) is open.

Chapter 4

Extension Property and Extension Operator for $\mathcal{E}(K(\gamma))$

4.1 Local version of Pawłucki-Pleśniak Extension Operator for $\mathcal{E}(K(\gamma))$

Here we consider rather small Cantor-type sets that are neither Markov nor local Markov. We follow [3] in our construction, so W is a local version of the Pawłucki-Pleśniak operator. It is interesting that, at least for small sets, W can be considered as an operator extending basis elements of the space. Thus, for such sets, the first method and a local version of the second coincide. Here, as in [3], we use the method of local Newton interpolations. We fix a nondecreasing sequence of natural numbers $(n_s)_{s=0}^{\infty}$ with $n_s \geq 2$ and $n_s \to \infty$. Given a function f on $K(\gamma)$, we interpolate f at 2^{n_0} points that are chosen by the rule of increase of the type on the whole set. A half of points are located on $K(\gamma) \cap I_{1,1}$. We continue interpolation on this set up to the degree 2^{n_1}. Separately we do the same on $K(\gamma) \cap I_{2,1}$. Continuing in this fashion, we interpolate f with higher and higher degrees on smaller and smaller basic intervals. At each step the additional points are chosen by the rule of increase of the type. Interpolation on $I_{j,s}$ does not affect other intervals of the same level due to the following function.

Let $t > 0$ and a compact set E on the line be given. Then $u(\cdot, t, E)$ is a $C^\infty-$ function with the following properties: $u(\cdot, t, E) \equiv 1$ on E, $u(x, t, E) = 0$ for $dist(x, E) > t$ and $\sup_{x\in E} |u^{(p)}_{x^p}(x, t, K)| \le c_p\, t^{-p}$, where the constant c_p depends only on p. Let c_p be increasing.

Given $N+1$ points $(z_k)_{k=1}^{N+1}$ on $K(\gamma)\cap I_{j,s}$, let $\Omega_{N+1}(x) = \prod_{k=1}^{N+1}(x - z_k)$. Define

$$L_N(f, x, I_{j,s}) = \sum_{k=1}^{N+1} f(z_k)\, \omega_k(x), \tag{4.1}$$

where

$$\omega_k(x) = \frac{\Omega_{N+1}(x)}{(x - z_k)\Omega'_{N+1}(z_k)}. \tag{4.2}$$

Let $N_s = 2^{n_s} - 1$ and $M_s = 2^{n_{s-1}-1} - 1$ for $s \ge 1, M_0 = 1$. Then, for fixed s, we take $M_s + 1 \le N \le N_s$, so $2^n \le N < 2^{n+1}$ with $n \in \{n_{s-1} - 1, \cdots, n_s - 1\}$. For such N and s we take $t_N := \delta_{s+n}$. Let, in addition, $1 \le j \le 2^s$ be fixed. Then we choose $N + 1$ points on the interval $I_{j,s}$ by the rule of increase of the type and consider for given f

$$A_{N,j,s} := [L_N(f, x, I_{j,s}) - L_{N-1}(f, x, I_{j,s})]\, u(x, t_N, I_{j,s} \cap K(\gamma)).$$

We call $A_{j,s}(f, x) := \sum_{N=M_s+1}^{N_s} A_{N,j,s}$ the *accumulation sum*. The last term here corresponds to the interpolation on $I_{j,s}$ at 2^{n_s} points. In order to continue interpolation on subintervals of $I_{j,s}$, let us consider the *transition sum*

$$T_{k,s}(f, x) := [L_{M_{s+1}}(f, x, I_{k,s+1}) - L_{N_s}(f, x, I_{j,\,s})]\, u(x, \delta_{s+n_s-1}, I_{k,\,s+1} \cap K),$$

where we suppose $1 \le k \le 2^{s+1}$, $j = [\frac{k+1}{2}]$ and $I_{j,\,s} \supset I_{k,s+1} \cup I_{i,s+1}$.

As above, we represent the difference in brackets in the telescoping form:

$$[L_{M_{s+1}} - L_{N_s}] = -\sum_{N=2^{n_s-1}}^{2^{n_s}-1} [L_N(f, x, I_{j,\,s}) - L_{N-1}(f, x, I_{j,\,s})].$$

Here, the interpolating set Z for L_N consists of $M_{s+1} + 1$ points of $Y_{s+n_s-1} \cap I_{k,s+1}$ and $N - M_{s+1}$ points, chosen by the rule of increase of the type on $I_{i,s+1}$. The second parameter of u is smaller than the mesh size of Z, so $T_{k,s}(f,x) \neq 0$ only near $I_{k,s+1}$.

Consider the linear operator

$$W(f,\cdot) = L_{M_0}(f,\cdot,I_{1,0})\, u(\cdot,1,K) + \sum_{s=0}^{\infty} \Big[\sum_{j=1}^{2^s} A_{j,s}(f,\cdot) + \sum_{k=1}^{2^{s+1}} T_{k,s}(f,\cdot) \Big].$$

We remark at the outset that, for fixed $x \in \mathbb{R}$ and s, because of the choice of parameters for the function u, at most one value $A_{j,s}$ does not vanish. The same is valid for $T_{k,s}$.

Let us show that W extends functions from $\mathcal{E}(K(\gamma))$, provided a suitable choice of $(n_s)_{s=0}^{\infty}$. Define $n_0 = n_1 = 2$ and $n_s = [\log_2 \log \frac{1}{\delta_s}]$ for $s \geq 2$. Then $n_s \leq n_{s+1}$ and

$$\frac{1}{2} \log \frac{1}{\delta_s} < 2^{n_s} \leq \log \frac{1}{\delta_s} \quad \text{for } s \geq 2. \tag{4.3}$$

Lemma 4.1.1. *Let $(n_s)_{s=0}^{\infty}$ be given as above. Then for any $f \in \mathcal{E}(K(\gamma))$ and $x \in K(\gamma)$ we have $W(f,x) = f(x)$.*

Proof. Let us fix a natural number q with $q > 2 + \log(8C_0/7)$, where C_0 is defined in (3.1). By the telescoping effect,

$$W(f,x) = \lim_{s\to\infty} L_{M_s}(f,x,I_{j,s}), \tag{4.4}$$

where $j = j(s,x)$ is chosen in a such way that $x \in I_{j,s}$. As in [10],

$$|\, L_{M_s}(f,x,I_{j,s}) - f(x)| \leq ||\, f\, ||_q \sum_{k=1}^{2^n} |\, x - z_k|^{\,q}\, |\, \omega_k(x)\, |. \tag{4.5}$$

Here n is shorthand for $n_{s-1} - 1$ and s is such that $M_s = 2^n - 1 > q$. The interpolating set $(z_k)_{k=1}^{2^n}$ for L_{M_s} consists of all points of the type $\leq s+n-1$ on $I_{j,s}$. Given point x, we consider the chain of basic intervals containing it: $x \in I_{j_n,s+n} \subset \cdots \subset I_{j_1,s+1} \subset I_{j,s}$. We see that $I_{j_n,s+n}$ contains one interpolating point, $I_{j_{n-1},s+n-1} \setminus I_{j_n,s+n}$ does one more z_i, $I_{j_{n-2},s+n-2} \setminus I_{j_{n-1},s+n-1}$ contains two such points, etc. Thus, for fixed k, we get

$$|\, x - z_k|^{\,q} \prod_{i=1, i\neq k}^{2^n} |x - z_i| \leq l_{j,s}^{\,q-1} \cdot l_{j_n,s+n} \cdot l_{j_{n-1},s+n-1} \cdot l_{j_{n-2},s+n-2}^2 \cdots l_{j,s}^{\,2^{\,n-1}}. \tag{4.6}$$

By (3.1), this does not exceed $C_0^{2^n+q-1}\delta_{s+n}\,\delta_{s+n-1}\,\delta_{s+n-2}^2\cdots\delta_{s+1}^{2^{n-2}}\,\delta_s^{2^{n-1}+q-1}$.

On the other hand, by a similar argument, for the denominator of $|\,\omega_k(x)\,|$ we have

$$|\,z_k-z_1|\cdots|\,z_k-z_{k-1}|\cdot|\,z_k-z_{k+1}|\cdots|\,z_k-z_{2^n}|\geq l_{q_{n-1},s+n-1}\cdot h^2_{q_{n-2},s+n-2}\cdots h_{j,s}^{2^{\,n-1}} \tag{4.7}$$

for some indices $q_{n-1},q_{n-2},\cdots$. The last product exceeds $(7/8)^{2^n-2}\delta_{s+n-1}\,\delta_{s+n-2}^2\cdots\delta_s^{2^{n-1}}$, by (3.3). It follows that

$$\text{LHS of (4.5)}\leq||\,f\,||_{\,q}\,2^n\,C_0^{q-1}(8C_0/7)^{\,2^{\,n}}\,\delta_{s+n}\,\delta_s^{q-1}.$$

The expression on the right side approaches zero as $s\to\infty$. Indeed, $2^n<\log(1/\delta_{s-1})$, by (4.3), and $2^n(8C_0/7)^{\,2^{\,n}}\,\delta_s^{q-1}<1$ due to the choice of q. Thus the limit in (4.4) exists and equals $f(x)$. □

4.2 Extension Property of Weakly Equilibrium Cantor-Type Sets

We need two more lemmas.

Lemma 4.2.1. *Let γ satisfy (3.4), $q=2^m$, $r=2^n$ with $m<n$ and $Z=(z_k)_{k=1}^r$ be all points of the type $\leq s+n-1$ on $I_{1,s}$ for some $s\in\mathbb{Z}_+$. Let $f(x)=\prod_{k=1}^r(x-z_k)$ for $x\in K(\gamma)\cap I_{1,s}$ and $f=0$ on $K(\gamma)\setminus I_{1,s}$. Then*

$$|f|_{0,K(\gamma)}\leq C_0^r\cdot\delta_{n+s}\cdot\delta_{n+s-1}\cdot\delta_{n+s-2}^2\cdots\delta_s^{2^{n-1}}, \tag{4.8}$$

$|f^{(q)}(0)|\geq q!\cdot(7/8)^{r-q}\cdot\delta_{n+s-m-1}^{2^m}\cdots\delta_s^{2^{n-1}}$ *and* $||f||_r\leq 2\cdot r!$.

Proof. Fix $\tilde{x}$ that realizes $|f|_{0,K(\gamma)}$ and a chain of basic intervals containing it: $\tilde{x}\in I_{j_0,n+s}\subset I_{j_1,n+s-1}\subset\cdots\subset I_{j_n,s}=I_{1,s}$. Arguing as in Lemma 4.1.1, we see that $|f|_{0,K(\gamma)}\leq l_{j_0,n+s}\cdot l_{j_1,n+s-1}\cdot l_{j_2,n+s-2}^2\cdots l_{1,s}^{2^{n-1}}$, which, by (3.1), gives the desired bound.

In order to estimate $|f^{(q)}(0)|$, let us remark that $f^{(q)}(x)$ is a sum of $\binom{r}{q}$ products, each product has a coefficient $q!$ and consists of $r-q$ terms $(x-z_k)$. One of these products is $g(x) := \prod_{k=q+1}^{r}(x-z_{i_k})$, where $z_{i_1} < z_{i_2} < \cdots < z_{i_r}$. All products are nonnegative at $x=0$, since $r-q$ is even. From here, $|f^{(q)}(0)| \geq q! \cdot g(0)$. Taking into account the location of points from Z, we get $g(0) = \prod_{k=q+1}^{r} z_{i_k} > h_{1,n+s-m-1}^{2^m} \cdots h_{1,s}^{2^{n-1}} > (7/8)^{r-q} \cdot \delta_{n+s-m-1}^{2^m} \cdots \delta_s^{2^{n-1}}$, by (3.3). The bound of $\|f\|_r$ is evident. $\square$

For given $2^n \leq N < 2^{n+1}$, we consider $\Omega_N(x) = \prod_{k=1}^{N}(x-z_k)$ with $Z_N = (z_k)_{k=1}^{N}$, where the points are chosen on $I_{j,s}$ by the rule of increase of the type. Let $u(x) = u(x, \delta_{s+n}, I_{j,s} \cap K(\gamma))$ and, as above, $d_i(x, Z_N) := |x - z_{k_i}| \nearrow$.

Lemma 4.2.2. *The bound* $|(\Omega_N \cdot u)^{(p)}(x)| \leq 2^p\,(C_0+1)\,c_p\,\delta_{s+n}^{-p+1}\,N^p \prod_{k=2}^{N} d_k(x, Z_N)$

is valid for each $p < N$ and $x \in \mathbb{R}$.

Proof. By Leibnitz's formula, $|(\Omega_N \cdot u)^{(p)}(x)| \leq \sum_{i=0}^{p} \binom{p}{i} |\,\Omega_N^{(i)}(x)|\, c_{p-i}\delta_{s+n}^{-p+i}$. Since d_k increases, we have $|\Omega_N^{(i)}(x)| \leq \frac{N!}{(N-i)!} \prod_{k=i+1}^{N} d_k(x, Z_N)$. This gives

$$|(\Omega_N \cdot u)^{(p)}(x)| \leq 2^p\, c_p\, \delta_{s+n}^{-p} \cdot \max_{0 \leq i \leq p}\, (N\,\delta_{s+n})^i \prod_{k=i+1}^{N} d_k(x, Z_N). \qquad (4.9)$$

The set Z_N consists of 2^n endpoints of subintervals of the level $s+n-1$ covered by $I_{j,s}$ and $N - 2^n$ points of the type $s+n$. Here, $\mathrm{dist}(x, I_{j,s} \cap K(\gamma)) = |x - x_0| \leq \delta_{s+n}$ for some x_0. Let $x_0 \in I_{i,s+n} \subset I_{m,s+n-1}$. Then $I_{m,s+n-1}$ contains from 2 to 4 points of Z_N. In all cases, $d_1(x, Z_N) \leq l_{i,s+n} + \delta_{s+n} \leq (C_0+1)\delta_{s+n}$, by (3.1). Also, $\delta_{s+n}/2 \leq d_2 \leq (C_0+1)\delta_{s+n-1}$. Here the lower bound corresponds to the case $\#(I_{i,s+n} \cap Z_N) = 2$, whereas the upper bound deals with $\#(I_{m,s+n-1} \cap Z_N) = 2$. Similarly, $d_3 \geq h_{m,s+n-1} - \delta_{s+n}$. From (3.3) and (3.4) it follows that $d_3 \geq 7/8\ \delta_{s+n-1} - \delta_{s+n} \geq 27\,\delta_{s+n}$. This gives $\delta_{s+n}^{i-1} d_{i+1} \cdots d_N \leq (C_0+1) d_2 \cdots d_N$ for $0 \leq i \leq p$ and, by (4.9), the lemma follows. $\square$

Theorem 4.2.3. *Suppose γ satisfies* (1.24)*. Then $K(\gamma)$ has the extension property if and only if*

$$(Y)\quad \frac{B_{n+s}}{\sum_{k=s}^{n+s} B_k} \to 0 \quad \text{as} \quad n \to \infty \quad \text{uniformly with respect to} \quad s. \qquad (4.10)$$

Proof. Recall that the extension property of a set is equivalent to the condition (DN) of the corresponding Whitney space. Due to L. Frerick ([5], Prop. 3.8), $\mathcal{E}(K)$ satisfies (DN) if and only if for any $\varepsilon > 0$ and for any $q \in \mathbb{N}$ there exist $r \in \mathbb{N}$ and $C > 0$ such that $|\cdot|_q^{1+\varepsilon} \leq C|\cdot|_{0,K} \ ||\cdot||_r^{\varepsilon}$. Hence, in order to prove that (Y) is necessary for EP of $K(\gamma)$, we can show that negation (1.40) implies the lack of (DN) for $\mathcal{E}(K(\gamma))$, that is there exist $\varepsilon > 0$ and q such that for any $r \in \mathbb{N}$ one can find a sequence $(f_j) \subset \mathcal{E}(K(\gamma))$ with

$$|\, f_j|_q^{1+\varepsilon}\, |\, f_j|_{0,K(\gamma)}^{-1}\ ||\, f_j||_r^{-\,\varepsilon} \ \to \infty \quad \text{as} \quad j \to \infty.$$

Let us fix ε and m from the condition (1.40) and take $q = 2^m$. For each fixed large r (clearly, we can take it in the form $r = 2^n$) and s_j defined by (1.40), we consider the function f_j given in Lemma 4.2.1 for $s = s_j$. Then

$$C\,|\, f_j|_q^{1+\varepsilon}\, |\, f_j|_{0,K(\gamma)}^{-1}\ ||\, f_j||_r^{-\,\varepsilon} \geq (\delta_{n+s}{\cdot}\delta_{n+s-1}{\cdot}\delta_{n+s-2}^2 \cdots \delta_{n+s-m)}^{2^{m-1}})^{-1}(\delta_{n+s-m-1}^{2^m} \cdots \delta_s^{2^{n-1}})^{\varepsilon},$$

where C does not depend on j. The right side here goes to infinity. Indeed, its logarithm is $2^{n+s}\,\{2B_{n+s} + B_{n+s-1} + \cdots + B_{n+s-m} - \varepsilon[B_{n+s-m-1} + \cdots + B_s]\}$ and the expression in braces exceeds B_{n+s} by (1.40). Therefore the whole value exceeds $2^{n+s}B_{n+s} = \frac{1}{2}\log\frac{1}{\delta_{s+n}}$, which goes to infinity when $s = s_j$ increases. Thus, EP of $K(\gamma)$ implies (1.36).

For the converse, we consider the extension operator W from previous section, where (n_s) are chosen as in (4.3). We proceed to show that W is bounded provided (1.42). Let us fix any natural number p. This p and C_1 from Lemma 3.1.4 define $M = 2p + 2 + \log(2C_1)$. We fix $m \in \mathbb{N}$ that corresponds to M it the sense of (1.42). Let $q = 2^m - 1$ and $r = r(q)$ be defined by (3.2). We will show that the bound $|(W(f,x))^{(p)}| \leq C\,||f||_r$ is valid for some constant $C = C(p)$ and all $f \in \mathcal{E}(K(\gamma))$, $x \in \mathbb{R}$.

Given f and x, let us consider terms of accumulation sums. For fixed $s \in \mathbb{N}$ we choose $j \leq 2^s$ such that $x \in I_{j,s}$. Fix N with $2^n \leq N < 2^{n+1}$ for $n_{s-1} - 1 \leq n \leq n_s - 1$, so $M_s + 1 \leq N \leq N_s$. For large enough s the value N exceeds p and q. As in Lemma 3.1.4, let $Z = (z_k)_{k=1}^{N+1} = Z_N \cup \{z_{N+1}\}$. By Newton's representation of interpolating operator in terms of divided differences, we have

$$A_{N,j,s}(f,x) = [z_1, \cdots, z_{N+1}]f \cdot \Omega_N(x)\, u(x),$$

where Ω_N and u are taken as in Lemma 4.2.2. We aim to show that

$$N_s\, |A^{(p)}_{N,j,s}(f,x)| \le s^{-2}\, ||f||_r \tag{4.11}$$

for large s. This gives convergence of the accumulation sums.

Since divided differences are symmetric in their arguments, we can use (4) from [12]:

$$|\ [z_1, \cdots, z_{N+1}]f\ | \le 2^{N-q}\, |||f|||_q\ (\Pi(J_0))^{-1}, \tag{4.12}$$

where $\Pi(J_0) = min_{1\le j\le N+1-q}\Pi(J)$ for $\Pi(J)$ defined in Lemma 3.1.6. Fix $\tilde{z} \in J_0$ that corresponds to this set in the sense of Lemma 3.1.6.

Applying Lemma 4.2.2 and Lemma 3.1.4 for $z = \tilde{z}$ yields

$$|(\Omega_N \cdot u)^{(p)}(x)| \le C\, \delta^{-p}_{s+n}\, N^p\, C_1^N \prod_{k=2}^{N+1} d_k(\tilde{z}, Z) \ \text{ with } C = 2^p(C_0+1)\, c_p.$$

On the other hand, (4.12) and Lemma 3.1.6 for J_0 give

$$|\ [z_1, \cdots, z_{N+1}]f\ | \le 2^{N-q}\, |||f|||_q \prod_{k=q+2}^{N+1} d_k^{-1}(\tilde{z}, Z).$$

Combining these we see that

$$|A^{(p)}_{N,j,s}(f,x)| \le C\ |||f|||_q\, \delta^{-p}_{s+n}\, N^p\, (2C_1)^N \prod_{k=2}^{q+1} d_k(\tilde{z}, Z).$$

Recall that the set Z includes all points of the type $\le s+n-1$ on $I_{j,s}$ and $N - 2^n$ points of the type $s+n$. We can only enlarge the product $\prod_{k=2}^{q+1} d_k(\tilde{z}, Z)$ if we will consider only distances from $\tilde{z}$ to points from $Y_{s+n-1} \cap I_{j,s}$. Arguing as in Lemma 4.1.1, we get $\prod_{k=2}^{q+1} d_k(\tilde{z}, Z) \le C_0^q \delta_{s+n-1}\, \delta^2_{s+n-2} \cdots \delta^{2^{m-1}}_{s+n-m}$. We observe that $d_1(\tilde{z}, Z) = 0$ is not included into the product on the left side. By (1.42), $\prod_{k=2}^{q+1} d_k(\tilde{z}, Z) \le C_0^q\ \delta^M_{s+n}$.

In order to get (4.11), it is enough to show that

$$s^2 \ N_s \, N^p \, (2C_1)^N \, \delta_{s+n}^{M-p} \to 0 \text{ as } s \to \infty. \tag{4.13}$$

Here, by (4.3), $N_s \, N^p < 2^{n_s(p+1)} \leq \log(1/\delta_s)^{p+1} < \delta_s^{-p-1}$. Also, $(2C_1)^N < \delta_s^{-\log(2C_1)}$. Clearly, we can replace δ_{s+n} in (4.13) by δ_s. Then, because of the choice of M, the product in (4.13) does not exceed $s^2 \, \delta_s$, which approaches 0 as $s \to \infty$, since, by (3.4), $\delta_s \leq 32^{-s}$.

Similar arguments are used for terms of the transition sums. □

4.3 Extension Operator by Suitable Extensions of Basis Elements

When the space $\mathcal{E}(K)$ has a topological basis for K with the extension property, the extension operator can be constructed by suitable individual extensions of basis elements.

Theorem 4.3.1. *If there exists a continuous linear extension operator for $\mathcal{E}(K(\gamma))$ then it can be given by replacing all basis elements in the basis expansion of a function by their extensions.*

Proof. As is shown in [15], the condition (1.36) is a characterization of the extension property for $K(\gamma)$, so we suppose that (1.36) is valid. We aim to show that the operator

$$W : \mathcal{E}(K(\gamma)) \to C^\infty(\mathbb{R}) : f = \sum_{s=0}^{\infty} \sum_{j=1}^{2^s} \sum_{N=M_s}^{N_s} \eta_{N,j,s}(f) \, e_{N,j,s} \mapsto \sum_{s=0}^{\infty} \sum_{j=1}^{2^s} \sum_{N=M_s}^{N_s} \eta_{N,j,s}(f) \, \tilde{e}_{N,j,s}$$

is bounded, provided an appropriate choice of extensions for basis elements given in Theorem 3.3.2. As in [15], given s, j and N with $2^n \leq N < 2^{n+1}$, we take a $C^\infty-$function $u(x) = u(x, \delta_{s+n}, I_{j,s} \cap K(\gamma))$ with $\sup_{x \in \mathbb{R}} |u^{(p)}(x)| \leq c_p \ \delta_{s+n}^{-p}$, where $c_p \nearrow$, $u \equiv 1$ on $I_{j,s} \cap K(\gamma)$ and $u(x) = 0$ if the distance from x to the set $I_{j,s} \cap K(\gamma)$

is larger that δ_{s+n}. Now we consider $e_{N,j,s}$ as a polynomial on the whole line and define $\tilde{e}_{N,j,s}$ as $e_{N,j,s} \cdot u$.

It suffices to show that for each $p \in \mathbb{N}$ there exists r such that for each $f \in \mathcal{E}(K(\gamma))$ and $x \in \mathbb{R}$ we have the bound

$$\sum_{s=0}^{\infty} \sum_{j=1}^{2^s} \sum_{N=M_s}^{N_s} |\eta_{N,j,s}(f)| \cdot |\tilde{e}^{(p)}_{N,j,s}(x)| \leq A_p \, ||f||_r, \tag{4.14}$$

where A_p depends only on p. Given p, let us take $M = \log C_2 + p + 3$ (with $C_2 = 5C_0^2$) and m that corresponds to M in the sense of (1.36). Next, we take $q = 2^{m+2} - 1$ and the corresponding r from (3.28).

By Lemma 5.2 in [15], $|\tilde{e}^{(p)}_{N,j,s}(x)| \leq C\, \delta_{s+n}^{-p+1}\, N^p \prod_{k=2}^N d_k(x, Z)$. As above, C stands for any coefficient that does not depend on s, j and N. As in Lemma 3.2.1, we have $d_k(x, Z) \leq C_0 \rho_{k,s}$, so $|\tilde{e}_{N,j,s}|_p \leq C\, C_0^N\, \delta_{s+n}^{-p+1}\, N^p \prod_{k=2}^N \rho_{k,s}$, where $\rho_{k,s}$ are defined after (3.39). By the proof of Theorem 3.3.2, $|\eta_{N,j,s}(f)| \leq 2\, N_s C_2^{N_s}\, |||f|||_q \, \prod_{k=q+1}^N \rho_{k,s}^{-1}$.

Because of the choice of parameters, the function $\tilde{e}_{N,j,s}$ vanishes on all intervals $I_{i,s}$ for $i \neq j$. Hence, for each fixed x, we get at most one nonzero term in the sum with respect to j in (4.14). Clearly, the sum with respect to N contains at most N_s terms. Thus it remains to show that $s^2\, N_s^{p+2}\, C_2^{N_s} \prod_{k=2}^q \rho_{k,s}$ is bounded. As in Theorem 3.2.4, we replace upper index q in the product with more convenient 2^{m+1}.

The product $\prod_{k=1}^{2^{m+1}} \rho_{k,s}$ takes its maximal possible value when the density of points $(x_{k,j,s})_{k=1}^N$ is minimal, that is in the case $N = M_s = 2^n$ with $n := n_{s-1} - 1$. Then $\prod_{k=2}^{2^{m+1}} \rho_{k,s} = \delta_{s+n-1}\, \delta_{s+n-2}^2 \cdots \delta_{s+n-m-1}^{2^m}$. By (1.36) and the choice of m, this does not exceed δ_{s+n-1}^M for large enough s. Also we have $n \geq 2$ and $\delta_{s+n-1} \leq \delta_s \leq e^{-N_s}$, by (3.39). Therefore, it is enough to show that $s^2\, N_s^{p+2}\, C_2^{N_s}\, e^{-M\, N_s}$ is bounded, which is valid due to the choice of M and the inequality $s^2\, \delta_s < 1$. $\square$

The condition (3.27) means that the set $K(\gamma)$ is so small that the space $\mathcal{E}(K(\gamma))$ possesses a Faber basis. On the other hand, (3.27) is compatible with

(1.36), which provides the extension property. A combination of both conditions, that is (3.26), we analyze separately. Thus we consider the operator W_{gl} which is W from the previous theorem for $s=0, j=1$. On the other hand, W_{gl} is equal to the accumulation part of the operator (11) from [15]. Therefore, W_{gl} coincides with the Pawłucki-Pleśniak operator, provided $(x_k)_{k=1}^N$ is the Fekete set. We conjecture that, at least for $N=2^n$, this is the case. Following Theorem 4.3.1, we take $u(x)=u(x,\delta_n,K(\gamma)), \tilde{e}_N=e_N\cdot u$ and

$$W_{gl}:\mathcal{E}(K(\gamma))\to C^\infty(\mathbb{R}): f=\sum_{N=0}^\infty \xi_N(f)\,e_N\mapsto \sum_{N=0}^\infty \xi_N(f)\,\tilde{e}_N.$$

Clearly, the sum $\sum_{N=0}^M \xi_N\, e_N$ is the interpolating operator L_M at the set $Z=(x_k)_{k=1}^{M+1}$. The parameter δ_n of u corresponds to the values $2^n\le N<2^{n+1}$. This gives the representation

$$W_{gl}(f,x)=f(0)\cdot u(x,\delta_0)+\sum_{n=0}^\infty\sum_{N=2^n}^{2^{n+1}-1}(L_N-L_{N-1})(f,x)\cdot u(x,\delta_n), \qquad (4.15)$$

as $L_0(f,x)=f(0)$. After cancellation of equal terms we get

$$W_{gl}(f,x)=\lim_{k\to\infty}\left\{\sum_{n=1}^k L_{2^n-1}(f,x)\cdot[u(x,\delta_{n-1})-u(x,\delta_n)]+L_{2^{k+1}-1}(f,x)\cdot u(x,\delta_k)\right\}.$$

Lemma 4.3.2. *If* (1.36) *is valid then there are* A *and* n_0 *such that* $\delta_{n-1}^A\le\delta_n$ *for* $n\ge n_0$.

Proof. We use (1.36) for $M=2:\delta_{n-m}^{2^m}\cdots\delta_{n-1}^2\le\delta_n$ for $n\ge n_0$. Clearly, LHS here is larger than $\delta_{n-1}^{2^m}$, so we can take $A=2^m$ for $m=m(2)$. □

Theorem 4.3.3. *The extension operator* W_{gl} *is bounded provided* (3.26).

Proof. We use the representation (4.15). Here, $(L_N-L_{N-1})(f,x)\cdot u(x,\delta_n)=A_{N,1,0}(f,x)$ in notations of [15]. For each p, we want to find r with $|A_{N,1,0}^{(p)}(f,x)|\le N^{-2}\,||f||_r$. Given p, we take $q=2^{m+1}-1$, where m will be defined later, and $r=r(q)$, by (3.28).

By (18) in [15], $|A^{(p)}_{N,1,0}(f,x)| \le C\, |||f|||_q\, \delta_n^{-p}\, N^p\, (2C_1)^N \prod_{k=2}^{q+1} \rho_k$. The product $\prod_{k=2}^{q+1} \rho_k$ can be handled as in Theorem 4.3.1. We are reduced to proving that the expression $\delta_n^{-p}\, N^{p+2}\, (2C_1)^N\, \delta_{n-1}\, \delta_{n-2}^2 \cdots \delta_{n-m-1}^{2m}$ is bounded. Since (3.26) is stronger than (1.36), we can apply Lemma 4.3.2. Hence, $\delta_n^{-p} < \delta_{n-1}^{-pA} = \exp(2^n\, p\, A\, B_{n-1})$. As before, $\delta_{n-1}\, \delta_{n-2}^2 \cdots \delta_{n-m-1}^{2m}$ is $\exp[-2^n(B_{n-1}+\cdots+B_{n-m-1})]$. The problem now reduces to establishing that

$$2^n\, p\, A\, B_{n-1} + (p+2)\log N + N\, \log(2C_1) < 2^n(B_{n-1} + \cdots + B_{n-m-1}),$$

which is valid for large n if we define m by (3.26) for $M = p\,A$ and $Q = 2\,log(2C_1) + 1$, as $N < 2^{n+1}$. $\square$

4.4 Two Examples

Recall the condition for extension property;

$$(Y)\quad \frac{B_{n+s}}{\sum_{k=s}^{n+s} B_k} \to 0 \text{ as } n \to \infty \text{ uniformly with respect to } s. \tag{4.16}$$

First we consider regular sequences $(B_k)_{k=1}^{\infty}$. Let $\beta_k = (\log B_k)/k$. We say that $(B_k)_{k=1}^{\infty}$ is *regular* if, for some k_0, both sequences $(B_k)_{k=k_0}^{\infty}$ and $(\beta_k)_{k=k_0}^{\infty}$ are monotone. Recall that $(B_k)_{k=1}^{\infty}$ has *subexponential growth* if $\beta_k \to 0$ as $k \to \infty$.

For example, given $a > 1$, let $\gamma_k^{(1)} = k^{-a}, \gamma_k^{(2)} = a^{-k}, \gamma_k^{(3)} = \exp(-a^k)$ for large enough k. Then $\gamma^{(j)}$ for $1 \le j \le 3$ generate regular $B^{(j)}$ with $B_k^{(1)} \sim 2^{-k-1}\, a\, k \log k, B_k^{(2)} \sim 2^{-k-2}\, k^2\, \log a, B_k^{(3)} \sim (a/2)^{k+1}/(a-1)$. Here, $\beta_k^{(1)}, \beta_k^{(2)} \nearrow -\log 2$ and $\beta_k^{(3)} \to -\log(a/2)$, so $B^{(j)}$ are not of subexponential growth, except $B^{(3)}$ for $a = 2$. We see that (4.16) is valid in the first two cases and in the third case with $a \le 2$.

More generally, (4.16) is valid for each monotone convergent $(B_k)_{k=1}^{\infty}$. Indeed, if $B_k \searrow B \ge 0$, then LHS of (4.16) does not exceed $(n+1)^{-1}$. If $B_k \nearrow B$, then we take s_0 with $B_s > B/2$ for $s \ge s_0$. Then $B_s + \cdots B_{s+n} \ge (n+1)B/2$ and LHS of (4.16) $< 2(n+1)^{-1}$. This covers the case of regular sequences $(B_k)_{k=1}^{\infty}$ when

β_k are negative. Let us show that (4.16) is valid as well for divergent regular sequences $(B_k)_{k=1}^\infty$ of subexponential growth.

Theorem 4.4.1. *Let $(B_k)_{k=1}^\infty$ be regular with positive values of β_k. Then (4.16) is valid if and only if $(B_k)_{k=1}^\infty$ has subexponential growth.*

Proof. A regular sequence $(B_k)_{k=1}^\infty$ is not of subexponential growth, provided $\beta_k > 0$, in the following three cases: $\beta_k \nearrow \beta < \infty$, $\beta_k \nearrow \infty$ and $\beta_k \searrow \varepsilon_0 > 0$. We aim to show that (4.16) is not valid under the circumstances.

In the first case, given s and n, let $b = \exp \beta_{s+n}$. Then $b - 1 \geq \exp \beta_1 - 1 > \beta_1 > 0$ and $b \leq \exp \beta$. Here, $\sum_{k=s}^{s+n} B_k < b^{s+n+1}/(b-1)$ as $B_k = \exp(k\beta_k) \leq b^k$ for such k. Therefore, $B_{s+n}/\sum_{k=s}^{s+n} B_k > (b-1)/b > \beta_1/\exp \beta$, which contradicts (4.16).

If $\beta_k \nearrow \infty$ then, by the same argument, $B_{s+n}/\sum_{k=s}^{s+n} B_k > (b-1)/b > 1/2$ for $s \geq s_0$, where s_0 is fixed with $\exp \beta_{s_0} > 2$.

Suppose $\beta_k \searrow \varepsilon_0$. We fix indices $s_1 < s_2 < \cdots$ such that the intervals I_j connecting points (s_j, β_{s_j}) and $(s_{j+1}, \beta_{s_{j+1}})$ form a convex envelope of the set (k, β_k) on the plane. We start from $s_1 = \max\{s : \beta_s = \beta_1\}$. If s_j is chosen, then we take s_{j+1} with he property: for each k with $s_j \leq k \leq s_{j+1}$ the point (k, β_k) is not over I_j. At any step we can take the next value so large that the slopes of I_j increases to zero. In addition, given s_j, we take s_{j+1} such that

$$(4 - 2\, s_j/s_{j+1})\beta_{s_{j+1}} \geq (3 - s_j/s_{j+1})\beta_{s_j}, \tag{4.17}$$

which is possible as β_k decreases to a positive limit.

For fixed j, we take $s = s_j$ and $s + n = s_{j+1}$. Let $\tilde{\beta}_k = ak + b$ with $a = -(\beta_s - \beta_{s+n})/n$ and $b = \beta_s + (\beta_s - \beta_{s+n})\, s/n$ for $s \leq k \leq s+n$, so the points $(k, \tilde{\beta}_k)$ are located just on the interval I_j. Also, let $g(x) = ax^2 + bx$ and $\tilde{B}_k = \exp g(k) = \exp(k\, \tilde{\beta}_k)$ on $[s, s+n]$. Of course, $\tilde{B}_s = B_s$ and $\tilde{B}_{s+n} = B_{s+n}$.

It is easy to check that the function g increases on this interval. Hence, $\sum_{k=s}^{s+n} B_k \leq \sum_{k=s}^{s+n} \tilde{B}_k < \int_s^{s+n} g(x)\, dx + B_{s+n}$. By integration by parts,

$\int_s^{s+n} g(x)\,dx = g(n+s)\cdot[2a(n+s)+b]^{-1} - g(s)\cdot[2as+b]^{-1} + 2a\int_s^{s+n} g(x)(2ax+b)^{-2}dx$. We neglect the last term, as $a<0$, and the second term, as $2as+b=g'(s)>0$. Also, $2a(n+s)+b=(2+s/n)\beta_{s+n}-(1+s/n)\beta_s \ge \beta_s/2 \ge \varepsilon_0/2$, by (4.17). Hence $\int_s^{s+n} g(x)\,dx < 2\,B_{s+n}/\varepsilon_0$ and $B_{s+n}/\sum_{k=s}^{s+n} B_k > \varepsilon_0/(2+\varepsilon_0)$, so (4.16) is not valid.

We proceed to show that (4.16) is valid for $\beta_k \searrow 0$, that is in the case of subexponential growth of $(B_k)_{k=1}^{\infty}$. Here, for fixed large s and n, we estimate $\sum_{k=s}^{s+n} B_k$ from below. Let $b=\exp\beta_{s+n}$. Then $B_k \ge b^k$ for $s \le k \le s+n$. Therefore, $B_{s+n}/\sum_{k=s}^{s+n} B_k \le \frac{b^n(b-1)}{b^{n+1}-1}$.

If $b^n<2$ for the given s and n then $b^{n+1}-1>(n+1)\beta_{s+n}$. On the other hand, $\exp\beta_{s+n}-1<2\,\beta_{s+n}$ for $\beta_{s+n}<1$. Thus the fraction above does not exceed $4/(n+1)$.

Otherwise, $b^n \ge 2$ and $b^n < 2(b^{n+1}-1)$. Here the fraction does not exceed $4\,\beta_{s+n}$. It follows that $B_{s+n}/\sum_{k=s}^{s+n} B_k \le \max\{4/(n+1), 4\,\beta_n\}$, which is the desired conclusion. $\square$

Our next objective is to consider irregular sequences $(B_k)_{k=1}^{\infty}$ (compare with Ex.6 in [14]). Given two sequences, $(k_j)_{j=1}^{\infty} \subset \mathbb{N}$ with $k_{j+1}-k_j \nearrow \infty$ and $(\varepsilon_j)_{j=1}^{\infty}$ with $\varepsilon_j \searrow 0$, let $\gamma_k=(k+5)^{-2}$ for $k \ne k_j$ and $\gamma_{k_j}=(k_j+5)^{-2}\varepsilon_j$. Then γ satisfies (3.4) with $\delta_k=(5!/(k+5)!)^2\,\varepsilon_1\varepsilon_2\cdots\varepsilon_j$ for $k_j \le k < k_{j+1}$. Let $A_j := \log\frac{1}{\varepsilon_1\varepsilon_2\cdots\varepsilon_j}$. We will consider only sequences with the property

$$k_{j+1}^2 \cdot A_j^{-1} \to 0 \ \text{ as } \ j \to \infty. \tag{4.18}$$

Provided this condition, $B_k = 2^{-k}\log\frac{(k+5)!}{5!} + 2^{-k-1}A_j \sim 2^{-k-1}A_j$ for $k_j \le k < k_{j+1}$. In addition, an easy computation shows that for large j,

$$B_{k_j} + B_{k_j+1} + \cdots + B_{k_{j+1}-1} < 3\,B_{k_j}. \tag{4.19}$$

Now we can construct different examples of compact sets $K(\gamma)$ without extension property.

Example 2. Let $A_j = 2^{k_j}$, so $\varepsilon_j = \exp(-2^{k_j} + 2^{k_{j-1}})$ for $j \geq 2$ and $\varepsilon_1 = \exp(-2^{k_1})$. In this case, (4.18) is valid under mild restriction $2^{-k_j}\, k_{j+1}^2 \to 0$ as $j \to \infty$. Let us take $s = k_j, n = k_{j+1} - k_j$. Then $B_{s+n} > 2^{-k_{j+1}-1}\, A_{j+1} = 1/2$ and, by (4.19), $B_s + \cdots + B_{s+n-1} < 3\, B_s < 4 \cdot 2^{-k_j-1}\, A_j = 2$. This gives (1.40) with $\varepsilon = 1/4$ and $m = 0$.

Chapter 5

Hausdorff Measures and Extension Property

R. Nevanlinna [19] and H. Ursell [29] proved that there is no complete characterization of polarity of compact sets in terms of Hausdorff measures. The scale of growth rate of functions h, which define the Hausdorff measure Λ_h, can be decomposed into three zones. For h from the first zone of small growth, if $0 < \Lambda_h(K)$ then the set K is not polar. For h from the zone of fast growth, if $\Lambda_h(K) < \infty$ then the set K is polar. But between them there is a zone of uncertainty. It is possible to take two functions with $h_2 \prec h_1$ from this zone and the corresponding Cantor-type sets K_j with $0 < \Lambda_{h_j}(K_j) < \infty$ for $j \in \{1, 2\}$, such that the larger (with respect to the Hausdorff measure) set K_2 is polar, whereas the smaller K_1 is not polar. Here we present a similar example of two Cantor-type sets: the smaller set has EP whereas the larger set does not have it.

Let h be a *dimension function*, which means that $h : (0, T) \to (0, \infty)$ is continuous, nondecreasing and $h(t) \to 0$ as $t \to 0$.

Definition 5.0.2. The $h-$Hausdorff content of $E \subset \mathbb{R}$ is defined as

$$\mathcal{M}_h(E) = \inf\{\sum h(|G_i|) : E \subset \cup G_i\}$$

and the $h-$Hausdorff measure of E is

$$\Lambda_h(E) = \liminf_{\delta\to 0}\{\sum h(|G_i|) : E \subset \cup G_i, |G_i| \le \delta\}.$$

Where $|G_i|$ is the diameter of the set. Here we consider at most countable coverings of E by intervals (open or closed).

It is easily seen that $\mathcal{M}_h(E) = 0$ if and only if $\Lambda_h(E) = 0$. We write $h_1 \prec h_2$ if $h_1(t) = o(h_2(t))$ as $t \to 0$. Let $h_1 \approx h_2$ if $C^{-1}h_1(t) \le h_2 \le Ch_1(t)$ for some constant $C \ge 1$ and $0 < t \le t_0 < T$. We will denote by h_0 the function

$$h_0(t) = \frac{1}{\log \frac{1}{t}} \tag{5.1}$$

with $0 < t < 1$, which defines the logarithmic measure of sets.

A set E is called *dimensional* if there is at least one dimension function h that makes E an $h-$set, that is $0 < \Lambda_h(E) < \infty$. In our case, the set $K(\gamma)$ is dimensional. In [1], following Nevanlinna [19], the corresponding dimension function was presented. Let $\eta(\delta_k) = k$ for $k \in \mathbb{Z}_+$ with $\delta_0 := 1$ ($\delta_k = \gamma_1 \cdot \gamma_2 \ldots \gamma_k$) and $\eta(t) = k + \log\frac{\delta_k}{t}/\log\frac{\delta_k}{\delta_{k+1}}$ for $\delta_{k+1} < t < \delta_k$. Then $h(t) := 2^{-\eta(t)}$ for $0 < t \le 1$. Clearly, $h(\delta_k) = 2^{-k}$.

Proposition 5.0.3. *Let γ satisfy* (3.4) *and h be defined as above. Then* $\Lambda_h(K(\gamma)) = 1$.

Proof. Take $t = C_0\,\delta_k$, where C_0 is given in (3.1). Then $\delta_k < t = C_0\,\gamma_k\,\delta_{k-1} < \delta_{k-1}$ for large enough k. Here, $\eta(t) = k - \log C_0/\log(1/\gamma_k)$ and $h(t) = 2^{-k}\,a_k$ with $a_k := \exp\frac{\log C_0 \cdot \log 2}{\log(1/\gamma_k)}$. Since $\gamma_k \to 0$, given $\varepsilon > 0$, there is k_0 such that $a_k < 1 + \varepsilon$ for $k \ge k_0$. From (3.1) it follows that

$$1 = 2^k\,h(\delta_k) < \sum_{j=1}^{2^k} h(l_{j,k}) < 2^k\,h(t) < 1 + \varepsilon \tag{5.2}$$

provided that $k \ge k_0$. Of course, $\Lambda_h(K(\gamma)) \le \sum_{j=1}^{2^k} h(l_{j,k})$ for each k. Since ε is arbitrary, we get $\Lambda_h(K(\gamma)) \le 1$.

Let us show that $\Lambda_h(K(\gamma)) \geq 1$. Fix $\varepsilon > 0$ and choose k_0 such that

$$\varepsilon \log 1/\gamma_k > -\log 2 \cdot \log(1 - 4\gamma_k) \text{ for } k \geq k_0. \tag{5.3}$$

This can be done as $\gamma_k \to 0$. Take any open covering $\cup G_i$ of $K(\gamma)$. Given ε, we can consider only coverings with $|G_i| < \delta_{k_0}$ for each i. We choose a finite subcover $\cup_{i=1}^N G_i$ of $K(\gamma)$.

Fix $i \leq N$ and k with $\delta_{k+1} < |G_i| \leq \delta_k$. Recall that, for each $j \leq 2^k$, we have $h_{j,k} > (1 - 4\gamma_{k+1})l_{j,k}$. Therefore the distance between any two basic intervals from E_{k+1} exceeds $(1 - 4\gamma_{k+1})\delta_k$. If $|G_i| < (1 - 4\gamma_{k+1})\delta_k$ then G_i can intersect at most one interval from E_{k+1}. In this case we can consider only $|G_i| \leq \max_{1\leq j\leq 2^{j+1}} l_{j,k+1} \leq C_0\delta_{k+1}$, by (3.1). Thus there are two possibilities: $\delta_{k+1} < |G_i| \leq C_0\delta_{k+1}$ or $(1 - 4\gamma_{k+1})\delta_k < |G_i| \leq \delta_k$.

In the first case we have $h(|G_i|) > 2^{-k-1}$. Here, G_i intersects at most one interval from E_{k+1} and, by construction, at most 2^{m-k-1} interval from E_m for $m > k$. In turn, in the latter case, $h(|G_i|) > 2^{-k}(1-\varepsilon)$. Indeed, here, $\eta(|G_i|) < k - \log(1 - 4\gamma_{k+1})/\log(1/\gamma_{k+1})$ and $h(|G_i|) > 2^{-k}\, a$, where $a = \exp \frac{\log(1-4\gamma_{k+1})\cdot\log 2}{\log(1/\gamma_{k+1})} > (1 - \varepsilon)$, by (5.3). Now G_i intersects at most two interval from E_{k+1} and at most 2^{m-k} interval from E_m.

Let us choose m so large that each basic interval from E_m belongs to some G_i, perhaps not to unique. We decompose all intervals from E_m into two groups corresponding to the cases considered above. Counting intervals gives $2^m \leq \sum_i' 2^{m-k-1} + \sum_i'' 2^{m-k} < 2^m[\sum_i' h(|G_i|) + \sum_i'' h(|G_i|)(1 - \varepsilon)^{-1}]$. From this we see that $\sum_i h(|G_i|) > 1 - \varepsilon$, which is the desired conclusion, as ε and (G_i) here are arbitrary. $\square$

The same reasoning applies to a part of $K(\gamma)$ on each basic interval.

Corollary 5.0.4. *Let γ and h be as in Proposition above, $k \in \mathbb{N}, 1 \leq j \leq 2^k$. Then $\Lambda_h(K(\gamma) \cap I_{j,k}) = 2^{-k}$.*

Suppose, in addition, that $K(\gamma)$ is not polar. Then, by Corollary 3.2 in [1],

$\omega_{K(\gamma)}(I_{j,k}) = 2^{-k}$, so the values of $\omega_{K(\gamma)}$ and the restriction of Λ_h on $K(\gamma)$ coincide on each basic interval. From here, by Lemma 3.3 in [1], these measures are equal on $K(\gamma)$. Thus, a non-polar set $K(\gamma)$ satisfying (3.4) is indeed *equilibrium* Cantor-type set if we accept for definition of this concept the condition $\omega_K = \Lambda_h|_{K(\gamma)}$, which is more natural than the definition suggested in (Section6, [13]). Let us rewrite the examples from previous sections.

Example 5.0.5. Let $\gamma_1 = \exp(-4B)$ and $\gamma_k = \exp(-2^k B)$ for $k \geq 2$, where $B \geq \frac{1}{4}\log 32$, so (1.24) is valid. Here, $B_k = B$ for all k. Hence (1.36) is satisfied and the set $K(\gamma)$ is polar.

Example 5.0.6. Let $A_j = 2^{k_j}$, so $\varepsilon_j = \exp(-2^{k_j} + 2^{k_{j-1}})$ for $j \geq 2$ and $\varepsilon_1 = \exp(-2^{k_1})$. In this case, (4.18) is valid under the mild restriction $2^{-k_j}\, k_{j+1}^2 \to 0$ as $j \to \infty$. Let us take $s = k_j, n = k_{j+1} - k_j$. Then $B_{s+n} > 2^{-k_{j+1}-1}\, A_{j+1} = 1/2$ and, by (4.19), $B_s + \cdots + B_{s+n-1} < 3\, B_s < 4 \cdot 2^{-k_j-1}\, A_j = 2$. This gives (1.40) with $\varepsilon = 1/4$ and $m = 0$.

Proposition 5.0.7. *There are two dimension functions $h_2 \prec h_1$ and two sets K_1, K_2, where K_j is an h_j-set for $j \in \{1, 2\}$, such that the smaller set K_1 has the extension property, whereas the larger set K_2 does not have.*

Proof. Take K_1 from Example 5.0.5. Let us show that the corresponding function $h_1 = 2^{-\eta_1}$ is equivalent to h_0. It is enough to find $C > 0$ such that $\eta_0(t) - C \leq \eta_1(t) \leq \eta_0(t) + C$ for small t. Here, $\eta_0(t) = (\log\log 1/t)/\log 2$, so $h_0(t) = 2^{-\eta_0(t)}$. For the set K_1 we have $\delta_k = \exp(-2^{k+1}B)$ and $\eta_0(\delta_k) = k + \log 2B/\log 2$. If $\delta_{k+1} < t \leq \delta_k$ for some k, then $k \leq \eta_1(t) < k+1$ and $k + \log 2B/\log 2 \leq \eta_0(t) < k + 1 + \log 2B/\log 2$, which gives $h_1 \approx h_0$.

In turn, let K_2 be as in Example 5.0.6 with $A_j = 2^{k_j}\, 2^{-j}$ and $\varepsilon_j = \exp(-A_j + A_{j+1})$ for $j \geq 2$. Here we suppose that $(k_j)_{j=1}^{\infty}$ satisfies $2^{-k_j} 2^j\, k_{j+1}^2 \to 0$ as $j \to \infty$. Then (4.18) and (4.19) are valid, which, as in Example 5.0.6, gives the lack of the extension property for K_2. Let us show that $h_2 \prec h_0$. It is enough to check that $\eta_2(t) - \eta_0(t) \to \infty$ as $t \to 0$. Let $\delta_k < t \leq \delta_{k-1}$ with $k_j \leq k < k_{j+1}$ for large enough

j. Then $\log 1/\delta_k = 2\log((k+5)!/5!)+A_j < 2\,A_j$ and $\eta_0(t) < \eta_0(\delta_k) < k_j+1-j$. On the other hand, $\eta_2(t) \geq \eta_2(\delta_{k-1}) = k-1 \geq k_j-1$. Therefore, $\eta_2(t)-\eta_0(t) > j-2$, which completes the proof. $\square$

One can suppose that, for the family of sets considered, the scale of growth rate of dimension functions can be decomposed as above into three zones. If $K(\gamma)$ is an $h-$set for a function h with moderate growth then the set has EP. If the corresponding function h is large enough, then EP fails. Proposition above shows that the zone of uncertainty here is not empty.

We see that $h = h_0$ is not the largest function which allows EP for $h-$sets $K(\gamma)$. If, as in the regular case, we take $B_k \nearrow \infty$ of subexponential growth, then $\delta_k = \exp(-2^{k+1}B_k)$ and $h_0(\delta_k) = 2^{-k-1}B_k^{-1}$, which is essentially smaller than $h(\delta_k) = 2^{-k}$ for the corresponding function h.

Example 3. Let $\log_{(m)} t$ denote the $m-$th iteration $\log \cdots \log t$ for large enough t. The sequence $B_k = \exp(k/\log_{(m)} k)$ has subexponential growth. Then the corresponding sequence $(\gamma_k)_{k=1}^{\infty}$ satisfies (3.4), as for large k we have $\gamma_k = \delta_k/\delta_{k-1} < \exp(-2^k B_k) < \exp(-2^k)$ and for the previous k we can take $\gamma_k = 1/32$. By Theorem 4.4.1, the set $K(\gamma)$ has EP. Let us find a dimension function h that corresponds to this set. We will search it in the form $h(t) = h_0^{\alpha(t)}(t)$. Let $t = \delta_k$. Then $\log 1/t = 2^{k+1}B_k$, so $k \sim (\log\log 1/t)/\log 2$. On the other hand, $h(t) = 2^{-k} = (2^{k+1}B_k)^{-\alpha(t)}$, which gives $\alpha(t) \sim 1-(\log 2 \cdot \log_{(m)} k)^{-1} \sim 1-(\log 2 \cdot \log_{(m+2)} 1/t)^{-1}$. Clearly, $h \succ h_0$.

The next Proposition generalizes Example 3. We restrict our attention to strictly increasing functions h of the form $h = h_0^{\alpha}$, where α is a monotone function on $[0, t_0]$. Since in the next section we shall be interested in dimension functions exceeding h_0, let us suppose that $\alpha(t) \leq 1$. Then $h \succ t^{\sigma}$ for each fixed $\sigma > 0$.

In addition we assume that asymptotically

$$h(t) \leq 2\, h(t^2), \tag{5.4}$$

which is valid for typical dimension functions corresponding to the cases
a) $\alpha(t) = \alpha_0 \in (0, 1]$,
b) $\alpha(t) = \alpha_0 + \varepsilon(t)$ with $\alpha_0 \in [0, 1)$,
c) $\alpha(t) = 1 - \varepsilon(t)$.

Here,

$$\varepsilon(t) \searrow 0 \quad \text{with} \quad \varepsilon(t) \log\log 1/t \nearrow \infty \;\text{ as }\; t \searrow 0, \tag{5.5}$$

since for slowly increasing ε we get $h^{\alpha_0 \pm \varepsilon} \approx h_0^{\alpha_0}$.

By (5.4), for the inverse function h^{-1}, we have $h^{-1}(\tau) \leq (h^{-1}(2\tau))^2$ and $h^{-1} \prec \tau^M$ for M given beforehand. From this, $\gamma_k = h^{-1}(2^{-k})/h^{-1}(2^{-k+1})$ defines a sequence satisfying (3.4). We denote the corresponding set by $K^\alpha(\gamma)$. Our aim is to check extension propertry for this set provided regularity of the sequence $B_k = 2^{-k-1} \log(1/h^{-1}(2^{-k}))$. We see at once that B_k increases. In its turn, $\beta_k \searrow 0$ if $\alpha_0 = 1$ in the case (a), $\beta_k \nearrow 1/\alpha_0 - 1$ in (b) and in (a) with $\alpha_0 < 1$. Concerning (c), the monotonicity of β_k requires additional rather technical restrictions on ε. At least for $\varepsilon(t) = \varepsilon_m(t) := (\log_{(m)} 1/t)^{-1}$ we have $\beta_k \searrow 0$. Here, $m \geq 3$, as $h \approx h_0$ for $m \in \{1, 2\}$.

Proposition 5.0.8. *Let $K^\alpha(\gamma)$ be defined by a function h, as above, with a regular sequence $(B_k)_{k=1}^\infty$. Then $K^\alpha(\gamma)$ has the extension property if and only if*

$$\left(\log \frac{1}{h^{-1}(2^{-k})}\right)^{1/k} \to 2 \;\; as \; k \to \infty.$$

Proof. Let us find h^{-1} for the case $\alpha(t) = 1 - \varepsilon(t)$. If $h(t) = \tau$ then $[1 - \varepsilon(t)] \log\log 1/t = \log 1/\tau$. Let us define a function δ by the condition $\log\log 1/t = [1 + \delta(\tau)] \log 1/\tau$. Then $[1 - \varepsilon(t)][1 + \delta(\tau)] = 1$, so $\delta(\tau) \searrow 0$ as $\tau \searrow 0$. Then $t = h^{-1}(\tau) = \exp[-(1/\tau)^{1+\delta(\tau)}]$ and $\log(1/h^{-1}(2^{-k})) = 2^{k(1+\delta(2^{-k}))}$. The $k-$th root of this expression tends to 2. On the other hand, $(B_k)_{k=1}^\infty$ here has subexponential growth as $\beta_k = (\delta(2^{-k}) - 1/k) \log 2 \to 0$. By Theorem 4.4.1, $K^\alpha(\gamma)$ has EP.

Similarly, if $\alpha(t) = \alpha_0 + \varepsilon(t)$ with $0 < \alpha_0 < 1$ then $h^{-1}(\tau) = \exp[-(1/\tau)^{1/\alpha_0 - \delta(\tau)}]$. Here, $(\log(1/h^{-1}(2^{-k})))^{1/k} = 2^{(1/\alpha_0 - \delta(2^{-k}))} \not\twoheadrightarrow 2$ and $\beta_k \not\twoheadrightarrow 0$, there is no EP. In the case (a), the function δ vanishes.

Lastly, $\alpha_0 = 0$ in (b) gives $h^{-1}(\tau) = \exp[-(1/\tau)^{\Delta(\tau)}]$ with $\Delta(\tau) \nearrow \infty$ as $\tau \searrow 0$. Here, $(\log(1/h^{-1}(2^{-k})))^{1/k} \to \infty$ and $\beta_k \to \infty$. $\square$

5.1 Extension Property and Densities of Hausdorff Contents

To decide whether a set K has EP, we have to consider a local structure of the most rarefied parts of K. Obviously, such global characteristics as Hausdorff measures or Hausdorff contents cannot be applied in general for this aim. Instead, one can suggest to describe EP in terms of lower densities of $\mathcal{M}_h$ or related functions. Given a dimension function h, a compact set K, $x \in K$ and $r > 0$, let $\varphi_{h,K}(x,r) := \mathcal{M}_h(K \cap B(x,r))$ and

$$\varphi_{h,K}(r) := \inf_{x \in K} \varphi_{h,K}(x,r), \tag{5.6}$$

where $B(x,r) = [x-r, x+r]$. One can suppose that K has EP if and only if the corresponding function $\varphi_{h,K}$ is not very small, in a sense, as $r \to 0$. Essentially, this is similar to analysis of the lower density of the Hausdorff content, which can be defined as

$$\phi_h(K) := \liminf_{r \to 0} \inf_{x \in K} \frac{M_h(K \cap B(x,r))}{M_h(B(x,r))}. \tag{5.7}$$

Indeed, $\mathcal{M}_h(B(x,r)) = h(2r)$ for h with $h(t) \succ t$ and the expression above is $\liminf_{r \to 0} \frac{\varphi_{h,K}(r)}{h(2r)}$.

In order to distinguish EP by means of ϕ_h, we have to consider large enough dimension functions h. Indeed, if for some h_1 with $h_1 \succ h$ there exists an h_1-set K_1 with EP, then h cannot be used for this aim, because $\Lambda_h(K_1) = 0$ implies $\mathcal{M}_h(K_1) = 0$ and the corresponding density vanishes contrary to our expectations. Therefore, we can consider only functions exceeding h_0.

We remark that Λ_h-analogs of $\varphi_{h,K}$ or ϕ_h cannot be applied in general for distinguishing EP, since for fat sets ($K = \overline{Int(K)}$) we have $\Lambda_h(K \cap B(x,r)) = \infty$ provided $h(t) \succ t$.

Interestingly, it turns out that the lower density ϕ_h can be used to characterize EP for the family of compact sets considered in [8].

Example 4. Given two sequences $b_k \searrow 0$ (for brevity, we take $b_k = e^{-k}$) and $Q_k \nearrow$ with $Q_k \geq 2$, let $K = \{0\} \cup \bigcup_{k=1}^{\infty} I_k$, where $I_k = [a_k, b_k], |I_k| = b_k^{Q_k}$. In what follows we will consider two cases: $Q_k \leq Q$ with some Q and $Q_k \nearrow \infty$ with $Q_k < \log k$ for large k. By Theorem 4 in [8], K has the extension property in the first case and does not have it for unbounded (Q_k).

In the next lemma we consider concave dimension functions $h = h_0^{\alpha}$ for the cases $(a), (b)$, as above, and for more general $c')$ $\alpha(t) = \alpha_0 - \varepsilon(t)$ with $\alpha_0 \in (0,1]$.

We suppose now that ε is a monotone differentiable function on $[0, t_0]$ with $0 < \varepsilon(t) < 1 - \alpha_0$ in (b) and $0 < \varepsilon(t) < \alpha_0/2$ in (c'). As before, we assume (5.5). A direct computation shows that

$$h'(t) < h(t)\, h_0(t)\, \alpha(t)/t \text{ for the cases } (a), (b) \text{ and } h'(t) < h(t)\, h_0(t)/t \text{ for } (c'). \tag{5.8}$$

Lemma 5.1.1. *Suppose intervals I_k are given as in Example 4 and n is large enough. Then $\mathcal{M}_h(\cup_{k=n}^{\infty} I_k) = h(b_n)$. This means that the covering of the set $\cup_{k=n}^{\infty} I_k$ by the interval $[0, b_n]$ is optimal in the sense of definition of $\mathcal{M}_h$.*

Proof. Let us fix any open covering of K, choose its finite subcovering $\cup_{i=1}^{M} G_i$ and enumerate sets G_i from left to right. We can suppose that G_1 covers $\cup_{k=N}^{\infty} I_k$ for some $N \geq n$. Indeed, if G_1 covers as well some part of I_{N-1}, then other part of I_{N-1} is covered by G_2. In this case, association of G_1 and G_1 into one interval will give better covering, since $h(b) \leq h(x)+h(b-x)$ for $0 \leq x \leq b$, by concavity of h. For the same reason, we suppose that each G_i covers entire number of I_k. After this we reduce each G_i to the minimal closed interval F_i containing the same intervals I_k. Thus, $F_1 = [0, b_N]$ and $F_2 = [a_{N-1}, b_q]$ with some $N-1 \leq q \leq n$. Our aim is to show that

$$h(b_q) < h(b_N) + h(b_q - a_{N-1}), \tag{5.9}$$

so replacing $F_1 \cup F_2$ with $[0, b_q]$ is preferable. We use the mean value theorem and the decrease of h'. Note that $h(b_k) = k^{-\alpha(b_k)}$.

Consider first the value $q = N-1$. We will show $h(b_{N-1}) - h(b_N) < h(|I_{N-1}|)$.

In the cases $(a), (b)$, by (5.8), $LHS < h'(b_N)e^{-N}(e-1) < N^{-1-\alpha(b_N)}\alpha(b_N)(e-1)$. On the other hand, $h(|I_{N-1}|) = [Q_{N-1}(N-1)]^{-\alpha(|I_{N-1}|)}$. Here, $\alpha(|I_{N-1}|) < \alpha(b_N)$, so we reduce the desired inequality to $(Q_{N-1}/N)^{\alpha(b_N)}\alpha(b_N)(e-1) < 1$. It is valid, since for $\alpha_0 > 0$ the first term on the left goes to zero, whereas for $\alpha_0 = 0$ in (b) we have $\alpha(b_N) = \varepsilon(b_N) \to 0$ as $N \to \infty$.

Similarly, in the case (c') the inequality $[Q_{N-1}(N-1)]^{\alpha_0-\varepsilon(|I_{N-1}|)}(e-1) < N^{1+\alpha_0-\varepsilon(b_N)}$ is valid, as is easy to check.

Suppose now that $q \leq N-2$. We write (5.9) as $h(b_q) - h(b_q - a_{N-1}) < h(b_N)$.

Here, in all cases, by (5.8), $LHS < h'(b_q - a_{N-1})\, a_{N-1} < h(b_q)h_0(b_q)\frac{a_{N-1}}{b_q - a_{N-1}}$, where the last fraction does not exceed $\frac{b_{N-1}}{b_q - b_{N-1}}$. On the other hand, $h(b_N) \geq N^{-1}$ as $\alpha(b_N) \leq 1$. Hence it is enough to show that $N < (e^{N-q-1}-1)\, q^{1+\alpha(b_q)}$. We neglect $\alpha(b_q)$ and notice that $(e^{N-q-1}-1)\, q \geq (e-1)(N-2)$, which completes the proof of (5.9).

Continuing in this manner, we see that $h(b_n) \leq \sum_{i=1}^{M} h(|F_i|)$. □

Corollary 5.1.2. *Suppose $b_{n+1} \leq r \leq b_n - b_{n+1}$. Then $\varphi_{h,K}(r) = h(|I_n|)$.*

Proof. Clearly, $\varphi_{h,K}(x,r) = h(|I_n|)$ for each $x \in I_n$. If $x \in K \cap [0, b_{n+1}]$ then $B(x,r)$ covers all intervals I_k with $k \geq n+1$. By previous Lemma, $\varphi_{h,K}(x,r) =$

$h(b_{n+1}) > h(|I_n|)$. Of course, for $x \in I_k$ with $k < n$ the value $\varphi_{h,K}(x,r)$ also exceed $h(|I_n|)$. $\square$

Remark 5.1.3. The covering of two (or smaller number of) intervals I_k by one interval is not optimal, since $\mathcal{M}_h(I_k \cup I_{k+1}) = h(|I_k|) + h(|I_{k+1}|) < h(b_k - a_{k+1})$.

We proceed to characterize EP for given compact sets in terms of lower densities ϕ_h for $h = h_0^\alpha$, where

$$\alpha(t) = \alpha_0 \in (0,1] \text{ or } \alpha(t) = \alpha_0 \pm \varepsilon_m(t) \tag{5.10}$$

with $0 < \alpha_0 < 1$ and $\varepsilon_m(t) = (\log_{(m)} 1/t)^{-1}$ for $m > 2$, so (5.5) is valid.

Proposition 5.1.4. *Let K be from the family of compact sets given in Example 4 and h be as above. Then K has the extension property if and only if $\phi_h(K) > 0$.*

Proof. Suppose first that $Q_k \le Q$ with some Q, so K has EP. We aim to show $\underline{\lim}_{r\to 0} \frac{\varphi_{h,K}(r)}{h(2r)} > 0$. Let $e^{-k-1} \le r < e^{-k}$ for some k. Then, as $\varphi_{h,K}$ increases, $\varphi_{h,K}(r) \ge \varphi_{h,K}(e^{-k-1})$, which is $h(|I_k|) = (k \cdot Q_k)^{-\alpha(|I_k|)}$, by Corollary 5.1.2. On the other hand, $h(2r) < h(2e^{-k}) = (k - \log 2)^{-\alpha(2e^{-k})}$. Therefore,

$$\varphi_{h,K}(r)/h(2r) > Q_k^{-\alpha(|I_k|)}\; k^{\alpha(2e^{-k})-\alpha(|I_k|)}\; (1 - \log 2/k)^{-\alpha(2e^{-k})}.$$

The first term on the right converges to $Q^{-\alpha_0}$ as $k \to \infty$. The second and the third terms converge to 1. Hence, $\phi_h(K) \ge Q^{-\alpha_0}$. Besides, this value is achieved in the case $Q_k = Q$ by the sequence $r_k = b_k - b_{k+1}$. Thus, $\phi_h(K) > 0$. In addition, for given $\sigma > 0$ we have a compact sets K with EP such that $0 < \phi_h(K) < \sigma$.

Similar arguments apply to the case $Q_k \nearrow \infty$, when K does not have EP. Here, $\phi_h(K) \le \lim_k \varphi_{h,K}(r_k)/h(2r_k)$ for r_k as above. By Corollary 5.1.2, $\varphi_{h,K}(r_k) = h(|I_k|)$. Also, $h(2r_k) > h(e^{-k})$. Hence, $\varphi_{h,K}(r_k)/h(2r_k) < Q_k^{-\alpha_0/2}\, k^{\alpha(e^{-k})-\alpha(|I_k|)}$, which converges to 0 as k increases. $\square$

Remark. For this family of sets, the extension property can be characterized alsol in terms of the Lebesgue linear measure λ. Let $\lambda(r) := \inf_{x\in K} \lambda(K \cap [x - r, x+r])$. Then K has the extension property if and only if $\liminf_{r\to 0} \lambda(r) \cdot r^{-Q} > 0$

for some Q.

Nevertheless, at least for dimension functions $h = h_0^\alpha$ with α as in (5.10), there is no general characterization of EP in terms of lower densities ϕ_h. In view of Example 3 and the discussion at the beginning of the section, the value $\alpha_0 = 1$ can be omitted from consideration

We treat now regular sets $K(\gamma)$ with $\delta_k = \exp(-b^k)$. Here, $B_k = 2^{-1}(b/2)^k$. By Theorem 4.4.1, $K(\gamma)$ has EP if $b = 2$ and does not have it for $b > 2$.

Lemma 5.1.5. *For each constants $C \geq 1$ and h, as above, there is $b > 2$ such that $h(C\delta_k) < 2\,h(\delta_{k+1})$ for large enough k. This inequality is valid also for $b = 2$.*

Proof. In all cases we have $h(\delta_k) = b^{-k\cdot\alpha(\delta_k)}$ and the desired inequality has the form

$$b^{(k+1)\alpha(\delta_{k+1})} < 2\,(b^k - \log C)^{\alpha(C\delta_k)}. \tag{5.11}$$

Suppose $\alpha \equiv \alpha_0$. Then (5.11) is valid as $b^{\alpha_0} < 2\,(1 - b^{-k}\log C)$ for large k and $b = 2 + \sigma$ with small enough σ. All the more it is valid for $b = 2$.

The same reasoning applies to the case $\alpha = \alpha_0 + \varepsilon(t)$ with $\varepsilon \nearrow$ as $\varepsilon(\delta_{k+1}) < \varepsilon(C\delta_k)$.

In the last case $\alpha(t) = \alpha_0 - \varepsilon_m(t)$ we use the following simple inequality

$$\log_{(m)}(Cx) - \log_{(m)}(x) < \log C \cdot [\log x\, \log_{(2)}(x) \cdots \log_{(m-1)}(x)]^{-1},$$

which is valid for all x from the domain of definition of $\log_{(m)}$. From this we have $k \cdot [\varepsilon(C\delta_k) - \varepsilon(\delta_{k+1})] \to 0$ as $k \to \infty$ and (5.11) can be treated as in the first case. $\square$

Corollary 5.1.6. *Let k be large enough. Then the covering of each basic interval $I_{j,k}$ of $K(\gamma)$ by one interval is better (in the sense of definition of $\mathcal{M}_h$) than covering by two adjacent subintervals.*

Indeed, by (3.1), $h(l_{j,k}) < h(C_0\delta_k) < 2\,h(\delta_{k+1}) < h(l_{2j-1,k+1}) + h(l_{2j,k+1})$.

Remark. It is essential that coverings of a whole basic interval are considered. For example, for the set $I_{1,k} \cup I_{3,k+1}$ we have $h(l_{1,k}) + h(l_{3,k+1}) < h(b_{3,k+1})$, which corresponds to the covering of the set by one interval.

Proposition 5.1.7. *Let $h = h_0^\alpha$ with α as in (5.10) and $K(\gamma)$ be defined by $\delta_k = \exp(-b^k)$. Then $\phi_h(K(\gamma)) = b^{-\alpha_0}$.*

Proof. For brevity, we denote here $K(\gamma)$ by K. Fix $x \in K$. Let $x \in I_{j,k} \subset I_{i,k-1}$ and $C_0\,\delta_k \le r \le 7/8 \cdot \delta_{k-1}$. Then, by (3.1), $l_{j,k} \le r < h_{i,k-1}$ and $K \cap [x-r, x+r] = K \cap I_{j,k}$. Arguing as in Lemma 5.1.1, by Lemma 5.1.5, we get $\varphi_{h,K}(x,r) = h(l_{j,k})$. Therefore, by monotonicity, $h(\delta_k) < \varphi_{h,K}(x,r) < h(C_0\delta_k)$ for each $x \in K$.

We proceed to estimate $\phi_h(K)$ from both sides. Suppose that $C_0\,\delta_k \le r \le C_0 \cdot \delta_{k-1}$ for some k. Then $h(\delta_k) < \varphi_{h,K}(r) < h(C_0\delta_{k-1})$ and

$$\frac{h(\delta_k)}{h(2C_0\delta_{k-1})} < \frac{\varphi_{h,K}(r)}{h(2r)} < \frac{h(C_0\delta_{k-1})}{h(2C_0\delta_k)}.$$

Here, $\delta_k = \delta_{k-1}^b$. Analysis similar to that in the proof of Lemma 5.1.5 shows that the first fraction above has the limit $b^{-\alpha_0}$, whereas the last fraction tends to b^{α_0} as $k \to \infty$. Moreover, the value $b^{-\alpha_0}$ can be achieved as $\lim_k \varphi_{h,K}(r_k)/h(2r_k)$ for $r_k = 7/8 \cdot \delta_{k-1}$. $\square$

Comparison of Propositions 5.1.4 and 5.1.7 shows that, for given dimension functions, lower densities of Hausdorff contents cannot be used in general to characterize the extension property.

Chapter 6

Extension Property and Growth of Markov's Factors

The representation $g_{\mathbb{C}\setminus K(\gamma)}(z) = \lim_{s\to\infty} 2^{-s} \log|P_{2^s}(z)/r_s|$ for $z \notin K(\gamma)$ allows to construct Green's functions with diverse moduli of continuity and sets with preassigned growth of subsequence of Markov's factors (see [13] for details). In this section, we show that, there is no complete characterization of EP in terms of growth rate of Markov's factors $M_n(\cdot)_{n=0}^{\infty}$ for sets.

Let $\mathcal{P}_n$ denote the set of all holomorphic polynomials of degree at most n.

Definition 6.0.8. For any infinite compact set $K \subset \mathbb{C}$ we consider the sequence of *Markov's factors*

$$M_n(K) = \inf\{M : |P'|_{0,K} \le M\,|P|_{0,K},\ P \in \mathcal{P}_n\}$$

for $n \in \mathbb{N}$.

We see that $M_n(K)$ is the norm of the operator of differentiation in the space $(\mathcal{P}_n, |\cdot|_{0,K})$. We say that a set K is *Markov* if the sequence $(M_n(K))$ is of polynomial growth. This class of sets is of interest to us, since, by W.Pleśniak [21], any Markov set has EP. On the other hand, there exist non-Markov compact sets with EP ([7],[3]). We guess that there is some extremal growth rate $(m_n)_{n=1}^{\infty}$ with the property: if, for some compact set K, $M_n(K)/m_n \to \infty$ as $n \to \infty$ then K does not have EP. The next proposition asserts that here, as above, there is a zone of uncertainty, in which growth rate of Markov's factors is not related with EP.

Proposition 6.0.9. *There are two sets K_1 with EP and K_2 without it such that $M_n(K_1)$ grows essentially faster than $M_n(K_2)$ as $n \to \infty$.*

Proof. By Theorem 6 in [13], $M_{2^k}(K(\gamma)) \sim 2/\delta_k$. By monotonicity, $\delta_k^{-1} < M_n(K(\gamma)) < 4\,\delta_{k+1}^{-1}$ for $2^k \le n < 2^{k+1}$ with large enough k. As in Proposition (5.0.7), we take K_1 from Example 5.0.5, so $\delta_k^{(1)} = \exp(-2^{k+1}B)$ with $B > 1$. Also, we use K_2 from Example 5.0.6 with $A_j = 2^{k_j}$. For simplicity, we fix $k_j = j^2$ that satisfies (4.18). Here, $\delta_k^{(2)} > k^{-2k}\,\varepsilon_1\varepsilon_2\cdots\varepsilon_j$ for $k_j \le k < k_{j+1}$. We aim to show that $M_n(K_2)/M_n(K_1) \to 0$ as $n \to \infty$. Let us fix large n with $2^k \le n < 2^{k+1}$. For this k we fix j with $k_j \le k < k_{j+1}$. Then

$$M_n(K_2)/M_n(K_1) < 4\,\delta_k^{(1)}/\delta_{k+1}^{(2)}. \tag{6.1}$$

Suppose first that $k \le k_{j+1} - 2$. Then right hand side of (6.1) does not exceed $4\,\exp[-2^{k+1}B + 2(k+1)\log(k+1) + A_j]$. The expression in brackets is smaller than $2^{k_j}(1-2B) + k_{j+1}^2$, which is $(j+1)^4 - (2B-1)\,2^{j^2}$, so it tends to $-\infty$ as $j \to \infty$.

If $k = k_{j+1} - 1$ then right hand side of (6.1) is smaller than $4\,\exp[-2^{k_{j+1}}B + 2k_{j+1}\log k_{j+1} + A_{j+1}]$, which goes to 0, since $B > 1$. This completes the proof. □

Existence of a zone of uncertainty (for the extension property) in the scale of growth rate of Markov's factors implicates the problem to find boundaries of this zone.

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