THREE ESSAYS ON SOCIAL CHOICE THEORY

A Ph.D. Dissertation

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THREE ESSAYS ON SOCIAL CHOICE THEORY

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by

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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy in Economics.

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Kemal Yıldız
Advisor

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy in Economics.

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ABSTRACT

THREE ESSAYS ON SOCIAL CHOICE THEORY

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Co-Advisor: Prof. Dr. Semih Koray

October 2025

This thesis consists of three essays on social choice theory.

In the first essay, we study implementation via two-stage rights structures, introduced by Yildiz (2013) as a multi-stage version of the rights structure framework of Koray and Yildiz (2018). While Yildiz established the implementability of the Nash bargaining solution, we consider the two-stage rights structure in the standard environment with a finite set of alternatives and linear order profiles. We identify necessary conditions for implementable social choice rules and provide a sufficient condition, contributing to a characterization of implementation via two-stage rights structures.

In the second essay, we revisit the notion of selection-closedness introduced by Koray and Senocak (2024) within the tops-only domain of social choice functions. We show that a subset of neutral functions on this domain is selection-closed if and only if all its

members are self-selective. We propose two weakenings of selection-closedness: one corrects Senocak's (2009) results on scoring correspondences, while the other yields a partial characterization of correspondences whose singleton-valued refinements form a weakly selection-closed family.

In the third essay, we study ordinal interdependent preferences and the problem of implementing a social choice correspondence in dominant-strategy equilibrium. We identify a monotonicity condition that is necessary and, in rich domains, sufficient for implementation. We characterize interdependence structures under which a counterpart of the Gibbard–Satterthwaite theorem holds. When interdependence is "weak," we show that dictatorship is the only dominant-strategy incentive compatible rule; when it is not, even dictatorship fails to be implementable. These results complement the robust mechanism design literature, which focuses on ex-post incentive compatibility, by highlighting the restrictiveness of dominant-strategy incentive compatibility.

Keywords: two-stage rights structures, implementation, selection-closedness, interdependent preferences, dominant-strategy incentive compatibility.

ÖZET

SOSYAL SEÇİM KURAMI ÜZERİNE ÜÇ MAKALE

Şenocak, Talat

Doktora, İktisat Bölümü

Tez Danışmanı: Doç. Dr. Kemal Yıldız

2. Tez Danışmanı: Prof. Dr. Semih Koray

Ekim 2025

Bu makale sosyal tercih teorisi üzerine üç çalışmadan oluşmaktadır.

İlk makalede, Yıldız (2013) tarafından Koray ve Yıldız (2018)'in haklar yapısı çerçevesinin çok-aşamalı bir versiyonu olarak sunulan iki aşamalı haklar yapısı aracılığıyla uygulanabilirliği inceliyoruz. Yıldız, Nash pazarlık çözümünün uygulanabilirliğini göstermişken, biz sonlu alternatifler kümesi ve lineer tercih profilleri içeren standart ortamda iki aşamalı haklar yapısını ele alıyoruz. Uygulanabilir sosyal tercih kuralları için gerekli koşulları belirliyor ve yeterli bir koşul sağlayarak, iki aşamalı haklar yapısı altında uygulanabilirliğin karakterizasyonuna katkı sunuyoruz.

İkinci makalede, Koray ve Şenocak (2024) tarafından ortaya konan seçmede-kapalılık kavramını, yalnızca en üst tercihlerin dikkate alındığı sosyal tercih fonksiyonları çerçevesinde yeniden inceliyoruz. Bu alanda tarafsız fonksiyonların bir alt kümesinin seçmede-kapalılık taşıdığını, ancak ve ancak tüm üyelerinin kendini-seçici olduğunu gösteriyoruz. Seçmede-kapalılığın iki zayıflatılmış versiyonunu öneriyoruz: bunlardan biri, Şenocak (2009)'un puanlama bağıntılarıyla ilgili sonuçlarını düzeltirken, diğeri tekil değerli inceltmelerinin zayıf seçmede-kapalılık oluşturup oluşturmadığına dair kısmi bir karakterizasyon sunuyor.

Üçüncü makalede, sıralı karşılıklı bağımlı tercihleri ve baskın strateji dengesinde bir sosyal tercih bağıntısının uygulanabilirliği problemini inceliyoruz. Uygulama için gerekli ve zengin alanlarda yeterli bir monotonluk koşulu belirliyoruz. Gibbard–Satterthwaite teoreminin bir karşılığının geçerli olduğu karşılıklı bağımlılık yapılarını karakterize ediyoruz. Karşılıklı bağımlılık "zayıf" olduğunda, diktatörlüğün baskın-strateji teşvik uyumlu tek kural olduğunu; güçlü olduğunda ise diktatörlüğün bile baskın-strateji uygulanabilirliğini sağlayamadığını gösteriyoruz. Bu sonuçlarla, esasen ex-post teşvik uyumluluğuna odaklanan sağlam mekanizma tasarımı literatürünü tamamlıyor ve baskın-strateji teşvik uyumluluğunun aşırı kısıtlayıcılığını vurguluyoruz.

Anahtar Kelimeler: iki aşamalı haklar yapısı, uygulanabilirlik, seçmede-kapalılık, karşılıklı bağımlı tercihler, baskın strateji teşvik uyumu

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There are many people, including my family, to whom I owe thanks for their help and encouragement during my graduate studies. I would like to thank them all. Since I would surely forget to mention someone (except Serhat Doğan — without whom this thesis might not even exist), I decided not to write an acknowledgement section at all.

If I had written one, it would have started with heartfelt thanks to Prof. Semih Koray, who taught me not only to be a better academic, researcher, and instructor, but also to appreciate the integrity and values that make these roles meaningful.

Finally, it would have ended with Kemal Yıldız — my friend, classmate, colleague, advisor, and chair — as a tribute of deepest gratitude:

This thesis is dedicated to Kemal, for bringing me back to where I belong: academics.

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CHAPTER 1

INTRODUCTION

The aim of this thesis is to study problems of implementation and self-selectivity in social choice theory. A central question in this field is under what conditions collective decisions can be represented as the outcome of a well-defined rule, and when these rules can be sustained by appropriate institutions or mechanisms. The three essays of the thesis approach this question from different angles: rights-based models of implementation, consistency of families of social choice functions, and environments with interdependent preferences. Each essay introduces a new concept or result that contributes to the understanding of implementation.

The first essay studies implementation through two-stage rights structures. The rights-structure framework models societies where admissible moves between states are determined by a code of rights. Earlier work showed that some solutions, such as the Nash bargaining solution, are implementable in this framework, but a general understanding of which rules are implementable with two stages remained incomplete. This essay fills part of this gap by identifying necessary and sufficient conditions for implementation. A key concept introduced is image consistency, which captures when violations of image monotonicity matter for implementation. The analysis shows that two-stage rights structures make possible the implementation of rules that fail under Nash equilibrium or one-stage rights structures. A notable example is the plurality rule with fixed tie-breaking, which cannot be implemented in either of the earlier frameworks but becomes implementable with two stages. These results both add to the class of implementable rules and describe what two-stage rights structures can and cannot achieve.

The second essay focuses on the idea of selection-closedness, which is a consistency requirement on families of social choice functions. Previous work established that under consequentialist reasoning, self-selectivity leads to strong impossibility results. Selection-closedness relaxes this requirement by testing the stability of entire families of rules rather than single rules. This essay contributes in three ways. First, it restricts attention to the tops-only domain and shows that, within this domain, selection-closedness coincides with self-selectivity. Second, it revisits earlier claims about scoring correspondences and corrects results in Senocak (2009). Third, it introduces another weaker version of selection-closedness and provides a partial characterization of correspondences whose singleton-valued refinements form a weakly selection-closed family. These results show how selection-closedness relates to natural domain restrictions and how using weaker forms of consistency gives larger families of admissible rules.

The third essay examines environments with interdependent preferences, where an agent's type affects not only her own ranking of outcomes but also the overall preference profile. Most of the literature on interdependent values has worked with quasilinear utilities and focused on ex-post incentive compatibility. By contrast, this essay studies dominant-strategy implementation under ordinal interdependent preferences without domain restrictions. The first step is to define a monotonicity condition that plays the role of Maskin monotonicity in this setting. Using this condition, the essay identifies interdependence structures under which impossibility results similar to the Gibbard–Satterthwaite theorem hold. When interdependence is weak, dictatorship is the only dominant-strategy implementable rule. When interdependence is stronger, even dictatorship cannot be implemented. These findings show how demanding dominant-strategy implementation is in interdependent settings and indicate what can and cannot be achieved in mechanism design under such environments.

Taken together, the three essays study implementation from different angles. The first develops new conditions in two-stage rights structures, the second examines consistency requirements on families of rules and shows what happens when they are relaxed, and the third extends impossibility results to interdependent preference domains. The common theme is to show what is implementable once we change the institutional setting, the notion of consistency, or the domain of preferences.

CHAPTER 2

IMPLEMENTATION VIA TWO-STAGE RIGHTS STRUCTURES

2.1. Introduction

Implementation theory seeks to understand under what conditions a social choice rule can be implemented as the equilibrium outcome of a suitably designed mechanism. The rights-structure framework of Koray and Yildiz (2018) provides one such approach, where a society's code of rights determines the admissible transitions between states. Yildiz (2013) extended this framework to two-stage rights structures and showed that the Nash bargaining solution is implementable in this richer environment. These contributions highlight rights-based models as an alternative to the classical mechanism design approach.

In this paper, we study implementation via two-stage rights structures in the standard setting with a finite set of alternatives and linear order preference profiles. We begin by establishing necessary conditions for a social choice rule to be implementable. In particular, we introduce *image consistency*, which serves as a necessary condition by ruling out violations of image monotonicity that depend only on changes outside the image of the rule. We also show that the *elimination property* is necessary: if an alternative is dropped even though all agents improve its position, this must be explained by the presence of another alternative whose position changes in a way that justifies the exclusion. These conditions allow us to focus on the essential sources of failure of monotonicity.

We then develop a stronger sufficient condition, the *replacement property*, which, together with Pareto consistency, guarantees Γ^2-implementability. This condition captures the idea that whenever an alternative is excluded despite being improved, it must be replaced by another alternative that was not previously selected. The replacement property, combined with Pareto consistency, thus ensures that two-stage rights structures can implement rules that cannot be implemented through Nash implementation or one-stage rights structures.

Along the way, we also obtain some useful side results. We show that rules can sometimes be decomposed into simpler image-monotonic components, which helps explain why two-stage structures can handle them. We also note that *image unanimity*, a natural strengthening of unanimity restricted to the image of a rule, plays a role in linking these conditions to existing results.

As an illustration, we show that the plurality rule with fixed tie-breaking—which fails both Nash implementability and one-stage rights-structure implementability—satisfies our sufficient condition and is therefore Γ^2-implementable. More generally, our analysis shows how two-stage rights structures enlarge the set of implementable rules while still respecting the monotonicity principles central to implementation theory.

Taken together, these results show both the new possibilities and the limits of two-stage rights structures. They help characterize which social choice rules can be implemented in rights-based models.

2.2. Definitions and Examples

Let $N = \{1, \ldots, n\}$ be a nonempty finite set of agents. We refer to each subset of N as a coalition, denoted by K. Let A be a nonempty finite set of alternatives. For each agent $i \in N$, let u_i denote the preference relation of agent i on A—that is, a complete, transitive, and antisymmetric binary relation (a linear order). A preference profile is given by $u = (u_1, \ldots, u_n)$. Let $\mathcal{P}$ denote the set of all preference profiles on A. A social choice rule is a correspondence $F : \mathcal{P} \to 2^A \setminus \{\emptyset\}$ that assigns to each preference profile $u \in \mathcal{P}$ a nonempty subset of alternatives $F(u) \subseteq A$.

Following Koray and Yildiz (2018), a **rights structure** is a triplet (S, h, γ), where S is a nonempty set, called the *state space*, whose elements are referred to as *states*, $h : S \to A$ is a function that maps each state to an alternative, called the *outcome function*, and γ is a *code of rights* that assigns, for each pair of distinct states $s, t \in S$, a family of coalitions $\gamma(s, t) \subseteq 2^N$. Each coalition in $\gamma(s, t)$ is entitled to approve a change from s to t. More formally:

A rights structure $\Gamma = (S, h, \gamma)$ consists of:

- a *state space* S,
- an *outcome function* $h : S \to A$,
- a *code of rights* $\gamma : \{(s, t) \in S \times S \mid s \neq t\} \to 2^{2^N}$.

Given h and a preference profile u, for every pair of states s, t with distinct outcomes $h(s)$ and $h(t)$, a coalition K **prefers** t to s if and only if, for every $i \in K$, $h(t)\, u_i\, h(s)$. We write $h(t)\, u_K\, h(s)$ to denote this.

Definition 1 *Given a rights structure $\Gamma = (S, h, \gamma)$ and a preference profile $u \in \mathcal{P}$, a state $s \in S$ is a Γ-**equilibrium** at u if for every $t \in S \setminus \{s\}$, there does not exist a coalition $K \in \gamma(s,t)$ such that $h(t)\, u_K\, h(s)$.*

A state s is a Γ-equilibrium at u if no coalition that is entitled to move from s to t actually prefers t to s. We denote the set of Γ-equilibria at u by $E(\Gamma, u)$.

Definition 2 *A social choice rule (SCR) F is Γ-**implementable** if there exists a rights structure $\Gamma = (S, h, \gamma)$ such that, for each $u \in \mathcal{P}$, $F(u) = h(E(\Gamma, u))$.*

Let $\mathrm{Im}(F)$ denote the set of all alternatives that appear in the outcome of F, i.e., $\mathrm{Im}(F) = \{\, a \in A : a \in F(u) \text{ for some } u \in \mathcal{P} \,\}$. Note that this set is not exactly the image of the correspondence F, since the formal image of a correspondence is the collection $\{F(u) : u \in \mathcal{P}\}$. With a slight abuse of terminology, however, we will refer to $\mathrm{Im}(F)$ as the image of F.

The lower contour set of u_i with respect to $a \in A$, denoted by $L(u_i, a)$, is the set of alternatives to which a is preferred by agent i at the preference profile u, i.e., $L(u_i, a) = \{\, b \in A : a\, u_i\, b \,\}$.

According to Koray and Yildiz (2018), the following monotonicity condition characterizes implementation by rights structures. We present the condition in formal terms.

Definition 3 *An SCR F is **image monotonic** if, for all $u, v \in \mathcal{P}$ and all $a \in F(u)$, we have $a \in F(v)$ whenever, for all $i \in N$, $L(u_i, a) \cap \mathrm{Im}(F) \subseteq L(v_i, a)$.*

Theorem 1 (Koray & Yildiz (2018)) *An SCR F is image monotonic if and only if it is Γ-implementable.*

We note that SCRs are taken to be nonempty-valued, both here and in Koray and Yildiz (2018). Nevertheless, their characterization remains valid even if the SCR F is allowed to be empty-valued at some profiles. In what follows, whenever we invoke this result, we shall—with a slight abuse of the nonempty-valued constraint—also apply it to image monotonic SCRs that may be empty-valued at some profiles.

Yildiz (2013) introduces implementation via two-stage rights structures. A **two-stage rights structure**, denoted by Γ^2, is a quadruple $(S, h, \gamma^1, \gamma^2)$. It extends a standard rights structure by introducing a second-stage code of rights γ^2, which may differ from the first-stage code of rights γ^1. At the first stage, the rights structure $\Gamma^1 = (S, h, \gamma^1)$ is applied, and the non-equilibrium states are eliminated, while the equilibrium states $E(\Gamma^1, u)$ proceed to the second stage. At the second stage, the rights structure $(E(\Gamma^1, u), h, \gamma^2)$ is applied, and the resulting equilibrium states determine the equilibria of Γ^2. The second-stage code of rights γ^2 serves to refine the set of equilibrium outcomes generated by γ^1 in the first stage, potentially eliminating some outcomes that would otherwise survive. A social choice rule is said to be implemented by a two-stage rights structure if, at each preference profile, it selects the equilibrium outcomes induced by the equilibrium states via the outcome function h.

Definition 4 *Let $\Gamma^2 = (S, h, \gamma^1, \gamma^2)$ be a two-stage rights structure and let u be a preference profile. A state $s \in S$ is a Γ^2-equilibrium at u, denoted by $s \in E(\Gamma^2, u)$, if*

(i) $s \in E(\Gamma^1, u)$, where $\Gamma^1 = (S, h, \gamma^1)$, and

(ii) for all $t \in E(\Gamma^1, u)$, there does not exist a coalition $K \in \gamma^2(s, t)$ such that $h(t)\, u_K\, h(s)$.

Definition 5 *An SCR F is implementable via a two-stage rights structure, called*

Γ^2***-implementable****, if there exists a two-stage rights structure* Γ^2 *such that, for all* $u \in \mathcal{P},\ F(u) = h(E(\Gamma^2, u))$.

Yildiz (2013) shows that the Nash bargaining solution is implementable via two-stage rights structures. The environment he considers involves two agents with preferences over lotteries on a compact and convex set of deterministic prizes in a finite-dimensional Euclidean space. In contrast, in this paper we focus on the standard environment with a finite set of alternatives and linear order profiles, and we provide a characterization of two-stage rights structures in this setting.

Let F be a Γ-implementable social choice rule, and let $\Gamma = (S, h, \gamma)$ be a rights structure that implements F. Then, trivially, F is also Γ^2-implementable by the two-stage rights structure $\Gamma^2 = (S, h, \gamma, \gamma^2)$, where $\gamma^2(s, t) = \emptyset$ for all $s, t \in S$.

We conclude this section with an example of an SCR that is not Γ^2-implementable, showing that the scope of two-stage rights structures is limited.

Example 1 *Let* $A = \{a, b, c\}$ *be the set of alternatives and* $N = \{1, 2, 3\}$ *the set of individuals. Define the social choice rule* F *by*

$$F(u) = \begin{cases} \{b\}, & \text{if } b \text{ is ranked first by all } i \in N \text{ under } u_i, \\ \{a\}, & \text{otherwise.} \end{cases}$$

Thus, F *selects* b *only under unanimous top ranking, and* a *in all other cases.*

Now consider the two preference profiles

u				v		
1	2	3		1	2	3
b	b	b		c	c	c
c	c	c		b	b	b
a	a	a		a	a	a

Since $F(v) = \{a\}$ while $F(u) = \{b\}$, the rule fails image monotonicity and is therefore not Γ-implementable. We now show that F is not Γ^2-implementable either. Suppose, to the contrary, that F is Γ^2-implementable by a two-stage rights structure $\Gamma^2 = (S, h, \gamma^1, \gamma^2)$.

First, suppose there exists $s \in S$ such that $h(s) = c$. Then s survives both stages at v, implying $s \in E(\Gamma^2, v)$. Hence $c \in h(E(\Gamma^2, v))$, contradicting $F(v) = \{a\}$.

Next, suppose there does not exist $s \in S$ such that $h(s) = c$. Since $F(u) = h(E(\Gamma^2, u)) = \{b\}$, there exists $t \in S$ with $h(t) = b$. At v, for any $s \neq t$ and any $K \in \gamma^1(t, s)$, we have $b\, v_K\, h(s)$, so $t \in E(\Gamma^1, v)$. Moreover, for any $s \in E(\Gamma^1, v) \setminus \{t\}$ and any $K \in \gamma^2(t, s)$, we again have $b\, v_K\, h(s)$, so $t \in E(\Gamma^2, v)$. Hence $b \in h(E(\Gamma^2, v))$, contradicting $F(v) = \{a\}$.

Thus, F is not Γ^2-implementable.

2.3. A Necessary Condition and Decomposability

In this section we introduce further definitions, establish a necessary condition for Γ^2-implementability, and present decomposability results.

Definition 6 *An SCR F satisfies **image unanimity** if, for all $a \in \mathrm{Im}(F)$ and all $u \in \mathcal{P}$, we have $a \in F(u)$ whenever a is ranked top by all $i \in N$ within* $\mathrm{Im}(F)$.

That is, any alternative in the image of F must be selected whenever it is unanimously ranked top within $\mathrm{Im}(F)$. This property is also a necessary condition for Γ^2-implementability.

Proposition 1 *If an SCR F is Γ^2-implementable, then F satisfies image unanimity.*

Proof. Let $a \in \mathrm{Im}(F)$ and $u \in \mathcal{P}$ be a preference profile such that a is ranked top within $\mathrm{Im}(F)$ by all individuals. Since F is Γ^2-implementable and $a \in \mathrm{Im}(F)$, there exists $s \in S$ such that $h(s) = a$.

Clearly, s cannot be eliminated at any stage by any state t with $h(t) \in \mathrm{Im}(F)$, since a is unanimously top-ranked in $\mathrm{Im}(F)$.

If s were eliminated at either stage by some state t with $h(t) \notin \mathrm{Im}(F)$, then at the profile v where every agent top-ranks $h(t)$, we would have $t \in E(\Gamma^2, v)$ and hence $h(t) \in F(v)$. But this contradicts the fact that $F(v) \subseteq \mathrm{Im}(F)$.

Therefore $s \in E(\Gamma^2, u)$, which implies $a \in F(u)$. Hence F satisfies image unanimity.

■

Definition 7 *An SCR F is **Pareto consistent** if, for all $u \in \mathcal{P}$ and all $a \in A$, we have $a \in F(u)$ whenever there exists $b \in F(u)$ such that a Pareto dominates b at u.*

That is, if some alternative in $F(u)$ is Pareto dominated, then the dominating alternatives must also be included in $F(u)$. However, Pareto consistency is not a necessary condition for two-stage implementation. There exist SCRs that satisfy image monotonicity and image unanimity but fail Pareto consistency. As an example, the constant social choice rule F, defined by $F(u) = \{a\}$ for all $u \in \mathcal{P}$ and for some fixed $a \in A$, is Γ-implementable, and hence Γ^2-implementable, but not Pareto consistent. We introduce Pareto consistency here because it will play a role in the decomposition results that follow.

We now present an example of an SCR F that is Γ^2-implementable. The following example will play a role in the decomposability results developed in the next part.

Example 2 *Let $A = \{a, b, c\}$ be the set of alternatives and $N = \{1, 2, 3\}$ the set of individuals. Define the social choice rule F by*

$$F(u) = \begin{cases} \{x\} & \textit{if } x \textit{ is the top choice of agents } 1 \textit{ and } 2, \\ \{y\} & \textit{otherwise, where } y \textit{ is the top choice of agent } 3. \end{cases}$$

Thus, F selects the common top alternative of agents 1 and 2 whenever they agree, and otherwise it selects the top alternative of agent 3.

Now consider the two preference profiles $u, v \in \mathcal{P}$:

u			v		
1	2	3	1	2	3
a	b	c	b	b	c
b	a	b	a	a	b
c	c	a	c	c	a

This SCR F satisfies image unanimity, but we show below that it does not satisfy image monotonicity, since $F(u) = \{c\}$ and $F(v) = \{b\}$. Let $S_1 = \{a, b, c\}$ and $S_2 =$

$\{(a,3),(b,3),(c,3)\}$. *The following two-stage rights structure,* $\Gamma^2 = (S,h,\gamma^1,\gamma^2)$, Γ^2*-implements the social choice rule* F*, where:*

- $S = S_1 \cup S_2$.
- h *is defined by*

$$h(s) = \begin{cases} s, & \textit{if } s \in S_1, \\ x, & \textit{if } s = (x,3) \in S_2. \end{cases}$$

- γ^1 *is defined by*

$$\gamma^1(s,t) = \begin{cases} \{\{1\},\{2\}\}, & \textit{if } s,t \in S_1, \\ \{\{3\}\}, & \textit{if } s,t \in S_2, \\ \emptyset, & \textit{otherwise.} \end{cases}$$

- γ^2 *is defined by*

$$\gamma^2(s,t) = \begin{cases} \{\{1\},\{2\}\}, & \textit{if } s \in S_2 \textit{ and } t \in S_1, \\ \emptyset, & \textit{otherwise.} \end{cases}$$

Now let u *be the preference profile such that agents* 1 *and* 2 *agree on their top choice* x*, i.e.,* $F(u) = \{x\}$. *At the first stage, from* S_1 *only* x *survives, and from* S_2 *only* $(y,3)$ *survives, where* y *is the top choice of agent* 3. *Since* $\{1\} \in \gamma^2((y,3),x)$ *and* $x\, u_1\, y$*, the state* $(y,3)$ *is eliminated at the second stage. Furthermore, since there is no transition from* S_1 *into* S_2 *at the second stage, we have* $E(\Gamma^2,u) = \{x\}$. *Thus,* $F(u) = h(E(\Gamma^2,u)) = \{x\}$.

Now let u *be the preference profile such that agents* 1 *and* 2 *do not agree on their top choice, so* $F(u) = \{y\}$*, where* y *is the top choice of agent* 3. *At the first stage, no state in* S_1 *survives: since agents* 1 *and* 2 *disagree, the top choice of agent* 1 *eliminates the top choice of agent 2 and vice versa. Only* $(y, 3)$ *from* S_2 *survives. Since* $(y, 3)$ *is the only state that survived from the first stage, it is also the only state that remains at the second stage. Thus,* $E(\Gamma^2, u) = \{(y, 3)\}$*, and therefore* $F(u) = h(E(\Gamma^2, u)) = \{y\}$.

The idea behind this example is as follows: the SCR F can be decomposed into two image-monotonic rules, one based on the unanimity of the first two agents and the other on the dictatorial rule of the third agent. Since both of these rules are Γ-implementable, we may use the rights structures that implement them. The following definition and theorem generalize this example.

Definition 8 *An SCR* F *is* ***decomposable into image-monotonic rules*** *if there exist image-monotonic rules* F_1, F_2 *such that, for all preference profiles* $u \in \mathcal{P}$*,*

- *If* $F_1(u) \neq \emptyset$*, then* $F(u) = F_1(u)$.
- *If* $F_1(u) = \emptyset$*, then* $F(u) = F_2(u)$.

That is, F coincides with F_1 whenever F_1 selects a nonempty set, and otherwise with F_2.

Theorem 2 *If an SCR* F *is Pareto consistent and decomposable into image-monotonic rules, then* F *is* Γ^2*-implementable.*

Proof. Assume that F is Pareto consistent and is decomposable into image-monotonic rules F_1 and F_2. Let $\Gamma_1 = (S_1, h_1, \gamma_1)$ and $\Gamma_2 = (S_2, h_2, \gamma_2)$ be rights structures that implement F_1 and F_2, respectively. Without loss of generality, if $S_1 \cap S_2 \neq \emptyset$, we may simply relabel the states in S_1 so that $S_1 \cap S_2 = \emptyset$.

The following two-stage rights structure, $\Gamma^2 = (S, h, \gamma^1, \gamma^2)$, implements the social choice rule F, where:

- $S = S_1 \cup S_2$
- h is defined by

$$h(s) = \begin{cases} h_1(s), & \text{if } s \in S_1, \\ h_2(s), & \text{if } s \in S_2. \end{cases}$$

- γ^1 is defined by

$$\gamma^1(s,t) = \begin{cases} \gamma_1(s,t), & \text{if } s,t \in S_1, \\ \gamma_2(s,t), & \text{if } s,t \in S_2, \\ \emptyset, & \text{otherwise.} \end{cases}$$

- γ^2 is defined by

$$\gamma^2(s,t) = \begin{cases} \{\{i\} : i \in N\}, & \text{if } s \in S_2 \text{ and } t \in S_1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Fix $u \in \mathcal{P}$. Suppose $F_1(u) = \emptyset$. By decomposability, $F(u) = F_2(u)$. At the first stage, no state in S_1 survives (since $F_1(u) = \emptyset$), while in S_2 the surviving states are exactly the equilibria of $\Gamma_2 = (S_2, h_2, \gamma_2)$; that is, $E(\Gamma^1, u) \cap S_2 = E(\Gamma_2, u)$ and $E(\Gamma^1, u) \cap S_1 = \emptyset$. At the second stage, γ^2 only permits moves from S_2 to S_1, but there are no surviving states in S_1, so the stage-one equilibria persist: $E(\Gamma^2, u) = E(\Gamma_2, u)$. Therefore, $h(E(\Gamma^2, u)) = h_2(E(\Gamma_2, u)) = F_2(u) = F(u)$.

Now suppose $F_1(u) \neq \emptyset$. Then, by decomposability, $F(u) = F_1(u)$. Any equilibrium state $s \in S_1$ of $\Gamma_1 = (S_1, h_1, \gamma_1)$ survives both stages of Γ^2, so $h(E(\Gamma^2, u)) \supseteq F(u)$.

Suppose that there exists $t \in S_2$ such that $t \in E(\Gamma^2, u)$ but $h(t) \notin F(u)$. Since

$F_1(u) \neq \emptyset$, there exists $s \in S_1$ with $s \in E(\Gamma^2, u)$.

If some agent i prefers $h(s)$ to $h(t)$, then because $\{i\} \in \gamma^2(t, s)$, state t would be eliminated at the second stage, contradicting $t \in E(\Gamma^2, u)$.

Otherwise, $h(t)$ strictly Pareto dominates $h(s)$. Since $h(s) \in F(u)$ and F is Pareto consistent, it follows that $h(t) \in F(u)$, again contradicting $h(t) \notin F(u)$.

Therefore no such t exists, so $h(E(\Gamma^2, u)) \subseteq F(u)$. Together with $h(E(\Gamma^2, u)) \supseteq F(u)$, we conclude $F(u) = h(E(\Gamma^2, u))$. ■

Now, the idea of decomposability and the above theorem can be extended to k image-monotonic rules in a recursive manner.

Definition 9 *An SCR F is **k-decomposable into image-monotonic rules** if there exist image-monotonic rules $F_1, F_2, \ldots, F_k$ such that, for all preference profiles $u \in \mathcal{P}$,*

$$F(u) = F_j(u) \quad \textit{where } j = \min\{\, m \in \{1, \ldots, k\} : F_m(u) \neq \emptyset \,\}.$$

That is, F coincides with F_1 if possible, otherwise with F_2, and so on, until F_k.

Theorem 3 *If an SCR F is Pareto consistent and k-decomposable into image-monotonic rules, then F is Γ^2-implementable.*

Proof. The argument mimics the proof of the preceding theorem, which corresponds to the case $k = 2$. The general case follows by applying the same reasoning recursively across $F_1, \ldots, F_k$. ■

We close this section with a trivial result concerning the union of Γ-implementable rules.

Theorem 4 *If there exist image-monotonic rules $F_1, F_2, \ldots, F_k$ such that $F(u) = \bigcup_{i=1}^{k} F_i(u)$ for all $u \in \mathcal{P}$, then F is Γ^2-implementable.*

Proof. Image monotonicity is preserved under union. $\blacksquare$

2.4. Non–Image-Monotonic Social Choice Rules

Since an SCR F is image monotonic if and only if it is Γ-implementable, we now turn our attention to non–image-monotonic social choice rules.

Image monotonicity requires that if $a \in F(u)$ and, for every $i \in N$, the position of a relative to the alternatives in $\mathrm{Im}(F)$ does not worsen when moving from u to v, then $a \in F(v)$. Therefore, if a social choice rule violates image monotonicity, there exist preference profiles u and v and an alternative $a \in A$ such that $a \in F(u)$ but $a \notin F(v)$, even though $L(u_i, a) \cap \mathrm{Im}(F) \subseteq L(v_i, a)$ for all $i \in N$. That is, for each $i \in N$, the only changes in preferences involve either rearranging the unchosen alternatives (those in $A \setminus \mathrm{Im}(F)$), or altering the relative order among alternatives in $\mathrm{Im}(F)$ while maintaining $L(u_i, a) \cap \mathrm{Im}(F) \subseteq L(v_i, a)$, or by both.

We now show that if there exists a violation of image monotonicity that results solely from reordering alternatives outside $\mathrm{Im}(F)$, then the social choice rule is not Γ^2-implementable.

Definition 10 *An SCR F is **image-consistent** if, for all $u, v \in \mathcal{P}$, we have $F(u) = F(v)$ whenever*

$$u|_{\mathrm{Im}(F)} = v|_{\mathrm{Im}(F)}.$$

Proposition 2 *If an SCR F is Γ^2-implementable, then F is image-consistent.*

Proof. Suppose, for contradiction, that there exists a rights structure $\Gamma^2 = (S, h, \gamma^1, \gamma^2)$ that Γ^2-implements F, but F is not image-consistent. Then there exist preference profiles $u, v \in \mathcal{P}$ and an alternative $a \in F(u)$ such that $u|_{\mathrm{Im}(F)} = v|_{\mathrm{Im}(F)}$, but $a \notin F(v)$.

Since $a \in F(u)$, there exists a state $s \in S$ such that $s \in E(\Gamma^2, u)$ and $h(s) = a$. But since $a \notin F(v)$, it follows that $s \notin E(\Gamma^2, v)$. Therefore, s must be eliminated at either the first or the second stage of Γ^2 under v.

Suppose s is eliminated by some $t \in S$ at the first stage. Then, there exists a coalition $K \in \gamma^1(s,t)$ such that $h(t)\, v_K\, h(s)$. Since F is Γ^2-implementable, it is image unanimous; hence, for the profile v where every agent top-ranks $h(t)$, we have $h(t) \in F(v) \subseteq \mathrm{Im}(F)$. Because $u_i|_{\mathrm{Im}(F)} = v_i|_{\mathrm{Im}(F)}$ for all $i \in K$, it follows that $h(t)\, u_K\, h(s)$, which contradicts the fact that $s \in E(\Gamma^2, u)$.

Suppose s is eliminated by some $t \in E(\Gamma^1, v)$ at the second stage. Then, there exists a coalition $\bar{K} \in \gamma^2(s,t)$ such that $h(t)\, v_{\bar{K}}\, h(s)$. Since $t \in E(\Gamma^1, v)$, we have that for each $w \in S \setminus \{t\}$, there is no coalition $K \in \gamma^1(t,w)$ such that $h(w)\, v_K\, h(t)$. Again, by image unanimity, for each such w we have $h(w) \in \mathrm{Im}(F)$. Because $u|_{\mathrm{Im}(F)} = v|_{\mathrm{Im}(F)}$, it follows that for each $w \in S \setminus \{t\}$, there is no coalition $K \in \gamma^1(t,w)$ such that $h(w)\, u_K\, h(t)$. Thus, $t \in E(\Gamma^1, u)$ as well. Since $u_i|_{\mathrm{Im}(F)} = v_i|_{\mathrm{Im}(F)}$ for all $i \in \bar{K}$, it also follows that $h(t)\, u_{\bar{K}}\, h(s)$. Therefore, s must be eliminated at the second stage under u by t, contradicting the fact that $s \in E(\Gamma^2, u)$. ■

The previous proposition establishes only a *necessary* condition for Γ^2-implementability: every Γ^2-implementable SCR must be image-consistent, but not every image-consistent SCR is Γ^2-implementable. The next result shows that combining image-consistency with Maskin monotonicity restores image monotonicity.

Definition 11 *An SCR F is* ***Maskin monotonic*** *if, for all $u, v \in \mathcal{P}$ and all $a \in F(u)$, we have $a \in F(v)$ whenever*

$$L(u_i, a) \subseteq L(v_i, a) \quad \textit{for all } i \in N.$$

Proposition 3 *If an SCR F is both Maskin monotonic and image-consistent, then F is image monotonic, and therefore Γ^2-implementable.*

Proof. Suppose, for contradiction, that F is both Maskin monotonic and image-consistent, but not image monotonic. Then there exist preference profiles $u, v \in \mathcal{P}$ and an alternative $a \in F(u)$ such that

$$L(u_i, a) \cap \mathrm{Im}(F) \subseteq L(v_i, a) \quad \text{for all } i \in N,$$

yet $a \notin F(v)$.

Now construct a new profile $u^* \in \mathcal{P}$ such that $u^*|_{\mathrm{Im}(F)} = u|_{\mathrm{Im}(F)}$, and for all $i \in N$, all $x \in A \setminus \mathrm{Im}(F)$, and all $y \in \mathrm{Im}(F)$, we have $x\, u_i^*\, y$. That is, every non-image alternative is ranked strictly above all image alternatives. Then, since $a \in F(u)$, by image-consistency we have $a \in F(u^*)$.

Now fix any $i \in N$, and take any $x \in L(u_i^*, a)$. If $x \in \mathrm{Im}(F)$, then since $u^*|_{\mathrm{Im}(F)} = u|_{\mathrm{Im}(F)}$, we have $x \in L(u_i, a)$, hence $x \in L(v_i, a)$ by assumption. If instead $x \notin \mathrm{Im}(F)$, then by construction $x\, u_i^*\, a$, so such an x cannot belong to $L(u_i^*, a)$. Thus every element of $L(u_i^*, a)$ lies in $\mathrm{Im}(F)$, and we conclude $L(u_i^*, a) \subseteq L(v_i, a)$ for all $i \in N$.

Since $a \in F(u^*)$, and $L(u_i^*, a) \subseteq L(v_i, a)$ for all $i \in N$, Maskin monotonicity implies $a \in F(v)$, contradicting our assumption. Thus F is image monotonic. ■

The following items illustrate the relationships between Maskin monotonicity (MM), image monotonicity (IM), image-consistency (IC), and Γ^2-implementability.

(a) MM, not IC (therefore not Γ^2-implementable and not IM). Let $|A| \geq 3$. Define F so that, for every $u \in \mathcal{P}$, $F(u) = \{a, b\}$ if b is unanimously top at u, and $F(u) = \{a\}$ otherwise.

(b) MM and IC, not Γ^2-implementable (therefore not IM). There does not exist such a rule, since MM and IC together imply IM.

(c) MM and Γ^2-implementable (therefore IC), but not IM. There does not exist such a rule, since MM and IC together imply IM.

(d) IM (therefore MM, Γ^2-implementable, and IC). Let $F(u) = A$ for all $u \in \mathcal{P}$.

(e) Γ^2-implementable (therefore IC), not MM, not IM. Let $A = \{a, b, c\}$, $N = \{1, 2, 3\}$, and

$$F(u) = \begin{cases} \{x\} & \text{if } x \text{ is the top choice of agents 1 and 2,} \\ \{y\} & \text{otherwise, where } y \text{ is the top choice of agent 3.} \end{cases}$$

(f) IC, not MM and not Γ^2-implementable (therefore not IM). Let $A = \{a, b, c\}$, $N = \{1, 2, 3\}$. Define F so that, for every $u \in \mathcal{P}$, $F(u) = A$ if a is unanimously bottom at u, and $F(u) = \{b, c\}$ otherwise.

2.5. Elimination and Replacement Properties

Since image-consistency is a necessary condition for Γ^2-implementability, we restrict our attention to social choice rules that satisfy this property. This allows us to focus exclusively on preferences over alternatives in $\text{Im}(F)$. Accordingly, when working with a social choice rule F, we may, without loss of generality, ignore alternatives outside $\text{Im}(F)$ and treat the set of alternatives A as equal to $\text{Im}(F)$ for the remainder of the paper. Under this assumption, the formulations of image monotonicity and Maskin monotonicity coincide. Thus, since every Maskin monotonic social choice rule is Γ^2-implementable, we can instead focus on those that fail to satisfy Maskin monotonicity.

Definition 12 *A social choice rule F satisfies the **elimination property** if, for all preference profiles $u, v \in \mathcal{P}$ and all alternatives $a \in A$,*

$$a \in F(u), \quad a \notin F(v), \quad \textit{and} \quad L(u_i, a) \subseteq L(v_i, a) \textit{ for all } i \in N$$

imply that there exists $b \in A$ such that $L(v_i, b) \not\subseteq L(u_i, b)$ for some $i \in N$.

The elimination property ensures that if an alternative a is dropped when moving from profile u to v, despite all agents improving its position (that is, $L(u_i, a) \subseteq L(v_i, a)$ for all $i \in N$), then this exclusion must be explained by a new pairwise reversal in favor of some other alternative b. In particular, there exist an agent $i \in N$ and an alternative $c \in A$ such that $c \in L(v_i, b)$ but $c \notin L(u_i, b)$; that is, b newly outranks c for agent i.

Before presenting the formal statement and proof, we first provide a brief sketch of the argument to illustrate why the elimination property is necessary for

Γ^2-implementability. If an SCR is not Maskin monotonic, then there exist $u, v \in \mathcal{P}$ and an alternative $a \in A$ such that $a \in F(u)$, $a \notin F(v)$, and $L(u_i, a) \subseteq L(v_i, a)$ for all $i \in N$. The equilibrium state $s \in E(\Gamma^2, u)$ with $h(s) = a$ would also remain an equilibrium at v in the first stage. But since $a \notin F(v)$, s must be eliminated in the second stage at v. This requires the existence of some alternative b and state t that eliminates s at v but not at u. If, however, $L(v_i, b) \subseteq L(u_i, b)$ for all $i \in N$, then t would also survive the first stage at u and eliminate s there as well, contradicting $a \in F(u)$.

We now show that the elimination property is necessary for Γ^2-implementability.

Theorem 5 *If an SCR F is Γ^2-implementable, then F satisfies the elimination property.*

Proof. Let F be a Γ^2-implementable SCR, and let $\Gamma^2 = (S, h, \gamma^1, \gamma^2)$ implement F. Suppose, for contradiction, that F violates the elimination property. Then there exist preference profiles $u, v \in \mathcal{P}$ and an alternative $a \in A$ such that $a \in F(u)$, $a \notin F(v)$, and $L(u_i, a) \subseteq L(v_i, a)$ for all $i \in N$, but for all $b \in A$ we have $L(v_i, b) \subseteq L(u_i, b)$ for all $i \in N$; that is, no b satisfies the conclusion of the elimination property.

Since F is Γ^2-implementable, we have $F(u) = h(E(\Gamma^2, u))$. Because $a \in F(u)$, there exists $s \in E(\Gamma^2, u)$ with $h(s) = a$.

We claim that s survives the first stage at v. Suppose not. Then some state t with outcome $h(t) = b$ and some coalition $K_1 \in \gamma^1(t, s)$ satisfy $b\, v_{K_1}\, a$. But since $L(u_i, a) \subseteq L(v_i, a)$ for all $i \in N$, it follows that $b\, u_{K_1}\, a$. Hence t would eliminate s at u in the first stage, contradicting the fact that $s \in E(\Gamma^2, u)$. Therefore, our claim holds.

Since s survives the first stage at v but $a \notin F(v)$, it must be eliminated in the second stage at v. Thus there exists some state t with outcome b and a coalition $K_2 \in \gamma^2(t, s)$ such that $b\, v_{K_2}\, a$. Note that t must itself survive the first stage at v in order to eliminate s later.

We claim that t also survives the first stage at u. Suppose not. Then some state w with outcome c and some coalition $K_3 \in \gamma^1(t, w)$ satisfy $c\, u_{K_3}\, b$. But since we assumed that for all $b \in A$, $L(v_i, b) \subseteq L(u_i, b)$ for all $i \in N$, it follows that $c\, v_{K_3}\, b$. Hence w would also eliminate t at v, contradicting the fact that t survives the first stage at v. Therefore, this claim holds as well.

Thus t survives the first stage at u. But since $K_2 \in \gamma^2(t, s)$ and $b\, v_{K_2}\, a$, and given that $L(u_i, a) \subseteq L(v_i, a)$ for all $i \in N$, we also have $b\, u_{K_2}\, a$. Hence t would eliminate s in the second stage at u, contradicting $s \in E(\Gamma^2, u)$.

Therefore, F must satisfy the elimination property. ■

This establishes that the elimination property is indeed necessary for Γ^2-implementability. However, it is not sufficient on its own: there exist social choice rules that satisfy the elimination property but are not Γ^2-implementable. We now introduce an additional condition, called the replacement property, which together with Pareto consistency is sufficient for Γ^2-implementability. Nevertheless, we could not provide an example of a rule that satisfies the elimination property but fails to be Γ^2-implementable while also satisfying Pareto consistency.

Definition 13 *A social choice rule F satisfies the **replacement property** if, for all $u, v \in \mathcal{P}$ and all $a \in A$,*

$$a \in F(u), \quad a \notin F(v), \quad L(u_i, a) \subseteq L(v_i, a) \textit{ for all } i \in N$$

imply that there exists $b \in A$ *such that*

$$b \in F(v), \quad b \notin F(u), \quad L(v_i, b) \nsubseteq L(u_i, b) \textit{ for some } i \in N.$$

It is clear that the replacement property is stronger than the elimination property. It requires not only that there exists an alternative b such that $L(v_i, b) \nsubseteq L(u_i, b)$ for some agent i, thereby accounting for why a is no longer selected, but also that this alternative b is actually selected at v. Moreover, it must be a genuinely new selection, in the sense that $b \notin F(u)$. In this way, b effectively replaces a among the alternatives selected by F at profile v.

We now show that the replacement property, combined with Pareto consistency, is sufficient for Γ^2-implementability.

Theorem 6 *Let* F *be a Pareto consistent social choice rule. If* F *satisfies the replacement property, then* F *is* Γ^2*-implementable.*

Proof. Assume that F is Pareto consistent and satisfies the replacement property. We construct a two-stage rights structure that implements F.

The two-stage rights structure $\Gamma^2 = (S, h, \gamma^1, \gamma^2)$ that Γ^2-implements F is designed as follows:

$$S_1 = \big\{(a, u) \in A \times \mathcal{P} \bigm| a \in F(u) \text{ and } \forall x \in F(u),\ \forall v \in \mathcal{P},$$
$$(\forall i \in N,\ L(u_i, x) \subseteq L(v_i, x)) \Rightarrow x \in F(v)\big\}.$$

Intuitively, the set S_1 contains states (a, u) such that $a \in F(u)$, and, more importantly, every alternative $x \in F(u)$ satisfies Maskin monotonicity: whenever all agents improve the position of x when moving from u to some profile v, the alternative x remains selected at v. In other words, each alternative chosen at u continues to be chosen under any such monotonic change. In this way, S_1 represents states in which the selected alternatives do not violate Maskin monotonicity.

$$S_2 = \big\{(a, u) \in A \times \mathcal{P} \;\big|\; a \in F(u),\ \exists v \in \mathcal{P},\ \exists x \in F(u),$$
$$\text{with } x \notin F(v) \text{ and } \forall i \in N,\ L(u_i, x) \subseteq L(v_i, x)\big\}$$

Intuitively, the set S_2 consists of states (a, u) where $a \in F(u)$, but at least one alternative $x \in F(u)$ fails Maskin monotonicity: there exists a profile v such that $x \notin F(v)$ even though all agents improve the position of x when moving from u to v. In this case, every alternative in $F(u)$ is placed in S_2.

Let $S = S_1 \cup S_2$.

The outcome function h maps each $(a, u) \in S$ to a.

The first-stage code of rights γ^1 is defined as follows:

$$\text{For all } ((a, u), (b, v)) \in S \times S \text{ and all } i \in N,$$

$$\{i\} \in \gamma^1((a, u), (b, v)) \quad \text{if and only if} \quad a\, u_i\, b.$$

Lastly, γ^2 is defined as follows:

- For all $((a,u),(b,v)) \in S_2 \times S_1$ with $u \neq v$,

$$\gamma^2((a,u),(b,v)) = \begin{cases} \emptyset & \text{if } (b,u) \in S_2, \\ \{\{i\} \mid i \in N\} & \text{otherwise.} \end{cases}$$

- For all $((a,u),(b,v)) \in S_2 \times S_2$ with $u \neq v$,

$$\gamma^2((a,u),(b,v)) = \begin{cases} \emptyset & \text{if } L(v_i,b) \subseteq L(u_i,b) \text{ for all } i \in N, \\ \{\{i\} \mid i \in N\} & \text{otherwise.} \end{cases}$$

Case I. Consider a preference profile $u \in \mathcal{P}$ such that all alternatives $x \in F(u)$ satisfy Maskin monotonicity, that is, none are dropped under monotonic transformations. Then $(a,u) \in S_1$ for all $a \in F(u)$.

By the construction of γ^1, for all states $(x,u) \in S_1$, we have that (x,u) survives the first stage at u. Moreover, in this case there is no state $(x,u) \in S_2$ with $x \in F(u)$. Now consider any $(b,v) \in S_1$ with $v \neq u$. If $b \in F(u)$, then (b,v) may survive the first stage at u. Suppose instead that $b \notin F(u)$. If $(b,v) \in E(\Gamma^1,u)$, that is, (b,v) survives the first stage at u, then by the structure of γ^1, for all $i \in N$ and all $c \in A$, $b\, v_i\, c \Rightarrow b\, u_i\, c$; equivalently, $L(v_i,b) \subseteq L(u_i,b)$ for all $i \in N$. Since $(b,v) \in S_1$, we have $b \in F(v)$; together with $b \notin F(u)$, this violates Maskin monotonicity. By construction, such a case is represented by a state $(b,v) \in S_2$. Hence no such (b,v) exists in S_1.

Thus, the states $(x,u) \in S_1$ with $x \in F(u)$ certainly survive the first stage at u. In addition, there may also survive states of the form $(b,v) \in S_1$ with $v \neq u$ and $b \in F(u)$, as well as states $(b,v) \in S_2$ with $v \neq u$.

Now consider the second stage. Since there are no transitions from states in S_1 to any other state in S, all states $(x, u) \in S_1$ with $x \in F(u)$ survive the second stage and are therefore equilibrium states Hence, $F(u) \subseteq h(E(\Gamma^2, u))$.

Now suppose there exists $t \in S$ such that $t \in E(\Gamma^1, u)$ (that is, t survives the first stage) but $h(t) \notin F(u)$. Since the only possible candidates for such a state are of the form $(b, v) \in S_1$ with $v \neq u$ and $b \in F(u)$, or $(b, v) \in S_2$ with $v \neq u$, it must be that $t = (b, v) \in S_2$ for some $v \neq u$ with $h(t) = b \notin F(u)$.

Since $(b, v) \in E(\Gamma^1, u)$, it follows that for all $i \in N$ and all $c \in A$, if $b\, v_i\, c$, then $b\, u_i\, c$; that is, $L(v_i, b) \subseteq L(u_i, b)$. By the replacement property, there exists an alternative $a \in A$ such that $a \in F(u)$ and $a \notin F(v)$. Hence, the state $(a, u) \in S_1$ exists, while $(a, v) \notin S_2$.

Since $(a, v) \notin S_2$, it follows that $\{i\} \in \gamma^2((b, v), (a, u))$ for all $i \in N$. If there exists an agent i such that $a\, u_i\, b$, then (b, v) is eliminated by (a, u) in the second stage. Otherwise, we must have $b\, u_i\, a$ for all $i \in N$, and since $a \in F(u)$, Pareto consistency implies $b \in F(u)$, a contradiction. Therefore, a state $(b, v) \in E(\Gamma^1, u)$ with $h(b, v) \notin F(u)$ cannot survive to the second stage of Γ^2.

Consequently, for all $x \in F(u)$, the state $(x, u) \in S_1$ is an equilibrium. Moreover, for every $s \in E(\Gamma^2, u)$, we have $h(s) \in F(u)$. Hence, $F(u) = h(E(\Gamma^2, u))$.

Case II. Consider a preference profile $u \in \mathcal{P}$ such that there exists an alternative $x \in F(u)$ that violates Maskin monotonicity, that is, there exists a profile v with $x \notin F(v)$ even though $L(u_i, x) \subseteq L(v_i, x)$ for all $i \in N$. Then, for such a profile u, we have $(a, u) \in S_2$ for all $a \in F(u)$.

By the construction of γ^1, any state $(x, u) \in S_2$ survives the first stage at u, and, in this case, there is no state $(x, u) \in S_1$ with $x \in F(u)$. Now consider any $(b, v) \in S_1$

with $v \neq u$. If $b \in F(u)$, then (b, v) may survive the first stage at u. Suppose instead that $b \notin F(u)$. If $(b, v) \in E(\Gamma^1, u)$, that is, (b, v) survives the first stage at u, then for all $i \in N$ and all $c \in A$, if $b\, v_i\, c$, then $b\, u_i\, c$; equivalently, $L(v_i, b) \subseteq L(u_i, b)$ for all $i \in N$. Since $(b, v) \in S_1$, we have $b \in F(v)$; together with $b \notin F(u)$, this would violate Maskin monotonicity. By construction, such a case is represented by a state $(b, v) \in S_2$. Hence, no such (b, v) exists in S_1.

Thus, the states $(x, u) \in S_2$ with $x \in F(u)$ certainly survive the first stage at u. In addition, there may also survive states of the form $(b, v) \in S_1$ with $v \neq u$ and $b \in F(u)$, as well as states $(b, v) \in S_2$ with $v \neq u$.

In the second stage, the states $(x, u) \in S_2$ with $x \in F(u)$ can only be eliminated by states $(b, v) \in S_1$ with $v \neq u$ and $b \in F(u)$, or by states $(b, v) \in S_2$. If $(b, v) \in E(\Gamma^1, u) \cap S_1$, then $b \in F(u)$ and the state $(b, u) \in S_2$ exists. By the construction of γ^2, this implies $\gamma^2((x, u), (b, v)) = \emptyset$. Hence, (x, u) cannot be eliminated by such $(b, v) \in S_1$.

If $(b, v) \in E(\Gamma^1, u) \cap S_2$, then for all $i \in N$ and all $c \in A$, if $b\, v_i\, c$, then $b\, u_i\, c$; that is, $L(v_i, b) \subseteq L(u_i, b)$. By the construction of γ^2, this implies $\gamma^2((x, u), (b, v)) = \emptyset$. Hence, (x, u) cannot be eliminated by such $(b, v) \in S_2$.

Thus, all the states $(x, u) \in S_2$ with $x \in F(u)$ survive in the second stage and thus are equilibrium states. Hence, $F(u) \subseteq h(E(\Gamma^2, u))$.

Now suppose there exists $t \in S$ such that $t \in E(\Gamma^1, u)$ (that is, t survives the first stage) but $h(t) \notin F(u)$. Since the only possible candidates for such a state are of the form $(b, v) \in S_1$ with $v \neq u$ and $b \in F(u)$, or $(b, v) \in S_2$ with $v \neq u$, it must be that $t = (b, v) \in S_2$ for some $v \neq u$ with $h(t) = b \notin F(u)$.

Since $(b, v) \in E(\Gamma^1, u)$, it follows that for all $i \in N$ and all $c \in A$, if $b\, v_i\, c$, then $b\, u_i\, c$;

that is, $L(v_i, b) \subseteq L(u_i, b)$. By the replacement property, there exists $a \in A$ such that $a \in F(u)$, and $L(u_i, a) \nsubseteq L(v_i, a)$ for some $i \in N$.

It follows that $\{i\} \in \gamma^2((b, v), (a, u))$. If there exists an agent i such that $a\, u_i\, b$, then (b, v) is eliminated by (a, u) in the second stage. Otherwise, we must have $b\, u_i\, a$ for all $i \in N$, and since $a \in F(u)$, Pareto consistency implies $b \in F(u)$, a contradiction. Therefore, a state $(b, v) \in E(\Gamma^1, u)$ with $h(b, v) \notin F(u)$ cannot survive to the second stage of Γ^2.

Consequently, for any $x \in F(u)$, $(x, u) \in S_2$ is an equilibrium outcome. For any $s \in E(\Gamma^2, u)$, we have $h(s) \in F(u)$. Hence, $F(u) = h(E(\Gamma^2, u))$. ■

We now turn to an example. The plurality rule with tie-breaking in favor of agent 1 is neither Nash implementable nor Γ-implementable. However, it satisfies the assumptions of the theorem and is therefore Γ^2-implementable.

Example 3 *For a profile u and alternative $a \in A$, define the* plurality score *of a as*

$$n(a, u) \;=\; \#\{\, i \in N \mid L(u_i, a) = A \,\}.$$

The set of plurality winners is

$$M(u) \;=\; \{\, a \in A \mid n(a, u) \geq n(b, u) \textit{ for all } b \in A \,\}.$$

The ***plurality rule with tie-breaking in favor of agent 1*** *is the social choice rule F given by*

$$F(u) = \begin{cases} \{a\}, & \textit{if } M(u) = \{a\}, \\ \{\, a \in M(u) \mid M(u) \subseteq L(u_1, a) \,\}, & \textit{if } |M(u)| > 1. \end{cases}$$

We now show that this SCR is Pareto consistent and satisfies the replacement property, and is therefore Γ^2*-implementable.*

Pareto consistency requires that if a Pareto dominated alternative a *is chosen, then all alternatives that Pareto dominate* a *must also be chosen. Since our SCR always selects the top choice of at least one agent, it never selects a Pareto dominated alternative. Hence,* F *is trivially Pareto consistent.*

Now, take any two preference profiles $u, v \in \mathcal{P}$ *and any alternative* $a \in A$ *such that*

$$a \in F(u), \quad a \notin F(v), \quad \textit{and} \quad L(u_i, a) \subseteq L(v_i, a) \textit{ for all } i \in N.$$

First note that $n(a, v) \geq n(a, u)$*, since* $L(u_i, a) \subseteq L(v_i, a)$ *for all* $i \in N$*. Let* $F(v) = \{b\}$*. Then* $n(b, v) \geq n(a, v)$*, since* $b \in F(v)$*. Moreover, from* $a \in F(u)$ *we have* $n(a, u) \geq n(b, u)$*. Now suppose* $L(v_i, b) \subseteq L(u_i, b)$ *for all* $i \in N$*. Then* $n(b, u) \geq n(b, v)$*. So we obtain*

$$n(b, v) \ \geq \ n(a, v) \ \geq \ n(a, u) \ \geq \ n(b, u) \ \geq \ n(b, v),$$

and therefore

$$n(a, v) = n(a, u) = n(b, u) = n(b, v).$$

This means that a *and* b *have the same plurality score both at* u *and at* v*. But in one case the tie is broken in favor of* a*, while in the other it is broken in favor of* b*. Contradiction. Therefore,* $L(v_i, b) \nsubseteq L(u_i, b)$ *for some* $i \in N$*. Hence there exists an alternative* $b \in A$ *such that* $b \in F(v)$*,* $b \notin F(u)$*, and* $L(v_i, b) \nsubseteq L(u_i, b)$ *for some* $i \in N$*. It follows that* F *satisfies the replacement property and is therefore* Γ^2*-implementable.*

Most equilibrium concepts in game theory are myopic: a joint strategy is excluded whenever a player finds a profitable unilateral deviation, and it is retained whenever no such deviation exists. Two-stage rights-structure implementation can be seen as myopic in a similar sense. In the first stage, agents may block a state whenever they prefer another, regardless of whether the resulting state will survive as an equilibrium. The second stage, however, requires more: elimination must be justified by the existence of a two-stage equilibrium state that both lies within the rights of some agent and is strictly preferred. This motivates the following definition.

Definition 14 *A social choice rule F is said to be* ***implemented via an externally stable two-stage rights structure*** $\Gamma^2 = (S, h, \gamma^1, \gamma^2)$ *if the following conditions hold:*

(i) Γ^2 *implements F;*

(ii) for each $u \in \mathcal{P}$ and each $s \in E(\Gamma^1, u)$ with $s \notin E(\Gamma^2, u)$, there exist $t \in E(\Gamma^2, u)$ and $i \in N$ such that

$$\{i\} \in \gamma^2(s,t) \quad \textit{and} \quad h(t)\, u_i\, h(s).$$

Intuitively, external stability requires that any state which survives the first stage but does not remain an equilibrium in the two-stage structure must be actively eliminated in the second stage by an actual equilibrium state. In other words, if a state is ruled out in the second stage, this exclusion is not arbitrary: there must exist an equilibrium outcome that both has the right and provides at least one agent with a strict incentive to move away from the non-equilibrium state. This ensures that the elimination process in the second stage is justified by stable alternatives, rather than by other non-equilibrium states.

We now establish the precise connection between the replacement property and external stability. Without external stability, the replacement property combined with Pareto consistency provides a sufficient condition for two-stage implementation. Once external stability is imposed, the replacement property becomes necessary as well, but only in the case of social choice functions. In this setting it is both necessary and sufficient, yielding a complete characterization.

Theorem 7 *Let F be a Pareto consistent social choice function. Then F is implemented via an externally stable two-stage rights structure if and only if F satisfies the replacement property.*

Proof. Let F be a Pareto consistent social choice function.

Assume first that F satisfies the replacement property. By the previous theorem, F is Γ^2-implementable. Moreover, by construction, the two-stage rights structure is externally stable.

Now assume that F is implemented via an externally stable two-stage rights structure. Let $u, v \in \mathcal{P}$ and $a \in A$ be such that

$$a \in F(u), \quad a \notin F(v), \quad \text{and} \quad L(u_i, a) \subseteq L(v_i, a) \text{ for all } i \in N.$$

Since $a \in F(u)$, there exists $s \in E(\Gamma^2, u)$ with $h(s) = a$. If $s \in E(\Gamma^1, v)$, then s must be eliminated in the second stage at v, since $a \notin F(v)$. By external stability, there exist $t \in E(\Gamma^2, v)$ and $i \in N$ such that

$$\{i\} \in \gamma^2(s,t) \quad \text{and} \quad h(t)\, v_i\, h(s).$$

Let $h(t) = b$. Since $t \in E(\Gamma^2, v)$, we have $b \in F(v)$. Since F is a function and $a \in F(u)$, it follows that $b \notin F(u)$.

Suppose $L(v_i, b) \subseteq L(u_i, b)$ for all $i \in N$. Then t would also survive the first stage at u. Since $\{i\} \in \gamma^2(s, t)$ and $h(t)\, u_i\, h(s)$, the state t would eliminate s in the second stage at u. This contradicts the fact that $s \in E(\Gamma^2, u)$. Therefore, $L(v_i, b) \nsubseteq L(u_i, b)$ for some $i \in N$.

It follows that there exists $b \in F(v)$ with $b \notin F(u)$ and $L(v_i, b) \nsubseteq L(u_i, b)$ for some $i \in N$. Hence, F satisfies the replacement property. ■

2.6. Conclusion

This paper has studied implementation through two-stage rights structures in the standard framework with finitely many alternatives and linear preference profiles. We established necessary conditions for Γ^2-implementability, namely image consistency and the elimination property, and we introduced the replacement property, which together with Pareto consistency provides a sufficient condition. These results extend the class of implementable social choice rules beyond those covered by Nash or one-stage rights-structure implementation.

A central contribution of the analysis is the notion of image consistency, which rules out violations of image monotonicity that depend only on changes outside the image of the rule. This refinement allows us to restrict attention to the image of the social choice rule, where image monotonicity and Maskin monotonicity coincide. The elimination and replacement properties then provide a framework for understanding how failures of monotonicity within the image can still be addressed in two-stage structures.

Together, these findings clarify the role of two-stage rights structures as a mechanism for enlarging the set of implementable social choice rules while remaining faithful to the monotonicity principles central to implementation theory. They also show that rules such as plurality with fixed tie-breaking, which fail both Nash and one-stage rights-structure implementation, can nonetheless be implemented in this richer framework. More generally, the analysis contributes to the ongoing exploration of rights-based approaches as an alternative to classical mechanism design, underscoring how additional institutional structure affects the set of rules that can be implemented.

An interesting question for future work concerns the status of the plurality rule. While plurality satisfies the necessary conditions identified here, it does not satisfy the sufficient condition. Whether plurality is Γ^2-implementable therefore remains unresolved and points to a natural direction for further research.

CHAPTER 3

SELECTION-CLOSEDNESS[1]

3.1. Introduction

We imagine that a society facing a choice problem from a finite set A of alternatives is also to choose the social choice function (SCF) that will be employed in making the choice from A. Given that the society is aware of the choice problem from A, it is not unnatural to assume that its members will rank the available SCFs in a consequentialist way. That is, the societal preferences on the set A of alternatives will induce a preference profile on the set $\mathcal{A}$ of available SCFs in accordance with what they will choose from A.

When our society selects an SCF F from $\mathcal{A}$, one can now question the consistency of this choice by asking whether F would choose itself if it were employed in choosing from $\mathcal{A}$. In case F does not choose itself from $\mathcal{A}$ at the preference profile induced on $\mathcal{A}$ consequentially, it seems quite natural to ascribe this phenomenon to a lack of consistency on the part of the SCF F, as it rejects itself according to its own rationale. On the other hand, if F turns out to choose itself in this context, then it deserves being referred to as self-selective. This forms the main ground on which

[1]This chapter extends Koray and Senocak (2024) and has been prepared to satisfy the publication requirement for graduation.

the self-referential notion of self-selectivity introduced by Koray (2000) is based on. Self-selectivity of F is regarded as universal if it applies to all finite nonempty sets $\mathcal{A}$ of available SCFs containing F and to all possible preference profiles on the set A of alternatives.

The preferences of an agent on a set of available SCFs may have two different sources. In case the choice problem, to which the chosen SCF will be applied, is known to the agent, it is only natural to assume that the agent will care about what alternative will be chosen when the chosen SCF is employed in choosing an alternative. In such a context, in the eyes of our agent, "the proof of the SCF is in what alternative it chooses, as the proof of the pudding is in the eating". An agent, however, may also be concerned about other aspects of an SCF such as fairness, efficiency or other social values in assessing different SCFs along with consequentialism that a myopic self-interest-seeking behavior is confined to. This will be especially true in an environment, where it is vague in what particular instances the chosen choice rule is to be employed.

The self-selectivity of an SCF introduced by Koray (2000) is purely consequentialist. The main result of Koray (2000) is that a neutral SCF is universally self-selective if and only if it is either dictatorial or anti-dictatorial. If we restrict ourselves to unanimous SCFs only by considering that unanimity is something universally accepted, the result gets reduced to that a neutral and unanimous SCF is self-selective if and only if it is dictatorial. Koray and Unel (2003) and Koray and Slinko (2008) do not abandon the principle of consequentialism. They both deal with the problem of whether the impossibility can be avoided by giving up universality, i.e., by restricting the domain of available social choice rules to certain subfamilies of SCFs. Koray and Unel (2003) end up showing that the impossibility survives in the tops-only domain. Koray and Slinko (2008) specify a family of correspondences, which includes the Pareto correspondence, and restrict the available SCFs to singleton-valued refinements of these correspondences. They manage to avoid the absolute impossibility thereby, though to a minimal extent, as the only SCFs added to the self-selective class turn out to be partial anti-dictatorships on a restricted set of alternatives, where the restriction is endogenously inherited from the correspondence whose singleton-valued refinements are regarded as available.

We can hardly deny the importance of combining consequentialism with the agents' stances towards the social value content of an SCF. The above attempt to possibly avoid the impossibility result by restricting the available SCFs slightly touches upon such a combination, although it does not adopt it as its main axis. For example, confining the available SCFs to the singleton-valued refinements of the Pareto correspondence can be interpreted to reflect a societal consensus on the value of efficiency. Considering only tops-only SCFs as legitimate may similarly originate from a societal stance, which only cares about the individual first bests. However, none of these interpretations can be regarded as a satisfactory treatment of combining consequentialism with the social value attached to an SCF, as they do not take into account the possible differences that may and usually do exist in assessing the particular social value in question among individuals.

On the other hand, it is clear that the set $\mathcal{A}$ of available SCFs and the set A from which an alternative will be chosen by employing an SCF from $\mathcal{A}$ correspond to two different levels, the former of which can be thought of as the "constitutional" level, while the latter corresponds to the level of "ordinary legislation". In real life, constitutional amendments are designed to be relatively more difficult to make for the sake of stability, while the rules concerning legislation are relatively more flexible. Thus, it is natural that the SCFs employed at these two levels differ in order to comply with these qualitative differences between them. The models of constitutional stability in Barbera and Jackson (2004) and Acemoglu (2012) are based on such an approach.

In the self-selectivity framework, the choice of an SCF to be employed at the "legislative level" can be thought as happening at the "constitutional level". Testing an SCF F for self-selectivity requires that the legislative level SCF F is also being used for its own choice at the constitutional level. One of the lessons taught by Koray (2000) is that, in case the constitutional level considers any SCF as legitimate as any other, one would do better by not insisting on self-selectivity of an SCF, for otherwise one ends up with dictatorship (or anti-dictatorship) at both levels, which is not only quite undesirable, but also renders any distinction between these two levels superfluous.

In real life, constitutions do not consider all SCFs as equally legitimate. In fact, they severely restrict the SCFs that may be employed at the legislative level, presumably based on certain principles widely accepted by the society. We have already noted that the main inconsistency associated with an SCF, which is not universally self-selective, is that it rejects itself according to its own rationale. In other words, its rationale is self-destroying. If we now think of a rationale as a more general principle reflected at the constitutional level and associated with a whole family of SCFs rather than just a particular SCF, then the test whether it is self-destroying or not will, of course, be different than self-selectivity. Pareto efficient, Condorcet-type or tops-only SCFs exemplify families of SCFs based on a common rationale. For such a family $\mathscr{F}$ of SCFs, the test now becomes whether an SCF from $\mathscr{F}$ always chooses some member of $\mathscr{F}$ rather than just itself in the presence of rival SCFs from outside $\mathscr{F}$. This is the main idea behind selection-closedness introduced in Koray and Senocak (2024).

Koray and Unel (2003) and Koray and Slinko (2008) also restrict the SCFs to be employed in a particular choice from an alternative set A, but test them for self-selectivity rather than selection-closedness. Those two papers are focused on whether the impossibility of Koray (2000) can be avoided by giving up universality. The main aim of Koray and Senocak (2024) on selection-closedness is, however, to introduce a criterion whether a rationale as a general principle underlying a whole class of social choice rules rather than just an individual SCF is self-destroying or not. The possibility to avoid the impossibility of Koray (2000) thereby is regarded as a byproduct under this novel approach.

In Koray and Senocak (2024), along with giving up universality by confining ourselves to a set $\mathscr{F}$ of constitutionally acceptable SCFs, they also relax the consistency test by substituting "self-selectivity of a family of SCFs" for "individual self-selectivity". Specifically, given any finite nonempty set $\mathcal{A}$ of available SCFs, which also contains some members of $\mathscr{F}$, they do not insist any more that an SCF in $\mathscr{F} \cap \mathcal{A}$ selects itself from $\mathcal{A}$, but only require that it selects some SCF in $\mathscr{F} \cap \mathcal{A}$ possibly other than itself, i.e., some SCF that is both available and constitutionally acceptable. This is the main idea behind what is referred to as selection-closedness of a set $\mathscr{F}$ of SCFs.

It turns out that this kind of relaxation does not suffice to avoid the impossibility result without any further modifications, the intuition behind which can easily be understood as follows. When the set $\mathcal{A}$ of available SCFs contains a unique SCF F from $\mathscr{F}$, the selection-closedness of $\mathscr{F}$ dictates that F selects itself, as no other member of $\mathscr{F}$ is available. In other words, individual self-selectivity of an SCF is forced by selection-closedness in this particular context. It then follows that this effect does not stay local, but prevails universally, implying that a selection-closed family of SCFs either consists of some dictatorships and anti-dictatorships or is equal to the class of all neutral SCFs, which is trivially selection-closed.

In the first part of this chapter, we combine the approaches of Koray and Senocak (2024) and Koray and Ünel (2003) by testing weak selection-closedness within the tops-only domain. Koray and Senocak (2024) proposed weak selection-closedness as a relaxed consistency criterion for families of SCFs, while Koray and Ünel (2003) explored whether the impossibility of self-selectivity could be avoided under the restriction to tops-only rules. Our analysis brings these two directions together by examining whether the relaxation of consistency through weak selection-closedness yields different insights once the domain is limited to tops-only SCFs.

We next weaken the notion of selection-closedness in such a way that a member of the set $\mathscr{F}$ of constitutionally acceptable SCFs is now enabled to seek the support of some chosen "constitutional companions" from $\mathscr{F}$ to outrival a given set $\mathcal{A}$ of constitutionally unacceptable competitors. That is, given an SCF F in $\mathscr{F}$ together with a set $\mathcal{A}$ of "rivals" from outside $\mathscr{F}$, we now allow the inclusion of a suitably chosen finite subset $\mathcal{A}'$ of $\mathscr{F}$ into the set of available SCFs such that F selects itself from $\mathcal{A} \cup \mathcal{A}'$. Although it may not be obvious at first glance, weak selection-closedness turns out to be indeed weaker than selection-closedness.

This notion of weak selection-closedness turns out to be useful on two different grounds. One is that we now obtain a variety of families of SCFs that are weakly selection-closed in addition to sets of dictatorships and anti-dictatorships and the entire set of neutral SCFs. In case a family $\mathscr{F}$ of constitutionally acceptable SCFs also turns out to be weakly selection-closed, this can be interpreted as indicating an internal robustness of $\mathscr{F}$ in competing with SCFs from outside the family in addition to its social desirability.

Secondly, as weak selection-closedness distinguishes between families of SCFs, it equips us with a tool to determine which of these families are based on a rationale that is self-destroying in some sense. If we refer to a correspondence as weakly selection-closed in case the family of its singleton-valued refinements is weakly selection-closed, it turns out that neither the Pareto nor the Condorcet correspondence is weakly selection-closed, while the scoring correspondences turn out to be weakly selection closed under certain mild restrictions, in case the scoring vectors are strict. As a scoring correspondence is defined for alternative sets of any finite size in our framework, we have an infinite score sequence associated with each such correspondence. The mild restrictions above are concerned with the sequence of increments in the score sequence and differ dually depending upon whether the finite score vectors associated with the correspondence are right or left extendable (or both).

Next, we introduced a further weakening of selection-closedness. Given an SCF $F \in \mathscr{F}$ together with a set $\mathcal{A}$ of rivals from outside $\mathscr{F}$, we now allow the inclusion of a suitably chosen finite subset $\mathcal{A}' \subseteq \mathscr{F}$ into the set of available SCFs, with the requirement that F selects some member of $\mathcal{A}'$ from $\mathcal{A} \cup \mathcal{A}'$. The intuition here is that F no longer insists on defending itself directly but may rely on an ally from within $\mathscr{F}$ to prevail against outsiders. This stands in contrast to weak selection-closedness, where F was still required to secure its own survival once its constitutional companions were introduced.

The Pareto correspondence turns out to be weakly selection-closed under this new framework. To further illuminate the scope of the notion, we draw on two well-known properties from choice theory, independence of irrelevant alternatives (IIA) and independence of rejected alternatives (IRA). These properties provide a partial characterization of weak selection-closedness: if a social choice correspondence C satisfies both IIA and IRA, then the set of all singleton-valued refinements of C is universally weakly selection-closed.

3.2. Basic Notions

Let N be a finite nonempty society that will be arbitrary but kept fixed throughout the chapter. For any finite nonempty set A, we denote $\mathscr{L}(A)$ for the set of all linear orders on A. Letting $\mathbb{N}$ stand for the set of natural numbers as usual, for each $m \in \mathbb{N}$, we represent a set of m alternatives by $I_m = \{1, \ldots, m\}$.

A *social choice function* (SCF) F is a function

$$F : \bigcup_{m \in \mathbb{N}} \mathscr{L}(I_m)^N \to \mathbb{N}$$

such that for all $m \in \mathbb{N}$ and for all $R \in \mathscr{L}(I_m)^N$, one has $F(R) \in I_m$.

In order to extend the domain of an SCF to also cover $\mathscr{L}(A)^N$ for any finite nonempty set A, we will first recall the definition of neutrality of an SCF. Given $m \in \mathbb{N}$ and $R \in \mathscr{L}(I_m)^N$, for each permutation σ on I_m, the permuted linear order profile R_σ on I_m is defined via the following bicondition: For each agent $\alpha \in N$ and for all k, $l \in I_m$, $kR_\sigma^\alpha l$ if and only if $\sigma(k)R^\alpha\sigma(l)$. An SCF F is said to be *neutral* if and only if for each $m \in \mathbb{N}$ and each permutation σ on I_m, one has

$$\sigma(F(R_\sigma)) = F(R)$$

at any profile $R \in \mathscr{L}(I_m)^N$. We will denote the class of all neutral SCFs by $\mathcal{G}$.

Now let any neutral SCF $F \in \mathcal{G}$ and any finite set A of alternatives with $|A| = m \in \mathbb{N}$ be given. Consider a bijection $\mu : I_m \to A$. For any linear order profile $L \in \mathscr{L}(A)^N$, μ induces a linear order profile L_μ on I_m such that for each agent $\alpha \in N$ and for all k, $l \in I_m$, $kL_\mu^\alpha l$ if and only if $\mu(k)L^\alpha\mu(l)$. Now, we define $F(L) = \mu(F(L_\mu))$. It is clear that, for any two bijections $\mu, v : I_m \to A$, we have $\mu(F(L_\mu)) = v(F(L_v))$ as F

is neutral. That is, $F(L)$ does not depend upon the bijection μ employed.

We imagine now that a finite nonempty set $\mathcal{A}$ of SCFs is available to our society N, which is to make a choice from I_m and is endowed with a preference profile $R \in \mathscr{L}(I_m)^N$. Assuming that the agents in N will rank the available SCFs in $\mathcal{A}$ in a consequentialist way, i.e. in accordance with what alternatives these SCFs choose at R, R will induce a complete preorder profile on $\mathcal{A}$, as different SCFs in $\mathcal{A}$ may choose the same alternative at R. Formally, given $m \in \mathbb{N}$ and $R \in \mathscr{L}(I_m)^N$, the complete preorder profile $R_{\mathcal{A}}$ on $\mathcal{A}$ is defined via the following biconditionals: For each agent $\alpha \in N$ and for any $F_1, F_2 \in \mathcal{A}$, we set $F_1 R^{\alpha}_{\mathcal{A}} F_2$ if and only if $F_1(R) R^{\alpha} F_2(R)$. We refer to $R_{\mathcal{A}}$ as *the preference profile on $\mathcal{A}$ induced by R.*

As the domain of our SCFs consists of linear order profiles, they do not act on complete preorder profiles on $\mathcal{A}$ in the presence of non-singleton indifference classes. We thus associate with each preference profile reference profile $R_{\mathcal{A}}$ on $\mathcal{A}$ induced by R, a set $\mathscr{L}(\mathcal{A}, R)$, which consists of linear order profiles on $\mathcal{A}$ that are consistent with $R_{\mathcal{A}}$. Formally, given a complete preorder ρ on a finite, nonempty set A, a linear order λ on A is said to be *compatible* with by ρ if and only if, for all $x, y \in A$, $x \lambda y \implies x \rho y$. Given $m \in \mathbb{N}$, $R \in \mathscr{L}(I_m)^N$ and a nonempty finite subset $\mathcal{A}$ of $\mathcal{G}$, we set $\mathscr{L}(\mathcal{A}, R) = \{L \in \mathscr{L}(\mathcal{A})^N \mid L^{\alpha}$ is a linear order on $\mathcal{A}$ compatible with $R^{\alpha}_{\mathcal{A}}$ for each $\alpha \in N\}$. We refer to $\mathscr{L}(\mathcal{A}, R)$ as *the set of all linear order profiles on $\mathcal{A}$ induced by R.*

We now define self-selectivity for a neutral SCF, following Koray (2000), and selection-closedness for a set of neutral SCFs, following Koray and Senocak (2024).

Definition 15 *Given $F \in \mathcal{G}$, $m \in \mathbb{N}$, $R \in \mathscr{L}(I_m)^N$ and a finite subset $\mathcal{A}$ of $\mathcal{G}$ with $F \in \mathcal{A}$, we say that F is* self-selective *at R relative to $\mathcal{A}$ if and only if there exists some $L \in \mathscr{L}(\mathcal{A}, R)$ such that $F = F(L)$. Moreover, F is said to be* self-selective *at R*

if and only if F is self-selective at R relative to any finite subset $\mathcal{A}$ of $\mathcal{G}$ with $F \in \mathcal{A}$. Lastly, we say that F is universally self-selective *if and only if F is self-selective at each $R \in \bigcup_{m \in \mathbb{N}} \mathscr{L}(I_m)^N$.*

In Koray (2000), it is shown that a neutral SCF is universally self-selective if and only if it is dictatorial or anti-dictatorial.

Definition 16 *Given $\mathscr{F} \subseteq \mathcal{G}$, $m \in \mathbb{N}$, $R \in \mathscr{L}(I_m)^N$ and a finite subset $\mathcal{A}$ of $\mathcal{G}$, we say that $\mathscr{F}$ is* selection-closed *at R relative to $\mathcal{A}$ if and only if for all $F \in \mathscr{F} \cap \mathcal{A}$, there exists some $L \in \mathscr{L}(\mathcal{A}, R)$ such that $F(L) \in \mathscr{F}$. Moreover, $\mathscr{F}$ is said to be* selection-closed *at R if and only if $\mathscr{F}$ is selection-closed at R relative to any finite subset $\mathcal{A}$ of $\mathcal{G}$. Lastly, we say that $\mathscr{F}$ is universally* selection-closed *if and only if $\mathscr{F}$ is selection-closed at each $R \in \bigcup_{m \in \mathbb{N}} \mathscr{L}(I_m)^N$.*

In Koray and Senocak (2024), it is shown that a universally selection-closed family of SCFs either consists of some dictatorships and anti-dictatorships or is equal to the class of all neutral SCFs, which is trivially selection-closed.

3.3. On the Tops-Only Domain

In this part, we confine the available social choice functions to the tops-only domain and examine whether such restriction allow us to escape the dictatoriality result in Koray and Senocak (2024).

Given $m \in \mathbb{N}$, a nonempty finite set of alternatives I_m, a preference profile $R \in \mathscr{L}(I_m)^N$, an alternative $a \in I_m$ and some agent $i \in N$, let $L_i(a, R) = \{b \in A \mid aR_ib\}$ and refer to $L_i(a, R)$ as the lower contour set of a at R_i. We write $\tau(R_i) = a$ if and only if $L_i(a, R) = A$ and call $\tau(R_i)$ the top alternative of agent i at R_i. Also let $T(R)$

denotes the set of all top ranked alternatives at R, i.e. $T(R) = \{\tau(R_i) \mid i \in N\}$.

A SCF F is called *unanimous* if and only if for any $m \in \mathbb{N}$, $R \in \mathscr{L}(I_m)^N$ and $a \in I_m$, one has

$$(\forall i \in N : \tau(R_i) = a) \Rightarrow F(R) = a.$$

A SCF F is called *tops-only* if and only if for any $m \in \mathbb{N}$, $R, R' \in \mathscr{L}(I_m)^N$, one has

$$[\forall i \in N : \tau(R_i) = \tau(R_i')] \Rightarrow F(R) = F(R').$$

Let $\theta \subseteq \mathcal{G}$ stands for the set of all neutral and tops-only SCFs.

Definition 17 Given a nonempty subclass $\mathcal{T} \subseteq \mathcal{G}$, a family $\mathscr{F} \subseteq \mathcal{T}$, $m \in \mathbb{N}$, $R \in \mathscr{L}(I_m)^N$, and a nonempty finite subset $\mathcal{A} \subseteq \mathcal{T}$, we say that $\mathscr{F}$ is $\mathcal{T}$*-selection-closed at* R *relative to* $\mathcal{A}$ if and only if, for every $F \in \mathscr{F} \cap \mathcal{A}$, there exists $L \in \mathscr{L}(\mathcal{A}, R)$ such that $F(L) \in \mathscr{F}$. Moreover, $\mathscr{F}$ is $\mathcal{T}$*-selection-closed at* R if and only if $\mathscr{F}$ is $\mathcal{T}$-selection-closed at R relative to every nonempty finite subset $\mathcal{A} \subseteq \mathcal{T}$. Lastly, $\mathscr{F}$ is $\mathcal{T}$*-selection-closed* if and only if $\mathscr{F}$ is $\mathcal{T}$-selection-closed at each $R \in \bigcup_{m \in \mathbb{N}} \mathscr{L}(I_m)^N$.

Note that, when $\mathcal{T}$=$\mathcal{G}$, $\mathcal{T}$-selection-closedness coincides with universally selection-closedness. In Koray and Unel (2003), it is shown that a neutral and unanimous SCF is Θ-self-selective if and only if it is dictatorial. We will denote the set of all neutral θ-self-selective SCFs by $\mathcal{U}_\theta$.

Remark 1 *By definition* $\emptyset$ *and* Θ *are* Θ*-selection-closed.*

Remark 2 *Any subset of* $\mathcal{U}_\theta$ *is* Θ*-selection-closed.*

3.3.1. Results

Note that our definition of a tops-only SCF does not guarantee the choice of an alternative which is top-ranked by at least one agent at a given linear order profile, but only requires the invariance of the chosen alternative so long as the list (N-tuple) of alternatives top-ranked by agents stays the same. So, we note the following fact which will used throughout the discussion: for any $m \leq N+1$, there exists some neutral, unanimous $F \in \theta$ such that $F(R) \notin T(R)$ for some $R \in \mathscr{L}(I_m)^N$.

Let $\alpha \in N$, and let D_α and $D_{-\alpha} \in \mathcal{G}$ stand for dictatoriality and anti-dictatoriality of agent α, respectively. For each $m \in \mathbb{N}$ and at each $R \in \mathscr{L}(I_m)^N$, define

$$F_\alpha(R) = \begin{cases} D_\alpha(R) & m \neq 2 \\ D_{-\alpha}(R) & m = 2 \end{cases}$$

Note that F_α is *tops-only* and for all odd $m \in \mathbb{N}$ and $R \in \mathscr{L}(I_m)^N$, one has $F_\alpha(R) \in T(R)$.

The SCF F_α, which is clearly neutral, will turn out to be useful in our characterization of Θ-selection-closedness below.

Lemma 1 If $\mathscr{F}$ is a Θ-selection-closed subset of Θ and $F_\alpha \in \mathscr{F}$ for some $\alpha \in N$, then $\mathscr{F} = \Theta$.

Proof. Assume that $\mathscr{F} \subset \Theta$ is θ-selection-closed and $F_\alpha \in \mathscr{F}$ for some $\alpha \in N$. Now set $C_1 = \{F_1 \in \Theta \mid F_1(R) = D_\alpha(R)$ for all $R \in \mathscr{L}(I_m)^N$ whenever m is odd$\}$ and $C_2 = \Theta \setminus C_1$. Moreover, for each $i \in \{1,2\}$, set $C_i' = C_i \cap (\Theta \setminus \mathscr{F})$. Clearly, $C_1' \cup C_2' = \Theta \setminus \mathscr{F}$.

We now claim that $C_2' = \emptyset$. Suppose not. Pick some $F_2 \in C_2'$. By definition of C_2', there is some odd $m \in \mathbb{N}$ such that F_2 does not coincide with D_α on $\mathscr{L}(I_m)^N$. That is, $F_2(R) \neq D_\alpha(R)$ for some $R \in \mathscr{L}(I_m)^N$. But $F_\alpha(R) = D_\alpha(R)$ since $m \neq 2$, implying that $F_2(R) \neq F_\alpha(R)$, and α prefers $F_\alpha(R)$ to $F_2(R)$ under R^α. Now set $\mathcal{A} = \{F_\alpha, F_2\}$. As $|\mathcal{A}| = 2$, for any $L \in \mathscr{L}(\mathcal{A}, R)$, we have that $F_\alpha(L) = D_{-\alpha}(L) = F_2 \notin \mathscr{F}$, contradicting that $\mathscr{F}$ is Θ-selection-closed. Thus, $C_2' = \emptyset$, implying that $C_2 \subset \mathscr{F}$.

We will now show that $C_1' = \emptyset$ as well. Suppose to the contrary that there is some $F_1 \in C_1'$. Clearly, there exist two distinct SCFs $F_2, F_3 \in \Theta$ such that $F_2(R) = F_3(R) = D_\alpha(R)$ for all $R \in \mathscr{L}(I_3)^N$, while $F_2(\overline{R})$, $F_3(\overline{R})$ and $D_\alpha(\overline{R})$ are pairwise distinct for some $\overline{R} \in \mathscr{L}(I_5)^N$. Note that $F_2, F_3 \in C_2 \subset \mathscr{F}$. Pick $\overline{R} \in \mathscr{L}(I_5)^N$, and set $\mathcal{A} = \{F_1, F_2, F_3\}$. Now $F_1(\overline{R}) = D_\alpha(\overline{R})$, while $F_2(\overline{R})$ and $F_3(\overline{R})$ are both different than $D_\alpha(\overline{R})$. So, for any $L \in \mathscr{L}(\mathcal{A}, \overline{R})$, F_1 is top-ranked under L^α, implying that $F_2(L) = F_1$, since $|\mathcal{A}| = 3$ is odd. As $F_1 \notin \mathscr{F}$, we conclude that $\mathscr{F}$ is not selection-closed at $\overline{R}$ relative to $\mathcal{A}$, in contradiction with the Θ-selection-closedness of $\mathscr{F}$. Hence $C_1' = \emptyset$, implying that $C_1 \subset \mathscr{F}$.

Finally, as $\emptyset = C_1' \cup C_2' = \Theta \setminus \mathscr{F}$, we conclude that $\mathscr{F} = \Theta$. ■

Theorem 8 If $\mathscr{F} \subset \Theta$ with $\mathscr{F} \not\subset \mathcal{U}_\theta$ is Θ-selection-closed, then $\mathscr{F} = \Theta$.

Proof. Assume that $\mathscr{F} \subset \Theta$ is θ-selection-closed, where $\mathscr{F}$ is not a subset of the set of all neutral Θ-self-selective SCFs, $\mathscr{F} \not\subset \mathcal{U}_\theta$. Let $F_1 \in \mathscr{F} \setminus \mathcal{U}_\theta$. As F_1 is not θ-self-selective, there exist $\overline{m} \in \mathbb{N}$, $\mathcal{A} = \{F_1, F_2, \ldots, F_k\} \subset \Theta$ and $\overline{R} \in \mathscr{L}(I_{\overline{m}})^N$ such that F_1 is not self-selective at $\overline{R}$ relative to $\mathcal{A}$, i.e., $F_1(\hat{L}) \neq F_1$ for any $\hat{L} \in \mathscr{L}(\mathcal{A}, \overline{R})$. On the other hand, $F_1(L) \in \mathscr{F}$ for some $L \in \mathscr{L}(\mathcal{A}, \overline{R})$ since $\mathscr{F}$ is θ-selection-closed. Since $F_1(\hat{L}) \neq F_1$ for any $\hat{L} \in \mathscr{L}(\mathcal{A}, \overline{R})$, F_1 must choose a different member of $\mathcal{A}$ at this profile L. Thus, $F_1(L) = F_l$ for some $l \in \{2, \ldots, k\}$.

For each $i \in \{2, \ldots, k\}$, define

$$F_i^{'}(R) = \begin{cases} F_i(R) & \text{if m} \in \{k, \overline{m}\} \\ G(R) & \text{otherwise} \end{cases}$$

for each $m \in \mathbb{N}$ and $R \in \mathscr{L}(I_m)^N$ where $G \in \theta$ is unanimous and has the following property; there exists an odd $m' \leq N+1$ and $m' \notin \{k, \overline{m}\}$ and $G(R') \notin T(R')$ for some $R' \in \mathscr{L}(I_{m'})^N$. Set $\mathscr{B} = \{F_1, F_2^{'}, \ldots, F_k^{'}\}$ and note that $\mathscr{B} \subset \Theta$. We thus conclude again that $F_1(L') \in \mathscr{F}$ for some $L' \in \mathscr{L}(\mathscr{B}, \overline{R})$, since $\mathscr{F}$ is Θ-selection-closed. Note that $|\mathscr{B}| = |\mathcal{A}| = k$ and $F_i^{'}$ coincides with F_i for all $i \in \{2, \ldots, k\}$ when $m \in \{k, \overline{m}\}$. Now by neutrality of F_1 it follows that $F_1(L') \neq F_1$ since we had $F_1(\hat{L}) \neq F_1$ for all $\hat{L} \in \mathscr{L}(\mathcal{A}, \overline{R})$. So $F_1(L') = F_j^{'}$ for some $j \in \{2, \ldots, k\}$.

Now pick some $\alpha \in N$ and we will complete the proof by showing that $F_\alpha \in \mathscr{F}$. Set $F_j^{'}(R') = G(R') = a \notin T(R')$. Note that $F_\alpha(R') = D_\alpha(R') \in T(R')$ since $m' \neq 2$. Now let $R'' \in \mathscr{L}(I_{m'})^N$ be the linear order profile where $\tau(R_i) = \tau(R_i'')$ and $L_i(a, R'') = a$ for each agent $i \in N$. Since $F_j^{'}$ and G are tops-only $F_j^{'}(R'') = G(R'') = a$ and $F_\alpha(R'') = D_\alpha(R')$. Now let $t \in \mathbb{N}$ be an odd number s.t. $t + 2 \neq k$ and $t + 2 \neq \overline{m}$ and set $\mathcal{A} = \{F_j^{'}, F_\alpha, \overline{F_1}, \ldots, \overline{F_t}\}$ where $\overline{F_i} \in \theta$ for all $i \in \{1, \ldots, t\}$ s.t. $\overline{F_i}(R'') = F_j^{'}(R'') = a \notin T(R'')$. Thus for all $L \in \mathscr{L}(\mathcal{A}, R'')$, each agent ranks F_α on top. We have $F_j^{'}(L) = F_\alpha$, since $|\mathcal{A}| = t + 2 \notin \{k, \overline{m}\}$, $F_j^{'}$ coincides with the unanimous G. Since $F_j^{'} \in \mathscr{F}$ and $\mathscr{F}$ is θ-*selection-closed*, we have $F_\alpha \in \mathscr{F}$.

Hence we observe that for any θ-*selection-closed* subset $\mathscr{F}$ of Θ s.t. $\mathscr{F} \nsubseteq \mathcal{U}_\theta$, we have $F_\alpha \in \mathscr{F}$, by previous lemma, we conclude that $\mathscr{F} = \Theta$. ■

3.4. Weak Selection-Closedness

In this section, we present two different versions of weakly selection-closedness. To avoid heavy notation, we will refer to both as weakly selection-closedness. Although we make our definitions for any family of SCFs in general, what we have in mind for the rest of the chapter is to restrict SCFs to be a singleton valued refinement of a social choice correspondence (SCC). We will start with a social choice correspondence C and confine our family of SCFs to be the singleton valued refinements of C without any restrictions on the set of rival functions. The social choice correspondence C can be considered as a constitutional rule reflecting the norms of the society.

3.4.1. Weak Selection-Closedness I

Definition 18 *Given $\mathscr{F} \subseteq \mathcal{G}$, $m \in \mathbb{N}$, $R \in \mathscr{L}(I_m)^N$ and a finite subset $\mathcal{A}$ of $\mathcal{G} \setminus \mathscr{F}$, we say that $\mathscr{F}$ is weakly* selection-closed *at R relative to $\mathcal{A}$ if and only if for each $F \in \mathscr{F}$, there exists some $\mathcal{A}' \subseteq \mathscr{F}$ with $F \in \mathcal{A}'$ such that F is self-selective at R relative to $\mathcal{A} \cup \mathcal{A}'$. Moreover, $\mathscr{F}$ is weakly* selection-closed *at R if and only if $\mathscr{F}$ is weakly selection-closed at R relative to any finite subset $\mathcal{A}$ of $\mathcal{G} \setminus \mathscr{F}$. Lastly, $\mathscr{F}$ is universally weakly* selection-closed *if and only if $\mathscr{F}$ is weakly selection-closed at each $R \in \bigcup_{m \in \mathbb{N}} \mathscr{L}(I_m)^N$.*

The requirement that F is self-selective at the preference profile R relative to $\mathcal{A} \cup \mathcal{A}'$ in the above definition may at first glance shed some shadow on the fact that weak selection-closedness is indeed weaker than selection-closedness. First notice that the sets of SCFs against which the two different versions of selection-closedness are to be tested have different structures. In case of selection-closedness, the relevant test is applied to any finite nonempty subset $\mathcal{A}$ of $\mathcal{G}$. As for weak selection-closedness, however, the arbitrary part $\mathcal{A}$ of the test set for an SCF $F \in \mathscr{F} \subseteq \mathcal{G}$ is supposed to belong $\mathcal{G} \setminus \mathscr{F}$, while its complementary component $\mathcal{A}'$ from $\mathscr{F}$ is to be chosen in an attempt to ensure the self-selectivity of $F \in \mathscr{F}$ in $\mathcal{A} \cup \mathcal{A}'$. Thus, a local comparison of the ordinary and weak versions of selection closedness is to be done at each preference profile R.

Proposition 4 *Given $\mathscr{F} \subseteq \mathcal{G}$, $m \in \mathbb{N}$, $R \in \mathscr{L}(I_m)^N$, if $\mathscr{F}$ is selection closed at R, then $\mathscr{F}$ is weakly selection-closed at R.*

Proof. Assume that $\mathscr{F}$ is selection closed at R. Let $\mathcal{A}$ be any finite subset of $\mathcal{G} \setminus \mathscr{F}$, and take any $F \in \mathscr{F}$. Set $\mathcal{A}' = \{F\}$. Now $\mathscr{F}$ is selection-closed at R relative to $\mathcal{A} \cup \mathcal{A}'$ by hypothesis. Thus, there is some $L \in \mathscr{L}(\mathcal{A} \cup \mathcal{A}', R)$ such that $F(L) \in \mathscr{F}$. But then $F(L) = F$. That is, F is self-selective at R relative to $\mathcal{A} \cup \mathcal{A}'$. As $\mathcal{A} \subset \mathcal{G} \setminus \mathscr{F}$ was arbitrary, $\mathscr{F}$ is weakly selection-closed at R. ■

Now it also follows that universal selection-closedness implies universal weak selection-closedness for any nonempty family of neutral SCFs.

Given a set $\mathcal{A}$ to represent the set of SCFs available to the society at the time of the choice, if an SCF from $\mathscr{F}$ is employed to resolve the society's underlying choice problem, our new weak consistency criterion requires that the chosen SCF selects itself with the appropriately chosen constitutionally prescribed companions in $\mathscr{F}$. When we consider the set of singleton valued refinements of a social choice correspondence as our family, the weak version of the selection-closedness is similar to Tideman's (1987) concept of independence of clones which requires the addition/deletion of clones not to affect the winning status of non-clones.

Senocak (2009) examined the Condorcet rule, the Pareto rule, and scoring correspondences, and characterized a special class of scoring correspondences through weak selection-closedness. He showed that the set of all singleton-valued refinements of the Pareto and Condorcet correspondences is not universally weakly selection-closed, whereas a special class of scoring correspondences is. However, his main result concerning the scoring correspondences is only partially correct. In the following section, we will point out this shortcoming and clarify the correct version.

3.4.2. Results

To state our results, which will pertain to scoring correspondences, we need some further definitions. For any $m \in \mathbb{N}$, an *m-score vector* is an m-tuple $s = (s_1, \ldots, s_m) \in \mathbb{R}^m$ with $s_i \geq s_{i+1}$ for all $i \in \{1, \ldots, m-1\}$ and $s_1 > s_m$.

Given $m \in \mathbb{N}$, $a \in I_m$, $R \in \mathscr{L}(I_m)^N$ and $\alpha \in N$, we define the *rank* $\sigma(a, R^\alpha)$ *of* a in R^α by $\sigma(a, R^\alpha) = |\{b \in A \,|\, b\, R^\alpha\, a\}|$. We say that F is a *scoring correspondence* if, for each $m \in \mathbb{N}$, there exists an m-score vector $s \in \mathbb{R}^m$ such that for any finite nonempty society N and any $R \in \mathscr{L}(I_m)^N$, one has

$$F(R) = \Big\{a \in I_m \,|\, \sigma_s^R(a) \geq \sigma_s^R(b) \quad \text{for any} \quad b \in I_m\Big\},$$

where, for each $x \in I_m$, $\sigma_s^R(x) = \sum_{\alpha \in N} s_{\sigma(x, R^\alpha)}$ denotes the total score of x under s at R. We denote $\mathcal{S}$ for the set of all scoring correspondences.

Note that a scoring correspondence in $\mathcal{S}$ is defined for all finite series of the alternative set. Clearly, any correspondence in $\mathcal{S}$ will be induced by an infinite variety of different score vectors.

A scoring correspondence $F \in \mathcal{S}$ is said to be *strict* if, for each $m \in \mathbb{N}$ and for any associated m-score vector s, we have $s_i > s_{i+1}$ for all $i \in \{1, \ldots, m-1\}$.

The definition of a scoring correspondence allows the m-score vectors to be arbitrarily independent of each other for different values of m, which is not the case for almost any well-known scoring correspondence. We will now introduce two different ways of relating score vectors corresponding to different sizes of alternative sets with each other in a natural fashion.

We will say that a scoring correspondence $F \in \mathcal{S}$ is *right-extendable* if, for any $m \in \mathbb{N}$, there are score vectors $s = (s_1, \ldots, s_m) \in \mathbb{R}^m$ and $s' = (s'_1, \ldots, s'_m, s'_{m+1}) \in \mathbb{R}^{m+1}$ associated with F such that $s'_i = s_i$ for all $i \in \{1, \ldots, m\}$.

A scoring correspondence $F \in \mathcal{S}$ will be said to be *left-extendable* if, for any $m \in \mathbb{N}$, there are score vectors $s = (s_1, \ldots, s_m) \in \mathbb{R}^m$ and $s' = (s'_1, \ldots, s'_m, s'_{m+1}) \in \mathbb{R}^{m+1}$ associated with F such that $s'_{i+1} = s_i$ for all $i \in \{1, \ldots, m\}$.

Note that most common scoring rules, e.g., Borda, plurality, inverse plurality, and any vote for k alternatives rule are either right-extendable or left-extendable (Borda is both right-extendable and left-extendable).

Given a right-extendable $F \in \mathcal{S}$, the m-score vectors associated with F can be inductively combined into a nonincreasing score sequence $(s_i)_{i \in \mathbb{N}}$, whose initial segment of length m is just an m-score vector associated with F for each $m \in \mathbb{N}$. As an example, for the plurality rule, the nonincreasing score sequence $(s_i)_{i \in \mathbb{N}}$ would be $(1, 0, 0, ...)$, where $s_1 = 1$ and $s_i = 0$ for all $i \geq 2$. When there are m alternatives, the m-score vector that is associated with the plurality rule would be the initial segment of the score sequence $(s_i)_{i \in \mathbb{N}}$, which is $(1, 0, ..., 0)$ with length of m.

As for the left-extendable $F \in \mathcal{S}$, the associated m-score vectors are again inductively combined into a sequence $(s_i)_{i \in \mathbb{N}}$, which is nondecreasing this time and whose initial segment of length m in reverse order yields an m-score vector for F for each $m \in \mathbb{N}$.

As an example, for the Borda rule, the nondecreasing score sequence $(s_i)_{i\in\mathbb{N}}$ would be $(1, 2, 3, ...)$, where $s_i = i$ for all $i \geq 1$. When there are m alternatives, the m-score vector that is associated with the Borda rule would be the initial segment of the score sequence $(s_i)_{i\in\mathbb{N}}$ in reverse order, which is $(m, m-1, ..., 1)$.

We will henceforth represent right- and left-extendable scoring correspondences via their score sequences. Also note that any sequence of a strict right-extendable scoring correspondence will be monotonically decreasing, while "decreasing" will be replaced by "increasing" in case of a strict left-extendable scoring correspondence.

Let F be a right- or left-extendable scoring correspondence. Now F is said to *have a lower bound in increments* if, for any score sequence $(s_i)_{i\in\mathbb{N}}$ associated with F, there is some $\delta > 0$ such that $|s_i - s_{i+1}| \geq \delta$ for all $i \in \mathbb{N}$. Similarly, we say that F *has an upper bound in increments* if, for any score sequence $(s_i)_{i\in\mathbb{N}}$ associated with F, there is some $\delta > 0$ such that $|s_i - s_{i+1}| \leq \delta$ for all $i \in \mathbb{N}$.

Lemma 2 (Senocak (2009)) Let $C \in \mathcal{S}$ be a scoring correspondence and let $\mathcal{C}$ denote the set of all singleton-valued refinements of this correspondence. If $\mathcal{C}$ is universally weakly selection-closed, then for any $m \in \mathbb{N}$ and for any $R \in \mathscr{L}(I_m)^N$, we have $C(R) \subseteq P(R)$.

Theorem 9 (Senocak (2009)) A right-extendable scoring correspondence is strict if and only if the set of all singleton-valued refinements of the correspondence is universally weakly selection-closed.

The equivalence stated in Theorem 9 is only partially correct. The implication from universal weak selection-closedness to strictness is valid, but the converse does not hold. In what follows, we address this shortcoming and present the correct version.

Theorem 10 If the set of all singleton-valued refinements of a right-extendable scoring correspondence is universally weakly selection-closed, then the correspondence is strict.

Proof. Same as in Senocak (2009). ■

Theorem 11 If a right-extendable scoring correspondence has a lower bound in increments then the set of all singleton valued refinements of the correspondence is universally weakly selection-closed.

Proof. Let C be a right-extendable scoring correspondence. Assume C has a lower bound in increments (therefore it is strict), then we will show that the set of all singleton valued refinements of the correspondence, $\mathcal{C}$, is universally weakly selection-closed.

Let $(s_i)_{i\in\mathbb{N}}$ be the score sequence associated with the scoring correspondence C. Take any $m \in \mathbb{N}$ and any $R \in \mathscr{L}(I_m)^N$. First, we claim that $C(R) \subseteq P(R)$. Suppose not, then there exists $a \in C(R)$ and $b \in I_m$ such that b Pareto dominates a. But, since C is strict, we must have $\sigma_s^R(b) > \sigma_s^R(a)$, contradicting with $a \in C(R)$.

Now, take any singleton valued refinement of the correspondence, $F \in \mathcal{C}$ and any $\mathcal{A} \subseteq \mathcal{G} \setminus \mathcal{C}$. Since $F(R) \in C(R) \subseteq P(R)$ (by previous claim), there exists a linear order profile $L \in \mathscr{L}(\mathcal{A} \cup \{F\}, R)$ such that $F \in P(L)$.

R_1	R_2	$\ldots$	$R_{\lvert N\rvert-1}$	$R_{\lvert N\rvert}$	*scores*
.	G	.	.	.	s_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	s_2
F	$\vdots$	$\vdots$	$\vdots$	F	$\vdots$
$\vdots$	F	$\vdots$	G	$\vdots$	$\vdots$
G	$\vdots$	$\vdots$	$\vdots$	G	$\vdots$
$\vdots$	$\vdots$	G	F	$\vdots$	$\vdots$
.	.	F	.	.	s_m

We will call a SCF F' a *replica* of F, if F' is a singleton valued refinement of the correspondence C, i.e. $F' \in \mathcal{C}$ and F' chooses the same alternative as F at the preference profile R, i.e. $F'(R) = F(R)$. We will construct the set $\mathcal{A}' \subseteq \mathcal{C}$ by adding replicas of F to the set $\{F\}$. Consider the set $\mathcal{A}' = \{F, F_1\}$ where F_1 is a replica of F. Now construct $L' \in \mathscr{L}(\mathcal{A} \cup \mathcal{A}', R)$ such that F_1 is just below F for all agents and the other alternatives remain in the same position as in L.

R_1	R_2	$\ldots$	$R_{\lvert N\rvert-1}$	$R_{\lvert N\rvert}$	*scores*
.	G	.	.	.	s_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	s_2
F	$\vdots$	$\vdots$	$\vdots$	F	$\vdots$
F_1	$\vdots$	$\vdots$	$\vdots$	F_1	$\vdots$
$\vdots$	F	$\vdots$	G	$\vdots$	$\vdots$
G	F_1	$\vdots$	$\vdots$	G	$\vdots$
$\vdots$	$\vdots$	G	F	$\vdots$	$\vdots$
$\vdots$	$\vdots$	F	F_1	$\vdots$	s_m
.	.	F_1	.	.	s_{m+1}

The position of the SCF F from the top has not been changed for all agents and since the correspondence C is right-extendable, we have $\sigma_s^L(F) = \sigma_s^{L'}(F)$. Since F is not Pareto dominated by any $G \in \mathcal{A}$, the position of all $G \in \mathcal{A}$ has been dropped by one rank for some agents. Since C is strict, we have $s_i > s_{i+1}$ for all $i \in \{1, \ldots, m+1\}$, therefore $\sigma_s^L(G) > \sigma_s^{L'}(G)$ for all $G \in \mathcal{A}$.

By adding more replicas to the set $\{F, F_1\}$, we can find $\mathcal{A}' = \{F, F_1, \ldots, F_k\}$ such that $F_i \in \mathcal{C}$ and $F(R) = F_i(R)$ for all $i \in \{1, \ldots, k\}$ and there exists $\hat{L} \in \mathscr{L}(\mathcal{A} \cup \mathcal{A}', R)$ such that we still have $\sigma_s^L(F) = \sigma_s^{\hat{L}}(F)$ and $\sigma_s^L(G) > \sigma_s^{L'}(G) > \sigma_s^{\hat{L}}(G)$ for all $G \in \mathcal{A}$. Since C has a lower bound in increments for the score sequence $(s_i)_{i \in \mathbb{N}}$ associated with C, there exist $\delta > 0$ such that $s_i - s_{i+1} \geq \delta$ for all $i \in \mathbb{N}$. Adding a replica each time makes a drop at least δ in the score of the SCF G for any $G \in \mathcal{A}$. Since $\sigma_s^L(F)$ is constant and $\sigma_s^{\hat{L}}(G)$ is decreasing at least δ for each replica, eventually we would have $\sigma_s^{\hat{L}}(F) > \sigma_s^{\hat{L}}(G)$ for any $G \in \mathcal{A}$.

Since C is strict and F Pareto dominates all F_i's for all $i \in \{1, \ldots, k\}$ at $\hat{L}$ we have $C(\hat{L}) = F$. Since F is a singleton valued refinement of C, we have $F(\hat{L}) = F$. Since F and $\mathcal{A}$ are arbitrary, we conclude that $\mathcal{C}$ is universally weakly selection-closed. ■

Having corrected the misstatement concerning right-extendable selection-closedness, we now turn to left-extendable scoring correspondences. We present new results for this case, relating universal weak selection-closedness of singleton-valued refinements to strictness and to upper bounds on increments.

Theorem 12 If the set of all singleton valued refinements of a left-extendable scoring correspondence is universally weakly selection-closed then it is strict.

Proof. Let C be a left-extendable scoring correspondence and assume the set of all singleton valued refinements of the correspondence, $\mathcal{C}$, is universally weakly selection-closed.

Let $(s_i)_{i\in\mathbb{N}}$ be the score sequence associated with the scoring correspondence C. Suppose to the contrary that C is not strict, then there exists $k \in \mathbb{N}$ such that $s_{k+1} = s_k$. Let $m = k+1$ and consider the profile $R \in \mathscr{L}(I_m)^N$ where everyone puts the same alternative b as their top choice and their second top ranked alternative is same, a. The m-score vector that is associated with the scoring correspondence C would be the initial segment of the score sequence $(s_i)_{i\in\mathbb{N}}$ in reverse order, so we have $s_1 = s_2$. Therefore, by construction we have $\{a, b\} \subseteq C(R)$. Take any $F \in \mathcal{C}$ such that $F(R) = a$ and let $\mathcal{A} = \{G\}$ with $G(R) = b$. For any $\mathcal{A}' \subseteq \mathcal{C}$, with $F \in \mathcal{A}'$ and for any linear order profile $L \in \mathscr{L}(\mathcal{A} \cup \mathcal{A}', R)$, F is Pareto dominated by G, so $F \notin C(L)$ by the previous lemma, contradicting with $F(L) = L$. ■

Theorem 13 If a left-extendable scoring correspondence is strict and has an upper bound in increments then the set of all singleton valued refinements of the correspondence is universally weakly selection-closed.

Proof. Let C be a left-extendable scoring correspondence. Assume C is strict and has an upper bound in increments, then we will show that the set of all singleton valued refinements of the correspondence, $\mathcal{C}$, is universally weakly selection-closed.

Let $(s_i)_{i\in\mathbb{N}}$ be the increasing score sequence associated with the scoring correspondence C. Take any $m \in \mathbb{N}$ and any $R \in \mathscr{L}(I_m)^N$. First, we claim that $C(R)$ is a subset of Pareto optimal outcomes, $C(R) \subseteq P(R)$. Suppose not, then there exists $a \in C(R)$ and $b \in I_m$ such that b Pareto dominates a. But, since C is strict, we have $s_i > s_{i+1}$ for all $i \in \{1, \ldots, m-1\}$ at the associated m-score vector, therefore we must have $\sigma_s^R(b) > \sigma_s^R(a)$, contradicting with $a \in C(R)$.

Now, take any $F \in \mathcal{C}$ and any $\mathcal{A} \subseteq \mathcal{G} \setminus \mathcal{C}$ where $\mathcal{G}$ is the set of all neutral SCFs. Since $F(R) \in C(R) \subseteq P(R)$ (by previous claim), there exists a linear order profile $L \in \mathscr{L}(\mathcal{A} \cup \{F\}, R)$ such that $F \in P(L)$. We will call a SCF F' a *replica* of F, if F' is a singleton valued refinement of the correspondence C, i.e. $F' \in \mathcal{C}$ and F' chooses the same alternative as F at the preference profile R, i.e. $F'(R) = F(R)$. We will construct the set $\mathcal{A}' \subseteq \mathcal{C}$ by adding replicas of F to the set $\{F\}$.

Denote the number of the social choice functions in the set $\mathcal{A} \cup \{F\}$ by m' and let $(s_{m'}, s_{m'-1}, ..., s_1)$ be the m'-score vector which is the initial segment (of length m') of the increasing score sequence $(s_i)_{i \in \mathbb{N}}$ in reverse order. Therefore we have $s_{m'} > s_{m'-1} > ... > s_1$. Consider any $G \in \mathcal{A}$. We will compare the scores of G and F in L. Since G does not Pareto dominate F in L, $\sigma_s^L(G)$ can be at most $(N-1)s_{m'} + s_j$ for some $j \in \{1, \ldots, m'-1\}$. Since F is not Pareto dominated by G in L, $\sigma_s^L(F)$ can be at least $(N-1)s_1 + s_k$ for some $k \in \{2, \ldots, m'\}$.

R_1	R_2	$\ldots$	$R_{\|N\|-1}$	$R_{\|N\|}$	*scores*
.	G	$\ldots$	G	G	$s_{m'}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
F	$\vdots$	$\vdots$	$\vdots$	$\vdots$	s_k
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
G	$\vdots$	$\vdots$	$\vdots$	$\vdots$	s_j
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
.	F	F	F	F	s_1

Since C is strict the score sequence $(s_i)_{i \in \mathbb{N}}$ is increasing; it is either bounded (therefore convergent) or diverges to infinity. We will consider these cases separately.

Case I : The score sequence $(s_i)_{i \in \mathbb{N}}$ diverges to infinity. The score difference between

F and G in L, $\sigma_s^L(F)-\sigma_s^L(G)$, is $s_k-s_j-(N-1)(s_{m'}-s_1)$. Note that $(N-1)(s_{m'}-s_1)$ is always less than or equal to $(N-1)(m'-1)\delta$ where δ is the upper bound in increments of C. Now add a replica F_1 of F to the set $\{F\}$ and consider the the linear order profile $\hat{L} \in \mathscr{L}(\mathcal{A} \cup \{F, F_1\}, R)$ such that F_1 is just below F for all agents and the other alternatives remain same.

R_1	R_2	$\dots$	$R_{\lvert N\rvert-1}$	$R_{\lvert N\rvert}$	*scores*
.	G	$\dots$	G	G	$s_{m'+1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
F	$\vdots$	$\vdots$	$\vdots$	$\vdots$	s_{k+1}
F_1	$\vdots$	$\vdots$	$\vdots$	$\vdots$	s_k
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
G	$\vdots$	$\vdots$	$\vdots$	$\vdots$	s_j
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
.	F	F	F	F	s_2
.	F_1	F_1	F_1	F_1	s_1

Note that the score of G in the first agent remains as s_j since G is below F. Also the number of alternatives between F and G, $(m'-1)$, remains same for the other agents since F is below G in their preferences. Now the score difference between F and G in $\hat{L}$, $\sigma_s^{\hat{L}}(F)-\sigma_s^{\hat{L}}(G)$, is $s_{k+1}-s_j-(N-1)(s_{m'+1}-s_2)$ which is greater than $s_{k+1}-s_j-(N-1)(m'-1)\delta$.

As we increase the number of replicas of F to a certain number T, $\{F_1, F_2, ..., F_T\}$ and considering the preference profile $\hat{L} \in \mathscr{L}(\mathcal{A} \cup \{F, F_1, \dots, F_T\}, R)$ such that such that the replicas are just below F for all agents and the other alternatives remain same, the score difference between F and G in $\hat{L}$, $\sigma_s^{\hat{L}}(F)-\sigma_s^{\hat{L}}(G)$ would be greater than than $s_{k+T}-s_j-(N-1)(m'-1)\delta$.

R_1	R_2	$\ldots$	$R_{\|N\|-1}$	$R_{\|N\|}$	*scores*
.	G	$\ldots$	G	G	$s_{m'+T}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
F	$\vdots$	$\vdots$	$\vdots$	$\vdots$	s_{k+T}
F_1	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
F_T	$\vdots$	$\vdots$	$\vdots$	$\vdots$	s_k
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
G	$\vdots$	$\vdots$	$\vdots$	$\vdots$	s_j
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
.	F	F	F	F	s_{T+1}
.	F_1	F_1	F_1	F_1	s_T
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
.	F_T	F_T	F_T	F_T	s_1

As the score sequence $(s_i)_{i\in\mathbb{N}}$ diverges to infinity we can find $\mathcal{A}' = \{F, F_1, \ldots, F_H\}$ such that $F_i \in \mathcal{C}$ and $F(R) = F_i(R)$ for all $i \in \{1, \ldots, H\}$ and there exists $\hat{L} \in \mathscr{L}(\mathcal{A} \cup \mathcal{A}', R)$ such that the score difference $s_{k+H} - s_j - (N-1)(m'-1)\delta > 0$.

Case II : The score sequence $(s_i)_{i\in\mathbb{N}}$ is convergent to real number M. Let $\epsilon > 0$ be such that $\frac{M-s_j}{N} > \epsilon$ (note that since C is strict we have $M - s_j > 0$). As we have seen above if we increase the number of replicas to a certain number T the score of F would be $s_{k+T} + (N-1)s_{T+1}$ and the score of G would be $s_j + (N-1)s_{m'+T}$. If we replicate F large enough the score of F will be at least $N(M-\epsilon)$ and the score of G will be at most $(N-1)M + s_j$. The difference between these two scores, $M - N\epsilon - s_j$, will be strictly greater than zero by construction of the ϵ.

In both cases, we can find replicas of F such that the score of F is strictly greater than

G. Since G is an arbitrary element of $\mathcal{A}$, we can find $\mathcal{A}' = \{F, F_1, \ldots, F_H\}$ such that $F_i \in \mathcal{C}$ and $F(R) = F_i(R)$ for all $i \in \{1, \ldots, H\}$ and there exists $\hat{L} \in \mathscr{L}(\mathcal{A} \cup \mathcal{A}', R)$ such that $\sigma_s^{\hat{L}}(F) > \sigma_s^{\hat{L}}(G)$ for all $G \in \mathcal{A}$.

Since C is strict and F Pareto dominates all F_i's for all $i \in \{1, \ldots, H\}$ at $\hat{L}$ we have $C(\hat{L}) = F$. Since F is a singleton valued refinement of C, we have $F(\hat{L}) = F$. Since F and $\mathcal{A}$ are arbitrary, we conclude that $\mathcal{C}$ is universally weakly selection-closed. ■

Example 4 There exists some left-extendable scoring correspondences which are strict and have a lower bound in increments but the set of all singleton valued refinements of the correspondence is not universally weakly selection-closed. Consider the following left-extendable scoring correspondence with the following score sequence $s = \{0, 1, \frac{3}{2}, 4, 9, 19, ...\}$, i.e. $s_1 = 0$, $s_2 = 1$, $s_3 = \frac{3}{2}$ and $s_{i+1} = 2s_i + 1$ for all $i \geq 3$. The scoring correspondence is strict and has a lower bound in increments with the increment $\delta = \frac{1}{2}$. Let F be a singleton valued refinement of this correspondence. Let $R \in \mathscr{L}(I_2)^N$ be the linear order profile represented by the matrix below:

R^α	R^β
2	3
1	1
3	2

Let $\mathcal{A} = \{G_1, G_2, G_3, G_4\}$ where $G_1(R) = G_2(R) = 2$, $G_3(R) = G_4(R) = 3$ and $G_1, G_2, G_3, G_4 \in \mathcal{G} \setminus \mathcal{S}$. Note that $F(R) = 1$ and $F_i(R) = 1$ for any $F_i \in \mathcal{S}$. So for any $\mathcal{A}' \subseteq \mathcal{S}$, with $F \in \mathcal{A}'$ and for any linear order profile $L \in \mathscr{L}(\mathcal{A} \cup \mathcal{A}', R)$, the score of the function F will be at most $2s_k$ which is strictly less than the score of the top choice of the agent α, which is either G_1 or G_2 since $s_{k+2} = 2s_{k+1} + 1 > 2s_k$. So, we conclude that F is not self-selective at R relative to $\mathcal{A} \cup \mathcal{A}'$ for any $\mathcal{A}' \subseteq \mathcal{S}$.

There exists some right-extendable scoring correspondences which are strict and have a upper bound in increments but the set of all singleton valued refinements of the correspondence is not universally weakly selection-closed. Consider the following right-extendable scoring correspondence with the following score sequence $s = \{\frac{5}{8}, \frac{1}{2}, \frac{1}{4}, ...\}$, i.e. $s_1 = \frac{5}{8}$, $s_2 = \frac{1}{2}$ and $s_{i+1} = \frac{s_i}{2}$ for all $i \geq 2$. The scoring correspondence is strict and has a upper bound in increments with the increment $\delta = \frac{1}{4}$. Let F be a singleton valued refinement of this correspondence. Let $R \in \mathscr{L}(I_2)^N$ be the linear order profile represented by the matrix below:

R^α	R^β
2	3
1	1
3	2

Let $\mathcal{A} = \{G_1, G_2, G_3, G_4\}$ where $G_1(R) = G_2(R) = 2$, $G_3(R) = G_4(R) = 3$ and $G_1, G_2, G_3, G_4 \in \mathcal{G} \setminus \mathcal{S}$. Note that $F(R) = 1$ and $F_i(R) = 1$ for any $F_i \in \mathcal{S}$. So for any $\mathcal{A}' \subseteq \mathcal{S}$, with $F \in \mathcal{A}'$ and for any linear order profile $L \in \mathscr{L}(\mathcal{A} \cup \mathcal{A}', R)$, the score of the function F will be at most $\frac{1}{2}$ which is strictly less than the score of the top choice of the agent α, which is either G_1 or G_2. So, we conclude that F is not self-selective at R relative to $\mathcal{A} \cup \mathcal{A}'$ for any $\mathcal{A}' \subseteq \mathcal{S}$.

3.4.3. Weak Selection-Closedness II

Now, we introduce a weaker variant of weakly selection-closedness. When an SCF $F \in \mathscr{F}$ is employed to resolve the society's underlying choice problem, the requirement is that F—rather than selecting itself—select one of the constitutionally prescribed companion rules from $\mathscr{F}$ appropriate to the context. Under this version, the set of all singleton-valued refinements of the Pareto correspondence P is universally weak selection-closed. After that example, we will provide a partial characterization.

Definition 19 *Given $\mathscr{F} \subseteq \mathcal{G}$, $m \in \mathbb{N}$, $R \in \mathscr{L}(I_m)^N$ and a nonempty finite subset $\mathcal{A}$ of $\mathcal{G} \setminus \mathscr{F}$, we say that $\mathscr{F}$ is weak* selection-closed *at R relative to $\mathcal{A}$ if and only if for each $F \in \mathscr{F}$, there exists some $\mathcal{A}' \subseteq \mathscr{F}$ with $F \in \mathcal{A}'$ and some $L \in \mathscr{L}(\mathcal{A} \cup \mathcal{A}', R)$ such that $F(L) \in \mathcal{A}'$. Moreover, $\mathscr{F}$ is said to be weak* selection-closed *at R if and only if $\mathscr{F}$ is weak selection-closed at R relative to any finite subset $\mathcal{A}$ of $\mathcal{G}$. Lastly, we say that $\mathscr{F}$ is universally weak* selection-closed *if and only if $\mathscr{F}$ is weak selection-closed at each $R \in \bigcup_{m \in \mathbb{N}} \mathscr{L}(I_m)^N$.*

Example 5 Let $\mathcal{P}$ stands for the set of all singleton-valued refinements of the Pareto Correspondence P. We will show that $\mathcal{P}$ is universally weak selection-closed. Take any $m \in \mathbb{N}$ and any $R \in \mathscr{L}(I_m)^N$ and any $F \in \mathcal{P}$ and consider a nonempty finite subset $\mathcal{A} = \{G_1, \cdots, G_k\}$ of $\mathcal{G} \setminus \mathcal{P}$. We will construct $\mathcal{A}' \subseteq \mathcal{P}$ and $L \in \mathscr{L}(\mathcal{A} \cup \mathcal{A}', R)$ such that any $G \in \mathcal{A}$ would be Pareto dominated so $F(L) \in \mathcal{A}'$.

Suppose $G_j(R) \in P(R)$ for some $j \in \{1, \cdots, k\}$. Choose a function $F_j \in \mathcal{P}$ such that $F_j(R) = G_j(R)$. Now let $\mathcal{A}' = \{F, F_j\}$ and construct $L' \in \mathscr{L}(\mathcal{A} \cup \mathcal{A}', R)$ such that G_j is just below F_j for all agents, so G_j would be Pareto dominated by F_j under L'. By continuing in this way, we can construct $\mathcal{A}' = \{F, F_1, \cdots F_{k_1}\}$ such that $F_i \in \mathcal{P}$ and $F_i(R) = G_i(R)$ and there exist some $\overline{L} \in \mathscr{L}(\mathcal{A} \cup \mathcal{A}', R)$ such that G_i is Pareto dominated by F_i for all $i \in \{1, \cdots, k_1\}$ where $k_1 \leq k$.

Now suppose $G'_j(R) \notin P(R)$ for some $j \in \{1, \cdots, k\}$. Then by finiteness there exists an alternative $a \in P(R)$ such that $G'_j(R)$ is Pareto dominated by a. Choose a function $F'_j \in \mathcal{P}$ such that $F'_j(R) = a$. Now let $\mathcal{A}' = \left\{F, F_1, \cdots F_{k_1}, F'_j\right\}$ and construct $L' \in \mathscr{L}(\mathcal{A} \cup \mathcal{A}', R)$ such that the other alternatives remain in the same position as in $\overline{L}$. By continuing in this way, we can construct $\mathcal{A}' = \left\{F, F_1, \cdots F_{k_1}, F'_1, \cdots F'_{k_2}, \right\}$

such that $F'_i \in \mathcal{P}$ and there exist some $L \in \mathscr{L}(\mathcal{A} \cup \mathcal{A}', R)$ such that G'_i is Pareto dominated by F'_i for all $i \in \{1, \cdots, k_2\}$ where $k_1 + k_2 \leq k$.

By construction any $G \in \mathcal{A}$ would be Pareto dominated under L, so $F(L) \in \mathcal{A}'$. Since F and $\mathcal{A}$ are arbitrary, we conclude that $\mathcal{P}$ is universally weak selection-closed.

Let $m \in \mathbb{N}$ and $R \in \mathscr{L}(I_m)^N$ and let $R_{|A}$ denote the restriction of the preference profile R to the set of alternatives $A \subset I_m$.

A social choice correspondence $C : \bigcup\limits_{m \in \mathbb{N}} \mathscr{L}(I_m)^N \to 2^{\mathbb{N}}$ satisfies

- **independence of irrelevant alternatives (IIA)** if for any $m \in \mathbb{N}$ and any $R \in \mathscr{L}(I_m)^N$, for each $S, T \subset I_m$ such that $a \in T \subset S$, if $a \in C(R_{|S})$, then $a \in C(R_{|T})$.
- **independence of rejected alternatives (IRA)** if for any $m \in \mathbb{N}$ and any $R \in \mathscr{L}(I_m)^N$, for each $S, T \subset I_m$, if $C(R_{|S}) \subset T \subset S$, then $C(R_{|T}) \subset C(R_{|S})$.

Theorem 14 If a social choice correspondence C satisfies IIA and IRA, then the set of all singleton-valued refinements of the correspondence C is universally weak selection-closed.

Proof. Let C be a social choice correspondence which satisfies IIA and IRA. Let $\mathscr{F}$ denotes the set of all singleton-valued refinements of this correspondence C. Suppose not, i.e. $\mathscr{F}$ is not universally weak selection-closed, then there exists $m \in \mathbb{N}$, $R \in$

$\mathscr{L}(I_m)^N$ and a nonempty finite subset $\mathcal{A}$ of $\mathcal{G} \setminus \mathscr{F}$, and $F \in \mathscr{F}$ such that for all $\mathcal{A}' \subseteq \mathscr{F}$ with $F \in \mathcal{A}'$ and for all $L \in \mathscr{L}(\mathcal{A} \cup \mathcal{A}', R)$ we have $F(L) \notin \mathcal{A}'$.

Let $\mathcal{A}' = \{F_1, \cdots, F_k\}$ where $F_i(R) \neq F_j(R)$ for all $i \neq j$ and $\bigcup_{i=1}^k F_i(R) = C(R)$. Pick an $L \in \mathscr{L}(\mathcal{A} \cup \mathcal{A}', R)$ and by supposition we have $F(L) \in \mathcal{A}$. Let $G \in \mathcal{A}$ such that $F(L) = G$. Note that $G(R) \neq F_i(R)$ for all $i \in \{1, \cdots, k\}$ since otherwise we can construct L' by changing the role of G and F_i while keeping the positions of the other remaining alternatives same, so that we have $F(L') = F_i$. Since $\bigcup_{i=1}^k F_i(R) = C(R)$, we have $G(R) \notin C(R)$ and since F is a singleton-valued refinements of C we have $F(L) = G \in C(L)$.

Now, let $B_1 = \{F_1, \cdots, F_k, G\}$ and consider the profile $L_{|B_1}$. Since $G \in C(L)$ and since C satisfies IIA we have $G \in C(L_{|B_1})$.

Now, let $B_2 = \{F_1(R), \cdots, F_k(R), G(R)\}$ and consider the profile $R_{|B_2}$. Since $G(R) \notin C(R)$ and $C(R_{|B_2}) \subset C(R)$ and C satisfies IRA, we have $G(R) \notin C(R_{|B_2})$.

So, we have $G(R) \notin C(R_{|B_2})$ and $G \in C(L_{|B_1})$. Check that $R_{|B_2}$ and $L_{|B_1}$ are the same preference profiles in terms of neutrality. ■

3.5. Conclusion

In this chapter, we analyzed selection-closedness within the tops-only domain. The proof of the impossibility result that selection-closedness ends up with is based on using F_α as an SCF that has to be outrivaled by each member of a set $\mathscr{F}$ of SCFs for $\mathscr{F}$ to be selection-closed, where F_α differs from anti-dictatorship of agent α only when the cardinality of the set of available SCFs is two. It is not only clear that F_α is an SCF that no human society would even think of employing to resolve any social choice problem, but jumping from dictatorship to anti-dictatorship as the number alternatives increases by one unit is very unnatural and seems impossible to be justified by any example whatsoever. It may, of course, be possible to provide another proof for the same result, where no such weird SCF is being used. Yet it seems worthwhile trying to restrict the rival SCFs outside $\mathscr{F}$ by introducing some other kind of "discrete smoothness" condition.

The two particular ways of weakening selection-closedness we have gone through in the chapter is just one out of several meaningful alternatives of doing so. One of the main aspects of weak selection-closedness is that it is "cloning-sensitive". Given an SCF F in $\mathscr{F}$, where $\mathscr{F}$ is the set of SCFs tested for selection-closedness, F is entitled to mobilize sufficiently many "quasi-clones" of itself to outrival certain competitors of its own. As the SCFs in our setting are defined on linear order profiles on any alternative set of finite size, two SCFs differing from each other at "irrelevant" preference profiles are formally different, while they act as perfect clones of each other at relevant preference profiles. Thus, it is no wonder that selection-closedness turns out to be useful in dealing and classifying scoring correspondences, as scoring correspondences are cloning-sensitive as well.

This observation gives rise to two different kinds of research questions. One is concerned with how different versions of self-selectivity or selection-closedness are related to "cloning-proofness", "cloning-friendliness" or "cloning-sensitivity" in general. The second is related to using "selection-closedness" as a measure of the "competitive power" of certain classes of SCFs, for a collection of SCFs deserves being called selection-closed, in case it can outrival outsiders in competing for the choice of the choice rule. As already noted, neither the Pareto correspondence nor the Condorcet correspondence is weakly selection-closed, whereas strict scoring correspondences are generally weakly selection-closed under one of the weakened forms of the notion. Moreover, if the definition is relaxed even further, the Pareto correspondence also turns out to be weakly selection-closed. Now a possibly interesting question seems to be whether one can weaken selection-closedness in such a way that now the competitive power intrinsic to the Condorcet correspondence gets thereby unleashed.

CHAPTER 4

DOMINANT STRATEGY IMPLEMENTATION FOR INTERDEPENDENT ORDINAL PREFERENCES

4.1. Introduction

One of the most fundamental results in the vast literature on social choice theory, implementation, and mechanism design is the well-known impossibility result, namely Gibbard-Satterthwaite Theorem for private value environments. It states that if there are at least three social alternatives and preferences of individuals belong to an unrestricted domain, then the only dominant-strategy implementable and onto social choice rule is dictatorial. When the domain of the preferences are restricted, there are positive results as well. For example, if preferences are quasi-linear, Vickrey-Clarke-Groves mechanism allows the implemention of efficient outcomes for private value choice problems.

Economists have long considered interdependent values as well (see Postlewaite,(1998)). However, research on incentive compatibility of the social choice rules in the direct mechanism with interdependent values has focused almost exclusively on quasilinear utilities as a domain restriction. Some examples are Bergemann and Morris (2008), Jehiel and Moldovanu (2001), Postlewaite and McLean (2015). In contrast, implementation without restricting the domain of preferences to quasilinear form while allowing for a general form of interdependence has not been studied in detail. We consider this problem, specifically under ordinal preferences.

For this purpose, we borrow a general form of interdependence notion of ordinal preferences from Gul and Pesendorfer 2016: agents have ex-post independent abstract types. Each type profile generates a preference profile, possibly interdependent via type profiles. When one agent's type changes it does not affect others' types but it does affect the entire profile of ordinal preferences over outcomes.

Our first contribution is to identify what notion of monotonicity, akin to Maskin monotonicity, under interdependent preferences yields truthfully implementability. Secondly, we move onto identifying appropriate conditions on the interdependence structure and establish a counterpart of Gibbard-Satterthwaite Theorem for dominant-strategy implementation of deterministic social choice functions in an interdependent preferences environment without restricting the domain of preference profiles. We show that if the preferences over alternatives, in particular high ranked alternatives, are not too interdependent then the impossibility result survives and the only possibility is dictatorship. Moreover, if high ranked alternatives are still not too interdependent, but now the preferences are not too independent, then there is even a stronger impossibility result. In this case there are no dominant strategy incentive compatible functions at all: even dictatorship is too permissive.

On another note, we believe our work is relevant for robust mechanism design (see Bergemann and Morris (2008). Researchers have focused on the ex-post incentive compatibility as a conveniently weaker concept than dominant-strategies incentive compatibility. We illustrate, under unrestricted domain of preference profiles, that dominant-strategies incentive compatibility is too restrictive to be useful.

This chapter is organized as follows. Each section starts with a subsection of relevant definitions that are used within the section, then follows with a subsection of results. In section 2 we lay out the framework and establish the revelation principle. In section 3, we give the monotonicity condition and establish implementation results under this condition. In section 4, provide impossibility results after introducing the appropriate restrictions on the interdependence structure.

4.2. Preliminaries and Revelation Principle

4.2.1. Definitions

Let $N = \{1, \ldots, n\}$ be a nonempty finite set of agents. Let A be a nonempty finite set of alternatives and $\mathscr{R}$ be a collection of preference profiles on A. For two alternatives $a, b \in A$ and a preference profile $R \in \mathscr{R}$, aR_ib represents: a is preferred to b with respect to R by agent i. Let $\mathscr{L}^N$ denote the set of all linear order preference profiles on A.

Definition 20 *An **interdependent preference model** (IPM) is a two-tuple $M = (\Theta, \gamma)$, where $\Theta = \Theta_1 \times \cdots \times \Theta_n$ is the type space, and $\gamma : \Theta \to \mathscr{R}$ is a function that assigns each profile of types to a preference profile.*

Notice that an agent's preference may depend on other agents' types. We fix an IPM throughout unless otherwise noted. Since the agents' preferences depend on the realizations of $\theta = (\theta_1, \ldots, \theta_n)$, we may want the collective decision to depend on Θ. To capture this dependence, we define social choice rules on the domain of interdependent type profiles rather than preference profiles, as follows.

Definition 21 *A* ***social choice rule*** *is a correspondence*

$$f : \Theta \to A$$

that assigns to each profile of agents' types $\theta = (\theta_1, \ldots, \theta_n)$ *a non-empty set* $f(\theta) \subseteq A$.

As in the classical mechanism design literature, the social planner's problem is to construct a deterministic mechanism (game form) for the agents to play in order to determine the social outcome.

Definition 22 *A* ***mechanism*** *is a tuple*

$$\Gamma = (S_1, \ldots, S_n, g),$$

where $(S_1, \ldots, S_n)$ *are the agents' strategy sets and*

$$g : S_1 \times \cdots \times S_n \to A$$

is the outcome function.

For the given environment, we combine the IPM $M = (\Theta, \gamma)$ with the mechanism Γ to obtain an incomplete information game. A realized type profile determines a preference profile over outcomes, which in turn induces preferences over message profiles via the outcome function g. Each player sends a message, and this structure defines a non-cooperative game. We are interested in the dominant strategy equilibria of this incomplete information game (without a prior). A pure strategy of an agnet i in this game is a function

$$s_i : \Theta_i \to S_i.$$

Definition 23 *A strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is a **dominant strategy equilibrium** for the mechanism $\Gamma = (S_1, \ldots, S_n, g)$ if, for all $i \in N$ and all $\theta_i \in \Theta_i$,*

$$g(s_i^*(\theta_i), s_{-i})\ \gamma_i(\theta_i, \theta_{-i})\ g(s_i', s_{-i})$$

for all $s_i' \in S_i$, all $s_{-i} \in S_{-i}$, and all $\theta_{-i} \in \Theta_{-i}$.

Similar to an ex-post equilibrium, a dominant strategy equilibrium satisfies an ex-post no-regret property within the game form: an agent would still prefer to play their dominant strategy message even if they knew their true preferences or the true type profile of the other agents.

Let E_g denote the set of dominant strategy equilibria of the mechanism $\Gamma = (S_1, \ldots, S_n, g)$, and define $E_g(\theta) := \{\, s^*(\theta) \mid s^* \in E_g \,\}$.

Definition 24 *The mechanism $\Gamma = (S_1, \ldots, S_n, g)$ is said to **implement** the social choice rule f in dominant strategies if, for every $\theta \in \Theta$,*

- *the set of outcomes induced by dominant strategy equilibria at θ,*

  $$g(E_g(\theta)) = \{\, g(s^*(\theta)) \mid s^* \in E_g \,\},$$

  is non-empty;

- *$g(E_g(\theta)) \subseteq f(\theta)$.*

Definition 25 *A **direct revelation mechanism** is a mechanism where $S_i = \Theta_i$ for every agent i.*

Definition 26 *The social choice rule f is **truthfully implementable** in dominant strategies by the outcome function g if the strategy profile*

$$(s_1^*(\theta_1), \ldots, s_n^*(\theta_n)) = (\theta_1, \ldots, \theta_n)$$

constitutes a dominant strategy equilibrium of the direct revelation mechanism $\Gamma = (\Theta_1, \ldots, \Theta_n, g)$.

That is, for all $i \in N$ and all $\theta_i \in \Theta_i$,

$$g(\theta_i, \theta'_{-i}) \; \gamma_i(\theta_i, \theta_{-i}) \; g(\theta'_i, \theta'_{-i})$$

for all $\theta'_i \in \Theta_i$ and all $\theta_{-i}, \theta'_{-i} \in \Theta_{-i}$, and moreover

$$g(\theta) \in f(\theta) \quad \textit{for all } \theta \in \Theta.$$

*A truthfully implementable social choice rule is also referred to as **dominant strategy incentive compatible**.*

4.2.2. Results

Proposition 5 (Revelation Principle for Dominant Strategy) *If there exists a mechanism $\Gamma = (S_1, \ldots, S_n, g)$ that implements the social choice rule f in dominant strategies, then f is truthfully implementable in dominant strategies.*

Proof. If $\Gamma = (S_1, \ldots, S_n, g)$ implements f in dominant strategies, then there exists a profile of strategies $s^* = (s_1^*, \ldots, s_n^*)$ such that

i) for every $\theta \in \Theta$, we have $g(s^*(\theta)) \in f(\theta)$, and

ii) for every $i \in N$ and every $\theta_i \in \Theta_i$,

$$g(s_i^*(\theta_i), s_{-i}) \ \gamma_i(\theta_i, \theta_{-i}) \ g(s_i', s_{-i})$$

for all $s_i' \in S_i$, all $s_{-i} \in S_{-i}$, and all $\theta_{-i} \in \Theta_{-i}$.

In particular, for every $i \in N$ and every $\theta_i \in \Theta_i$,

$$g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}')) \ \gamma_i(\theta_i, \theta_{-i}) \ g(s_i^*(\theta_i'), s_{-i}^*(\theta_{-i}'))$$

for all $\theta_i' \in \Theta_i$ and all $\theta_{-i}, \theta_{-i}' \in \Theta_{-i}$.

Define the direct revelation mechanism with outcome function $h : \Theta \to A$ by

$$h(\theta) := g(s^*(\theta)) \quad \text{for all } \theta \in \Theta.$$

Since $g(s^*(\theta)) = h(\theta)$ for all θ by construction, we have, for every $i \in N$ and every $\theta_i \in \Theta_i$,

$$h(\theta_i, \theta_{-i}') \ \gamma_i(\theta_i, \theta_{-i}) \ h(\theta_i', \theta_{-i}')$$

for all $\theta_i' \in \Theta_i$ and all $\theta_{-i}, \theta_{-i}' \in \Theta_{-i}$.

Thus, in the mechanism h, truth-telling is always a dominant strategy. Hence, by definition, f is truthfully implementable in dominant strategies. ■

The Revelation Principle states that if a mechanism g implements a social choice rule f in dominant strategies, then there exists a direct mechanism h that implements it truthfully. In truthful implementation we require only that truth-telling in the direct mechanism be a dominant strategy, not necessarily the unique one. The direct mechanism h may also have equilibria other than the truthful equilibrium, and these equilibria may not correspond to those in the game form g, nor do they necessarily yield outcomes selected by f. In other words, such additional equilibria may lead to situations where the direct mechanism h does not implement the social choice rule f, even though the indirect mechanism g does. We conclude this section by giving the definition of full implementation.

Definition 27 *The mechanism $\Gamma = (S_1, \ldots, S_n, g)$ **fully implements** the social choice rule f in dominant strategies if, for every $\theta \in \Theta$,*

$$g(E_g(\theta)) = f(\theta).$$

Proposition 6 *Let $\gamma(\Theta) \subseteq \mathscr{L}^N$. If the social choice rule f is fully implementable in dominant strategies, then f is a function.*

Proof. Suppose not. Then there exists a profile $\theta \in \Theta$ such that $a, b \in f(\theta)$ with $a \neq b$. If $g : S \to A$ implements f in dominant strategies, there exist dominant strategy equilibria $s, s' \in S$ for θ such that $g(s) = a$ and $g(s') = b$.

Since both s_1 and s'_1 are dominant strategies for agent 1, we have

$$g(s'_1, s_{-1})\ \gamma_1(\theta_1, \theta_{-1})\ g(s_1, s_{-1}),$$

and

$$g(s_1, s_{-1})\ \gamma_1(\theta_1, \theta_{-1})\ g(s'_1, s_{-1}).$$

By the linear order assumption, this implies

$$g(s'_1, s_{-1}) = g(s).$$

Similarly,

$$g(s'_1, s'_2, s_{-\{1,2\}}) = g(s'_1, s_{-1}).$$

Proceeding iteratively in this way, we obtain $g(s') = a$.

But this contradicts $g(s') = b$ with $a \neq b$. Hence $f(\theta)$ must be a singleton for every $\theta \in \Theta$, and f is a function. ∎

Corollary 1 *The social choice rule f is fully implementable in dominant strategies if and only if f is a social choice function that is implementable.*

This result means that focusing on full implementation of social choice rules is equivalent to focusing on implementation of social choice functions. We will make use of this result later.

4.3. Characterization with Monotonicity

4.3.1. Definitions

Maskin introduced a monotonicity notion which is a necessary and almost sufficient condition for both complete information Nash implementation and dominant strategy implementation. In a complete information setup, each agent i is assumed to know the entire type profile θ rather than only their private type θ_i. We now state the notion of monotonicity in a similar form. Our contribution in this section is to identify the lower contour set notion that is appropriate for our framework.

Definition 28 *For agent i with type θ_i, the* ***lower contour set*** *of an alternative*

a is

$$L_i(a, \theta_i) = \bigcap_{\theta_{-i} \in \Theta_{-i}} \{\, b \in A \mid b \neq a \text{ and } a \ \gamma_i(\theta_i, \theta_{-i}) \ b \,\}.$$

The lower contour set of an alternative a for agent i with type θ_i is the set of all alternatives $b \neq a$ such that agent i weakly prefers a to b for every possible profile θ_{-i} of the other agents' types.

Definition 29 *The social choice rule f is **monotonic** if, for any $\theta \in \Theta$ and any $a \in A$, the following holds: if $a \in f(\theta)$ and θ' satisfies*

$$L_i(a, \theta_i) \subseteq L_i(a, \theta'_i) \quad \text{for all } i,$$

then $a \in f(\theta')$.

Monotonicity is a very strong restriction. For instance, there exist interdependent preference models in which even dictatorial social choice functions are not monotonic.

Example 6 *Consider the following IPM where $\Theta_i = \{\theta_i^1, \theta_i^2\}$ for all $i \in \{1, 2\}$, with the preference function γ represented in the matrix below:*

	θ_2^1	θ_2^2
θ_1^1	$\begin{array}{cc} \gamma_1 & \gamma_2 \\ \hline a & a \\ \vdots & \vdots \end{array}$	$\begin{array}{cc} \gamma_1 & \gamma_2 \\ \hline \vdots & \vdots \\ a & b \end{array}$
θ_1^2	$\begin{array}{cc} \gamma_1 & \gamma_2 \\ \hline \vdots & \vdots \\ b & a \end{array}$	$\begin{array}{cc} \gamma_1 & \gamma_2 \\ \hline \vdots & \vdots \\ a & b \end{array}$

Let f be dictatorial with agent 1 as the dictator, so $f(\theta_1^1, \theta_2^1) = a$. Note that $L_1(a, \theta_1^1) = L_2(a, \theta_2^1) = \emptyset$, so by monotonicity we must have $f(\theta_1^2, \theta_2^2) = a$, contradicting the fact that agent 1 is the dictator.

To establish sufficiency of monotonicity for truthful implementability, we will use a rich domain assumption. In classical mechanism design with private preferences, sufficiency requires only that the domain allow us to rank any pair of alternatives first and second. We therefore impose a condition similar to the rich domain assumption in Dasgupta, Hammond, and Maskin (1979). To define a rich domain, we first introduce the concept of improvement.

Definition 30 *If $a \ \gamma_i(\theta_i, \theta_{-i}) \ b$ and $b \ \gamma_i(\theta_i', \theta_{-i}) \ a$, with at least one of these preferences strict, then we say that b **improves** with respect to a for agent i as the state changes from (θ_i, θ_{-i}) to (θ_i', θ_{-i}).*

Definition 31 *The domain is **rich** if the following holds: whenever there exist $i \in N$, $a, b \in A$, $\theta \in \Theta$, and $\theta_i' \in \Theta_i$ such that b does not improve with respect to a for agent i as the state changes from (θ_i, θ_{-i}) to (θ_i', θ_{-i}), then there exists $\theta_i'' \in \Theta_i$ such that*

$$L_i(a, \theta_i) \subseteq L_i(a, \theta_i'') \quad \text{and} \quad L_i(b, \theta_i') \subseteq L_i(b, \theta_i'').$$

4.3.2. Results

Proposition 7 (Necessity) *Let $\gamma(\Theta) \subseteq \mathscr{L}^N$. If the social choice function f is truthfully implementable in dominant strategies, then it is monotonic.*

Proof. Consider two type profiles θ and θ' such that

$$L_i(f(\theta), \theta_i) \subseteq L_i(f(\theta), \theta_i') \quad \text{for all } i.$$

We must show that $f(\theta') = f(\theta)$.

Since f is truthfully implementable in dominant strategies and is singleton-valued, for every $\theta''_{-1} \in \Theta_{-1}$ we have

$$f(\theta_1, \theta_{-1}) \; \gamma_1(\theta_1, \theta''_{-1}) \; f(\theta_1', \theta_{-1}),$$

which implies

$$f(\theta_1', \theta_{-1}) \in L_1(f(\theta), \theta_1).$$

By assumption, $L_1(f(\theta), \theta_1) \subseteq L_1(f(\theta), \theta_1')$, so

$$f(\theta_1', \theta_{-1}) \in L_1(f(\theta), \theta_1').$$

By truthful implementability in dominant strategies, we also have for every $\theta''_{-1} \in \Theta_{-1}$,

$$f(\theta_1', \theta_{-1}) \; \gamma_1(\theta_1', \theta''_{-1}) \; f(\theta_1, \theta_{-1}),$$

which implies

$$f(\theta) \in L_1(f(\theta_1', \theta_{-1}), \theta_1').$$

Since $\gamma(\theta) \in \mathscr{L}^N$, this yields

$$f(\theta_1', \theta_{-1}) = f(\theta).$$

Proceeding inductively over agents $i = 1, \ldots, N$, we obtain

$$f(\theta') = f(\theta).$$

■

Proposition 8 (Sufficiency) *Suppose f is a monotonic social choice function and the domain is rich. Then f is truthfully implementable in dominant strategies.*

Proof. Let f be as hypothesized. We must show that for all $i \in N$ and all $\theta_i \in \Theta_i$,

$$f(\theta_i, \theta'_{-i}) \; \gamma_i(\theta_i, \theta_{-i}) \; f(\theta'_i, \theta'_{-i})$$

for all $\theta'_i \in \Theta_i$ and all $\theta_{-i}, \theta'_{-i} \in \Theta_{-i}$.

If $f(\theta_i, \theta'_{-i}) = f(\theta'_i, \theta'_{-i})$, then the condition is satisfied. Suppose instead that

$$f(\theta_i, \theta'_{-i}) \neq f(\theta'_i, \theta'_{-i})$$

for some i, $\theta_i \in \Theta_i$, $\theta'_i \in \Theta_i$, and $\theta'_{-i} \in \Theta_{-i}$.

Claim: $f(\theta'_i, \theta'_{-i})$ improves with respect to $f(\theta_i, \theta'_{-i})$ for agent i as the state changes from (θ_i, θ_{-i}) to (θ'_i, θ_{-i}).

If the claim were false, the definition of a rich domain implies that there exists $\theta''_i \in \Theta_i$ such that

$$L_i(f(\theta_i, \theta'_{-i}), \theta_i) \subseteq L_i(f(\theta_i, \theta'_{-i}), \theta''_i) \quad \text{and} \quad L_i(f(\theta'_i, \theta'_{-i}), \theta'_i) \subseteq L_i(f(\theta'_i, \theta'_{-i}), \theta''_i).$$

By monotonicity, this would imply

$$f(\theta_i'', \theta_{-i}') = f(\theta_i, \theta_{-i}') \quad \text{and} \quad f(\theta_i'', \theta_{-i}') = f(\theta_i', \theta_{-i}'),$$

contradicting the fact that f is a function. Hence the claim holds.

Therefore, since $f(\theta_i', \theta_{-i}')$ improves with respect to $f(\theta_i, \theta_{-i}')$, we have

$$f(\theta_i, \theta_{-i}') \; \gamma_i(\theta_i, \theta_{-i}) \; f(\theta_i', \theta_{-i}').$$

Thus f is truthfully implementable in dominant strategies. ■

Theorem 15 *Let $\gamma(\Theta) \subseteq \mathscr{L}^N$ and suppose the domain is rich. A social choice function f is truthfully implementable in dominant strategies if and only if it is monotonic.*

The following two corollaries summarize the necessary and sufficient conditions for a social choice rule to be truthfully and fully implementable.

Corollary 2 *Let $\gamma(\Theta) \subseteq \mathscr{L}^N$ and suppose the domain is rich. A social choice rule f is truthfully implementable in dominant strategies if and only if there exists a monotonic social choice function f^* such that $f^*(\theta) \in f(\theta)$ for all $\theta \in \Theta$.*

Proof. Let $g : \Theta \to A$ be the truthful implementation of f. Define $f^*(\theta) = g(\theta)$ for all $\theta \in \Theta$. Then g implements f^* truthfully, and hence f^* is monotonic.

Conversely, if f^* is monotonic, then it is truthfully implementable in dominant strategies. Since $f^*(\theta) \in f(\theta)$ for all θ, it follows that f is also truthfully implementable in dominant strategies. ■

Corollary 3 *Let $\gamma(\Theta) \subseteq \mathscr{L}^N$ and suppose the domain is rich. A social choice rule f is fully implementable in dominant strategies if and only if f is a monotonic social choice function.*

4.4. Impossibility Results

4.4.1. Interdependent Preference Model

Let $T(R)$ stand for the top two alternatives of the strict preference ordering R. Let $Q(\theta_i)$ stands for all possible top-ranked alternatives of agent i at θ_i, i.e. $Q(\theta_i) = \bigcup_{\theta_{-i} \in \Theta_{-i}} \{a \in A | a\gamma_i(\theta_i, \theta_{-i})b$ for all $b \in A\}$. Let $W(\theta_{-i})$ stands for all possible top-ranked alternatives of agent i whenever the other agents have θ_{-i}, i.e. $W(\theta_{-i}) = \bigcup_{\theta_i \in \Theta_i} \{a \in A | a\gamma_i(\theta_i, \theta_{-i})b$ for all $b \in A\}$.

Definition 32 *The IPM $M = (\Theta, \gamma)$ satisfies* ***top two alternatives immunity*** *if for any $\theta \in \Theta$ and for any $\{a, b\} \subset A$, if we have $T(\gamma_i(\theta)) = \{a, b\}$ for all i, then there exists $\theta' \in \Theta$ such that $T(\gamma_i(\theta_i', \theta_{-i}''))) = \{a, b\}$ and $\gamma_i(\theta_i', \theta_{-i}'')|\{a, b\} = \gamma_i(\theta)|\{a, b\}$ for all $\theta_{-i}'' \in \Theta_{-i}$ and for all i.*

The definition states that high-ranked alternatives are close to being independent. If there exists a type profile where all agents have the same two alternatives at the top of their preferences, then for each agent i there exists a type such that agent i still keeps the same two alternatives in his preferences regardless of the other agents' type profile.

Definition 33 *The IPM $M = (\Theta, \gamma)$ satisfies* ***top independence*** *if there exists $i \in N$ such that $Q(\theta_i)$ is a singleton for every $\theta_i \in \Theta_i$.*

4.4.2. Social Choice Rules

Now we define some properties of social choice rules that will be used repeatedly in what follows, and we give an example to relate these properties to implementation.

Definition 34 *The social choice rule $f(\cdot)$ is* ***Pareto efficient*** *if, for no profile $\theta \in \Theta$, there exists an $a \in A$ such that $a\gamma_i(\theta)f(\theta)$ for every i, with strict preference for some $i \in N$.*

Definition 35 *The social choice rule $f(\cdot)$ is* ***onto*** *if, for every $a \in A$, there exists a profile of types $\theta \in \Theta$ such that $a \in f(\theta)$.*

Example 7 *Consider the IPM below where $\Theta_i = \{\theta_i^1, \theta_i^2\}$ for all $i \in \{1, 2\}$, and the γ function is represented in the matrix:*

	θ_2^1		θ_2^2	
	γ_1	γ_2	γ_1	γ_2
θ_1^1	c	d	$\boldsymbol{c}$	$\boldsymbol{c}$
	a	a	a	b
	$\boldsymbol{b}$	$\boldsymbol{b}$	b	a
	d	c	d	d
	γ_1	γ_2	γ_1	γ_2
θ_1^2	$\boldsymbol{d}$	b	$\boldsymbol{a}$	c
	b	c	c	b
	a	$\boldsymbol{d}$	d	$\boldsymbol{a}$
	c	a	b	d

Let $f(\cdot)$ be a social choice function such that

$$f(\theta_1^1, \theta_2^1) = b, \quad f(\theta_1^1, \theta_2^2) = c, \quad f(\theta_1^2, \theta_2^1) = d, \quad f(\theta_1^2, \theta_2^2) = a.$$

Then $f(\cdot)$ is truthfully implementable and onto, but not efficient.

Definition 36 *The social choice rule $f(\cdot)$ is* ***unanimous*** *if, whenever an alternative a is at the top of every individual i's ranking $\gamma_i(\theta)$, then $a \in f(\theta)$.*

Definition 37 *The social choice rule $f(\cdot)$ is* ***dictatorial*** *if there exists an agent i such that, for all $\theta \in \Theta$ and for all $a \in A$,*

$$a \in f(\theta) \quad \text{only if} \quad \forall b \in A,\ a\gamma_i(\theta)b.$$

4.4.3. Social Welfare Functions

Definition 38 *A* ***social welfare function*** *is a mapping $F : \Theta_1 \times \cdots \times \Theta_n \to \mathscr{R}$ that assigns to each profile of agents' types $(\theta_1, \theta_2, \ldots, \theta_n)$ a preference relation $F(\theta_1, \theta_2, \ldots, \theta_n) \in \mathscr{R}$.*

Definition 39 *The social welfare function $F(\cdot)$ satisfies* ***Pareto efficiency*** *if, for any pair of alternatives $\{a, b\} \subset A$ and any type profile $\theta \in \Theta$, we have $aF(\theta)b$ whenever $a\gamma_i(\theta)b$ for every $i \in N$.*

Definition 40 *The social welfare function $F(\cdot)$ satisfies* ***independence of irrelevant alternatives*** *(IIA) if the social preference between any two alternatives $\{a, b\} \subset A$ depends only on the profile of individual preferences over those alternatives. Formally, for any $\{a, b\} \subset A$, if $\gamma_i(\theta)|\{a, b\} = \gamma_i(\theta')|\{a, b\}$ for all $i \in N$,* *then $F(\theta)|\{a,b\} = F(\theta')|\{a,b\}$, where $R|\{a,b\}$ denotes the restriction of the preference ordering R to the set $\{a,b\}$.*

Definition 41 *The social welfare function $F(\cdot)$ is* ***dictatorial*** *if there exists an agent $i \in N$ such that, for all $\theta \in \Theta$ and for all $\{a,b\} \subset A$, we have $aF(\theta)b$ whenever $a\gamma_i(\theta)b$.*

4.4.4. Results

Theorem 16 (Arrow's Impossibility Theorem) *Let $\gamma(\Theta) = \mathscr{L}^N$ and suppose $|A| \geq 3$. If the social welfare function $F(\cdot)$ satisfies Pareto efficiency and IIA, then $F(\cdot)$ is dictatorial.*

Proposition 9 *Let $\gamma(\Theta) \subseteq \mathscr{L}^N$. If the social choice function $f(\cdot)$ is truthfully implementable in dominant strategies and onto, then it is unanimous.*

Proof. Suppose not. Then there exists a profile $\theta \in \Theta$ and $a \in A$ such that $a\gamma_i(\theta)b$ for every $b \in A$ and every i, but $f(\theta) \neq a$. Since $f(\cdot)$ is onto, there exists a profile $\theta' \in \Theta$ such that $f(\theta') = a$.

By truthfully implementability in dominant strategies,

$$f(\theta_1, \theta'_{-1})\, \gamma_1(\theta_1, \theta_{-1})\, f(\theta'_1, \theta'_{-1}).$$

Since $a\gamma_1(\theta)b$ for every $b \in A$ and $\gamma(\Theta) \subseteq \mathscr{L}^N$, it follows that $f(\theta_1, \theta'_{-1}) = a$, which differs from $f(\theta)$.

By truthfully implementability again,

$$f(\theta_1,\theta_2,\theta^{'}_{-\{1,2\}})\,\gamma_2(\theta_2,\theta_{-2})\,f(\theta_1,\theta^{'}_{-1}).$$

By the same reasoning, $f(\theta_1,\theta_2,\theta^{'}_{-\{1,2\}}) = a$, again differing from $f(\theta)$.

Proceeding iteratively over agents yields the contradiction that $f(\theta) = a \neq f(\theta)$. Hence, $f(\cdot)$ must be unanimous. ■

Proposition 10 *Let $M = (\Theta, \gamma)$ satisfy top two alternatives immunity and $\gamma(\Theta) = \mathscr{L}^N$. If the social choice function $f(\cdot)$ is truthfully implementable in dominant strategies and onto, then it is Pareto efficient.*

Proof. Suppose not. Then there exists a profile $\theta \in \Theta$ and a pair $\{a,b\} \subset A$ such that $a\gamma_i(\theta)b$ for every i, but $f(\theta) = b$.

Since $\gamma(\Theta) = \mathscr{L}^N$, there exists a profile $\theta^{'} \in \Theta$ such that $T(\gamma_i(\theta^{'})) = \{a,b\}$ and $a\gamma_i(\theta)b$ for all i. Clearly $\theta^{'} \neq \theta$. Because $f(\cdot)$ is truthfully implementable in dominant strategies and onto, unanimity implies $f(\theta^{'}) = a$.

Now, since $M = (\Theta, \gamma)$ satisfies top two alternatives immunity, there exists $\theta^{''} \in \Theta$ such that

$$L_i(b,\theta) \subseteq L_i(b,\theta^{''}) \quad \text{and} \quad L_i(a,\theta^{'}) \subseteq L_i(a,\theta^{''})$$

for all i.

By truthfully implementability in dominant strategies, monotonicity applies, yielding $f(\theta) = f(\theta^{'})$, a contradiction since $f(\theta) = b \neq a = f(\theta^{'})$. Hence, $f(\cdot)$ must be Pareto efficient. ■

Theorem 17 (Gibbard–Satterthwaite) *Let $M = (\Theta, \gamma)$ satisfy top two alternatives immunity, and suppose $\gamma(\Theta) = \mathscr{L}^N$ with $|A| \geq 3$. If the social choice function $f(\cdot)$ is truthfully implementable in dominant strategies and onto, then it is dictatorial.*

Proof. We derive a social welfare function $F(\cdot)$ that rationalizes $f(\cdot)$, i.e., for every profile $\theta \in \Theta$, $f(\theta)$ is the most preferred alternative according to $F(\theta)$ in A. For every profile $\theta \in \Theta$ and every pair of alternatives $\{a, b\} \subset A$, define $aF(\theta)b$ if $a = f(\theta')$, where $T(\gamma_i(\theta')) = \{a, b\}$ and $\gamma_i(\theta')|\{a, b\} = \gamma_i(\theta)|\{a, b\}$ for all i. Since $f(\cdot)$ is Pareto efficient and monotonic, the preference relation $F(\theta)$ is well defined and complete: by efficiency, $f(\theta') \in \{a, b\}$, and by monotonicity, $f(\theta'') = f(\theta')$ for every $\theta'' \in \Theta$ with $T(\gamma_i(\theta'')) = \{a, b\}$ and $\gamma_i(\theta'')|\{a, b\} = \gamma_i(\theta)|\{a, b\}$ for all i.

Next, we verify that $F(\theta)$ is transitive. Suppose $aF(\theta)b$ and $bF(\theta)c$, where $\{a, b, c\} \subset A$ are, without loss of generality, all distinct. Consider θ'' where the top three alternatives in each agent's preference profile are $\{a, b, c\}$ and $\gamma_i(\theta'')|\{a, b, c\} = \gamma_i(\theta)|\{a, b, c\}$ for all i. We claim that $f(\theta'') \neq b$. Suppose instead that $f(\theta'') = b$. Then consider the profile θ' where $T(\gamma_i(\theta')) = \{a, b\}$ and $\gamma_i(\theta')|\{a, b\} = \gamma_i(\theta)|\{a, b\}$ for all i. Since $aF(\theta)b$, we have $f(\theta') = a$, but monotonicity together with top two alternatives immunity implies $f(\theta') = b$. Contradiction. Hence $f(\theta'') \neq b$. By a similar argument, $f(\theta'') \neq c$. Since $f(\cdot)$ is efficient, $f(\theta'') \in \{a, b, c\}$, so $f(\theta'') = a$. Now consider the profile θ' where $T(\gamma_i(\theta')) = \{a, c\}$ and $\gamma_i(\theta')|\{a, c\} = \gamma_i(\theta'')|\{a, c\}$ for all i. Monotonicity and top two alternatives immunity imply $f(\theta') = a$. But by construction $\gamma_i(\theta')|\{a, c\} = \gamma_i(\theta)|\{a, c\}$ for all i, hence $aF(\theta)c$. Thus $F(\theta)$ is transitive, and so $F(\theta)$ is a linear order on A for every $\theta \in \Theta$.

We now show that $F(\cdot)$ rationalizes $f(\cdot)$. Suppose not: let $f(\theta) = b$ but $aF(\theta)b$. Consider θ' with $T(\gamma_i(\theta')) = \{a, b\}$ and $\gamma_i(\theta')|\{a, b\} = \gamma_i(\theta)|\{a, b\}$ for all i. By monotonicity and top two alternatives immunity, $f(\theta') = b$, contradicting $aF(\theta)b$. Hence $f(\theta)$ is indeed the most preferred alternative for $F(\theta)$.

Next, consider a profile θ such that $a\gamma_i(\theta)b$ for every $i \in N$. Take θ' with $T(\gamma_i(\theta')) = \{a, b\}$ and $\gamma_i(\theta')|\{a, b\} = \gamma_i(\theta)|\{a, b\}$ for all $i \in N$. By unanimity, $f(\theta') = a$, so $aF(\theta)b$. Therefore, $F(\cdot)$ satisfies Pareto efficiency.

Now consider profiles θ and θ' such that $\gamma_i(\theta)|\{a, b\} = \gamma_i(\theta')|\{a, b\}$ for all i. By construction, $F(\theta)|\{a, b\} = F(\theta')|\{a, b\}$, so $F(\cdot)$ satisfies IIA.

By Arrow's Impossibility Theorem, $F(\cdot)$ is dictatorial: there exists an agent i such that, for all $\theta \in \Theta$ and all $\{a, b\} \subset A$, we have $aF(\theta)b$ whenever $a\gamma_i(\theta)b$. Since $F(\cdot)$ rationalizes $f(\cdot)$, it follows that $f(\theta)\gamma_i(\theta)b$ for all $b \in A$. Hence agent i is a dictator for $f(\cdot)$. ■

Lemma 3 *Let $\gamma(\Theta) \subseteq \mathscr{L}^N$. If there exists a social choice function $f(\cdot)$ which is truthfully implementable in dominant strategies and dictatorial for some agent i, then for every $\theta \in \Theta$, $Q(\theta_i) \cap W(\theta_{-i})$ is a singleton.*

Proof. Suppose not. Then there exists $\theta \in \Theta$ such that $Q(\theta_i) \cap W(\theta_{-i})$ is not a singleton. Let $a = f(\theta) = TR_i(\theta)$, where $TR_i(\theta)$ denotes the top-ranked alternative of agent i at the type profile θ. Clearly, $a \in Q(\theta_i) \cap W(\theta_{-i})$.

By assumption, there exists $b \in A$, $b \neq a$, such that $b \in Q(\theta_i) \cap W(\theta_{-i})$. By definition, there exist $\theta_i^{'} \in \Theta_i$ and $\theta_{-i}^{'} \in \Theta_{-i}$ such that

$$b = TR(\theta_i, \theta_{-i}^{'}) = TR(\theta_i^{'}, \theta_{-i}),$$

which implies

$$b = f(\theta_i, \theta'_{-i}) = f(\theta'_i, \theta_{-i})$$

by the dictatorship of agent i.

Now, when the true type profile is (θ_i, θ'_{-i}) and the agents other than i announce θ_{-i}, agent i has an incentive to deviate from θ_i to θ'_i, i.e.

$$f(\theta'_i, \theta_{-i}) \; \gamma_i(\theta_i, \theta'_{-i}) \; f(\theta_i, \theta_{-i}),$$

which contradicts $f(\cdot)$ being truthfully implementable in dominant strategies. ■

Theorem 18 *Let $M = (\Theta, \gamma)$ satisfy top two alternatives immunity, $\gamma(\Theta) = \mathscr{L}^N$, and $|A| \geq 3$. Then M satisfies top independence if and only if there exists a social choice function $f(\cdot)$ which is truthfully implementable in dominant strategies and onto (hence dictatorial).*

Proof. Suppose there exists a social choice function $f(\cdot)$ which is truthfully implementable in dominant strategies and onto. By the Gibbard–Satterthwaite theorem, $f(\cdot)$ must be dictatorial. Let i be the dictator. We show that $Q(\theta_i)$ is a singleton for every $\theta_i \in \Theta_i$.

Take any $a \in A$ and any $b \in A$. Since $\gamma(\Theta) = \mathscr{L}^N$, there exists a profile $\theta \in \Theta$ such that every agent ranks a first and b second, i.e. $TR_i(\theta) = a$ for all $i \in N$ and $T(\gamma_i(\theta)) = \{a, b\}$. Since M satisfies top two alternatives immunity, there exists $\theta' \in \Theta$ such that

$$T(\gamma_i(\theta'_i, \theta''_{-i})) = \{a, b\} \quad \text{and} \quad \gamma_i(\theta'_i, \theta''_{-i})|_{\{a,b\}} = \gamma_i(\theta)|_{\{a,b\}}$$

for all $\theta''_{-i} \in \Theta_{-i}$ and for all i. Thus $Q(\theta'_i) = \{a\}$. Since a was arbitrary, for every

$a \in A$ there exists θ_i^a such that $Q(\theta_i^a) = \{a\}$.

Now fix any $\theta_{-i} \in \Theta_{-i}$. For every $a \in A$, we have $TR_i(\theta_i^a, \theta_{-i}) = a$. Hence, as a ranges over A, we obtain $W(\theta_{-i}) = A$.

Finally, take any $\theta_i \in \Theta_i$. By the lemma, $Q(\theta_i) \cap W(\theta_{-i})$ is a singleton for every $\theta \in \Theta$. Since $W(\theta_{-i}) = A$, this implies $Q(\theta_i)$ itself is a singleton. Therefore, M satisfies top independence.

Conversely, suppose M satisfies top independence. Then there exists $i \in N$ such that $Q(\theta_i)$ is a singleton for every $\theta_i \in \Theta_i$. Define $f(\cdot)$ to be the dictatorial rule of agent i. Since $\gamma(\Theta) = \mathscr{L}^N$, the rule $f(\cdot)$ is onto.

Consider any profile $\theta \in \Theta$. Because $Q(\theta_i)$ is a singleton and $f(\cdot)$ is dictatorial, the agents other than i cannot affect the outcome by misreporting. Moreover, since $Q(\theta_i)$ is a singleton, agent i cannot gain by announcing a type different from his true type. Thus $f(\cdot)$ is truthfully implementable in dominant strategies. ■

4.5. Conclusion

Along the lines of Maskin monotonicity, we have introduced an appropriate notion of monotonicity for interdependent preferences. Since an agent's preferences depend on the types of others, the relevant lower contour set at a given type must account for all possible less-preferred alternatives across other agents' types. We show that this notion of monotonicity is necessary and, in rich domains, sufficient for dominant-strategy incentive compatibility.

Under the conditions on interdependence structures we have pinned down, counterparts of fundamental results have been presented as the ordinal preferences are assumed to be interdependent. If the top two alternatives are not too interdependent, then only possibility is dictatorship. Yet it is not a priori clear when even dictatorship is incentive compatible with dominant strategies. We show that if the lower-ranked parts of preferences are not too interdependent, dictatorship is indeed dominant-strategy implementable; however, if those parts are sufficiently interdependent, then even dictatorship becomes too permissive.

Furthermore, we think our results also connect to the literature on robust mechanism design. Dominant strategy incentive compatibility which is considered but not studied much in the recent literature, is extremely restrictive and this is suggestive of why it is possibly not fruitful to use it as opposed to ex-post incentive compatibility.

Investigating how domain restrictions and interdependence structure interplay with regards to implementation is an avenue for future work.

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