EXPLORATIONS ON MONOTONICITY IN SOCIAL CHOICE THEORY

A Master's Thesis

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of
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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

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ABSTRACT

EXPLORATIONS ON MONOTONICITY IN SOCIAL CHOICE THEORY

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Due to Maskin (1977), Maskin-monotonicity is known to be a necessary condition for Nash-implementability. Once one classifies social choice rules as the ones which are Maskin-monotonic and those which are not, a natural question one may ask is whether it is possible to further classify the Maskin-monotonic social choice rules according to how strongly monotonic they are. This study utilizes two key notions , namely *self-monotonicity* and *center*, which enable us to compare Maskin-monotonic social choice rules among themselves according to the strength of their monotonicities. Moreover, Nash-implementable two-person social choice rules are now characterized via the notion of center, in line with the conjecture that Implementation Theory can be rewritten in terms of monotonicity.

Keywords: Social Choice, Monotonicity, Self-monotonicity, Center, Implementation.

ÖZET

SOSYAL SEÇİM KURAMI'NDA TEKDÜZELİK ÜZERİNE BAZI İNCELEMELER

DOĞAN, Battal

Yüksek Lisans, Ekonomi Bölümü

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Maskin'den (1977) dolayı, Maskin-tekdüzelik Nash-uygulanabilirlik için bir gerek koşul olarak bilinmektedir. Sosyal Seçim Kuralları'nı Maskin-tekdüze olanlar ve olmayanlar olarak sınıflandırdıktan sonra sorulabilecek doğal bir soru, Maskin-tekdüze Sosyal Seçim Kuralları'nın tekdüzelik derecelerine göre sınıflandırılıp sınıflandırılamayacağıdır. Bu çalışmada, *Öz-Tekdüzelik* ve *Merkez* kavramları kullanılarak Maskin-tekdüze Sosyal Seçim Kuralları tekdüzelik derecelerine göre karşılaştırılmış ve yine bu kavramlar kullanılarak, Uygulama Kuramı'nın tekdüzelik cinsinden yeniden yazılabileceği tahminini destekler nitelikte olduğunu düşündüğümüz yeni bir Nash-uygulanabilirlik karakterizasyonu sunulmuştur.

Anahtar Kelimeler: Sosyal Seçim, Tekdüzelik, Öz-Tekdüzelik, Merkez, Uygulama.

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CHAPTER 1

INTRODUCTION

Given a society where each individual is endowed with preferences over a set of alternatives, the problem of aggregating these individual preferences into a social preference or a social choice has been the main topic of Social Choice Theory. Depending on what a society regards as desirable, it can decide on a Social Choice Rule (SCR) to achieve this aim. However, in most of cases, a central authority who is to enforce this SCR will not be able to observe the actual preferences of the individuals. It is also well-known that trying to elicit the true preference profile by directly asking the individuals about their preferences is hopeless except for in some trivial situations. This situation gives rise to Implementation Theory whose main question can be summarized as follows: ”Is it possible to design a mechanism which, for each particular state of the society, leads to a game whose equilibrium outcomes will coincide with the ones prescribed by the Social Choice Rule?” Of course, the notion of equilibrium employed is to reflect the mode of behavior of the individuals in the society. Here we will try to shed some further light on the relationship between Nash implementability and monotonicity by utilizing a more refined approach to the latter.

Maskin (1977) showed that any SCR that is implementable in Nash-equilibrium has to be monotonic in the sense that, if some alternative is selected by an SCR under some preference profile, it will continue to get selected under any preference profile where every agent continues to rank alternatives that were not better than the chosen alternative under the former profile below that alternative. Maskin also introduced sufficient conditions for Nash-implementability, leaving a full characterization as an open question. It was Moore and Repullo (1990) who first introduced conditions which are both necessary and sufficient for Nash implementability. But their characterization is based on the existence of a system of sets satisfying certain conditions which are rather complicated to check. Danilov (1992) simplified Moore and Repullo's (1990) work considerably by introducing the notion of essential monotonicity which in the presence of at least three agents turned out to be equivalent to Nash implementability. Danilov (1990) also gave a characterization for the two-agent case by conjoining essential monotonicity with certain other properties.

Kaya and Koray (2000) characterize the solution concepts which only implement Maskin-monotonic social choice rules. They find that it is simply the monotonicities of a solution concept σ that get inherited by all the SCR's which are σ-implementable. Thus, a natural question is now whether it is possible to classify social choice rules according to the monotonicity conditions they satisfy. The idea of Self-Monotonicity, which was introduced by Koray (2002) departing from this question, refers to the strongest monotonicities satisfied by a social choice rule and allows us to compare social choice rules in accordance with how monotonic they are. Also, the notions of a critical profile and center, which originated from a study of Koray, Adali, Erol and Ordulu (2001), will turn out to be telling about the implementability of an SCR.

The thesis starts with definitions of some basic notions including Self-Monotonicity, critical profile and center. The second chapter deals with the problem of how social choice rules can be compared regarding the strength of the monotonicity conditions they satisfy. The last chapter is devoted to the main result of the thesis, namely a characterization of Nash-implementability for the two-agent case via the notions of critical profile and center. Our characterization is different from the existing characterizations in the literature, possibly also regarding its clarity and simplicity.

CHAPTER 2

SELF-MONOTONICITY AND NOTION OF A CENTER

2.1 Notation and Definitions

Let N denote a finite set of participants, A denote a finite set of alternatives and $\mathscr{L}(A)^N$ be the set of all family of linear orders (preference profiles) on A. A *Social Choice Rule(SCR)* F is a function $F : \mathscr{L}(A)^N \to 2^A$. We will denote the set of all SCR's as $\mathscr{F}$.

For a preference profile $R \in \mathscr{L}(A)^N$, an alternative $a \in A$ and some agent $i \in N$, let $L_i(a, R) = \{b \in A \mid aR_ib\}$. Given $a \in A$, $\rho(a)$ will denote the following partition of $\mathscr{L}(A)^N$ induced by a;

$$\rho(a) = \Big\{\{R' \in \mathscr{L}(A)^N \mid \forall i \in N : L_i(a, R') = L_i(a, R)\} \mid R \in \mathscr{L}(A)^N\Big\}$$

Definition. Let $R, R' \in \mathscr{L}(A)^N$ be preference profiles. We say that R' is a *refinement* of R with respect to an alternative $a \in A$ if for any participant $i \in N$, we have $L_i(a, R') \subset L_i(a, R)$. We say that R' is a *strict refinement* of R if for at least one agent, the inclusion is strict.

Definition. An SCR $F \in \mathscr{F}$ is *Maskin-monotonic* if for any $R \in \mathscr{L}(A)^N$, any $a \in F(R)$ and for any $R' \in \mathscr{L}(A)^N$ such that R is a refinement of R'

with respect to a, we have $a \in F(R')$. We will denote the set of all Maskin-monotonic SCR's as $\mathscr{M}$.

Definition. Let $F \in \mathscr{M}$. Define $GrF = \{(a, R) \in A \times \mathscr{L}(A)^N | a \in F(R)\}$. Let $h : GrF \to (2^A)^N$ be a function. We say that F is *h-monotonic* if for any $R, R' \in \mathscr{L}(A)^N$ and any $a \in F(R)$ we have;

$$[\forall i \in N : L_i(a, R) \cap h_i(a, R) \subset L_i(a, R')] \Rightarrow a \in F(R')$$

Definition. We say that $h : GrF \to (2^A)^N$ is a *self-monotonicity* of an SCR F if F is h-monotonic and there is no $h' : GrF \to (2^A)^N$ with $h' \subsetneq h$ such that F is h'-monotonic.

Remark 1. Note that, if h is a self-monotonicity of F, then for any $(a, R) \in Gr(F)$ and any $i \in N$, we have $h_i(a, R) \subset L_i(a, R)$.

Definition. A profile $R \in \mathscr{L}(A)^N$ is an *a-critical profile* for some $a \in A$ relative to an SCR $F \in \mathscr{F}$ if $a \in F(R)$ and for any strict refinement R' of R with respect to a, we have $a \notin F(R')$. We will denote the set of a-critical profiles relative to F by $C_a(F)$.

Remark 2. Let $a \in A$ and $F \in \mathscr{M}$. Note that $C_a(F)$ is empty if and only if $a \notin F(R)$ for all $R \in \mathscr{L}(A)^N$. If $C_a(F)$ is not empty, it is a union of some members of $\rho(a)$, i.e.

$$C_a(F) = \bigcup_{i \in \{1,\dots,k\}} S_i \quad \text{for some} \quad S_1, \dots, S_k \in \rho(a), \quad k \in \mathbb{N}.$$

Definition. Let $F \in \mathscr{M}$ and $S_1, \dots, S_k$ be distinct members of $\rho(a)$ such that $\bigcup_{i \in \{1,\dots,k\}} S_i = C_a(F)$. We will refer to a set $\{R_1, \dots R_k\}$ such that $R_i \in S_i$ for each $i \in \{1, \dots, k\}$ as an *a-center* of F. Let for each $a \in A$, $CE_a(F)$ be an a-center of F. We will refer to a set $\bigcup_{a \in A} CE_a(F)$ as a *center* of F.

Remark 3. Center of an SCR F is not unique.

Remark 4. A preference profile $R \in \mathscr{L}(A)^N$ may belong to both an a-center and a b-center of F where $a, b \in A$ and $a \neq b$.

Remark 5. Two different centers of F may have different cardinalities.

Remark 6. Two different SCR's may have the same center. As an example, let $A = \{a, b, c\}$. Consider the SCR's F and G where $F(R) = a$ for any $R \in \mathscr{L}(A)^N$ and $G(R) = \{b\}$ if all agents top-rank alternative b in profile R, $G(R) = \emptyset$ otherwise. Note that, the profile $R \in \mathscr{L}(A)^N$ where every agent bottom-ranks alternative a and every agent top-ranks alternative b is a center for both F and G.

2.2 Examples

Example 1. (Dictatoriality) Let F be a dictatorial SCR where agent i is the dictator, i.e.

$$\forall R \in \mathscr{L}(A)^N : F(R) = \{a \in A | aR_i b \quad \text{for any} \quad b \in A\}$$

Consider $h : GrF \to (2^A)^N$ with;

$$\forall (a, R) \in GrF : h_i(a, R) = A \backslash \{a\} \quad \text{and} \quad h_j(a, R) = \emptyset \quad \text{for any} \quad j \in N \backslash \{i\}$$

That is, $h(a, R) = (\emptyset, \ldots, \emptyset, \underbrace{A \setminus \{a\}}_{\text{agent i}}, \emptyset, \ldots, \emptyset)$. Observe that F is h-monotonic and there is no $h' : GrF \to (2^A)^N$ with $h' \subsetneqq h$ such that F is h'-monotonic. Thus, h is a self-monotonicity of F.

For some alternative $a \in A$, let $R^a \in \mathscr{L}(A)^N$ be a profile such that alternative a is top-ranked by agent i and bottom-ranked by every other agent, i.e. $aR_i^a b$ and $bR_j^a a$ for any $b \in A \setminus \{a\}$ and $j \in N \setminus \{i\}$. Now, $CE(F) = \bigcup_{a \in A} \{R^a\}$ is a center of F. As an illustrative example, let $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$. The following set of profiles R^a, R^b and R^c is a center of the dictatorial SCR F with agent 2 being the dictator.

R_1^a	R_2^a	R_3^a	R_1^b	R_2^b	R_3^b	R_1^c	R_2^c	R_3^c
b	$\boxed{a}$	b	a	$\boxed{b}$	a	a	$\boxed{c}$	a
c	b	c	c	a	c	b	a	b
$\boxed{a}$	c	$\boxed{a}$	$\boxed{b}$	c	$\boxed{b}$	$\boxed{c}$	b	$\boxed{c}$

Example 2. (Constant SCR) Let F be a constant SCR, i.e. there is some alternative $a \in A$ such that for any profile R, we have $F(R) = a$. Observe that, $h : GrF \to (2^A)^N$ with for any $(a, R) \in Gr(F)$, $h(a, R) = (\emptyset, \emptyset, \ldots, \emptyset)$ is a self-monotonicity of F.

Let $R \in \mathscr{L}(A)^N$ be a profile such that alternative a is bottom-ranked by all agents, i.e. bR_ia for any $i \in N$ and $b \in A \setminus \{a\}$. Note that, we have $a \in F(R)$ and R is an a-critical profile. Now, $CE(F) = \{R\}$ is a center of F.

Example 3. (Unanimity SCR) Let F be the unanimity SCR, i.e. for any profile R, we have $a \in F(R)$ if and only if aR_ib for any $i \in N$ and $b \in A$. Observe that, $h : GrF \to (2^A)^N$ with $h(a, R) = (A \setminus \{a\}, \ldots, A \setminus \{a\})$ for any $(a, R) \in Gr(F)$ is a self-monotonicity of F.

For some alternative $a \in A$, let $R^a \in \mathscr{L}(A)^N$ be a profile such that alternative a is top-ranked by all agents. Now, $CE(F) = \bigcup_{a \in A}\{R^a\}$ is a center of F. As an illustrative example, $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$. The following set of profiles R^a, R^b and R^c is a center of the Unanimity SCR F.

R_1^a	R_2^a	R_3^a	R_1^b	R_2^b	R_3^b	R_1^c	R_2^c	R_3^c
$\boxed{a}$	$\boxed{a}$	$\boxed{a}$	$\boxed{b}$	$\boxed{b}$	$\boxed{b}$	$\boxed{c}$	$\boxed{c}$	$\boxed{c}$
b	b	b	a	a	a	a	a	a
c	c	c	c	c	c	b	b	b

Example 4. (Pareto Correspondence) Given some profile R, we say that an alternative $a \in A$ is Pareto dominated by some other alternative $b \in A$ if for any agent $i \in N$, we have bR_ia. Let F be the Pareto correspondence, i.e.

for any profile R, we have $a \in F(R)$ if and only if there is no $b \in A$ which Pareto dominates a. Consider $h : GrF \to (2^A)^N$ with for any $(a, R) \in GrF$, $h(a, R) = (\overline{L}_1(a, R), \overline{L}_2(a, R) \backslash \overline{L}_1(a, R), \ldots, \overline{L}_i(a, R) \backslash \bigcup_{j<i} \overline{L}_j(a, R), \ldots)$ where $\overline{L}_i(a, R) = L_i(a, R) \setminus \{a\}$. Now, we will show that h is a self-monotonicity of F.

Take some $(a, R) \in GrF$. Take any profile R' such that for any $i \in N$, we have $L_i(a, R) \cap h_i(R, a) \subset L_i(a, R')$. Since for any $i \in N$ $h_i(a, R) \subset L_i(a, R)$, we have $h_i(a, R) \subset L_i(a, R')$. Now, take any $b \in A \backslash \{a\}$. Since a is not Pareto dominated, we have $b \in \overline{L}_i(a, R)$ and also $b \in h_i(a, R)$ for some $i \in N$. But then, we have $b \in L_i(a, R')$, since $h_i(R, a) \subset L_i(a, R')$. So, a is not Pareto dominated in profile R', either. Thus, we have $a \in F(R')$, implying that F is h-monotonic.

Now, consider any $h' : GrF \to (2^A)^N$ with $h' \subsetneqq h$. Take some $(a, R) \in GrF$. Consider some profile R' such that for any $i \in N$ we have $h'_i = L_i(a, R')$. From definition of h and from $h' \subsetneqq h$, there should exist some $b \in A$ with $b \notin h'_i(a, R)$ for any $i \in N$. So, $b \notin L_i(a, R')$ for any $i \in N$. But then, b Pareto dominates a, implying that $a \notin F(R')$. So, F is not h'-monotonic. Hence, h is a self-monotonicity of F.

Now, we will define a center for the Pareto correspondence. First observe that, given some alternative $a \in A$, a profile R is an a-critical profile if and only if $\bigcup_{i \in N} L_i(a, R) = A$ and for any $i, j \in N$ with $i \neq j$, we have $L_i(a, R) \cap L_j(a, R) = \{a\}$. That is, for every alternative other than a, there should exist exactly 1 agent who prefers a to that alternative. So, the problem is assigning the alternatives in $A \setminus \{a\}$ to agents in N. Each of these assignments will constitute a different profile in an a-center of F. The set of all such profiles will be an a-center of F. Note that, an a-center of F consists of $|N|^{|A|-1}$ preference profiles. If we do it for all alternatives in A, we will obtain a center of F. As an illustrative example, let $N = \{1, 2\}$ and $A = \{a, b, c\}$. The following set of profiles R^1, R^2 R^3 and R^4 is an a-center of the Pareto correspondence F.

R_1^1	R_2^1	R_1^2	R_2^2	R_1^3	R_2^3	R_1^4	R_2^4
$\boxed{a}$	c	c	$\boxed{a}$	c	b	b	c
b	b	b	b	$\boxed{a}$	$\boxed{a}$	$\boxed{a}$	$\boxed{a}$
c	$\boxed{a}$	$\boxed{a}$	c	b	c	c	b

CHAPTER 3

DIFFERENT DEGREES OF MONOTONICITY

3.1 Preliminaries

Definition. Let $F, G \in \mathscr{M}$. We say that F satisfies a *stronger monotonicity condition* than G if $GrG \subset GrF$ and there exist self-monotonicities h^f, h^g of F and G, respectively, such that for any $(a, R) \in GrG$, we have $h^f(a, R) \subset h^g(a, R)$.

Lemma 1. *Let $F \in \mathscr{M}$. For any $a \in A$ with $a \in F(R)$ for some $R \in \mathscr{L}(A)^N$, there exists some $R' \in C_a(F)$ such that R' is a refinement of R with respect to a.*

Proof. Let $a \in F(R)$ for some $R \in \mathscr{L}(A)^N$. If R is an a-critical profile relative to F, we are done. So, suppose R is not an a-critical profile relative to F. Then, there is some strict refinement R_1 of R with respect to a such that $a \in F(R_1)$. Suppose R_1 is not an a-critical profile. Then, there should exist some strict refinement R_2 of R_1 with respect to a such that $a \in F(R_2)$. If we continue in this fashion, since we have finite number of alternatives, for some $t \in \mathbb{N}$ we will have some $R_t \in \mathscr{L}(A)^N$ so that for any strict refinement R_k of R_t, we have $a \notin F(R_k)$, i.e. R_t is an a-critical profile relative to F. But now, R_t is a refinement of R with respect to a. $\square$

Theorem 1. *Let $F, G \in \mathscr{M}$. F satisfies a stronger monotonicity condition than G if and only if for any $a \in A$ and $R \in C_a(G)$, there exists some $R' \in C_a(F)$ such that R' is a refinement of R with respect to a.*

Proof. Suppose F satisfies a stronger monotonicity condition than G. Take any $a \in A$ and $R \in C_a(G)$. Then, we have $(a, R) \in GrG$ and $(a, R) \in GrF$, since $GrG \subset GrF$. Now, from Lemma 1, there exists some $R' \in C_a(F)$ such that R' is a refinement of R with respect to a.

Suppose, for any $a \in A$ and $R \in C_a(G)$, there is some $R' \in C_a(F)$ such that R' is a refinement of R with respect to a. Note that, for any $(a, R) \in GrG$, we have $(a, R) \in GrF$, implying that $GrG \subset GrF$. Now, take some $(a, R) \in GrG$. From Lemma 1, we know that there is some $R' \in C_a(G)$ such that R' is a refinement of R with respect to a. Define $h_i^g(a, R') = L_i(a, R') \setminus \{a\}$ for any $i \in N$. Note that, there is some $R'' \in C_a(F)$ such that R'' is a refinement of R' with respect to a. Since R'' is also a refinement of R with respect to a, define $h_i^f(a, R'') = L_i(a, R'') \setminus \{a\}$ for any $i \in N$. Note that, $h^f(a, R) \subset h^g(a, R)$. Define $h^f(a, R)$ and $h^g(a, R)$ similarly for any $(a, R) \in GrG$. Also define $h^f(a, R)$ for any $(a, R) \in GrF \setminus GrG$, appropriately. Now, note that h^f and h^g constitute self-monotonicities of F and G such that for any $(a, R) \in GrG$, we have $h^f(a, R) \subset h^g(a, R)$. Since we also have $GrG \subset GrF$, F satisfies a stronger monotonicity condition than G. $\square$

3.2 An Illustrative Example

For a preference profile $R \in \mathscr{L}(A)^N$, an alternative $a \in A$ and some agent $i \in N$, let

$$L_i(a, R) = \{b \in A \mid aR_ib\} \quad , \quad U_i(a, R) = \{b \in A \mid bR_ia\}$$

and let $\sigma(i, k, R)$ denote the k'th best alternative for agent i in preference profile R, i.e. $|\{a \in A \mid aR_i\sigma(i, k, R)\}| = k$.

Definition. Let $k \in \{1, \ldots, |A|\}$. An SCR $F \in \mathscr{F}$ is called the *k-plurality SCR* if, for any $a \in A$, one has $a \in F(R)$ if and only if

$$|\{i \in N \mid a \in U_i(\sigma(i,k,R),R)\}| \geq |\{i \in N \mid b \in U_i(\sigma(i,k,R),R)\}|$$

for any $b \in A$.

Proposition 1. *Let $|N| = n \geq 3$ and $|A| = m \geq 3$. Let $F \in \mathscr{F}$ be the k-plurality SCR for some $k \in \{1, \ldots, |A|\}$. Now, F is Maskin-monotonic if and only if $k > \frac{m(n-1)}{n}$.*

Proof. Throughout the proof, we will assume that $k \geq 2$, since we know that for $k = 1$, we have the well-known plurality SCR which is not Maskin-monotonic for $m \geq 3$, $n \geq 3$.

Suppose $k > \frac{m(n-1)}{n}$, i.e. $m > n(m-k)$ since $n > 0$. Note that, for any profile $R \in \mathscr{L}(A)^N$, we must have some $a \in A$ with $|\{i \in N \mid a \in U_i(R_{ik}, R)\}| = n$, which is the total number of participants. Thus, for any $R \in \mathscr{L}(A)^N$ and $a \in A$, one has $a \in F(R)$ if and only if $|\{i \in N \mid a \in U_i(\sigma(i,k,R),R)\}| = n$. Now, take any $R \in \mathscr{L}(A)^N$, any $a \in F(R)$ and any $R' \in \mathscr{L}(A)^N$ with $L_i(a,R) \subset L_i(a,R')$ for all $i \in N$. We then have $|\{i \in N \mid a \in U_i(R'_{ik}, R')\}| = n$ and so $a \in F(R')$. Thus, F is Maskin-monotonic.

Now, suppose $k \leq \frac{m(n-1)}{n}$, i.e. $m \leq n(m-k)$. Let $A = \{a_1, a_2, \ldots, a_m\}$. First, suppose that $m = n(m-k)$. Consider the following preference profile $R \in \mathscr{L}(A)^N$ with;

$\sigma(1,k+1,R) = a_1, \sigma(1,k+2,R) = a_2, \ldots, \sigma(1,m,R) = a_{m-k}, \sigma(2,k+1,R) = a_{m-k+1}, \ldots, \sigma(n,m,R) = a_m$ and $\sigma(1,1,R) = a_m$.

Note that, for all $a \in A$, we have $|\{i \in N \mid a \in U_i(\sigma(i,k,R),R)\}| = n-1$, since $m = n(m-k)$. Thus, we have $F(R) = \{a_1, \ldots, a_m\}$. Now, consider the profile R' which is obtained from profile R by only interchanging $\sigma(1,k,R)$ with $\sigma(i,k+1,R)$, leaving everything else the same, i.e. $\sigma(1,k,R) =$

$\sigma(1, k+1, R')$ and $\sigma(1, k+1, R) = \sigma(1, k, R')$. Note that $\sigma(1, k, R) \neq a_m$ since $k \geq 2$. Now, we have $|\{i \in N \mid a_1 \in U_i(\sigma(i, k, R'), R')\}| = n$ while $|\{i \in N \mid b \in U_i(\sigma(i, k, R'), R')\}| < n$ for any $b \in A \setminus \{a_1\}$. Note that, $L_i(a_m, R) \subset L_i(a_m, R')$ for any $i \in N$ and $a_m \notin F(R')$ while $a_m \in F(R)$. Thus, F is not Maskin-monotonic.

Now, suppose $m < n(m-k)$. Consider some profile $R \in \mathscr{L}(A)^N$ with;

$\sigma(1, k+1, R) = a_1, \sigma(1, k+2, R) = a_2, \ldots, \sigma(1, m, R) = a_{m-k}, \sigma(2, k+1, R) = a_{m-k+1}, \ldots$

Now, let R' be the preference profile obtained from R by applying the following procedure;

i. for any agent $i \in N$ with $a_m \in U_i(\sigma(i, k, R), R)$, interchange a_m with $\sigma(i, 1, R)$

ii. for any agent $i \in N$ with $a_1 \in L_i(\sigma(i, k+1, R), R)$, interchange a_1 with $\sigma(i, k+1, R)$

leaving everything else the same. Note that $a_m \in F(R')$ and if m divides $n(m-k)$, we have $|\{i \in N \mid a_m \in U_i(\sigma(i, k, R'), R')\}| = |\{i \in N \mid a_1 \in U_i(\sigma(i, k, R'), R')\}|$, if m does not divide $n(m-k)$, we have $|\{i \in N \mid a_m \in U_i(\sigma(i, k, R'), R')\}| = |\{i \in N \mid a_1 \in U_i(\sigma(i, k, R'), R')\}| + 1$. Now, since $m < n(m-k)$, there should exist two different agents $i, j \in N$ with $\sigma(i, k+1, R') = \sigma(j, k+1, R') = a_1$. Consider the profile R'' obtained from R' by only interchanging $\sigma(i, k, R')$ with $\sigma(i, k+1, R')$ and $\sigma(j, k, R')$ with $\sigma(j, k+1, R')$, that is we move the alternative a_1 one level up in the orderings of agents i and j. Now we have $|\{i \in N \mid a_1 \in U_i(\sigma(i, k, R''), R'')\}| > |\{i \in N \mid a_m \in U_i(\sigma(i, k, R''), R'')\}|$. That is, $a_m \notin F(R'')$ while $L_i(a_m, R') \subset L_i(a_m, R'')$ for any $i \in N$. Thus, F is not Maskin-monotonic.

$\square$

Proposition 2. *Let N denote a finite set of participants and A denote a finite set of alternatives with $|N| = n$ and $|A| = m$. Let F be the p-plurality*

SCR and G be the q-plurality SCR for some $m \geq p > q > \frac{m(n-1)}{n}$. Now, F satisfies a stronger monotonicity condition than G.

Proof. First note that, from *Proposition 1*, both p-plurality and q-plurality SCR's are Maskin-monotonic. Now, for any alternative $a \in A$, let $R \in \mathscr{L}(A)^N$ be a profile where every agent ranks alternative a as the q'th best alternative; i.e. $\forall i \in N : \sigma(i, q, R) = a$. Note that, the set of all such profiles constitutes the set $C_a(G)$ of all a-critical profiles of G. Now, take any profile $R \in C_a(G)$. Consider the profile R' where every agent ranks a as the p'th best alternative; i.e. $\forall i \in N : \sigma(i, p, R') = a$ and for any agent $i \in N$, we have $\sigma(i, t, R) = \sigma(i, q, R')$ for any $t > p$. Note that, R' is a refinement of R with respect to a and we have $a \in F(R')$. Now, from Lemma 1, there should exist some $R'' \in CE_a(F)$ such that R'' is a refinement of R with respect to a. But now, R'' is also a refinement of R with respect to alternative a. Thus, from Theorem 1, F satisfies a stronger monotonicity condition than G. $\square$

CHAPTER 4

AN IMPLEMENTATION RESULT

4.1 Preliminaries

Definition. Let $i \in N$ be a participant, $a, b \in A$ be alternatives and $F \in \mathscr{F}$ be an SCR. Let $I(F)$ denote the image of F, i.e.

$$I(F) = \bigcup_{R \in \mathscr{L}(A)^N} F(R)$$

We say that b is *essential* for i with respect to a in some profile $R \in \mathscr{L}(A)^N$ relative to F if for any profile $R' \in \mathscr{L}(A)^N$;

$$L_i(a, R) \subset L_i(b, R') \quad \text{and} \quad I(F) \subset L_j(b, R') \quad \text{for any} \quad j \in N \setminus \{i\}$$

imply $b \in F(R')$.

Lemma 2. *Let $F \in \mathscr{F}$ be a Maskin-monotonic SCR and $a \in A$ be an alternative. We have $a \in I(F)$ if and only if $C_a(F) \neq \emptyset$.*

Proof. Suppose $C_a(F) \neq \emptyset$. Then, there is some $R \in C_a(F)$ and by definition we have $a \in F(R)$ implying that $a \in I(F)$.

Suppose $a \in I(F)$. Then, there is some profile $R \in \mathscr{L}(A)^N$ with $a \in F(R)$. But from *Lemma 1*, we must have some $R' \in CE_a(F)$ such that R' is a refinement of R with respect to a. Thus, $C_a(F) \neq \emptyset$. $\square$

Definition. A *solution concept* σ for normal form games is a function which associates with each normal form game $g = (N, A, R)$ a subset $\sigma(g)$ of A.

A *mechanism* is an ordered pair $G = (M, \pi)$ where $M = \Pi_{i \in N} M_i$ is a nonempty joint strategy space and $\pi : M \to A$ an outcome function.

Given some $R \in \mathscr{L}(A)^N$ and some mechanism $G = (M, \pi)$, we define the normal form game $G[R] = (N, M, \tilde{R})$, where for each $i \in N$ and $m, m' \in M$ we have $m\tilde{R}_i m'$ if and only if $\pi(m) R_i \pi(m')$. We say that a mechanism σ-*implements* an SCR $F \in \mathscr{F}$ for some solution concept σ if for any $R \in \mathscr{L}(A)^N$, $\pi(\sigma(G[R]))$ coincides with $F(R)$. F is said to be σ-implementable if and only if there is some mechanism $G = (M, \pi)$ which σ-implements F.

4.2 Implementation Result

Theorem 2. *(Characterization of Nash-implementability for the two-agent case)*

Let $N = \{1, 2\}$ and $F \in \mathscr{F}$. Then, F is Nash-implementable if and only if F is Maskin-monotonic and for any $i \in \{1, 2\}$ and any $a \in A$ one has;

i. for any $R \in C_a(F)$ and any $b \in L_i(a, R)$, b is essential for i with respect to a relative to F;

ii. for any $R \in C_a(F)$, $b \in A$ and $R' \in C_b(F)$, there should exist some $c \in L_i(a, R) \cap L_{N\setminus\{i\}}(b, R')$ such that for any $R'' \in \mathscr{L}(A)^N$ with $L_i(a, R) \subset L_i(c, R'')$ and $L_{N\setminus\{i\}}(b, R') \subset L_{N\setminus\{i\}}(c, R'')$, we have $c \in F(R'')$.

Proof. Suppose F is Nash-implementable. We know that $F \in \mathscr{M}$ and there exists a mechanism $G = (M, \pi)$ which Nash-implements F. Take any $i \in N$ and $a \in A$. Consider any profile $R \in C_a(F)$ and any $b \in L_i(a, R)$. From definition of a critical profile, we have $a \in F(R)$. So, there exist $m \in M$ with $m \in \sigma_0(G[R])$ and $a = \pi(m)$ where σ_0 stands for the Nash equilibrium notion.

Then, we have $\pi(\overline{m}_i, m_{N\setminus\{i\}}) \subset L_i(a, R)$ for all $\overline{m}_i \in M_i$ and $\pi(m_i, \overline{m}_{N\setminus\{i\}}) \subset L_{N\setminus\{i\}}(a, R)$ for all $\overline{m}_{N\setminus\{i\}} \in M_{N\setminus\{i\}}$. Now, suppose $b \notin \pi(\overline{m}_i, m_{N\setminus\{i\}})$ for all $\overline{m}_i \in M_i$. Consider the profile $\overline{R} \in \mathscr{L}(A)^N$ obtained from R by moving the alternative b to just above the alternative a in agent i's ordering, leaving everything else the same. Now, we still have $m \in \sigma_0(G[\overline{R}])$ and $\pi(m) = a \in F(\overline{R})$, contradicting with R being an a-critical profile, since $\overline{R}$ is a strict refinement of R with respect to a. Thus, there exists $m'_i \in M_i$ with $\pi(m'_i, m_{N\setminus\{i\}}) = b$ and $\pi(\overline{m}_i, m_{N\setminus\{i\}}) \subset L_i(a, R)$ for any $\overline{m}_i \in M_i$. Thus, for any profile $R' \in \mathscr{L}(A)^N$ with $L_i(b, R') = L_i(a, R)$ and $L_{N\setminus\{i\}}(b, R') = I(F)$, we have $(m'_i, m_{N\setminus\{i\}}) \in \sigma_0(G[R'])$, implying that $\pi(m'_i, m_{N\setminus\{i\}}) = b \in F(R')$. Hence, b is essential for i with respect to a in profile R, which proves the necessity of condition (i). Now, take any $b \in A$. Consider any $R \in C_a(F)$ and $R' \in C_b(F)$. By definition of a critical profile, we have $a \in F(R)$ and $b \in F(R')$. So, there exist $m, m' \in M$ with $m \in \sigma_0(G[R])$, $a = \pi(m)$ and $m' \in \sigma_0(G[R'])$, $b = \pi(m')$. Then, we have $\forall \overline{m}_i \in M_i : \pi(\overline{m}_i, m_{N\setminus\{i\}}) \in L_i(a, R)$ and $\forall \overline{m}_{N\setminus\{i\}} \in M_{N\setminus\{i\}} : \pi(m'_i, \overline{m}_{N\setminus\{i\}}) \in L_{N\setminus\{i\}}(b, R')$. But then, we have $\pi(m'_i, m_{N\setminus\{i\}}) \in L_i(a, R)$ and $\pi(m'_i, m_{N\setminus\{i\}}) \in L_{N\setminus\{i\}}(a, R')$. Let $\pi(m'_i, m_{N\setminus\{i\}}) = c$. Note that $c \in L_i(a, R) \cap L_{N\setminus\{i\}}(b, R')$ and since F is Nash-implementable, for any $R'' \in \mathscr{L}(A)^N$ with $L_i(a, R) \subset L_i(c, R'')$ and $L_{N\setminus\{i\}}(b, R') \subset L_{N\setminus\{i\}}(c, R'')$, we have $c \in F(R'')$, which proves the necessity of condition (ii).

Suppose F is Maskin-monotonic and for any $i \in \{1, 2\}$ and $a \in A$, conditions (i) and (ii) are satisfied. Now, we will define a mechanism which will Nash-implement F. Let X and Y denote the strategy spaces of agents 1 and 2, with generic elements x_i and y_i, $i \in \mathbb{N}$, respectively. Take any center $CE(F)$ of F. Let for any $c \in A$, $CE_c(F)$ be the corresponding c-center of F. Consider the set $S = \{(a, R) \in I(F) \times CE(F) \mid R \in CE_a(F)\}$. Take some $(a, R) \in S$. Let $|I(F)| = I$. Define $\pi(x_1, y_1) = a$. Note that from condition (i), for any $i \in \{1, 2\}$ and for any $b \in L_i(a, R)$, we have $b \in I(F)$.

So, we have $|L_i(a,R)| \leq I$. Now, define $\pi(x_1,y_1),\ldots,\pi(x_{I!+1},y_1)$ such that $\bigcup_{i\in\{1,\ldots,I!+1\}} \pi(x_i,y_1) = L_1(a,R)$ and also define $\pi(x_1,y_1),\ldots,\pi(x_1,y_{I+1})$ such that $\bigcup_{i\in\{1,\ldots,I+1\}} \pi(x_1,y_i) = L_2(a,R)$. Consider the box with coordinates (x_2,y_2), (x_2,y_{I+1}), $(x_{I!+1},y_2)$, $(x_{I!+1},y_{I+1})$. Fill in the rows of this box such that each row corresponds to a different permutation of the members of $I(F)$. Now, suppose there is some $(b,R') \in S$ with $(b,R') \neq (a,R)$. Define $\pi(x_{I!+2},y_{I+2}) = b$ and define the box with coordinates $(x_{I!+2},y_{I+2})$, $(x_{I!+2},y_{2I+2})$, $(x_{2I!+2},y_{I+2})$, $(x_{2I!+2},y_{2I+2})$ similarly by only changing a with b and R with R'. Now, since condition (ii) is satisfied, we know that $L_2(a,R)\cap L_1(b,R') \neq \emptyset$ and $L_1(a,R)\cap L_2(b,R') \neq \emptyset$. Fill in the box with coordinates (x_1,y_{I+2}), (x_1,y_{2I+2}), $(x_{I!+1},y_{I+2})$, $(x_{I!+2},y_{2I+2})$ with some alternative in $L_2(a,R)\cap L_1(b,R')$ satisfying the prescribed property in condition (ii). Also, fill in the box with coordinates $(x_{I!+2},y_1)$, $(x_{I!+2},y_{I+1})$, $(x_{2I!+2},y_1)$, $(x_{2I!+2},y_{I+1})$ with some alternative in $L_1(a,R)\cap L_2(b,R')$ satisfying the prescribed property in condition (ii). Now, suppose there is some $(c,R'') \in S$ with $(c,R'') \neq (b,R')$ and $(c,R'') \neq (a,R)$. Define $\pi(x_{2I!+3},y_{2I+3}) = c$ and define the box with coordinates $(x_{2I!+3},y_{2I+3})$, $(x_{2I!+3},y_{3I+3})$, $(x_{3I!+3},y_{2I+3})$, $(x_{3I!+3},y_{3I+3})$ similarly by only changing R' with R'' and b with c. Now, since condition (ii) is satisfied, we know that $L_2(a,R)\cap L_1(c,R'') \neq \emptyset$, $L_2(b,R')\cap L_1(c,R'') \neq \emptyset$, $L_1(a,R)\cap L_2(c,R'') \neq \emptyset$ and $L_1(a,R)\cap L_2(c,R'') \neq \emptyset$. Fill in the box with coordinates (x_1,y_{2I+3}), (x_1,y_{3I+3}), $(x_{I!+1},y_{2I+3})$, $(x_{I!+1},y_{3I+3})$ with some alternative in $L_2(a,R)\cap L_1(c,R'')$, fill in the box with coordinates $(x_{I!+2},y_{2I+3})$, $(x_{I!+2},y_{3I+3})$, $(x_{2I!+2},y_{2I+3})$, $(x_{2I!+2},y_{3I+3})$ with some alternative in $L_2(b,R')\cap L_1(c,R'')$, fill in the box with coordinates $(x_{2I!+3},y_1)$, $(x_{2I!+3},y_{I+1})$, $(x_{3I!+3},y_1)$, $(x_{3I!+3},y_{I+1})$ with some alternative in $L_1(a,R)\cap L_2(c,R'')$ and fill in the box with coordinates $(x_{2I!+3},y_{I+2})$, $(x_{2I!+3},y_{2I+2})$, $(x_{3I!+3},y_{I+2})$, $(x_{3I!+3},y_{2I+2})$ with some alternative in $L_1(b,R')\cap L_2(c,R'')$, all satisfying the prescribed property in condition (ii). Now, if there is any other $(d,R''') \in S$, the mechanism can be extended accordingly. But, w.l.o.g. suppose there is no such $(d, R''') \in S$. We will show that F can be Nash-implemented via mechanism $G = (M, \pi)$, where $M = X \times Y$, $X = \{x_1, \ldots, x_{3I!+3}\}$, $Y = \{y_1, \ldots, y_{3I+3}\}$.

First, we will show that for any $\overline{R} \in \mathscr{L}(A)^N$ and $a \in A$, $a \in F(\overline{R})$ implies $a = \pi(x, y)$ for some $(x, y) \in \sigma_0(G[\overline{R}])$. Take some profile $\overline{R} \in \mathscr{L}(A)^N$. Suppose $a \in F(\overline{R})$ for some $a \in A$. From *Lemma 1*, there should exist some profile in $C_a(F)$ which is a refinement of $\overline{R}$ with respect to a, thus there should also exist some $\overline{\overline{R}} \in CE_a(F)$ such that $\overline{\overline{R}}$ is a refinement of $\overline{R}$ with respect to a. Note that $(a, \overline{\overline{R}}) \in S$ and also note that there exists some $(x, y) \in M$ with $\pi(x, y) = a$ such that $\forall x' \in X : \pi(x', y) \subset L_1(a, \overline{\overline{R}})$ and $\forall y' \in Y : \pi(x, y') \subset L_2(a, \overline{\overline{R}})$. But since $\overline{\overline{R}}$ is a refinement of $\overline{R}$ with respect to a, we have $\forall x' \in X : \pi(x', y) \subset L_1(a, \overline{R})$ and $\forall y' \in Y : \pi(x, y') \subset L_2(a, \overline{R})$, implying that $(x, y) \in \sigma_0(G[\overline{R}])$.

Now, we will show that for any $\overline{R} \in \mathscr{L}(A)^N$, having some $(x, y) \in \sigma_0(G[\overline{R}])$ with $\pi(x, y) = a$ implies $a \in F(\overline{R})$. Take some profile $\overline{R} \in \mathscr{L}(A)^N$. Suppose $(x_i, y_j) \in \sigma_0(G[\overline{R}])$ for some $x_i \in X$ and $y_j \in Y$.

Case 1. Suppose $i \in \{1, I! + 2, 2I! + 3\}$ and $j \in \{1, I + 2, 2I + 3\}$.

First suppose $(i, j) \in \{(1, 1), (I! + 2, I + 2), (2I! + 3, 2I + 3)\}$. W.l.o.g suppose $(i, j) = (1, 1)$. We have $\pi(x_1, y_1) = a$, $(a, R) \in S$ and from construction of mechanism G, we know that;

$$L_1(a, R) \subset \bigcup_{x_i \in X} \pi(x_i, y_1) \quad \text{and} \quad L_2(a, R) \subset \bigcup_{y_i \in Y} \pi(x_1, y_i)$$

Since $(x_1, y_1) \in \sigma_0(G[\overline{R}])$, we also have;

$$\bigcup_{x_i \in X} \pi(x_i, y_1) \subset L_1(a, \overline{R}) \quad \text{and} \quad \bigcup_{y_i \in Y} \pi(x_1, y_i) \subset L_2(a, \overline{R})$$

Thus, we have $L_1(a, R) \subset L_1(a, \overline{R})$ and $L_2(a, R) \subset L_2(a, \overline{R})$, that is R is a refinement of $\overline{R}$ with respect to a. But then, since F is Maskin-monotonic and $a \in F(R)$, we have $a \in F(\overline{R})$. So, suppose $(i,j) \notin \{(1,1),(I!+2,I+2),(2I!+3,2I+3)\}$. W.l.o.g. suppose $(i,j)=(I!+2,1)$. Let $\pi(x_{I!+2},y_1)=c$. From construction of mechanism G, we know that;

$$L_1(a,R) \subset \bigcup_{x_i \in X} \pi(x_i,y_1) \quad \text{and} \quad L_2(b,R') \subset \bigcup_{y_i \in Y} \pi(x_{I!+2},y_i)$$

Since $(x_{I!+2},y_1) \in \sigma_0(G[\overline{R}])$, we also have;

$$\bigcup_{x_i \in X} \pi(x_i,y_1) \subset L_1(c,\overline{R}) \quad \text{and} \quad \bigcup_{y_i \in Y} \pi(x_{I!+2},y_i) \subset L_2(c,\overline{R})$$

Thus, we have $L_1(a,R) \subset L_1(c,\overline{R})$ and $L_2(b,R') \subset L_2(c,\overline{R})$. But from construction of mechanism G, from condition (ii) and from F being Maskin-monotonic, we have $c \in F(\overline{R})$.

Case 2. Suppose $[i \in \{1,I!+2,2I!+3\}$ and $j \notin \{1,I+2,2I+3\}]$ or $[i \notin \{1,I!+2,2I!+3\}$ and $j \in \{1,I+2,2I+3\}]$.

W.l.o.g. suppose $(i,j)=(2,1)$. Let $\pi(x_2,y_1)=b$. First note that, $b \in L_1(a,R)$ where $R \in CE_a(F)$. So, from condition (i), b is an essential element for agent 1 with respect to a. That is, for any profile R' with $L_1(a,R) \subset L_1(b,R')$ and $I(F) \subset L_2(b,R')$, we must have $b \in F(R')$. Now, note that from construction of mechanism G, we have;

$$L_1(a,R) \subset \bigcup_{x_i \in X} \pi(x_i,y_1) \quad \text{and} \quad I(F) \subset \bigcup_{y_i \in Y} \pi(x_2,y_i)$$

Since $(x_2,y_1) \in \sigma_0(G[\overline{R}])$, we also have;

$$\bigcup_{x_i \in X} \pi(x_i,y_1) \subset L_1(b,\overline{R}) \quad \text{and} \quad \bigcup_{y_i \in Y} \pi(x_2,y_i) \subset L_2(b,\overline{R})$$

But then, we have $L_1(a,R) \subset L_1(b,\overline{R})$ and $I(F) \subset L_2(b,\overline{R})$, implying that $b \in F(\overline{R})$.

Case 3. Suppose $i \notin \{1, I! + 2, 2I! + 3\}$ and $j \notin \{1, I + 2, 2I + 3\}$.

Let $\pi(x_i, y_j) = a$. First note that $a \in I(F)$. Now, from *Lemma 2*, we know that $CE_a(F) \neq \emptyset$. That is, there is some $R \in CE_a(F)$ with $a \in F(R)$. Also note that, since condition (i) is satisfied, we have $b \in I(F)$ for any $b \in L_i(a, R)$, $i \in \{1, 2\}$. That is, we have $L_1(a, R) \subset I(F)$ and $L_2(a, R) \subset I(F)$. Then, since F is Maskin-monotonic, we have $a \in F(R')$ for any $R' \in \mathscr{L}(A)^N$ with $I(F) \subset L_1(a, R)$ and $I(F) \subset L_2(a, R)$. Now, note that from construction of mechanism G, we have;

$$\bigcup_{x \in X} \pi(x, y_j) = I(F) \quad \text{and} \quad \bigcup_{y \in Y} \pi(x_i, y) = I(F)$$

Now, since $(x_i, y_j) \in \sigma_0(G[\overline{R}])$, we have;

$$\bigcup_{x \in X} \pi(x, y_j) \subset L_1(a, \overline{R}) \quad \text{and} \quad \bigcup_{y \in Y} \pi(x_i, y) \subset L_2(a, \overline{R})$$

Then, we have;

$$I(F) \subset L_1(a, \overline{R}) \quad \text{and} \quad I(F) \subset L_2(a, \overline{R})$$

implying that $a \in F(\overline{R})$.

Thus, $G = (M, \pi)$ Nash-implements F, completing the proof. □

4.3 Examples

Example 5. (Dictatoriality) Let $N = \{1, 2\}$ and F be a Dictatorial SCR where agent 1 is the dictator. F is clearly Maskin-monotonic. Note that, $I(F) = A$. Take any $a \in A$ and any $i \in \{1, 2\}$.

i. Take any $R \in C_a(F)$ and any $b \in L_i(a, R)$. Remember that, R must be such that a is top-ranked in agent 1's ordering and bottom-ranked in agent 2's ordering. Take any profile $R' \in \mathscr{L}(A)^N$ with;

$$L_i(a,R) \subset L_i(b,R') \quad \text{and} \quad I(F) = A \subset L_j(b,R') \quad \text{for any} \quad j \in N \backslash \{i\}$$

Now, we must have $b \in F(R')$, implying that b is essential for i with respect to a. So, condition (i) is satisfied.

ii. Take any $R \in C_a(F)$, $b \in A$ and $R' \in C_b(F)$. W.l.o.g. consider $L_1(a,R) \cap L_2(b,R')$. Note that, we have $b \in L_1(a,R) \cap L_2(b,R')$ and for any $R'' \in \mathscr{L}(A)^N$ with $L_1(a,R) = A \subset L_i(b,R'')$ and $L_2(b,R') = \{b\} \subset L_2(b,R'')$, we have $b \in F(R'')$. So, condition (ii) is also satisfied.

Hence, F is Nash-implementable.

Example 6. (Pareto Correspondence) Let $N = \{1,2\}$, $A = \{a,b,c\}$ and F be the pareto correspondence. Note that F is Maskin-monotonic. Now, consider the profiles $R \in C_a(F)$ and $R' \in C_b(F)$ given as follows;

R_1	R_2	$R_1^{'}$	$R_2^{'}$
b	c	$\boxed{b}$	c
$\boxed{a}$	$\boxed{a}$	a	a
c	b	c	$\boxed{b}$

We have $L_1(a,R) = \{a,c\}$ and $L_2(b,R') = \{b\}$, implying that $L_1(a,R) \cap L_2(b,R') = \emptyset$. Thus, F does not satisfy condition (ii) and hence not Nash-implementable.

Example 7. Let $N = \{1,2\}$, $A = \{a,b,c\}$ and F be a SCR such that F chooses alternative a whenever both agents rank a above c, F chooses alternative b whenever both agents rank b above c and F chooses alternative c whenever at least one agent top-ranks c. That is, for some profile $R \in \mathscr{L}(A)^N$;

- $a \in F(R)$ if $\forall i \in N : aR_ic$
- $b \in F(R)$ if $\forall i \in N : bR_ic$

- $c \in F(R)$ if $\exists i \in N : cR_i a$ and $cR_i b$.

F is clearly Maskin-monotonic. Now, note that the following profiles R^1 and R^2 are the only a-critical and b-critical profiles, respectively. Any profile where one of the agents top-ranks c and the other agent bottom-ranks c is a c-critical profile, including the following profile R^3.

R_1^1	R_2^1	R_1^2	R_2^2	R_1^3	R_2^3
b	b	a	a	$\boxed{\text{c}}$	a
$\boxed{\text{a}}$	$\boxed{\text{a}}$	$\boxed{\text{b}}$	$\boxed{\text{b}}$	a	b
c	c	c	c	b	$\boxed{\text{c}}$

First, observe that for any $a \in A$, $i \in N$, $R \in C_a(F)$ and any alternative $b \in L_i(a, R)$, b is essential for i with respect to alternative a. So, condition (i) is satisfied. Also note that, for any $a, b \in A$, $R \in C_a(F)$ and $R' \in C_b(F)$, we have $c \in L_i(a, R) \cap L_{N \setminus \{i\}}(b, R')$. Now, consider the profiles $R^1 \in C_a(F)$ and $R^3 \in C_c(F)$. We have $L_1(a, R^1) \cap L_2(c, R^3) = c$. Take some profile $R' \in \mathscr{L}(A)^N$ with $L_1(c, R') = L_1(a, R^1) = \{a, c\}$ and $L_2(c, R') = L_2(c, R^3) = c$. But now, we have $c \notin F(R')$, implying that condition (ii) is not satisfied. Thus, F is not Nash-implementable.

Example 8. Let $N = \{1, 2\}$, $A = \{a, b, c, d\}$ and F be a SCR such that F chooses alternative a whenever both agents rank a above b and c, F chooses alternative b whenever both agents top-rank b, F chooses alternative c in every profile and F chooses alternative d whenever both agents rank d above c. That is, for any profile R, we have;

- $a \in F(R)$ if $\forall i \in N : aR_i b$ and $aR_i c$

- $b \in F(R)$ if $\forall i \in N : bR_i a$ for any $a \in A$

- $c \in F(R)$

- $d \in F(R)$ if $\forall i \in N : dR_i c$

Check that F is Maskin-monotonic. Now, any profile where both agents top-rank b and second-rank a is an a-critical profile, any profile where both agents top-rank b is a b-critical profile, any profile where both agents bottom-rank c is a c-critical profile and any profile where both agents third-rank d and bottom-rank c is a d-critical profile. Observe that the following profiles $R^1 \in C_a(F)$, $R^2 \in C_b(F)$, $R^3 \in C_c(F)$ and $R^4 \in C_d(F)$ constitute a Center of F.

R_1^1	R_2^1	R_1^2	R_2^2	R_1^3	R_2^3	R_1^4	R_2^4
d	d	$\boxed{\text{b}}$	$\boxed{\text{b}}$	a	a	a	a
$\boxed{\text{a}}$	$\boxed{\text{a}}$	a	a	d	d	b	b
b	b	c	c	b	b	$\boxed{\text{d}}$	$\boxed{\text{d}}$
c	c	d	d	$\boxed{\text{c}}$	$\boxed{\text{c}}$	c	c

First note that, for any $i \in N$, $a, b \in A$, $R \in C_a(F)$ and $R' \in C_b(F)$, we have $c \in L_i(a, R) \cap L_{N\setminus\{i\}}(b, R')$. Also, since we have $c \in F(R)$ for any $R \in \mathscr{L}(A)^N$, the condition (ii) is clearly satisfied. Now, consider the profile $R^1 \in C_a(F)$. Consider the alternative $b \in L_1(a, R^1)$. Let R' be a preference profile with $L_1(b, R') = L_1(a, R^1) = \{a, b, c\}$ and $L_2(b, R') = I(F) = \{a, b, c, d\}$. Now, we have $b \notin F(R')$, implying that b is not essential for agent 1 with respect to alternative a in profile R^1. So, condition (i) is not satisfied. Thus, F is not Nash-implementable.

CHAPTER 5

CONCLUSION

The main result of the present study was a new characterization of Nash implementability for the two-agent case. It is true that there exist several different characterizations of the same phenomenon in the implementation literature. The previous characterizations all originate, however, from common roots. In fact, they all start with a set of sufficient conditions which are not necessary for Nash implementability, weaken these conditions appropriately so that the weakened conditions also become necessary and modify the entire construct to make it fit the two-agent case. Thus, the mechanisms used in this literature for Nash-implementation are similar in nature and are referred to as Maskin-Vind mechanisms.

The fact that we start from the notions of center and critical profile underlying Maskin-monotonicity gets also reflected in the mechanisms we construct to achieve Nash implementation. Thus, it sheds some further light into what the interdependence of Nash-implementability and monotonicity is. Moreover, the characterization that we end up with seems to be clearer and simpler. The kind of mechanisms we employ in the present work are also promising for the many-agent case. Given the results in a companion paper by Koray and Pasin (2005), we expect to find a family of mechanisms such that every Nash-implementable SCR can be implemented via a mechanism in that family.

Given that the interplay between implementability and monotonicity is not confined to Nash implementability only as illustrated in Kaya and Koray (2000) and shown in Koray (2002), the approach employed in this work seems to be extendable to the general field of implementation via an arbitrary solution concept, leading to a variety of open questions that are yet to be dealt with.

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