DYNAMICS OF A TWO LEG LADDER UNDER A TIME DEPENDENT ARTIFICIAL MAGNETIC FIELD

A THESIS SUBMITTED TO

THE GRADUATE SCHOOL OF ENGINEERING AND SCIENCE

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR

THE DEGREE OF

MASTER OF SCIENCE

IN

PHYSICS

By

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September 2019

Dynamics of a two leg ladder under a time dependent artificial magnetic field
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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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September 2019

Especially in the last two decades, there have been intensive theoretical and experimental studies on the periodically driven quantum systems. The reasons for these intensive studies are the emergence of new topological effects and to be able to control these systems by periodic forcing. Another important reason is that systems governed by time periodic Hamiltonian can be also described by an infinite dimensional time-independent Hamiltonian in the Floquet formalism. Reducing this time-independent Hamiltonian into an effective $N \times N$ dimensional Hamiltonian provides great convenience to analyze the periodic quantum systems.

We study the dynamics of a particle in a two leg ladder subject to a time dependent artificial magnetic flux for two cases. In the first case, we consider that artificial magnetic flux varying linearly with time, and in the second case artificial magnetic flux oscillates in time. In both cases, we obtain a time periodic Hamiltonian in k-space at each k-point . Therefore, we use Floquet formalism to get eigensolutions of the Hamiltonian in k-space numerically, and then we construct the wave function of the particle in real space as a superposition of the Floquet states. We first analyze the quasi-energy bands for different values of the driving frequency and examine the effect of the field amplitude on the quasi-energy bands for oscillating magnetic field. Secondly, we examine the behavior of the particle in the real space for the different values of the driving frequency and the field amplitude.

Keywords: Time dependent artificial magnetic field , Time independent Floquet Hamiltonian, Time-periodic Hamiltonian.

ÖZET

SÖZDE MANYETİK ALAN ALTINDA BULUNAN İKİ BACAKLİ MERDİVENİN DİNAMİĞİ

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Eylül 2019

Özellikle son yirmi yılda periyodik olarak sürülen kuantum sistemler üzerine teorik ve deneysel olarak yoğun çalışmalar yapıdı. Yeni topolojik etkilerin ortaya çıkması ve bu sistemlerin kontrol edilebilir olması bu yoğun çalışmalara neden olmuştur. Zamana göre periyodik olan Hamiltoniyen tarafından tanımlanan bu sistemlerin zamandan bağımsız sonsuz boyuta sahip Hamiltoniyen tarafından da tanımlanması bu yoğun çalışmaların bir başka önemli nedenidir. Bu sonsuz boyuta sahip Hamiltoniyenin $N \times N$ boyutuna sahip etkili bir Hamiltoniyene indirgenmesi periyodik kuantum sistemlerin analizinde büyük kolaylık sağlar.

Zamana göre lineer değişen sözde manyetik akıyı birinci durumda, zamana göre salınım yapan sözde manyetik akıyı ise ikinci durumda değerlendirdik. Her iki durumda da k-uzayında her k noktası için zamana göre periyodik Hamiltoniyen elde ettik. Bundan dolayı k-uzayında Hamiltoniyenin öz-çözümlerini elde etmek için Floquet formalizmini kullandık, ve sonra parçacığın dalga fonksiyonunu Floquet durumlarının bir süperpozisyonu olarak oluşturduk. İlk olarak Kuasi enerji bandlarını farklı frekans değerleri için analiz ettik ve salınım yapan manyetik akı için manyetik alan genliğinin kuasi enerji bandlara etkisini inceledik. Daha sonra, parçacığın gerçek uzayda davranışını farklı frekans ve manyetik alan genliği için inceledik.

Anahtar sözcükler: Zamana bağlı sözde manyetik alan, zamandan bağımsız Floquet Hamiltoniyeni, Zamana göre periyodik Hamiltoniyen .

Acknowledgement

I gratefully acknowledge Prof. Dr. Mehmet Özgür Oktel for his guidance, interest, and patience.
I am also grateful to my friends Enes Aybar, and Fırat Yılmaz for useful discussions.

This project is supported by Türkiye Bilimsel ve Teknolojik Araştırma Kurumu (TÜBİTAK) Grant No. 116F215, I would like to thank TÜBİTAK for supporting me financially.

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Chapter 1

Introduction

Controlling and manipulating the quantum systems in order to understand their deep properties has become possible with the help of the powerful technologies resulting from quantum concepts. Recent experiments [1, 2, 3, 4, 5, 6, 7] show that time-periodic driving is a versatile tool for manipulating quantum systems and their dynamics. Therefore, over the last decades periodically time-dependent systems known as Floquet syatems have attracted a great deal of attention. These systems have brought interesting and non-intuitive phenomena, which are absent in static systems, to light such as dynamic stabilization and dynamic localization[8, 9, 10, 11], dynamical phase transition[12, 13, 14], coherent destruction of tunnelling[15, 16].

The fascinating phenomenon of dynamical stabilization , which has a wide range of applications including charged particle traps, optical resonators, and mass spectrometers, was firstly demonstrated by Kapitza. In classical physics, it emerges as a result of the shaking the suspension point of a rigid pendulum periodically along the vertical line with a sufficiently high frequency, which makes the pendulum stable at the inverted position[17].

The concept of controlling lattice systems by means of periodic driving was introduced by Dunlap and Kenkre[8]. They studied the effect of periodic forcing on a charged particle and found that a moving charged particle in a lattice structure under the influence of an oscillating electric field can be localized whenever the ratio of the field amplitude to the field frequency takes certain values. This phenomenon known as dynamic localization is the quantum mechanical version of the above mechanism and was observed with dilute Bose-Einstein condensates in a shaken optical lattices[1].

One of the most remarkable results of time-periodic systems is the quantized charged pump. Thouless in 1983 investigated the effect of the periodically time-dependent potential, which changes adiabatically, on an electronic system and he demonstrated that it causes charge transport to be quantized. This quantization of charge transport is topological invariant over an adiabatic cycle. This mechanism known as Thouless' pump[18]. Thouless pump was observed experimentally by using ultracold atoms in optical lattice[19, 20]. In fact, topological structure in periodically time-dependent systems is richer than in static systems. References [21, 22, 23, 24, 25] include some topological phenomena which can be realized only in time-periodic systems.

This thesis is organized as follows. Chapter 2 gives an outline of the Floquet theory. We develop perturbation theory in Chapter 3, and examine the dynamics of the particle in two-leg ladder subject to time dependent artificial magnetic field in Chapter 4. We summarize our results in Chapter 5.

Chapter 2

Floquet Theory

2.1 Floquet Theory and The Extended Hilbert Space

Floquet theory is a powerful tool for the study of dynamical systems described by a time periodic Hamiltonian. These systems have steady states $|\psi_\alpha(t)\rangle$ called Floquet states. According to the Floquet theorem [26], the solution of the Schrödinger equation with time periodic Hamiltonian can be written as a time periodic function multiplied by a phase factor. The Schrödinger equation with time periodic Hamiltonian is (for convenience, we set $\hbar = 1$ up to Chapter 4) ;

$$i\frac{\partial\psi(t)}{\partial t} = H(t)\psi(t) \tag{2.1}$$

where $H(t+T) = H(t) = H_0 + V(t)$ with T, defined by $T = \frac{2\pi}{\omega}$, being the period of the drive, and associated driving frequency ω. The form of solutions of the Schrödinger equation is as follows;

$$\psi_\alpha(t) = e^{-i\epsilon_\alpha t}\phi_\alpha(t), \tag{2.2}$$

with $\phi_\alpha(t+T) = \phi_\alpha(t)$. Here, $\psi_\alpha(t)$, $\phi_\alpha(t)$, ϵ_α are called Floquet state (steady-state), Floquet mode, and quasienergy, respectively. When we substitute Eq.(2.2) into the Eq.(2.1), we obtain an eigenvalue problem

$$\big(H(t) - i\frac{\partial}{\partial t}\big)\phi_\alpha(t) = \epsilon_\alpha \phi_\alpha(t) \tag{2.3}$$

is defined in an extented Hilbert space $\mathcal{F} = \mathcal{H} \otimes \mathcal{T}$. The tensor product of the state space of $\mathcal{H}$ of a quantum space and the space of $\mathcal{T}$ of functions which are periodic in time with period $T = \frac{2\pi}{\omega}$.
$\mathcal{H}_F = H(t) - i\frac{\partial}{\partial t}$ is known as the Floquet Hamiltonian and it is also time periodic. Due to the fact that the Floquet state depends on time we need to solve eigenvalue problem for all t $\in$ [0,T). The eigenvalue problems includes plenty of Floquet states which are physically equivalent. For instance, $|\phi_{\alpha n}(t)\rangle = e^{in\omega t}|\phi_\alpha(t)\rangle$ is a new solution with the shifted quasienergy $\epsilon_{\alpha n} = \epsilon_\alpha + n\omega$,

$$|\psi_\alpha(t)\rangle = e^{-i\epsilon_\alpha t}|\phi_\alpha(t)\rangle = e^{-i\epsilon_{\alpha n} t}|\phi_{\alpha n}(t)\rangle \tag{2.4}$$

Therefore, there appear an infinite number of replicas of Floquet states indexed by α. This redundancy provides a periodicity of ω for the quasienergy spectrum. Thus, we can reduce each quasienergy to a point in a zone which we can call the first Floquet Brillouin zone, $\frac{-\omega}{2} \leq \epsilon_\alpha \leq \frac{\omega}{2}$. This zone helps us to define the quasienergies correctly for the physical states. That's why, each physically different Floquet states can be characterized by these quasienergies that lie in the first Floquet Brillouin zone . Moreover, because of the time periodicity of Floquet modes, we can expand them in a Fourier series;

$$|\phi_\alpha(t)\rangle = \sum_m e^{im\omega t}|\phi_\alpha^m\rangle \tag{2.5}$$

$$|\psi_\alpha(t)\rangle = e^{-i\epsilon_\alpha t}\sum_m e^{im\omega t}|\phi_\alpha^m\rangle \tag{2.6}$$

The steady states can be interpreted as the linear combination of stationary states with energies $\epsilon_\alpha + m\omega$. Besides, the Fourier functions e^{imwt} encode the time dependence of Floquet state and the orthonormal set of these functions $\langle t|m\rangle = e^{im\omega t}$ spans vector space $\mathcal{T}$. The inner product in $\mathcal{T}$ space can be shown as;

$$(n, m) = \frac{1}{T}\int_0^T dt\ e^{i(m-n)\omega t} \tag{2.7}$$

Furthermore, the Floquet modes also form an orthonormal basis set in Hilbert space,

$$\sum_\alpha |\phi_\alpha(t)\rangle \langle\phi_\alpha(t')| = \delta(t-t') \tag{2.8}$$

Thus, the inner product in the extented Hilbert space $\mathcal{F}$ is defined as;

$$\langle\langle\phi|\psi\rangle\rangle = \frac{1}{T}\int_0^T dt \, \langle\phi(t)|\psi(t)\rangle \tag{2.9}$$

This is the time average of inner product of states in $\mathcal{H}$ over a single period T. The double ket notation $|\phi\rangle\rangle$ is used to represent the elements of the extended Hilbert space $\mathcal{F}$; the corresponding state in physical Hilbert space H at time t is denoted $|\phi(t)\rangle$.

2.2 Time-independent Floquet Hamiltonian

A time-dependent Schrödinger equation with a Hamiltonian periodic in time can be reformulated as an equivalent time-independent infinite-dimensional matrix eigenvalue problem by the help of Floquet theory[27].

When we substitute the Eqn.(2.5) into the Eqn.(2.3), then we get

$$H(t)\sum_n |\phi_\alpha^n\rangle e^{in\omega t} + n\omega |\phi_\alpha^n\rangle e^{in\omega t} \;\; = \;\; \epsilon_\alpha \sum_n |\phi_\alpha^n\rangle e^{in\omega t} \tag{2.10}$$

If we multiply above equation by $e^{-im\omega t}$ and integrate with respect to time, then we get

$$\sum_{m=-\infty}^{\infty} H^{n-m} |\phi_\alpha^m\rangle + n\omega |\phi_\alpha^n\rangle = \epsilon_\alpha |\phi_\alpha^n\rangle \tag{2.11}$$

Where, $H^{(n-m)} = \frac{1}{T}\int_0^T dt e^{i(n-m)\omega t} H(t)$ is the (n-m)-th Fourier component of the Hamiltonian. We can think the Fourier mode index n as a lattice index in an extra fictitious dimension. Therefore, a d-dimensional time-dependent system can be transformed into a d+1 dimensional time-independent system by the aid of the Fourier expansion. Since n runs from $-\infty$ to ∞, we arrive at an infinite dimensional time-independent Hamiltonian which is equivalent to the time periodic Hamiltonian.

If we use the Floquet-state nomenclature introduced in the reference [28]

$$|\gamma n\rangle = |\gamma\rangle \otimes |n\rangle \tag{2.12}$$

where γ refers to the system index and n refers to the Fourier index and $n \in (-\infty, \infty)$, then we obtain a matrix eigenvalue equation as;

$$\sum_{\gamma n} \langle \beta m|\mathcal{H}_F|\gamma n\rangle \, \phi^n_{\alpha,\gamma} = \epsilon_\alpha \phi^m_{\alpha,\beta} \tag{2.13}$$

with the Floquet matrix defined by

$$\langle \beta m|\mathcal{H}_F|\gamma n\rangle = \langle \beta|H^{(n-m)}|\gamma\rangle + n\omega\delta_{n,m}\delta_{\gamma,\beta} \tag{2.14}$$

where $\phi^n_{\alpha,\gamma} = \langle\gamma|\phi^n_\alpha\rangle$. The matrix elements of $\mathcal{H}_F$ are provided by the orthonormal basis $|\gamma n\rangle$, which can be considered as column matrices of all zeros, except for one in the position of γ, n. $\mathcal{H}_F$ has periodic structure due to Eq.(2.14),

$$\langle \beta m + q|\mathcal{H}_F|\gamma n + q\rangle = \langle \beta m|\mathcal{H}_F|\gamma n\rangle + q\omega\delta_{n,m}\delta_{\gamma,\beta} \tag{2.15}$$

where, q is an arbitrary integer. The quasienergies are obtained by solving the following secular equation;

$$\det|\mathcal{H}_F - \epsilon\mathbb{1}| = 0 \tag{2.16}$$

If we look at Table 2.1 a bit closer, we see that when we replace ϵ in secular equation by $\epsilon + q\omega$, q is any integer, the determinant doesn't change so both ϵ and $\epsilon + q\omega$ are the eigenvalues of $\mathcal{H}_F$. Let $\epsilon_{\alpha m}$ and $|\phi_{\alpha m}\rangle\rangle$ be eigenvalues and normalized eigenvectors of $\mathcal{H}_F$, respectively

$$\mathcal{H}_F\,|\phi_{\alpha m}\rangle\rangle = \epsilon_{\alpha m}\,|\phi_{\alpha m}\rangle\rangle \tag{2.17}$$

then $\epsilon_{\alpha m} = \epsilon_{\alpha 0} + m\omega$, and clearly the orthonormality condition is satisfied in the extended space $\mathcal{F}$,

$$\langle\langle\phi_{\alpha m}|\phi_{\beta n}\rangle\rangle = \frac{1}{T}\int_0^T dt\,\langle\phi_{\alpha m}(t)|\phi_{\beta n}(t)\rangle = \delta_{\alpha\beta}\delta_{mn} \tag{2.18}$$

Table 2.1 and Table 2.2 are the pictorial representation of the Floquet time independent matrix and its eigenstates, respectively;

Note that the Floquet Hilbert space have absorbed all temporal dependencies into its structure, therefore, the eigenstates $|\phi_\alpha\rangle\rangle$ of Floquet space do not depend on time.

Table 2.1: Floquet infinite matrix form

$$\mathcal{H}_F = \left(\begin{array}{c|c|c|c|c|c|c}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \hline
\dots & H_0 - 2\omega & H_1 & H_2 & H_3 & H_4 & \dots \\ \hline
\dots & H_{-1} & H_0 - \omega & H_1 & H_2 & H_3 & \dots \\ \hline
\dots & H_{-2} & H_{-1} & H_0 & H_1 & H_2 & \dots \\ \hline
\dots & H_{-3} & H_{-2} & H_{-1} & H_0 + \omega & H_1 & \dots \\ \hline
\dots & H_{-4} & H_{-3} & H_{-2} & H_{-1} & H_0 + 2\omega & \dots \\ \hline
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)$$

Table 2.2: Eigenvectors

$$|\phi_\alpha\rangle\rangle = \begin{pmatrix} \vdots \\ |\phi_\alpha^2\rangle\rangle \\ |\phi_\alpha^1\rangle\rangle \\ |\phi_\alpha^0\rangle\rangle \\ |\phi_\alpha^{-1}\rangle\rangle \\ |\phi_\alpha^{-2}\rangle\rangle \\ \vdots \end{pmatrix}$$

2.2.1 The Unitary Evolution operator

The evolution operator defined as;

$$U(t, t_0) = \mathcal{T} exp\Big[- i \int_{t_0}^{t} H(t')dt' \Big] \approx \prod_{t_1 \le t_i \le t_2} e^{-iH(t_i)(t_{i+1} - t_i)} \tag{2.19}$$

$$|\psi_\alpha(t)\rangle = U(t, t_0) \, |\psi_\alpha(t_0)\rangle \quad \text{with} \quad U(t_0, t_0) = \mathbb{1} \tag{2.20}$$

$$U(t, t_0) = \sum_\alpha |\psi_\alpha(t)\rangle \, \langle\psi_\alpha(t_0)| \tag{2.21}$$

where $\mathcal{T}$ is the time-ordering operator. The matrix elements of $U(t, t_0)$ can be written as;

$$\begin{aligned}
\langle\beta|U(t,t_0)|\gamma\rangle &= U_{\beta\gamma(t-t_0)} = \sum_\alpha \langle\beta|\phi_\alpha(t)\rangle\,\langle\phi_\alpha(t_0)|\gamma\rangle \\
&= \sum_\alpha\sum_n \langle\beta n|\phi_{\alpha 0}\rangle\, e^{in\omega t} e^{-i\epsilon_\alpha(t-t_0)} \sum_m \langle\phi_{\alpha m}|\gamma 0\rangle\, e^{-im\omega t_0} \\
&= \sum_{\alpha m}\sum_n \langle\beta n|\phi_{\alpha m}\rangle\, e^{in\omega t} e^{-i\epsilon_{\alpha m}(t-t_0)}\,\langle\phi_{\alpha m}|\gamma 0\rangle \\
&= \sum_n \langle\beta n|e^{-i\mathcal{H}_F(t-t_0)}|\gamma 0\rangle\, e^{in\omega t} \qquad (2.22)
\end{aligned}$$

We can consider $U_{\beta\gamma}(t-t_0)$ as the amplitude that the system is evolved according to the time independent Hamiltonian $\mathcal{H}_F$ from initial Floquet state $|\gamma 0\rangle$ at time t_0 to the Floquet state $|\beta n\rangle$ by time t , summed over n with weighting factor $e^{in\omega t}$, also it is interpreted as the amplitude that the system is evolved according to the time dependent Hamiltonian $H(t)$ from initial state $|\gamma\rangle$ at time t_0 to the state $|\beta\rangle$ by time t . The transition probability corresponding to the initial state $|\gamma\rangle$ to the final state $|\beta\rangle$, summed over all final field states, is;

$$P_{\beta\gamma}(t-t_0) = |U_{\beta\gamma}(t-t_0)|^2 = \sum_{nm} \langle\beta n|e^{-i\mathcal{H}_F(t-t_0)}|\gamma 0\rangle\, e^{im\omega t_0}\,\langle\gamma m|e^{-i\mathcal{H}_F(t-t_0)}|\beta n\rangle \qquad (2.23)$$

We assume here that initial state is not well-defined. The transition probability averaged over initial times t_0 by keeping the time $t-t_0$ fixed is written as;

$$P_{\beta\gamma}(t-t_0) = \sum_n |\,\langle\beta n|e^{-i\mathcal{H}_F(t-t_0)}|\gamma 0\rangle\,|^2 \qquad (2.24)$$

Finally, if we average $t-t_0$, then the long time average probability is obtained as;

$$\bar{P}_{\beta\gamma}(t-t_0) = \sum_{n\alpha k} |\,\langle\beta n|\phi_{\alpha k}\rangle\,\langle\phi_{\alpha k}|\gamma 0\rangle\,|^2 \qquad (2.25)$$

Let me summarize this chapter. First of all, since the Floquet states have infinite replicas, we choose a zone of size ω to characterize the physically different Floquet states by the quasi-energies lie in this zone. Secondly, we use the Fourier expansion show the equivalence between a d-dimensional time dependent system and a d+1-dimensional time-independent system by considering the Floquet index n as a label of lattice sites in an extra fictitious dimension. Then, we show that an infinite dimensional time-independent Hamiltonian can describe the system, which is governed by the time periodic Hamiltonian, in the extended Hilbert space. Finally, we see that a time averaged transition probability between the Floquet states can be obtained by the infinite dimensional time-independent Hamiltonian.

Chapter 3

Perturbation Theory in the Extended Hilbert Space

In this chapter, we develop perturbation theory in the extended Hilbert space to obtain approximate solutions for the Floquet states to analyze them analytically and also introduce two time formalism and adiabatic Floquet perturbation theory for our future work.

Sambe showed that the Floquet states , he called them as steady states, and their quasienergies behave in the extended Hilbert space like stationary states and energies of the time-independent systems[29]. Therefore, we can use time-independent perturbation theory to find approximate eigensolutions of the Hamiltonian when it includes a time periodic perturbation. Let the time-dependent Hamiltonian $H(t, \lambda)$ of system be given by

$$H(t, \lambda) = H^0 + \lambda V(t) \tag{3.1}$$

where H^0 is the time independent Hamiltonian and $V(t)$ is time periodic perturbation with period $T = \frac{2\pi}{\omega}$, and λ is an expansion parameter.

The Floquet-type Schröndiger equation is

$$H_F(t)\,|\phi(t)\rangle = (H_F^0(t) + \lambda V(t))\,|\phi(t)\rangle = \epsilon\,|\phi(t)\rangle \tag{3.2}$$

where $H_F^0(t) = H^0 - i\frac{\partial}{\partial t}$ is the unperturbed operator and its eigensolutions $(\epsilon_0, |\phi^0(t)\rangle)$ are well known. Before we develop perturbation theory, let's outline the differences between $H_F^0(t)$ and H^0.

3.1 The Difference Between Unperturbed Hamiltonians $\mathcal{H}_F^{(0)}$ and $H^{(0)}$

In order to show the difference between unperturbed Hamiltonian $H_F^{(0)}(t)$, defined as $H_F^{(0)} = H^{(0)} - i\hbar\frac{\partial}{\partial t}$, in the extended Hilbert space F and unperturbed Hamiltonian $H^{(0)}$ in the physical Hilbert space H, assume that the operator $H^{(0)}$ has discrete eigenvalues E_m and eigenkets $|\phi_m\rangle$, namely, $H^{(0)}\,|\phi_m\rangle = E_m\,|\phi_m\rangle$. Now if we consider the extended Hilbert space, then we can find the solutions of following equation;

$$H_F^{(0)}\,|\phi_{mn}\rangle\rangle = \epsilon_{mn}\,|\phi_m\rangle\rangle$$

where n is arbitrary integer. The eigensolutions of the above equation become $|\phi_{mn}\rangle\rangle := |\phi_m\rangle\, e^{in\omega t}$, and $\epsilon_{mn} = E_m + n\omega$. Since the $|\phi_{mn}\rangle\rangle$ and $|\phi_{mn+q}\rangle\rangle$ with an arbitrary integer q can construct same physically Floquet state with their corresponding eigenvalues. We have infinite eigenenergies of a physical Floquet states, $\epsilon_{mn} = E_m + n\omega$.

Suppose that $E_m, E_k, E_r, ...$ are the eigenvalues of $H^{(0)}$ and the relation between them is; $E_m = E_k + p\omega = E_r + q\omega = ...$ for some integers p, q, ..., then the eigenstates $|\phi_m\rangle, |\phi_k\rangle\, e^{ip\omega t}, |\phi_r\rangle\, e^{iq\omega t}, ...$ are the eigenstates of $H_F^{(0)}$. This proves that the eigenvalue of $H_F^{(0)}$ could be degenerate even if the eigenvalues of $H^{(0)}$ are non-degenerate.

3.2 Non-Degenerate Perturbation Theory in the Extended Hilbert Space

Let $(E_\alpha, E_\beta, ...)$ be eigenvalues of H^0 and the difference between any two eigenvalues be not equal to integer multiples of frequency, $E_\alpha \neq E_\beta + q\omega$, where q is an arbitrary integer. The steady-state Schrodinger equation for the system is;

$$\mathcal{H}_F \left|\phi_{\alpha n}\right\rangle\rangle = \epsilon_{\alpha n} \left|\phi_{\alpha n}\right\rangle\rangle \tag{3.3}$$

If we expand $\epsilon_{\alpha m}$ and $|\phi_{\alpha m}\rangle\rangle$ in power series of λ

$$\begin{aligned} \epsilon_{\alpha m} &= \epsilon^{(0)}_{\alpha m} + \lambda\epsilon^{(1)}_{\alpha m} + \lambda^2\epsilon^{(2)}_{\alpha m} + ..., & (3.4) \\ |\phi_{\alpha m}\rangle\rangle &= |\phi^{(0)}_{\alpha m}\rangle\rangle + \lambda\,|\phi^{(1)}_{\alpha m}\rangle\rangle + \lambda^2\,|\phi^{(2)}_{\alpha m}\rangle\rangle + ..., & (3.5) \end{aligned}$$

and equate the coefficients of the same power of λ, we obtain the following sequence of equations;

$$\begin{aligned} \left[\mathcal{H}^{(0)}_F \right. &- \left.\epsilon^{(0)}_{\alpha m}\right] |\phi^{(0)}_{\alpha m}\rangle\rangle = 0, & (3.6) \\ \left[\mathcal{H}^{(0)}_F \right. &- \left.\epsilon^{(0)}_{\alpha m}\right] |\phi^{(1)}_{\alpha m}\rangle\rangle + \left[V - \epsilon^{(1)}_{\alpha m}\right] |\phi^{(0)}_{\alpha m}\rangle\rangle = 0, & (3.7) \\ \left[\mathcal{H}^{(0)}_F \right. &- \left.\epsilon^{(0)}_{\alpha m}\right] |\phi^{(2)}_{\alpha m}\rangle\rangle + \left[V - \epsilon^{(1)}_{\alpha m}\right] |\phi^{(1)}_{\alpha m}\rangle\rangle - \epsilon^{(2)}_{\alpha m}\,|\phi^{(0)}_{\alpha m}\rangle\rangle = 0, & (3.8) \end{aligned}$$

With the normalization condition;

$$\begin{aligned} \langle\langle\phi_{\alpha m}(\lambda)|\phi_{\alpha m}(\lambda)\rangle\rangle &= 1 = \langle\langle\phi^{(0)}_{\alpha m}|\phi^{(0)}_{\alpha m}\rangle\rangle \Rightarrow \langle\langle\phi^{(0)}_{\alpha m}|\phi^{(1)}_{\alpha m}\rangle\rangle = 0 \Rightarrow & \\ \epsilon^{(1)}_{\alpha m} &= \langle\langle\phi^{(0)}_{\alpha m}|V|\phi^{(0)}_{\alpha m}\rangle\rangle & (3.9) \\ \epsilon^{(2)}_{\alpha m} &= \langle\langle\phi^{(0)}_{\alpha m}|V|\phi^{(1)}_{\alpha m}\rangle\rangle & (3.10) \end{aligned}$$

Since the set of the unperturbed states $|\phi^{(0)}_{\alpha m}\rangle\rangle$ form a complete and orthonormal basis, we can expand $|\phi^{(1)}_{\alpha m}\rangle\rangle$ in the $|\phi^{(0)}_{\alpha m}\rangle\rangle$ basis.

$$\begin{aligned} |\phi^{(1)}_{\alpha m}\rangle\rangle &= \sum_{\beta n} |\phi^{(0)}_{\beta n}\rangle\rangle\,\langle\langle\phi^{(0)}_{\beta n}|\,|\phi^{(1)}_{\alpha m}\rangle\rangle & (3.11) \\ \left[\mathcal{H}^{(0)}_F \right. &- \left.\epsilon^{(0)}_{\alpha m}\right] \sum_{\beta n} |\phi^{(0)}_{\beta n}\rangle\rangle\,\langle\langle\phi^{(0)}_{\beta n}|\,|\phi^{(1)}_{\alpha m}\rangle\rangle + \left[V - \epsilon^{(1)}_{\alpha m}\right] |\phi^{(0)}_{\alpha m}\rangle\rangle = 0, & (3.12) \end{aligned}$$

multiply by $\langle\langle\phi^{(0)}_{\beta n}|$, we obtain following relations;

$$\left[\epsilon^{(0)}_{\alpha m} \quad - \quad \epsilon^{(0)}_{\beta n}\right] \langle\langle\phi^{(0)}_{\beta n}|\phi^{(1)}_{\alpha m}\rangle\rangle = \langle\langle\phi^{(0)}_{\beta n}|V|\phi^{(0)}_{\alpha m}\rangle\rangle \Rightarrow \tag{3.13}$$

$$|\phi^{(1)}_{\alpha m}\rangle\rangle \quad = \quad \sum_{\beta n\neq\alpha m} \frac{\langle\langle\phi^{(0)}_{\beta n}|V|\phi^{(0)}_{\alpha m}\rangle\rangle}{\epsilon^{(0)}_{\alpha m} - \epsilon^{(0)}_{\beta n}} |\phi^{(0)}_{\alpha m}\rangle\rangle \tag{3.14}$$

Consider the following Hamiltonian as an example;
$H(t) = H^{(0)} + \lambda V(t)$, where

$$H^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad and \quad V(t) = \begin{pmatrix} 0 & f(t) \\ f(t) & 0 \end{pmatrix} \tag{3.15}$$

where $f(t+T) = f(t) = \sum_k f_k e^{ik\omega t}$. Assume that the time dependent part is sufficiently small compared to the time independent part. Suppose that at t$=0$ the particle is in the state $|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We want to find the state of the particle at time t. Since the system is two level system , there are only two physically different states. We can call them as up state by $|\psi_{\uparrow m}(t)\rangle = e^{-i\epsilon_{\uparrow m}t}|\phi_{\uparrow m}(t)\rangle$ and down state by $|\psi_{\downarrow m}(t)\rangle = e^{-i\epsilon_{\downarrow m}t}|\phi_{\downarrow m}(t)\rangle$. Now we can apply non-degenerate perturbation theory ($\hbar = 1$);
The zeroth order;

$$\left[\mathcal{H}^{(0)}_F - \epsilon^{(0)}_{\uparrow m}\right] |\phi^{(0)}_{\uparrow m}\rangle\rangle = 0, \Rightarrow$$

$$|\phi^{(0)}_{\uparrow m}(t)\rangle = |\phi^{(0)}_{\uparrow m}\rangle\, e^{im\omega t}, \quad \text{where} \quad |\phi^{(0)}_{\uparrow m}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.16}$$

$$\epsilon_{\uparrow m} = 1 + m\omega \tag{3.17}$$

The first order;

$$|\phi^{(1)}_{\uparrow m}\rangle\rangle = \sum_{n\neq m} |\phi^{(0)}_{\uparrow n}\rangle\rangle\,\langle\langle\phi^{(0)}_{\uparrow n}|\,|\phi^{(1)}_{\uparrow m}\rangle\rangle \quad + \quad \sum_{n} |\phi^{(0)}_{\downarrow n}\rangle\rangle\,\langle\langle\phi^{(0)}_{\downarrow n}|\,|\phi^{(1)}_{\uparrow m}\rangle\rangle \tag{3.18}$$

$$\begin{aligned}\left[\mathcal{H}^{(0)}_F - \epsilon^{(0)}_{\uparrow m}\right]\{\sum_{n\neq m} |\phi^{(0)}_{\uparrow n}\rangle\rangle\,\langle\langle\phi^{(0)}_{\uparrow n}|\,|\phi^{(1)}_{\uparrow m}\rangle\rangle \quad &+ \quad \sum_{n} |\phi^{(0)}_{\downarrow n}\rangle\rangle\,\langle\langle\phi^{(0)}_{\downarrow n}|\,|\phi^{(1)}_{\uparrow m}\rangle\rangle\} \\ &+ \quad \left[V - \epsilon^{(1)}_{\uparrow m}\right] |\phi^{(0)}_{\uparrow m}\rangle\rangle = 0,\end{aligned} \tag{3.19}$$

multiply by $\langle\langle\phi^{(0)}_{\uparrow n}|$, we obtain;

$$\left[\epsilon^{(0)}_{\uparrow n} \quad - \quad \epsilon^{(0)}_{\uparrow m}\right]\langle\langle\phi^{(0)}_{\uparrow n}|\phi^{(1)}_{\uparrow m}\rangle\rangle + \langle\langle\phi^{(0)}_{\uparrow n}|V|\phi^{(0)}_{\uparrow m}\rangle\rangle - \epsilon^{(1)}_{\uparrow m}\langle\langle\phi^{(0)}_{\uparrow n}|\phi^{(0)}_{\uparrow m}\rangle\rangle = 0, \quad (3.20)$$

the second and third terms give no contribution so $\langle\langle\phi^{(0)}_{\uparrow n}|\phi^{(1)}_{\uparrow m}\rangle\rangle = 0$
and $\epsilon^{(1)}_{\uparrow m} = \langle\langle\phi^{(0)}_{\uparrow m}|V|\phi^{(0)}_{\uparrow m}\rangle\rangle = 0$. multiply by $\langle\langle\phi^{(0)}_{\uparrow n}|$, we obtain;

$$\left[\epsilon^{(0)}_{\downarrow n} \quad - \quad \epsilon^{(0)}_{\uparrow m}\right]\langle\langle\phi^{(0)}_{\uparrow n}|\phi^{(1)}_{\uparrow m}\rangle\rangle + \langle\langle\phi^{(0)}_{\downarrow n}|V|\phi^{(0)}_{\uparrow m}\rangle\rangle - \epsilon^{(1)}_{\uparrow m}\langle\langle\phi^{(0)}_{\downarrow n}|\phi^{(0)}_{\uparrow m}\rangle\rangle = 0, \quad (3.21)$$

the third term is zero, thus,

$$\begin{aligned}
\langle\langle\phi^{(0)}_{\uparrow n}|\phi^{(1)}_{\uparrow m}\rangle\rangle &= \frac{\langle\langle\phi^{(0)}_{\downarrow n}|V|\phi^{(0)}_{\uparrow m}\rangle\rangle}{\epsilon^{(0)}_{\downarrow n} - \epsilon^{(0)}_{\uparrow m}} \\
|\phi^{(1)}_{\uparrow m}\rangle\rangle &= \sum_n \frac{\langle\langle\phi^{(0)}_{\downarrow n}|V|\phi^{(0)}_{\uparrow m}\rangle\rangle}{\epsilon^{(0)}_{\uparrow m} - \epsilon^{(0)}_{\downarrow n}}|\phi^{(0)}_{\downarrow n}\rangle\rangle \\
\text{similarly;} & \\
\epsilon^{(1)}_{\downarrow m} &= \langle\langle\phi^{(0)}_{\downarrow m}|V|\phi^{(0)}_{\downarrow m}\rangle\rangle = 0 \\
|\phi^{(1)}_{\downarrow m}\rangle\rangle &= \sum_n \frac{\langle\langle\phi^{(0)}_{\uparrow n}|V|\phi^{(0)}_{\downarrow m}\rangle\rangle}{\epsilon^{(0)}_{\downarrow m} - \epsilon^{(0)}_{\uparrow n}}|\phi^{(0)}_{\uparrow n}\rangle\rangle \\
\text{and} & \\
\langle\langle\phi^{(0)}_{\downarrow n}|V|\phi^{(0)}_{\uparrow m}\rangle\rangle &= \langle\langle\phi^{(0)}_{\uparrow n}|V|\phi^{(0)}_{\downarrow m}\rangle\rangle \\
&= \frac{1}{T}\int_0^T dt\; e^{-i(n-m)\omega t} f(t) \\
&= \tilde{f}_{n-m} \qquad (3.22) \\
\epsilon^{(0)}_{\uparrow m} - \epsilon^{(0)}_{\downarrow n} &= 2 - (n-m)\omega \\
\epsilon^{(0)}_{\downarrow m} - \epsilon^{(0)}_{\uparrow n} &= -2 - (n-m)\omega
\end{aligned}$$

In this example, we find the first order corrections for both eigenstates and quasienergies, and now we can write the complete wave function as follows,

$$
\begin{aligned}
|\psi(t)\rangle &= c_1\{|\phi^{(0)}_{\uparrow m}(t)\rangle + |\phi^{(1)}_{\uparrow m}(t)\rangle\}e^{-i(\epsilon^{(0)}_{\uparrow m}+\epsilon^{(1)}_{\uparrow m}+\dots)t} \\
&+ c_2\{|\phi^{(0)}_{\downarrow m}(t)\rangle + |\phi^{(1)}_{\uparrow}(t)\rangle\}e^{-i(\epsilon^{(0)}_{\downarrow m}+\epsilon^{(1)}_{\downarrow m}+\dots)t} \\
&= c_1\{|\uparrow\rangle\, e^{im\omega t} + \sum_n \frac{\tilde{f}_{n-m}}{2-(n-m)\omega}\,|\downarrow\rangle\, e^{in\omega t}\}e^{-i(1+m\omega)t} \\
&+ c_2\{|\downarrow\rangle\, e^{im\omega t} + \sum_n \frac{\tilde{f}_{n-m}}{-2-(n-m)\omega}\,|\uparrow\rangle\, e^{in\omega t}\}e^{-i(-1+m\omega)t} \\
&= c_1\{|\uparrow\rangle\, e^{-it} + \sum_n \frac{\tilde{f}_{n-m}}{2-(n-m)\omega}\,|\downarrow\rangle\, e^{i(n-m)\omega t}e^{-it}\} \\
&+ c_2\{|\downarrow\rangle\, e^{it} + \sum_n \frac{\tilde{f}_{n-m}}{-2-(n-m)\omega}\,|\uparrow\rangle\, e^{i(n-m)\omega t}e^{it}\} \\
&= \{c_1 e^{-it} + c_2 \sum_n \frac{\tilde{f}_n}{-2-n\omega}e^{in\omega t}e^{it}\}\,|\uparrow\rangle\} \\
&+ \{c_2 e^{it} + c_1 \sum_n \frac{\tilde{f}_n}{2-n\omega}e^{in\omega t}e^{-it}\}\,|\downarrow\rangle
\end{aligned} \tag{3.23}
$$

We can find c_1 and c_2 from the initial conditions

$$
|\psi(0)\rangle = \{c_1 + c_2 \sum_n \frac{\tilde{f}_n}{-2-n\omega}\}\,|\uparrow\rangle\} + \{c_2 + c_1 \sum_n \frac{\tilde{f}_n}{2-n\omega}\}\,|\downarrow\rangle \tag{3.24}
$$

This example is an application for the non-degenerate perturbation theory in the extended Hilbert space. If conditions which we mention above are satisfied, then we can apply non-degenerate perturbation theory to find approximate solutions for any system.

3.3 Degenerate Perturbation Theory in the Extended Hilbert Space

Suppose that $|\phi^{(0)}_{\alpha 0}\rangle\rangle$ and $|\phi^{(0)}_{\beta q}\rangle\rangle$ are nearly degenerate states, $E_\alpha \sim E_\beta + q\omega$, of the unperturbed Hamiltonian $H^{(0)}_F$. We want to determine the eigenkets $(|\phi_{\alpha 0}\rangle\rangle\,, |\phi_{\beta q}\rangle\rangle)$ and their energies $(\epsilon_{\alpha 0}, \epsilon_{\beta q})$ of the full Hamiltonian H_F with the aid of almost degenerate perturbation theory developed by J. O. Hirschfilder[30].

For convenience, let $(|\phi_1\rangle\rangle\,, |\phi_2\rangle\rangle)$ and (ϵ_1, ϵ_2) represent $(|\phi_{\alpha 0}\rangle\rangle\,, |\phi_{\beta q}\rangle\rangle)$,$(\epsilon_{\alpha 0}, \epsilon_{\beta q})$, respectively. We can make the following transformation by the aid of a nonsingular matrix M,

$$|\tilde{\phi}_k\rangle\rangle = \sum_{l=1}^{2} |\phi_l\rangle\rangle\, M_{lk} \tag{3.25}$$

where M_{lk} are the elements of a nonsingular transformation. If M^{-1} is the inverse of the matrix M, then

$$|\phi_k\rangle\rangle = \sum_{s=1}^{2} |\tilde{\phi}_s\rangle\rangle\, M^{-1}_{sl} \tag{3.26}$$

After transformation, we have following eigenvalue equations

$$\begin{aligned} H_F\,|\tilde{\phi}_k\rangle\rangle &= \sum_{l=1}^{2} \epsilon_l\,|\phi_l\rangle\rangle\, M_{lk} \\ &= \sum_{s=1}^{2} |\tilde{\phi}_s\rangle\rangle\, \epsilon_{sk} \end{aligned} \tag{3.27}$$

where

$$\epsilon_{sk} = \sum_{l=1}^{2} M^{-1}_{sl}\epsilon_l M_{lk} \tag{3.28}$$

The eigensolutions $(\epsilon_k, |\phi_k\rangle\rangle)$ are the eigensolutions of the 2-dimensional secular equation,

$$|\,\langle\langle\tilde{\phi}_k|H_F - \epsilon|\tilde{\phi}_l\rangle\rangle\,| = 0 \tag{3.29}$$

If we use expansions in Eqn.(4,5) for eigensolutions $(\epsilon_{sk}, |\tilde{\phi}_k\rangle\rangle)$ and substitute them into Eqn.(3.27) and equate the coefficient of each power of λ to zero, then we obtain following equations;

for $n = 0$:

$$(H_F^0 - \epsilon_k^{(0)})\,|\tilde{\phi}_k^{(0)}\rangle\rangle = 0 \tag{3.30}$$

for $n \geq 1$:

$$(H_F^0 - \epsilon_k^{(0)})\,|\tilde{\phi}_k^{(n)}\rangle\rangle = -V\,|\tilde{\phi}_k^{(n-1)}\rangle\rangle + \sum_{s=1}^{2}\sum_{m=0}^{n-1}|\tilde{\phi}_s^{(m)}\rangle\rangle\,\epsilon_{sk}^{(n-m)} \tag{3.31}$$

where $\epsilon_k^{(0)} = \epsilon_{sk}^{(0)}\delta_{sk}$ and $|\tilde{\phi}_k^{(0)}\rangle\rangle$ are the well known eigensolutions of H_F^0.

If we multiply Eqn.(3.27) with $\langle\langle\tilde{\phi}_l^{(0)}|$, where $l = 1, 2$, then we get

$$\epsilon_{lk}^{(n)} = \langle\langle\tilde{\phi}_l^{(0)}|V|\tilde{\phi}_k^{(0)}\rangle\rangle + (\epsilon_l^{(0)} - \epsilon_k^{(0)})\,\langle\langle\tilde{\phi}_l^{(0)}|\tilde{\phi}_k^{(n)}\rangle\rangle - \sum_{s=1}^{2}\sum_{m=1}^{n-1}\langle\langle\tilde{\phi}_l^{(0)}|\tilde{\phi}_s^{(m)}\rangle\rangle\,\epsilon_{sk}^{(n-m)} \tag{3.32}$$

Let $(|\tilde{\phi}_{\alpha x}^{(0)}\rangle\rangle\,, |\tilde{\phi}_{\beta y}^{(0)}\rangle\rangle\,, \{|\tilde{\phi}_{\gamma m}^{(0)}\rangle\rangle\})$ be the nondegenerate states with $x \neq 0$, and $y \neq q$, and an operator be defined as

$$R_k^0 = \sum_p \frac{|\tilde{\phi}_p^{(0)}\rangle\rangle\,\langle\langle\tilde{\phi}_p^{(0)}|}{\epsilon_k^{(0)} - \epsilon_s^{(0)}} \tag{3.33}$$

where p excludes just αx and βy with $x = 0$ and $y = q$. If we apply this operator to Eqn.(3.31), we get

$$|\tilde{\phi}_k^{(n)}\rangle\rangle = R_k^0 V\,|\tilde{\phi}_k^{(n-1)}\rangle\rangle - \sum_{s=1}^{2}\sum_{m=0}^{n-1} R_k^0\,|\tilde{\phi}_s^{(m)}\rangle\rangle\,\epsilon_{sk}^{(n-m)} + \sum_{s=1}^{2}|\tilde{\phi}_s^{(0)}\rangle\rangle\,\langle\langle\tilde{\phi}_s^{(0)}|\tilde{\phi}_k^{(n)}\rangle\rangle \tag{3.34}$$

with the Certain's type of normalization [30]

$$\langle\langle\tilde{\phi}_l^{(0)}|\tilde{\phi}_k^{(2n+1)}\rangle\rangle = -\sum_{m=1}^{n}\langle\langle\tilde{\phi}_l^{(m)}|\tilde{\phi}_k^{(2n+1-m)}\rangle\rangle \tag{3.35}$$

and

$$\langle\langle\tilde{\phi}_l^{(0)}|\tilde{\phi}_k^{(2n)}\rangle\rangle = -\frac{1}{2}\,\langle\langle\tilde{\phi}_l^{(n)}|\tilde{\phi}_k^{(n)}\rangle\rangle - \sum_{m=1}^{n-1}\langle\langle\tilde{\phi}_l^{(m)}|\tilde{\phi}_k^{(2n-m)}\rangle\rangle \tag{3.36}$$

we can obtain

$$|\tilde{\phi}_k^{(1)}\rangle\rangle = R_k^0 V\,|\tilde{\phi}_k^{(0)}\rangle\rangle \tag{3.37}$$

$$|\tilde{\phi}_k^{(2)}\rangle\rangle = R_k^0\big\{V\,|\tilde{\phi}_k^{(1)}\rangle\rangle - \sum_{s=1}^{2}|\tilde{\phi}_s^{(1)}\rangle\rangle\,\epsilon_{sk}^{(1)}\big\} - \frac{1}{2}\sum_{s=1}^{2}|\tilde{\phi}_s^{(0)}\rangle\rangle\,\langle\langle\tilde{\phi}_s^{(1)}|\tilde{\phi}_k^{(1)}\rangle\rangle \tag{3.38}$$

$$|\tilde{\phi}_k^{(3)}\rangle\rangle = R_k^0\big\{V\,|\tilde{\phi}_k^{(2)}\rangle\rangle - \sum_{s=1}^{2}\big\{\,|\tilde{\phi}_s^{(1)}\rangle\rangle\,\epsilon_{sk}^{(2)} + |\tilde{\phi}_s^{(1)}\rangle\rangle\,\epsilon_{sk}^{(2)}\big\}\big\} - \sum_{s=1}^{2}|\tilde{\phi}_s^{(0)}\rangle\rangle\,\langle\langle\tilde{\phi}_s^{(1)}|\tilde{\phi}_k^{(2)}\rangle\rangle \tag{3.39}$$

For practical purpose, we define the sequence of partial sums instead of expansions in Eqn.(3.4, and 3.5),

$$\begin{aligned} |\tilde{\phi}_k(N)\rangle\rangle &= \sum_{n=1}^{N} \lambda^n \, |\tilde{\phi}_k^{(n)}(N)\rangle\rangle \\ \epsilon_{sk}(N) &= \sum_{n=1}^{N} \lambda^n \epsilon_{sk}^n \end{aligned} \tag{3.40}$$

By doing so, we can obtain the approximate eigensolutions $(\epsilon_k(N), |\phi_k(N)\rangle\rangle)$ which are the eigensolutions of the following 2-dimensional secular equation,

$$|\,\langle\langle\tilde{\phi}_k(N)|H_F - \epsilon(N)|\tilde{\phi}_l(N)\rangle\,| = 0 \tag{3.41}$$

3.4 Floquet Adiabatic Perturbation Theory

If a term s(t), which is varying slowly, is added to the periodic Hamiltonian at t=0, then the Hamiltonian will not be periodic anymore. However, if we consider s(t) as almost constant during one period, we can apply the Floquet theory. By using (t,t') formalism [31, 32], we can develop the Floquet adiabaatic theory.

3.4.1 (t,t') formalism

The time dependent solution of the Schrödinger equation

$$i\frac{\partial}{\partial t}\Psi(t) = H(t)\Psi(t) \tag{3.42}$$

with the initial state

$$\Psi(t) = \psi_0 \tag{3.43}$$

can be written as

$$\Psi(t) = \int_{-\infty}^{\infty} dt' \delta(t'-t)\tilde{\psi}(t',t) \tag{3.44}$$

where $\tilde{\psi}(t',t)$ is the solution of the following two time Schrödinger equation,

$$i\frac{\partial}{\partial t}\tilde{\psi}(t',t) = \mathcal{K}(t')\tilde{\psi}(t',t) \tag{3.45}$$

where $\mathcal{K}(t')$ is defined as

$$\mathcal{K}(t') = H(t') - i\frac{\partial}{\partial t'} \tag{3.46}$$

Pfiffer and Levine [33] proved the validity of the Eqn.(3.44) by using time ordering operator. An alternative proof was developed by Peskin and Moisyev[32] as follows: from the Eqn.(3.45)

$$i(\frac{\partial}{\partial t} + \frac{\partial}{\partial t'})\tilde{\psi}(t',t) = H(t')\tilde{\psi}(t',t) \tag{3.47}$$

Since the contour $t = t'$ is the only contour we are interested in, and $\frac{\partial t}{\partial t'} = 1$, thus we can find that

$$\frac{\partial}{\partial t}\tilde{\psi}(t',t)|_{t=t'} + \frac{\partial}{\partial t'}\tilde{\psi}(t',t)|_{t=t'} = \frac{\partial}{\partial t}\Psi(t)|_{t=t'} \tag{3.48}$$

therefore, Eqn.(3.47) reduces to Eqn.(3.42) at t=t'.

3.4.2 Adiabatic Floquet Model

Now we are ready to introduce two time variables t' and t, which correspond to "slow" time scale and "fast" time scale respectively, to develop Floquet adiabatic perturbation theory[34].

$$\mathcal{K}(s(t'),t)\,|\tilde{\psi}(s(t'),t)\rangle = i\frac{\partial}{\partial t'}\,|\tilde{\psi}(s(t'),t)\rangle \tag{3.49}$$

where

$$\mathcal{K}(s(t'),t) = H(s(t'),t) - i\frac{\partial}{\partial t} \tag{3.50}$$

The solutions of the Schröndiger equation for any constant value of s are the Floquet states;

$$|\psi_\alpha(s,t)\rangle = e^{-i\epsilon_\alpha(s)t}\,|u_\alpha(s,t)\rangle \tag{3.51}$$

We can write the solution of the Eqn.(3.49) as

$$|\tilde{\psi}(s(t'),t)\rangle = \sum_\alpha c_\alpha(s(t'))e^{-i\int_0^{t'}\epsilon_\alpha(s(\tau))d\tau}\,|u_\alpha(s,t)\rangle \tag{3.52}$$

Substituting Eqn.(3.52) into Eqn.(3.49), and taking inner product with $\langle\langle u_\beta(s(t'),t)|$, we can obtain following equation;

$$\frac{\partial}{\partial t'}c_\beta(s(t')) = -\sum_\alpha c_\alpha(s(t'))e^{-i\int_0^{t'}(\epsilon_\alpha(\tau)-\epsilon_\beta(\tau))d\tau}\,\langle\langle u_\beta(s(t'),t)|\frac{\partial}{\partial t'}u_\alpha(s(t'),t)\rangle\rangle \tag{3.53}$$

Now performing a transformation,

$$a_\beta(t') = e^{i\gamma_\beta(t')} c_\beta(t') \tag{3.54}$$

where $\gamma_\beta(t') = i\int_0^{t'} d\tau \, \langle\langle u_\beta(s(\tau), t)| \frac{\partial}{\partial\tau} u_\alpha(s(\tau), t)\rangle\rangle$ known as Berry phase[35], then the equation becomes

$$\frac{\partial}{\partial t'} a_\beta(s(t')) = -\sum_{\alpha\neq\beta} a_\alpha(s(t')) e^{-i\int_0^{t'}(\epsilon_\alpha(\tau)-\epsilon_\beta(\tau))d\tau} A_{\alpha\beta}(t') \tag{3.55}$$

Here $A_{\alpha\beta}(t') = e^{i\gamma_\beta(t') - i\gamma_\alpha(t')} \langle\langle u_\beta(s(t'), t)| \frac{\partial}{\partial s(t')} u_\alpha(s(t'), t)\rangle\rangle \frac{\partial s(t')}{\partial t'}$. Let the system be initially in an arbitrary Floquet state r, $c_r(s(0)) = 1$, then

$$\frac{\partial}{\partial t'} a_\beta(s(t')) = -e^{-i\int_0^{t'}(\epsilon_\alpha(\tau)-\epsilon_\beta(\tau))d\tau} A_{r\beta} \tag{3.56}$$

If we use integration by parts, then we can obtain the first order correction in $\frac{\partial s(t')}{\partial t'}$

$$\begin{aligned}
a_r(s(t')) &\approx 1 \Rightarrow c_r(s(t')) \approx e^{-i\gamma_r(t')} \\
a_{\alpha\neq r}(s(t')) &\approx i\frac{e^{i\int_0^{t'}(\epsilon_\alpha(\tau)-\epsilon_r(\tau))d\tau}}{\epsilon_\alpha(t') - \epsilon_r(t')} A_{\alpha\beta} \Rightarrow \\
c_{\alpha\neq r}(s(t')) &\approx i\frac{e^{i\int_0^{t'}(\epsilon_\alpha(\tau)-\epsilon_r(\tau))d\tau}}{\epsilon_\alpha(t') - \epsilon_r(t')} e^{-i\gamma_r(t')} \langle\langle u_\beta(s(t'), t)| \frac{\partial}{\partial s(t')} u_\alpha(s(t'), t)\rangle\rangle \frac{\partial s(t')}{\partial t'} \\
|\tilde{\psi}(s(t'), t)\rangle &\approx e^{-i\gamma_r(t')} e^{-i\int_0^{t'}\epsilon_r(\tau)d\tau} \{ |u_r(s(t'), t)\rangle \\
&\quad + i\sum_{\alpha\neq r} |u_\alpha(s(t'), t)\rangle \frac{\langle\langle u_\alpha(s(t'), t)| \frac{\partial}{\partial s(t')} u_r(s(t'), t)\rangle\rangle}{\epsilon_\alpha(t') - \epsilon_r(t')} \frac{\partial s(t')}{\partial t'} \}
\end{aligned} \tag{3.57}$$

Since $|\tilde{\psi}(s(t'), t)\rangle|_{t=t'} = \Psi(t)$, our initial Floquet state (Eqn.(3.51)) will evolve as;

$$
\begin{aligned}
|\psi_\alpha(s(t),t)\rangle \;\; &\approx \;\; e^{-i\gamma_\alpha(t)} e^{-i\int_0^t \epsilon_\alpha(\tau)d\tau} \{\, |u_\alpha(s(t),t)\rangle \\
&\quad + i\sum_{\beta\neq\alpha} |u_\beta(s(t),t)\rangle \, \frac{\langle\langle u_\beta(s(t),t)|\frac{\partial}{\partial s(t)} u_\alpha(s(t),t)\rangle\rangle}{\epsilon_\beta(t)-\epsilon_\alpha(t)} \frac{\partial s(t)}{\partial t} \}
\end{aligned}
\tag{3.58}
$$

In this chapter, we see the difference between the unperturbed Hamiltonians $H_F^{(0)}$ and H_0, we prove that the degeneracy may occur due to the structure of $H_F^{(0)}$ even if H_0 has no degenerate eigenvalues. Then, we develop non-degenerate perturbation theory in the extended Hilbert space to show that the Floquet states behaves in the extended Hilbert space like the stationary states in the Hilbert space.

The almost degenerate perturbation theory which we develop converges rapidly near resonance, and if we determine the eigenstates up to n-th order $O(\lambda^n)$, then it allows us to calculate the energies up to 2n+1-th order $O(\lambda^{2n+1})$. This provides us the transition probability between the Floquet states to observe (2n+1) photons process.

Finally, we develop two time formalism in order to obtain the Floquet adiabatic model. We develop this part for our near future work, that is, we will be interested in the topological charge pumping which is related to adiabaticity.

Chapter 4

Two Leg Ladder Subject to Time Dependent Artificial Magnetic Field

4.1 Two-Leg Ladder

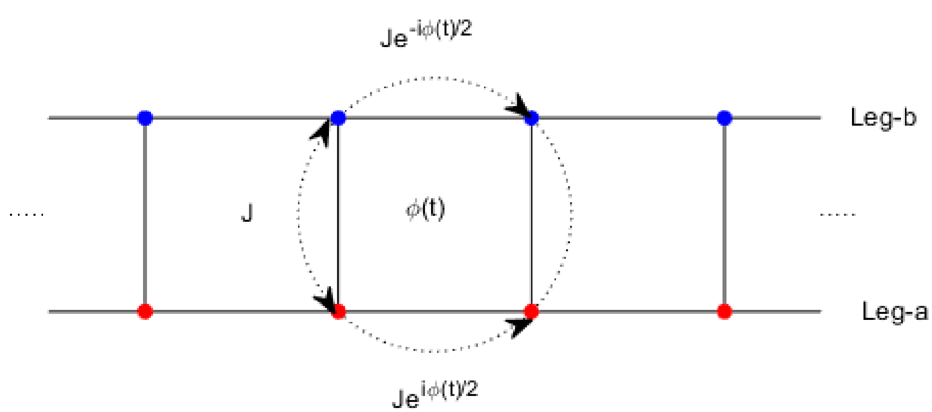


Figure 4.1: Two Leg-Ladder.

In order to apply the Floquet formalism, which we develop in Chapter 2, to a physical system, we choose a simple model. Since the tight-binding model was studied for static case and experimentally, we use it to analyze the dynamics of the two-leg ladder under the time-dependent artificial magnetic field. We assume that a particle is allowed for tunnelling or hopping between the nearest neighbouring sites. We can write our Hamiltonian without magnetic field as:

$$H = J\sum_i a_i^\dagger a_{i+1} + b_i^\dagger b_{i+1} + a_i^\dagger b_i + h.c \tag{4.1}$$

where $a_i^\dagger(a_i)$ are creation(annihilation) operators for leg-a and $b_i^\dagger(b_i)$ are creation(annihilation) operators for leg-b and J is the hopping parameter. We apply a time-dependent magnetic field in the z-direction and choose the vector potential $A = (-yB_0(t), 0, 0)$ as the Landau Gauge. Due to the vector potential, the wave-function of the particle picks a phase when the particle moves from a lattice site to a different lattice site. The phase, which is described as an integral $\frac{e}{\hbar c}\int_{R_n}^{R_m} \vec{A}.d\vec{l}$, enters the Hamiltonian via Peierls substitution

$$J \rightarrow Jexp\{\frac{-ie}{\hbar c}\int_{R_n}^{R_m} \vec{A}.d\vec{l}\} \tag{4.2}$$

then our Hamiltonian becomes

$$H(t) = J\sum_i \{e^{i\frac{\phi(t)}{2}} a_i^\dagger a_{i+1} + e^{-i\frac{\phi(t)}{2}} b_i^\dagger b_{i+1} + a_i^\dagger b_i + h.c\} \tag{4.3}$$

where

$$\phi(t) = \frac{e}{\hbar c}\oint \vec{A}.d\vec{l} = \frac{e}{\hbar c}\int (\vec{\bigtriangledown} \times \vec{A}).d\vec{s} = \frac{e}{\hbar c}\Phi(t) = \frac{2\pi}{\phi_0}\Phi(t) \tag{4.4}$$

where $\phi_0 = \frac{hc}{e}$ is the flux quantum and $\Phi(t)$ is the magnetic flux passing through each plaquette. The general wave function can be written as the linear combination of the all localized states $|i, j\rangle$ as :

$$|\Psi(t)\rangle = \sum_i \psi_{i,A}(t) |i, A\rangle + \psi_{i,B}(t) |i, B\rangle \tag{4.5}$$

We can write our Hamiltonian and eigenstate in k-space by using Fourier transform;

$$\begin{aligned} a_j &= \frac{1}{\sqrt{L}} \sum_k e^{ikja} a_k \\ b_j &= \frac{1}{\sqrt{L}} \sum_k e^{ikja} b_k \end{aligned} \tag{4.6}$$

where L is the length of the system and $[a_k, a^\dagger_{k'}] = \delta_{kk'}$,and $[b_k, b^\dagger_{k'}] = \delta_{kk'}$.
Since we assume the periodic boundary condition $a_{j+L} = a_j$, $\sum_k e^{ikja} e^{ikLa} a_k = \sum_k e^{ikja} a_k$, we can find k as $k = \frac{2\pi n}{La}$, where n is an arbitrary integer. Our Hamitonian and wave function becomes

$$H(t) = \sum_k 2J\cos(ka + \frac{\phi(t)}{2}) a^\dagger_k a_k + 2J\cos(ka - \frac{\phi(t)}{2}) b^\dagger_k b_k + J a^\dagger_k b_k + J b^\dagger_k a_k \tag{4.7}$$

$$= \sum_k \begin{pmatrix} a^\dagger_k & , & b^\dagger_k \end{pmatrix} \begin{pmatrix} 2J\cos(ka + \frac{\phi(t)}{2}) & J \\ J & 2J\cos(ka - \frac{\phi(t)}{2}) \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \tag{4.8}$$

$$\begin{aligned} |\Psi(t)\rangle &= \sum_k \frac{1}{\sqrt{L}} \sum_j \{\psi_{j,A}(t) e^{-ikja} |k, A\rangle + \psi_{j,B}(t) e^{-ikja} |k, B\rangle\} \\ &= \sum_k \psi_{k,A}(t) |k, A\rangle + \psi_{k,B}(t) |k, B\rangle \end{aligned} \tag{4.9}$$

where

$$\begin{pmatrix} \psi_{k,A}(t) \\ \psi_{k,B}(t) \end{pmatrix} = \frac{1}{\sqrt{L}} \sum_j e^{-ikja} \begin{pmatrix} \psi_{j,A}(t) \\ \psi_{j,B}(t) \end{pmatrix} \tag{4.10}$$

and

$$\begin{pmatrix}\psi_{j,A}(t)\\ \psi_{j,B}(t)\end{pmatrix} = \frac{1}{\sqrt{L}}\sum_k e^{ikja}\begin{pmatrix}\psi_{k,A}(t)\\ \psi_{k,B}(t)\end{pmatrix} \tag{4.11}$$

Then the Schrödinger equation reads

$$\begin{aligned} H(t)\left|\Psi(t)\right\rangle &= \sum_k 2J\cos(ka+\frac{\phi(t)}{2})\psi_{k,A}(t)\left|k,A\right\rangle + J\psi_{k,A}(t)\left|k,B\right\rangle \\ &+ J\psi_{k,B}(t)\left|k,A\right\rangle + 2J\cos(ka-\frac{\phi(t)}{2})\psi_{k,B}(t)\left|k,B\right\rangle \\ &= i\hbar\frac{\partial}{\partial t}(\sum_k \psi_{k,A}(t)\left|k,A\right\rangle + \psi_{k,B}(t)\left|k,B\right\rangle) \end{aligned} \tag{4.12}$$

projection by $\langle k,A|$ gives

$$i\hbar\frac{\partial}{\partial t}\psi_{k,A}(t) = 2J\cos(ka+\frac{\phi(t)}{2})\psi_{k,A}(t) + J\psi_{k,B}(t) \tag{4.13}$$

and projection by $\langle k,B|$ gives

$$i\hbar\frac{\partial}{\partial t}\psi_{k,B}(t) = J\psi_{k,B}(t) + 2J\cos(ka-\frac{\phi(t)}{2})\psi_{k,B}(t) \tag{4.14}$$

Let $\tau = \frac{Jt}{\hbar}$, then we can write above equations as:

$$i\frac{\partial}{\partial\tau}\begin{pmatrix}\psi_{k,A}(\tau)\\ \psi_{k,B}(\tau)\end{pmatrix} = \begin{pmatrix}2\cos(ka+\frac{\phi(\tau)}{2}) & 1\\ 1 & 2\cos(ka-\frac{\phi(\tau)}{2})\end{pmatrix}\begin{pmatrix}\psi_{k,A}(\tau)\\ \psi_{k,B}(\tau)\end{pmatrix} \tag{4.15}$$

For each k point we obtain a Hamiltonian as

$$H_k(\tau) = \begin{pmatrix}2\cos(ka+\frac{\phi(\tau)}{2}) & 1\\ 1 & 2\cos(ka-\frac{\phi(\tau)}{2})\end{pmatrix} \tag{4.16}$$

4.1.1 Case I: Artificial Magnetic Flux Varying Linearly With Time

If artificial magnetic flux varies with time linearly, $\phi(\tau) = 2\Omega\tau$, then Eqn.(4.16) becomes

$$i\frac{\partial}{\partial\tau}\begin{pmatrix}\psi_{k,A}(\tau)\\ \psi_{k,B}(\tau)\end{pmatrix} = \begin{pmatrix}2\cos(ka+\Omega\tau) & 1\\ 1 & 2\cos(ka-\Omega\tau)\end{pmatrix}\begin{pmatrix}\psi_{k,A}(\tau)\\ \psi_{k,B}(\tau)\end{pmatrix} \tag{4.17}$$

Although our field is not time periodic, the Hamiltonian for each k-point becomes time periodic due to the lattice structure.

$$H_k(\tau+T) = H_k(\tau) = H_0+V_k(\tau) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \begin{pmatrix}2\cos(ka+\Omega\tau) & 0\\ 0 & 2\cos(ka-\Omega\tau)\end{pmatrix}$$

where

$$T = \frac{2\pi}{\Omega}$$

Therefore we can write our solutions as;

$$|\psi_{k,1}(\tau)\rangle = \begin{pmatrix}\psi_{k,A,1}(\tau)\\ \psi_{k,B,1}(\tau)\end{pmatrix} = e^{-i\epsilon_1(k)\tau}\begin{pmatrix}\phi_{k,A,1}(\tau)\\ \phi_{k,B,1}(\tau)\end{pmatrix} \tag{4.18}$$

and

$$|\psi_{k,2}(\tau)\rangle = \begin{pmatrix}\psi_{k,A,2}(\tau)\\ \psi_{k,B,2}(\tau)\end{pmatrix} = e^{-i\epsilon_2(k)\tau}\begin{pmatrix}\phi_{k,A,2}(\tau)\\ \phi_{k,B,2}(\tau)\end{pmatrix} \tag{4.19}$$

where

$$\begin{pmatrix}\phi_{k,A,r}(\tau+T)\\ \phi_{k,B,r}(\tau+T)\end{pmatrix} = \begin{pmatrix}\phi_{k,A,r}(\tau)\\ \phi_{k,B,r}(\tau)\end{pmatrix} \tag{4.20}$$

where $r = 1, 2$ and $\epsilon_1(k), \epsilon_1(k)$ lie in the first Floquet Brillouin zone. The general solution for the Hamiltonian at k-point can be written as;

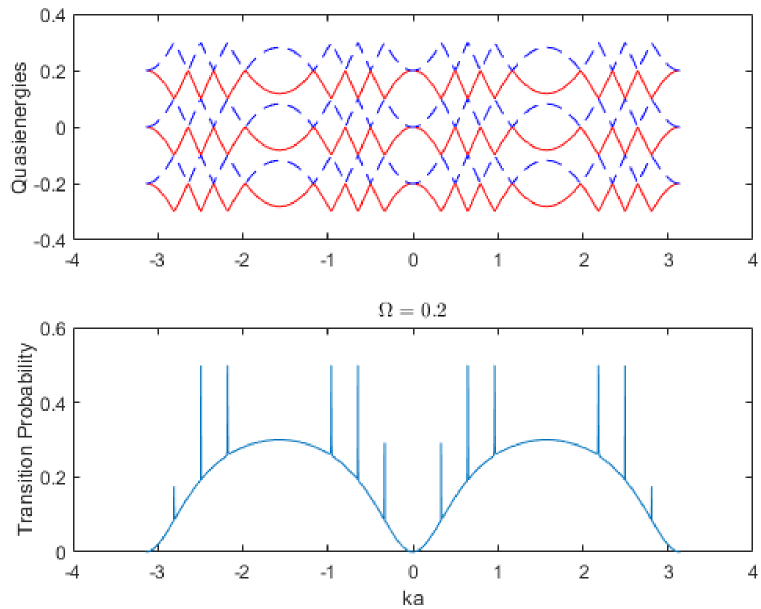
$$
\begin{aligned}
|\psi_k(\tau)\rangle = \begin{pmatrix} \psi_{k,A}(\tau) \\ \psi_{k,B}(\tau) \end{pmatrix} &= c_1(k)e^{-i\epsilon_1(k)\tau} \begin{pmatrix} \phi_{k,A,1}(\tau) \\ \phi_{k,B,1}(\tau) \end{pmatrix} \\
&+ c_2(k)e^{-i\epsilon_2(k)\tau} \begin{pmatrix} \phi_{k,A,2}(\tau) \\ \phi_{k,B,2}(\tau) \end{pmatrix} \\
&= \sum_m c_1(k)e^{-i\epsilon_1(k)\tau} \begin{pmatrix} \phi^m_{k,A,1} \\ \phi^m_{k,B,1} \end{pmatrix} e^{-im\Omega\tau} \\
&+ c_2(k)e^{-i\epsilon_2(k)\tau} \begin{pmatrix} \phi^m_{k,A,2} \\ \phi^m_{k,B,2} \end{pmatrix} e^{-im\Omega\tau} \qquad (4.21)
\end{aligned}
$$

Since the Floquet states can be decomposed in any basis $|\alpha\rangle$ of the Hilbert space H we are working in as

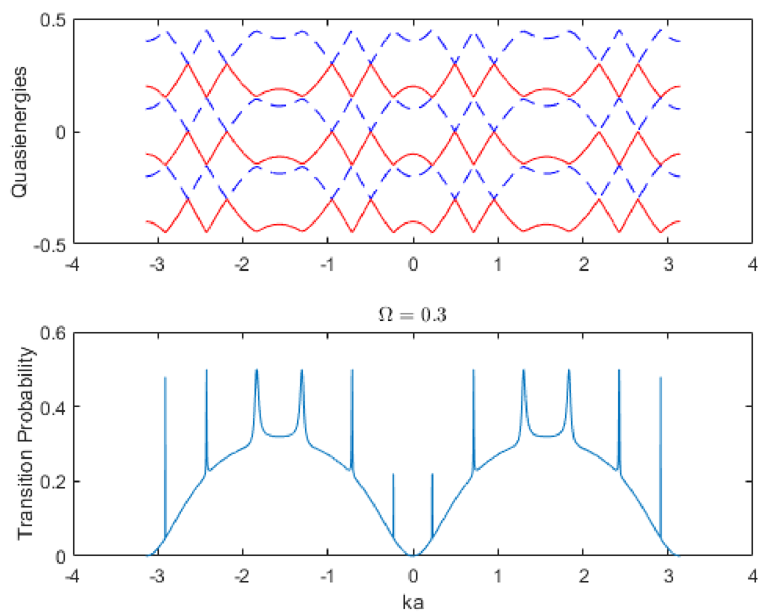
$$
|\psi_{k,r}(\tau)\rangle = e^{-i\epsilon_r(k)\tau} \sum_m \phi^m_{k,r}(\alpha)\, |\alpha\rangle\, e^{-im\tau} \qquad (4.22)
$$

where $\phi^m_{k,r}(\alpha) = \langle\alpha|\phi^m_{k,r}\rangle$, we can construct the infinite Floquet matrix by the aid of $|\alpha\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\beta\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\langle\alpha n|H_F|\beta m\rangle = H^{n-m}_{\alpha\beta} + n\Omega\delta_{nm}\delta_{\alpha\beta}$, as;

$$
H_F = \begin{array}{cc}
\begin{array}{cccccccccc}
|\alpha,-2\rangle & |\beta,-2\rangle & |\alpha,-1\rangle & |\beta,-1\rangle & |\alpha,0\rangle & |\beta,0\rangle & |\alpha,+1\rangle & |\beta,+1\rangle & |\alpha,+2\rangle & |\beta,+2\rangle \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow
\end{array} & \\
\left(\begin{array}{cc|cc|cc|cc|cc}
\ddots & & & & & & & & & \cdot^{\cdot^{\cdot}} \\
\hline
-2\Omega & 1 & e^{ika} & & & & & & & \\
1 & -2\Omega & 0 & e^{-ika} & & & & & & \\
\hline
e^{-ika} & 0 & -\Omega & 1 & e^{ika} & & & & & \\
 & e^{ika} & 1 & -\Omega & 0 & e^{-ika} & & & & \\
\hline
 & & e^{-ika} & 0 & 0 & 1 & e^{ika} & & & \\
 & & & e^{ika} & 1 & 0 & 0 & e^{-ika} & & \\
\hline
 & & & & e^{-ika} & 0 & \Omega & 1 & e^{ika} & \\
 & & & & & e^{ika} & 1 & \Omega & 0 & e^{-ika} \\
\hline
 & & & & & & e^{-ika} & 0 & 2\Omega & 1 \\
 & & & & & & & e^{ika} & 1 & 2\Omega \\
\hline
\cdot^{\cdot^{\cdot}} & & & & & & & & & \ddots
\end{array}\right) &
\begin{array}{c}
\\
\leftarrow |\alpha,-2\rangle \\
\leftarrow |\beta,-2\rangle \\
\leftarrow |\alpha,-1\rangle \\
\leftarrow |\beta,-1\rangle \\
\leftarrow |\alpha,0\rangle \\
\leftarrow |\beta,0\rangle \\
\leftarrow |\alpha,+1\rangle \\
\leftarrow |\beta,+1\rangle \\
\leftarrow |\alpha,+2\rangle \\
\leftarrow |\beta,+2\rangle \\
\\
\end{array}
\end{array}
\qquad (4.23)
$$

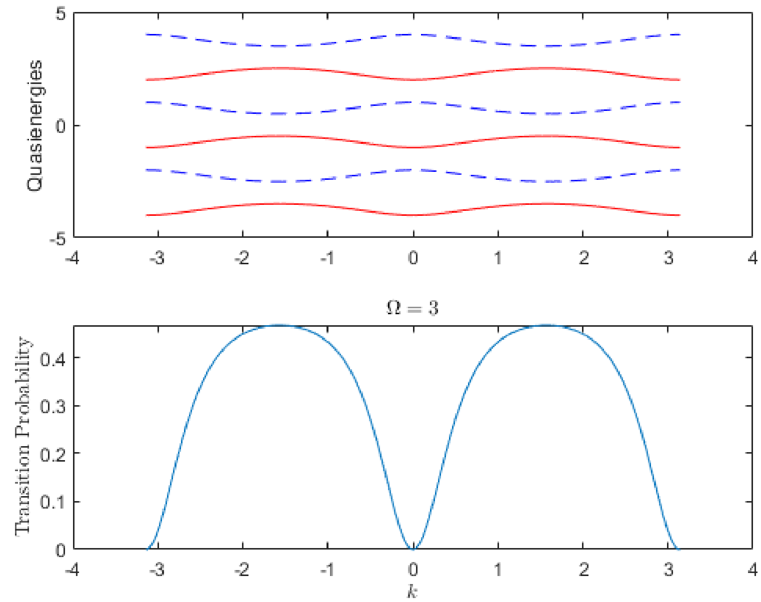


(a) $\Omega = 0.2$

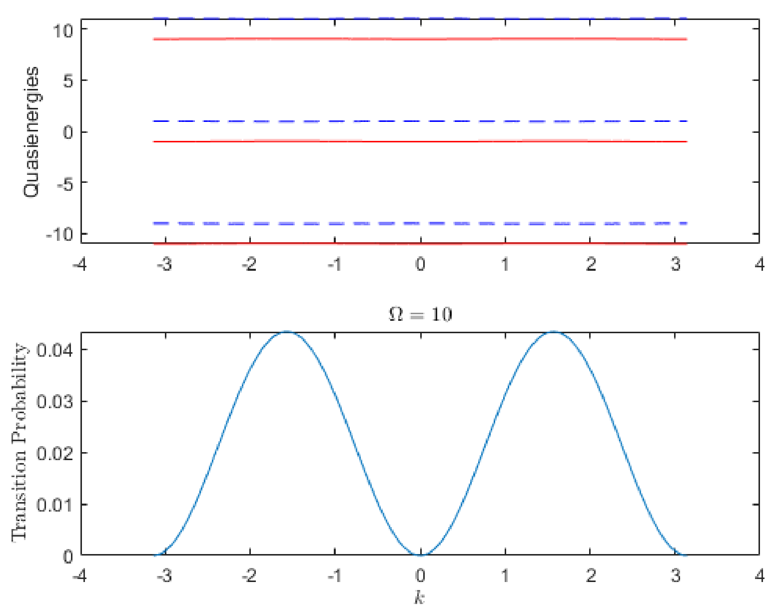


(b) $\Omega = 0.3$

Figure 4.2: Quasi-Energy band diagram and transition probability between Floquet states for values of $\Omega = 0.2, \quad 0.3$



(a) $\Omega = 3$



(b) $\Omega = 10$

Figure 4.3: Quasi-Energy band diagram and transition probability between Floquet states for for values of $\Omega = 3, \quad 10$

Figure(4.2) and Figure(4.3) show the quasi-energies and corresponding the long time averaged transition probabilities as a function of ka with $\Omega = 0.2$, $\Omega = 0.3$, $\Omega = 3$, $\Omega = 10$ .The blue and red lines in the middle represent the quasi-energies that lie in the first Floquet Brillouin zone which we define in Chapter 2. The two lines at the top and the two lines at the bottom represent the shifted quasi-energies $\epsilon_{1,2}(k) + \Omega$ and $\epsilon_{1,2}(k) - \Omega$, respectively. Note that the quasi-energy bands become flat as frequency increases, actually, in the high frequencies the system is governed by an effective time-independent Hamiltonian obtained by high frequency expansion[36]. In the low frequency vicinity as in Figure(2a)and Figure(2b) we observe the n-photon time averaged transition probability from $|\phi_-\rangle = \frac{1}{\sqrt{2}}\Big(1, -1\Big)^T$ to $|\phi_+\rangle = \frac{1}{\sqrt{2}}\Big(1, 1\Big)^T$. The red lines represent upper Floquet states($|\alpha n\rangle$) and the blues corresponding to the lower Floquet states ($|\beta, n\rangle$). In this figure we can easily observe the periodicity of the quasienergies with Ω and there are resonance transition between $|\alpha\rangle$ and $|\beta\rangle$ at the some of the avoided crossings between Floquet states. Resonance occurs when two quasinergies,each of them represents different physical state, in different Brillioun zone come close to each other. At the resonance points the Floquet states are strongly mixed and the condition $\epsilon_1 - \epsilon_2 = n\Omega$ with integer n is satisfied, thus, the particle absorbs and emits n photons. In other words, n-photon transitions occurs at the resonance points. To understand multi-photon resonance process analytically we can apply quasi-degenerate perturbation theory.

To apply quasi-degenerate perturbation theory, let first perform a unitary transformation;

$$\begin{aligned}\tilde{H}(\tau) &= \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}2\cos(ka-\Omega\tau) & 1\\ 1 & 2\cos(ka+\Omega\tau)\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\\ &= \begin{pmatrix}1+2\cos(ka)\cos(\Omega\tau) & 2\sin(ka)\sin(\Omega\tau)\\ 2\sin(ka)\sin(\Omega\tau) & -1+2\cos(ka)cos(\Omega\tau)\end{pmatrix}\end{aligned} \tag{4.24}$$

$$
\tilde{H}_F = \begin{array}{cc}
\begin{array}{cccccccccc}
|\alpha,-2\rangle & |\beta,-2\rangle & |\alpha,-1\rangle & |\beta,-1\rangle & |\alpha,0\rangle & |\beta,0\rangle & |\alpha,+1\rangle & |\beta,+1\rangle & |\alpha,+2\rangle & |\beta,+2\rangle \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow
\end{array} & \\
\begin{pmatrix}
\ddots & & & & & & & & & \cdot^{\cdot^{\cdot}} \\
1-2\Omega & 0 & c & is & & & & & & \\
0 & -1-2\Omega & is & c & & & & & & \\
c & -is & 1-\Omega & 0 & c & is & & & & \\
-is & c & 0 & -1-\Omega & is & c & & & & \\
 & & c & -is & 1 & 0 & c & is & & \\
 & & -is & c & 0 & -1 & is & c & & \\
 & & & & c & -is & 1+\Omega & 0 & c & is \\
 & & & & -is & c & 0 & -1+\Omega & is & c \\
 & & & & & & c & -is & 1+2\Omega & 0 \\
 & & & & & & -is & c & 0 & -1+2\Omega \\
\cdot^{\cdot^{\cdot}} & & & & & & & & & \ddots
\end{pmatrix} &
\begin{array}{l}
\\
\leftarrow |\alpha,-2\rangle \\
\leftarrow |\beta,-2\rangle \\
\leftarrow |\alpha,-1\rangle \\
\leftarrow |\beta,-1\rangle \\
\leftarrow |\alpha,0\rangle \\
\leftarrow |\beta,0\rangle \\
\leftarrow |\alpha,+1\rangle \\
\leftarrow |\beta,+1\rangle \\
\leftarrow |\alpha,+2\rangle \\
\leftarrow |\beta,+2\rangle \\
\\
\end{array}
\end{array}
\tag{4.25}
$$

where $c = \cos ka$ and $is = i \sin ka$. Now if we consider this transformed Hamiltonian as sum of unperturbed Hamiltion, $\tilde{H}^0_F$, and perturbation,V,

$$
\tilde{H}^0_F = \begin{array}{cc}
\begin{array}{cccccccccc}
|\alpha,-2\rangle & |\beta,-2\rangle & |\alpha,-1\rangle & |\beta,-1\rangle & |\alpha,0\rangle & |\beta,0\rangle & |\alpha,+1\rangle & |\beta,+1\rangle & |\alpha,+2\rangle & |\beta,+2\rangle \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow
\end{array} & \\
\begin{pmatrix}
\ddots & & & & & & & & & \cdot^{\cdot^{\cdot}} \\
1-2\Omega & 0 & c & 0 & & & & & & \\
0 & -1-2\Omega & 0 & c & & & & & & \\
c & 0 & 1-\Omega & 0 & c & 0 & & & & \\
0 & c & 0 & -1-\Omega & 0 & c & & & & \\
 & & c & 0 & 1 & 0 & c & 0 & & \\
 & & 0 & c & 0 & -1 & 0 & c & & \\
 & & & & c & 0 & 1+\Omega & 0 & c & 0 \\
 & & & & 0 & c & 0 & -1+\Omega & 0 & c \\
 & & & & & & c & 0 & 1+2\Omega & 0 \\
 & & & & & & 0 & c & 0 & -1+2\Omega \\
\cdot^{\cdot^{\cdot}} & & & & & & & & & \ddots
\end{pmatrix} &
\begin{array}{l}
\\
\leftarrow |\alpha,-2\rangle \\
\leftarrow |\beta,-2\rangle \\
\leftarrow |\alpha,-1\rangle \\
\leftarrow |\beta,-1\rangle \\
\leftarrow |\alpha,0\rangle \\
\leftarrow |\beta,0\rangle \\
\leftarrow |\alpha,+1\rangle \\
\leftarrow |\beta,+1\rangle \\
\leftarrow |\alpha,+2\rangle \\
\leftarrow |\beta,+2\rangle \\
\\
\end{array}
\end{array}
\tag{4.26}
$$

$$
V = \begin{array}{c}
\begin{array}{cccccccccc}
|\alpha,-2\rangle & |\beta,-2\rangle & |\alpha,-1\rangle & |\beta,-1\rangle & |\alpha,0\rangle & |\beta,0\rangle & |\alpha,+1\rangle & |\beta,+1\rangle & |\alpha,+2\rangle & |\beta,+2\rangle \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow
\end{array} \\
\left(\begin{array}{cc|cc|cc|cc|cc}
\ddots & & & & & & & & & \\
\hline
0 & 0 & 0 & is & & & & & & \\
0 & 0 & is & 0 & & & & & & \\
\hline
0 & -is & 0 & 0 & 0 & is & & & & \\
-is & 0 & 0 & 0 & is & 0 & & & & \\
\hline
 & & 0 & -is & 0 & 0 & 0 & is & & \\
 & & -is & 0 & 0 & 0 & is & 0 & & \\
\hline
 & & & & 0 & -is & 0 & 0 & 0 & is \\
 & & & & -is & 0 & 0 & 0 & is & 0 \\
\hline
 & & & & & & 0 & -is & 0 & 0 \\
 & & & & & & -is & 0 & 0 & 0 \\
\hline
 & & & & & & & & & \ddots
\end{array}\right)
\end{array}
\begin{array}{l}
\\ \\ \\
\leftarrow |\alpha,-2\rangle \\
\leftarrow |\beta,-2\rangle \\
\leftarrow |\alpha,-1\rangle \\
\leftarrow |\beta,-1\rangle \\
\leftarrow |\alpha,0\rangle \\
\leftarrow |\beta,0\rangle \\
\leftarrow |\alpha,+1\rangle \\
\leftarrow |\beta,+1\rangle \\
\leftarrow |\alpha,+2\rangle \\
\leftarrow |\beta,+2\rangle \\
\\
\end{array}
\tag{4.27}
$$

then since the unperturbed eigensolutions are well known we can find approximate solutions of the full Hamiltonian.

If we write our Hamiltonian in terms of new bases $|\tilde{\alpha}n\rangle = \sum_{k=-\infty}^{\infty} J_{k-n}(-2\cos ka/\Omega)\,|\alpha k\rangle$ and $|\tilde{\beta}n\rangle = \sum_{k=-\infty}^{\infty} J_{k-n}(-2\cos ka/\Omega)\,|\beta k\rangle$, where J_n is the Bessel function (We can obtain these new bases by performing unitary transformation $\psi(\tau) = e^{-i2\cos(ka)\int_0^\tau \cos(\Omega\tau')d\tau'}\phi(\tau))$, then we get

$$
\tilde{H}_F = \begin{array}{c}
\begin{array}{cccccccccc}
|\tilde{\alpha},-2\rangle & |\tilde{\beta},-2\rangle & |\tilde{\alpha},-1\rangle & |\tilde{\beta},-1\rangle & |\tilde{\alpha},0\rangle & |\tilde{\beta},0\rangle & |\tilde{\alpha},+1\rangle & |\tilde{\beta},+1\rangle & |\tilde{\alpha},+2\rangle & |\tilde{\beta},+2\rangle \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow
\end{array} \\
\left(\begin{array}{cc|cc|cc|cc|cc}
\ddots & & & & & & & & & \\
\hline
1-2\Omega & 0 & 0 & is & & & & & & \\
0 & -1-2\Omega & is & 0 & & & & & & \\
\hline
0 & -is & 1-\Omega & 0 & 0 & is & & & & \\
-is & 0 & 0 & -1-\Omega & is & 0 & & & & \\
\hline
 & & 0 & -is & 1 & 0 & 0 & is & & \\
 & & -is & 0 & 0 & -1 & is & 0 & & \\
\hline
 & & & & 0 & -is & 1+\Omega & 0 & 0 & is \\
 & & & & -is & 0 & 0 & -1+\Omega & is & 0 \\
\hline
 & & & & & & 0 & -is & 1+2\Omega & 0 \\
 & & & & & & -is & 0 & 0 & -1+2\Omega \\
\hline
 & & & & & & & & & \ddots
\end{array}\right)
\end{array}
\begin{array}{l}
\\ \\ \\
\leftarrow |\tilde{\alpha},-2\rangle \\
\leftarrow |\tilde{\beta},-2\rangle \\
\leftarrow |\tilde{\alpha},-1\rangle \\
\leftarrow |\tilde{\beta},-1\rangle \\
\leftarrow |\tilde{\alpha},0\rangle \\
\leftarrow |\tilde{\beta},0\rangle \\
\leftarrow |\tilde{\alpha},+1\rangle \\
\leftarrow |\tilde{\beta},+1\rangle \\
\leftarrow |\tilde{\alpha},+2\rangle \\
\leftarrow |\tilde{\beta},+2\rangle \\
\\
\end{array}
\tag{4.28}
$$

We see that $|\tilde{\alpha}, n\rangle$ is coupled to $|\tilde{\beta}, m\rangle$ when $n = m - 1$ or $n = m + 1$ from the matrix structure of Eq.(4.28).

As an example, let assume that the states $\phi_\alpha^{(0)} = |\tilde{\alpha}, 0\rangle$ and $\phi_\beta^{(0)} = |\tilde{\beta}, +3\rangle$ are nearly degenerate($2 \approx 3\Omega$). Then Eq.(4.28) is reduced to 2×2 matrix.

$$H = \begin{pmatrix} 1+\mathcal{U}_{\alpha\alpha} & \mathcal{U}_{\alpha\beta} \\ \mathcal{U}^*_{\beta\alpha} & -1+n\Omega+\mathcal{U}_{\beta\beta} \end{pmatrix} \tag{4.29}$$

where the elements of matrix $\mathcal{U}$ are obtained by following equation

$$\begin{aligned} \mathcal{U} = (\mathcal{U}_{\alpha\beta}) &= \langle\phi^{(0)}|V|\phi^{(0)}\rangle + \langle\phi^{(0)}|V|\phi^{(1)}\rangle - \langle\phi^{(0)}|V|\phi^{(0)}\rangle \langle\phi^{(0)}|\phi^{(1)}\rangle \\ &+ \langle\phi^{(1)}|V|\phi^{(1)}\rangle - \langle\phi^{(1)}|\phi^{(1)}\rangle \langle\phi^{(0)}|V|\phi^{(0)}\rangle \end{aligned} \tag{4.30}$$

with $\phi = (\phi_\alpha, \phi_\beta)$ and $\phi_\alpha^{(1)} = \frac{A}{2-\Omega}|\beta, 1\rangle - \frac{A}{2+\Omega}|\beta, -1\rangle$, $\phi_\beta^{(1)} = \frac{A}{2-\Omega}|\alpha, 2\rangle - \frac{A}{2+\Omega}|\alpha, 4\rangle$, where $A = i\sin(ka)$.

$$\begin{aligned} \mathcal{U}_{\alpha\alpha} &= -\mathcal{U}_{\beta\beta} = -A^2\frac{4}{4-\Omega^2} \\ \mathcal{U}_{\alpha\beta} &= \mathcal{U}^*_{\beta\alpha} = \frac{A^3}{(2-\Omega)^3} \end{aligned} \tag{4.31}$$

Let $\Omega_{res.} = 2 + \mathcal{U}_{\alpha\alpha} - \mathcal{U}_{\beta\beta}$, $d = \Omega_{res.} - n\Omega$, and $E_0 = \frac{2+\mathcal{U}_{\alpha\alpha}+\mathcal{U}_{\beta\beta}+n\Omega}{2}$. Then we can write H as;

$$H = E_0\mathbb{1} + \frac{d}{2}\sigma_z + Re\{\mathcal{U}_{\alpha\beta}\}\sigma_x - Im\{\mathcal{U}_{\alpha\beta}\}\sigma_y \tag{4.32}$$

where σ are the Pauli matrices. Let $E = \frac{d}{2}\sigma_z + Re\{\mathcal{U}_{\alpha\beta}\}\sigma_x - Im\{\mathcal{U}_{\alpha\beta}\}\sigma_y$. The Pauli matrices anticommute so

$$E^2 = \frac{d^2}{4} + |\mathcal{U}_{\alpha\beta}|^2 = r^2 \tag{4.33}$$

$$\begin{aligned} e^{-iHt} &= e^{-iE_0t}e^{-iEt} = e^{-iE_0t}(\mathbb{1} - iEt - \frac{E^2t^2}{2!} + ...) \\ &= e^{-iE_0t}(\cos(rt)\mathbb{1} - i\frac{E}{r}\sin(rt)) \end{aligned} \tag{4.34}$$

If the initial state is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then the transition probability is

$$P(t) = |\begin{pmatrix} 0 & 1 \end{pmatrix} e^{-iHt} \begin{pmatrix} 1 \\ 0 \end{pmatrix}|^2 = \frac{|\mathcal{V}_{pq}|^2}{r^2}\sin^2(rt) \tag{4.35}$$

and the time averaged transition probability becomes

$$\hat{P} = \frac{1}{2}\frac{|\mathcal{V}_{pq}|^2}{r^2} = \frac{2|\mathcal{V}_{pq}|^2}{(\omega_{res} - n\Omega)^2 + 4|\mathcal{V}_{pq}|^2} \tag{4.36}$$

whenever $\Omega_{res} = n\Omega$ resonance transition occurs.

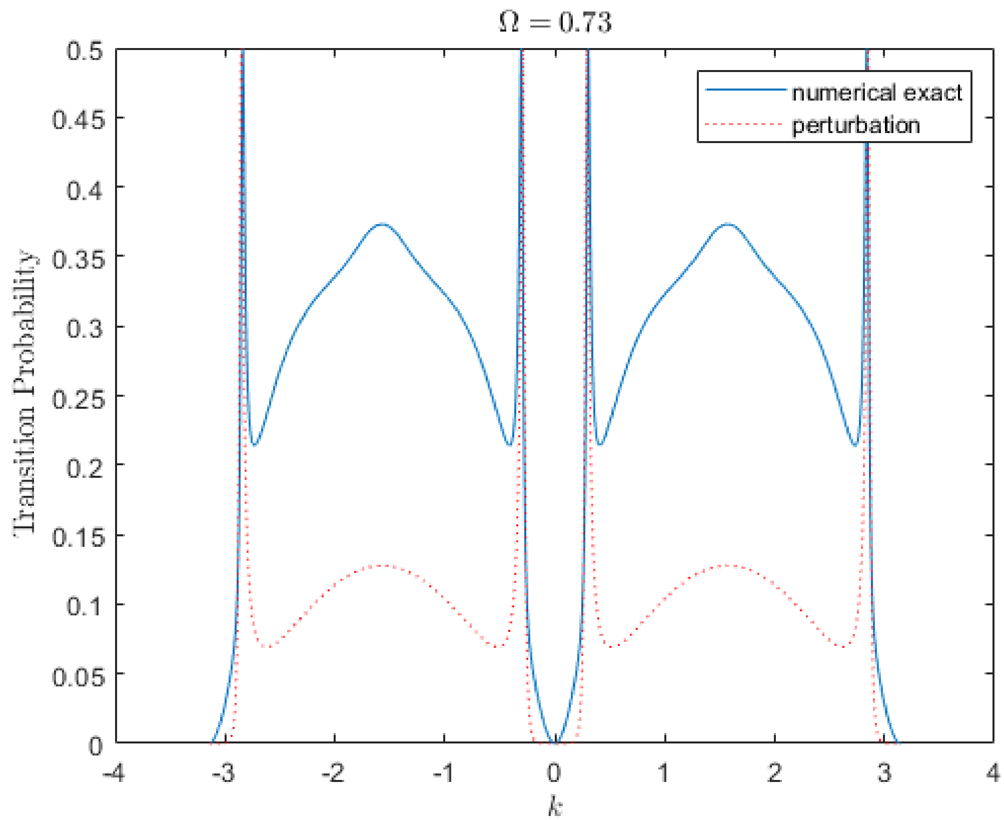
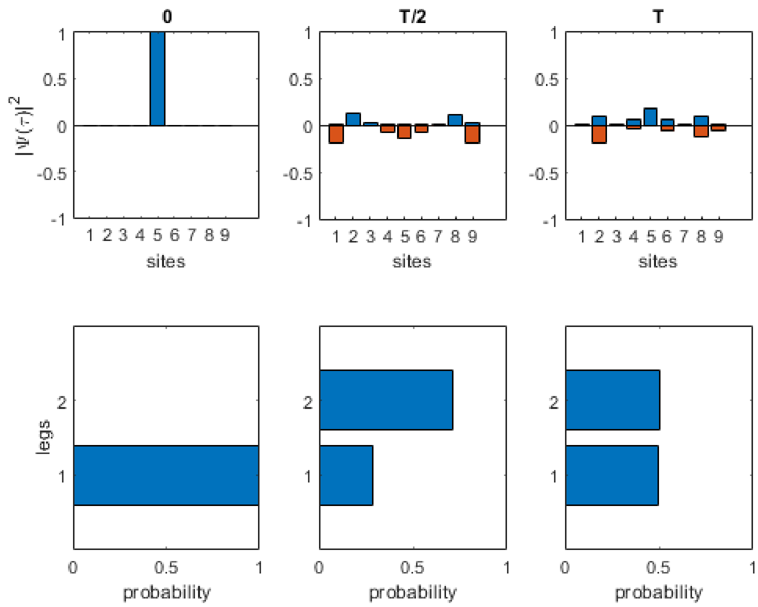
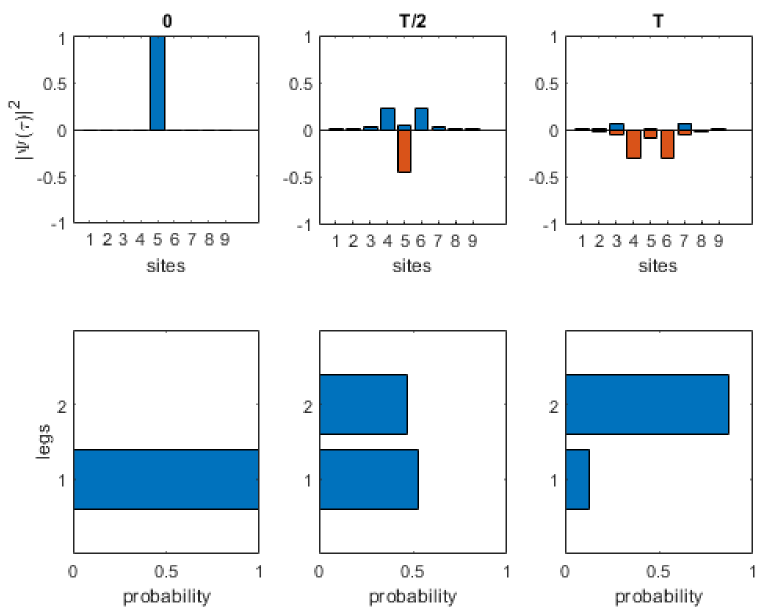


Figure 4.4: Transition probability between Floquet states for $\Omega = 0.73$

Note that the components of the eigenvectors of the infinite Floquet matrix in Eqn.(4.23) are the Fourier coefficients of the Floquet states, we pick these values to construct the general solution for the Hamiltonian at k-point in Eqn.(4.21), and by the Eqn.(4.11) we can obtain the wave function of the particle in real space and can observe how it behaves over time.

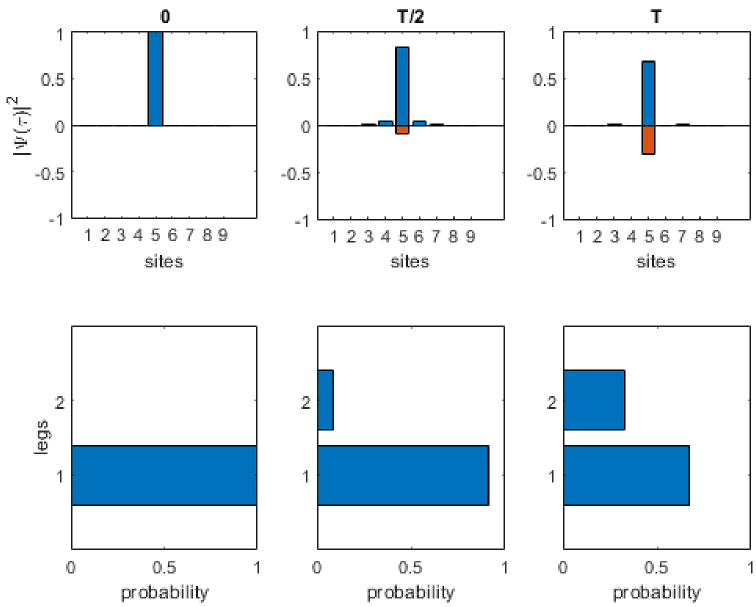


(a) $\Omega = 0.3$

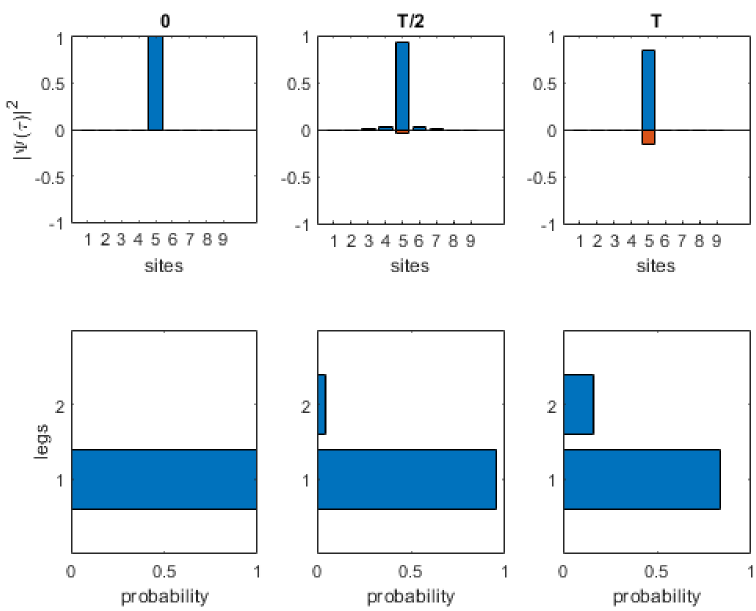


(b) $\Omega = 3$

Figure 4.5: Evolution of the particle for values of $\Omega = 0.3, \quad 3$



(a) $\Omega = 10$



(b) $\Omega = 15$

Figure 4.6: Evolution of the particle for values of $\Omega = 10, \quad 15$

In this figures leg-1(leg-2) refers to leg-a(leg-b) and these figures show how particle spreads out for different values of frequency. Fig.(4.5a), Fig.(4.5b), Fig.(4.6a), and Fig.(4.6b) are obtained with the frequencies 0.3, 3, 10 and 15, respectively. To observe the spreading of the particle very well, we set our two-leg ladder such that each leg has 9 sites. In real space we observe that center of mass doesn't change with time, in other words, the particle spreads out in left direction and right direction equally. In addition, we see that particle with low frequencies prefers to spread to other sites. However when we have sufficiently high frequency, the particle remains its initial site. This means dynamic localization.

4.1.2 Case II: Artificial Magnetic Flux Oscillating In Time

In this case we consider an oscillating artificial magnetic flux, $\phi(\tau) = 2\alpha\cos(\Omega\tau)$. For each k point the Hamiltonian is

$$H_k(\tau) = \begin{pmatrix} 2\cos(ka+\alpha\cos(\Omega\tau)) & 1 \\ 1 & 2\cos(ka-\alpha\cos(\Omega\tau)) \end{pmatrix} \tag{4.37}$$

If we perform a unitary transformation

$$|\psi_k(\tau)\rangle = e^{-i2\cos(ka)\int_0^\tau \cos(\alpha\cos(\Omega\tau'))d\tau'}\,|\tilde{\psi}_k(\tau)\rangle \tag{4.38}$$

then the Schrödinger equation reads

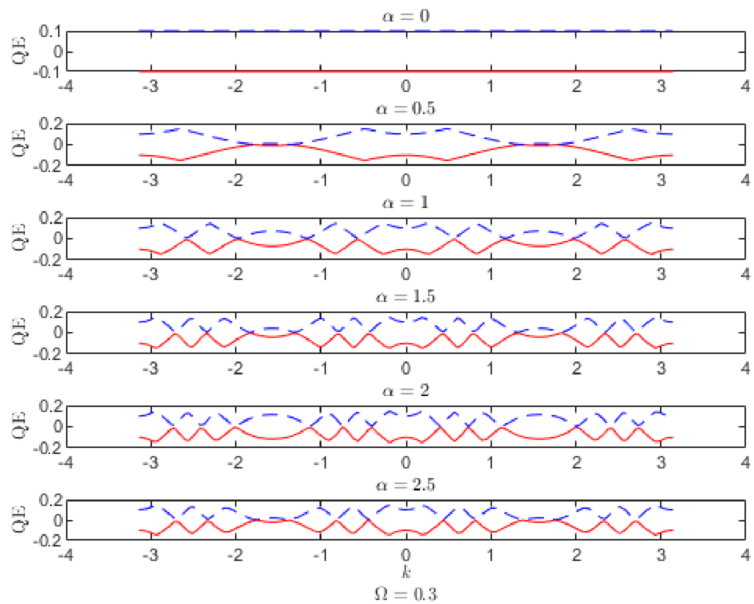
$$i\frac{\partial}{\partial\tau}\begin{pmatrix} \tilde{\psi}_{k,A}(\tau) \\ \tilde{\psi}_{k,B}(\tau) \end{pmatrix} = \begin{pmatrix} -2\sin(ka)\sin(\alpha\cos(\Omega\tau)) & 1 \\ 1 & 2\sin(ka)\sin(\alpha\cos(\Omega\tau)) \end{pmatrix}\begin{pmatrix} \tilde{\psi}_{k,A}(\tau) \\ \tilde{\psi}_{k,B}(\tau) \end{pmatrix} \tag{4.39}$$

we can write Hamiltonian as

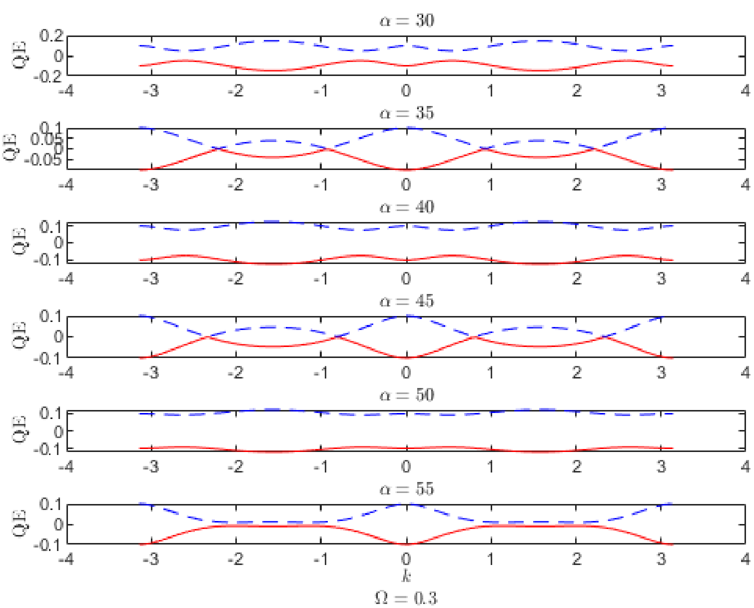
$$\begin{aligned} \tilde{H}(\tau) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + e^{i\alpha\cos(\Omega\tau)}\begin{pmatrix} i\sin(ka) & 0 \\ 0 & -i\sin(ka) \end{pmatrix} + e^{-i\alpha\cos(\Omega\tau)}\begin{pmatrix} -i\sin(ka) & 0 \\ 0 & i\sin(ka) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sum_s J_s(\alpha)e^{is\Omega\tau}\begin{pmatrix} (i^{s+1}+(-i)^{s+1})\sin(ka) & 0 \\ 0 & (i^{s-1}+(-i)^{s-1})\sin(ka) \end{pmatrix} \end{aligned} \tag{4.40}$$

$$
\begin{array}{cc}
\begin{array}{cccccccccc}
|\alpha,-2\rangle & |\beta,-2\rangle & |\alpha,-1\rangle & |\beta,-1\rangle & |\alpha,0\rangle & |\beta,0\rangle & |\alpha,+1\rangle & |\beta,+1\rangle & |\alpha,+2\rangle & |\beta,+2\rangle \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow
\end{array} & \\
\left(\begin{array}{c|cc|cc|cc|cc|cc|c}
\ddots & & & & & & & & & & & \iddots \\
\hline
 & -2\Omega & 1 & -2J_1 & 0 & 0 & 0 & 2J_3 & & & & \\
 & 1 & -2\Omega & 0 & 2J_1 & & & & -2J_3 & & & \\
\hline
 & 2J_{-1} & 0 & -\Omega & 1 & -2J_1 & & & & 2J_3 & & \\
 & & -2J_{-1} & 1 & -\Omega & 0 & 2J_1 & & & & -2J_3 & \\
\hline
 & & & 2J_{-1} & 0 & 0 & 1 & -2J_1 & & & & \\
 & & & & -2J_{-1} & 1 & 0 & 0 & 2J_1 & & & \\
\hline
 & -2J_{-3} & & & & 2J_{-1} & 0 & \Omega & 1 & -2J_1 & & \\
 & & 2J_{-3} & & & & -2J_{-1} & 1 & \Omega & 0 & 2J_1 & \\
\hline
 & & & -2J_{-3} & & & & 2J_{-1} & 0 & 2\Omega & 1 & \\
 & & & & 2J_{-3} & & & & -2J_{-1} & 1 & 2\Omega & \\
\hline
\iddots & & & & & & & & & & & \ddots
\end{array}\right) &
\begin{array}{c}
\\ \\
\leftarrow |\alpha,-2\rangle \\
\leftarrow |\beta,-2\rangle \\
\leftarrow |\alpha,-1\rangle \\
\leftarrow |\beta,-1\rangle \\
\leftarrow |\alpha,0\rangle \\
\leftarrow |\beta,0\rangle \\
\leftarrow |\alpha,+1\rangle \\
\leftarrow |\beta,+1\rangle \\
\leftarrow |\alpha,+2\rangle \\
\leftarrow |\beta,+2\rangle \\
\\
\end{array}
\end{array}
\tag{4.41}
$$

where $J_n = J_n(\alpha)\sin(ka)$. If we follow same procedure as in case-I, we obtain following figures,

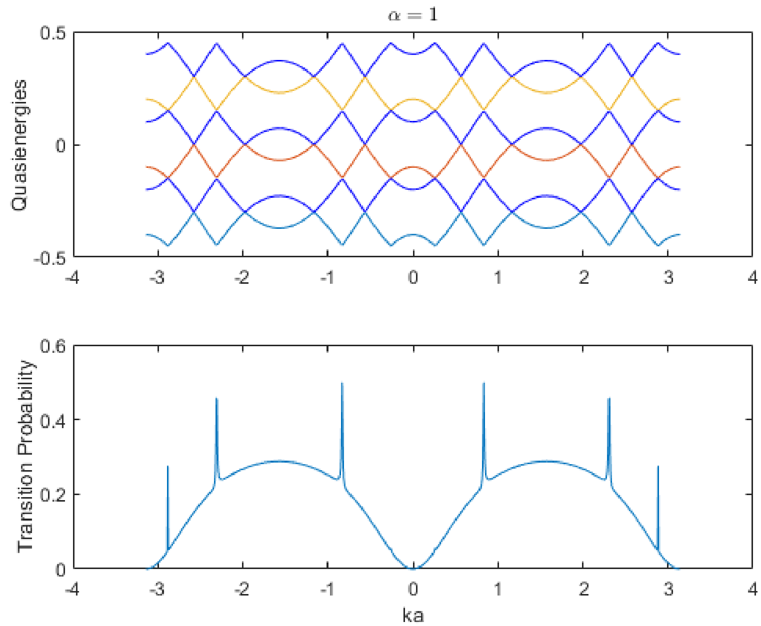


(a)

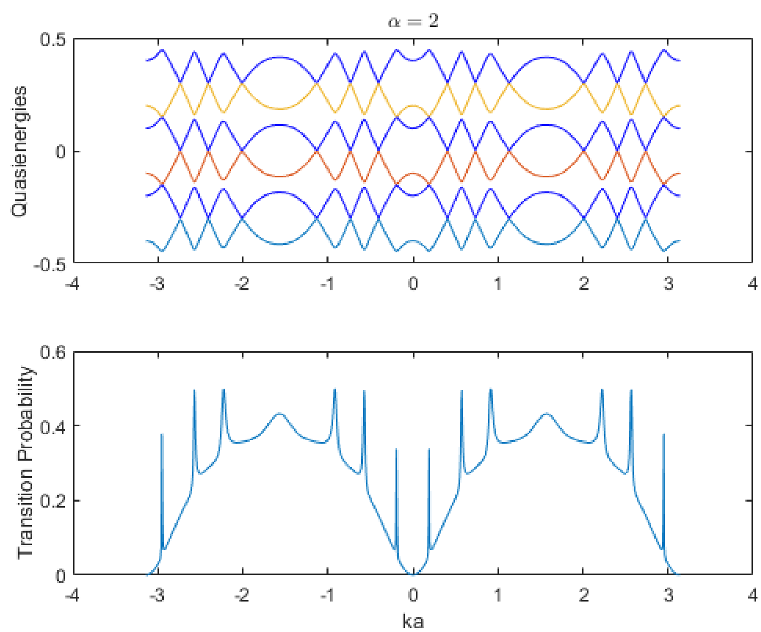


(b)

Figure 4.7: Effect of the α on the Quasi-Energy bands

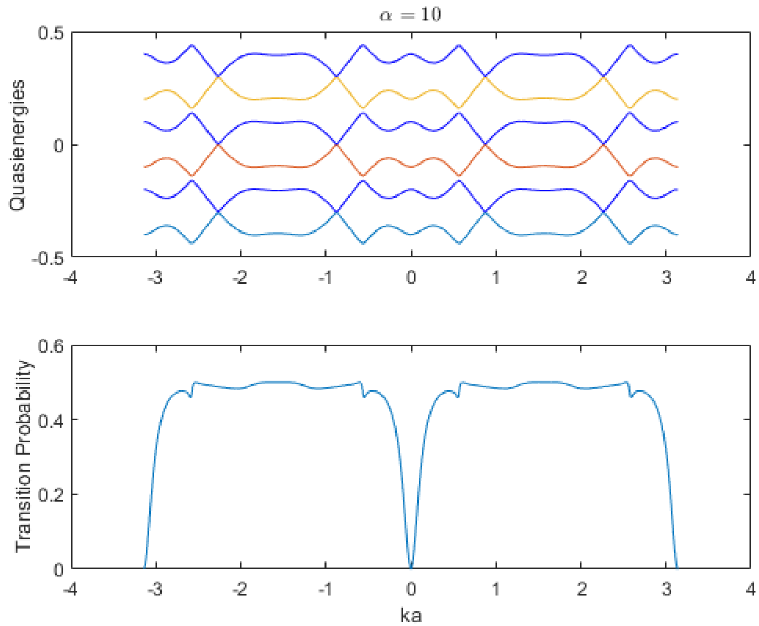


(a)

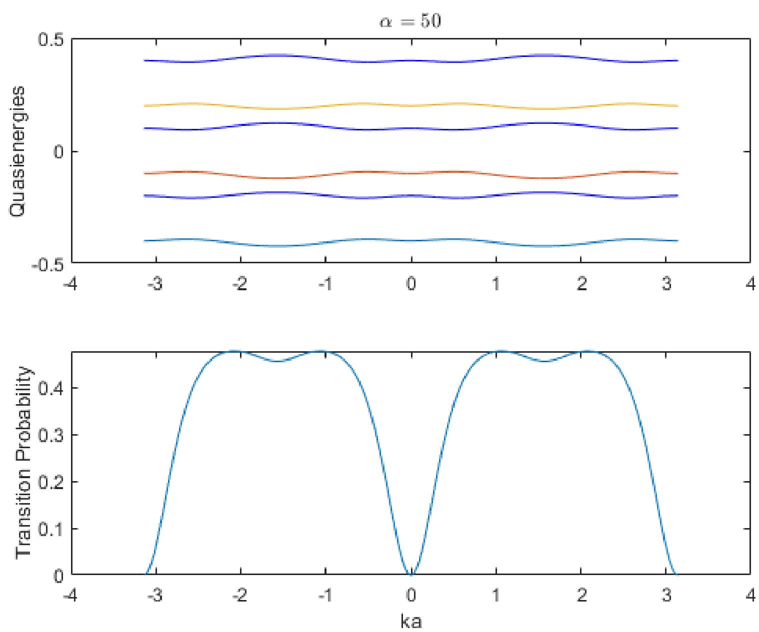


(b)

Figure 4.8: Transition probability between Floquet states for values of $\alpha = 1, 2$ with frequency $\Omega = 0.3$

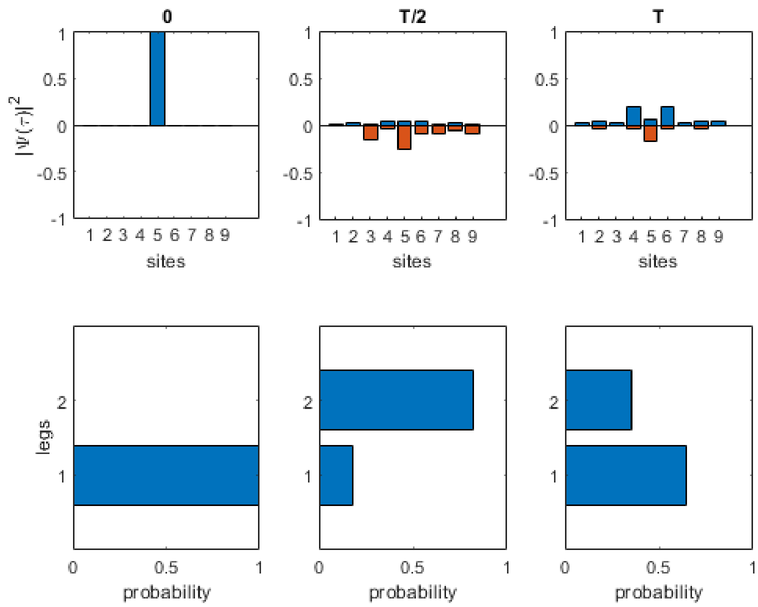


(a)

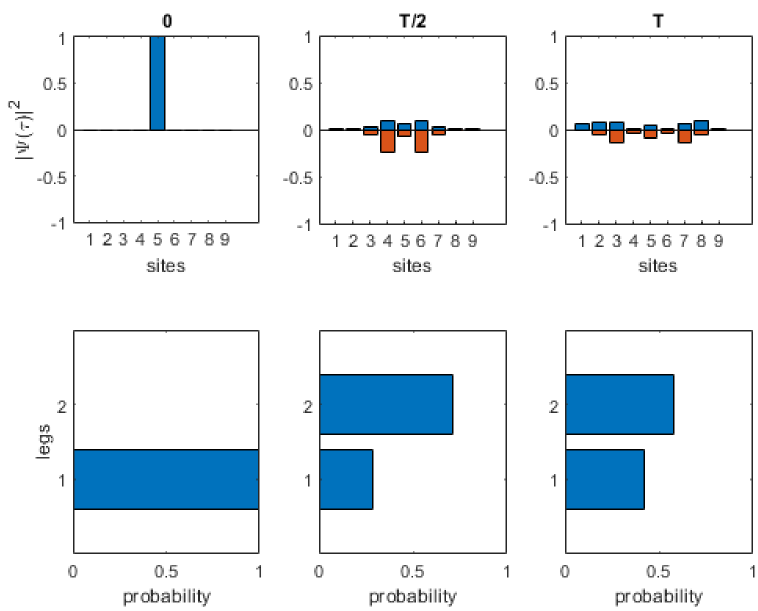


(b)

Figure 4.9: Transition probability between Floquet states for values of $\alpha = 10, 50$ with frequency $\Omega = 0.3$

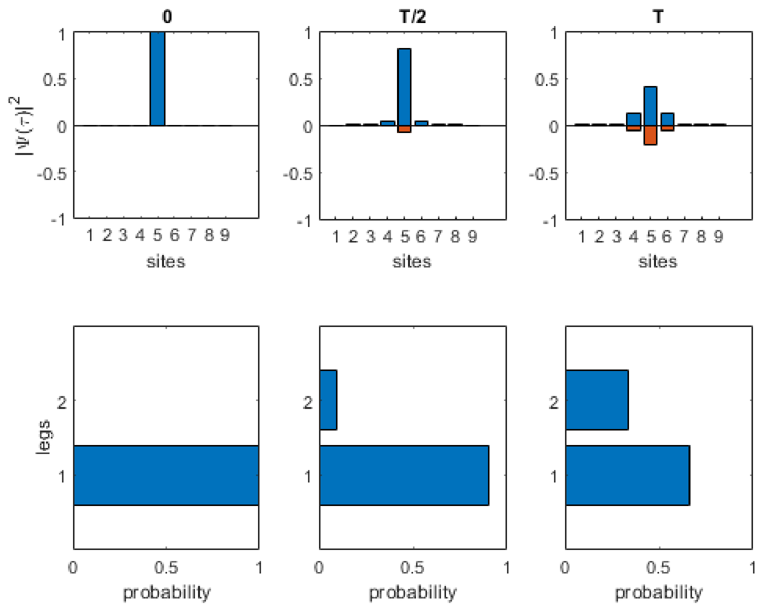


(a) $\Omega = 0.3$

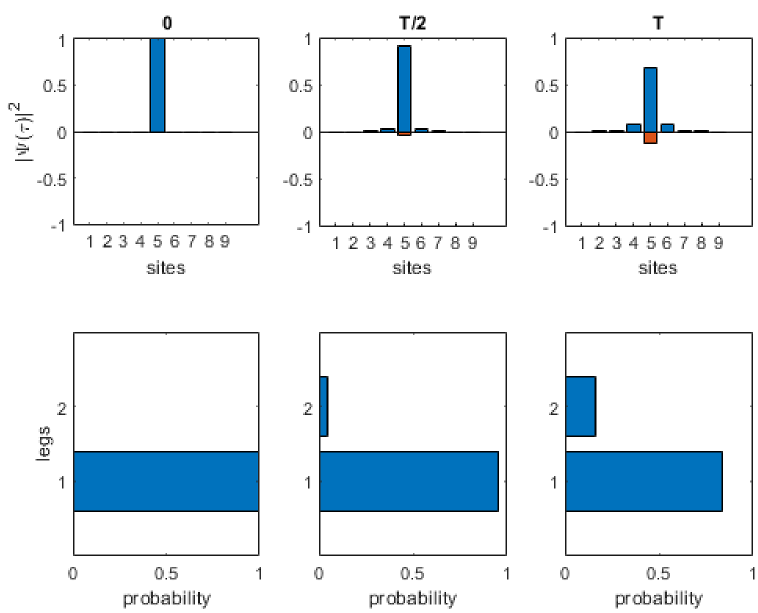


(b) $\Omega = 3$

Figure 4.10: Evolution of the particle for the values of $\Omega = 0.3,\ 3$ when $\alpha = 1$



(a) $\Omega = 10$



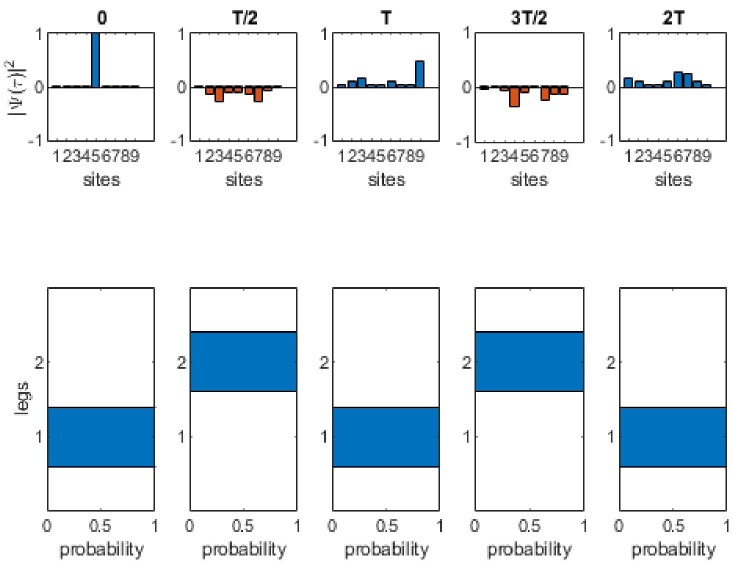
(b) $\Omega = 15$

Figure 4.11: Evolution of the particle for the values of $\Omega = 10,\ 15$ when $\alpha = 1$

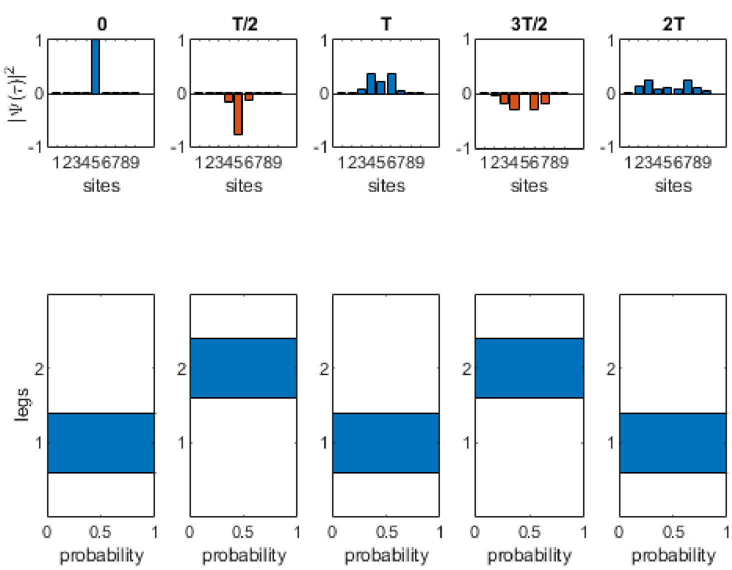
Fig.(4.7) shows how the field amplitude affects the quasi-energy bands with frequency $\Omega = 0.3$. It changes the zig-zag shape of the bands and affects the number of resonance points in the range of the k as shown in Fig.(4.8) and Fig.(4.9). Actually, there is no linear relationship between α and the number of resonance points. However, when α takes high values, the width of resonance curves expands and energy bands approach each other.

The spreading of the particle in the real space is shown for the values of frequencies $\Omega = 0.3$ and $\Omega = 3$ in Fig.(4.10) and for the values of frequencies $\Omega = 10$ and $\Omega = 15$ in Fig.(4.11) with the value of the field amplitude $\alpha = 1$. we see again that the particle spreads less at high frequencies.

In this case, we have a chance to control the system. By finding suitable values for the field amplitude and the driving frequency, we can send the particle to a site with a desired probability. Let examine the following figure as an example.

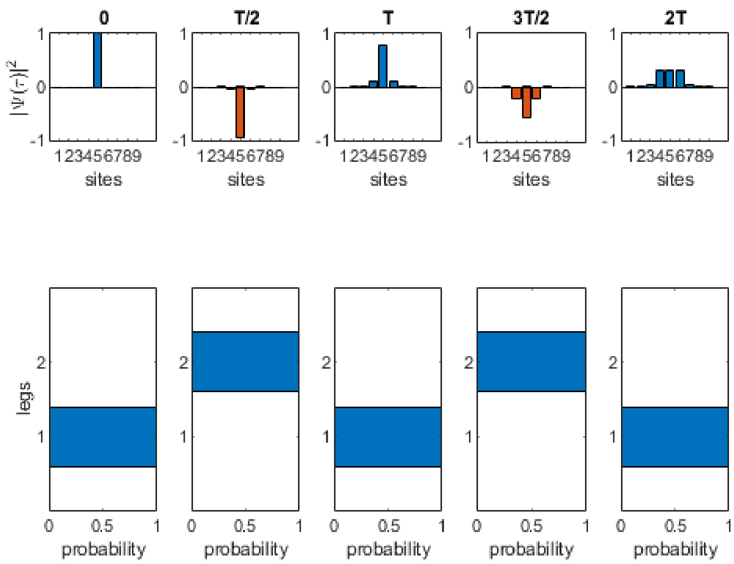


(a) $\alpha = 0.5$ for $\Omega = 2$

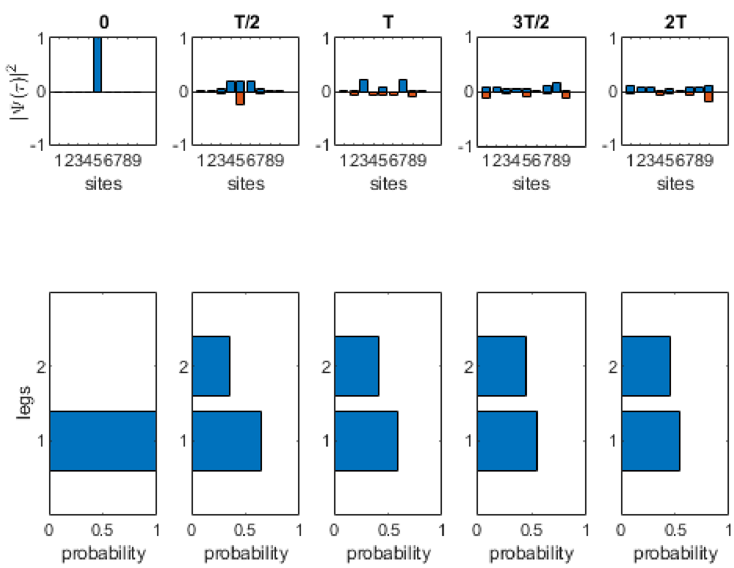


(b) $\alpha = 10$ for $\Omega = 2$

Figure 4.12: Evolution of the particle for different values of α when $\Omega = 2$



(a) $\alpha = 50$ for $\Omega = 2$



(b) case-I for $\Omega = 2$

Figure 4.13: Evolution of the particle for different values of α when $\Omega = 2$

Fig.(4.12) and Fig.(4.13a) shows the spreading of the particles for different values of the field amplitude with the value of the frequency $\Omega = 2$. Fig.(4.13b) shows the spreading of the particle in the case-I for the value of the frequency $\Omega = 2$ We see that the particle is completely found in just one leg and spread out on it, and every half period later it prefers the other leg with a different configuration when $\Omega = 2$ in the case-II, and the population of the sites is changing according to the value of the field amplitude. Therefore, we can determine the value of the field's amplitude to obtain a desired configuration. However, we have no such a opportunity in the case-I as shown in Fig.(4.13b).

Chapter 5

Conclusion

In this thesis, we developed the Floquet formalism and examined some properties of the Floquet states and their corresponding quasienergies. We showed that Floquet sytems can be described by an infinite dimensional time-independent Hamiltonian. In Chapter 3, we developed time-independent perturbation theory in the extended Hilbert space and the Floquet adiabatic model.

In Chapter 4, we studied the dynamics of a particle subject to time dependent artificial magnetic field in two cases by using tight-binding model. In the first case, the time-dependent artificial magnetic field varied linearly with time, and in the second case the field oscillated in time. We solved the problem numerically by using the time-independent Floquet matrix.

In both cases, we obtained quasi-energy bands for different values of the driving frequency. When the driving frequency is less than or equal to the difference between the eigenvalues of the time-independent part of the time-periodic Hamiltonian, we realized resonance transition at some k-point between the Floquet states. We applied the almost degenerate perturbation theory to understand multi-photon process better at these resonance point. In the case-I, the smaller value of the driving frequency is, the more resonance points occurs. The differences between quasi-energies are an integer multiple of the driving frequency at these points. As the driving frequency increased, we observed that the quasi-energy bands became more flat. In the real space, we observed that as the driving frequency increased, the spreading of the particle slowed down and the particle remained at its initial position with sufficiently high frequency.

In the case-II, we first observed the effect of the field amplitude on the quasi-energy bands. When we change the value of the field amplitude, the shape of quasi-energy bands changes, and this also change the position of the resonance transition and the number of the resonance points. In this case we have an opportunity to populate any site by changing the value of field amplitude for a given value of the driving frequency.

For future work, we will study adiabatic charge pumping for the case-II by changing the field amplitude slowly. Then we will study the physics of the time-dependent topological insulators.

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