T.R.

BOLU ABANT İZZET BAYSAL UNIVERSITY

INSTITUTE OF GRADUATE STUDIES

ON A SPECIAL CASE OF CLEAN RINGS

MASTER OF SCIENCE

RUQAYAH MUSTAFA SALIH ALFAQI

BOLU, AĞUSTOS - 2021

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APPROVAL OF THE THESIS

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RUQAYAH MUSTAFA SALIH ALFAQI

ABSTRACT

ONA SPECIAL CASE OF CLEAN RINGS

MSC THESIS
RUQAYAH MUSTAFA SALIH ALFAQI
BOLU ABANT IZZET BAYSAL UNIVERSITY
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Throughout this thesis, let any ring be an associative ring with identity. We define a class of clean rings with idempotent not identity which is a special case of the class of clean rings. We obtain some characterizations of these rings. Moreover, we extend some known concepts to this structure.

KEYWORDS: Clean rings, Unit element, Idempotent element, Jacobson radical.

ÖZET

CLEAN HALKALARIN ÖZEL BİR DURUMU ÜZERİNE
YÜKSEK LİSANS TEZİ

RUQAYAH MUSTAFA SALIH ALFAQI
BOLU ABANT İZZET BAYSAL ÜNİVERSİTESİ
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Bu tezde herhangi bir halka, birimli ve birleşmelidir. Clean halkaların sınıfının özel bir hali olan, idempotenti birim eleman olmayan clean halkaların sınıfını tanımladık. Bu halkaları karakterize ettik. Ayrıca bazı bilinen kavramları bu yapıya genişlettik.

ANAHTAR KELİMELER: Clean halkalar, Birim eleman, İdempotent eleman, Jacobson radikali.

TABLE OF CONTENTS

LIST OF ABBREVIATIONS AND SYMBOLS

T	**:** A ring with identity
D	**:** Division ring
J(T)	**:** Jacobson radical of T
$\boldsymbol{M_n}$**(T)**	**:** The ring of all n $\times$ n matrices over T
$\boldsymbol{U_n}$**(T)**	**:** The ring of all n $\times$ n upper triangular matrices over T
$\boldsymbol{U}$**(T)**	**:** The set of all unit elements of T
End($\boldsymbol{W_D}$**)**	**:** The ring of endomorphisms of a vector space W over D
$\boldsymbol{Z_n}$	**:** The ring of integers modulo n
T[x]	**:** The ring of polynomials in a varible x with coefficients in T
T[[x]]	**:** The formal power series ring over T with elements a_0+ a_1 x + …+ a_i x +… where a_i ∈ T for all i ∈ N
$\boldsymbol{r_T}$**(a)**	**:** The right annihilator of a in T

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1. INTRODUCTION

Let **T** be a ring with identity and J(T) (= J) be the Jacobson radical of **T**. We study on the class of clean rings with idempotent not identity. These rings are a special case of clean rings which are very important structures in associative rings and modules theory.

Clean rings were introduced by Nicholson (1977). Every semisimple, semiperfect rings and finite rings are clean rings. We can see that a clean ring may not be semiperfect. Indeed, $Z_2 \times Z_2 \times \ldots$ is a clean ring, but it is not semiperfect (see Immormino (2013)). Moreover, many special cases of clean rings have been studied such as strongly clean (Nicholson (1999)), weakly clean (Ahn and Anderson (2006)), J-clean (Chen (2010)), uniquely clean (Chen (2011)) and etc... Unit regular rings were introduced by Ehrlich (1968). These rings are also proved to be clean rings by Camillo and Khurana (2001). Furthermore, the class of clean rings is a subclass of exchange and semipotent rings (see Immormino (2013)). This thesis is organized as follow:

In section **3**, we give some unknown basic concepts to understand this thesis.

In section **4**, we define clean rings with idempotent not identity and we give examples on these rings. We prove that **T** is a clean ring with idempotent not identity if and only if so is $\boldsymbol{U_n}$**(T)**. And we also prove that the formal power series and the formal triangular matrix rings are clean rings with idempotent not identity.

In section **5**, we prove that the ring of linear transformations of a nonzero vector space $\mathbf{W_D}$ is a clean ring with idempotent not identity where **D** is a division ring. Furthermore, we prove that unit regular rings are clean rings with idempotent not identity. Thus, we also obtain that every semisimple, semiperfect, artinian and finite rings are clean rings with idempotent not identity.

2. AIM AND SCOPE OF THE STUDY

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The main purpose of this thesis is to investigate the class of clean rings with idempotent not identity which is a special case of the class of clean rings. Clean rings are very important structures in associative rings and modules theory. In general, the aims of this thesis are itemized as follow:

1. Using the properties in literature for some very known rings such as semiperfect, unit regular, clean and etc…, we determine some common properties of rings in this class.
2. We investigate the relationships between the classes of some important rings in literature and the class of this rings.
3. We aim to determine when any clean ring is a clean ring with idempotent not identity.

3. PRELIMINARIES

In this section**,** we give some unknown concepts which is used in this thesis (see Anderson and Fuller (1992) for more details).

Definition 3.1. (Jacobson Radical) J(T) is the intersection of all maximal right (or left) ideals of the ring **T**. Then **J(T)** is an ideal of **T**. Moreover**, a** $\in$ **J(T)** if and only if **1- ta** (or **1- at**) is invertible in **T** for all **t** $\in$ **T**. (See Anderson and Fuller (1992).)

Definition 3.2. (Idempotent, Unit Element and Abelian Ring) If $\mathbf{f^2 = f} \in \mathbf{T}$, then **f** is called idempotent. For example, the only idempotents of a field **S** are **0** and **1**. A ring is abelian if every idempotent is central. **u** $\in$ **T** is called a unit element in **T** if there exists $\mathbf{u'} \in \mathbf{T}$ such that $\mathbf{uu' = u'u = 1_T}$.

For example, we know that

$Z_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$.

Idempotent elements of Z_6:

$(\bar{0})^2 = \bar{0}$, $(\bar{1})^2 = \bar{1}$, $(\bar{3})^2 = \bar{3}$, $(\bar{4})^2 = \bar{4}$ and hence $\bar{0}$, $\bar{1}$, $\bar{3}$, and $\bar{4}$ are idempotents in Z_6.

Unit elements of Z_6:

$\bar{1}.\ \bar{1} = \bar{1}$, $\bar{5}.\ \bar{5} = \bar{1}$ and hence, only unit elements of Z_6 are $\bar{1}$ and $\bar{5}$.

Notice that in a division ring **T**, every non$-$zero element is unit and there exist no idempotent elements except for $\mathbf{0_T}$ and $\mathbf{1_T}$.

Definition 3.3. (Orthogonal Idempotent and Orthogonally Finite Rings) A pair of idempotents $\mathbf{f_1}$ and $\mathbf{f_2}$ are called to be orthogonal if $\mathbf{f_1\ f_2 = f_2\ f_1 = 0}$. For example, $\bar{3}$ and $\bar{4}$ are orthogonal idempotents of $\mathbf{Z_6}$. A ring **T** is called an orthogonally finite **(**or **I- finite**) ring if there does not exist an infinite set of orthogonal idempotents in **T**.

Definition 3.4. (Primitive Idempotent) A non-zero idempotent **e** is called primitive if it can not be written as a sum $\mathbf{e} = \mathbf{e_1} + \mathbf{e_2}$ such that $\mathbf{e_1}$ and $\mathbf{e_2}$ are non-zero orthogonal idempotents (if and only if the corner ring **eTe** has exactly one nonzero idempotent**,** namely **e**). For example**,** since $\bar{1} = \bar{3} + \bar{4}$ and $\bar{3}$ and $\bar{4}$ are non-zero orthogonal idempotents**,** $\bar{1}$ is not a primitive idempotent**,** but $\bar{3}$ is a primitive idempotent in $\mathbf{Z_6}$.

Definition 3.5. (Local Ring and Local Idempotent) A ring **T** is local if for every **x** ∈ **T**, either **x** or **x-1** is invertible. If a corner ring **eTe** is a local ring where **e** is idempotent**,** then **e** is called a local idempotent. For example, let $\mathbf{T} = \mathbf{Z_4} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ is a local ring.

Definition 3.6. (Lifting Idempotent) Let **I** be an ideal of a ring **T**. We say that idempotents can be lifted modulo **I** if for every element **a** ∈ **I** such that $\mathbf{a^2- a} \in \mathbf{I}$ there exists an idempotent **e** ∈ **T** such that **a-e** ∈ **I**. For example, $\mathbf{I} = \langle \bar{\mathbf{2}} \rangle$ is an ideal of $\mathbf{Z_4}$. Let $\mathbf{a^2- a} \in \mathbf{I}$. Then $\mathbf{a^2 - a} = \bar{\mathbf{0}}$ or $\bar{\mathbf{2}}$. If $\mathbf{a^2- a} = \bar{\mathbf{0}}$, then $\mathbf{a} = \bar{\mathbf{0}}$ or $\bar{\mathbf{1}}$. Hence $\bar{\mathbf{0}} = \mathbf{a- a} \in \mathbf{I}$ where **e= a** is idempotent in $\mathbf{Z_4}$. If $\mathbf{a^2- a} = \bar{\mathbf{2}}$, then $\mathbf{a} = \bar{\mathbf{2}}$ **or** $\mathbf{a} = \bar{\mathbf{3}}$. Let $\mathbf{a} = \bar{\mathbf{2}}$, then $\mathbf{a} = \mathbf{a} - \bar{\mathbf{0}} \in \mathbf{I}$ where $\bar{\mathbf{0}} \in \mathbf{Z_4}$. Let $\mathbf{a} = \bar{\mathbf{3}}$, then $\bar{\mathbf{2}} = \mathbf{a} - \bar{\mathbf{1}} \in \mathbf{I}$. Thus, idempotents can be lifted modulo **I** in $\mathbf{Z_4}$.

Definition 3.7. (Semisimple Left T-Module and Semisimple Ring) Let **N** be **a left T-modulo**. **N** is called a semisimple left **T** module if **N** is a direct sum of simple submodules. For example, $\mathbf{Z_6} = \langle \bar{\mathbf{2}} \rangle \oplus \langle \bar{\mathbf{3}} \rangle$ is a semisimple **Z-module**. If **T** is a semisimple ring, then $\mathbf{T} \cong \oplus_{i=1}^{k} \boldsymbol{M_{ni}}\ (\boldsymbol{D_i})$ where all the $\boldsymbol{D_i}$ are division rings (see Wedderburn-Artin Theorem in Anderson-Fuller (1992)).

Definition 3.8. (Semiperfect Ring) A ring **T** is called a semiperfect ring if $\boldsymbol{T}/\boldsymbol{J}$ is a semisimple ring and idempotents lift module **J (T)** (⟺ $\mathbf{1_T}$ is the sum of orthogonal local idempotents ⟺ every primitive idempotent is local and **T** is orthogonally finite).

Definition 3.9. (Regular Ring and Unit Regular Ring) A ring **T** is regular if for all **a** ∈ **T** there exists **b** ∈ **T** such that **aba= a**. A ring T is unit regular if for all **a** ∈ **T** there exists **u** ∈ $\boldsymbol{U}$**(T)** such that **aua= a** (see Goodearl (1991) and Ehrlich (1968)).

Definition 3.10. (Clean Ring*)* Let **T** be a ring. An element of **T** is called clean if it is a sum of a unit and idempotent element. The ring **T** is called a clean ring if every element is clean.

For example, take $Z_6 = \{ \overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5} \}$. Then from above, we know that all idempotents of Z_6 are $\overline{0}, \overline{1}, \overline{3}, \overline{4}$ and all units of Z_6 are $\overline{1}, \overline{5}$. Thus, $\overline{0} = \overline{1} + \overline{5}$, $\overline{1} = \overline{0} + \overline{1}$, $\overline{2} = \overline{1} + \overline{1}$, $\overline{3} = \overline{4} + \overline{5}$, $\overline{4} = \overline{3} + \overline{1}$ and $\overline{5} = \overline{0} + \overline{5}$. Therefore, Z_6 is a clean ring. Furthermore, every field and finite ring are clean rings.

Definition 3.11. (Exchange and Semipotent Ring) If there exists an idempotent $\mathbf{e} \in \mathbf{aT}$ such that $(1- \mathbf{e}) \in \mathbf{aT}$ for all $\mathbf{a} \in \mathbf{T}$, then $\mathbf{T}$ is called an exchange ring. Every clean ring is an exchange ring. If for all $\mathbf{a} \notin \boldsymbol{J}(\boldsymbol{T})$, there exist $\mathbf{0} \neq \mathbf{e}^2 = \mathbf{e}$ and $\mathbf{t} \in \mathbf{T}$ such that $\mathbf{at} = \mathbf{e}$, then $\mathbf{T}$ is called a semipotent ring. Every exchange ring is semipotent. (See Immormino (2013) for clean, exchange and semipotent rings.)

4. ON A SPECIAL CLEAN RING

In this section, we define and characterize clean rings with idempotent not identity.

Definition 4.1. (Clean Element with Idempotent not Identity) Let **T** be a ring and **t** ∈ **T**. If **t** = **e** + **u** where **1** ≠**e** idempotent and **u** unit, then **t** ∈ **T** is called a clean element with idempotent not identity. Then every clean element with idempotent not identity is a clean element.

Let **T** be a ring and **t** ∈ **J(T)**. Then **t** is a clean element, because **1-t** is unit since **t** ∈ **J(T)** and hence **t = 1+ u** where **u** ∈ **T** is unit. But **t** ∈ **J(T)** can not be a clean element with idempotent not identity. Otherwise, if **t** = **e** + **u** where $\mathbf{e^2}$ = **e**≠**1** and **u** unit, then (**1-e**) **t** = (**1-e**) **u** where **1-e** ≠ **0**. So (**1-e**) **t** $\boldsymbol{u^{-1}}$= (**1-e**) ∈**J(T)**. Thus **1-(1-e) = e** is invertible, and so **e =1**. This gives a contradiction. Then **t** ∈ **J(T)** can not be a clean element with idempotent not identity.

Let t be a unit element in **T**, then **t** = **0** + **t** where **0** ≠ **1**. Thus, we see that if **t** is unit, then t is a clean element with idempotent not identity.

Now, let **t** be idempotent such that **t** ≠ **0**. Then **t** = (**1-t**) + (**2t-1**) where **1-t** ≠ **1** and $(\mathbf{2t-1})^{\mathbf{2}}$ = **4t** $^{\mathbf{2}}$- **4t** +**1** = **1** and hence **2t-1** is a unit. Thus, **t** is a clean element with idempotent not identity.

Definition 4.2. (Clean Ring with Idempotent not Identity) Let **T** be a ring. If every element not in **J(T)** is a clean element with idempotent not identity, then **T** is called a clean ring with idempotent not identity. Then every clean ring with idempotent not identity is a clean ring.

Lemma 4.3. Let $\boldsymbol{f}$**:** $\boldsymbol{T_1} \longrightarrow \boldsymbol{T_2}$ be a ring isomorphism such that $\boldsymbol{f}\,(\mathbf{1}_{\boldsymbol{T_1}}) = \mathbf{1}_{\boldsymbol{T_2}}$. Then

$\boldsymbol{f}\,(\mathrm{J}(\boldsymbol{T_1})) = \mathbf{J}(\boldsymbol{T_2})$.

Proof. Since f is isomorphism and hence it is onto, ∃ x ∈ $\boldsymbol{T_1}$ such that $\boldsymbol{f}$ **(x) = y,** ∀ y ∈ $\mathbf{J}(\boldsymbol{T_2})$. In addition, ∀ $\boldsymbol{t_2} \in \boldsymbol{T_2}$, ∃ $\boldsymbol{t_1} \in \boldsymbol{T_1}$ such that $\boldsymbol{f}\,(\boldsymbol{t_1}) = \boldsymbol{t_2}$.

Claim: **x** ∈ $\mathbf{J}(\boldsymbol{T_1})$. Let's prove this!

$\forall\, t_1 \in T_1$ we will prove that 1_{T_1}- t_1x is invertible. Since $f\,(1_{T_1}$- $t_1x)$ =$f\,(1_{T_1})$ – $f\,(t_1)f$ (x) = $1_{T_2} - f\,(t_1)$y, y ∈ J(T_2) and $f(t_1) \in T_2$, $1_{T_2} - f\,(t_1)$**y** is invertible in T_2. Thus, $f\,(1_{T_1}$- $t_1x)$ is invertible and hence there exists **s** ∈ T_2 such that $f\,(1_{T_1}$- $t_1x)$ **s** = **s f** $(1_{T_1}$- $t_1x)$ = 1_{T_2}. Since f is onto, ∃ $s_1 \in T_1$ such that **f** (s_1) **= s**. Then $f\ (1_{T_1}$- $t_1x)\ f\ (s_1)$ = $f\ (s_1)\ f\ (1_{T_1}$- $t_1x)$ = 1_{T_2} = $f\ (1_{T_1})$. Since f is a 1-1 homomorphism, we have that $(1_{T_1}$- $t_1x)\, s_1$ = $s_1(1_{T_1}$- $t_1x)$ = 1_{T_1}. Therefore, **x** ∈ J(T_1) and so, y = f **(x)** ∈ f(J(T_1)). Thus, J(T_2) ⊆ f**(J**(T_1)**) … (1)**

Now, let f **(x)** ∈ f(J(T_1)) where **x** ∈ J(T_1). Since f is onto, , ∃ $t_1 \in T_1$ such that $f\,(t_1) = t_2$, $\forall\, t_2 \in T_2$. Thus, 1_{T_2} - t_2 f **(x)** = $f\,(1_{T_1})$ – $f\,(t_1)$ f **(x)** = $f\,(1_{T_1}$- $t_1x)$. Since **x** ∈ J(T_1), 1_{T_1}- t_1x is unit, and so $f\,(1_{T_1}$- $t_1x)$ is unit, hence, f (**x**) ∈ J(T_2), and then f**(J**(T_1)**)** ⊆ J(T_2) **… (2)**

By (1) and (2) f(J(T_1)) = J(T_2).

Proposition. 4.4. Let f: $T_1 \longrightarrow T_2$ be a ring isomorphism such that $f\,(1_{T_1})$ =1_{T_2}. If T_1 is a clean ring with idempotent not identity, then so is T_2.

Proof. Let $t_2 \notin$ **J**(T_2). Since f is onto, ∃ $t_1 \in T_1$ such that $f\,(t_1) = t_2$. Then $t_1 \notin$ J (T_1). (Otherwise, if $t_1 \in$ J(T_1), then $f\,(t_1) \in f\big(\mathrm{J}(T_1)\big)$. Since f **(J**(T_1)**)** = **J**(T_2) by above lemma. t_2 = $f\,(t_1) \in$ J(T_2). This gives a contradiction.) Since T_1 is a clean ring with idempotent not identity, t_1 = **e** + **w** where **e**^{2} **= e** ≠ 1_{T_1} and w unit. Then t_2 = $f\,(t_1)$ = **f (e)** + f (**w**) where $(f(e))^2$ = $f(e)$ and f **(w)** is unit. In addition, $f(e) \neq 1_{T_2}$, otherwise if $f(e)$ =1_{T_2}, then $f(e)$ =1_{T_2} = $f(1_{T_1})$. Since f is **1-1**, **e** =1_{T_1}. This gives contradiction. Therefore, T_2 is also a clean ring with idempotent not identity.

Lemma 4.5. Let $\{T_i\}_{i \in I}$ be a family of some rings. Then $\prod_{i\in I} J(T_i)$ =$J(\prod_{i \in I} T_i)$.

Proof. Let **x** =$(x_i)_{i\in I} \in \prod_{i\in I} J(T_i)$ be any element. Then ∀ **i** ∈ **I,** $x_i \in$ **J**(T_i).

Let **y** =$(y_i)_{i\in I} \in \prod_{i\in I} T_i$ be any element and 1 = $(1_{T_i})_{i\in I}$ be the identity of $\prod_{i\in I} T_i$.

Now, we prove that **1** – **yx** is invertible in $\prod_{i\in I} T_i$. Then **1** – **yx** = $(1_{T_i} - y_i x_i)_{i\in I}$ and since $x_i \in$ **J**(T_i), ∀ **i** ∈ **I** $1_{T_i} - y_i x_i$ is invertible. Then there exists $u_i \in T_i$ such that $(1_{T_i} - y_i x_i)u_i = u_i(1_{T_i} - y_i x_i)$ =1_{T_i}, ∀ **i** ∈ **I**. Therefore, if you take as

$\mathbf{u} = (\boldsymbol{u_i})_{i\in I} \in \prod_{i\in I} \boldsymbol{T_i}$, then **u (1 – yx) = (1 – yx) u =1** and hence **1 – yx** is invertible. Thus, **x** $\in \boldsymbol{J}(\prod_{i\in I} \boldsymbol{T_i})$ and hence

$\prod_{i\in I} \boldsymbol{J}(\boldsymbol{T_i}) \subseteq \boldsymbol{J}(\prod_{i\in I} \boldsymbol{T_i})$ **… (1)**

Conversely, let $\mathbf{x} = (\boldsymbol{x_i})_{i\in I} \in \boldsymbol{J}(\prod_{i\in I} \boldsymbol{T_i})$. Let $\mathbf{y} = (\boldsymbol{y_i})_{i\in I} \in \prod_{i\in I} \boldsymbol{T_i}$ be any element. Then **1 – yx** = $(\mathbf{1}_{T_i} - \boldsymbol{y_i x_i})_{i\in I}$ is invertible since $\mathbf{x} \in \boldsymbol{J}(\prod_{i\in I} \boldsymbol{T_i})$. So there exists **u** $= (\boldsymbol{u_i})_{i\in I} \in \prod_{i\in I} \boldsymbol{T_i}$ such that **u (1 – yx) = (1 – yx) u =1**. Thus, we have that $\forall$ **i** $\in$ **I** $\boldsymbol{u_i}\,(\mathbf{1}_{T_i} - \boldsymbol{y_i x_i}) = (\mathbf{1}_{T_i} - \boldsymbol{y_i x_i})\,\boldsymbol{u_i} = \mathbf{1}_{T_i}$. This gives to us $\forall$ **i** $\in$ **I** $\boldsymbol{x_i} \in \mathbf{J}(\boldsymbol{T_i})$, and hence $\mathbf{x} = (\boldsymbol{x_i})_{i\in I} \in \prod_{i\in I} \boldsymbol{J}(\boldsymbol{T_i})$. Then

$\boldsymbol{J}(\prod_{i\in I} \boldsymbol{T_i}) \subseteq \prod_{i\in I} \boldsymbol{J}(\boldsymbol{T_i})$ ... **(2)**

By (1) and **(2)** $\prod_{i\in I} \boldsymbol{J}(\boldsymbol{T_i}) = \boldsymbol{J}(\prod_{i\in I} \boldsymbol{T_i})$.

Proposition 4.6. Let $\{\boldsymbol{T_i}\}_{i\in I}$ be a family of some rings. All the $\boldsymbol{T_i}$ are clean rings with idempotent not identity, $\forall$ **i** $\in$ **I** if and only if $\prod_{i\in I} \boldsymbol{T_i}$ is a clean ring with idempotent not identity.

Proof. ($\Rightarrow$): Let all the $\boldsymbol{T_i}$ be clean rings with idempotent not identity and $\mathbf{t} = (\boldsymbol{t_i})_{i\in I} \in \prod_{i\in I} \boldsymbol{T_i}$ but $\mathbf{t} \notin \boldsymbol{J}(\prod_{i\in I} \boldsymbol{T_i})$. Then $\exists\, \boldsymbol{i_0} \in \mathbf{I}$ $\boldsymbol{t}_{i_0} \notin \boldsymbol{J}(\boldsymbol{T}_{i_0})$, since $\boldsymbol{J}(\prod_{i\in I} \boldsymbol{T_i}) = \prod_{i\in I} \boldsymbol{J}(\boldsymbol{T_i})$. Since $\boldsymbol{T}_{i_0}$ is a clean element with idempotent not identity, $\boldsymbol{t}_{i_0} = \boldsymbol{e}_{i_0} + \boldsymbol{u}_{i_0}$ where $\boldsymbol{e}_{i_0} \neq \mathbf{1}_{T_{i_0}}$ idempotent and $\boldsymbol{u}_{i_0}$ unit. Since all the $\boldsymbol{T_i}$ are clean rings with idempotent not identity, all the $\boldsymbol{T_i}$ are clean rings. Then for $\boldsymbol{i_0} \neq \boldsymbol{i} \in \mathbf{I}$ $\boldsymbol{t_i} = \boldsymbol{e_i} + \boldsymbol{u_i}$ where e_i idempotent and u_i unit. So $\mathbf{t} = (\boldsymbol{t_i})_{i\in I} = (\boldsymbol{e_i})_{i\in I} + (\boldsymbol{u_i})_{i\in I}$ where $\mathbf{1}_{T_{i_0}} \neq \boldsymbol{e}_{i_0} \in \boldsymbol{T}_{i_0}$ for $\boldsymbol{i_0} \in \boldsymbol{I}$ and then, $(\boldsymbol{e_i})_{i\in I} \neq \mathbf{1}$ in $\prod_{i\in I} \boldsymbol{T_i}$. Therefore, t is a clean element with idempotent not identity and hence, $\prod_{i\in I} \boldsymbol{T_i}$ is a clean ring with idempotent not identity.

($\Leftarrow$): Let $\prod_{i\in I} \boldsymbol{T_i}$ be a clean ring with idempotent not identity. Assume that there exists at least one $\boldsymbol{i_0} \in \boldsymbol{I}$ such that $\boldsymbol{T}_{i_0}$ is not a clean ring with idempotent not identity. Then there exists $\boldsymbol{t}_{i_0} \notin \boldsymbol{J}(\boldsymbol{T}_{i_0})$ such that $\boldsymbol{t}_{i_0}$ is not clean element with idempotent not identity. Let $\mathbf{t} = (\boldsymbol{t_i})_{i\in I}$ where $\boldsymbol{t_i} = \mathbf{0}_{T_i}$ for all $\boldsymbol{i} \neq \boldsymbol{i_0}$. Since $\prod_{i\in I} \boldsymbol{T_i}$ is a clean ring with idempotent not identity, $\mathbf{t} = \mathbf{e} + \boldsymbol{u}$ such that $\mathbf{e^2} = \mathbf{e} \neq \mathbf{1}$ idempotent and $\boldsymbol{u}$ unit in $\prod_{i\in I} \boldsymbol{T_i}$. Let $\boldsymbol{e} = (\boldsymbol{e_i})_{i\in I}$ and $\boldsymbol{u} = (\boldsymbol{u_i})_{i\in I}$ hence **t = e+u** where for all $\boldsymbol{i} \in \boldsymbol{I}$, $\boldsymbol{t_i} = \boldsymbol{e_i} + \boldsymbol{u_i}$. Let $\boldsymbol{i} \neq \boldsymbol{i_0}$. Then we know that $\boldsymbol{t_i} = \mathbf{0}$ and hence $\mathbf{0} = \boldsymbol{e_i} + \boldsymbol{u_i}$ gives that $\boldsymbol{u_i}$ = -$\boldsymbol{e_i}$. Since **u** unit, all the $\boldsymbol{u_i}$ are unit, hence $\boldsymbol{u_i}^{-1}$ exist. Thus, (1-$\boldsymbol{e_i}$) $\boldsymbol{u_i}$ $\boldsymbol{u_i}^{-1}$= -(1-$\boldsymbol{e_i}$) $\boldsymbol{e_i}$ $\boldsymbol{u_i}^{-1}$, 1-$\boldsymbol{e_i}$ = **0** and so $\boldsymbol{e_i}$ = **1** for all $\boldsymbol{i} \neq \boldsymbol{i_0}$. Since **e** ≠**1**, $\mathbf{e}_{i_0}$ ≠**1**. Therefore, $\boldsymbol{t}_{i_0}$ = $\boldsymbol{e}_{i_0}$ +$\boldsymbol{u}_{i_0}$ where 1≠ $\boldsymbol{e}_{i_0}$ idempotent and $\boldsymbol{u}_{i_0}$ unit. Thus $\boldsymbol{t}_{i_0}$ is a clean element with idempotent not identity. This gives a contradiction. By the contradiction, for all $\boldsymbol{i} \in \boldsymbol{I}, \boldsymbol{T_i}$ are clean rings with idempotent not identity.

Definition 4.7. (The Ring of Morita Context) Let $_{\mathbf{S}}\mathbf{C}_{\mathbf{T}}$ and $_{\mathbf{T}}\mathbf{D}_{\mathbf{S}}$ be bimodules over the rings **S** and **T** and let α: C × D $\xrightarrow[(x,y)\to xy]{}$S and β: D × C $\xrightarrow[(y,x)\to yx]{}$T be some binary functions. Then (S, T, C, D, α, β) is called a Morita context if α and β preserve addition and satisfy that s(xy) =(sx)y, (xt)y =x(ty), t(yx) =(ty)x, (ys)x =y(sx),(yx)t =y(xt). If (S, T, C, D, α, β) is a Morita context, then L =$\begin{bmatrix} S & C \\ D & T \end{bmatrix}$ = {$\begin{bmatrix} s & x \\ y & t \end{bmatrix}$: s ∈ S, t ∈ T, x ∈ C, y ∈ D} is a ring with respect to the usual operations. Then L is called the ring of a Morita context.

Proposition. 4.8. Let (S, T, C, D, α, β) be a Morita context. If **S** and **T** are clean rings with idempotent not identity, then L_0=$\begin{bmatrix} S & C_0 \\ D_0 & T \end{bmatrix}$ is also a clean ring with idempotent not identity where C_0 = {x ∈ C: xD ⊆ $\boldsymbol{J}(\boldsymbol{S})$} and D_0 = {x ∈ D: xC ⊆ $\boldsymbol{J}(\boldsymbol{T})$}

Proof. Let A = $\begin{bmatrix} a & x \\ y & b \end{bmatrix}$ ∉ $\boldsymbol{J}$($\boldsymbol{L_0}$) where $\boldsymbol{J}$(**L**) = $\begin{bmatrix} \boldsymbol{J}(\boldsymbol{S}) & C_0 \\ D_0 & \boldsymbol{J}(\boldsymbol{T}) \end{bmatrix}$. Then **a** ∉ $\boldsymbol{J}(\boldsymbol{S})$ or **b** ∉ $\boldsymbol{J}(\boldsymbol{T})$.

Case 1. Let **a** ∉ **J**($\boldsymbol{S}$). Since **S** is a clean ring with idempotent not identity. **a = f +u** where $\mathbf{f}^{2}$=**f**≠**1** idempotent and **u** unit in **S**. Since **T** is a clean ring with idempotent not identity, **T** is a clean ring and hence **b - y**$\boldsymbol{u}^{-1}$ $\boldsymbol{x}$ **= g + v** where $\mathbf{g}^{2}$=**g** idempotent and **v** unit in **T**. Thus, $\begin{bmatrix} a & x \\ y & b \end{bmatrix}$ = $\begin{bmatrix} \mathbf{f} + \mathbf{u} & x \\ y & \mathbf{g} + \mathbf{v} + \mathbf{y}\boldsymbol{u}^{-1}\boldsymbol{x} \end{bmatrix}$ = $\begin{bmatrix} \mathbf{f} & 0 \\ 0 & \mathbf{g} \end{bmatrix}$ + $\begin{bmatrix} u & x \\ y & \mathbf{v} + \mathbf{y}\boldsymbol{u}^{-1}\boldsymbol{x} \end{bmatrix}$. Since $\begin{bmatrix} 1 & 0 \\ -\mathbf{y}\boldsymbol{u}^{-1} & 1 \end{bmatrix}\begin{bmatrix} u & x \\ y & \mathbf{v} + \mathbf{y}\boldsymbol{u}^{-1}\boldsymbol{x} \end{bmatrix}\begin{bmatrix} 1 & -\boldsymbol{u}^{-1}\boldsymbol{x} \\ 0 & 1 \end{bmatrix}$ = $\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}$, $\begin{bmatrix} u & x \\ y & \mathbf{v} + \mathbf{y}\boldsymbol{u}^{-1}\boldsymbol{x} \end{bmatrix}$ is invertible. In addition, $\begin{bmatrix} \mathbf{f} & 0 \\ 0 & \mathbf{g} \end{bmatrix}$ ≠**1** idempotent and hence A is a clean element with idempotent not identity.

Case 2. Now**,** let $\mathbf{b} \notin \boldsymbol{J(T)}$. Since **T** is a clean ring with idempotent not identity, $\mathbf{b} = \mathbf{f} + \mathbf{u}$ where $\mathbf{f^2}=\mathbf{f}\neq\mathbf{1}$ idempotent and **u** unit in **T**. Since **S** is a clean ring with idempotent not identity, and then it is a clean ring and hence $\mathbf{b} - \mathbf{x}\boldsymbol{u}^{-1}\boldsymbol{y} = \mathbf{g} + \mathbf{v}$ where $\mathbf{g^2} = \mathbf{g}$ idempotent and **v** unit in **S**. Thus, $\begin{bmatrix} a & x \\ y & b \end{bmatrix} = \begin{bmatrix} \mathbf{g} + \mathbf{v} + \mathbf{x}\boldsymbol{u}^{-1}\boldsymbol{y} & x \\ y & \mathbf{f} + \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{g} & 0 \\ 0 & \mathbf{f} \end{bmatrix} + \begin{bmatrix} \mathbf{v} + \mathbf{x}\boldsymbol{u}^{-1}\boldsymbol{y} & x \\ y & \mathbf{u} \end{bmatrix}$. Since $\begin{bmatrix} 1 & -\mathbf{x}\boldsymbol{u}^{-1} \\ \mathbf{0} & 1 \end{bmatrix}\begin{bmatrix} \mathbf{v} + \mathbf{x}\boldsymbol{u}^{-1}\boldsymbol{y} & x \\ y & \mathbf{u} \end{bmatrix}\begin{bmatrix} 1 & \mathbf{0} \\ -\boldsymbol{u}^{-1}\boldsymbol{y} & 1 \end{bmatrix} = \begin{bmatrix} v & 0 \\ y & u \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\boldsymbol{u}^{-1}\boldsymbol{y} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} v & 0 \\ 0 & u \end{bmatrix}$, $\begin{bmatrix} \mathbf{v} + \mathbf{x}\boldsymbol{u}^{-1}\boldsymbol{y} & x \\ y & \mathbf{u} \end{bmatrix}$ is invertible. In addition, $\begin{bmatrix} \mathbf{g} & 0 \\ 0 & \mathbf{f} \end{bmatrix} \neq \mathbf{1}$ idempotent and hence A is a clean element with idempotent not identity.

By case (1) and (2), $\boldsymbol{L_0}$ is a clean ring with idempotent with idempotent not identity.

Proposition. 4.9. Let ${}_S\mathbf{M}_T$ be a bimodule. Then K $= \begin{bmatrix} S & M \\ 0 & T \end{bmatrix} = \{\begin{bmatrix} s & x \\ 0 & t \end{bmatrix} : s \in S, t \in T, x \in M\}$ is a clean ring with idempotent not identity if and only if **S** and **T** are clean rings with idempotent not identity.

Proof. (⇒): Let **K** be a clean ring with idempotent not identity and $\mathbf{s} \notin \mathbf{J}(\boldsymbol{S})$. Then $\begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix} \notin \boldsymbol{J}(\mathbf{K}) = \begin{bmatrix} \boldsymbol{J(S)} & M \\ 0 & \boldsymbol{J(T)} \end{bmatrix}$ Thus, $\begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{e_1} & m \\ 0 & \boldsymbol{e_2} \end{bmatrix} + \begin{bmatrix} u & -m \\ 0 & -\boldsymbol{e_2} \end{bmatrix}$ where $\begin{bmatrix} \boldsymbol{e_1} & m \\ 0 & \boldsymbol{e_2} \end{bmatrix}$ idempotent not identity and $\begin{bmatrix} u & -m \\ 0 & -\boldsymbol{e_2} \end{bmatrix}$ is unit. Since $\begin{bmatrix} u & -m \\ 0 & -\boldsymbol{e_2} \end{bmatrix}$ is invertible, there exists $\mathbf{t} \in \mathbf{T}$ such that $-\boldsymbol{e_2}\,\mathbf{t} = \mathbf{1}_{\boldsymbol{T}}$. Since $\begin{bmatrix} \boldsymbol{e_1} & m \\ 0 & \boldsymbol{e_2} \end{bmatrix}$ is idempotent, $\boldsymbol{e_2}^{\mathbf{2}}=\boldsymbol{e_2}$. Thus, $-(1-\boldsymbol{e_2})\,\boldsymbol{e_2}\mathbf{t} = (1-\boldsymbol{e_2})$ and hence $\boldsymbol{e_2} = \mathbf{1}_{\boldsymbol{T}}$.

Case 1. Let $\mathbf{m} = \mathbf{0}$. Then $\boldsymbol{e_1} \neq \mathbf{1}_{\boldsymbol{S}}$ and $\mathbf{s} = \boldsymbol{e_1} + \mathbf{u}$ where $\mathbf{1} \neq \boldsymbol{e_1}^{\mathbf{2}} = \boldsymbol{e_1}$ and **u** unit.

Case 2. Let $\mathbf{m} \neq 0$. Since $\begin{bmatrix} \boldsymbol{e_1} & m \\ 0 & \mathbf{1} \end{bmatrix}\begin{bmatrix} \boldsymbol{e_1} & m \\ 0 & \mathbf{1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{e_1} & m \\ 0 & \mathbf{1} \end{bmatrix}$, we have that $\boldsymbol{e_1}^{\mathbf{2}} = \boldsymbol{e_1}$ and $\boldsymbol{e_1}\mathbf{m} + \mathbf{m} = \mathbf{m}$ and then $\boldsymbol{e_1}\mathbf{m} = \mathbf{0}$. This show that $\boldsymbol{e_1} \neq \mathbf{1}$. Thus, $\mathbf{s} = \boldsymbol{e_1} + \mathbf{u}$ where $\mathbf{1} \neq \boldsymbol{e_1}^{\mathbf{2}} = \boldsymbol{e_1}$ and **u** unit.

In each case, we see that *s* is a clean element with idempotent not identity. Thus, **S** is a clean ring with idempotent not identity. Similarly, we can obtain that so is **T**.

(⇐): let **S** and **T** be clean rings with idempotent not identity and $\begin{bmatrix} s & m \\ 0 & t \end{bmatrix} \notin \boldsymbol{J}(\mathbf{K})$. Then **s** $\notin$ **J**($\boldsymbol{S}$) or **t** $\notin$ **J**($\boldsymbol{T}$).

Case 1. Let **s** $\notin$ $\boldsymbol{J}(\boldsymbol{S})$. Since **S** is a clean ring with idempotent not identity, **s** = $\boldsymbol{e}$ + **u** where $\mathbf{e^2}$= **e** $\neq$ $\mathbf{1}_{\boldsymbol{S}}$ and **u** unit. Since, T is a clean ring with idempotent not identity, so it is clean and then **t** = $\boldsymbol{g} + \boldsymbol{u_1}$ where $\mathbf{g^2}$= **g** and $\boldsymbol{u_1}$ unit. Then we can write that $\begin{bmatrix} s & m \\ 0 & t \end{bmatrix} = \begin{bmatrix} e & 0 \\ 0 & g \end{bmatrix} + \begin{bmatrix} u & m \\ 0 & \boldsymbol{u_1} \end{bmatrix}$ where $\begin{bmatrix} e & 0 \\ 0 & g \end{bmatrix}$ idempotent not identity and $\begin{bmatrix} u & m \\ 0 & \boldsymbol{u_1} \end{bmatrix}$ unit where $\begin{bmatrix} u & m \\ 0 & \boldsymbol{u_1} \end{bmatrix}^{-1} = \begin{bmatrix} u^{-1} & -u^{-1}m\,\boldsymbol{u_1}^{-1} \\ 0 & \boldsymbol{u_1}^{-1} \end{bmatrix}$. Therefore, $\begin{bmatrix} s & m \\ 0 & t \end{bmatrix}$ is a clean element with idempotent not identity.

Case 2. Let **t** $\notin$ $\boldsymbol{J}(\boldsymbol{T})$. Similarly, we can prove $\begin{bmatrix} s & m \\ 0 & t \end{bmatrix}$ is a clean element with idempotent not identity.

By these cases, we see that $\begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$ is a clean ring with idempotent not identity.

Lemma 4.10. Let **U** = $[\boldsymbol{u_{ij}}]_{\,\boldsymbol{n\times n}} \in \boldsymbol{U_n}$**(T)**. **U** is invertible if and only if ∀ **i** ∈ {1, …, n} $\boldsymbol{u_{ii}}$ are invertible in **T**.

Proof. (⇒): Let **U** be invertible. Then $\boldsymbol{UU^{-1}} = \boldsymbol{U^{-1}}\mathbf{U} = \boldsymbol{I_n}$ and hence $\boldsymbol{u_{ii}}\boldsymbol{u_{ii}}^{-1} = \boldsymbol{u_{ii}}^{-1}\boldsymbol{u_{ii}} = 1$.

(⇐): let ∀ **i** ∈ {1, …, n} all the $\boldsymbol{u_{ii}}$ are invertible. We will use an induction on **n**. Let

n = 2. Then **U** = $\begin{bmatrix} \boldsymbol{u_{11}} & x \\ 0 & \boldsymbol{u_{22}} \end{bmatrix}$ where $\boldsymbol{u_{11}}\boldsymbol{u_{11}}^{-1} = \boldsymbol{u_{11}}^{-1}\boldsymbol{u_{11}} = 1$ and $\boldsymbol{u_{22}}\boldsymbol{u_{22}}^{-1} = \boldsymbol{u_{22}}^{-1}\boldsymbol{u_{22}} = 1$. Then $\begin{bmatrix} \boldsymbol{u_{11}} & x \\ 0 & \boldsymbol{u_{22}} \end{bmatrix}\begin{bmatrix} \boldsymbol{u_{11}}^{-1} & -\boldsymbol{u_{11}}^{-1}x\,\boldsymbol{u_{22}}^{-1} \\ 0 & \boldsymbol{u_{22}}^{-1} \end{bmatrix} = \boldsymbol{I_2}$ and $\begin{bmatrix} \boldsymbol{u_{11}}^{-1} & -\boldsymbol{u_{11}}^{-1}x\,\boldsymbol{u_{22}}^{-1} \\ 0 & \boldsymbol{u_{22}}^{-1} \end{bmatrix}\begin{bmatrix} \boldsymbol{u_{11}} & x \\ 0 & \boldsymbol{u_{22}} \end{bmatrix} = \boldsymbol{I_2}$. Thus, for **n = 2** the induction is true.

Now, suppose that the induction is true for **n-1**. Let **U** = $\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \mathbf{0} & \boldsymbol{u_{nn}} \end{bmatrix}$ where **B**= $\begin{pmatrix} \boldsymbol{u_{1n}} \\ \boldsymbol{u_{2n}} \\ \vdots \\ \boldsymbol{u_{(n-1)n}} \end{pmatrix}$. Since **A** ∈ $\boldsymbol{U_{n-1}}$**(T)** and $\boldsymbol{u_{ii}}$ is invertible for all **i** ∈ {1,…, n-1}, **A** is invertible, $\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{u_{nn}} \end{bmatrix}\begin{bmatrix} A^{-1} & -A^{-1}B\,\boldsymbol{u_{nn}}^{-1} \\ 0 & \boldsymbol{u_{nn}}^{-1} \end{bmatrix} = \begin{bmatrix} A^{-1} & -A^{-1}B\,\boldsymbol{u_{nn}}^{-1} \\ 0 & \boldsymbol{u_{nn}}^{-1} \end{bmatrix}\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{u_{nn}} \end{bmatrix}$ $= \boldsymbol{I_n}$. Thus, the induction is true for **n**. Therefore, **U** is invertible if $\forall$ **i** $\in \{1, \ldots, n\}$ all the $\boldsymbol{u_{ii}}$ are invertible.

Lemma 4.11. Let **E** = $[\boldsymbol{e_{ij}}]_{\,\boldsymbol{n\times n}} \in \boldsymbol{U_n}$**(T)** be idempotent such that $\boldsymbol{e_{ii}}$ **= 1** for all **i** $\in \{1, \ldots, n\}$. Then $\boldsymbol{e_{ij}}$ **= 0** for all **i**$\neq$**j**.

Proof. We will use an induction on $\boldsymbol{n}$. Let **n = 2**. Then **E** = $\begin{bmatrix} \mathbf{1} & \boldsymbol{e_{12}} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$ and $\mathbf{E^2} = \mathbf{E}$ gives that $\begin{bmatrix} \mathbf{1} & \boldsymbol{e_{12}} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}\begin{bmatrix} \mathbf{1} & \boldsymbol{e_{12}} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \boldsymbol{e_{12}} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$ and hence $\begin{bmatrix} \mathbf{1} & \boldsymbol{e_{12}} + \boldsymbol{e_{12}} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \boldsymbol{e_{12}} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$. Then $\boldsymbol{e_{12}} = \mathbf{0}$. Thus, the induction is true for **n = 2**. Now, suppose that the induction is true for **n-1**. If **E** = $\begin{bmatrix} \boldsymbol{F} & \boldsymbol{G} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$, then $\begin{bmatrix} \boldsymbol{F} & \boldsymbol{G} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}\begin{bmatrix} \boldsymbol{F} & \boldsymbol{G} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F} & \boldsymbol{G} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$ implies that $\mathbf{F^2} = \mathbf{F}$ such that $\boldsymbol{e_{ii}}$ **= 1** for all **i** $\in \{1, \ldots, n\}$. By the induction hypothesis, **F** = $\boldsymbol{I_{n-1}}$ and so $\begin{bmatrix} \boldsymbol{I_{n-1}} & \boldsymbol{G} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}\begin{bmatrix} \boldsymbol{I_{n-1}} & \boldsymbol{G} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I_{n-1}} & \boldsymbol{G} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$ and hence $\boldsymbol{G} + \boldsymbol{G} = \boldsymbol{G}$ gives that $\boldsymbol{G} = \mathbf{0}$ and so $\boldsymbol{E} = \boldsymbol{I_n}$.

Theorem 4.12. T is a clean ring with idempotent not identity if and only if $\boldsymbol{U_n}$**(T)** is a clean ring with idempotent not identity, too.

Proof. ($\Rightarrow$): Let **T** be a clean ring with idempotent not identity. Let **A** = $[\boldsymbol{a_{ij}}] \notin \boldsymbol{J}(\boldsymbol{U_n}(\mathbf{T}))$. Then $\exists\ \boldsymbol{a_{ii}} \notin \boldsymbol{J(T)}$ where **i** $\in \{1, \ldots, n\}$. Since **T** is a clean ring with idempotent not identity, $\boldsymbol{a_{ii}} = \boldsymbol{e_i} + \boldsymbol{u_i}$ where $\boldsymbol{e_i} \neq \mathbf{1}_{\boldsymbol{T}}$ idempotent and $\boldsymbol{u_i}$ unit. Since **T** is a clean ring for all **t** $\in \{1, \ldots, n\}$-{i} $\boldsymbol{a_{tt}} = \boldsymbol{e_t} + \boldsymbol{u_t}$ where $\boldsymbol{e_t}$ idempotent and $\boldsymbol{u_t}$ unit. Then **A = E + U** where E = $[\boldsymbol{e_{ij}}] \in \boldsymbol{U_n}$**(T)** with $\boldsymbol{e_{jj}} = \boldsymbol{e_j}$ and $\boldsymbol{e_{ij}} = 0$ for **i**$\neq$**j** and **U** = $[\boldsymbol{u_{ij}}]$ with $\boldsymbol{u_{jj}} = \boldsymbol{u_j}$ for **i** $\in \{1, \ldots, n\}$. Thus, $\boldsymbol{E} \neq \boldsymbol{I_n}$ idempotent and **U** is unit by **Lemma 4.10.** So $\boldsymbol{U_n}$(**T**) is a clean ring with idempotent not identity.

($\Leftarrow$): Let $\boldsymbol{U_n}$(**T**) be a clean ring with idempotent not identity. Let **a** $\notin \boldsymbol{J(T)}$. Let **A** = $[\boldsymbol{a_{ij}}] \in \boldsymbol{U_n}$**(T)** such that $\boldsymbol{a_{ii}} = \boldsymbol{a}$ and otherwise, $\boldsymbol{a_{ij}} = 0$ for **i**$\neq$**j**. Then **A** $\notin \boldsymbol{J}(\boldsymbol{U_n}(\mathbf{T}))$, and since $\boldsymbol{U_n}$**(T)** is a clean ring with idempotent not identity, **A = E + U** where $\boldsymbol{I_n} \neq \mathbf{E^2} = \mathbf{E} = [\boldsymbol{e_{ij}}]$ and **U** = $[\boldsymbol{u_{ij}}]$ unit and hence all the $\boldsymbol{u_{ii}}$ are invertible. Suppose that $\boldsymbol{e_{ii}}$ **= 1** for all **i** $\in \{1, \ldots, n\}$. By **Lemma. 4.11. E** = $\boldsymbol{I_n}$**.** And hence there exists at least one **i** ∈ {1, …, n} such that $\boldsymbol{e_{ii}} \neq \mathbf{1}$. Then $\boldsymbol{a_{ii}} = \boldsymbol{e_i} + \boldsymbol{u_i}$ where $\boldsymbol{e_{ii}} \neq \mathbf{1}$ idempotent and $\boldsymbol{u_i}$ unit. Then T is a clean ring with idempotent not identity.

Remark. Notice that $Z/_{3Z} \cong Z_3$ is a clean ring with idempotent not identity where idempotents lift modulo **3Z**. But **Z** is not a clean ring with idempotent not identity.

Lemma 4.13. **T** is a clean ring with idempotent not identity if and only if idempotents lift modulo ***J*(T)** and $\boldsymbol{T}/_{\boldsymbol{J}(\mathbf{T})}$ is a clean ring with idempotent not identity.

Proof. (⇒): ∀ **x** + ***J*(T)** ∉ $J(\boldsymbol{T}/_{\boldsymbol{J}(\mathbf{T})})$ = **0**, since **T** is a clean with idempotent not identity and **x** ∉ ***J*(T)**, **x = f + w** where $\mathbf{f^2}$= **f≠1** and **w** unit. Thus, we have that **f +** ***J*(T)** ≠ **1 +** ***J*(T)**, otherwise; if **f +** ***J*(T)** = **1 +** ***J*(T),** then **1-f** ∈ ***J*(T)** and hence **1-(1-f) = f** is unit and then **f = 1**. This gives a contradiction. Furthermore, **w +** ***J*(T)** is a unit in $\boldsymbol{T}/_{\boldsymbol{J}(\mathbf{T})}$. So, we are done.

Now, we will prove that idempotents lift modulo ***J*(T)**. Let $\mathbf{t^2}$**- t** ∈ ***J*(T)**. If **t** ∈ ***J*(T)**, then **t +** ***J*(T)** = **0 +** ***J*(T)**, and hence **t – 0** ∈ ***J*(T)** where **0** is idempotent in **T**. Now, let **t** ∉ ***J*(T)**. Since **T** is a clean with idempotent not identity, **t = f + w** where 1 ≠ **f** = $\mathbf{f^2}$ and w unit. Then **t** - $\mathbf{w^{-1}}$**(1 - f) w** = $\mathbf{w^{-1}}$($\mathbf{t^2}$- **t**) ∈ ***J*(T)**. Thus, we have that an idempotent $\mathbf{w^{-1}}$**(1 - f) w** ∈ **T** such that **t** - $\mathbf{w^{-1}}$ **(1 - f) w** ∈ ***J*(T)**. Then we are done.

(⇐): ∀ **x** ∉ ***J*(T)**, **x** + ***J*(T)** ∉ J $(\boldsymbol{T}/_{\boldsymbol{J}(\mathbf{T})})$ = **0**. Since $\boldsymbol{T}/_{\boldsymbol{J}(\mathbf{T})}$ is a clean with idempotent not identity, **x +** ***J*(T)** = **(f +** ***J*(T)) + (w +** ***J*(T))** where $(\mathbf{f} + \boldsymbol{J}(\mathbf{T}))^2$= **f +** ***J*(T)**≠ **1 +** ***J*(T)** and **w +** ***J*(T)** is unit. Since idempotents lift modulo ***J*(T)**, there exists $\mathbf{g^2}$= **g** ∈ **T** such that **g +** ***J*(T)** = **f +** ***J*(T))** ≠ **1 +** ***J*(T)** and so we have that $\mathbf{g^2}$= **g** ≠ **1**. Then **x +** ***J*(T)** = **(g +** ***J*(T)) + (w +** ***J*(T))** and hence we can write that **x- g = w + j** where **j** ∈ ***J*(T)**. Since **w +** ***J*(T)** is unit, we have that **(w +** ***J*(T)) ($\boldsymbol{w_1}$ +** ***J*(T))** = **1 +** ***J*(T)** and then **w**$\boldsymbol{w_1}$ **– 1** ∈ ***J*(T)** and so **1** – (**w**$\boldsymbol{w_1}$ **– 1**) = **w**$\boldsymbol{w_1}$ is unit, that is, **w** is unit in **T** with inveres of **w**, $\boldsymbol{w_2}$. Since **j** ∈ ***J*(T)**, 1 + $\boldsymbol{w_2}$ **j** is unit and so **w** **(**1 + $\boldsymbol{w_2}$ **j) = w + j** is unit. Then **x = g + w +j** where **g** ≠ **1** idempotent and **w +j** is unit.

Lemma 4.14. Let **T [[x]]** be the formal power series. Then the following are satisfied:

i) ***u(x)*** **=** $u_0 + u_1x + \ldots \in$ **T [[x]]** is invertible if and only if u_0 is invertible.

ii) ***J*****(T [[x]]) =** ***J*****(T) + T [[x]] x**.

iii) Let ***e(x)*** **=** $e_0 + e_1x + \ldots \neq 1$ be idempotent**.** Then $e_0 \neq 1$.

Proof. i) (⇒): Since ***u(x)*** is invertible, there exists ***v(x)*** **=** $v_0 + v_1x + \ldots \in$ **T [[x]]** such that ***u(x) v(x)*** **= 1**. Then $(u_0 + u_1x + \ldots)(v_0 + v_1x + \ldots) = 1$ and hence $u_0 v_0 = 1$, u_0 is invertible in **T**.

(⇐): Let u_0 be invertible. Let ***t(x)*** **=** $(u_0 + u_1x + u_2x^2 + \ldots)(v_0 + v_1x + v_2x^2 + \ldots) = 1 = t_0 + t_1x + \ldots$ where $v_0 = u_0^{\ -1}$. Then $t_0 = 1 = u_0u_0^{\ -1}$, $t_1 = u_0v_1 + u_1v_0 = 0$, and hence $v_1 = -u_0^{\ -1}u_1v_0$, $t_2 = u_0v_2 + u_1v_1 + u_2v_0 = 0$, $v_2 = -u_0^{\ -1}(u_1v_1 + u_2v_0)$. If we continue in this way, then we find that $v_0, v_1, \ldots, v_{n-1} \in$ **T**. Let's find $v_n \in$ **T**. $t_n = u_0v_n + u_1v_{n-1} + u_2v_{n-2} + \ldots + u_nv_0 = 0$. $v_n = -u_0^{\ -1}(u_1v_{n-1} + \ldots + u_nv_0)$. Then there exists ***v(x)*** ∈ **T [[x]]** such that ***u(x) v(x)*** **= 1** where ***v(x)*** **=** $v_0 + v_1x + \ldots$ Similarly, we can find that ***v(x) u(x)*** **= 1**. So, ***u(x)*** is also invertible.

ii) Let ***t(x)*** ∈ ***J*****(T [[x]])**. Then ∀ ***r(x)*** **=** $r_0 + r_1x + \ldots \in$ **T [[x]]**, 1- ***r(x) t(x)*** is invertible, that is, $1 - (r_0 + r_1x + \ldots)(t_0 + t_1x + \ldots)$ is invertible By the part (i) $1 - r_0t_0$ is invertible, namely $t_0 \in$ ***J*****(T)**. And hence ***t(x)*** **=** $t_0 + (t_1 + t_2x + \ldots)$ **x** ∈ ***J*****(T) + T [[x]] x**.

Now, let ***t(x)*** ∈ ***J*****(T) + T [[x]] x**. Then ***t(x)*** **=** $t_0 + t_1x + \ldots =$ **a** + $(b_0 + b_1x + \ldots)$ x where **a** ∈ ***J*****(T)** and so **a** = $t_0 \in$ ***J*****(T)**. ∀ ***r (x)*** = $r_0 + r_1x + \ldots \in$ **T [[x]]**, 1 - ***r(x) t(x)*** **=** $1 - (r_0 + r_1x + \ldots)(t_0 + t_1x + \ldots) = 1 - (r_0t_0 + (r_0t_1 + r_1t_0)x + \ldots) = 1 - r_0t_0 + [r_0t_1 + r_1t_0 + (r_0t_2 + r_1t_1 + r_2t_0)x + \cdots]$**x.** Since $t_0 \in$ ***J*****(T)**, $1 - r_0t_0$ is invertible and hence ***t(x)*** ∈ ***J*****(T [[x]])**.

iii) Since ***e(x)*** is idempotent, $(e(x))^2 =$ ***e*****(*****x*****)**. Assume that $e_0 = 1$ where ***e(x)*** **=** $e_0 + e_1x + \ldots$. Then $(1 + e_1x + \ldots)(1 + e_1x + \ldots) = (1 + e_1x + \ldots)$. Thus $1\,e_1 + e_1\,1 = e_1$ implies $e_1 = 0$, $1\,e_2 + e_1e_1 + e_2\,1 = e_2$ implies $e_2 = 0$. If we continue in this way, then we have that $e_1, e_2, \ldots, e_{n-1} = 0$, so $e_n = 1\,e_n + e_1e_{n-1} + \ldots + e_{n-1}e_1 + e_n1$ implies $e_n = 0$. Then $e(x) = 1$. This gives a contradiction.

Proposition 4.15. **T** is a clean ring with idempotent not identity if and only if **T [[x]]** is a clean ring with idempotent not identity.

Proof. (⇒): Let $\boldsymbol{t(x) = t_0 + t_1(x) + \ldots \notin J(\mathrm{T}\,[[x]])}$. Then $\boldsymbol{t_0} \notin \boldsymbol{J(\mathrm{T})}$ and hence $\boldsymbol{t_0 = e_0 + u_0}$ with $\boldsymbol{e_0}^{\mathbf{2}} = \boldsymbol{e_0} \neq \mathbf{1}$ and $\boldsymbol{u_0}$ unit. Thus, we have that $\boldsymbol{t(x) = e_0 + (u_0 + t_1 x + \ldots)}$ **By Lemma 4.14,** $\boldsymbol{u_0 + t_1 x + \ldots}$ is unit and $\boldsymbol{e_0}^{\mathbf{2}} = \boldsymbol{e_0} \neq \mathbf{1}$ idempotent. Then **T [[x]]** is a clean ring with idempotent not identity.

(⇐): Let **T [[x]]** be a clean ring with idempotent not identity and $\boldsymbol{t} \notin \boldsymbol{J(\mathrm{T})}$. Then $\boldsymbol{t} \notin \boldsymbol{J(\mathrm{T}\,[[x]])}$, and hence t = $(\boldsymbol{e_0 + e_1 x + \ldots}) + (\boldsymbol{u_0 + u_1 x + \ldots})$ where $\boldsymbol{e(x) = e_0 + e_1 x + \ldots \neq 1}$ idempotent and $\boldsymbol{u(x) = u_0 + u_1 x + \ldots}$ unit. **By Lemma 4.14,** $\boldsymbol{e_0}^{\mathbf{2}} = \boldsymbol{e_0} \neq \mathbf{1}$ and hence $\boldsymbol{t_0 = e_0 + u_0}$ where $\boldsymbol{e_0}^{\mathbf{2}} = \boldsymbol{e_0} \neq \mathbf{1}$ idempotent and $\boldsymbol{u_0}$ is unit. Thus, the proof is completed.

We know that every primitive idempotent is local in a semipotent ring and every orthogonally finite semipotent ring is semiperfect (see Immormino (2013)). Then we can give the following proposition since every clean ring with idempotent not identity is semipotent.

Proposition 4.16. i) Every primitive idempotent in a clean ring with idempotent not identity is local.

ii) Every orthogonally finite clean ring with idempotent not identity is semiperfect.

5. SOME EXAMPLES

Now, we will prove that End ($\mathbf{W_T}$) is a clean ring with idempotent not identity where $\mathbf{W_T}$ is a nonzero vector space over the division ring **T** by using Nicholson and Varadarajan (1998).

Definition 5.1 (Shift operator) Let $\mathbf{W_T}$ be a vector space over the division ring **T** such that $\{y_1, y_2, \ldots\}$ is a basis of $\mathbf{W_T}$. The linear transformation $\boldsymbol{\beta}$**:** $\mathbf{W_T} \longrightarrow \mathbf{W_T}$ given by $\boldsymbol{\beta}\ (\mathbf{y_i}) = \mathbf{y_{i+1}}$, for each i, is called a shift operator relative to the basis $\{ y_1, y_2, \ldots\}$ on $\mathbf{W_T}$.

Lemma 5.2. Every shift operator on $\mathbf{W_T}$ is a clean ring with idempotent not identity in End($\mathbf{W_T}$) where $\mathbf{W_T}$ is a vector space of countably infinite dimension over the division ring **T**.

Proof. Let $\boldsymbol{\beta}$**:** $\mathbf{W_T} \longrightarrow \mathbf{W_T}$ be a shift operator on $\mathbf{W_T}$ where $\{y_1, y_2, \ldots\}$ is a basis of $\mathbf{W_T}$ and $\boldsymbol{\beta}(\ \mathbf{y_i}\ ) = \mathbf{y_{i+1}}$ for each $\mathbf{i} \geq \mathbf{1}$. Our goal is to find an idempotent γ not identity such that β−γ is unit (that is, β−γ is a 1-1 and onto linear transformation on $\mathbf{W_T}$).

Let $\boldsymbol{\gamma}$**:** $\mathbf{W_T} \longrightarrow \mathbf{W_T}$ given by

$\gamma(y_1) = y_1 + y_2$, and then

$\gamma^2(y_1) = \gamma(y_1 + y_2)$

$\gamma\ (y_1) = \gamma(y_1) + \gamma(y_2)$

$0_{W_T} = \gamma\ (y_2)$

and then $\gamma\ (y_2) = 0_{W_T}$.

$\gamma(y_{2k}) = ?$ where $k \geq 1$.

Case 1. Let **k** be even. Then take as

$\boldsymbol{\gamma}(y_{4k}) = y_{4k} + y_2$ where $\mathbf{k \geq 1}$.

Thus, $\gamma^2\ (y_{4k}) = \gamma\ (y_{4k} + y_2\ )$

$\gamma^2(y_{4k}) = \gamma\ (y_{4k}\ ) + \gamma\ (y_2) = \gamma\ (y_{4k}\ ) + 0_{W_T}$

$\gamma^2(y_{4k}) = \gamma(y_{4k})$.

Case 2. Let **k** be odd. Then take as

$\gamma(y_{4k+2}) = y_{4k+2} - y_2 \quad \forall$ k$\geq$ 1.

$\gamma^2(y_{4k+2}) = \gamma(y_{4k+2} - y_2)$

$= \gamma(y_{4k+2}) - \gamma(y_2)$

$= \gamma(y_{4k+2}) - 0_{W_T}$

$\gamma^2(y_{4k+2}) = \gamma(y_{4k+2})$.

By case **1** and case **2** we see that

$\gamma^2(y_{2k}) = \gamma(y_{2k}) \quad \forall \ k \geq 1$.

Now, let $\gamma\,(y_{2k+1}) = y_{2k} + y_{2k+2}$

Let's prove that $\boldsymbol{\gamma^2(y_{2k+1}) = \gamma(y_{2k+1})}$

Case 1. Let k be even. Then

$\gamma(y_{4k+1}) = y_{4k} + y_{4k+2}$

$\gamma^2(y_{4k+1}) = \gamma(y_{4k}) + \gamma(y_{4k+2})$

$= y_{4k} + y_2 + y_{4k+2} - y_2$

$= y_{4k} + y_{4k+2} = \gamma(y_{4k+1})$.

Case 2 let **k** be odd. Then

$\gamma(y_{4k+3}) = y_{4k+2} + y_{4k+4}$

$\gamma^2(y_{4k+3}) = \gamma(y_{4k+2}) + \gamma(y_{4k+4})$

$= y_{4k+2} - y_2 + y_{4k+4} + y_2$

$= y_{4k+2} + y_{4k+4}$

$\boldsymbol{\gamma^2(y_{4k+3}) = \gamma(y_{4k+3})}$

By case **1** and case **2,**

$\gamma^2(\boldsymbol{y_{2k+1}}) = \gamma\,(\boldsymbol{y_{2k+1}})$.

As a result, $\boldsymbol{\gamma^2 = \gamma}$, that is, $\boldsymbol{\gamma}$ is idempotent not identity.

Now, let's prove that $\boldsymbol{\beta-\gamma: W_T \longrightarrow W_T}$ is onto.

$(\beta-\gamma)\,(-y_1) = y_1$,

$(\beta-\gamma)\,(y_2) = y_3$,

$\boldsymbol{\gamma}\,(y_{2k+1}) = y_{2k} + y_{2k+2}$, $\forall$ $\mathbf{k \geq 1}$.

$= y_{2k} + \boldsymbol{\beta\,(y_{2k+1})}$

$\boldsymbol{(\beta-\gamma)\,(-\,y_{2k+1}) = y_{2k}}$

Let' s find that $\boldsymbol{(\beta-\gamma)\,(?) = y_{2k+1}}$, $\forall$ $\mathbf{k \geq 1}$.

Case 1. Let **k** be even.

$(\beta-\gamma)\,(y_{4k}) = y_{4k+1} - y_{4k} - y_2$

$(\beta-\gamma)\,(y_{4k+1}) = y_{4k+2} - y_{4k} - y_{4k+2}$

$(\beta-\gamma)\,(y_3) = y_4 - y_2 - y_4$

Then $\boldsymbol{(\beta-\gamma)\,(y_{4k} - y_{4k+1} - y_3) = y_{4k+1}}$

Case 2. Let **k** be odd.

$(\beta-\gamma)\,(y_{4k+2}) = y_{4k+3} - y_{4k+2} + y_2$

$(\beta-\gamma)\,(y_{4k+3}) = y_{4k+4} - y_{4k+2} - y_{4k+4}$

$(\beta-\gamma)\,(y_3) = y_4 - y_2 - y_4$

Thus, $\boldsymbol{(\beta-\gamma)\,(y_{4k+2} - y_{4k+3} - y_3) = y_{4k+3}}$

Consequently, $\boldsymbol{\beta-\gamma}$ **is onto.**

Finally, we will prove that $\boldsymbol{\beta-\gamma}$ is **1-1**.

Let $a_1\, y_1 + a_2\, y_2 + \ldots + a_n\, y_n \in \ker(\beta-\gamma)$

where $a_1, \ldots, a_n \in T$.

Then $(\beta-\gamma)(a_1 y_1 + a_2 \ y_2 + \ldots + a_n y_n) = 0_{W_T}$

Using the definitions of β and γ, we can find the following:

$(\beta-\gamma)(y_1) = -y_1$,

$(\beta-\gamma)(y_2) = y_3$,

$(\beta-\gamma)(y_{2k+1}) = -y_{2k} \ \forall \ k \geq 1$,

$(\beta-\gamma)(y_{4k}) = y_{4k+1} - y_{4k} - y_2 \quad \forall \ k \geq 1$,

$(\beta-\gamma)(y_{4k+2}) = y_{4k+3} - y_{4k+2} + y_2 \quad \forall \ k \geq 1$.

Using these equalities, we can see that,

Coefficient of y_1: $-a_1 = 0_T$.

Coefficient of y_2: $-a_3 - a_4 + a_6 - a_8 + a_{10} - a_{12} + a_{14} - a_{16} + \ldots = 0_T$.

Coefficient of y_3: $a_2 = 0_T$.

Coefficient of y_{4k}: $-a_{4k+1} - a_{4k} = 0_T$.

Coefficient of y_{4k+1}: $a_{4k} = 0_T$.

Coefficient of y_{4k+2}: $-a_{4k+3} - a_{4k+2} = 0_T$.

Coefficient of y_{4k+3}: $a_{4k+2} = 0_T$.

Therefore, $a_1 = a_2 = 0_T$.

$a_{4k} = 0_T \qquad \forall \ k \geq 1$ (that is, $a_4 = a_8 = \ldots = 0_T$).

$a_{4k+2} = 0_T \qquad \forall \ k \geq 1$ (that is, $a_6 = a_{10} = \ldots = 0_T$).

Then $a_{4k+1} = 0_T \quad \forall \ k \geq 1$ and hence $a_3 = 0_T$ and so $a_{4k+3} = 0_T$.

Therefore, $a_1 y_1 + a_2 \ y_2 + \ldots + a_n y_n = 0_{W_T}$ and hence $\ker(\beta-\gamma) = \{0_{W_T}\}$, and so

β−γ is 1-1. Thus, β−γ is unit where γ is idempotent not identity and then β is a clean element with idempotent not identity.

Lemma 5.3. Let $\boldsymbol{\theta} \in$ **End** ($\mathbf{W_T}$) where $\mathbf{W_T}$ is spanned by $\{y, \theta(y), \theta^2(y), \ldots\}$ for some $\mathbf{y} \in \mathbf{W_T}$ such that $\theta(y) \neq 0_T$. Then $\boldsymbol{\theta}$ is a clean element with idempotent not identity.

Proof. Let $\theta^n(y) \notin \langle y, \theta(y), \ldots, \theta^{n-1}(y) \rangle$, $\forall\ n \geq 1$. Then $\{y, \theta(y), \theta^2(y), \ldots\}$ is linearly independent and hence it is a basis for W_T. Therefore, $\boldsymbol{\theta}$ is a shift operator relative to this basis on $\mathbf{W_T}$. By Lemma **5.2.** $\boldsymbol{\theta}$ is a clean element with idempotent not identity.

Let $\exists\ n \geq 1$, $\theta^n(y) \in \langle y, \theta(y), \ldots, \theta^{n-1}(y) \rangle$. Then we can choose that *n* is the smallest number in this way. Then $\{y, \theta(y), \ldots, \theta^{n-1}(y)\}$ is a basis for W_T. Let n = 1. Then $\theta(y) \in \langle y \rangle$ and so, $\theta(y) = ay$ and it gives that $\theta = aI$ ($a \neq 0$) is unit.

Let $n \geq 2$ and $A = [a_{ij}]$ be the matrix with respect to this basis. Thus, we have that

$$\theta(y) = 0_T\, y + 1_T\, \theta(y) + 0_T\, \theta^2(y) + \ldots + 0_T\, \theta^{n-1}(y)$$

$$\theta(\theta(y)) = \theta^2(y) = 0_T\, y + 0_T\, \theta(y) + 1_T\, \theta^2(y) + \ldots + 0_T\, \theta^{n-1}(y)$$

.

.

.

$$\theta(\theta^{n-2}(y)) = \theta^{n-1}(y) = 0_T\, y + 0_T\, \theta(y) + \ldots + 0_T\, \theta^{n-2}(y) + 1_T\, \theta^{n-1}(y)$$

$$\theta(\theta^{n-1}(y)) = \theta^n(y) = a_{n1}\, y + a_{n2}\theta(y) + \ldots + a_{nn}\theta^{n-1}(y)$$

Therefore, if we take as $0_T = 0$, then we have that

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & a_{n1} \\ 1 & 0 & 0 & \ldots & 0 & a_{n2} \\ 0 & 1 & 0 & \ldots & 0 & a_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & a_{nn} \end{bmatrix}$$

Since T is a clean ring, we have an idempotent **e** and a unit **u** such that $\mathbf{a_{n1}} = \mathbf{e} + \mathbf{u}$. Then we can obtain that

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & e \\ 0 & 0 & 0 & \dots & 0 & e \\ 0 & 0 & 0 & \dots & 0 & e \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & e \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & u \\ 1 & 0 & 0 & \dots & 0 & b_2 \\ 0 & 1 & 0 & \dots & 0 & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & b_n \end{bmatrix}$$

where $\boldsymbol{a_{n2}}$ = **e +b$_2$, …,** $\boldsymbol{a_{nn}}$ **= e + b$_n$,**

The first matrix is an idempotent matrix not identity and the second matrix is a unit element in M_n (T) such that the inverse of second matrix is as in the following:

$$\begin{bmatrix} -b_2u^{-1} & 1 & 0 & \dots & 0 & 0 \\ -b_3u^{-1} & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -b_nu^{-1} & 0 & 0 & \vdots & 0 & 1 \\ u^{-1} & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

Thus, A is a clean element with idempotent not identity and hence, θ is a clean element with idempotent not identity.

Theorem 5.4. Let $\mathbf{W_T} \neq 0$ be a vector space over a division ring **T**. Then **End** (**W$_T$**) is a clean ring with idempotent not identity.

Proof. Let $\boldsymbol{\alpha} \in$ End (**W$_T$**) and $y_0 \in \mathbf{W_T}$ with $\alpha(y_0) \neq 0_T$. If $U_0 = y_0T + \alpha(y_0)T + \alpha^2(y_0)T + \dots = W_T$, then $\boldsymbol{\alpha}$ **= e + u** (where **e** is an idempotent not 1 and **u** is a unit) by Lemma **5.3.**

Let $U_0 = y_0T + \alpha(y_0)T + \boldsymbol{\alpha}^2(y_0)T + \dots \neq W_T$.

Let Y = {(U, δ, π): $U_0 \subseteq U \subseteq W_T$ is α −invariant such that $\alpha|_{\boldsymbol{U}} = \delta + \pi$ where

$\delta \in$ End (U) unit, $\pi^2 = \pi \in$ End (U) and $\delta|_{U_0}$ = u and $\pi|_{U_0}$ = e}.

Let the partial ordering on Y as follow:

(U, $\boldsymbol{\delta}$, $\boldsymbol{\pi}$) ≤ $(\boldsymbol{U'}, \boldsymbol{\delta'}, \boldsymbol{\pi'})$ if **U** ⊆ $\boldsymbol{U'}$, $\boldsymbol{\delta'}|_{\boldsymbol{U}} = \boldsymbol{\delta}$ and $\boldsymbol{\pi'}|_{\boldsymbol{U}} = \boldsymbol{\pi}$.

Let **C** be any chain in **Y**. (Thus, any two elements in **C** are comparable). Let $(\bigcup_{\boldsymbol{U}\in\boldsymbol{C}} \boldsymbol{U}, \beta, \psi)$ be defined as follow: $\forall\, x, y \in \bigcup_{\boldsymbol{U}\in\boldsymbol{C}} \boldsymbol{U}$, $\exists$ $(\mathbf{U_1}, \boldsymbol{\delta_1}, \boldsymbol{\pi_1})$ and $(\mathbf{U_2}, \boldsymbol{\delta_2}, \boldsymbol{\pi_2}) \in$ C such that x $\in U_1$ and y $\in U_2$. Since C is a chain, we can suppose that $(U_1, \delta_1, \pi_1) \leq (\mathbf{U_2}, \delta_2, \pi_2)$ and, we can define β(x) = δ_2(x), β(y) = δ_2(y), ψ(x) = π_2(x), ψ(y) = π_2(y). Then we have β(x + y) = δ_2(x + y) = δ_2(x) + δ_2(y) = β(x)

+ β(y) and ψ(x + y) = π_2(x + y) = π_2(x) + π_2(y) = ψ(x) +ψ(y). Thus, β and ψ are additive homomorphisms. In addition, we can say that β and ψ are linear transformations such that $\alpha|_D$ = β + ψ where β is a unit in **End** $(\boldsymbol{D})$ and $\boldsymbol{\psi^2}$ = $\boldsymbol{\psi}$ ∈ **End** $(\boldsymbol{D})$ and $\beta|_{U_0}$= u and $\psi|_{U_0}$= e ≠ 1 where D = $\bigcup_{U\in C}$ U. Then (D, β, ψ) ∈ Y is an upper bound of C in Y. By Zorn's Lemma, Y has a maximal element. Say (U_1, δ, π) for this maximal element.

It is enough to prove that $\mathbf{U_1} = \mathbf{W_T}$.

Assume that $\mathbf{y} \in \mathbf{W_T} - \mathbf{U_1}$. If $\alpha($ y$) \in U_1$, then $\boldsymbol{\pi'}$**:** $\mathbf{yT \oplus U_1} \longrightarrow \mathbf{yT \oplus U_1}$ with $\boldsymbol{\pi'}(\mathbf{u_1}) = \boldsymbol{\pi}(\mathbf{u_1})$ and $\boldsymbol{\pi'}(\boldsymbol{y}) = \mathbf{y}$ and $\boldsymbol{\delta'}$**:** $\mathbf{yT \oplus U_1} \longrightarrow \mathbf{yT \oplus U_1}$ with $\boldsymbol{\delta'}(\mathbf{u_1}) = \boldsymbol{\delta}(\mathbf{u_1})$ and $\boldsymbol{\delta'}(\boldsymbol{y}) = \boldsymbol{\delta}(\boldsymbol{\alpha}(\boldsymbol{y}))$– **y** where $u_1 \in U_1$. Then π' is idempotent on $\mathbf{yT \oplus U_1}$. Now, let' s prove that $\boldsymbol{\delta'}$is unit.

Let $\boldsymbol{\delta'}(\boldsymbol{yt_1} + \boldsymbol{u_1}) = \mathbf{0}$**.** Then we can write that

$\boldsymbol{\delta'}(\boldsymbol{y})\boldsymbol{t_1}$ + $\boldsymbol{\delta}(\mathbf{u_1})$ = **0**

$\boldsymbol{\delta}(\boldsymbol{\alpha}(\boldsymbol{y}))\boldsymbol{t_1}$ – $\mathbf{y}\boldsymbol{t_1}$ + $\boldsymbol{\delta}(\mathbf{u_1})$ = **0**

$\boldsymbol{\delta}(\boldsymbol{\alpha}(\boldsymbol{y}))\boldsymbol{t_1}$ + $\boldsymbol{\delta}(\mathbf{u_1})$ = $\mathbf{y}\boldsymbol{t_1} \in \mathbf{U_1} \cap \boldsymbol{yT}$ = **0** and so $\mathbf{y} = \mathbf{0}$ **or** $\boldsymbol{t_1} = \mathbf{0}$. Then $\boldsymbol{\delta}(\mathbf{u_1})$ = **0**, so $\mathbf{u_1} = \mathbf{0}$, and $\mathbf{y}\boldsymbol{t_1} + \mathbf{u_1} = \mathbf{0}$. Thus, $\boldsymbol{\delta'}$ is **1-1**.

Now, let' s prove that $\boldsymbol{\delta'}$ is onto. Since $\boldsymbol{\delta'}(\boldsymbol{\alpha}(\boldsymbol{y}) - \mathbf{y}) = \boldsymbol{\delta}(\boldsymbol{\alpha}(\boldsymbol{y}) - \boldsymbol{\delta}(\boldsymbol{\alpha}(\boldsymbol{y}) + \mathbf{y} = \mathbf{y}$. Then $\boldsymbol{\delta'}$ is onto.

Let $\mathbf{W_0} = \mathbf{yT \oplus U_1}$, then $\mathbf{W_0}$ is α −invariant. Thus $(\mathbf{U_1}, \mathbf{U}, \boldsymbol{e}) < (\boldsymbol{W_0}, \boldsymbol{\delta'}, \boldsymbol{\pi'})$ and it gives a contradiction.

Now, let $\boldsymbol{\alpha}(\boldsymbol{y}) \notin \mathbf{U_1}$**. Let** K= y T + α(y) T+ … and $\boldsymbol{W_0} = \mathbf{U_1} + \mathbf{K}$ where $\mathbf{W_0}$ is $\boldsymbol{\alpha}$ −invariant.

Claim: $\boldsymbol{\alpha}$ is a clean element with idempotent not identity in **End** ($\mathbf{W_0}$). Let' s prove this !

We can find that $\mathbf{W_0} = \mathbf{M} \oplus \mathbf{U_1}$ where M is a subspace containing y.

Since $\mathbf{U_1}$ is $\boldsymbol{\alpha}$ −invariant,

$\bar{\alpha}$: $\mathbf{W_0}/\mathbf{U_1} \to \mathbf{W_0}/\mathbf{U_1}$ given by

$\bar{\alpha}(w_0 + \mathrm{U_1}) = \alpha\,(w_0) + \mathrm{U_1}$ is a linear transformation.

Then $\overline{\alpha^n}(w_0 + U_1) = \bar{\alpha}^{n}(w_0 + U_1)\ \forall\ n \geq 1$.

Let $w_0 + U_1 = \overline{w_0}$.

Let θ:W$_0 \xrightarrow[m+u_1 \to m]{}$ M. Then $\theta^2 = \theta \in$ End (W$_0$) where θ(W$_0$) = M and ker $(\theta) = U_1$.

Let θ_0: $\mathrm{W_0}/U_1 \xrightarrow[w_0+U_1 \to \theta(w_0)]{}$ M

Then θ_0 is a 1-1 and onto linear transformation.

Write $\beta = \theta_0 \bar{\alpha} \theta_0^{-1} \in$ End (M).

Then $\forall\ w_0 \in$ W$_0$ where w_0 = m + u_1

$\beta\theta(w_0) = \theta_0\bar{\alpha}\theta_0^{-1}(\theta(w_0))$

$= \theta_0\bar{\alpha}(w_0 + U_1) = \theta_0(\alpha(w_0) + U_1) = \theta\alpha(w_0)$

And hence $\beta\theta(w_0) = \theta\alpha(w_0)\ \forall\ w_0 \in$ W$_0$.

Thus, $\beta\theta$(m) = $\theta\alpha$(m)

$\theta\alpha$(m) = $\beta\theta$(m) = β (m)

and hence $\theta\beta$(m) = $\theta\theta(\alpha$(m)) = $\theta(\alpha$(m))

Since ker $(\theta) = U_1$, α(m) - β (m)$\in U_1$, $\forall$ m $\in$ M.

Since $\{\bar{y}, \bar{\alpha}(\bar{y}), \bar{\alpha}^2(\bar{y}), \ldots\}$ spans $\mathrm{W_0}/U_1$ where $\bar{\alpha}(\bar{y}) \neq 0$,

$\{\theta_0(\bar{y}), \theta_0\bar{\alpha}(\bar{y}), \theta_0\bar{\alpha}^2(\bar{y}), \ldots\} = \{\theta_0(\bar{y}), \beta\theta_0(\bar{y}), \beta^2\,\theta_0(\bar{y}), \ldots\}$ spans M where $\beta\theta_0(\bar{y}) \neq 0$.

Thus, by Lemma **5.3.** $\boldsymbol{\beta = \delta_0 + \pi_0}$ where $\boldsymbol{\pi_0^2 = \pi_0}$ $\in$ **End (M)** and $\boldsymbol{\delta_0}$ $\in$ **End (M)** is a unit.

We have that $\boldsymbol{\alpha|_{U_1} = \delta + \pi}$ where $\boldsymbol{\pi^2 = \pi}$ $\in$ **End** $\boldsymbol{(U_1)}$ and $\boldsymbol{\delta}$ $\in$**End** $\boldsymbol{(U_1)}$ is a unit.

Since $\mathbf{W_0} = \mathbf{M} \oplus \mathbf{U_1}$, let's define

$\varphi(\mathrm{m} + u_1) = \pi_0(\mathrm{m}) + \pi(u_1)$

$\Gamma(\mathrm{m} + u_1) = \delta_0(\mathrm{m}) + (\underbrace{\alpha(\mathrm{m}) - \beta(\mathrm{m})}_{\in U_1} + \delta(u_1))$

Then $\varphi|_{\mathrm{U_1}} = \pi$ and $\Gamma|_{\mathrm{U_1}} = \delta$.

In addition,

$(\varphi + \Gamma)(\mathrm{m} + u_1) = \pi_0(\mathrm{m}) + \delta_0(\mathrm{m}) + \pi(u_1) + \delta(u_1) + \alpha(\mathrm{m}) - \beta(\mathrm{m}) = \beta(\mathrm{m}) + \alpha(u_1) + \alpha(\mathrm{m}) - \beta(\mathrm{m}) = \alpha(\mathrm{m} + u_1) \Rightarrow \alpha = \varphi + \Gamma$.

$\varphi^2(\mathrm{m} + u_1) = \varphi(\pi_0(\mathrm{m}) + \pi(u_1)) = \pi_0^2(\mathrm{m}) + \pi^2(u_1) = \pi_0(\mathrm{m}) + \pi(u_1) = \varphi(\mathrm{m} + u_1)$ and hence

$\varphi^2 = \varphi$.

Now, let' s prove that Γ is 1-1 and onto.

Let $\Gamma(\mathrm{m} + u_1) = 0$.

$\delta_0(\mathrm{m}) + ((\underbrace{\alpha(\mathrm{m}) - \beta(\mathrm{m})}_{\in U_1} + \delta(u_1)) = 0 \Longrightarrow \mathrm{m} = 0$ and $u_1 = 0$. Then Γ is 1-1.

If $\boldsymbol{u'} \in \boldsymbol{U_1}$, then there exists $\boldsymbol{u_0} \in \boldsymbol{U_1}$ such that $\boldsymbol{\delta(u_0) = u' = \Gamma\,(0 + u_0)}$.

$\forall$ m $\in$ M, there exists $\mathrm{m}_1 \in$ M such that $\delta_0(\mathrm{m}_1) = \mathrm{m}$.

Then $\boldsymbol{\alpha(\mathrm{m}_1) - \beta(\mathrm{m}_1) = -\delta(u_1)}$ where $\boldsymbol{u_1 \in U_1}$.

Thus, $\Gamma(\mathrm{m}_1 + u_1) = \delta_0(\mathrm{m}_1) + (\alpha(\mathrm{m}_1) - \beta(\mathrm{m}_1) + \delta(u_1)) = \mathrm{m}$ and so Γ is onto.

Therefore, α is clean in **End** ($\mathbf{W_0}$) where $\mathbf{U_1} < \mathbf{W_0}$ and $\alpha = \varphi + \Gamma$ such that $\varphi^2 = \varphi$ and Γ is a unit and $\varphi|_{\mathrm{U_1}} = \pi$ and $\Gamma|_{\mathrm{U_1}} = \delta$.Then $(\mathrm{U_1}, \delta, \pi) < (\mathrm{W_0}, \Gamma, \varphi) \in$ Y. This gives a contradiction. Because $(\mathbf{U_1}, \boldsymbol{\delta}, \boldsymbol{\pi})$ is maximal in Y in this way. By the contradiction, $\mathbf{U_0} \subseteq \mathbf{U_1} = \mathbf{W_T}$. As a result, **End** ($\mathbf{W_T}$) is clean ring with idempotent not identity.

We give the following theorem using Camillo and Khurana (2001).

Theorem 5.5. Let **T** be a unit regular ring. Then **T** is a clean ring with idempotent not identity.

Proof. Let $\mathbf{0} \neq \mathbf{a} \in \mathbf{T}$. Then $\mathbf{aT}$, $\boldsymbol{r_T}$ **(a)**, $\mathbf{aT} + \boldsymbol{r_T}$ **(a)** and $\mathbf{K} = \mathbf{aT} \cap \boldsymbol{r_T}$ **(a)** are direct summands of $\mathbf{T_T}$. Moreover, we have $\boldsymbol{X} \subseteq \boldsymbol{r_T}$ **(a)** with $\mathbf{aT} + \boldsymbol{r_T}\,\mathbf{(a)} = \mathbf{aT} \oplus \mathbf{X}$. Put $\mathbf{T} = \mathbf{aT} \oplus \mathbf{X} \oplus \mathbf{Y}$ and $\mathbf{aT} = \mathbf{K} \oplus \mathbf{Z}$ where $\mathbf{Y}, \mathbf{Z} \leq \mathbf{T_T}$. Thus, we have that $\mathbf{T} = \mathbf{K} \oplus \mathbf{X} \oplus \mathbf{Y} \oplus \mathbf{Z}$. We have $\mathbf{X} \oplus \mathbf{K} = \boldsymbol{r_T}\,\mathbf{(a)} \cong \boldsymbol{T}/\boldsymbol{aT} \cong \mathbf{X} \oplus \mathbf{Y}$. By Goodearl (1991), $\mathbf{K} \cong \mathbf{Y}$. Let $\boldsymbol{\phi}\mathbf{: K} \longrightarrow \mathbf{Y}$ and $\boldsymbol{h}\mathbf{:}\ \boldsymbol{r_T}\,\mathbf{(a)} \longrightarrow \mathbf{X} \oplus \mathbf{Y}$ be isomorphism with $\mathbf{w} = \boldsymbol{h^{-1}}\mathbf{: X} \oplus \mathbf{Y} \longrightarrow \boldsymbol{r_T}\,\mathbf{(a)}$. We can extend $\boldsymbol{h}$ and **w** to $\mathbf{T_T}$ by definitions $\mathbf{h(y) = y}$, $\mathbf{w\,(z) = h\,(z) = 0}$, $\mathbf{w\,(k)} = \boldsymbol{\phi}\,\mathbf{(k)}$ where $\mathbf{y} \in \mathbf{Y}$, $\mathbf{z} \in \mathbf{Z}$, $\mathbf{k} \in \mathbf{K}$.

Now, we will prove that $\mathbf{hwh = h}$. $\forall\, \mathbf{t} \in \mathbf{T}\ \mathbf{t = k + x + y + z}$ since $\mathbf{T} = \mathbf{K} \oplus \mathbf{X} \oplus \mathbf{Y} \oplus \mathbf{Z}$ and then $\mathbf{(hwh)(t) = (hwh)(\,k + x + y + z) = (hw)\,(h(k)) + (hw)\,(h(x)) + (hw)\,(h(y)) + (hw)\,(h(z)) = h(k) + h(x) + h(y) = h\,(k + x + y + z) = h(t)}$. Then $\mathbf{hwh = h}$, and hence $\mathbf{hwhw = hw}$ and we have that **hw** is idempotent. Then $\mathbf{(hw)\,(1)\,(hw)\,(1) = (hw)((hw)\,(1)) = (hw)\,(1)}$, and hence **(hw) (1)** is idempotent.

Now, we will prove that $\boldsymbol{r_T}\,\mathbf{(a - (hw)\,(1)) = 0}$. Let $\mathbf{t} \in \boldsymbol{r_T}\,\mathbf{(a - (hw)\,(1))}$, then $\mathbf{at - (hw)(t) = 0}$. Then $\mathbf{at = (hw)(t)} \in \mathbf{aT} \cap \mathbf{(X \oplus Y) = 0}$. Since $\mathbf{at = 0}$, $\mathbf{t} \in \boldsymbol{r_T}\,\mathbf{(a)} = \mathbf{X \oplus K}$. So, $\mathbf{t = x + k}$ where $\mathbf{x} \in \mathbf{X}, \mathbf{k} \in \mathbf{K}$. **(hw)(t) = 0 imples that (hw) (x + k) = 0 and hence (hw)(x) +(hw) (k) = 0. Then** $\mathbf{x + h}\,\boldsymbol{\phi}\,\mathbf{(k) = 0} \Longrightarrow \mathbf{x} + \boldsymbol{\phi}\,\mathbf{(k) = 0}$, so $\mathbf{x} = \boldsymbol{\phi}\,\mathbf{(k) = 0}$, and hence $\mathbf{k = 0}$. Then **t = x + k = 0 and hence**, $\boldsymbol{r_T}\,\mathbf{(a - (hw)\,(1)) = 0}$. Since **T** is unit regular, there is $\boldsymbol{u} \in \boldsymbol{U(T)}$ such that $\mathbf{(a - (hw)\,(1))\,u\,(a - (hw)\,(1)) = a - (hw)\,(1)}$. Then $\mathbf{(a - (hw)\,(1))\,[u\,(a - (hw)\,(1)) - 1] = 0}$, so $\mathbf{u\,(a - (hw)\,(1)) = 1}$. Thus, **a – (hw) (1)** is unit.

Assume that $\mathbf{(hw)\,(1) = 1}$. Since $\mathbf{T} = \mathbf{K} \oplus \mathbf{X} \oplus \mathbf{Y} \oplus \mathbf{Z}$, $1 = \mathbf{k + x + y + z}$. $\mathbf{(hw)\,(k + x + y + z) = k + x + y + z} \Longrightarrow \mathbf{h}\,(\boldsymbol{\phi}\,\mathbf{(k)}) + \mathbf{x + y + 0 = k + x + y + z} \Longrightarrow \boldsymbol{\phi}\,\mathbf{(k) + x + y = k + x + y + z}$. Then, $\boldsymbol{\phi}\,\mathbf{(k) = 0}$, $\mathbf{z = 0}$ and hence $\mathbf{k = z = 0}$. Thus, $\mathbf{1_T = x + y}$ and hence $\mathbf{T = X \oplus Y = K \oplus X \oplus Y \oplus Z}$ gives to us $\mathbf{k = z = 0}$, so $\mathbf{aT = 0}$. This gives a contradiction. Thus, $\mathbf{(hw)\,(1) \neq 1}$. Then **a** is a clean element with idempotent not identity.

Corollary 5.6. The following are satisfied:

i) Every semisimple ring is a clean ring with idempotent not identity.

ii) Every semiperfect ring is a clean ring with idempotent not identity.

iii) Every artinian ring is a clean ring with idempotent not identity.

iv) Every finite ring is a clean ring with idempotent not identity.

Proof. i) A division ring is unit regular. If **T** is a semisimple ring, then $\mathbf{T} \cong \prod_{i=1}^{n} \mathbf{M}_{ki}\,(\mathbf{D}_i)$ by Wedderburn – Artin Theorem. If $\mathbf{f} \in \mathbf{M}_{ki}\,(\mathbf{D}_i)$, then it is equivalent to a diagonal matrix with idempotent entries by **Theorem 3.7** in Lok and Shum (2003). Then **f** is unit regular and then **T** is a clean ring with idempotent not identity.

ii) Since T is semiperfect, $T/J(\mathbf{T})$ is semisimple and idempotent lift modulo $J(\mathbf{T})$. By the part (i) and **Lemma 4.13.** we are done.

iii) Since **T** is artinian, $T/J(\mathbf{T})$ is artinian. By Anderson, Fuller (1992), $T/J(\mathbf{T})$ is semisimple and then it is a clean ring with idempotent not identity. Since $J(\mathbf{T})$ **is** nilpotent, idempotents lift modulo $J(\mathbf{T})$. By **Lemma 4.13. T** is a clean ring with idempotent not identity.

iv) Since every finite ring is artinian, by the part (iii) it is a clean ring with idempotent not identity.

6. CONCLUSIONS AND RECOMMENDATIONS

In this thesis, we define clean rings with idempotents not identity which are a special case of clean rings. We will continue to solve when a clean ring is a clean ring with idempotent not identity. We have some important conclusions on these rings as follow:

1. T is a clean ring with idempotent not identity if and only if so is U_n(T).
2. We obtain when the ring of formal upper triangular matrices and a subring of Morita context are clean rings with idempotent not identity.
3. Unit regular rings are clean rings with idempotent not identity.
4. The formal power series ring T[[x]] is a clean ring with idempotent not identity if T is a clean ring with idempotent not identity.
5. Endomorphism rings of a vector space is a clean ring with idempotent not identity.
6. Every semisimple , semiperfect, artinian and finite rings are clean rings with idempotent not identity.

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