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# SPECTRAL PROPERTIES OF IMPULSIVE QUANTUM DIFFERENCE EQUATIONS

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SPECTRAL PROPERTIES OF IMPULSIVE QUANTUM DIFFERENCE EQUATIONS

By Yasir Naser Hasan ALQATRANI

September 2021

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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**Yasir Naser Hasan ALQATRANI**

**ABSTRACT**

SPECTRAL PROPERTIES OF IMPULSIVE QUANTUM DIFFERENCE EQUATIONS

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This thesis consists of five chapters.
The first chapter is devoted to the introduction.
In the second chapter, some well known concepts of functional analysis and spectral theory are mentioned and an example of an eigenvalue problem is presented.
In the third chapter, some generalizations and consequences are given for dynamic equations on time scales containing $q$–difference equations as a special case. Later, the $q$–analogous of the Sturm–Liouville equation is obtained and some spectral properties of the second order $q$–difference operator corresponding to that second–order equation are mentioned.
The fourth chapter consists of two sections. In the first section, the impulsive $q$–difference operator is defined, theory on the eigenvalues and the spectral singularities of this operator is given. In the second section, some special cases are examined, where the impulsive condition at the point $t = 1$ has certain symmetries.
The fifth and last chapter is devoted to the conclusion and recommendations.

**2021, 46 pages**

**Keywords:** $q$–difference equation, Impulsive $q$–difference operator, Jost solution, Eigenvalue, Spectral singularity, Symmetry.

**ÖZET**

İMPALSİF KUANTUM FARK DENKLEMLERİNİN SPEKTRAL ÖZELLİKLERİ

Yasir Naser Hasan ALQATRANI
Matematik, Yüksek Lisans Tezi
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Bu tez beş bölümden oluşmaktadır.
Birinci bölüm giriş kısmına ayrılmıştır.
İkinci bölümde, fonksiyonel analiz ve spektral teoride bilinen bazı temel kavramlardan bahsedilmiş ve bir özdeğer problemi örneği sunulmuştur.
Üçüncü bölümde, özel durumda $q$–fark denklemini de içeren zaman skalasındaki dinamik denklemler için bazı genelleştirmeler ve sonuçlar verilmiştir. Daha sonra, Sturm–Liouville denkleminin $q$–analoğu elde edilmiş ve bu ikinci basamaktan denkleme karşılık gelen $q$–fark operatörünün bazı spektral özelliklerinden bahsedilmiştir.
Dördüncü bölüm, iki kısımdan oluşmaktadır. İlk kısımda, impalsif $q$–fark operatörü tanımlanmış, bu operatörün özdeğerleri ve spektral tekillikleri üzerindeki teori verilmiştir. İkinci kısımda ise, $t = 1$ noktasındaki impalsif koşulun sahip olduğu bazı simetrilere ait özel durumlar incelenmiştir.
Beşinci ve son bölüm ise sonuç ve öneriler için ayrılmıştır.

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**Anahtar Kelimeler:** $q$–fark denklemi, İmpalsif $q$–fark operatörü, Jost çözümü, Özdeğer, Spektral tekillik, Simetri.

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**CONTENTS**

## LIST OF SYMBOLS

| | |
|---|---|
| $\mathbb{R}$ | Set of real numbers |
| $\mathbb{C}$ | Set of complex numbers |
| $\mathbb{Z}$ | Set of integers |
| $\mathbb{N}$ | Set of natural numbers |
| $\mathbb{T}$ | Any time scale |
| $\lambda$ | Spectral parameter |
| $\Delta$ | Delta derivative on time scale |
| $D_q y$ | $q$-derivative of $y$ |
| $\mu(t)$ | Graininess function on time scale |
| $q^{\mathbb{Z}}$ | $\{q^n : \ n \in \mathbb{Z}\}$ |
| $\overline{q^{\mathbb{Z}}}$ | $q^{\mathbb{Z}} \cup \{0\}$ |
| $\mathrm{Re} z$ | Real part of the complex number $z$ |
| $\mathrm{Im} z$ | Imaginary part of the complex number $z$ |
| $\mathbb{C}_{\pm}$ | $\{z \in \mathbb{C} : \pm \mathrm{Im} z > 0\}$ |
| $\overline{\mathbb{C}}_{\pm}$ | $\{z \in \mathbb{C} : \pm \mathrm{Im} z \geq 0\}$ |
| $\sigma(L)$ | Spectrum of the operator $L$ |
| $\sigma_c(L)$ | Continuous spectrum of the operator $L$ |
| $\sigma_d(L)$ | Set of eigenvalues of the operator $L$ |
| $\sigma_{ss}(L)$ | Set of spectral singularities of the operator $L$ |
| $L_p(-\infty, \infty)$ | $\left\{f : \int_{-\infty}^{\infty} \lvert f(x)\rvert^p dx < \infty\right\}$ |
| $\ell_2(q^{\mathbb{Z}})$ | $\left\{y : q^{\mathbb{Z}} \to \mathbb{C} : \sum_{t \in q^{\mathbb{Z}}} \mu(t) \lvert y(t)\rvert^2 < \infty\right\}$ |
| $\Vert f \Vert$ | Norm of the function $f$ |
| $< f, g >$ | Inner product of $f$ and $g$ |
| $D(L)$ | Domain of the operator $L$ |
| $W[y, z]$ | Wronskian of the solutions $y$ and $z$. |

# 1. INTRODUCTION

A differential equation together with the initial conditions or boundary conditions at certain moments is called an initial value problem *(IVP)* or a boundary value problem *(BVP)*, respectively. If all the associated supplementary conditions relate to one independent variable value, the problem is called an *IVP*, if the conditions relate more than one independent variable values, the problem is called a *BVP* (Ross 1989). The investigations of such problems first initiated by Birkoff (Birkoff 1908). In applied mathematics, mathematicians are interested in dealing with practical computational procedures of such problems. For instance, boundary value problems are often needed in calculations made in some fields of mechanics, acoustics, seismic movements of the ground, transmission of sound in layers, vibration of a string, sound waves in a pipe, etc. Among the boundary value problems, Sturm–Liouville problems are the first that comes to mind. The second order differential equation

$$-\left[p(x)y'(x)\right]' + q(x)y(x) = \lambda\rho(x)y(x) \tag{1.1}$$

that puts forth with some boundary conditions at the end points of a certain bounded interval is said to be a Sturm–Liouville problem. Here $p,q$ and $\rho$ are the functions of independent variable and $\lambda$ is a spectral parameter. Sturm–Liouville problems constitute the research fields of spectral theory which is a very important sub–branch of applied mathematics and functional analysis.

The foundations of Sturm–Liouville theory were laid by two French mathematicians Charles Sturm and Joseph Liouville who gave their names to such study. In 1836, in their own article, they studied the boundary value problem

$$-y'' + q(x)y = \lambda y, \quad a \le x \le b \tag{1.2}$$

$$\begin{cases} y(a)\cos\alpha + y'(a)\sin\alpha = 0 \\ y(b)\cos\beta + y'(b)\sin\beta = 0 \end{cases} \tag{1.3}$$

on a finite interval (Sturm 1836, Liouville 1836). However, it originally appeared only to study heat dissipation, Sturm–Liouville theory have been applied various problems encountered in other natural sciences besides physics and mathematics. For example, many problems modeled by the help of ordinary differential equations or partial differential equations can be solved by Fourier method and the eigenvalues and eigenfunctions of this Sturm–Liouville problem are needed to be investigated for the solutions. The term "eigen" was first used by David Hilbert (1862–1943) in his proceedings on integral equations. Hilbert called the non-zero $\lambda$ solutions of the homogeneous part of the equation $(I-\lambda A)x = y$ as "eigenvalues" and concluded that $\lambda$ values correspond to characteristic roots of the matrix $A$. On the other hand, the term "eigenvector", namely "eigenfunction", was first used by Courant and Hilbert in order to explain the finite dimension. In one of his study, John Von Neumann (1903–1957) named the $\lambda$ number in the expression $Rf = \lambda f$ as eigenvalue under the condition $f \neq 0$, and named the function $f$ as eigenfunction. Over the years, the concepts of eigenvalues and eigenfunctions have become widely used.

In quantum mechanics, eigenvalues are known as bound states and bound states of a quantum mechanical system correspond to the energy. As mentioned before, investigations of eigenvalues and eigenfunctions are main subjects of spectral analysis in which "operator theory" is mostly used to do such researches. At first, a compatible operator with the differential equation is determined, then spectral properties of the operator are examined. Operator theory is no doubt a diverse area that has grown out of linear algebra and complex analysis and is often described as the branch of functional analysis that deals with bounded linear operators and their spectral properties. In spectral theory, some of the objectives of operators are to interest with the existence, uniqueness and solutions of boundary value problems, to determine the asymptotic behaviours of the solutions, to investigate the stability of the solutions, to interpret eigenvalues and eigenfunctions (if exist), to examine the distribution of eigenvalues and eigenfunctions to the real or complex plane.

The early works concerned with operator theory were about differential operators, in particular Sturm–Liouville operators. The operator generated by (1.1) together with some boundary conditions is called a Sturm–Liouville operator, which is known as one dimensional Schrödinger operator in literature.

There has been various investigations about Sturm–Liouville operators since the beginning of 18th century. It is worth mentioning here that if the domain region of the operator is finite or the coefficients are continuous, operator is said to be *regular*, if the domain region is infinite or the coefficients are discontinuous functions, operator is said to be *singular*. Moreover, if the potential of the *BVP* is real valued, the operator corresponding to such *BVP* is said to be *selfadjoint*, whereas the operator is said to be *non–selfadjoint* corresponding to the *BVP* having a complex valued potential. Selfadjoint operators are equal to their adjoint operators, while non–selfadjoint operators are not.

Study of the spectral analysis of non–selfadjoint Sturm–Liouville operators was begun by Naimark. Naimark considered the non–selfadjoint singular Sturm–Liouville operator generated by the Sturm–Liouville equation

$$-y'' + q(x)y = \lambda y, \quad x \in \mathbb{R}_+ := [0,\infty)$$

and the initial condition

$$y'(0) - hy(0) = 0$$

in the Hilbert space

$$L_2(0,\infty) := \left\{ f : \int_0^\infty |f(x)|^2 dx < \infty \right\},$$

where $q$ is a complex potential and $\lambda$ is a spectral parameter (Naimark 1960). Naimark proved that the spectrum of non–selfadjoint operator consists of the continuous spectrum, the eigenvalues and the spectral singularities. Keep in mind that, all eigenvalues of non–selfadjoint operators have to be real, but the eigenvalues of non–selfadjoint operators may not be real in general. In non–selfadjoint case, operators have spectral singularities. The spectral singularities are the poles of the kernel of the resolvent operator and are also embedded in the continuous spectrum, but they are not eigenvalues. The spectral analysis of differential operators with spectral singularities was investigated in detail by several mathematicians. Schwartz studied the spectral singularities of a certain class of abstract linear operator in a Hilbert space and provided a new definition to the literature (Schwartz 1960). The sets of the spectral singularities for closed linear operators on a Banach space was given by Nagy (Nagy 1986). Nagy showed that the set of spectral singularities defined according to his general definition coincides in the case of differential operators as defined by Naimark and Lyance (Naimark 1960, Lyance 1967). Lyance considered the effects of spectral singularities in the spectral expansion of Sturm–Liouville operators in terms of the principal functions. Pavlov established the dependence of the structure of the spectral singularities of Sturm–Liouville operators on the behaviour of the potential function at infinity (Pavlov 1967). For some of the other techniques for the differential and abstract operators, we may consult with the studies of Naimark, Gasymov, Maksudov, Allahverdiev and Kemp (Kemp 1958, Naimark 1968, Gasymov 1968, Gasymov and Maksudov 1972, Maksudov and Allahverdiev 1984). Another approach for the discussion of the spectral analysis of the non–selfadjoint differential operators has been given by Marchenko (Marchenko 1963). One of the greatest contribution to the spectral theory of differential equations is that Sturm–Liouville equations have some specific solutions, defined by Marchenko. Let us consider the *IVP* created by the Sturm–Liouville equation

$$-y'' + q(x)y = \lambda^2 y, \quad 0 \le x < \infty \tag{1.4}$$

and the boundary condition

$$y(0) = 0, \tag{1.5}$$

where $q$ is a real valued function and $\lambda \in \mathbb{C}$ is a spectral parameter. The bounded solution of (1.4) satisfying the condition

$$\lim_{x\to\infty} y(x,\lambda)e^{-i\lambda x} = 1, \quad \lambda \in \overline{\mathbb{C}}_+ := \{\lambda : \lambda \in \mathbb{C},\ \mathrm{Im}\lambda \ge 0\}$$

will be denoted by $e(x,\lambda)$. The solution $e(x,\lambda)$ is called Jost solution of (1.4). Under the condition

$$\int_0^\infty x|q(x)|dx < \infty,$$

the Jost solution has the representation

$$e(x,\lambda) = e^{i\lambda x} + \int_x^\infty K(x,t)e^{i\lambda t}dt,$$

where the function $K(x,t)$ is kernel defined by $q$. The function $e(x,\lambda)$ is analytic with respect to $\lambda$ in $\mathbb{C}_+ := \{\lambda : \lambda \in \mathbb{C},\ \mathrm{Im}\lambda > 0\}$, continuous in $\overline{\mathbb{C}}_+$ and the asymptotic equation

$$e(x,\lambda) = e^{i\lambda x}[1+o(1)], \quad \lambda \in \overline{\mathbb{C}}_+, \quad x \to \infty$$

is satisfied (Marchenko 1986). The functions $e(x,\lambda)$ and $e(\lambda) := e(0,\lambda)$ are called Jost solution and Jost function of the *IVP* (1.4)–(1.5), respectively. These functions play an important role in the solution of inverse problems of quantum scattering theory (Chadan and Sabatier 1977, Marchenko 1986, Levitan 1987). Such results of differential operators have been extended to the operators on the whole axis (Blashak 1966, Marchenko 1986). Similar to the half axis, there exist the definitions of Jost solutions on the entire axis as follows.

Consider the differential equation

$$-y'' + q(x)y = \lambda y, \quad x \in \mathbb{R} := (-\infty,\infty) \tag{1.6}$$

in the Hilbert space

$$L_2(-\infty,\infty) := \left\{ f : \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty \right\}.$$

Two bounded solutions of (1.6) fulfilling the conditions

$$\lim_{x\to\infty} y(x,\lambda)e^{-i\lambda x} = 1, \quad \lambda \in \overline{\mathbb{C}}_+$$

and

$$\lim_{x\to-\infty} y(x,\lambda)e^{i\lambda x} = 1, \quad \lambda \in \overline{\mathbb{C}}_+$$

are called the Jost solutions of (1.6) and denoted by $e^+(x,\lambda)$ and $e^-(x,\lambda)$, respectively. Moreover, under the condition

$$\int_{-\infty}^{\infty} |1+x||q(x)|dx < \infty,$$

they satisfy the integral equations

$$e^{+}(x,\lambda)=e^{i\lambda x}+\int\limits_{x}^{\infty}K^{+}(x,t)e^{i\lambda t}dt,\quad \lambda\in\overline{\mathbb{C}}_{+},$$

$$e^{-}(x,\lambda)=e^{-i\lambda x}+\int\limits_{-\infty}^{x}K^{-}(x,t)e^{i\lambda t}dt,\quad \lambda\in\overline{\mathbb{C}}_{+},$$

respectively, where $K^{\mp}(x,t)$ are the kernel functions defined in terms of the potential.

The set of eigenvalues and the set of spectral singularities can be characterized in terms of the zeros of the Jost function so that many crucial properties such as the finiteness and multiplicities of eigenvalues and the spectral singularities can be obtained. Since the Jost solutions are of great importance for the literature, various author have been dealt with these solutions in their studies (Krall 1965, Gasymov and Levitan 1966, Jaulent and Jean 1972, Levitan and Sargsjan 1975, Tunca and Bairamov 1999, etc.).

As a result of developing technology, differential equations have become very insufficient to overcome the requirements of new modellings. So, the theory of difference equations has grown at an accelerated pace in the past decade. For the studies of the spectral and scattering theory of difference equations, we refer to various references (Atkinson 1964, Guseinov 1976, Agarwal and Wong 1997, Krall *et al.* 2001, Bairamov *et al.* 2001, Adıvar and Bairamov 2001, Adıvar and Bairamov 2003).

In addition to differential and difference equations, quantum calculus has led to a rapid development in recent years, which appeared as a connection between mathematics and physics. The study of quantum difference equations has recently attracted interest and some problems of $q$–difference equations have been treated by scientists due to their application in several areas such as combinatorics, orthogonal polynomials, control theories and other fields of quantum theory such as mechanics and the theory of relativity. In the course of developing quantum calculus along the traditional lines of ordinary calculus, Kac and Cheung discovered many important notions and results (Kac and Cheung 2002). Moreover, some useful generalizations and consequences were given for dynamic equations on time scales, which contain $q$–difference equations as a special case (Bohner and Peterson 2001, Bohner and Peterson 2003). Furthermore, scattering and spectral theory of $q$–difference equations having spectral singularities with both exponential type and polynomial type Jost solutions has been investigated so far (Adıvar and Bohner 2006a, Adıvar and Bohner 2006b, Aygar and Bohner 2015, Aygar and Bohner 2016).

In all the problems of differential, difference and $q$–difference equations mentioned above, the processes in which mathematical modelling is done do not always continue uninterruptedly and continuously. While a physical or chemical phenomena takes place, the instantaneous and sharp situation changes may occur at some stages of these processes. When compared with the total process, period of this sharp change is neglectable, however, the functioning of the system changes. These short–term effects are called "impulse (jump)" effects, the problems that occur as a result of impulse effects are called "impulsive problems". Naturally, in an impulsive problem, additional conditions arise in discontinuity points called "impulsive conditions", "transmission conditions" or "point interactions". Impulsive problems can be seen in various scientific facts. To be more clear, many biological phenomena involving thresholds, bursting rhythm models in medicine, optimal control models in economics, pharmacokinetics, and frequency modulated systems do exhibit impulsive effects (Lakshmikantham *et al.* 1989). As a natural response to developing technology, the interest in impulsive equations has grown and impulsive equations have been a subject of both theoretical and experimental researchs. The theory is improved very well by the differential equations (Lakshmikantham *et al.* 1989, Samoilenko and Perestyuk 1995, Bainov and Simeonov 1998) and in difference equations as well (Zhang 2002, He and Zhang 2004, Wang and Wang 2006). Notably, spectral theory of singular and regular Sturm–Liouville operators with impulsive effects has been studied extensively in previous papers (Mukhtarov *et al.* 2004, Mostafazadeh and Dehnavi 2009, Mostafazadeh 2011, Ugurlu and Bairamov 2013).

This master thesis is concerned with the investigation of the impulsive quantum difference operator. In the next section, some well known concepts of spectral analysis are mentioned and required preliminary informations are given. Then, the $q$–difference equation is obtained on the whole axis and related operator is introduced in the Hilbert space $\ell_2(q^{\mathbb{Z}})$. The main results and theorems of this thesis are given at Chapter 4 in which the locations of eigenvalues and spectral singularities of an impulsive operator corresponding to a $q$–difference impulsive equation are investigated. Moreover, some particular occasions, where the point interaction at a single $t = 1$ point possess certain symmetries are examined. Throughout this master thesis, especially in Chapter 4, the paper of Bohner and Cebesoy is used as a main reference and investigated in detail (Bohner and Cebesoy 2019).

## 2. BASIC CONCEPTS AND DEFINITIONS

In this section, we establish some basic properties of spectral theory and recall some definitions from functional analysis. The results of following sections are mostly based on the concepts of this section.

**Definition 2.1.** (Vector space) A vector space (or linear space) over a field $K$ is a nonempty set $X$ of elements $x, y, \ldots$ (called vectors) together with two algebraic operations. First of these operations is vector addition having the following properties for all $x, y, z \in X$:

(i) $x + y \in X$ (Closure property)

(ii) $x + y = y + x$ (Commutative property)

(iii) $x + (y + z) = (x + y) + z$ (Associative property)

(iv) There exists a vector $\theta$, called the zero vector such that $x + \theta = x$ (Identity element property)

(v) There exists a vector $-x$ for every vector $x$ such that $x + (-x) = \theta$ (Inverse element property).

The other operation is multiplication of vectors by scalars having the following properties for all $x, y \in X$ and for all $\alpha, \beta$ scalars:

(i) $\alpha x \in X$ (Closure property)

(ii) $\alpha(\beta x) = (\alpha\beta)x$ (Associative property)

(iii) $\alpha(x + y) = \alpha x + \alpha y$ (Distributive property)

(iv) $(\alpha + \beta)x = \alpha x + \beta x$ (Distributive property)

(v) There exists a vector $1 \in X$ such that $1x = x$ (Identity element property)

(vi) $0x = \theta$, where "0" is the zero scalar (Absorbing element property).

From the definition, we conclude that vector addition is a mapping $X \times X \to X$, whereas multiplication by scalars is a mapping $K \times X \to X$. The vector space $X$ is denoted by $(X, +, .)$ and $K$ is called the scalar field of $X$. Moreover, $X$ is called a real vector space if $K = \mathbb{R}$ (the field of real numbers), and a complex vector space if $K = \mathbb{C}$ (the field of complex numbers).

Throughout this master thesis, we consider an impulsive $q$–difference operator. As known, operators are certain transformations between two vector spaces. But a vector space may not supply the needs of calculations, that's why we need to mention the definitions of linear operator, normed space, Hilbert space and inner product of two functions in the given Hilbert space.

**Definition 2.2.** (Linear operator ) A linear operator $T$ is an operator such that

(i) the domain $D(T)$ of $T$ is a vector space and the range $R(T)$ lies in a vector space over the same field,

(ii) for all $x, y \in D(T)$ and scalars $\alpha, \beta$

$$T(\alpha x+\beta y) = \alpha Tx+\beta Ty$$

satisfies.

Note that, the notation $Tx$ is used instead of $T(x)$ which is a standard simplification in functional analysis (Kreyszig 1978).

**Definition 2.3.** (Normed space ) A norm on a (real or complex) vector space $X$ is a real valued function represented as

$$\|.\| : X \to \mathbb{R},$$

which has the properties

(i) $\| x \| \geqslant 0$

(ii) $\| x \| = 0 \Leftrightarrow x = 0$

(iii) $\| \alpha x \| = | \alpha | \| x \|$

(iv) $\| x+y \| \leqslant \| x \| + \| y \|$ (Triangle inequality),

here $x$ and $y$ are arbitrary vectors in $X$ and $\alpha$ is any scalar.

A normed space $X$ is a vector space with a norm function defined on it and denoted by $(X, \| \, . \, \|)$ (Kreyszig 1978).

**Definition 2.4.** (Inner product space) An inner product on a (real or complex) vector space $X$ is a mapping represented as

$$<,>: X \times X \to K,$$

where $K$ is a scalar field of $X$; that is, with every pair of vectors $x$ and $y$, there is associated a scalar which is written as

$$< x, y >$$

and is called the inner product of $x$ and $y$ such that for all vectors $x, y, z$ and scalars $\alpha$, we have

(i) $< x+y, z > = < x, z > + < y, z >$

(ii) $<\alpha x,y>=\alpha<x,y>$

(iii) $<x,y>=\overline{<y,x>}$

(iv) $<x,x>\geqslant 0 \quad (<x,x>=0 \Leftrightarrow x=0)$.

An inner product space is a vector space $X$ with an inner product on it and denoted by $(X,<,>)$. An inner product on $X$ defines a norm on $X$ given by

$$\| x \|=\sqrt{<x,x>}$$

(Kreyszig 1978).

**Definition 2.5.** The metric space $X$ is said to be *complete* if every Cauchy sequence in $X$ converges. A *Banach* space is a complete normed space (complete in the metric defined by the norm), a *Hilbert* space is a complete inner product space (complete in the metric defined by $\| x \|=\sqrt{<x,x>}$ norm).

The theory of Hilbert spaces is richer than that of general Banach spaces (Kreyszig 1978).

**Definition 2.6.** ($L_p(-\infty,\infty)$ space) The vector space of measurable functions for which the $p$-th power of the absolute value is Lebesgue integrable on the interval $(-\infty,\infty)$ is said to be $L_p$ space and represented as follows:

$$L_p(-\infty,\infty):=\left\{f:\int\limits_{-\infty}^{\infty}|f(x)|^p dx<\infty\right\}.$$

Note that, a $L_p$ space forms a Banach space for $p\geqslant 1$. In particular, for $p=2$, $L_2$ space forms a Hilbert space. Obviously,

$$L_1(-\infty,\infty):=\left\{f:\int\limits_{-\infty}^{\infty}|f(x)|dx<\infty\right\}$$

is a Banach space with the norm

$$\| f \|_1:=\int\limits_{-\infty}^{\infty}|f(x)|dx, \quad f\in L_1(-\infty,\infty).$$

On the other hand,

$$L_2(-\infty,\infty) := \left\{ f : \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty \right\}$$

is a Hilbert space with the norm and inner product

$$\| f \|_2 := \left( \int_{-\infty}^{\infty} |f(x)|^2 dx \right)^{\frac{1}{2}}, \quad f \in L_2(-\infty,\infty), \tag{2.1}$$

$$< f,g > := \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx, \quad f,g \in L_2(-\infty,\infty), \tag{2.2}$$

respectively.

**Example 2.7.** Let $f,g : (-\infty,\infty) \to \mathbb{R}$ are real valued functions such that

$$f(x) = e^{-|x|} \quad \text{and} \quad g(x) = \frac{1}{1+x}.$$

The function $f$ belongs to both $L_1(-\infty,\infty)$ and $L_2(-\infty,\infty)$ spaces, whereas the function $g$ only belongs to $L_2(-\infty,\infty)$, that is, $g \notin L_1(-\infty,\infty)$.

The remainder of this section is devoted to some basic definitions of spectral theory which have certainly contributed significantly to the rapid development of the theory of Sturm–Liouville operators, even $q$–difference operators that we investigate in this master thesis.

**Definition 2.8.** Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ is a linear operator with domain $D(T) \subset X$. If the operator

$$T_\lambda := T - \lambda I,$$

where $\lambda$ is a complex number and $I$ is the identity operator on $D(T)$, has an inverse, we denote it by $R_\lambda(T)$, that is,

$$R_\lambda(T) = T_\lambda^{-1} = (T - \lambda I)^{-1}$$

and call it the resolvent operator of $T$ or simply, the resolvent of $T$ (Lusternik and Sobolev 1974).
Many properties of $T_\lambda$ and $R_\lambda$ depend on $\lambda$ and spectral theory is concerned with those properties such as existence, boundedness, density, etc.

**Definition 2.9.** Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ is a linear operator with domain $D(T) \subset X$. A regular value $\lambda$ of $T$ is a complex number such that $R_\lambda(T)$ exists, bounded and is defined on a set which is dense in $X$. The resolvent set $\rho(T)$ of $T$ is the set of all regular values $\lambda$ of $T$ and represented as follows:

$$\rho(T) := \left\{ \lambda : \lambda \in \mathbb{C}, \quad \begin{array}{l} R_\lambda(T) \text{ exists} \\ R_\lambda(T) \text{ is bounded} \\ \overline{D(R_\lambda(T))} = X \end{array} \right\}$$

(Lusternik and Sobolev 1974).

**Definition 2.10.** The complement of the resolvent set in the complex plane, that is, the set

$$\sigma(T) = \mathbb{C} \setminus \rho(T)$$

is called the spectrum of $T$ and every $\lambda \in \sigma(T)$ number is called a spectral value of $T$ (Lusternik and Sobolev 1974).

**Definition 2.11.** The set of $\lambda$ complex numbers of the spectrum such that $R_\lambda(T)$ does not exist is said to be point spectrum or discrete spectrum and represented as follows:

$$\sigma_d(T) := \{\lambda : \lambda \in \mathbb{C},\ R_\lambda(T) \text{ does not exist}\}$$

(Lusternik and Sobolev 1974).
It is time to mention in passing that the set $\sigma_d(T)$ is called the set of eigenvalues of the operator $T$ and all $\lambda \in \sigma_d(T)$ is called an eigenvalue of $T$.

**Definition 2.12.** The set of $\lambda$ complex numbers of the spectrum such that $R_\lambda(T)$ exists, is defined on a dense set in $X$, but unbounded is said to be the continuous spectrum and represented as follows:

$$\sigma_c(T) := \left\{ \lambda : \lambda \in \mathbb{C}, \quad \begin{array}{l} R_\lambda(T) \text{ exists} \\ R_\lambda(T) \text{ is unbounded} \\ \overline{D(R_\lambda(T))} = X \end{array} \right\}$$

(Lusternik and Sobolev 1974).

**Definition 2.13.** The set of $\lambda$ complex numbers of the spectrum such that $R_\lambda(T)$ exists (and may be bounded or not) but does not satisfy the last condition, that is, the domain of $R_\lambda(T)$ is not dense in $X$ is said to be the residual spectrum and represented as follows:

$$\sigma_r(T) := \left\{ \lambda : \lambda \in \mathbb{C}, \quad \begin{array}{l} R_\lambda(T) \text{ exists} \\ R_\lambda(T) \text{ is bounded} \\ \overline{D(R_\lambda(T))} \neq X \end{array} \right\}$$

(Lusternik and Sobolev 1974).

The sets given in Definition 2.11, Definition 2.12 and Definition 2.13 are pairwise disjoint and union of them generates the spectrum, then we write

$$\sigma(T) = \sigma_d(T) \cup \sigma_c(T) \cup \sigma_r(T).$$

Now, let us introduce eigenvalues and spectral singularities that we frequently use in the main section of this thesis.

**Definition 2.14.** Let $X$ be a vector space and $T : X \to X$ be a linear operator. If the equation

$$Ty = \lambda y, \quad y \in D(T)$$

has a non-zero $y(x,\lambda_0)$ solution for $\lambda = \lambda_0$ complex number, then $\lambda_0$ is said to be an *eigenvalue* of $T$, the function $y(x,\lambda_0)$ is said to be the *eigenfunction* or the *eigenvector* of $T$ corresponding to $\lambda_0$ value (Naimark 1968).

Recall that, the set of eigenvalues is written in Definition 2.11.

**Example 2.15.** In $L_2(0,\pi)$, consider the differential operator $L$ generated by the differential expression

$$\ell(y) := -y'', \qquad 0 \leqslant x \leqslant \pi$$

and the boundary conditions

$$y'(0) = y(\pi) = 0.$$

Obviously,

$$D(L) = \left\{ y : y \in L_2(0,\pi), \quad \begin{array}{l} (i)\, y'' \text{ exists} \\ (ii)\, y'' \in L_2(0,\pi) \\ (iii)\, y'(0) = y(\pi) = 0 \end{array} \right\}$$

denotes the domain of $L$. Let $\lambda$ be an eigenvalue of $L$ and $y$ be an eigenfunction of $L$ corresponding to $\lambda$. Then, we write

$$Ly = \lambda y, \quad y \neq 0, \; y \in D(L)$$

which implies the boundary value problem

$$-y'' = \lambda y \tag{2.3}$$

$$y'(0) = y(\pi) = 0. \tag{2.4}$$

Determining the eigenvalues of the operator $L$ means determining the eigenvalues of the *BVP* (2.3)–(2.4). Hence, we need to find the $\lambda$ values satisfying (2.3)–(2.4). Since the roots of the characteristic equation $r^2 + \lambda = 0$ of (2.3) are $r_{1,2} = \mp i\sqrt{\lambda}$, $y_1(x) = \cos(\sqrt{\lambda}x)$ and $y_2(x) = \sin(\sqrt{\lambda}x)$ are two linearly independent solutions of the homogeneous differential equation (2.3) for $\lambda > 0$. Then, the general solution of (2.3) can be written as

$$y(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x), \quad |c_1| + |c_2| \neq 0. \tag{2.5}$$

Moreover, since

$$y'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x),$$

using the boundary condition $y'(0) = 0$, we have

$$y'(0) = \sqrt{\lambda}c_2 = 0.$$

Since $\lambda > 0$, we get

$$c_2 = 0.$$

Regulating (2.5), we rewrite the general solution as

$$y(x) = c_1 \cos(\sqrt{\lambda} x).$$

Similarly, substituting the other boundary condition $y(\pi) = 0$ into the last equation, we get

$$y(\pi) = c_1 \cos(\sqrt{\lambda}\pi) = 0.$$

Since $c_1 \neq 0$, we conclude that

$$\cos(\sqrt{\lambda}\pi) = 0,$$

which yields

$$\lambda_k = \left(k + \frac{1}{2}\right)^2, \quad k = 0, 1, 2, \ldots$$

On the other hand, in the case of $\lambda = 0$, Equation (2.3) turns into

$$y'' = 0. \tag{2.6}$$

Since the roots of the characteristic equation are repeated real zeros, we can write the general solution of (2.6) as

$$y(x) = c_1 x + c_2, \qquad |c_1| + |c_2| \neq 0.$$

Since

$$y'(x) = c_1,$$

we find

$$y'(0) = c_2 = 0$$

from the boundary condition. Moreover, using the next boundary condition, we see

$$y(\pi) = c_1 = 0,$$

this means that $y = 0$ is the only solution satisfying the boundary condition. It contradicts the definition of eigenfunction given in Definition 2.14. Therefore, $\lambda = 0$ can not be an eigenvalue, $y=0$ can not be an eigenfunction corresponding to, $\lambda=0$. Consequently, the eigenvalues and the eigenfunctions of the operator $L$ can be listed respectively as follows:

$$\lambda_k = \left(k+\frac{1}{2}\right)^2, \quad k=0,1,2,...$$
$$y_k(x) = \cos\left[\left(k+\frac{1}{2}\right)x\right], \quad k=0,1,2,...$$

**Definition 2.16.** A *spectral singularity* is said to be the $\lambda$ complex number belonging continuous spectrum but not an eigenvalue, which is the pole of resolvent operator. The set of spectral singularities of $T$ is denoted by $\sigma_{ss}(T)$.

## 3. QUANTUM DIFFERENCE EQUATIONS AND RELATED OPERATOR

In this section, we first give brief informations on quantum calculus in which we use quantum difference ($q$–difference) equations. Later, we obtain the $q$-analogue of the Sturm–Liouville equation and focus on the spectral properties of the second order $q$–difference operator corresponding to that second order equation. The results given in this section become for us a very effective tool to create the impulsive $q$–difference operator that is used in the main section of this thesis.

### 3.1 Introduction to Quantum Calculus

Suppose that $q > 1$ is a real fixed number and introduce the sets

$$
\begin{aligned}
q^{\mathbb{N}} &:= \{q^n : \ n \in \mathbb{N}\}, \\
q^{-\mathbb{N}} &:= \left\{q^{-n} : \ n \in \mathbb{N}\right\}, \\
q^{\mathbb{Z}} &:= \{q^n : \ n \in \mathbb{Z}\}, \\
\overline{q^{\mathbb{Z}}} &:= q^{\mathbb{Z}} \cup \{0\},
\end{aligned}
$$

where $\mathbb{N}$ and $\mathbb{Z}$ denote the sets of natural numbers and integers, respectively. A $q$–difference equation is an equation that contains $q$–derivatives of a function defined on one of the sets above.

Before giving the definition of $q$–derivative of any function, we first shall mention about dynamic equations on arbitrary time scales which include $q$–difference equations as a special case. In other words, we shall extend the definition of derivative to other larger and more interesting domains.

An arbitrary nonempty closed subset of the real numbers is said to be a "time scale" and denoted by the symbol $\mathbb{T}$. For instance, the real numbers, the integers, the natural numbers, the interval $[-1,1] \cup [2,5]$ and the set $[-1,1] \cup \mathbb{Z}$ are some examples of time scales, whereas the rational numbers, the irrational numbers, the complex numbers or the open intervals of real numbers are not time scales. The calculus of time scales was first introduced by Stefan Hilger in 1988 to generate a new theory that can unify continuous and discrete analysis (Hilger 1988). The general idea of this theory is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale $\mathbb{T}$.

Development of the study of dynamic equations on time scales helps avoid proving results twice, once for differential equations and once for difference equations. By choosing the time scale to be the set of real numbers, the general result yields a result concerning on ordinary differential equations. On the other hand, by choosing the time scale to be the set of integers, the same general result yields a result for difference equations. Similarly, choosing the time scale to be the set $q^{\mathbb{Z}}$, the result for $q$–difference equation is achieved from the same general result for arbitrary $\mathbb{T}$. To sum up, the time scale calculus aims two main features: unification and extension. Indeed, if we denote the delta derivative $f^{\Delta}$ for a function $f$ defined on $\mathbb{T}$, this delta derivative turns into

(i) $f^{\Delta} = f'$, where $f'$ is the usual derivative for $\mathbb{T} = \mathbb{R}$.
(ii) $f^{\Delta} = \Delta f$, where $\Delta f$ is the usual forward difference operator for $\mathbb{T} = \mathbb{Z}$.

Since there are many other time scales than just $\mathbb{R}$ or $\mathbb{Z}$, there are much more derivative definitions.

Now, let us give some well known definitions of time scales theory to understand easy relationships concerning the delta derivative.

**Definition 3.1.** Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : \ s < t\}$$

(Bohner and Peterson 2001).

In Definition 3.1, if $\mathbb{T}$ has a maximum at $t$, then $\sigma(t) = t$, so we arrive at $\inf\emptyset = \sup\mathbb{T}$, where $\emptyset$ denotes the empty set. On the other hand, if $\mathbb{T}$ has a minimum at $t$, then $\rho(t) = t$, so we arrive at $\sup\emptyset = \inf\mathbb{T}$. Furthermore, $t$ is said to be right–scattered if $\sigma(t) > t$ and $t$ is said to be left-scattered if $\rho(t) < t$. Points that are both left and right–scattered are said to be isolated. Also, if $t < \sup\mathbb{T}$ and $\sigma(t) = t$, then $t$ is called right–dense, and if $t > \inf\mathbb{T}$ and $\rho(t) = t$, then $t$ is called left–dense. At last, the function $\mu : \mathbb{T} \to [0,\infty)$ defined as

$$\mu(t) := \sigma(t) - t$$

is said to be the graininess function.

Note that, there exists a set $\mathbb{T}^*$ that should be defined as $\mathbb{T}^* = \mathbb{T} - \{m\}$, if $\mathbb{T}$ has a left-scattered maximum $m$, otherwise defined as $\mathbb{T}^* = \mathbb{T}$. For clarity, we can write

$$\mathbb{T}^* = \begin{cases} \mathbb{T} \backslash (\rho(\sup\mathbb{T}), \sup\mathbb{T}], & \sup\mathbb{T} < \infty \\ \mathbb{T}, & \sup\mathbb{T} = \infty. \end{cases}$$

Finally, if $f : \mathbb{T} \to \mathbb{R}$ is a function, then we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by

$$f^\sigma(t) = f(\sigma(t)), \quad \forall t \in \mathbb{T}.$$

**Example 3.2.** Consider the three cases whenever $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \overline{q^{\mathbb{Z}}}$.

**Case I** If $\mathbb{T} = \mathbb{R}$, then for any $t \in \mathbb{R}$, we find

$$\sigma(t) = \inf\{s \in \mathbb{R} : \ s > t\} = \inf(t,\infty) = t$$

and similarly $\rho(t) = t$. So, every point $t \in \mathbb{R}$ is dense. At last, the graininess function $\mu$ turns out to be $\mu(t) \equiv 0$ for all $t \in \mathbb{T}$.

**Case II** If $\mathbb{T} = \mathbb{Z}$, then for every $t \in \mathbb{Z}$, we find

$$\sigma(t) = \inf\{s \in \mathbb{Z} : \ s > t\} = \inf\{t+1, t+2, \dots\} = t+1$$

and similarly $\rho(t) = t - 1$. So, every point $t \in \mathbb{Z}$ is isolated. Finally, the graininess function $\mu$ in this case turns out to be $\mu(t) \equiv 1$ for all $t \in \mathbb{T}$.

**Case III** If $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, then we have

$$\sigma(t) = \inf\{q^n : \ n \in [m+1,\infty)\} = q^{m+1} = q.q^m = qt.$$

If $t = q^m \in \mathbb{T}$ and obviously $\sigma(0) = 0$. Then, we find $\rho(t) = \dfrac{t}{q}$ for all $t \in \mathbb{T}$.
Finally, we write

$$\mu(t) = \sigma(t) - t = (q-1)t \tag{3.1}$$

for all $t \in \mathbb{T}$.

Now, let us consider a function $f : \mathbb{T} \to \mathbb{R}$ and define the delta derivative of $f$ at a point $t \in \mathbb{T}^*$.

**Definition 3.3.** Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^*$. We call $f^{\Delta}(t)$ the delta derivative of $f$ at $t$ if for any $\varepsilon > 0$, there is a neighbourhood $U$ of $\mathbb{T}$ such that

$$\left| [f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s] \right| \leq \varepsilon \, |\sigma(t) - s|$$

for all $s \in U$ (Bohner and Peterson 2001).

**Example 3.4.** Let $f : \mathbb{T} \to \mathbb{R}$ be a function. If $f(t) = \alpha$ for all $t \in \mathbb{T}$, where $\alpha$ is a constant, then $f^{\Delta}(t) = 0$. If $f(t) = t$ for all $t \in \mathbb{T}$, then $f^{\Delta}(t) = 1$.

**Theorem 3.5.** Suppose that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^*$. Then, we arrive at the followings:

(i) If $f$ is differentiable at $t$, then $f$ is continuous at $t$.

(ii) If $f$ is continuous at $t$ and $t$ is right–scattered, then $f$ is differentiable at $t$ with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) If $t$ is right–dense, then $f$ is differentiable at $t$ if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) If $f$ is differentiable at $t$, then

$$f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t)$$

is satisfied.

(Bohner and Peterson 2001).

**Theorem 3.6.** Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^{*}$. Then, we arrive at the followings:

(i) $f+g : \mathbb{T} \to \mathbb{R}$ is differentiable at $t$ and

$$(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$

(ii) $\alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at $t$, where $\alpha$ is any constant and

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t).$$

(iii) $f.g : \mathbb{T} \to \mathbb{R}$ is differentiable at $t$ and

$$\begin{aligned}(fg)^{\Delta}(t) &= f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) \\ &= f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).\end{aligned}$$

(iv) If $f(t).f(\sigma(t)) \neq 0$, then $\dfrac{1}{f}$ is differentiable at $t$ and

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}.$$

(v) If $g(t).g(\sigma(t)) \neq 0$, then $\dfrac{f}{g}$ is differentiable at $t$ and

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

(Bohner and Peterson 2001).

**Example 3.7.** Consider the time scales in Example 3.2 again.

**Case I** If $\mathbb{T} = \mathbb{R}$, then by Theorem 3.5, $f : \mathbb{R} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ if and only if

$$f'(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists, i.e., if and only if $f$ is differentiable at $t$. In this case, we have

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} = f'(t)$$

by Theorem 3.5(iii).

Note that, last equation gives the same definition with the usual derivative of ordinary differential equations.

**Case II** If $\mathbb{T} = \mathbb{Z}$, then by Theorem 3.5, $f : \mathbb{Z} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{Z}$ with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(t+1) - f(t)}{1} = f(t+1) - f(t) = \Delta f(t),$$

where $\Delta$ is the usual forward difference operator defined by last equation above.

**Case III** If $\mathbb{T} = q^{\mathbb{Z}}$, then 0 is a right–dense minimum and every point in $\mathbb{T}$ is isolated. Then, by Theorem 3.5, for the function $f : \overline{q^{\mathbb{Z}}} \to \mathbb{R}$, we have

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(qt) - f(t)}{(q-1)t}, \quad \forall t \in q^{\mathbb{Z}} \tag{3.2}$$

and

$$f^{\Delta}(0) = \lim_{s \to 0} \frac{f(0) - f(s)}{0 - s} = \lim_{s \to 0} \frac{f(s) - f(0)}{s}$$

provided this limit exists.

In view of the theorems and definitions of time scale calculus given above, we can specialize the delta derivative definition into quantum calculus and give some remarks as follows:

**Definition 3.8.** Let $q > 1$ be a fixed real number and $q^{\mathbb{Z}} = \{\dots, \frac{1}{q^2}, \frac{1}{q}, 1, q, q^2, \dots\}$.

For the function $y : q^{\mathbb{Z}} \to \mathbb{C}$, the $q$–difference operator is introduced as

$$D_q y = \frac{y^\sigma - y}{\mu},$$

where

$$y^\sigma := y \circ \sigma,$$

$$\sigma(t) = qt, \ \ t \in q^{\mathbb{Z}}$$

and

$$\mu(t) = (q-1)t, \ \ t \in q^{\mathbb{Z}}.$$

$D_q$ is said to be $q$–derivative of $y$ (Bohner and Cebesoy 2019).

**Remark 3.9.** For any $f, g : q^{\mathbb{Z}} \to \mathbb{C}$, the $q$–difference operator satisfies the followings.

(i)

$$D_q(\alpha f(t) + \beta g(t)) = \alpha D_q(f(t)) + \beta D_q(g(t)),$$

for any constants $\alpha$ and $\beta$.

(ii)

$$\begin{aligned} D_q(f(t)g(t)) &= f(qt)D_q g(t) + g(t)D_q f(t) \\ &= f(t)D_q g(t) + g(qt)D_q f(t). \end{aligned}$$

(iii)

$$\begin{aligned} D_q\left(\frac{f(t)}{g(t)}\right) &= \frac{g(t)D_q f(t) - f(t)D_q g(t)}{g(t)g(qt)} \\ &= \frac{g(qt)D_q f(t) - f(qt)D_q g(t)}{g(t)g(qt)}. \end{aligned}$$

(Kac and Cheung 2002).

### 3.2 Quantum Difference Operator

In this subsection, we shall get the $q$–analogous of the Sturm–Liouville equation given by (1.6).

Now, let the functions $p$ and $r$ be defined on $q^{\mathbb{Z}}$ with $p(t) \neq 0$ for all $t \in q^{\mathbb{Z}}$. The second order selfadjoint linear homogenous $q$–difference equation is defined to be

$$D_q\left[p\left(\frac{t}{q}\right)D_q y\left(\frac{t}{q}\right)\right]+r(t)y(t)=0, \quad t\in q^{\mathbb{Z}}. \tag{3.3}$$

After making required calculations using Definition 3.8, the Equation (3.3) turns out to be

$$qp\left(\frac{t}{q}\right)y\left(\frac{t}{q}\right)+\left[t^2(q-1)^2r(t)-p(t)-qp\left(\frac{t}{q}\right)\right]y(t)+p(t)y(qt)=0. \tag{3.4}$$

Choosing

$$p(t)=t^2q(q-1)^2\gamma(t) \quad \text{and} \quad r(t)=\beta(t)+q\gamma(t)+\gamma\left(\frac{t}{q}\right)$$

in (3.4), we arrive at the equation

$$q\gamma(t)y(qt)+\beta(t)y(t)+\gamma\left(\frac{t}{q}\right)y\left(\frac{t}{q}\right)=0, \quad t\in q^{\mathbb{Z}}. \tag{3.5}$$

Adıvar and Bohner (2006) considered the $q$–difference expression

$$q\gamma(t)y(qt)+\beta(t)y(t)+\gamma\left(\frac{t}{q}\right)y\left(\frac{t}{q}\right)=\lambda y(t), \quad t\in q^{\mathbb{Z}}, \tag{3.6}$$

where $\lambda=2\sqrt{q}\cos z$ is a spectral parameter and $\gamma(t)\neq 0$ for all $t\in q^{\mathbb{Z}}$. On the other hand, Aygar and Bohner (2016) considered the same equation, i.e. Equation 3.6, by assuming $\lambda=\sqrt{q}\left(z+z^{-1}\right)$. In both cases, the authors investigated the analytic properties and asymptotic behaviours of the Jost solution and obtained the continuous spectrum of the operator. The primary differences between these two studies are that the first one is for nonselfadjoint case and it is on the whole axis. Therefore, there exist spectral singularities and eigenvalues. But in the second paper, the operator generated by (3.6) is selfadjoint and the problem is given on the half axis in which there only exist eigenvalues. Moreover, different spectral parameter selection leads to some structural changes in Jost solutions. More clearly, the Jost solutions are of exponential type in the case that $\lambda=2\sqrt{q}\cos z$, whereas the Jost solution is of polynomial type in the case that $\lambda=\sqrt{q}\left(z+z^{-1}\right)$.

In the light of previous studies on quantum difference operators, in this thesis, we focus on an impulsive operator generated by a simpler $q$–difference equation obtained from (3.4).

By choosing $\lambda=\sqrt{q}\left(z+z^{-1}\right)\in\mathbb{C}$ and putting

$$p(t) := t^2 q(q-1)^2 \quad \text{and} \quad r(t) := q - \lambda + 1,$$

then Equation (3.4) turns into

$$y\left(\frac{t}{q}\right) + qy(qt) = \sqrt{q}\left(z + z^{-1}\right) y(t), \quad t \in q^{\mathbb{Z}}. \tag{3.7}$$

In the next section, we will consider (3.7) together with an impulsive condition and examine some spectral properties of related operator.

# 4. IMPULSIVE QUANTUM DIFFERENCE OPERATOR

Let the space $\ell_2(q^{\mathbb{Z}})$ be a Hilbert space containing of all complex valued functions having the inner product

$$< u, v >_q = \sum_{t\in q^{\mathbb{Z}}} \mu(t)u(t)\overline{v(t)}$$

for all $u, v : q^{\mathbb{Z}} \to \mathbb{C}$ and the norm

$$\| u \|_q := \left( \sum_{t\in q^{\mathbb{Z}}} \mu(t)|u(t)|^2 \right)^{\frac{1}{2}},$$

where $\mu$ is the graininess function as given in (3.1).

Assume that $\mathbf{L}$ denotes the $q$–difference operator in $\ell_2(q^{\mathbb{Z}})$ created by the equation

$$y\left(\frac{t}{q}\right) + qy(qt) = \lambda y(t), \quad t \in q^{\mathbb{Z}} \setminus \left\{\frac{1}{q}, 1, q\right\} \tag{4.1}$$

and the impulsive condition

$$\begin{pmatrix} y(q) \\ y(q^2) \end{pmatrix} = B \begin{pmatrix} y(\frac{1}{q}) \\ y(\frac{1}{q^2}) \end{pmatrix}, \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{C} \tag{4.2}$$

where $ad - bc \neq 0,\ q > 1$ and $\lambda = \sqrt{q}\left(z + z^{-1}\right)$ is a spectral parameter.

Note that, Equation (4.2) is said to be the impulsive condition at $t = 1$, and also the matrix $B$ works out to continue the solution of Equation (4.1) for $t = q^{-n},\ n \in \mathbb{N}$ to $t = q^n,\ n \in \mathbb{N}$.

## 4.1 Eigenvalues and Spectral Singularities

Let $y(t, z)$ be the solution of Equation (4.1) and denote $y_+(t, z)$ and $y_-(t, z)$, respectively, as follows:

$$\begin{cases} y_+(t, z) := y(t, z), & t = q^n,\ n \in \mathbb{N}, \\ y_-(t, z) := y(t, z), & t = q^{-n},\ n \in \mathbb{N}. \end{cases}$$

Throughout this section, let $\lambda = \sqrt{q}\left(z+z^{-1}\right)$ and $z \in \mathbb{C}\setminus\{-1,1\}$. Then it is easy to see that

$$u(t,z) = \frac{z^{\log_q t}}{\sqrt{\mu(t)}}, \qquad v(t,z) = \frac{z^{-\log_q t}}{\sqrt{\mu(t)}}$$

are two linearly independent solutions of Equation (4.1). It is known that the Wronskian of $u$ and $v$ can be written as

$$W[u,v] := \frac{1}{\mu}\det\begin{pmatrix} u & v \\ u^{\sigma} & v^{\sigma} \end{pmatrix} = \frac{uv^{\sigma} - vu^{\sigma}}{\mu} = \frac{\frac{1}{z} - z}{\sqrt{q}\mu^2} \tag{4.3}$$

in quantum difference calculus.

Therefore, the general solution of Equation (4.1) can be expressed by

$$\begin{cases} y_+(t,z) = A_+u(t,z) + B_+v(t,z), & t = q^n,\ n \in \mathbb{N} \\ y_-(t,z) = A_-u(t,z) + B_-v(t,z), & t = q^{-n},\ n \in \mathbb{N}, \end{cases}$$

where $A_{\mp}$ and $B_{\mp}$ are constant coefficients. By the help of impulsive condition (4.2), we have

$$\begin{pmatrix} A_+u(q,z) + B_+v(q,z) \\ A_+u(q^2,z) + B_+v(q^2,z) \end{pmatrix} = B\begin{pmatrix} A_-u(\frac{1}{q},z) + B_-v(\frac{1}{q},z) \\ A_-u(\frac{1}{q^2},z) + B_-v(\frac{1}{q^2},z) \end{pmatrix}.$$

After direct calculations, impulsive condition implies that

$$\begin{pmatrix} A_+\frac{z}{\sqrt{(q-1)q}} + B_+\frac{z^{-1}}{\sqrt{(q-1)q}} \\ A_+\frac{z^2}{\sqrt{(q-1)q^2}} + B_+\frac{z^{-2}}{\sqrt{(q-1)q^2}} \end{pmatrix} = B\begin{pmatrix} A_-\frac{z^{-1}}{\sqrt{\frac{q-1}{q}}} + B_-\frac{z}{\sqrt{\frac{q-1}{q}}} \\ A_-\frac{z^{-2}}{\sqrt{\frac{q-1}{q^2}}} + B_-\frac{z^2}{\sqrt{\frac{q-1}{q^2}}} \end{pmatrix}.$$

By the previous equation, it is clear to arrive at

$$\begin{pmatrix} \frac{z}{\sqrt{(q-1)q}} & \frac{z^{-1}}{\sqrt{(q-1)q}} \\ \frac{z^2}{\sqrt{(q-1)q^2}} & \frac{z^{-2}}{\sqrt{(q-1)q^2}} \end{pmatrix}\begin{pmatrix} A_+ \\ B_+ \end{pmatrix} = B\begin{pmatrix} \frac{\sqrt{q}z^{-1}}{\sqrt{(q-1)}} & \frac{\sqrt{q}z}{\sqrt{(q-1)}} \\ \frac{qz^{-2}}{\sqrt{(q-1)}} & \frac{qz^2}{\sqrt{(q-1)}} \end{pmatrix}\begin{pmatrix} A_- \\ B_- \end{pmatrix}.$$

Then, the last equation turns into

$$\frac{1}{\sqrt{(q-1)q}}\begin{pmatrix} z & z^{-1} \\ \frac{z^2}{\sqrt{q}} & \frac{z^{-2}}{\sqrt{q}} \end{pmatrix}\begin{pmatrix} A_+ \\ B_+ \end{pmatrix} = \sqrt{\frac{q}{q-1}}\,B\begin{pmatrix} z^{-1} & z \\ z^{-2}\sqrt{q} & z^2\sqrt{q} \end{pmatrix}\begin{pmatrix} A_- \\ B_- \end{pmatrix}.$$

Choosing

$$N_1 := \begin{pmatrix} z & z^{-1} \\ \frac{z^2}{\sqrt{q}} & \frac{z^{-2}}{\sqrt{q}} \end{pmatrix}, \quad N_2 := \begin{pmatrix} z^{-1} & z \\ z^{-2}\sqrt{q} & z^2\sqrt{q} \end{pmatrix} \tag{4.4}$$

we have transfer matrix **M** satisfying

$$\begin{pmatrix} A_+ \\ B_+ \end{pmatrix} = \mathbf{M}\begin{pmatrix} A_- \\ B_- \end{pmatrix}, \tag{4.5}$$

where

$$\mathbf{M} := \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} := N_1^{-1} qBN_2. \tag{4.6}$$

Since

$$N_1^{-1} = \frac{\sqrt{q}}{z^{-1}-z}\begin{pmatrix} \frac{z^{-2}}{\sqrt{q}} & -z^{-1} \\ -\frac{z^2}{\sqrt{q}} & z \end{pmatrix},$$

we have the following representation for the matrix **M** using (4.4) and (4.6),

$$\begin{aligned} \mathbf{M} &= \frac{\sqrt{q}}{z^{-1}-z}\,q\begin{pmatrix} \frac{z^{-2}}{\sqrt{q}} & -z^{-1} \\ -\frac{z^2}{\sqrt{q}} & z \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z^{-1} & z \\ z^{-2}\sqrt{q} & z^2\sqrt{q} \end{pmatrix} \\ &= \frac{q\sqrt{q}}{\frac{1}{z}-z}\begin{pmatrix} -\frac{c}{z^2}+\frac{\frac{a}{\sqrt{q}}-d\sqrt{q}}{z^3}+\frac{b}{z^4} & -d\sqrt{q}z+\frac{a}{\sqrt{q}z}+b-c \\ -\frac{az}{\sqrt{q}}+\frac{d\sqrt{q}}{z}-b+c & -bz^4+(d\sqrt{q}-\frac{a}{\sqrt{q}})z^3+cz^2 \end{pmatrix} \end{aligned} \tag{4.7}$$

and thus

$$M_{11}\left(z^{-1}\right) = M_{22}(z), \qquad M_{12}\left(z^{-1}\right) = M_{21}(z).$$

Furthermore, there exist a pair of bounded $y_-^\bullet(t,z)$ and $y_+^\bullet(t,z)$ solutions satisfying the Equation (4.1) in $z \in D_0 \cup D_1$ as $t \to 0^+$ and $t \to +\infty$, respectively, where

$$D_1 := \{z : 0 < |z| < 1\} \qquad \text{and} \qquad D_0 := \{z : |z| = 1\}.$$

These two solutions are known as Jost solutions of Equation (4.1) and in terms of their asymptotics, they satisfy the following conditions.

$$\lim_{t\to 0^+} \frac{y_-^\bullet(t,z)}{v(t,z)} = 1, \quad \lim_{t\to\infty} \frac{y_+^\bullet(t,z)}{u(t,z)} = 1.$$

Let $\psi$ and $\varphi$ be any two solutions of impulsive problem (4.1)–(4.2). Substituting the coefficients $A_\mp$ and $B_\mp$ by $A_\mp^\mp$ and $B_\mp^\mp$, these solutions can be expressed as

$$\psi(t,z) = \begin{cases} A_+^- u(t,z) + B_+^- v(t,z), & t = q^n \\ A_-^- u(t,z) + B_-^- v(t,z), & t = q^{-n} \end{cases} \tag{4.8}$$

and

$$\varphi(t,z) = \begin{cases} A_+^+ u(t,z) + B_+^+ v(t,z), & t = q^n \\ A_-^+ u(t,z) + B_-^+ v(t,z), & t = q^{-n}, \end{cases} \tag{4.9}$$

where $A_\mp^\mp$ and $B_\mp^\mp$ are complex coefficients and $n \in \mathbb{N}$. Let $\psi$ and $\varphi$ be associated with the Jost solutions $y_-^\bullet$ and $y_+^\bullet$, respectively. Then, the asymptotics turn into

$$\psi(t,z) \to v(t,z),\ t \to 0^+ \quad \text{and} \quad \varphi(t,z) \to u(t,z),\ t \to \infty.$$

According to these asymptotics, we get uniquely

$$A_-^- = B_+^+ = 0, \qquad A_+^+ = B_-^- = 1. \tag{4.10}$$

Moreover, for the solution $\psi$, we obtain

$$\begin{pmatrix} A_+^- \\ B_+^- \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

using (4.5) and (4.10). This implies that

$$A_+^- = M_{12}, \qquad B_+^- = M_{22}. \tag{4.11}$$

Similarly, using (4.5) and (4.10) again, we also have

$$A_-^+ = \frac{M_{22}}{\det \mathbf{M}}, \qquad B_-^+ = -\frac{M_{21}}{\det \mathbf{M}} \tag{4.12}$$

for the solution $\varphi$, by solving

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} A_-^+ \\ B_-^+ \end{pmatrix}.$$

Substituting all these 8 coefficients into (4.8) and (4.9), also considering the Jost solutions $y_+^\bullet$ and $y_-^\bullet$ in view of their asymptotics, we arrive at two new $y^r$ and $y^\ell$ solutions, known as *right-going scattering solution* and *left-going scattering solution*, respectively, in quantum theory.

To be more clear, if we insert $A_+^-, A_-^+, B_+^-, B_-^+$ in Equation (4.8), the right-going scattering solution $y^r$ of impulsive boundary value problem (4.1)–(4.2) is obtained providing the asymptotic

$$y^r(t,z) = \begin{cases} M_{12}u(t,z) + M_{22}v(t,z), & t \to +\infty \\ v(t,z) & t \to 0^+. \end{cases} \tag{4.13}$$

In a similar way, if we insert $A_-^+, A_+^+, B_-^+, B_+^+$ in Equation (4.9), the left-going scattering solution $y^\ell$ of impulsive boundary value problem (4.1)–(4.2) is obtained fulfilling the asymptotic

$$y^\ell(t,z) = \begin{cases} u(t,z), & t \to +\infty \\ \dfrac{M_{22}}{\det \mathbf{M}} u(t,z) - \dfrac{M_{21}}{\det \mathbf{M}} v(t,z), & t \to 0^+. \end{cases} \tag{4.14}$$

Physically, the scattering solutions are mostly compared with the Jost solutions. Paying attention, it can be seen that the Jost solutions are proportional to the scattering solutions. In brief, the scattering solutions $y^r$ and $y^\ell$ can be regarded as Jost solutions of the impulsive boundary value problem (4.1)–(4.2). Therefore, the left-going scattering solution $y^\ell$ and the right-going scattering solution $y^r$ become linearly dependent at any spectral singularity point, that is, the Wronskian of them become zero.

Next theorem can be given considering the Wronskian definition (4.3) of Jost solutios of the quantum difference $\mathbf{L}$ operator acting in $\ell_2(q^{\mathbb{Z}})$.

**Theorem 4.1.** The Wronskian of scattering solutions satisfy the following asymptotic equations.

$$W[y^\ell,y^r](z) = \frac{M_{22}}{\det\mathbf{M}}W[u,v](z) = \frac{M_{22}}{\det\mathbf{M}}\frac{\frac{1}{z}-z}{\sqrt{q}\,\mu^2}, \quad t\to 0^+, \tag{4.15}$$

$$W[y^\ell,y^r](z) = M_{22}W[u,v](z) = M_{22}\frac{\frac{1}{z}-z}{\sqrt{q}\,\mu^2}, \quad t\to +\infty. \tag{4.16}$$

*Proof.* Owing to the fact that Wronskian is independent of $t$, the Wronskian of the scattering solutions can be calculated by (4.3), (4.13) and (4.14) for $t\to 0^+$ and $t\to+\infty$.

For $t\to 0^+$, we have

$$\begin{aligned}
W[y^\ell,y^r](z) &= \frac{1}{\mu}\Big(y^\ell(t)y^r(qt)-y^r(t)y^\ell(qt)\Big)\\
&= \frac{1}{\mu}\left\{\left(\frac{M_{22}}{\det\mathbf{M}}u(t,z)-\frac{M_{21}}{\det\mathbf{M}}v(t,z)\right)v(qt,z)\right.\\
&\qquad\left.-v(t,z)\left(\frac{M_{22}}{\det\mathbf{M}}u(qt,z)-\frac{M_{21}}{\det\mathbf{M}}v(qt,z)\right)\right\}\\
&= \frac{1}{(q-1)t}\,\frac{M_{22}}{\det\mathbf{M}}\,\frac{z^{\log_q t}}{\sqrt{(q-1)t}}\,\frac{z^{-\log_q qt}}{\sqrt{(q-1)qt}}\\
&\qquad-\frac{1}{(q-1)t}\,\frac{z^{-\log_q t}}{\sqrt{(q-1)t}}\,\frac{M_{22}}{\det\mathbf{M}}\,\frac{z^{\log_q qt}}{\sqrt{(q-1)qt}}\\
&= \frac{1}{(q-1)t}\,\frac{M_{22}}{\det\mathbf{M}}\left(\frac{z^{-1}}{(q-1)t\sqrt{q}}-\frac{z}{(q-1)t\sqrt{q}}\right)\\
&= \frac{M_{22}}{\det\mathbf{M}}\,\frac{\frac{1}{z}-z}{\mu^2\sqrt{q}}\\
&= \frac{M_{22}}{\det\mathbf{M}}\,W[u,v](z),
\end{aligned}$$

which implies (4.15).

For $t \to +\infty$, we see

$$\begin{aligned}
W[y^\ell, y^r](z) &= \frac{1}{\mu}\Big(y^\ell(t)y^r(qt) - y^r(t)y^\ell(qt)\Big) \\
&= \frac{1}{\mu}\Big[u(t,z)\Big(M_{12}u(qt,z) + M_{22}v(qt,z)\Big) - \Big(M_{12}u(t,z) + M_{22}v(t,z)\Big)u(qt,z)\Big] \\
&= \frac{1}{(q-1)t}\left(\frac{z^{\log_q t}}{\sqrt{(q-1)t}}\, M_{22}\, \frac{z^{-\log_q qt}}{\sqrt{(q-1)qt}} - \frac{z^{\log_q qt}}{\sqrt{(q-1)qt}}\, M_{22}\, \frac{z^{-\log_q t}}{\sqrt{(q-1)t}}\right) \\
&= \frac{1}{(q-1)t}\, M_{22}\left(\frac{z^{-1}}{(q-1)t\sqrt{q}} - \frac{z}{(q-1)t\sqrt{q}}\right) \\
&= M_{22}\frac{\frac{1}{z} - z}{\mu^2\sqrt{q}} \\
&= M_{22}W[u,v](z),
\end{aligned}$$

which also implies (4.16). Then the proof is completed. □

A direct consequence of (4.15) and (4.16) is

$$\det \mathbf{M} = 1 \tag{4.17}$$

that is the transfer matrix $\mathbf{M}$ has unit determinant. The impulsive conditions violating this condition is called *anomalous impulsive conditions*.

Now, the following result can be given using Theorem 4.1 and (4.17).

**Corollary 4.2.** A necessary and sufficient condition to examine the spectral singularities and eigenvalues of the impulsive quantum difference operator $\mathbf{L}$ is to examine the zeros of the function $M_{22}$.

By Corollary 4.2, we need to investigate the zeros of the function $M_{22}$, that is

$$b\sqrt{q}z^2 - (dq - a)z - c\sqrt{q} = 0. \tag{4.18}$$

Indeed, the eigenvalues and spectral singularities of operator $\mathbf{L}$ correspond to $\lambda$ values, where $M_{22}(z) = 0$.

Let $\sigma_d(\mathbf{L})$ and $\sigma_{ss}(\mathbf{L})$ denote the sets of eigenvalues and spectral singularities of impulsive quantum difference operator $\mathbf{L}$, respectively. Therefore, using the well known definitions in spectral theory, we introduce these sets by

$$\sigma_d(\mathbf{L}) = \left\{\lambda \in \mathbb{C} : \lambda = \sqrt{q}\left(z + \frac{1}{z}\right),\, z \in D_1,\ M_{22}(z) = 0\right\} \tag{4.19}$$

and

$$\sigma_{ss}(\mathbf{L}) = \left\{\lambda \in \mathbb{C} : \lambda = \sqrt{q}\left(z + \frac{1}{z}\right),\, z \in D_0,\ M_{22}(z) = 0\right\}. \tag{4.20}$$

At first stage, we aim to examine the zeros of Equation (4.18), for this purpose, we will discuss all the cases and look for the circumstances for the existence of eigenvalues and spectral singularities.

**Case I** Let $b \neq 0$. Under this circumstance, it follows from (4.18) that

$$z_{1,2} = \frac{dq-a}{2b\sqrt{q}} \mp \sqrt{\left(\frac{dq-a}{2b\sqrt{q}}\right)^2 + \left(\frac{c}{b}\right)}.$$

Then, the last equality can be written

$$z_{1,2} = \alpha \mp \sqrt{\alpha^2 + \beta},$$

where

$$\alpha := \frac{dq-a}{2b\sqrt{q}}, \quad \beta := \frac{c}{b}.$$

Thus, eigenvalues exist whenever

$$0 < \left|\alpha \mp \sqrt{\alpha^2+\beta}\right| < 1$$

and spectral singularities appear if

$$\left|\alpha \mp \sqrt{\alpha^2+\beta}\right| = 1.$$

**Case II** Let $b = 0$. In this case, (4.18) gives

$$(a - dq)z - c\sqrt{q} = 0$$

and so there emerge two extra subcases.

**(IIa)** Let $a \neq dq$. Then, an eigenvalue exists whenever

$$\sqrt{q}\,|c| = |a - dq|$$

and a spectral singularity appears if

$$0 < \sqrt{q}\,|c| < |a - dq|.$$

**(IIb)** Let $a = dq$. Then, the condition of the existence of eigenvalues and spectral singularities, namely $M_{22} = 0$, yields that $c = 0$. In this case, (4.7) gives

$$B = d\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{M} = aq\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $\det\mathbf{M} = -a^2q^2$. In particular, $M_{22}$ vanishes identically, $\mathbf{M}$ is independent of $z$, and the impulsive condition is anomalous for $a \neq \mp\frac{i}{q}$.

### 4.2 P, T and PT–Symmetries

Hermitian, in other words, self-adjoint operators have a substantial importance in quantum mechanics, mostly these operators are said to be *symmetric operators*. One of the most important feature of these operators is having a real spectrum without any spectral singularity. Disconcertingly, later on, it was detected that having a real spectrum does not necessarily mean selfadjointness. As a novel example, PT–symmetric operators, a special class of non-Hermitian operators, have real spectrum, however, they are not symmetric.

In this subsection, we investigate the outcomes of applyind P, T and PT–symmetries to the impulsive condition (4.2) pertain to the quantum difference operator **L** and we will particularize situations to have eigenvalues and spectral singularities.

### 4.2.1 P–Symmmetry

**Definition 4.3.** The operator P acting on complex valued functions $y : q^{\mathbb{Z}} \to \mathbb{C}$ defined by

$$(\mathrm{P}y)(t) := y\left(\frac{1}{t}\right), \quad t \in q^{\mathbb{Z}}$$

is called the *parity (reflection) operator*.

**Definition 4.4.** The impulsive condition (4.2) has P–symmetry (or (4.2) is P–invariant) if

$$\mathrm{P}\begin{pmatrix} y(q) \\ y(q^2) \end{pmatrix} = B\,\mathrm{P}\begin{pmatrix} y(\frac{1}{q}) \\ y(\frac{1}{q^2}) \end{pmatrix}. \tag{4.21}$$

**Theorem 4.5.** The necessary and sufficient condition for the impulsive condition (4.2) to have a P–symmetry is that

$$B = B^{-1} \tag{4.22}$$

holds.

*Proof.* By the help of (4.21), definition of parity operator P and (4.2), we obtain

$$\begin{aligned}
\mathrm{P}\begin{pmatrix} y(q) \\ y(q^2) \end{pmatrix} = B\,\mathrm{P}\begin{pmatrix} y(\frac{1}{q}) \\ y(\frac{1}{q^2}) \end{pmatrix} &\Leftrightarrow \begin{pmatrix} y(\frac{1}{q}) \\ y(\frac{1}{q^2}) \end{pmatrix} = B\begin{pmatrix} y(q) \\ y(q^2) \end{pmatrix} \\
&\Leftrightarrow B^{-1}\begin{pmatrix} y(\frac{1}{q}) \\ y(\frac{1}{q^2}) \end{pmatrix} = \begin{pmatrix} y(q) \\ y(q^2) \end{pmatrix} \\
&\Leftrightarrow B^{-1}\begin{pmatrix} y(\frac{1}{q}) \\ y(\frac{1}{q^2}) \end{pmatrix} = B\begin{pmatrix} y(\frac{1}{q}) \\ y(\frac{1}{q^2}) \end{pmatrix} \\
&\Leftrightarrow B^{-1} = B.
\end{aligned}$$

□

Some detailed specifications of having P–symmetry for the impulsive condition (4.2) can be expressed as follows, in terms of the components of B.

**Theorem 4.6.**

(i) Assume $b \neq 0$. Then, the impulsive condition (4.2) has P–symmetry if and only if

$$a^2 + bc = 1 \quad \text{and} \quad a = -d.$$

(ii) Assume $b = c = 0$. Then, the impulsive condition (4.2) has P–symmetry if and only if

$$a^2 = d^2 = 1.$$

(iii) Assume $b = 0$ and $c \neq 0$. Then the impulsive condition (4.2) has P–symmetry if and only if

$$a^2 = d^2 = 1 \quad \text{and} \quad a = -d.$$

*Proof.* Assume that the impulsive condition (4.2) has P–symmetry. In this case (4.22) gives

$$a = \frac{d}{ad - bc}, \quad b = \frac{-b}{ad - bc}, \quad c = \frac{-c}{ad - bc}, \quad d = \frac{a}{ad - bc} \tag{4.23}$$

(i) Let $b \neq 0$. It follows from (4.23) that $ad - bc = -1$. Using the last equality and (4.23), we have $a = -d$. Inserting $a = -d$ in $ad - bc = -1$, we obtain $a^2 + bc = 1$.

(ii) Let $b = c = 0$. In this case, we have

$$a = \frac{d}{ad}, \quad d = \frac{a}{ad}$$

using (4.23) and $ad \neq 0$. Using the last equalities, we obtain $a^2 = d^2 = 1$.

(iii) Let $b = 0$ and $c \neq 0$. By the help of (4.23), we have

$$a = \frac{d}{ad}, \quad c = \frac{-c}{ad}, \quad d = \frac{a}{ad}.$$

Since $c \neq 0$, we obtain $a^2 = d^2 = 1$ and $a = -d$.

□

### 4.2.2 T–Symmetry

**Definition 4.7.** The operator T acting on complex valued functions $y : q^{\mathbb{Z}} \to \mathbb{C}$ defined by

$$(\mathrm{T}y)(t) = y^*(t), \quad t \in q^{\mathbb{Z}}$$

is called the *time-reversal operator*, where "$*$" denotes the complex conjugate of $y$.

**Definition 4.8.** The impulsive condition (4.2) has T–symmetry (or (4.2) is T–invariant) if

$$\mathrm{T}\begin{pmatrix} y(q) \\ y(q^2) \end{pmatrix} = B\,\mathrm{T}\begin{pmatrix} y(\frac{1}{q}) \\ y(\frac{1}{q^2}) \end{pmatrix}. \tag{4.24}$$

**Theorem 4.9.** The necessary and sufficient condition for the impulsive condition (4.2) to have a T–symmetry is that

$$B^* = B$$

holds, that is B is real.

*Proof.* Using (4.24), Definition 4.8 and impulsive condition (4.2), we have

$$\begin{aligned}
\mathrm{T}\begin{pmatrix} y(q) \\ y(q^2) \end{pmatrix} = B\,\mathrm{T}\begin{pmatrix} y(\frac{1}{q}) \\ y(\frac{1}{q^2}) \end{pmatrix} &\Leftrightarrow \begin{pmatrix} y^*(q) \\ y^*(q^2) \end{pmatrix} = B\begin{pmatrix} y^*(\frac{1}{q}) \\ y^*(\frac{1}{q^2}) \end{pmatrix} \\
&\Leftrightarrow \left[B\begin{pmatrix} y(\frac{1}{q}) \\ y(\frac{1}{q^2}) \end{pmatrix}\right]^* = B\begin{pmatrix} y^*(\frac{1}{q}) \\ y^*(\frac{1}{q^2}) \end{pmatrix} \\
&\Leftrightarrow B^*\begin{pmatrix} y^*(\frac{1}{q}) \\ y^*(\frac{1}{q^2}) \end{pmatrix} = B\begin{pmatrix} y^*(\frac{1}{q}) \\ y^*(\frac{1}{q^2}) \end{pmatrix} \\
&\Leftrightarrow B^* = B.
\end{aligned}$$

This implies that the matrix $B$ has to be real, namely $a, b, c, d$ has to be real. $\square$

**Theorem 4.10.** If the impulsive condition (4.2) has T–symmetry, then the situations for the appearence of eigenvalues and spectral singularities can be listed as follows:

(i) Assume $b \neq 0$ and $\alpha^2 > -\beta$. In this case, there exist spectral singularities when

$$\mp(dq - a) = \sqrt{q}(c - b).$$

(ii) Assume $b \neq 0$ and $\alpha^2 < -\beta$. In this case, there exist spectral singularities when $b = -c$ and also there exist eigenvalues when $c > b$ and $a \neq dq$.

(iii) Assume $b = 0$. In this case, there exists an eigenvalue when $-1 < \frac{c\sqrt{q}}{a-dq} < 1$ and $c \neq 0$, and also there exists a spectral singularity whenever $\mp\sqrt{q}\, c = a - dq$.

#### 4.2.3 PT–Symmetry

**Definition 4.11.** The impulsive condition (4.2) has PT–symmetry (or (4.2) is PT–invariant) if

$$\mathrm{PT}\begin{pmatrix} y(q) \\ y(q^2) \end{pmatrix} = B\,\mathrm{PT}\begin{pmatrix} y(\frac{1}{q}) \\ y(\frac{1}{q^2}) \end{pmatrix}. \tag{4.25}$$

**Theorem 4.12.** The necessary and sufficient condition for the impulsive condition (4.2) to have a PT–symmetry is that

$$B^* = B^{-1} \tag{4.26}$$

holds.

*Proof.* Assume that impulsive condition (4.2) has PT–symmetry. Using (4.2), we have

$$\begin{pmatrix} y^*(q) \\ y^*(q^2) \end{pmatrix} = B^* \begin{pmatrix} y^*(\frac{1}{q}) \\ y^*(\frac{1}{q^2}) \end{pmatrix}$$

and it follows from Definition 4.7 that

$$\begin{pmatrix} (\mathrm{T}y)(q) \\ (\mathrm{T}y)(q^2) \end{pmatrix} = B^* \begin{pmatrix} (\mathrm{T}y)(\frac{1}{q}) \\ (\mathrm{T}y)(\frac{1}{q^2}) \end{pmatrix}.$$

Then, Definition 4.3 implies the equation

$$\mathrm{P}\begin{pmatrix}(\mathrm{T}y)(\frac{1}{q})\\(\mathrm{T}y)(\frac{1}{q^2})\end{pmatrix}=B^*\,\mathrm{P}\begin{pmatrix}(\mathrm{T}y)(q)\\(\mathrm{T}y)(q^2)\end{pmatrix}.$$

After direct calculations, the last equation turns into

$$(B^*)^{-1}\,\mathrm{PT}\begin{pmatrix}y(\frac{1}{q})\\y(\frac{1}{q^2})\end{pmatrix}=\mathrm{PT}\begin{pmatrix}y(q)\\y(q^2)\end{pmatrix}$$

and using (4.25), we obtain

$$(B^*)^{-1}\,\mathrm{PT}\begin{pmatrix}y(\frac{1}{q})\\y(\frac{1}{q^2})\end{pmatrix}=B\,\mathrm{PT}\begin{pmatrix}y(\frac{1}{q})\\y(\frac{1}{q^2})\end{pmatrix}.$$

The last equation implies that $(B^*)^{-1}=B$, that is, $B^*=B^{-1}$.
The contrary part of the proof proceeds similarly. □

**Theorem 4.13.** If the impulsive condition (4.2) has PT–symmetry, in terms of the components of $B$, (4.26) is equivalent to

$$a+d=2\mathrm{Re}(a),\quad \mathrm{Re}(b)=0,\quad \mathrm{Re}(c)=0.$$

*Proof.* Supposing that the impulsive condition (4.2) has PT–symmetry, we get $B^*=B^{-1}$ if and only if

$$\begin{pmatrix}a^* & b^*\\ c^* & d^*\end{pmatrix}=\frac{1}{ad-bc}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix},$$

holds, i.e.,

$$ad-bc=1,\quad a^*=d,\quad b^*=-b,\quad c^*=-c.$$

Thus, it follows from the last equalities that

$$\mathrm{Re}(b)=0,\quad \mathrm{Re}(c)=0,\quad a+d=2\mathrm{Re}(a).$$

□

# 5. CONCLUSIONS AND RECOMMENDATION

In the last decade, the equations under impulse effect attract the attention of scientists due to the expanding application area in various disciplines and necessity in modeling real processes, this leads to a requirement of developing the theory. As in other fields of applied mathematics, the theory of impulsive equations has been studied by various mathematicians so far and it has been well developed for differential and difference equations. Even though impulsive equations have been the topic of many research papers in the existing literature, there is still a lack on the investigation of some spectral problems in quantum calculus. Considering this inadequacy, studies on this subject have become substantial for us.

In this master thesis, we study the locations of eigenvalues and spectral singularities of an impulsive $q$–difference operator on the whole axis concerning an interior impulsive point. We point out that if a discontinuity appears in a $q$–difference equation, this may create some structural changes on the solutions. Thus, the locations of eigenvalues and spectral singularities (if exist), depend on the choice of the constants given in impulsive condition. Unlike traditional methods, a transfer matrix is determined first, then some special cases of eigenvalues and spectral singularites are investigated by the help of the zeros of a quadratic polynomial obtained from this matrix. At last, we deduce some consequences about P, T and PT–symmetries which are directly related with mathematical physics.

Throughout this master thesis, the paper (Bohner and Cebesoy 2019) is used as a main reference. Possible studies in the future, the theory may be searched for more general opearators having a compatible impulsive condition, and the number of impulsive points may be increased in order to check the validity of proofs. Morever, by determining a transfer matrix, one can study spectral analysis of a boundary value problem with one interior impulsive point located anywhere on its domain, which is still an uninvestigated problem in the literature.

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