TÜRKİYE
FIRAT UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

SPECTRAL STRUCTURE OF CONFORMABLE DERİVATİVE EİGENVALUE PROBLEMS

Isam Najemadeen ARAB

Master's Thesis

DEPARTMENT OF MATHEMATİCS

JUNE 2020

Türkiye
FIRAT UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

Department of Mathematics

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Title:	Spectral Structure of Conformable Derivative Eigenvalue Problems
Author:	Isam Najemadeen ARAB
Submission Date:	10 June 2020
Defense Date:	26 June 2020

THESİS APPROVAL

This thesis, which was prepared according to the thesis writing rules of the Graduate School of Natural and Applied Sciences, Fırat University, was evaluated by the committee members who have signed the following signatures and was unanimously approved after the defense exam made open to the academic audience.

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DECLARATION

I hereby declare that I wrote this Master's Thesis titled “Spectral Structure of Conformable Derivative Eigenvalue Problems ” in consistent with the thesis writing guide of the Graduate School of Natural and Applied Sciences, Firat University. I also declare that all information in it is correct, that I acted according to scientific ethics in producing and presenting the findings, cited all the references I used, express all institutions or organizations or persons who supported the thesis financially. I have never used the data and information I provide here in order to get a degree in any way.

26 June 2020

Isam Najemadeen ARAB

Preface

First of all, praise and thanks to God, the Almighty for his showers of blessings throughout my thesis study to complete the thesis successfully.

I have to express my deepest gratitude to my dear parents and my whole family for their continuous support, motivation, and encouragement throughout my years of study, and they had a great role and always supported me to continue my studies. I am extremely grateful for their love, prayers caring and sacrifices for educating and preparing me for my future.

I would like to express my deep and sincere gratitude to my thesis supervisor, Prof. Dr. Erdal BAS, for allowing me to do a thesis and for providing valuable advice throughout this thesis. His dynamism, his vision, his sincerity, and his motivation deeply inspired me. Who has always helped me and his door was always open. He taught me the methodology to carry out the thesis and to present the thesis study as clearly as possible. It was a great privilege and an honor to work and study under his leadership. I am extremely grateful for what he has taught me. I also want to thank Dr. Ramazan Özarslan for his continuous help and support. Besides my adviser, I would like to thanks the committee of the mathematics department at Firat University for their assistance and their unique classy treatment with me during my study.

Isam Najemadeen ARAB

ELAZIG, 2020

TABLE OF CONTENTS

ABSTRACT

Spectral Structure of Conformable Derivative Eigenvalue Problems

Isam Najemadeen ARAB

Master's Thesis

FIRAT UNIVERSITY
Graduate School of Natural and Applied Sciences

Department of Mathematics
[Select an Item for Technology Faculty]
June 2020, Page: ix + 52

The thesis consists of six chapters.

In the first chapter, a general history of fractional calculations and conformable derivatives are given. Different between classical and conformable derivatives is explained.

In the second chapter, fundamental definitions and theorems are given. Fractional power series and conformable Laplace transform are given.

In the third chapter, the spectral theory of the conformable Sturm-Liouville problem with normal boundary conditions is analyzed.

The fourth and fifth chapters form the original chapters of the thesis.

In the fourth chapter, by using the Frobenius method, solutions of the conformable Sturm-Liouville equation having modified Coulomb and hydrogen atom potential are obtained. Additionally, these solutions are analyzed with graphics.

In the fifth chapter, the fundamental spectral theory of conformable Sturm-Liouville problem with Coulomb potential is proven. The conformable Lagrange identity theorem is proven. The eigenfunctions corresponding to distinct eigenvalues are α-orthogonal and the representation of the solution are obtained.

In the sixth chapter, results obtained from thesis are analyzed in detail.

Keywords: Conformable calculus, Sturm-Liouville eigenvalue problem, Spectral theory, Coulomb potential, Singular.

ÖZET

Uyumlu Türevli Özdeğer Problemlerinin Spektral Yapısı

Isam Najemadeen ARAB

Yüksek Lisans Tezi

FIRAT ÜNİVERSİTESİ
Fen Bilimleri Enstitüsü

Matematik Anabilim Dalı
[Program Seçiniz]
Haziran 2020, Sayfa: ix + 52

Tez altı bölümden oluşmaktadır.

İlk bölümde, kesirli hesaplamalar ve uyumlu türev hakkında genel tarihçe verilmiştir. Klasik ile uyumlu türev arasındaki fark açıklanmıştır.

İkinci bölümde, temel tanım ve teoremler verilmiştir. Kesirli kuvvet serileri ve uyumlu Laplace dönüşümleri verilmiştir.

Üçüncü bölümde, uyumlu türevli Sturm-Liouville problemi için normal sınır şartları ile spektral teorisi incelenmiştir.

Dördüncü ve beşinci bölümler tezin orijinal kısmını oluşturmaktadır.

Dördüncü bölümde, uyumlu türevli singüler Sturm-Liouville problemi için Frobenius metodu kullanılarak çözümler elde edilmiştir. Bu çözümlerin grafiklerle analizi yapılmıştır.

Beşinci bölümde, Coulomb potensiyelli conformable Sturm-Liouville problemi için temel spektral teori ispatlanmıştır. Uyumlu Lagrange özdeşliği teoremi ispatlanmıştır. Farklı özdeğerlere karşılık gelen özfonksiyonların α-ortogonal olduğu ve çözümlerinin görüntüsü elde edilmiştir.

Altıncı bölümde, tezden elde edilen sonuçlar detaylı bir şekilde analiz edilmiştir.

Anahtar Kelimeler: Uyumlu hesap, Sturm-Liouville özdeğer problemi, Spektral teori, Coulomb potansiyel, Tekil.

LIST OF FIGURES

Page

SYMBOLS

$\frac{d^n}{dx^n}$	: A derivative of order n concerning x.
D_a^α	: The fractional derivative operator with order α.
${}^cD_a^\alpha$	: Fractional derivative operator for Caputo definition with order α.
T_α^a	: Left conformable derivative with order α.
${}_\alpha^bT$	: Right conformable derivative with order α.
$\mathbb{R}$	: Set of real numbers.
$\mathbb{N}$	: Set of natural numbers.
I_α^a	: Left conformable integral with order α.
${}_\alpha^bI$	: Right conformable integral with order α.
Γ	: Gamma function which is defined in [1].
$L_\alpha^{t_0}$	: Conformable Laplace transform of order α, where $t_0 \in \mathbb{R}$.
$\mathcal{L}$	: Normal Laplace transform.
λ	: Lambda.
$\Longrightarrow$	: Implies that.

1. INTRODUCTION

Fractional calculus is old as the ordinary calculus. In 1695 L'Hospital asked if we have $n = \frac{1}{2}$, so what does it mean of $\frac{d^n f}{dx^n}$. From that time on, it attracted many types of research [3–6]. The fractional name was given to express differentiation and integration in an arbitrary order. The impact of this fractional calculation in the branches of pure applied science and engineering has started to increase considerably in the last time [5–9]. During the last and present centuries many scientists have attempted to find a definition of the fractional derivative and fractional integral, by using their notations and methodology, they found appropriate definitions for the notion of integral or derivative with non-integer order. Many definitions have been popularized around the world of fractional calculus like Riemann-Liouville, Caputo, Weyl, Fourier, Abel, Leibniz, Grünwald, and Letnikov definitions [10–14]. Furthermore, the most popular definitions that observed are

1) *Riemann-Liouville definition*. Here it is to iterate the integral concerning to the certain weight function and simply replace iterated integral with single integral direct Leibniz-Cauchy formula and replace fractional function to the Gamma function. Also, the arbitrary order Riemann-Liouville outcomes from the integration of the metric dt. For $\alpha \in [n-1, n), n \in \mathbb{N}$, the derivative of the function f with order α on the Riemann-Liouville definition is given by

$$(D_a^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(t)}{(t-x)^{\alpha-n+1}} dx, \tag{1.1}$$

which D_a^α is an operator of the left fractional derivative with order is not integer number, also $\frac{d^n}{dt^n}$ is an operator of the derivative with integer number.

2) *Caputo definition*. This definition utilized an integral form of the fractional derivative. Let $\alpha \in [n-1, n), n \in \mathbb{N}$, then the fractional derivative of f is defined as

$${}^cD_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{(f)^n(t)}{(t-x)^{\alpha-n+1}} dx. \tag{1.2}$$

3) *Grünwald –Letnikov definition*. In this law, the limit definition of the derivative should be iterated to obtain a quantity with a certain binomial coefficient and then fractionalize by utilizing the Gamma function rather than the fractional in the binomial coefficient. Fractional derivative by Grunwald-Letnikov is defined as

$$(D_a^\alpha f)(t) = \lim_{h\to 0} h^{-\alpha} \sum_{j\to 0}^{\frac{x-a}{h}} (-1)^j \binom{\alpha}{j} f(x - jh). \tag{1.3}$$

The acquired fractional derivatives in this analytic appeared to be confounded and lost a potation of the essential properties that standard derivatives have, for example, the Riemann-Liouville derivative does not fulfill $D_a^\alpha(1) = 0$ whereas ${}^cD_a^\alpha(1) = 0$ for Caputo derivative, where α is not an integer number [12,14–18]. Additionally, in these definitions, all fractional derivatives do not fulfill the following residences as in the usual derivative

- $\mathrm{D}_\alpha^\alpha(\mathrm{fg}) = \mathrm{fD}_\alpha^\mathrm{a}(\mathrm{g}) + \mathrm{gD}_\alpha^\mathrm{a}(\mathrm{f})$. The recognized formula of derivative of the product of two functions.
- $D_\alpha^a\left(\frac{f}{g}\right) = \frac{gD_\alpha^a(f) - fD_\alpha^a(g)}{g^2}$. The recognized method of derivative of the quotient for two functions.
- $D_\alpha^a(fog) = f^{(\alpha)}(g(t))g^{(\alpha)}(t)$. chain rule.
- $D^\alpha D^\beta f = D^{\alpha+\beta} f$. In general.
- Let f be a given function, then the Caputo definition assumes f is differentiable.

In recent times, Khalil and associates introduced a new definition of non-integer number order derivative which is a very simple called conformable derivative. This new basic definition simply takes a limit method as the normal derivatives such that satisfy all the previous properties. The conformable integral of order $1 < \alpha \leq 1$ only, is defined by Khalil at al. They also really tested and proved the conformable Rolle's theorem and mean value theorem, additionally, if the conformable exponential function $e^{\frac{t^\alpha}{\alpha}}$ played an important rule, then some conformable differential equations can be solved.

Levitan et al., in [19] give some important definitions and theorems on the spectral theory. Depending on that definitions and theorems, the process in this study gives out of conformable Sturm-Liouville equation and some applications on this equation.

2. BASIC DEFINITIONS AND THEOREMS

2.1. Conformable derivative

Definition 2.1.1. [20] The left conformable derivative of the function $f: (0, \infty) \to \mathbb{R}$ of order α, where $0 < \alpha \leq 1$ is defined as follows

$$(T_\alpha^a f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t+\varepsilon(t-a)^{1-\alpha}\right)-f(t)}{\varepsilon}, \tag{2.1}$$

for every $t > a$. If $a = 0$, then we write T_α. From this definition if $(T_\alpha^a f)(t)$ exists on the interval (a, b), then we can say that

$$(T_\alpha^a f)(a) = \lim_{t \to a^+} (T_\alpha^a f)(t). \tag{2.2}$$

Note: From [21] suppose that f be a differentiable function and $0 < \alpha \leq 1$ then we can say that

$$(T_\alpha^a f)(t) = (t-a)^{1-\alpha} f'(t). \tag{2.3}$$

Definition 2.1.2. [20] The right conformable derivative of the function $f: (0, \infty) \to \mathbb{R}$ of order α, where $0 < \alpha \leq 1$ is defined as follows

$$\left({}_\alpha^b T f\right)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t+\varepsilon(b-t)^{1-\alpha}\right)-f(t)}{\varepsilon}, \tag{2.4}$$

for every $t < b$. If $b = 0$. From this definition if $({}_\alpha^b T f)(t)$ exists on the interval (a, b), then we can say that

$$({}_\alpha^b T f)(b) = \lim_{t \to b^-} ({}_\alpha^b T f)(t). \tag{2.5}$$

Note that: [21] We can define the right conformable derivative as

$$\left({}_\alpha^b T f\right)(t) = -(b-t)^{1-\alpha} f'(t), \tag{2.6}$$

when $0 < \alpha \leq 1$ and f is differentiable. Suppose that f be a given function and conformable derivative of f is exists, then f is said to be α-differentiable function [22].

Let $0 < \alpha \leq 1$, f and g are two α-differentiable functions at $t > 0$, then the definition of conformable derivative satisfies all the following properties

- $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, for every $a, b \in \mathbb{R}$.
- $T_\alpha(fg) = gT_\alpha(f) + fT_\alpha(g)$.

- $T_\alpha(t^p) = pt^{p-\alpha}$, for every $p \epsilon \mathbb{R}$.
- $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.
- $T_\alpha(\lambda) = 0$, for every constant function $f(t) = \lambda$.

The conformable derivative of some functions

1) $T_\alpha(t^p) = pt^{p-\alpha}$, for every $p \epsilon \mathbb{R}$.
2) $T_\alpha(1) = 0$.
3) $T_\alpha(e^{cx}) = cx^{1-\alpha}e^{cx}$, where $c \epsilon \mathbb{R}$.
4) $T_\alpha(sinbx) = bx^{1-\alpha}cosbx$, $b \epsilon \mathbb{R}$.
5) $T_\alpha(cosbx) = -bx^{1-\alpha}sinbx$, $b \epsilon \mathbb{R}$.

Notwithstanding this, we noted the following conformable derivative of specific functions

I. $T_\alpha\left(sin\frac{1}{\alpha}t^\alpha\right) = cos\frac{1}{\alpha}t^\alpha$.

II. $T_\alpha\left(cos\frac{1}{\alpha}t^\alpha\right) = -sin\frac{1}{\alpha}t^\alpha$.

III. $T_\alpha\left(e^{\frac{1}{\alpha}t^\alpha}\right) = e^{\frac{1}{\alpha}t^\alpha}$.

Definition 2.1.3. [22] Let f be an n-differentiable at t, where $t > 0$. Then the conformable derivative of f with order α where $\alpha \epsilon (n, n+1], n \in \mathbb{N}$ is given by

$$(T_\alpha f)(t) = \lim_{\varepsilon \to 0} \frac{f^{(\lceil\alpha\rceil-1)}(t+\varepsilon t^{\lceil\alpha\rceil-\alpha}) - f^{(\lceil\alpha\rceil-1)}(t)}{\varepsilon}, \tag{2.7}$$

where $\lceil\alpha\rceil$ is the smallest integer number that greater than or equal to the number α. Also, we can easily show that

$$(T_\alpha f)(t) = t^{(\lceil\alpha\rceil-\alpha)} f^{\lceil\alpha\rceil}(t), \tag{2.8}$$

where $\alpha \epsilon (n, n+1], n$ is a positive integer, and we say f is $(n+1)$-differentiable at non-negative number t.

Definition 2.1.4 [21] Let $f: (0, \infty) \to \mathbb{R}$ be a given function and $f^{(n)}(t)$ exists, then the left conformable derivative starting from a of the function f of order $\alpha \epsilon (n, n+1], n \in \mathbb{N}$ where $\beta = \alpha - n$ is defined as

$$(\boldsymbol{T}_\alpha^a f)(t) = (T_\beta^a f^{(n)})(t), \tag{2.9}$$

we write T_α and T_β when $a = 0$.

Definition 2.1.5. [21] The right conformable derivative ended at b of the function $f:(0,\infty)\to\mathbb{R}$ with order α where $\alpha \epsilon (n, n+1], n \in \mathbb{N}$ and $\beta = \alpha - n$ is given by

$$\left({}_{\alpha}^{b}\boldsymbol{T}f\right)(t) = (-1)^{n+1}\left({}_{\beta}^{b}Tf^{n}\right)(t). \tag{2.10}$$

Note that if $\alpha = n+1$ then $\beta = \alpha - n = n + 1 - n = 1$ and the conformable derivative will become $f^{(n+1)}(t)$. Also, the Definition 2.1.4 and Definition 2.1.5 are respectively the same as Definition 2.1.1 and Definition 2.1.2 when $n = 0$ or $\alpha \epsilon (0,1)$.

Theorem 2.1.6. (Rolle's theorem for Conformable differentiable functions). Let $a > 0$, if the function $f:(0,\infty)\to\mathbb{R}$ satisfies

i. f is continuous on the interval (a, b).
ii. f is α-differentiable for some $\alpha \in (0,1)$.
iii. $f(a) = f(b)$.

Then, there exist $c \epsilon (a, b)$, such that $(T_\alpha f)(c) = 0$.

Proof. By conditions (i) and (iii) we have $c \epsilon (a, b)$, which is extrema point. Without loss generality, suppose that c is a local minimum point. So

$$(T_\alpha f)(c) = \lim_{\varepsilon\to 0^+}\frac{f\left(c+\varepsilon c^{1-\alpha}\right)-f(c)}{\varepsilon} = \lim_{\varepsilon\to 0^-}\frac{f\left(c+\varepsilon c^{1-\alpha}\right)-f(c)}{\varepsilon}, \tag{2.11}$$

hence $(T_\alpha f)(c) = 0$ because the limit is non-negative when $\varepsilon \to 0^+$ and the limit is non-positive when $\varepsilon \to 0^-$.

Theorem 2.1.7 (Mean value theorem for Conformable differentiable functions). Let $a > 0$ and the function $f:(0,\infty)\to\mathbb{R}$ satisfies

i. f is continuous on the interval (a, b).
ii. f is α $-$differentiable for some $\alpha \in (0,1)$.

Then, there exist $c \epsilon (a, b)$, such that $(T_\alpha f)(c) = \frac{f(b)-f(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}$.

Proof. Take the function

$$g(c) = f(x) - f(\alpha) - \frac{f(b)-f(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}\left(\frac{1}{\alpha}x^\alpha - \frac{1}{\alpha}a^\alpha\right), \tag{2.12}$$

$$(T_\alpha g)(x) = (T_\alpha f)(x) - 0 - \frac{f(b)-f(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}, \tag{2.13}$$

since the function g satisfies the condition of Rolle's theorem, then there exists $c \epsilon (a, b)$ such that

$(T_\alpha g)(c) = 0,$

so that

$$(T_\alpha f)(c) = \frac{f(b)-f(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}. \tag{2.14}$$

We know that if the function f is differentiable and $\alpha\epsilon(0,1)$, then $(T_\alpha^a f)(t) = (t-a)^{1-\alpha} f'(x)$ and $\left({}_\alpha^b T f\right)(t) = -(b-t)^{1-\alpha} f'(t)$. It is obvious, the conformable derivative of the constant function is zero. Conversely, if $(T_\alpha^a f)(t) = 0$ on the interval (a,b) then by the mean value theorem of conformable differentiable function for all $t \,\epsilon\, (a,b)$ we can easily show that $f(t) = 0$. Also, by the help of a conformable mean value theorem there we can prove that if the conformable derivative of the function f on (a,b) is negative (positive) then the graph of f is decreasing (increasing) there [21].

Lemma 2.1.8. [21] Assume that $f, h: [a,\infty) \to \mathbb{R}$ be functions such that T_α^a is exists for all $t > a$. If f is a differentiable function on the interval (a,∞) and $T_\alpha^a f(t) = (t-a)^{1-\alpha} h(t)$, so that $h(t) = f'(t)$ for all $t > 0$.

The proof is based on the definition and putting in place $h = \varepsilon(t-a)^{1-\alpha}$ then $h \to 0$ as $\varepsilon \to 0$. We can state this lemma as a result.

Theorem 2.1.9. (Chain Rule). Suppose $0 < \alpha \le 1$ and $f, g: [a,\infty) \to \mathbb{R}$ be (left) α-differentiable functions. If we let $h(t) = f\big(g(t)\big)$. So $h(t)$ will be (left) α-differentiable function and for all t when $t \neq 0$ and $g(t) \neq 0$ we have

$$(T_\alpha^a h)(t) = (T_\alpha^a f)\big(g(t)\big) \cdot (T_\alpha^a g)(t) g(t)^{\alpha-1}, \tag{2.15}$$

if $t = a$ then we have

$$(T_\alpha^a h)(a) = \lim_{t \to a^+} (T_\alpha^a f)\big(g(t)\big) \cdot (T_\alpha^a g)(t) \cdot g(t)^{\alpha-1}. \tag{2.16}$$

Proof. Since g is a continuous function and by putting $u = t + \varepsilon(t-a)^{1-\alpha}$ in definition then we have

$$(T_\alpha^a h)(t) = (t-a)^{1-\alpha} \frac{d}{dt}\big(h(t)\big)$$

$$= \lim_{u \to t} \frac{f\big(g(u)\big) - f\big(g(t)\big)}{(u-t)} (t-a)^{1-\alpha}$$

$$= \lim_{u \to t} \frac{f\big(g(u)\big) - f\big(g(t)\big)}{g(u)-g(t)} \cdot \lim_{u \to t} \frac{g(u)-g(t)}{(u-t)} (t-a)^{1-\alpha}$$

$$= \lim_{g(u)\to g(t)} \frac{f(g(u))-f(g(t))}{g(u)-g(t)} \cdot g(t)^{1-\alpha} . T_\alpha^a g(t) . g(t)^{\alpha-1}$$

$$= (T_\alpha^a f)\big(g(t)\big) \cdot (T_\alpha^a g)(t) \cdot g(t)^{\alpha-1}.$$

Proposition 2.1.10. Let $f: [a, \infty) \to \mathbb{R}$ be given and $0 < \alpha, \beta \le 1$ such that $1 < \alpha + \beta \le 2$. If f is twice differentiable on (a, ∞) then

$$(T_\alpha^a T_\beta^a f)(t) = T_{\alpha+\beta}^a f(t) + (1-\beta)(t-\alpha)^{-\beta} T_\alpha^a f(t). \tag{2.17}$$

Proof. Since f is twice differentiable and by conformable product rule we see that

$$(T_\alpha^a T_\beta^a f)(t) = (t-a)^{1-\alpha} \frac{d}{dt}\big[T_\beta^a f(t)\big]$$

$$= (t-a)^{1-\alpha} \frac{d}{dt}\big[(t-a)^{1-\beta} f'(t)\big]$$

$$= (t-a)^{1-\alpha}\big[(t-a)^{1-\beta} f''(t) + (1-\beta)(t-a)^{-\beta} f'(t)\big]$$

$$= T_{\alpha+\beta}^a f(t) + (1-\beta)(t-\alpha)^{-\beta} T_\alpha^a f(t).$$

Also, we have $(T_\alpha^a T_\beta^a f)(t) = T_2 f(t) = f''(t)$ when $\alpha, \beta \to 1$.

Finally, in this section, we explore the conformable derivative for some smooth functions at b in the right case also at a in the left case. From [21] let $n-1 < \alpha < n, n \in \mathbb{N}$ and suppose $f: [a, \infty) \to \mathbb{R}$ be given such that $f^{(n)}(t)$ exists and continuous, then, by Theorem 2.1.4 we see that

$$\begin{aligned}(\boldsymbol{T}_\alpha^a f)(t) &= \big(T_{\alpha+1-n}^a f^{(n-1)}\big)(t) \\ &= (t-a)^{n-\alpha} f^{(n)}(t),\end{aligned} \tag{2.18}$$

therefore

$$(\boldsymbol{T}_\alpha^a f)(a) = \lim_{t\to a^+} (t-a)^{n-\alpha} f^{(n)}(t) = 0.$$

Similarly, in the right case when $f: (-\infty, b) \to \mathbb{R}$ such that $f^{(n)}(t)$ exists and continuous, then we have

$$\big({}_\alpha^b\boldsymbol{T} f\big)(b) = \lim_{t\to b^+} (b-t)^{n-\alpha} f^{(n)}(t) = 0.$$

Now, let $0 < \alpha \le 1$ and $n = 1,2,3,\cdots$. Then the left sequential conformable derivative of order n is defined as

$$ {}^{(n)}T_\alpha^a f(t) = T_\alpha^a T_\alpha^a \dots T_\alpha^a f(f), \quad n-times. \tag{2.19} $$

Also, in the right case defined by

$$ {}_\alpha^b T^{(n)} f(t) = {}_\alpha^b T\, {}_\alpha^b T \dots {}_\alpha^b T f(t), \quad n-times. \tag{2.20} $$

Assume that $f: [a,\infty) \to \mathbb{R}$ be second continuously differentiable and $0 < \alpha \leq \frac{1}{2}$ then the collections directly demonstrate that

$$ T_\alpha^a T_\alpha^a f(t) = \begin{cases} (1-\alpha)(t-a)^{1-2\alpha} f'(t) + (t-a)^{2-2\alpha} f''(t), & if \quad t > a, \\ 0, & if \quad t = a. \end{cases} \tag{2.21} $$

Similarly, when $f: (-\infty, b) \to \mathbb{R}$ is second continuously differentiable and $0 < \alpha \leq \frac{1}{2}$ then the direct collections in the right case show that

$$ {}_\alpha^b T\, {}_\alpha^b T f(t) == \begin{cases} (1-\alpha)(b-t)^{1-2\alpha} f'(t) + (b-t)^{2-2\alpha} f''(t), & if \quad t < b, \\ 0, & if \quad t = b. \end{cases} \tag{2.22} $$

From this process, we know that the second-order sequential conformable derivative my not be continuous when f is second continuously differentiable for $\frac{1}{2} < \alpha < 1$. If we take an inductive step, then we can obtain n^{th}-order sequential conformable derivative which is continuous and disappears at the endpoints a in the left case and b in the right case when f is n-continuously differentiable for $0 < \alpha \leq \frac{1}{n}$.

2.2. Conformable integral

Definition 2.2.1 [21] Let $f: (0,\infty) \to \mathbb{R}$ be a given function, then the left conformable integral of f of order α where $\alpha \in (0,1]$ is defined as follows

$$ \begin{aligned} (I_\alpha^a f)(t) &= \int_a^t f(x) d_\alpha(x,a) \\ &= \int_a^t (x-a)^{\alpha-1} f(x)\, dx. \end{aligned} \tag{2.23} $$

If $a = 0$, then in I_α^a and $d_\alpha(x,a)$ we write I_α and $d_\alpha x$.

Definition 2.2.2. [21] The right conformable integral of f of order α where $f: (0,\infty) \to \mathbb{R}$ and $\alpha \in (0,1]$ is defined by

$$\left({}_{\alpha}^{b}If\right)(t) = \int_t^b f(x) d_\alpha(b,x)$$
$$= \int_t^b (b-x)^{\alpha-1} f(x)\, dx. \tag{2.24}$$

If $b = 0$ then we write $d_\alpha x$.

Theorem 2.2.3. Suppose that $f\colon(0,\infty) \to \mathbb{R}$ be a continuous function and $\alpha \in (0,1]$, then for all $t > a$

$$T_\alpha^a I_\alpha^a f(t) = f(t). \tag{2.25}$$

Proof. $I_\alpha^a f(t)$ is differentiable because f is continuous. Hence

$$T_\alpha^a (I_\alpha^a f)(t) = (t-a)^{1-\alpha} \frac{d}{dt} (I_\alpha^a f)(t)$$
$$= (t-a)^{1-\alpha} \frac{d}{dt} \int_a^t (x-a)^{\alpha-1} f(x)\, dx$$
$$= (t-a)^{1-\alpha} (x-a)^{\alpha-1} f(t)$$
$$= f(t).$$

Theorem 2.2.4. Suppose that $f\colon(0,\infty) \to \mathbb{R}$ be a continuous function and $\alpha \in (0,1]$, then for all t greater than a we have

$${}_{\alpha}^{b}T\, {}_{\alpha}^{b}If(t) = f(t). \tag{2.26}$$

We can get prove same as theorem 2.2.3 easily.

Definition 2.2.5. [21] Let $\beta = \alpha - n$ and f be a given function, then the left conformable integral starting from $a > 0$ of f of order $\alpha \in (n, n+1]$ is defined by

$$(I_\alpha^a f)(t) = \boldsymbol{I}_{n+1}^a((t-a)^{\beta-1} f) = \frac{1}{n!} \int_a^t (t-x)^n (x-a)^{\beta-1} f(x) dx. \tag{2.27}$$

Recognize that if $\alpha = n+1$ then we get $\beta = 1$, hence employing Cauchy formula the iterative integral of f with $n+1$ times over $(a,t]$ is given by

$$(I_\alpha^a f)(t) = (\boldsymbol{I}_{n+1}^a f)(t) = \frac{1}{n!} \int_a^t (t-x)^n f(x) dx. \tag{2.28}$$

And remember that the left Riemann-Liouville conformable integral of order any positive number α starting from a is defined as follows

$$(\boldsymbol{I}_{\alpha}^{a}f)(t)=\frac{1}{\Gamma(\alpha)}\int_{a}^{t}(t-s)^{\alpha-1}f(s)ds. \tag{2.29}$$

Notice that in gamma function we have $\Gamma(n+1)=n!$ for $n\in\mathbb{N}$. then from (2.28) and (2.29), we see that

$$(I_{\alpha}^{a}f)(t)=(\boldsymbol{I}_{\alpha}^{a}f)(t),$$

where $\alpha=n+1$ for $n=0,1,2,\cdots$

Application 2.2.6. Recalling that in [21], if we have

$$(\boldsymbol{I}_{\alpha}^{a}(t-a)^{\mu-1})(x)=\frac{\Gamma(\mu)}{\Gamma(\mu+\alpha)}(x-a)^{\alpha+\mu-1}, \tag{2.30}$$

where $\alpha,\mu>0$, then we can easily calculate left fractional integral of $(t-a)^{\mu}$ of order $\alpha\in(n,n+1]$ and $\mu\in\mathbb{R}$ such that $\alpha+\mu-n>0$ as

$$(I_{\alpha}^{a}(t-a)^{\mu})(x)=(\boldsymbol{I}_{n+1}^{a}(t-a)^{\mu+\alpha-n-1})(x)=\frac{\Gamma(\alpha+\mu-n)}{\Gamma(\alpha+\mu+1)}(x-a)^{\alpha+\mu}.$$

Also, the right fractional integral of $(t-a)^{\mu}$ of order $\alpha\in(n,n+1]$ and $\mu\in\mathbb{R}$ such that $\alpha+\mu-n>0$ where $\alpha,\mu>0$ is given by

$$\left({}_{\boldsymbol{\alpha}}^{\boldsymbol{b}}\boldsymbol{I}(b-t)^{\mu}\right)(x)=\left({}_{\boldsymbol{n+1}}^{\boldsymbol{b}}\boldsymbol{I}(b-t)^{\mu+\alpha-n-1}\right)(x)=\frac{\Gamma(\alpha+\mu-n)}{\Gamma(\alpha+\mu+1)}(b-x)^{\alpha+\mu}.$$

From the discussion above, we notice that the fractional integrals of Riemann and the conformable integrals of polynomial functions are different from a constant multiple and coincide with natural orders [21].

Proposition 2.2.7. Assume that $f:[a,\infty)\to\mathbb{R}$ be a given function and $0<\alpha,\mu\le 1$ such that $1<\alpha+\mu\le 2$. Then

$$(I_{\alpha}I_{\mu}f)(t)=\frac{t^{\mu}}{\mu}(I_{\alpha}f)(t)+\frac{1}{\mu}\left(I_{\alpha+\mu}f\right)(t)-\frac{t}{\mu}\int_{0}^{t}s^{\alpha+\mu-2}f(s)ds. \tag{2.31}$$

Proof. Interchange the integral order and note that

$$\begin{aligned}\left(I_{\alpha+\mu}f\right)(t)&=\left(I_{2}s^{\alpha+\mu-2}f(s)\right)(t)\\&=\int_{0}^{t}(t-s)s^{\alpha+\mu-2}f(s)ds,\end{aligned} \tag{2.32}$$

also, we see that

$$(I_{\alpha}I_{\mu}f)(t)=\int_{0}^{t}\left(\int_{0}^{t_1}f(s)s^{\alpha-1}ds\right)t_1{}^{\mu-1}dt_1$$

$$= \int_0^t f(s)s^{\alpha-1}\left(\int_s^t {t_1}^{\mu-1}\, dt_1\right) ds$$

$$= \int_0^t f(s)s^{\alpha-1}\left[\frac{t^\mu}{\mu} - \frac{s^\mu}{\mu}\right] ds$$

$$= \int_0^t \frac{t^\mu}{\mu} s^{\alpha-1} f(s)\, ds - \int_0^t \frac{s^\mu}{\mu} s^{\alpha-1} f(s)\, ds$$

$$= \int_0^t \frac{t^\mu}{\mu} s^{\alpha-1} f(s)\, ds + \int_0^t \frac{t}{\mu} s^{\alpha+\mu-2} f(s)\, ds - \int_0^t \frac{t}{\mu} s^{\alpha+\mu-2} f(s)\, ds - \int_0^t \frac{s^\mu}{\mu} s^{\alpha-1} f(s)\, ds$$

$$= \int_0^t \frac{t^\mu}{\mu} s^{\alpha-1} f(s)\, ds + \frac{1}{\mu}\left[\int_0^t t s^{\alpha+\mu-2} f(s)\, ds - \int_0^t s^{\alpha+\mu-1} f(s)\, ds - t\int_0^t s^{\alpha+\mu-2} f(s)\, ds\right]$$

$$= \frac{t^\mu}{\mu}(I_\alpha f)(t) + \frac{1}{\mu}\left[\left(I_{\alpha+\mu} f\right)(t) - t\int_0^t s^{\alpha+\mu-2}\, f(s) ds\right].$$

Lemma 2.2.8. Suppose that $f: [a, \infty) \to \mathbb{R}$ be a function such that $f^n(t)$ is continuous and $\alpha \in (n, n+1]$ where $\beta = \alpha - n$ then for all $t > a$ we have

$$\boldsymbol{T}_\alpha^a I_\alpha^a f(t) = f(t). \tag{2.33}$$

Proof. From the definitions of conformable derivative and conformable integral we have

$$T_\alpha^a I_\alpha^a f(t) = T_\beta^a\left(\frac{d^n}{dt^n} I_\alpha^a f(t)\right)$$

$$= T_\beta^a\left(\frac{d^n}{dt^n} I_{n+1}^a((t-a)^{\beta-1} f(t))\right),$$

there is a derivative of order n and integral with order $n+1$ when $n = 1,2,3,\cdots$. Then

$$T_\alpha^a I_\alpha^a f(t) = T_\beta^a\left(I_1^a((t-a)^{\beta-1} f(t))\right),$$

since $\beta = \alpha - n$ where $\alpha \in (n, n+1]$, $n \in \mathbb{N}$ there for $\beta \in (0,1]$ so that

$$I_1^a\left((t-a)^{\beta-1} f(t)\right) = I_\beta^a f(t).$$

$$T_\alpha^a I_\alpha^a f(t) = T_\beta^a I_\beta^a f(t).$$

hence by Theorem 2.2.3, we have

$$T_\alpha^a I_\alpha^a f(t) = f(t).$$

Lemma 2.2.9. Suppose that $f: (-\infty, b] \to \mathbb{R}$ be a function such that $f^n(t)$ is continuous and $\alpha \in (n, n+1]$ where $\beta = \alpha - n$ then for all $t < b$ we have

$${}_{\alpha}^{b}T_{\alpha}^{b}If(t) = f(t).$$

Similarly, we can prove the same as the above lemma.

Corollary 2.2.10. [21] Let $f:[a,b) \to \mathbb{R}$ be a given function. If $I_{\alpha}^{a}T_{\alpha}^{a}f(t)$ exists for $a < t < b$, then $f(t)$ is a differentiable function on (a,b).

Lemma 2.2.11. Let $\alpha \in (0,1]$ and $f:[a,b) \to \mathbb{R}$ be a differentiable function, then for all $t > 0$

$$I_{\alpha}^{a}T_{\alpha}^{a}f(t) = f(t) - f(a). \tag{2.34}$$

Proof. Since f is differentiable then by the help notation in Theorem 2.1.7 and Definition 2.2.1 we have

$$\begin{aligned} I_{\alpha}^{a}T_{\alpha}^{a}f(t) &= \int_{a}^{t}(x-a)^{\alpha-1}T_{\alpha}(f)(x)\,dx \\ &= \int_{a}^{t}(x-a)^{\alpha-1}(x-a)^{1-\alpha}f'(x)\,dx \\ &= f(t) - f(a). \end{aligned}$$

Now for the higher-order, Lemma 2.2.11 can be generalized as follows

Proposition 2.2.12. Suppose that $f:[a,b) \to \mathbb{R}$ be $(n+1)$ times differentiable and $\alpha \in (n, n+1]$, then for all $t > a$ we have that

$$I_{\alpha}^{a}\boldsymbol{T}_{\alpha}^{a}f(t) = f(t) - \sum_{k=0}^{n}\frac{f^{(k)}(a)(t-a)^{k}}{k!}. \tag{2.35}$$

Proof. From definitions of conformable derivative and conformable integral we have

$$\begin{aligned} I_{\alpha}^{a}\boldsymbol{T}_{\alpha}^{a}f(t) &= I_{n+1}^{a}\left((t-a)^{\beta-1}T_{\beta}^{a}f^{(n)}(t)\right) \\ &= I_{n+1}^{a}((t-a)^{\beta-1}(t-a)^{1-\beta}f^{(n+1)}(t)) \\ &= I_{n+1}^{a}f^{(n+1)}(t) \\ &= f(t) - \sum_{k=0}^{n}\frac{f^{(k)}(a)(t-a)^{k}}{k!}. \end{aligned}$$

In the same way, we have the right case of the form as follows.

Proposition 2.2.13. [21] Assume that $\alpha \in (n, n+1]$ and $f:[a,b) \to \mathbb{R}$ be $(n+1)$ times differentiable, then for all $t > a$ we have that

$${}_{\alpha}^{b}I_{\alpha}^{b}\boldsymbol{T}(f)(t) = f(t) - \sum_{k=0}^{n}\frac{(-1)^{k}f^{(k)}(b)(b-t)^{k}}{k!}. \tag{2.36}$$

But if $n = 0$ or $0 < \alpha \leq 1$ then

$${}^{b}_{\alpha}I\,{}^{b}_{\alpha}T(f)(t) = f(t) - f(b). \tag{2.37}$$

Theorem 2.2.14. Suppose that r be a non-negative continuous function at interval $j = [a, b]$ also, δ and k be positive constant such that

$$r(t) \leq \delta + \int_0^t kr(s)(s-a)^{\alpha-1}ds, \quad (t \in j), \tag{2.38}$$

then for $t \in j$

$$r(t) \leq \delta e^{k\frac{(t-a)^{\alpha}}{\alpha}}. \tag{2.39}$$

Proof. Redefine that

$$\begin{aligned} R(t) &= \delta + \int_0^t kr(s)(s-a)^{\alpha-1}ds \\ &= \delta + I_{\alpha}^{a}\big(kr(s)\big)(t), \end{aligned} \tag{2.40}$$

so that $r(t) \leq$ R(t) and by theorem 2.2.3 $T_{\alpha}^{a}R(t) = kr(t)$. Now

$$\begin{aligned} (t) - kR(t) &= kr(t) - kR(t) \\ &\leq kr(t) - kr(t) \\ &= 0, \end{aligned} \tag{2.41}$$

multiply (2.41) by $h(t) = e^{-k\frac{(t-a)^{\alpha}}{\alpha}}$, using product rule of conformable derivative, we have

$$T_{\alpha}^{a}\big(h(t)R(t)\big) \leq 0, \tag{2.42}$$

where $T_{\alpha}^{a}h(t) = -kh(t)$. By Lemma 2.2.11 and in (2.42) by integrating both sides with order α we see that

$$\begin{aligned} &\mathrm{I}_{\alpha}^{a}\mathrm{T}_{\alpha}^{a}\big(h(t)R(t)\big) \leq 0, \\ &h(t)R(t) - h(a)R(a) \leq 0, \end{aligned} \tag{2.43}$$

it is clear that $h(a) = 1$, and $R(a) = \delta$, then we obtain

$$h(t)R(t) - \delta \leq 0.$$

Since $r(t) \leq$ R(t) then the proof is complete.

2.3. Conformable Integration by parts.

Theorem 2.3.1 Assume f and g be two differentiable functions such that $f, g: [a,b] \to \mathbb{R}$, then

$$\int_a^b f(x)\, T_\alpha^a(g)(x) d_\alpha(x,a) = fg|_a^b - \int_a^b g(x)\, T_\alpha^a(f)(x) d_\alpha(x,a). \tag{2.44}$$

Proof. By Definition 2.1.1 and Definition 2.2.1 we can say that

$$\begin{aligned} \int_a^b f(x)\, T_\alpha^a(g)(x) d_\alpha(x,a) &= \int_a^b (x-a)^{\alpha-1} f(x)\, (x-a)^{1-\alpha} g'(x) dx \\ &= \int_a^b f(x)\, g'(x) dx, \end{aligned} \tag{2.45}$$

from usual calculus by integration by parts of (2.45) we get

$$\begin{aligned} \int_a^b f(x)\, T_\alpha^a(g)(x) d_\alpha(x,a) &= fg|_a^b - \int_a^b g(x)\, f'(x) dx \\ &= fg|_a^b - \int_a^b (x-a)^{1-\alpha} (x-a)^{\alpha-1} g(x)\, f'(x) dx \\ &= fg|_a^b - \int_a^b g(x)\, T_\alpha^a(f)(x) d_\alpha(x,a). \end{aligned}$$

The following integration by parts formula is specified through the left and right conformable integral.

Proposition 2.3.2 Let $f, g: [a,b] \to \mathbb{R}$ be a function, then for $\alpha \in (0,1]$ we have

$$\int_a^b I_\alpha^a f(t)\, (g)(t) d_\alpha(b,t) = \int_a^b f(t)\, {}_\alpha^a I(g)(t) d_\alpha(t,a). \tag{2.46}$$

Proof. From Definitions 2.2.1 and 2.2.2 we have

$$\int_a^b I_\alpha^a f(t)\, (g)(t) d_\alpha(b,t) = \int_t^b \left(\int_a^t (x-a)^{\alpha-1} f(x)\, dx\right) (b-t)^{\alpha-1} g(t) dt|_a^b, \tag{2.47}$$

by interchanging order of integral then we get

$$\begin{aligned} \int_a^b I_\alpha^a f(t)\, (g)(t) d\alpha(b,t) &= \int_a^t (x-a)^{\alpha-1} f(x) \left(\int_b^x (b-t)^{\alpha-1} g(t) dt\right) dx|_a^b \\ &= \int_a^b f(t)\, {}_\alpha^a I(g)(t) d_\alpha(t,a). \end{aligned}$$

In the following theorem, we use Proposition 2.3.2 to prove the integration by parts formula utilizing the left and right conformable derivatives.

Theorem 2.3.3 Suppose $f, g: [a,b] \to \mathbb{R}$ be differentiable functions and $\alpha \in (0,1]$ then we have

$$\int_a^b (T_\alpha^a f)(t)\, g(t) d_\alpha(t,a) = \int_a^b f(t)\, ({}_\alpha^a T g)(t) d_\alpha(b,t) + f(t)g(t)|_a^b. \qquad (2.48)$$

Proof. Since f and g are differentiable functions, then by Proposition 2.2.13 we get

$$\int_a^b (T_\alpha^a f)(t)\, g(t) d_\alpha(t,a) = \int_a^b (T_\alpha^a f)(t)\, {}_\alpha^b I\, {}_\alpha^b T g(t) d_\alpha(t,a) + g(b) \int_a^b (T_\alpha^a f)(t)\, d_\alpha(t,a).$$

By using Proposition 2.3.2 we obtain

$$\int_a^b (T_\alpha^a f)(t)\, g(t) d_\alpha(t,a) = \int_a^b (I_\alpha^a T_\alpha^a f)(t)\, {}_\alpha^b T g(t) d_\alpha(b,t) + g(b) \int_a^b (T_\alpha^a f)(t)\, d_\alpha(t,a). \quad (2.49)$$

From Lemma 2.2.11, we can see that

$$(I_\alpha^a T_\alpha^a f)(t) = f(t) - f(a),$$

therefore in (2.49), we have

$$\int_a^b (T_\alpha^a f)(t)\, g(t) d_\alpha(t,a) = \int_a^b f(t)\, {}_\alpha^b T g(t) d_\alpha(b,t) - f(a) \int_a^b {}_\alpha^b T g(t)\, d_\alpha(b,t)$$

$$+ g(b) \int_a^b (T_\alpha^a f)(t)\, d_\alpha(t,a). \qquad (2.50)$$

Applying that f and g are differentiable functions, then by Definitions 2.1.1 and 2.1.2 also Definitions 2.2.1 and 2.2.2 we can say

$$\int_a^b (T_\alpha^a f)(t)\, d_\alpha(t,a) = \int_a^b f'(t) dt$$

$$= f(b) - f(a).$$

And

$$\int_a^b {}_\alpha^b T g(t)\, d_\alpha(b,t) = \int_a^b -g'(t) dt$$

$$= g(a) - g(b).$$

Now in (2.50), we obtain

$$\int_a^b (T_\alpha^a f)(t)\, g(t) d_\alpha(t,a) = \int_a^b f(t)\, ({}_\alpha^a T g)(t) d_\alpha(b,t) + f(t)g(t)|_a^b.$$

Remark 2.3.4. Remember that if in the Theorem 2.3.1 or Theorem 2.3.3 we assume $\alpha \to 1$ then we get the integration by parts formula in usual calculus, when we have to take note of that $d_\alpha(t,a) \to dt$, $d_\alpha(b,t) \to dt$, $T_\alpha^a f(t) \to f'(t)$ and ${}_\alpha^a T f \to f'(t)$ as $\alpha \to 1$.

Definition 2.3.5. [21] Some function spaces such that the integration by parts formulas acquired are still valid yet for $0 < \alpha \leq 1$ on the interval $[a, b]$ define as

$$I_\alpha([a,b]) = \{f: [a,b] \to \mathbb{R}: f(x) = (I_\alpha^a \Phi)(x) + f(a), \text{ for some } \Phi \in L_\alpha(a)\}. \tag{2.51}$$

And

$$^{\alpha}I([a,b]) = \{g: [a,b] \to \mathbb{R}: g(x) = \left({}_{\alpha}^{b}I\varphi\right)(x) + f(b), \text{ for some } \varphi \in L_\alpha(b)\}. \tag{2.52}$$

Where

$$L_\alpha(a) = \{\Phi: [a,b] \to \mathbb{R} : (I_\alpha^a \Phi)(x) \text{ exists for all } x \in \{a,b\}\}. \tag{2.53}$$

And

$$L_\alpha(b) = \{\varphi: [a,b] \to \mathbb{R} : \left({}_{\alpha}^{b}I\varphi\right)(x) \text{ exists for all } x \in \{a,b\}\}. \tag{2.54}$$

Lemma 2.3.6. Assume that $f, g: [a,b] \to \mathbb{R}$ be function and $0 < \alpha \leq 1$, then

a) $f \in I_\alpha([a,b])$ and $g \in {}^{\alpha}I([a,b])$ when f is left and g is right α-differentiable.

b) If ψ is continuous and $f \in I_\alpha([a,b])$ with $f(x) = (I_\alpha^a \Phi)(x) + f(a)$ then $\Phi(t) = T_\alpha^a f(x)$ and $I_\alpha^a T_\alpha^a f(t) = f(t) - f(a)$.

c) If φ is continuous and $g \in {}^{\alpha}I([a,b])$ with $g(x) = \left({}_{\alpha}^{b}I\varphi\right)(x) + f(b)$ then $\varphi(t) = {}_{\alpha}^{a}Tg(t)$ and ${}_{\alpha}^{a}I\,{}_{\alpha}^{a}Tg(t) = g(t) - g(a)$.

Proof. At the beginning proof of (a) follows by putting $\Phi(t) = T_\alpha^a f$ in Lemma 2.2.11 we get $I_\alpha^a \Phi(t) = f(t) - f(a) \Longrightarrow \left(f(t) = I_\alpha^a \Phi(t) - f(a)\right) \in I_\alpha([a,b])$. Also, by putting $\varphi(t) = {}_{\alpha}^{a}Tg$ in Proposition 2.2.13 we obtain $(f(t) = ({}_{\alpha}^{b}I\varphi)(t) + f(b)) \in {}^{\alpha}I([a,b])$. Now the proof of (b) by hypothesis we have $f(t) = (I_\alpha^a \Phi)(t) + f(a) \Longrightarrow (I_\alpha^a \Phi)(t) = f(t) - f(a)$, then take left α-derivative of both sides we get $T_\alpha^a f(t) = \Phi(t)$, because the left α-derivative of a constant function is zero and from Theorem 2.2.3 there is $T_\alpha^a I_\alpha^a \mathrm{f(t)} = \mathrm{f(t)}$. Also, by taking left integral with order α of $T_\alpha^a f(t) = \Phi(t)$ and applying Definition 2.3.5 we get $I_\alpha^a T_\alpha^a f(t) = f(t) - f(a)$. Proof of (c) is similar to (b) follows by Theorem 2.2.4, Definition 2.3.5 and the right α-derivative of a constant function is zero.

Theorem 2.3.7. Suppose $f, g: [a,b] \to \mathbb{R}$ be functions such that $f \in I_\alpha([a,b])$ with Φ is continuous also $g \in {}^{\alpha}I([a,b])$ with φ is continuous then for $0 < \alpha \leq 1$ we have

$$\int_a^b (T_\alpha^a f)(t)\, g(t) d_\alpha(t,a) = \int_a^b f(t)\, ({}_{\alpha}^{a}Tg)(t) d_\alpha(b,t) + f(t)g(t)|_a^b. \tag{2.55}$$

Proof. The proof is comparable to that in Theorem 2.3.3 where (b) and (c) are used in Lemma 2.3.6.

2.4. Fractional power series expansion.

Theorem 2.4.1. Let $0 < \alpha \leq 1$ and f is an infinitely α-differentiable function at a neighborhood of a point t_0, then the power series expansion of the function f is defined as

$$f(t) = \sum_{k=0}^{\infty} \frac{(T_\alpha^{t_0} f)^{(k)}(t_0)(t-t_0)^{k\alpha}}{\alpha^k k!}, \tag{2.56}$$

where $t_0 < t < t_0 + R^{\frac{1}{\alpha}}, R > 0$ and $(T_\alpha^{t_0})^{(k)}(t_0)$ means the application of the fractional derivatives k times.

Proof. Assume that for $t_0 < t < t_0 + R^{\frac{1}{\alpha}}, R > 0$ the function f is defined by

$$f(t) = c_0 + c_1(t-t_0)^{\alpha} + c_2(t-t_0)^{2\alpha} + c_3(t-t_0)^{3\alpha} + \cdots.$$

Then, $f(t_0) = c_0$. Taking $T_\alpha^{t_0}$ of f and evaluate at t_0 we get

$$\left(T_\alpha^{t_0} f\right)(t_0) = c_1 \alpha \Longrightarrow c_1 = \frac{\left(T_\alpha^{t_0} f\right)(t_0)}{\alpha}.$$

$$\left(T_\alpha^{t_0} T_\alpha^{t_0} f\right)(t_0) = 2\alpha^2 c_2 \Longrightarrow c_2 = \frac{\left(T_\alpha^{t_0} T_\alpha^{t_0} f\right)(t_0)}{2\alpha^2}.$$

$$\left(T_\alpha^{t_0} T_\alpha^{t_0} T_\alpha^{t_0} f\right)(t_0) = 6\alpha^3 c_3 \Longrightarrow c_3 = \frac{\left(T_\alpha^{t_0} T_\alpha^{t_0} T_\alpha^{t_0} f\right)(t_0)}{6\alpha^3}. \ldots$$

Inductively proceed and applying $T_\alpha^{t_0}$ of f, n-times and evaluating at t_0 we obtain

$$c_n = \frac{\left(T_\alpha^{t_0} f\right)^{(n)}(t_0)}{\alpha^n n!}.$$

Therefore (2.56) is acquired and the proof is finished.

Proposition 2.4.2. [21] (Fractional Taylor inequality). Assume $0 < \alpha \leq 1$ and f is α-differentiable at a neighborhood of a point t_0 also has a Taylor power series representation (2.56) such that $|(T_\alpha^{\alpha} f)^{n+1}| \leq M, \ M > 0$ for some $n \in \mathbb{N}$. If

$$R_n^{\alpha}(t) = \sum_{k=n+1}^{\infty} \frac{(T_\alpha^{t_0})^{(k)}(t_0)(t-t_0)^{k\alpha}}{\alpha^k k!}$$

$$= f(x) - \sum_{k=0}^{n} \frac{(T_\alpha^{t_0})^{(k)}(t_0)(t-t_0)^{k\alpha}}{\alpha^k k!},$$

then we have

$$|R_n^\alpha(t)| \leq \frac{M}{\alpha^{n+1}(n+1)!}(t-t_0)^{\alpha(n+1)}.$$

Application 2.4.3. For $0 < \alpha < 1$ consider the fractional exponential function $f(t) = e^{\frac{(t-t_0)^\alpha}{\alpha}}$. The function $f(t)$ dos not have a Taylor power series representation about t_0 because it is not differentiable at a point t_0. We know that $\left(T_\alpha^{t_0} f\right)^{(n)}(t_0) = 1$ for all n, therefore

$$e^{\frac{(t-t_0)^\alpha}{\alpha}} = \sum_{k=0}^{\infty} \frac{(t-t_0)^{k\alpha}}{\alpha^k k!}. \tag{2.57}$$

The ratio test demonstrates that this series convergent to f on interval $[t_0, \infty)$.

Application 2.4.4. Let $f(t) = sin\frac{(t-t_0)^\alpha}{\alpha}$ and $g(t) = cos\frac{(t-t_0)^\alpha}{\alpha}$, know that f and g do not have Taylor power series expansions about $t = t_0$ for $0 < \alpha < 1$ because they are not differentiable there. By the help of (2.56),

$$T_\alpha^{t_0} sin\frac{(t-t_0)^\alpha}{\alpha} = cos\frac{(t-t_0)^\alpha}{\alpha},$$

$$T_\alpha^{t_0} cos\frac{(t-t_0)^\alpha}{\alpha} = -sin\frac{(t-t_0)^\alpha}{\alpha},$$

we have

$$sin\frac{(t-t_0)^\alpha}{\alpha} = \sum_{k=0}^{\infty}(-1)^k \frac{(t-t_0)^{(2k+1)\alpha}}{\alpha^{(2k+1)}(2k+1)!}, \quad t \in [t_0, \infty). \tag{2.58}$$

And

$$cos\frac{(t-t_0)^\alpha}{\alpha} = \sum_{k=0}^{\infty}(-1)^k \frac{(t-t_0)^{(2k)\alpha}}{\alpha^{(2k)}(2k)!}, \quad t \in [t_0, \infty). \tag{2.59}$$

Application 2.4.5. Let $0 < \alpha < 1$ then the function $f(t) = \frac{1}{1-\frac{t^\alpha}{\alpha}}$ is not differentiable at $t = 0$ hence it does not have Taylor power series representation about $t = 0$. Now by the help of (2.56), we can see that

$$\frac{1}{1-\frac{t^\alpha}{\alpha}} = \sum_{k=0}^{\infty} t^{\alpha k}, \quad t \in [0,1). \tag{2.60}$$

For more generally we have

$$\frac{1}{1-\frac{(t-t_0)^\alpha}{\alpha}} = \sum_{k=0}^{\infty}(t-t_0)^{\alpha k}, \quad t \in [t_0, t_0+1). \tag{2.61}$$

Remark 2.4.6. In case the function f is not differentiable at a where f be a function such that defined on $(-\infty, a)$, then for some $0 < \alpha \leq 1$ we are looking for its conformable right fractional-order derivatives ${}_{\alpha}^{a}T$ at a and apply it for our fractional Taylor series on some $(a-R, a), R > 0$. For example, the functions $\frac{(t-t_0)^\alpha}{\alpha}$ and $\sin\left(\frac{(t-t_0)^\alpha}{\alpha}\right)$ and so on.

2.5. Conformable Laplace Transform.

In this part, we are going to define the conformable Laplace transform and apply it to solve some linear fractional equation to generate the fractional exponential function. After that, we use the method for progressive estimation to check the solution by taking advantage of the fractional power series representation discussed in the section above. However, we will calculate the Laplace transform for some certain fractional type functions.

Definition 2.5.1. [21] Suppose that $f: [t_0, \infty) \to \mathbb{R}$ be a real all valued function for $t_0 \in \mathbb{R}$ and $\alpha \in (0,1]$, then the conformable Laplace transform starting from a of order α of f is defined by.

$$\begin{aligned} L_\alpha^{t_0}\{f(t)\}(s) &= F_\alpha^{t_0}(s) \\ &= \int_{t_0}^{\infty} e^{-s\frac{(t-t_0)^\alpha}{\alpha}} f(t) d_\alpha(t, t_0) \\ &= \int_{t_0}^{\infty} e^{-s\frac{(t-t_0)^\alpha}{\alpha}} f(t)(t-t_0)^{\alpha-1} dt. \end{aligned} \tag{2.62}$$

Theorem 2.5.2. Let $f: [t_0, \infty) \to \mathbb{R}$ is a real-valued function for $a \in \mathbb{R}$ and $0 < \alpha \leq 1$. Then

$$L_\alpha^{t_0}\{T_\alpha^a(f)(t)\}(s) = sF_\alpha^a(s) - f(a). \tag{2.63}$$

Proof. From the definition of conformable Laplace transformation and Definition 2.1.1 also by integration by parts in usual calculus we see that

$$\begin{aligned} L_\alpha^{t_0}\{T_\alpha^a(f)(t)\}(s) &= \int_a^{\infty} e^{-s\frac{(t-a)^\alpha}{\alpha}} (t-a)^{1-\alpha} f'(t)(t-a)^{\alpha-1} dt \\ &= \int_a^{\infty} e^{-s\frac{(t-a)^\alpha}{\alpha}} f'(t) dt \\ &= f(t) e^{-s\frac{(t-a)^\alpha}{\alpha}} \Big|_a^{\infty} + s \int_a^{\infty} (t-a)^{\alpha-1} e^{-s\frac{(t-a)^\alpha}{\alpha}} f(t) dt \\ &= -f(a) + sF_\alpha^a(s). \end{aligned}$$

Application 2.5.3. To find the fractional initial value problem of

$$(T_\alpha^a y)(t) = \lambda y(t), \qquad y(a) = y_0, \;\; t > a. \tag{2.64}$$

When it is assumed that the solution is differentiable on (a, ∞). Use the operator I_α^a to both sides in (2.64), we obtain

$$y(t) = y_0 + \lambda (I_\alpha^a y)(t). \tag{2.65}$$

Then

$$y_{n+1} = y_0 + \lambda (I_\alpha^a y_n)(t). \quad n = 0,1,2,\cdots. \tag{2.66}$$

When $n = 0$ we see that

$$\begin{aligned} y_1 &= y_0 + \lambda y_0 \frac{(t-a)^\alpha}{\alpha} \\ &= y_0 \left(1 + \lambda \frac{(t-a)^\alpha}{\alpha}\right). \end{aligned} \tag{2.67}$$

Also, when $n = 1$ we have

$$y_2 = y_0 \left[1 + \lambda \frac{(t-a)^\alpha}{\alpha} + \lambda^2 \frac{(t-a)^{2\alpha}}{\alpha(2\alpha)}\right]. \tag{2.68}$$

If we continue to proceed inductively, we will conclude that

$$y_n = y_0 \sum_{k=0}^{n} \frac{\lambda^k \, (t-a)^{k\alpha}}{\alpha^k k!}. \tag{2.69}$$

Allowing $n \to \infty$ we get

$$y(t) = y_0 \sum_{k=0}^{\infty} \frac{\lambda^k \, (t-a)^{k\alpha}}{\alpha^k k!}. \tag{2.70}$$

That is obvious the fractional Taylor power series representation of the fractional exponential function $y_0 e^{\lambda \frac{(t-t_0)^\alpha}{\alpha}}$.

Lemma 2.5.4. This lemma relates the conformable Laplace transform to the normal Laplace transform in this way, assume that $f: [t_0, \infty) \to \mathbb{R}$ be function and $L_\alpha^{t_0}\{f(t)\}(s) = F_\alpha^{t_0}(s)$ exists then we have

$$F_\alpha^{t_0}(s) = \mathcal{L}\{(f(t_0 + (\alpha t)^{\frac{1}{\alpha}})\}(s). \tag{2.71}$$

Where for $g: [0, \infty) \to \mathbb{R}$

$$\mathcal{L}\{g(t)\}(s) = \int_0^\infty e^{-st}\, g(t)dt. \tag{2.72}$$

Proof. Let $u = \frac{(t-t_0)^\alpha}{\alpha} \Longrightarrow t - t_0 = (\alpha u)^{\frac{1}{\alpha}}$ then $dt = (\alpha u)^{\frac{1}{\alpha}-1}du$. From (2.62) we see that

$$\begin{aligned} F_\alpha^{t_0}(s) &= \int_{t_0}^\infty e^{-s\frac{(t-t_0)^\alpha}{\alpha}} f(t)(t-t_0)^{\alpha-1}dt \\ &= \int_{t_0}^\infty e^{-su} f(t_0 + (\alpha u)^{\frac{1}{\alpha}}) \left((\alpha u)^{\frac{1}{\alpha}}\right)^{\alpha-1} (\alpha u)^{\frac{1}{\alpha}-1}du \\ &= \mathcal{L}\{f(t_0 + (\alpha u)^{\frac{1}{\alpha}})\}(s). \end{aligned}$$

Application 2.5.5. In this application, we give the conformable Laplace transform for certain functions.

- $L_\alpha^{t_0}\{1\}(s) = \frac{1}{s}, \;\; s > 0.$
- $L_\alpha^{t_0}\{t\}(s) = \mathcal{L}\{f(t_0 + (\alpha u)^{\frac{1}{\alpha}})\}(s)$

  $= \frac{t_0}{s} + \alpha^{\frac{1}{\alpha}} \frac{\Gamma(1+\frac{1}{\alpha})}{s^{1+\frac{1}{\alpha}}}, \quad s > 0.$
- $L_\alpha^0\{t^p\}(s) = \frac{\alpha^{\frac{p}{\alpha}}}{s^{1+\frac{p}{\alpha}}}\Gamma(1 + \frac{1}{\alpha}), \quad s > 0.$
- $L_\alpha^0\{e^{\frac{t^\alpha}{\alpha}}\}(s) = \frac{1}{s-1}, \quad s > 1.$
- $L_\alpha^0\{sin\omega\frac{t^\alpha}{\alpha}\}(s) = \mathcal{L}\{sin\omega t\}(s)$

  $= \frac{1}{\omega^2+s^2}.$
- $L_\alpha^0\{cos\omega\frac{t^\alpha}{\alpha}\}(s) = \mathcal{L}\{cos\omega t\}(s)$

  $= \frac{s}{\omega^2+s^2}.$
- $L_\alpha^{t_0}\{e^{-k\frac{(t-t_0)^\alpha}{\alpha}}f(t)\}(s) = \mathcal{L}\{e^{-kt}f(t_0 + (\alpha u)^{\frac{1}{\alpha}})\}.$

  For example

  $L_\alpha^0\{e^{-k\frac{t^\alpha}{\alpha}}sin\frac{t^\alpha}{\alpha}\}(s) = \mathcal{L}\{e^{-kt}sint\}(s)$

  $= \frac{1}{(s+k)^2+1}.$

$$L_{\alpha}^{t_0}\{e^{\lambda\frac{(t-t_0)^{\alpha}}{\alpha}}\} = \mathcal{L}\{e^{\lambda t}\}$$

$$= \frac{1}{s-\lambda}.$$

Application 2.5.6. In this application, we apply the conformable Laplace transform to evaluate the solution of the fractional initial value problem

$$T_{\alpha}^{a} y(t) = \lambda y(t), \qquad y(a) = y_0, \ \ t > a. \tag{2.73}$$

When the solution is supposed to be differentiable on the interval (a, ∞). By applying L_{α}^{a} and use (2.63) we obtain

$$sL_{\alpha}^{a}\{y(t)\} - y(a) = \lambda L_{\alpha}^{a}\{y(t)\},$$

$$L_{\alpha}^{a}\{y(t)\} = \frac{y_0}{s-\lambda}.$$

Hence from the above example, we get

$$y(t) = y_0 e^{\lambda\frac{(t-t_0)^{\alpha}}{\alpha}}.$$

3. CONFORMABLE STURM-LIOUVILLE EIGENVALUE PROBLEM

In this chapter, from [13] we consider some conformable applications of the Sturm-Liouville eigenvalue problem

$$T_\alpha^a(p(x)T_\alpha^a y) + q(x)y = -\lambda w(x)y, \tag{3.1}$$

where $\frac{1}{2} < \alpha \leq 1, a < x < b$. Also, p, $T_\alpha^a y$, q, and the weight functions $w(x)$ are continuous on interval (a, b), $p(x) > 0$, $w(x) > 0$ on $[a, b]$ and T_α^a is the conformable derivative with order α. We analyze (3.1) with boundary conditions

$$\begin{aligned} &c_1 y(a) + c_2 y'(a) = 0, \quad {c_1}^2 + {c_2}^2 > 0, \\ &r_1 y(b) + r_2 y'(b) = 0, \quad {r_1}^2 + {r_2}^2 > 0. \end{aligned} \tag{3.2}$$

If $T_\alpha^a T_\alpha^a y$ is continuous on the interval $[a.b]$ then we say that y is 2α-continuously differentiable on $[a, b]$. Also, $y \in C^{2\alpha}[a, b]$, if $y \in C^1[a, b]$ and is 2α- continuously differentiable on $[a, b]$. Let

$$L(y, \alpha) = T_\alpha^a(p(x)T_\alpha^a y) + q(x)y, \tag{3.3}$$

then we can write the conformable Sturm-Liouville eigenvalue problem as

$$L(y, \alpha) = -\lambda w(x)y. \tag{3.4}$$

3.1. Conformable Lagrange identity

In this section, we generalize the easy result of a well-known Lagrange identity.

Theorem 3.1.1. (The conformable Lagrange identity theorem). Suppose y_1, y_2 be 2α- continuously differentiable on the interval $[a, b]$, then we can say the following is true:

$$\int_a^b (y_2\, L(y_1, \alpha) - y_1 L(y_2, \alpha)) d_\alpha(x) = [p(x)(y_2 T_\alpha^a y_1 - y_1 T_\alpha^a y_2)]|_a^b. \tag{3.5}$$

Proof. By using (3.3) we have

$$\begin{aligned} y_2 L(y_1, \alpha) - y_1 L(y_2, \alpha) &= y_2 T_\alpha^a(p(x)T_\alpha^a y_1) + q(x)y_1 y_2 \\ &\quad -y_1 T_\alpha^a(p(x)T_\alpha^a y_2) - q(x)y_1 y_2 \\ &= y_2 T_\alpha^a(p(x)T_\alpha^a y_1) - y_1 T_\alpha^a(p(x)T_\alpha^a y_2). \end{aligned} \tag{3.6}$$

Taking conformable integration with order α of (3.6) and using integration by parts from Theorem 2.3.1 we see that

$$\int_a^b (y_2\, L(y_1,\alpha) - y_1 L(y_2,\alpha)) d_\alpha(x)$$

$$\begin{aligned} &= \int_a^b y_2 T_\alpha^a (p(x) T_\alpha^a y_1)\, d_\alpha(x) - \int_a^b y_1 T_\alpha^a (p(x) T_\alpha^a y_2)\, d_\alpha(x) \\ &= p(x) y_2 T_\alpha^a y_1 |_a^b - \int_a^b p(x)\, T_\alpha^a y_1 T_\alpha^a y_2 d_\alpha(x) \\ &\quad - p(x) y_1 T_\alpha^a y_2 |_a^b + \int_a^b p(x)\, T_\alpha^a y_1 T_\alpha^a y_2 d_\alpha(x) \\ &= [p(x)(y_2 T_\alpha^a y_1 - y_1 T_\alpha^a y_2)]|_a^b. \end{aligned} \tag{3.7}$$

Proposition 3.1.2. If $y \in C^1[0,1]$ and $y'(x_0) = 0$, then $(T_\alpha^a y)(\, x_0) = 0$ for some $x_0 \in [a,b]$.

Proof. Since $y \in C^1[0,1]$ then y is a differentiable function on $[0,1]$ and by Theorem 2.1.1 we have $(T_\alpha^a y)(x_0) = (x_0 - a)^{1-\alpha} y'(x_0)$, but this result follows for $x_0 \in (a,b]$. If $x_0 = a$, then we have $(T_\alpha^a)(\, a) = \lim_{x \to a^+} (x-a)^{1-\alpha} y'(x) = 0$.

Proposition 3.1.3. If $y_1, y_2 \in C^1[a,b]$ and satisfy the boundary conditions (3.2), then It maintains that

$$[p(x)(y_2 T_\alpha^a y_1 - y_1 T_\alpha^a y_2)]|_a^b = 0. \tag{3.8}$$

Proof. From (3.8) we have

$$\begin{aligned} [\mathrm{p(x)}(\mathrm{y}_2 \mathrm{T}_\alpha^\mathrm{a} \mathrm{y}_1 - \mathrm{y}_1 \mathrm{T}_\alpha^\mathrm{a} \mathrm{y}_2)]|_\mathrm{a}^\mathrm{b} &= \mathrm{p(b)}[\mathrm{y}_2(\mathrm{b})(\mathrm{T}_\alpha^\mathrm{a} \mathrm{y}_1)(\mathrm{b}) - \mathrm{y}_1(\mathrm{b})(\mathrm{T}_\alpha^\mathrm{a} \mathrm{y}_2)(\mathrm{b})] \\ &\quad - \mathrm{p(a)}[\mathrm{y}_2(\mathrm{a})(\mathrm{T}_\alpha^\mathrm{a} \mathrm{y}_1)(a) + y_1(a)(T_\alpha^a y_2)(a)]. \end{aligned} \tag{3.9}$$

Since ${c_1}^2 + {c_2}^2 > 0$, and ${r_1}^2 + {r_2}^2 > 0$, in the beginning, we suppose that $c_1 \neq 0$ and $r_1 \neq 0$, without loss of generalization and the proof of other cases going to be gotten similarly. Now in (3.2), we have

$$\begin{aligned} y(a) &= -\frac{c_2}{c_1} y'(a), \\ y(b) &= -\frac{r_2}{r_1} y'(b). \end{aligned} \tag{3.10}$$

Also, since $y_1, y_2 \in C^1[a,b]$, then we have $(T_\alpha^a y_1)(x) = (x-a)^{1-\alpha} {y'}_1(x)$ and $(T_\alpha^a y_2)(x) = (x-a)^{1-\alpha} {y'}_2(x)$. Now by putting that and (3.10) in equation (3.9), we acquire

$$p(b)(y_2(b)(T_\alpha^a y_1)(b) - y_1(b)(T_\alpha^a y_2)(b))$$

$$= -\frac{r_2}{r_1} y'_2(b)(T_\alpha^a y_1)(b) + \frac{r_2}{r_1} y'_1(b)(T_\alpha^a y_2)(b)$$

$$= -\frac{r_2}{r_1}\left(y'_2(b)(b-a)^{1-\alpha} y'_1(b) - y'_1(b)(b-a)^{1-\alpha} y'_2(b)\right) \tag{3.11}$$

$$= 0.$$

Similarly

$$y_2(a)(T_\alpha^a y_1)(a) + y_1(a)(T_\alpha^a y_2)(a) = 0. \tag{3.12}$$

Which demonstrated the Proof.

3.2. Orthogonality regarding the weight function

Definition 3.2.1. [13] Let f and g be two given function then we say f and g are α-orthogonal with respect to the weight function $\mu(x) \geq 0$, where

$$\int_a^b \mu(x) f(x) g(x) d_\alpha(x) = 0. \tag{3.13}$$

Theorem 3.2.2. Let y_1 and y_2 be two eigenfunctions of the conformable eigenvalue problem $(3.1) - (3.2)$ corresponding to distinct eigenvalues λ_1 and λ_2 respectively, then we can say that y_1 and y_2 are α −orthogonal concerning the weight function $w(x)$.

Proof. Since λ_1 and λ_2 are two distinct eigenvalues of eigenfunctions y_1 and y_2 then by using (3.4), we have

$$L(y_1, \alpha) = -\lambda_1 w(x) y_1, \tag{3.14}$$

$$L(y_2, \alpha) = -\lambda_2 w(x) y_2. \tag{3.15}$$

Through multiplying (3.14) by y_2 and (3.15) by y_1 and subtracting the two equations then we get

$$-(\lambda_1 - \lambda_2) w(x) y_1 y_2 = y_2 L(y_1, \alpha) - y_1 L(y_2, \alpha) \tag{3.16}$$

Performing the conformable integration of order α and applying the conformable Lagrange identity we receive that

$$-(\lambda_1-\lambda_2)\int_a^b w(x)y_1y_2\,d_\alpha(x)=\int_a^b(y_2\,L(y_1,\alpha)-y_1L(y_2,\alpha))d_\alpha(x)$$

$$=[p(x)(y_2T_\alpha^a y_1-y_1T_\alpha^a y_2)]|_a^b \tag{3.17}$$

$$=0,$$

with the righteousness of Proposition 3.1.3. Since the eigenvalues λ_1 and λ_2 are distinct, then in (3.17) we have $\int_a^b w(x)y_1y_2\,d_\alpha(x)=0$. Hence by Definition 3.2.1, the result is received.

Notation. Let $z=a+bi$, be any complex number with $a,b\in\mathbb{R}$ then the complex conjugate of z is denoted by $\bar{z}$ and defined as $\bar{z}=a-bi$. Also, we have $z\bar{z}=|z|^2$, because $|z|=\sqrt{z\bar{z}}$.

Theorem 3.2.3. Assume that y is a solution of the conformable Sturm-Liouville eigenvalue problem $(3.1)-(3.2)$, then the eigenvalues of y are real.

Proof. Taking the complex conjugate of the Sturm-Liouville eigenvalue problem $(3.1)-(3.2)$ and utilizing the way that the functions $p(x),q(x)$ and $w(x)$ are real-valued functions, we get

$$L(\bar{y},\alpha)=T_\alpha^a\big(p(x)\overline{T_\alpha^a y}\big)+q(x)\bar{y}$$

$$=-\bar{\lambda}w(x)\bar{y}. \tag{3.18}$$

$$c_1\bar{y}(a)+c_2\overline{y'}(a)=0,\quad {c_1}^2+\ {c_2}^2>0,$$

$$r_1\bar{y}(b)+r_2\overline{y'}(b)=0,\quad {r_1}^2+\ {r_2}^2>0. \tag{3.19}$$

Applying similar measures to the evidence of Proposition 3.1.3 and Theorem 3.2.2 for $y_1=y$ and $y_2=\bar{y}$, we receive that

$$-\big(\lambda-\bar{\lambda}\big)\int_a^b w(x)|y(x)|^2\,d_\alpha(x)=\int_a^b(\bar{y}\,L(y,\alpha)-yL(\bar{y},\alpha))d_\alpha(x)$$

$$=[p(x)(\bar{y}T_\alpha^a y-y\overline{T_\alpha^a \mathrm{y}})]|_a^b \tag{3.20}$$

$$=0\,,$$

since $w(x)$ is a real-valued function and $|y(x)|^2>0$, then $\int_a^b w(x)|y(x)|^2\,d_\alpha(x)\neq 0$. Hence $\lambda-\bar{\lambda}=0\Longrightarrow\lambda=\bar{\lambda}$. So, the eigenvalues must be real and completed the Proof.

3.3. The Conformable Wronskian function

Definition 3.3.1. [13] Assume f and g be two givens α-differentiable functions, then the conformable Wronskian function of f and g is denoted by $W_\alpha(f,g)$ and defined as

$$W_\alpha(f,g) = fT_\alpha^a g - gT_\alpha^a f. \tag{3.21}$$

Theorem 3.3.2. Suppose that $y_1$and y_2 be 2α-continuously differentiable on the interval $[a,b]$, then we have

$$W_\alpha(y_1,y_2) = \frac{W_\alpha(y_1,y_2)(a)p(a)}{p(x)}, \tag{3.22}$$

where y_1 and y_2 are the linearly independent solution of (3.1).

Proof. Since y_1 and y_2 are linearly independent then we can say

$$W_\alpha(y_1,y_2) = y_1 T_\alpha^a y_2 - y_2 T_\alpha^a y_1 \neq 0, \tag{3.23}$$

by taking the conformable derivative and using product rule in (3.23) we see that

$$T_\alpha^a W_\alpha(y_1,y_2) = y_1 T_\alpha^a T_\alpha^a y_2 - y_2 T_\alpha^a T_\alpha^a y_1 \, . \tag{3.24}$$

Similarly, using the product rule of (3.1) returns

$$T_\alpha^a T_\alpha^a y = -\frac{1}{p}\left(T_\alpha^a p T_\alpha^a y + (q+\lambda w)y\right) . \tag{3.25}$$

Replacing equation (3.25) in (3.24) yields

$$\begin{aligned} T_\alpha^a W_\alpha(y_1,y_2) &= -\frac{y_1}{p}\left(T_\alpha^a p T_\alpha^a y_2 + (q+\lambda w)y_2\right) \\ &\quad + \frac{y_2}{p}\left(T_\alpha^a p T_\alpha^a y_1 + (q+\lambda w)y_1\right) \\ &= \frac{T_\alpha^a p}{p}\left(y_2 T_\alpha^a y_1 - y_1 T_\alpha^a y_2\right) \\ &= -\frac{T_\alpha^a p}{p} W_\alpha(y_1,y_2) \end{aligned} \tag{3.26}$$

$$pT_\alpha^a W_\alpha(y_1,y_2) + T_\alpha^a p W_\alpha(y_1,y_2) = 0$$

$$T_\alpha^a\big(pW_\alpha(y_1,y_2)\big) = 0, \tag{3.27}$$

applying conformable integral of (3.27) we get

$$\begin{aligned} &\big(pW_\alpha(y_1,y_2)\big)(x) - \big(pW_\alpha(y_1,y_2)\big)(a) = 0 \\ &W_\alpha(y_1,y_2) = \frac{W_\alpha(y_1,y_2)(a)p(a)}{p(x)}. \end{aligned} \tag{3.28}$$

Theorem 3.3.3. The eigenvalues of the conformable eigenvalue problem $(3.1) - (3.2)$ are simple as follows.

Proof. Assume that y_1 and y_2 be two eigenfunctions for the eigenvalue λ, then in (3.16) we can get that

$$
\begin{aligned}
0 &= y_2 L(y_1,\alpha) - y_1 L(y_2,\alpha) \\
&= y_2 T_\alpha^a(p(x)T_\alpha^a y_1) - y_1 T_\alpha^a(p(x)T_\alpha^a y_2) \\
&= y_2(T_\alpha^a p T_\alpha^a y_1 + pT_\alpha^a T_\alpha^a y_1) - y_1(T_\alpha^a p T_\alpha^a y_2 + pT_\alpha^a T_\alpha^a y_2) \\
&= p(y_2 T_\alpha^a T_\alpha^a y_1 - y_1 T_\alpha^a T_\alpha^a y_2) + T_\alpha^a p(y_2 T_\alpha^a y_1 - y_1 T_\alpha^a y_2) \\
&= T_\alpha^a\big(p(y_2 T_\alpha^a y_1 - y_1 T_\alpha^a y_2)\big).
\end{aligned} \tag{3.29}
$$

Take conformable integral of (3.29) and by letting $\big(p(x)(y_2 T_\alpha^a y_1 - y_1 T_\alpha^a y_2)\big)(a) = c$, where c is constant then

$$p(y_2 T_\alpha^a y_1 - y_1 T_\alpha^a y_2) = c\ . \tag{3.30}$$

But here, the constant $c = 0$ because y_1 and y_2 are satisfy the same boundary conditions and $p(x) > 0$, therefore, we get

$$y_2 T_\alpha^a y_1 - y_1 T_\alpha^a y_2 = 0. \tag{3.31}$$

Hence, y_1 and y_2 are linearly dependent, since both of y_1 and y_2 are solutions to the conformable eigenvalue problem $(3.1) - (3.2)$ and $W_\alpha(y_1, y_2) = 0$, so proof obtained.

Theorem 3.3.4. (conformable Rayleigh Quotient). The eigenvalues λ of the problem (3.1) satisfy

$$\lambda = \frac{\int_a^b p(T_\alpha^a y)^2 d_\alpha(x) - \int_a^b q y^2 d_\alpha(x) - pyT_\alpha^a y|_a^b}{\int_a^b w y^2 d_\alpha(x)}. \tag{3.32}$$

Proof. Multiplying (3.1) by y and integrating with order α yields

$$\int_a^b y\, T_\alpha^a(p(x)T_\alpha^a y) d_\alpha(x) + \int_a^b q(x) y^2 d_\alpha(x) = -\lambda \int_a^b w y^2 d_\alpha(x), \tag{3.33}$$

using conformable integration by parts formula of the first integral, then we obtain

$$pyT_\alpha^a y|_a^b - \int_a^b p(T_\alpha^a y)^2 d_\alpha(x) + \int_a^b q(x) y^2 d_\alpha(x) = -\lambda \int_a^b w y^2 d_\alpha(x)\ , \tag{3.34}$$

which received the result.

Corollary 3.3.5. Letting $y_1 \in C^1[a,b]$ and $q(x) \leq 0$, then the eigenvalues of the problem (3.1) such that connected with homogeneous boundary conditions of Dirichlet or Neumann type are not negative.

Proof. Since the boundary conditions are the type of Dirichlet or Neumann then we have

$$yT_\alpha^a y|_a^b = 0. \tag{3.35}$$

So, from conformable Rayleigh Quotient theorem we have

$$\lambda = \frac{\int_a^b p(T_\alpha^a y)^2 d_\alpha(x) - \int_a^b qy^2 d_\alpha(x)}{\int_a^b wy^2 d_\alpha(x)}. \tag{3.36}$$

The result is obtained directly, since $q(x) \leq 0, p(x) > 0, w(x) > 0$ and we know that $a < b$, then the results of integrations are positive from a to b.

4. SINGULAR EIGENVALUE PROBLEM WITH MODIFIED FROBENIUS METHOD

Let $0 < \alpha \leq 1$, then the conformable differential equation of order 2α is defined as

$$T_\alpha T_\alpha y + p(x) T_\alpha y + q(x) y = 0, \tag{4.1}$$

where p and q are two functions of x defined on $[a, b]$ [23].

Definition 4.1. [23] If both functions $p(x)$ and $q(x)$ are α-analytic at a point $x_0 \epsilon [a, b]$ in the equation (4.1), then we say that x_0 is an α-ordinary point. Otherwise, x_0 is said to be α singular point.

Definition 4.2. [23] If $x_0 \epsilon [a, b]$ be α singular point of the function $p(x)$ or $q(x)$ and both functions $(x - x_0)^\alpha p(x)$ and $(x - x_0)^{2\alpha} q(x)$ are α-analytic at x_0 in the equation (4.1), then we say that x_0 is a regular α singular point. In the opposite case, we say that x_0 is an essential α singular point or irregular.

Application 4.3. a) We shall reflect on consideration on the following conformable differential equations

$$x^\alpha T_\alpha y - y = 0.$$

$$x^{2\alpha} T_\alpha T_\alpha y - 2x^\alpha y = 0.$$

$$x^{2\alpha} T_\alpha T_\alpha y - 2x^\alpha T_\alpha y + x^{2\alpha} y = 0.$$

The argument $x = 0$ is a regular α singular point for the equations mentioned above.

b) In the following, we shall consider more conformable differential equations that have another regular α singular points

$$(x - 1)^\alpha T_\alpha y - y = 0.$$

$$(x - 1)^{2\alpha} T_\alpha T_\alpha y - 2(x - 1)^\alpha T_\alpha y + (x - 1)^{2\alpha} y = 0.$$

For these equations, it is evident that the point $x = 1$ is a regular α singular point. To prove these processes, we can use Definition 3.1 and Definition 3.2 easily.

Theorem 4.4. (Frobenius' Theorem). [20] Suppose that $x_0 \epsilon [a, b]$ is a regular α singular point of equation (4.1), and $\alpha \epsilon (0,1]$, then the solution of this equation is defined as follows

$$y(x; r) = \sum_{n=0}^{\infty} c_n(r)(x - x_0)^{(n+r)\alpha}. \tag{4.2}$$

Let us $c_0 \neq 0$, and r is a number that must be recognized.

Theorem 4.5. 12] Assume x_0 is a regular α singular point of the equation (4.1), if r_1 and r_2 be two real roots of the conformable indicial equation, then the solution of the equation (4.1) has three cases.

Case 1. If r_1 and r_2 be two distinct real roots such that $r_1 - r_2 \neq n, n \epsilon \mathbb{N}$, then equation (4.1) has two linearly independent solutions of the form

$$y_1(x; r_1) = \sum_{n=0}^{\infty} c_n(r_1)(x - x_0)^{(n+r_1)\alpha}, \tag{4.3}$$

$$y_2(x; r_2) = \sum_{n=0}^{\infty} c_n(r_2)(x - x_0)^{(n+r_2)\alpha}, \tag{4.4}$$

where $x \epsilon (x_0, x_0 + \delta)$ with $\delta > 0$ and initial conditions are $c_0 = y(x_0), \alpha c_0 = T_\alpha y(x_0)$.

Case 2. If $r_1 = r_2$, then there are two linearly independent solutions of the equation (4.1) for $x \epsilon (x_0, x_0 + \delta)$ defined as follows

$$y_1(x; r_1) = \sum_{n=0}^{\infty} c_n(x - x_0)^{(n+r_1)\alpha}, \quad (c_0 \neq 0). \tag{4.5}$$

$$y_2(x; r_1) = ln|x - x_0| y_1(x; r_1) + \sum_{n=0}^{\infty} b_n(x - x_0)^{(n+r_1+1)\alpha}. \tag{4.6}$$

Case 3. If r_1 and r_2 be two distinct real roots and $r_1 - r_2 = n, n \epsilon \mathbb{N}$, then there are two linearly independent solutions of equation (4.1) defined by

$$y_1(x; r_1) = \sum_{n=0}^{\infty} c_n(x - x_0)^{(n+r_1)\alpha}, \quad (c_0 \neq 0). \tag{4.7}$$

$$y_2(x; r_2) = C. ln|x - x_0| y_1(x; r_1) + \sum_{n=0}^{\infty} b_n(x - x_0)^{(n+r_2)\alpha}, \tag{4.8}$$

For $x \epsilon (x_0, x_0 + \delta)$, where C is a constant number also sometimes it is equal to zero.

Application 4.6. A solution of the conformable Sturm-Liouville equation having modified Coulomb potential. We shall consider the following equation

$$T_\alpha T_\alpha y - \alpha^2 \frac{2}{x^{2\alpha}} y = \alpha^2 \lambda y, \tag{4.9}$$

where $\lambda \epsilon \mathbb{R}$ and $\alpha \epsilon (0,1]$. If $\alpha = 1$, then the equation (4.9) can be solved in an ordinary differential equation easily. Now for $\alpha \epsilon (0,1)$, in equation (4.9) we have $p(x) = 0$ and $q(x) = -\alpha^2(\frac{1}{x^{2\alpha}} - \lambda)$, it is clear that $x_0 = 0$ is α singular point of the above equation. Since both $(x - x_0)^\alpha p(x)$ and $(x - x_0)^{2\alpha} q(x)$ are α −analytic at $x_0 = 0$, then we say the point $x_0 = 0$ is a regular α singular point and we can use Frobenius method to solve equation (4.9)

$$y(x;r) = \sum_{n=0}^{\infty} c_n(r)(x-x_0)^{(n+r)\alpha}, \quad c_0 = 0$$
$$= \sum_{n=0}^{\infty} c_n x^{(n+r)\alpha}, \tag{4.10}$$

where $x > 0$. Now we obtain $T_\alpha y$ and $T_\alpha T_\alpha y$ of the form

$$T_\alpha y = \sum_{n=0}^{\infty} \alpha(n+r)c_n x^{(n+r-1)\alpha}. \tag{4.11}$$

$$T_\alpha T_\alpha y = \sum_{n=0}^{\infty} \alpha^2(n+r)(n+r-1)c_n x^{(n+r-2)\alpha}. \tag{4.12}$$

By Putting y and $T_\alpha T_\alpha y$ in equation (4.9), we get

$$\sum_{n=0}^{\infty} \alpha^2(n+r)(n+r-1)c_n x^{(n+r-2)\alpha} - \sum_{n=0}^{\infty} 2\alpha^2 c_n x^{(n+r-2)\alpha}$$
$$-\sum_{n=0}^{\infty} \alpha^2 \lambda c_n x^{(n+r)\alpha} = 0$$
$$\Longrightarrow \sum_{n=0}^{\infty} \alpha^2(n+r)(n+r-1)c_n x^{(n+r-2)\alpha} - \sum_{n=0}^{\infty} 2\alpha^2 c_n x^{(n+r-2)\alpha}$$
$$-\sum_{n=2}^{\infty} \alpha^2 \lambda c_{n-2} x^{(n+r-2)\alpha} = 0 \tag{4.13}$$

$$\Longrightarrow \alpha^2 r(r-1)c_0 x^{(r-2)\alpha} + \alpha^2 r(r+1)c_1 x^{(r-1)\alpha} - 2\alpha^2 c_0 x^{(r-2)\alpha}$$
$$-2\alpha^2 c_1 x^{(r-1)\alpha} + \sum_{n=2}^{\infty} \alpha^2(((n+r)(n+r-1)-2)c_n$$
$$-\lambda c_{n-2})x^{(n+r-2)\alpha} = 0 \tag{4.14}$$

$$\Longrightarrow \alpha^2(r(r-1)-2)c_0 x^{(r-2)\alpha} + \alpha^2(r(r+1)-2)c_1 x^{(r-1)\alpha}$$
$$+\sum_{n=2}^{\infty} \alpha^2(((n+r)(n+r-1)-2)c_n - \lambda c_{n-2})x^{(n+r-2)\alpha} = 0. \tag{4.15}$$

From (4.15) we have $\alpha^2(r(r-1)-2)c_0 = 0$, we know that $\alpha^2 \neq 0$ and $c_0 \neq 0$, hence $r(r-1)-2 = 0$ then we get $r_1 = 2$ and $r_2 = -1$. Also, we have that

$$\alpha^2(r(r+1)-2)c_1 = 0, \tag{4.16}$$

$$\alpha^2(((n+r)(n+r-1)-2)c_n - \lambda c_{n-2}) = 0, \tag{4.17}$$

since r_1 and r_2 are two distinct real roots and $r_1 - r_2 = 3$, then equation (4.9) has two linearly independent solutions. For the first solution when $r_1 = 2$, then in (4.16) it is obvious that $r(r+1)-2 \neq 0$ where $r_1 = 2$, then we have $c_1 = 0$. Also, in (4.17) we have

$$c_n = \frac{\lambda c_{n-2}}{n(n+3)} \;, \; n \geq 2. \tag{4.18}$$

$$c_2 = \frac{\lambda c_0}{2.5}, \; c_3 = 0, \; c_4 = \frac{\lambda^2 c_0}{2.5.4.7}, \; c_5 = 0, \; c_6 = \frac{\lambda^3 c_0}{2.5.4.7.6.9}, \; c_7 = 0, \; c_8 = \frac{\lambda^4 c_0}{2.5.4.7.6.9.8.11}, \cdots.$$

Hence the first solution of the equation (4.9) is

$$y_1 = c_0 x^{2\alpha} + \sum_{n=1}^{\infty} \frac{\lambda^n c_0}{\prod_{i=1}^{n} 2i(2i+3)} x^{(n+2)\alpha}. \tag{4.19}$$

The second solution by Theorem 4.5 we have

$$\begin{aligned} y_2(x; r_2) &= C.\, ln|x - x_0| y_1(x; r_1) + \sum_{n=0}^{\infty} b_n (x - x_0)^{(n+r_2)\alpha} \\ &= C.\, lnx y_1 + \sum_{n=0}^{\infty} b_n x^{(n+r_2)\alpha}. \end{aligned} \tag{4.20}$$

$$T_\alpha y_2 = C x^{-\alpha} y_1 + C lnx T_\alpha y_1 + \sum_{n=0}^{\infty} \alpha (n + r_2) b_n x^{(n+r_2-1)\alpha}.$$

$$\begin{aligned} T_\alpha T_\alpha y_2 = &-C\alpha x^{-2\alpha} y_1 + 2C x^{-\alpha} T_\alpha y_1 + C lnx T_\alpha T_\alpha y_1 \\ &+ \sum_{n=0}^{\infty} \alpha^2 (n + r_2)(n + r_2 - 1) b_n x^{(n+r_2-2)\alpha}. \end{aligned}$$

By using y_2 and $T_\alpha T_\alpha y$ in equation (4.9), we see that

$$-C\alpha x^{-2\alpha} y_1 + 2C x^{-\alpha} T_\alpha y_1 + C lnx T_\alpha T_\alpha y_1 + \sum_{n=0}^{\infty} \alpha^2 (n + r_2)(n + r_2 - 1) b_n x^{(n+r_2-2)\alpha}$$

$$C\alpha^2 \frac{2}{x^{2\alpha}} lnx y_1 - \alpha^2 \frac{2}{x^{2\alpha}} \sum_{n=0}^{\infty} b_n x^{(n+r_2)\alpha} - \alpha^2 \lambda C.\, lnx y_1 - \alpha^2 \lambda \sum_{n=0}^{\infty} b_n x^{(n+r_2)\alpha} = 0$$

$$\begin{aligned} \Rightarrow C \cdot lnx \cdot \left(T_\alpha T_\alpha y_1 - \alpha^2 \frac{2}{x^{2\alpha}} y_1 - \alpha^2 \lambda y_1 \right) - C\alpha x^{-2\alpha} y_1 + 2C x^{-\alpha} T_\alpha y_1 \\ + \sum_{n=0}^{\infty} \alpha^2 (n + r_2)(n + r_2 - 1) b_n x^{(n+r_2-2)\alpha} - \sum_{n=0}^{\infty} 2\alpha^2 b_n x^{(n+r_2-2)\alpha} \\ - \sum_{n=2}^{\infty} \alpha^2 \lambda b_{n-2} x^{(n+r_2-2)\alpha} = 0. \end{aligned}$$

$$\begin{aligned} \Rightarrow -C\alpha c_0 - C\alpha \sum_{n=1}^{\infty} \frac{\lambda^n c_0}{\prod_{i=1}^{n} 2i(2i+3)} x^{(n+2)\alpha} + 2C \cdot 2\alpha c_0 + 2C \sum_{n=1}^{\infty} \frac{(n+2)\alpha \lambda^n c_0}{\prod_{i=1}^{n} 2i(2i+3)} x^{(n+1)\alpha} \\ + \sum_{n=0}^{\infty} \alpha^2 (n + r_2)(n + r_2 - 1) b_n x^{(n+r_2-2)\alpha} - \sum_{n=0}^{\infty} 2\alpha^2 b_n x^{(n+r_2-2)\alpha} \\ - \sum_{n=2}^{\infty} \alpha^2 \lambda b_{n-2} x^{(n+r_2-2)\alpha} = 0. \end{aligned}$$

From this equation $-C\alpha c_0 + 2C2\alpha c_0 = 0 \Longrightarrow 3C\alpha c_0 = 0$, since $c_0 \neq 0$, then $C = 0$ and in (4.20) we have Frobenius method, known as in there the first term is not equal to zero. Therefore, $b_0 \neq 0$. Now in the above equation, we see that

$$\alpha^2(r_2(r_2-1)-2)b_0 x^{(r_2-2)\alpha} + \alpha^2(r_2(r_2+1)-2)b_1 x^{(r_2-1)\alpha} + \sum_{n=2}^{\infty} \alpha^2(((n+r_2)(n+r_2-1)-2)b_n - \lambda b_{n-2})x^{(n+r_2-2)\alpha} = 0. \quad (4.21)$$

By the same previous technique and take $r_2 = -1$, in (4.21) we have that $b_1 = 0$ and

$$b_n = \frac{\lambda b_{n-2}}{n(n-3)} \;,\; n \geq 2. \quad (4.22)$$

$$b_2 = \frac{\lambda c_0}{-2},\; b_3 = 0, b_4 = \frac{\lambda^2 c_0}{-2.4}, b_5 = 0, b_6 = \frac{\lambda^3 c_0}{-2.4.6.3}, b_7 = 0, b_8 = \frac{\lambda^4 c_0}{-2.4.6.3.8.5}, \cdots.$$

$$y_2 = b_0 x^{-\alpha} + \sum_{n=1}^{\infty} \frac{\lambda^n b_0}{\prod_{i=1}^{n} 2i(2i-3)} x^{(n-1)\alpha}. \quad (4.23)$$

For example, if $\lambda = 0$, then the first solution of equation (4.9) is

$$y_1 = c_0 x^{2\alpha}. \quad (4.24)$$

And the second solution is

$$y_2 = b_0 x^{-\alpha}. \quad (4.25)$$

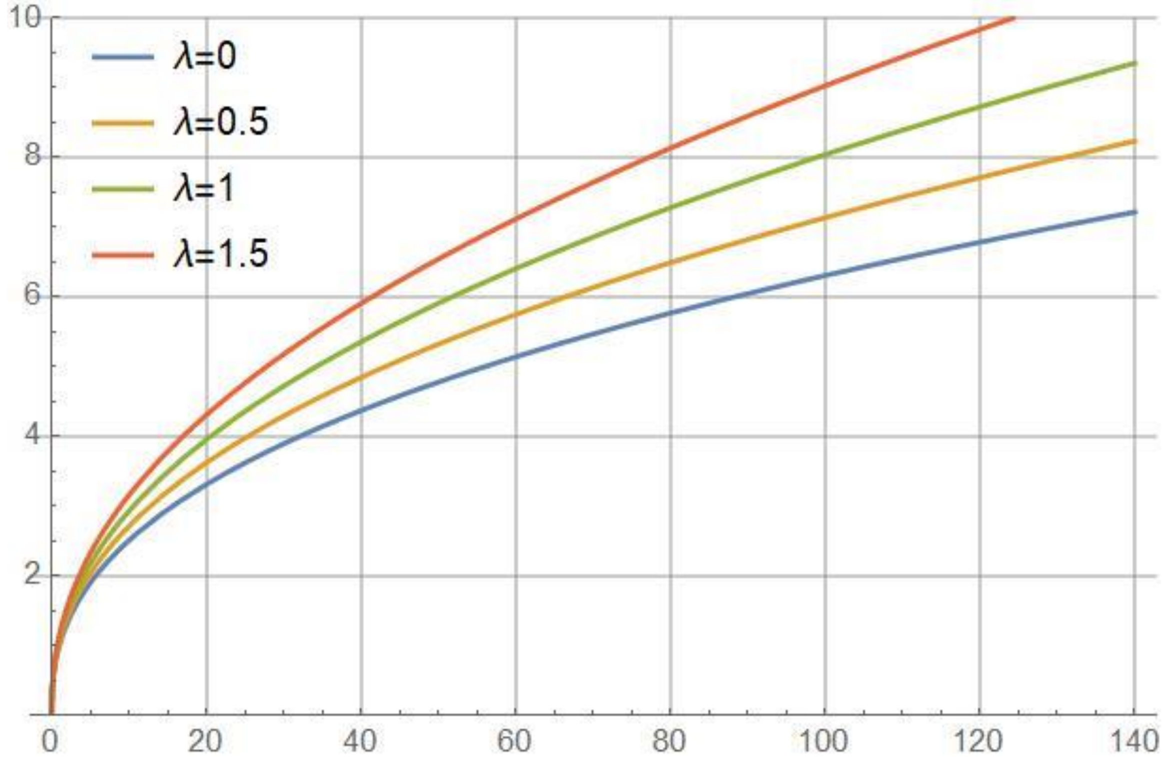

Figure 4.1. The first solution of the conformable Sturm-Liouville equation having modified Coulomb potential with order $\alpha = 0.2$ and $c_0 = 1$.

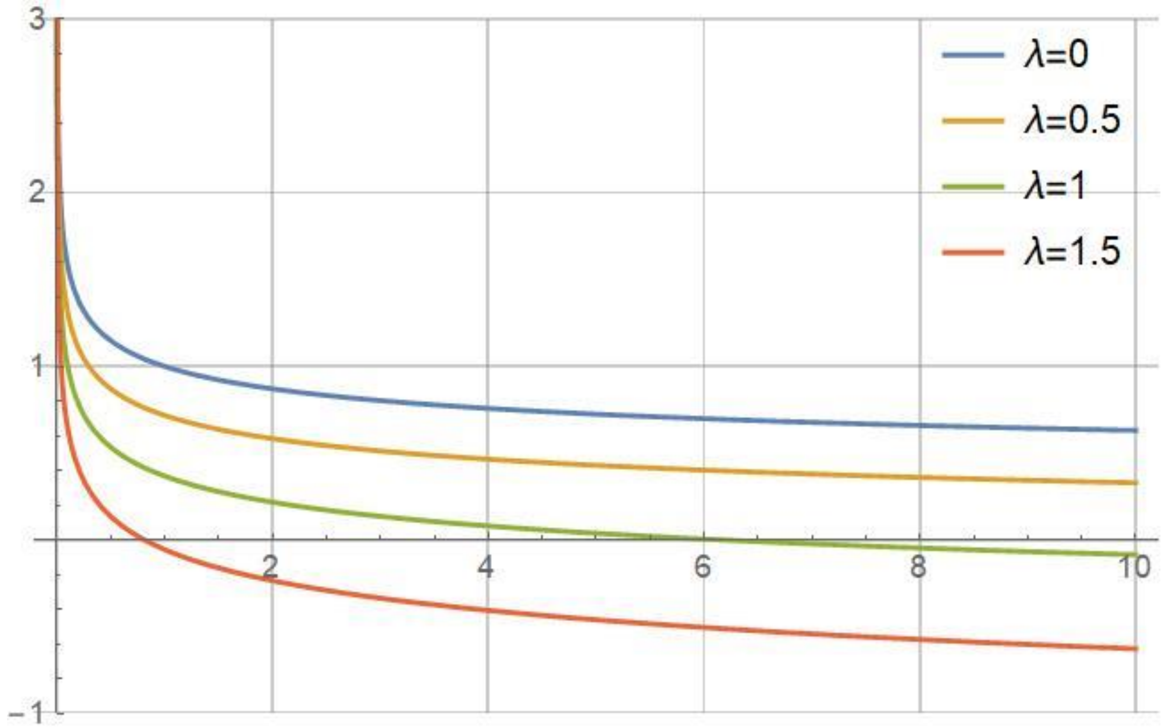

Figure 4.2. The second solution of the conformable Sturm-Liouville equation having modified Coulomb potential with order $\alpha = 0.2$ and $b_0 = 1$.

Application 4.7. A solution of the Conformable Sturm Liouville equation having modified hydrogen atom potential. We shall take the following equation into practice

$$T_\alpha T_\alpha y + \alpha^2 \left(\frac{2}{x^\alpha} - \frac{L(L+1)}{x^{2\alpha}}\right) y = 0, \tag{4.26}$$

where $\alpha \in (0,1]$ and $L \in \mathbb{R}$. In this equation we have $p(x) = 0$ and $q(x) = \alpha^2 \left(\frac{2}{x^\alpha} - \frac{L(L+1)}{x^{2\alpha}}\right)$. So, $x_0 = 0$ is α singular point. Also $x_0 = 0$ is a regular α singular point because in this point both $(x-x_0)^\alpha p(x)$ and $(x-x_0)^{2\alpha} q(x)$ are α −analytic. To solve equation (4.26) we use the Frobenius method

$$\begin{aligned} y(x;r) &= \sum_{n=0}^{\infty} c_n(r)(x-x_0)^{(n+r)\alpha} \\ &= \sum_{n=0}^{\infty} c_n x^{(n+r)\alpha}. \end{aligned} \tag{4.27}$$

$$T_\alpha y = \sum_{n=0}^{\infty} \alpha(n+r) c_n x^{(n+r-1)\alpha}. \tag{4.28}$$

$$T_\alpha T_\alpha y = \sum_{n=0}^{\infty} \alpha^2 (n+r)(n+r-1) c_n x^{(n+r-2)\alpha}. \tag{4.29}$$

Using (4.28) and (4.29) in equation (4.26) we get

$$\sum_{n=0}^{\infty} \alpha^2 (n+r)(n+r-1) c_n x^{(n+r-2)\alpha} + \sum_{n=0}^{\infty} 2\alpha^2 c_n x^{(n+r-1)\alpha}$$

$$-\sum_{n=0}^{\infty} L(L+1) c_n x^{(n+r-2)\alpha} \quad = 0$$

$$\begin{aligned} &\Longrightarrow \alpha^2 r(r-1) c_0 x^{(r-2)\alpha} - L(L+1) c_0 x^{(r-2)\alpha} \\ &+ \sum_{n=1}^{\infty} \alpha^2 (((n+r)(n+r-1) - L(L+1)) c_n + 2c_{n-1}) x^{(n+r-2)\alpha} = 0 \end{aligned} \tag{4.30}$$

In (4.30), we have $\alpha^2\big(r(r-1) - L(L+1)\big) = 0$. Since $\alpha^2 \neq 0$, then $r_1 = L+1$ and $r_2 = -L$. And

$$c_n = \frac{-2c_{n-1}}{(n+r-1)(n+r) - L(L+1)}, \qquad n \geq 1 \tag{4.31}$$

$$c_1 = \frac{-2c_0}{r(r+1) - L(L+1)},$$

$$c_2 = \frac{2^2 c_0}{(r(r+1) - L(L+1))(r+1)(r+2) - L(L+1)},$$

$$c_3 = \frac{-2^3 c_0}{\big(r(r+1) - L(L+1)\big)\big((r+1)(r+2) - L(L+1)\big)\big((r+2)(r+3) - L(L+1)\big)},$$

$$\vdots$$

$$c_n = \frac{(-1)^n 2^n c_0}{\prod_{i=1}^{n}\left((r+i-1)(r+i)-L(L+1)\right)}. \tag{4.32}$$

Since $L \in \mathbb{R}$, so we have four cases of solutions of equation (4.26) as following

Case 1. If $L \neq n,\ n \in \{\dots, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$, then $r_1 - r_2 \neq n \in \mathbb{N}$ then the equation (4.26) has two linearly independent solutions. By applying $r_1 = L+1$ in the equation (4.32), then the first solution is provided by

$$y_1(x; r_1) = c_0 x^{(L+1)\alpha} + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n c_0}{\prod_{i=1}^{n} i(i+2L+1)} x^{(n+L+1)\alpha}. \tag{4.33}$$

And the second solution for $r_2 = -L$ is given by

$$y_2(x; r_2) = c_0 x^{-L\alpha} + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n c_0}{\prod_{i=1}^{n} i(i-2L-1)} x^{(n-L)\alpha}. \tag{4.34}$$

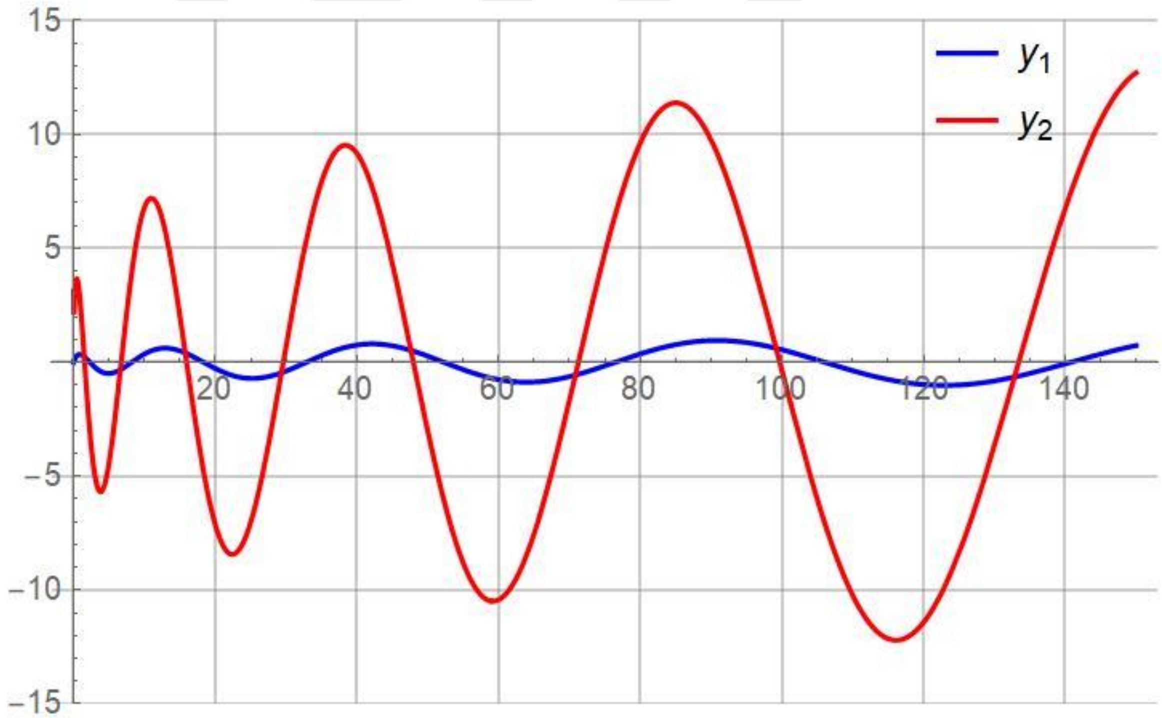

Figure 4.3. The first and second solutions of the conformable Sturm Liouville equation having modified hydrogen atom potential with order $\alpha = 0.9$ for $L = 0.1$ and $c_0 = 1$.

Case 2. If $L = -\frac{1}{2}$, then $r_1 = r_2 = \frac{1}{2}$. We have two linearly independent solutions of the equation (4.26). By applying $L = -\frac{1}{2}$ and $r_1 = \frac{1}{2}$ in (4.32) so we get the first solution as follows

$$y_1(x;r_1) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n c_0}{n!\,n!} x^{(n+\frac{1}{2})\alpha}. \quad (4.35)$$

For the second solution by Theorem 4.5, we have

$$\begin{aligned} y_2(x;r_1) &= ln|x-x_0|y_1(x;r_1) + \sum_{n=0}^{\infty} b_n (x-x_0)^{(n+r_1+1)\alpha} \\ &= lnx \cdot y_1 + \sum_{n=0}^{\infty} b_n x^{(n+\frac{3}{2})\alpha}. \end{aligned} \quad (4.36)$$

$$T_\alpha y_2 = x^{-\alpha} y_1 + lnx \cdot T_\alpha y_1 + \sum_{n=0}^{\infty} \alpha(n+\tfrac{3}{2}) b_n x^{(n+\frac{1}{2})\alpha}$$

$$T_\alpha T_\alpha y_2 = -\alpha x^{-2\alpha} y_1 + 2x^{-\alpha} T_\alpha y_1 + lnx \cdot T_\alpha T_\alpha y_1 + \sum_{n=0}^{\infty} \alpha^2 (n+\tfrac{3}{2})(n+\tfrac{1}{2}) b_n x^{(n-\frac{1}{2})\alpha}.$$

Using y_2 and $T_\alpha T_\alpha y_2$ in (4.26) we get

$$-\alpha x^{-2\alpha} y_1 + 2x^{-\alpha} T_\alpha y_1 + lnx \cdot T_\alpha T_\alpha y_1 + \sum_{n=0}^{\infty} \alpha^2 (n+\tfrac{3}{2})(n+\tfrac{1}{2}) b_n x^{(n-\frac{1}{2})\alpha}$$

$$+lnx \cdot \alpha^2 \left(\frac{2}{x^\alpha} - \frac{L(L+1)}{x^{2\alpha}}\right) y_1 + \sum_{n=0}^{\infty} 2\alpha^2 b_n x^{(n+\frac{1}{2})\alpha} - \sum_{n=0}^{\infty} \alpha^2 L(L+1) b_n x^{(n-\frac{1}{2})\alpha} = 0$$

$$\Longrightarrow \sum_{n=0}^{\infty} \frac{-\alpha(-1)^n 2^n c_0}{n!\,n!} x^{(n-\frac{3}{2})\alpha} + \sum_{n=0}^{\infty} \frac{2\alpha\left(n+\frac{1}{2}\right)(-1)^n 2^n c_0}{n!\,n!} x^{(n-\frac{3}{2})\alpha}$$

$$+\sum_{n=0}^{\infty} \alpha^2 (n+\tfrac{3}{2})(n+\tfrac{1}{2}) b_n x^{(n-\frac{1}{2})\alpha} + \sum_{n=0}^{\infty} 2\alpha^2 b_n x^{(n+\frac{1}{2})\alpha} + \sum_{n=0}^{\infty} \alpha^2 \tfrac{1}{4} b_n x^{(n-\frac{1}{2})\alpha} = 0$$

$$\Longrightarrow \sum_{n=1}^{\infty} \frac{2\alpha n(-1)^n 2^n c_0}{n!\,n!} x^{(n-\frac{3}{2})\alpha} + \sum_{n=1}^{\infty} \alpha^2 (n+\tfrac{1}{2})(n-\tfrac{1}{2}) b_{n-1} x^{(n-\frac{3}{2})\alpha}$$

$$+\sum_{n=2}^{\infty} 2\alpha^2 b_{n-2} x^{(n-\frac{3}{2})\alpha} + \sum_{n=1}^{\infty} \alpha^2 \tfrac{1}{4} b_{n-1} x^{(n-\frac{3}{2})\alpha} = 0$$

$$\Longrightarrow -4\alpha c_0 x^{-\frac{1}{2}\alpha} + \alpha^2 b_0 x^{-\frac{1}{2}\alpha} + \sum_{n=2}^{\infty} \left(\frac{2\alpha n(-1)^n 2^n c_0}{n!\,n!} + \alpha^2 (n^2 b_{n-1} + 2b_{n-2})\right) x^{(n-\frac{3}{2})\alpha} = 0$$

$-4\alpha c_0 + \alpha^2 b_0 = 0$, since $\alpha \neq 0$, then $b_0 = \frac{1}{\alpha} 4c_0 \neq 0$. Also, we have

$$\frac{2n(-1)^n 2^n c_0}{n!\,n!} + \alpha(n^2 b_{n-1} + 2b_{n-2}) = 0$$

$$b_{n-1} = \frac{-2b_{n-2}}{n^2} + \frac{(-1)^{n+1} 2^{n+1} c_0}{\alpha n n!\,n!}, \quad n \geq 2. \quad (4.37)$$

$$b_1 = -\frac{1}{2!2!}\left(1+\frac{1}{2}\right)\frac{1}{\alpha} 2^3 c_0,$$

$$b_2 = \frac{1}{3!3!}\left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{1}{\alpha} 2^4 c_0,$$

$$b_3 = -\frac{1}{4!4!}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\frac{1}{\alpha}2^5 c_0, \cdots,$$

$$\vdots$$

$$b_n = \frac{(-1)^n}{(n+1)!(n+1)!}\left(\sum_{i=0}^{n}\frac{1}{(i+1)}\right)\frac{1}{\alpha}2^{n+2}c_0.$$

Therefore, when $L = -\frac{1}{2}$ the second solution of equation (4.26) is given by

$$y_2(x;r_1) = lnx \cdot y_1 + \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!(n+1)!}\left(\sum_{i=0}^{n}\frac{1}{(i+1)}\right)\frac{1}{\alpha}2^{n+2}c_0 x^{\left(n+\frac{3}{2}\right)\alpha}. \tag{4.38}$$

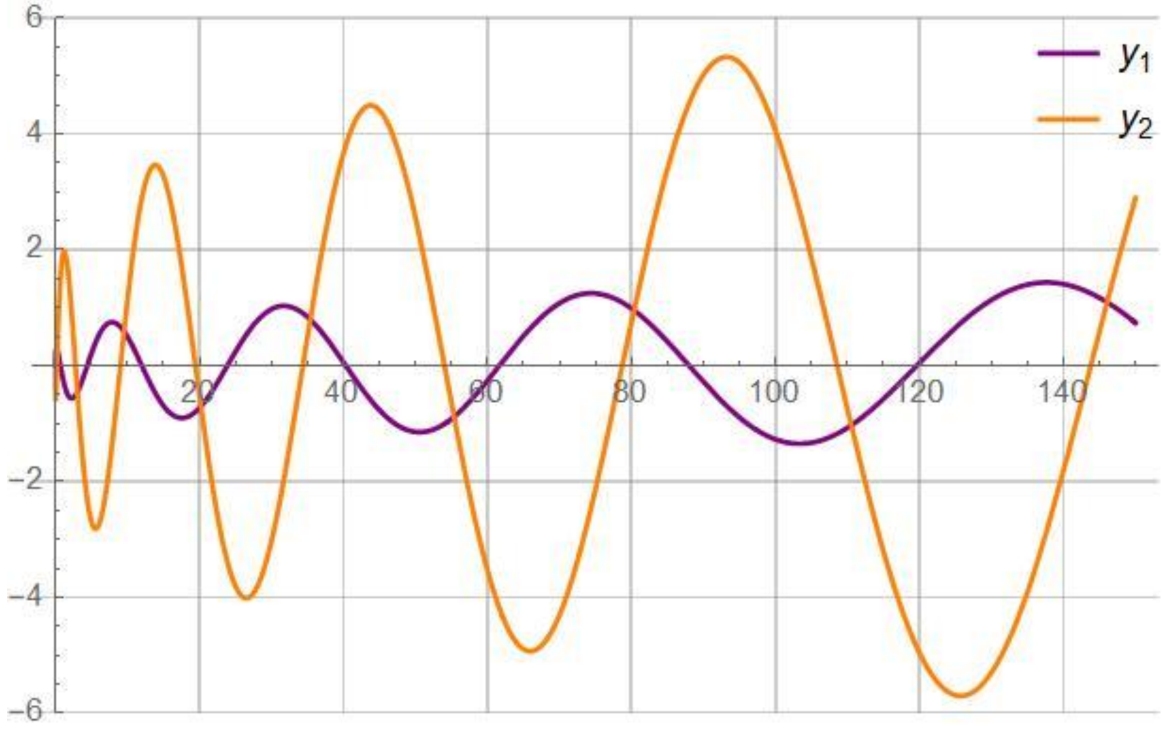

Figure 4.4. The first and second solutions of the conformable Sturm Liouville equation having modified hydrogen atom potential with order $\alpha = 0.9$ for $L = -0.5$ and $c_0 = 1$.

Case 3. If $L = n, n \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$, then $r_1 - r_2 = n \in \mathbb{N}$ and the equation (4.26) has two linearly independent solutions. The first solution by using $r_1 = L + 1$ in (4.32) is provided by

$$y_1(x;r_1) = c_0 x^{(L+1)\alpha} + \sum_{n=1}^{\infty}\frac{(-1)^n 2^n c_0}{\prod_{i=1}^{n} i(i+2L+1)} x^{(n+L+1)\alpha}. \tag{4.39}$$

Now for the second solution where $r_2 = -L$ we have

$$y_2(x;r_2) = C.\,ln|x - x_0| y_1(x;r_1) + \sum_{n=0}^{\infty} b_n (x - x_0)^{(n+r_2)\alpha} \tag{4.40}$$

$$= C.\,lnx \cdot y_1 + \sum_{n=0}^{\infty} b_n x^{(n-L)\alpha}.$$

$$T_\alpha y_2 = Cx^{-\alpha} y_1 + Clnx \cdot T_\alpha y_1 + \sum_{n=0}^{\infty} \alpha(n-L) b_n x^{(n-L-1)\alpha}.$$

$$T_\alpha T_\alpha y_2 = -C\alpha x^{-2\alpha} y_1 + 2Cx^{-\alpha} T_\alpha y_1 + Clnx \cdot T_\alpha T_\alpha y_1$$

$$+ \sum_{n=0}^{\infty} \alpha^2 (n-L)(n-L-1) b_n x^{(n-L-2)\alpha}.$$

By using y_2 and $T_\alpha T_\alpha y$ in equation (4.9), we see that

$$-C\alpha x^{-2\alpha} y_1 + 2Cx^{-\alpha} T_\alpha y_1 + Clnx \cdot T_\alpha T_\alpha y_1 + \sum_{n=0}^{\infty} \alpha^2 (n-L)(n-L-1) b_n x^{(n-L-2)\alpha}$$

$$+C.\,lnx \cdot \alpha^2 \left(\frac{2}{x^\alpha} - \frac{L(L+1)}{x^{2\alpha}} \right) y_1 + \sum_{n=0}^{\infty} 2\alpha^2 b_n x^{(n-L-1)\alpha} - \sum_{n=0}^{\infty} L(L+1) b_n x^{(n-L-2)\alpha} = 0$$

$$\Rightarrow -C\alpha c_0 x^{(L-1)\alpha} - \sum_{n=1}^{\infty} \frac{C\alpha(-1)^n 2^n c_0}{\prod_{i=1}^{n} i(i+2L+1)} x^{(n+L-1)\alpha} + (L+1)\alpha 2Cc_0 x^{(L-1)\alpha}$$

$$+ \sum_{n=1}^{\infty} \frac{(n+L+1)\alpha 2C(-1)^n 2^n c_0}{\prod_{i=1}^{n} i(i+2L+1)} x^{(n+L-1)\alpha} + \sum_{n=0}^{\infty} \alpha^2 (n-L)(n-L-1) b_n x^{(n-L-2)\alpha}$$

$$+ \sum_{n=0}^{\infty} 2\alpha^2 b_n x^{(n-L-1)\alpha} - \sum_{n=0}^{\infty} \alpha^2 L(L+1) b_n x^{(n-L-2)\alpha} = 0$$

$-C\alpha c_0 + (L+1)\alpha 2Cc_0 = 0 \Rightarrow C\alpha c_0(2L+1) = 0$. So, we get $C = 0$ and in above equation we obtain

$$\sum_{n=1}^{\infty} \alpha^2 (n(n-2L-1) b_n + 2\alpha^2 b_{n-1}) x^{(n-L-2)\alpha} = 0,$$

$$n(n-2L-1) b_n + 2\alpha^2 b_{n-1} = 0, \qquad n \geq 1.$$

By the same previous process, the second solution is given by

$$y_2(x; r_2) = b_0 x^{-L\alpha} + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n b_0}{\prod_{i=1}^{n} i(i-2L-1)} x^{(n-L)\alpha}, \quad i \neq (2L+1). \tag{4.41}$$

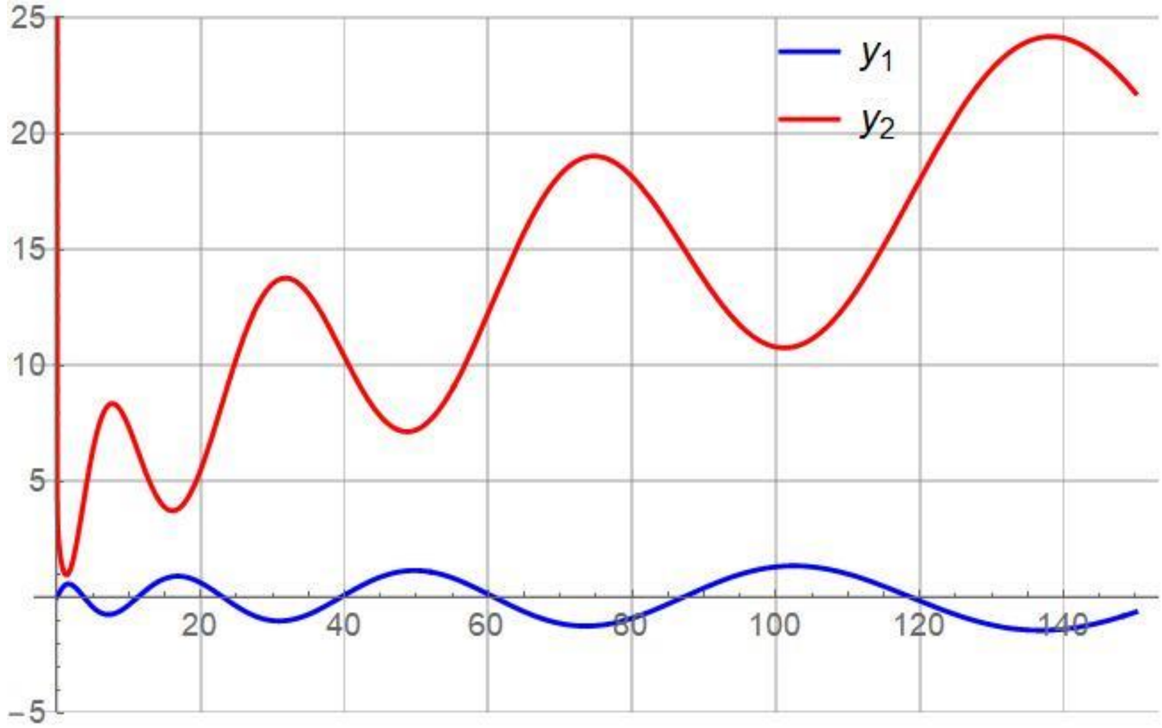

Figure 4.5. First and second solutions of the conformable Sturm Liouville equation having modified hydrogen atom potential with order $\alpha = 0.9$ for $L = 0.5$, $b_0 = 1$ and $c_0 = 1$.

Case 4. If $L = n, n \in \{\dots, -\frac{5}{2}, -2, -\frac{3}{2}, -1\}$, then $r_1 - r_2 = n \in \mathbb{N}$, where $r_1 = -L$ and $r_2 = L + 1$ then equation (4.26) has two linearly independent solutions. The first solution by taking $r_1 = -L$ in (4.32) is given by

$$y_1(x; r_1) = c_0 x^{-L\alpha} + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n c_0}{\prod_{i=1}^{n} i(i-2L-1)} x^{(n-L)\alpha}. \tag{4.42}$$

Now for the second solution where $r_2 = L + 1$ we have

$$\begin{aligned} y_2(x; r_2) &= C.\, ln|x - x_0| y_1(x; r_1) + \sum_{n=0}^{\infty} b_n (x - x_0)^{(n+r_2)\alpha} \\ &= C.\, lnx \cdot y_1 + \sum_{n=0}^{\infty} b_n x^{(n+L+1)\alpha}. \end{aligned} \tag{4.43}$$

$$\begin{aligned} T_\alpha T_\alpha y_2 = &-C\alpha x^{-2\alpha} y_1 + 2C x^{-\alpha} T_\alpha y_1 + C lnx \cdot T_\alpha T_\alpha y_1 \\ &+ \sum_{n=0}^{\infty} \alpha^2 (n + L)(n + L + 1) b_n x^{(n+L-1)\alpha}. \end{aligned}$$

Using y_2 and $T_\alpha T_\alpha y_2$ in (4.26) we obtain that

$$-C\alpha c_0 x^{(-2-L)\alpha} - \sum_{n=1}^{\infty} \frac{C\alpha(-1)^n 2^n c_0}{\prod_{i=1}^{n} i(i-2L-1)} x^{(n-L-2)\alpha} - 2CL\alpha c_0 x^{(-L-2)\alpha}$$

$$+\sum_{n=1}^{\infty} \frac{2C(n-L)\alpha(-1)^n 2^n c_0}{\prod_{i=1}^{n} i(i-2L-1)} x^{(n-L-2)\alpha} + \sum_{n=0}^{\infty} \alpha^2 (n+L)(n+L+1) b_n x^{(n+L-1)\alpha}$$

$$+\sum_{n=0}^{\infty} 2\alpha^2 b_n x^{(n+L)\alpha} - \sum_{n=0}^{\infty} \alpha^2 L(L+1) b_n x^{(n+L-1)\alpha} = 0,$$

$$\Rightarrow -C\alpha c_0 - -2CL\alpha c_0 = 0 \Rightarrow -C\alpha c_0(1+2L) = 0,$$

hence, C=0 and from the above equation we get

$$\sum_{n=1}^{\infty} \alpha^2 (n(n+2L+1)b_n + 2b_{n-1}) x^{(n+L-1)\alpha} = 0.$$

$$n(n+2L+1)b_n + 2b_{n-1} = 0, \quad n \geq 1.$$

Therefore, the second solution is

$$y_2(x; r_2) = b_0 x^{(L+1)\alpha} + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n b_0}{\prod_{i=1}^{n} i(i+2L+1)} x^{(n+L+1)\alpha}, \quad i \neq -(2L+1). \tag{4.44}$$

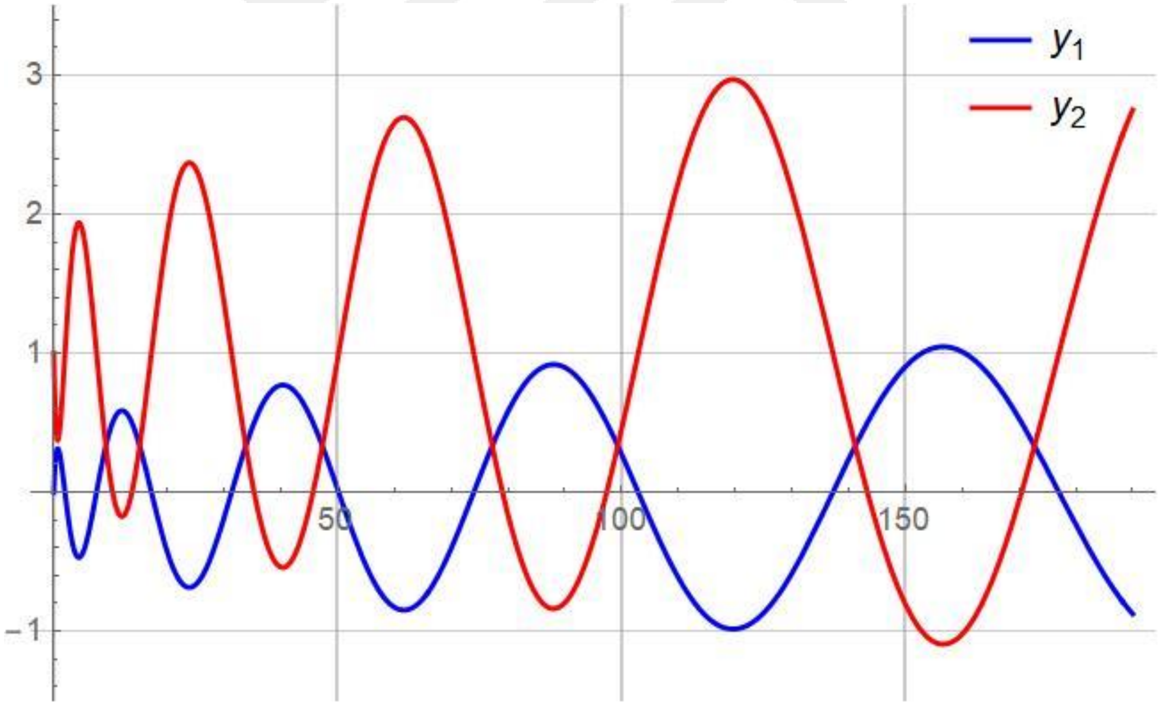

Figure 4.6. First and second solutions of the conformable Sturm Liouville equation having modified hydrogen atom potential with order $\alpha = 0.9$ for $L = -1$, $b_0 = 1$ and $c_0 = 1$.

5. CONFORMABLE STURM-LIOUVILLE PROBLEM WITH MODIFIED COULOMB POTENTIAL

In this chapter, we consider the conformable extension into account of the singular Sturm-Liouville eigenvalue problem having modified Coulomb potential

$$-T_\alpha^a T_\alpha^a y + \alpha^2 \left(\frac{A}{x^\alpha} + q(x)\right) y = \alpha^2 \lambda y. \tag{5.1}$$

Where $\frac{y(x)}{x^\alpha} \in C^{2\alpha}[0,\pi]$. We examine (5.1) with boundary conditions

$$\begin{aligned} &c_1 y(0) + c_2 y'(0) = 0, \quad {c_1}^2 + {c_2}^2 > 0. \\ &r_1 y(\pi) + r_2 y'(\pi) = 0, \quad {r_1}^2 + {r_2}^2 > 0. \end{aligned} \tag{5.2}$$

Assume that L be a linear operator described on certain elements and

$$L(y,\alpha) = -T_\alpha^a T_\alpha^a y + \alpha^2 \left(\frac{A}{x^\alpha} + q(x)\right) y, \tag{5.3}$$

then (5.1) can be written as

$$L(y,\alpha) = \alpha^2 \lambda y. \tag{5.4}$$

Theorem 5.1. Assume that y_1, y_2 be 2α-continuously differentiable functions on $[0,\lambda]$, then the following equation holds true

$$\int_0^\pi (y_2\, L(y_1,\alpha) - y_1 L(y_2,\alpha)) d_\alpha(x) = (y_2 T_\alpha^a y_1 - y_1 T_\alpha^a y_2)|_0^\pi. \tag{5.5}$$

This is said to be a theorem of the conformable Lagrange identity.

Proof. By using (5.3), we have

$$\begin{aligned} y_2 L(y_1,\alpha) - y_1 L(y_2,\alpha) &= -y_2 T_\alpha^a T_\alpha^a y_1 + \alpha^2 \left(\frac{A}{x^\alpha} + q(x)\right) y_1 y_2 \\ &\quad + y_1 T_\alpha^a T_\alpha^a y_2 - \alpha^2 \left(\frac{A}{x^\alpha} + q(x)\right) y_1 y_2 \\ &= y_1 T_\alpha^a T_\alpha^a y_2 - y_2 T_\alpha^a T_\alpha^a y_1. \end{aligned} \tag{5.6}$$

From conformable integration with order α of (5.6) and by applying conformable integration by parts in Theorem 2.3.1, we obtain

$$\int_0^\pi (y_2\, L(y_1,\alpha) - y_1 L(y_2,\alpha)) d_\alpha(x)$$

$$= \int_0^\pi y_1 T_\alpha^a T_\alpha^a y_2\, d_\alpha(x) - \int_0^\pi y_2 T_\alpha^a T_\alpha^a y_1\, d_\alpha(x)$$

$$= y_1 T_\alpha^a y_2|_0^\pi - \int_0^\pi T_\alpha^a y_1 T_\alpha^a y_2\, d_\alpha(x) \tag{5.7}$$

$$-y_2 T_\alpha^a y_1|_0^\pi + \int_0^\pi T_\alpha^a y_1 T_\alpha^a y_2\, d_\alpha(x)$$

$$= (y_1 T_\alpha^a y_2 - y_2 T_\alpha^a y_1)|_0^\pi.$$

Proposition 5.2. Suppose that y_1 and y_2 in $C^1[0,\lambda]$ that satisfies the boundary conditions (5.2), then the following holds true

$$(y_2 T_\alpha^a y_1 - y_1 T_\alpha^a y_2)|_0^\pi = 0. \tag{5.8}$$

Proof. From (5.7) we have

$$\begin{aligned}(y_2 T_\alpha^a y_1 - y_1 T_\alpha^a y_2)|_0^\pi &= y_2(\pi)(T_\alpha^a y_1)(\pi) - y_1(\pi)(T_\alpha^a y_2)(\pi) \\ &\quad -(y_2(0)(T_\alpha^a y_1)(0) + y_1(0)(T_\alpha^a y_2)(0)).\end{aligned} \tag{5.9}$$

Since ${c_1}^2 + {c_2}^2 > 0$, and ${r_1}^2 + {r_2}^2 > 0$, in the beginning, we suppose that $c_1 \neq 0$ and $r_1 \neq 0$, without loss of generalization and the proof of other cases, will be gotten similarly. Now in (5.2), we obtain

$$\begin{aligned} y(0) &= -\frac{c_2}{c_1} y'(0), \\ y(\pi) &= -\frac{r_2}{r_1} y'(\pi), \end{aligned} \tag{5.10}$$

since $y_1, y_2 \in C^1[0,\pi]$, then we have $(T_\alpha^a y_1)(x) = (x-a)^{1-\alpha} {y'}_1(x)$ and $(T_\alpha^a y_2)(x) = (x-a)^{1-\alpha} {y'}_2(x)$. Now by using that and (5.10) in equation (5.9), we receive

$$\begin{aligned} &(y_2(\pi)(T_\alpha^a y_1)(\pi) - y_1(\pi)(T_\alpha^a y_2)(\pi)) \\ &= -\frac{r_2}{r_1} {y'}_2(\pi)(T_\alpha^a y_1)(\pi) + \frac{r_2}{r_1} {y'}_1(\pi)(T_\alpha^a y_2)(\pi) \\ &= -\frac{r_2}{r_1}\left({y'}_2(\pi)(\pi-a)^{1-\alpha}{y'}_1(\pi) - {y'}_1(\pi)(\pi-a)^{1-\alpha}{y'}_2(\pi)\right) \\ &= 0. \end{aligned} \tag{5.11}$$

Similarly

$$y_2(0)(T_\alpha^a y_1)(0) + y_1(0)(T_\alpha^a y_2)(0) = 0. \tag{5.12}$$

Hence, Proof has been shown.

Theorem 2.3. [25] Let p be a point in real number, then for $p \geq 1$ the set $L_\alpha^p([a,b],\mathbb{R})$, $(a \geq 0)$ be a Banach space along with the norm such that defined for $f \in L_\alpha^p([a,b],\mathbb{R})$ as

$$\|f\|_{L_\alpha^p([a,b],\mathbb{R})} = \left(\int_a^b |f(t)|^p \, d_\alpha t\right)^{1/p}. \tag{5.13}$$

Furthermore, space $L_\alpha^2([a,b],\mathbb{R})$ is a Hilbert space along with inner product provided for all $(f,g) \in L_\alpha^p([a,b],\mathbb{R}) \times L_\alpha^p([a,b],\mathbb{R})$

$$\langle f,g\rangle_{L_\alpha^2([a,b],\mathbb{R})} = \int_a^b f(t)g(t)\, d_\alpha t. \tag{5.14}$$

Theorem 5.4. The eigenfunctions $y_1(x)$ and $y_2(x)$ of the singular Sturm-Liouville eigenvalue problem $(5.1) - (5.2)$ Matching to distinct eigenvalues λ_1 and λ_2 respectively are α-orthogonal where $(y_1, y_2) \in L_\alpha^2([0,\pi],\mathbb{R}) \times L_\alpha^2([0,\pi],\mathbb{R})$.

Proof. From (5.4) we have

$$L(y_1,\alpha) = \alpha^2\lambda_1 y_1. \tag{5.15}$$

$$L(y_2,\alpha) = \alpha^2\lambda_2 y_2. \tag{5.16}$$

Through multiplying (5.15) by y_2 and (5.16) by y_1 and subtracting the two equations then we obtain

$$(\lambda_1 - \lambda_2)\alpha^2 y_1 y_2 = y_2 L(y_1,\alpha) - y_1 L(y_2,\alpha). \tag{5.17}$$

By applying the conformable integration with order α and the conformable Lagrange identity we see that

$$\begin{aligned}(\lambda_1 - \lambda_2)\alpha^2 \int_0^\pi y_1 y_2 \, d_\alpha(x) &= \int_0^\pi (y_2\, L(y_1,\alpha) - y_1 L(y_2,\alpha)) d_\alpha(x) \\ &= [y_2 T_\alpha^a y_1 - y_1 T_\alpha^a y_2]|_0^\pi \\ &= 0.\end{aligned} \tag{5.18}$$

It is under Proposition 5.2. Since the eigenvalues λ_1 and λ_2 are distinct and $\alpha^2 \neq 0$, then in (5.18) we have $\int_a^b y_1 y_2 \, d_\alpha(x) = 0$. The result is received.

Theorem 5.5. The eigenvalues of the conformable singular Sturm-Liouville eigenvalue problem $(5.1) - (5.2)$ are real.

Proof. Suppose that y be a solution of the conformable Sturm-Liouville eigenvalue problem $(5.1) - (5.2)$ and $\frac{y(x)}{x^\alpha} \in C^{2\alpha}[0,\pi]$. By applying the complex conjugate of $(5.1) - (5.2)$, then we have

$$\begin{aligned} L(\bar{y},\alpha) &= -T_\alpha^\alpha T_\alpha^\alpha \bar{y} + \alpha^2 \left(\frac{A}{x^\alpha} + q(x)\right) \bar{y} \\ &= \alpha^2 \lambda \bar{y}. \end{aligned} \tag{5.19}$$

$$\begin{aligned} &c_1 \bar{y}(0) + c_2 \overline{y}'(0) = 0, \quad {c_1}^2 + {c_2}^2 > 0. \\ &r_1 \bar{y}(\pi) + r_2 \overline{y}'(\pi) = 0, \quad {r_1}^2 + {r_2}^2 > 0. \end{aligned} \tag{5.20}$$

Applying comparable measures to the proof of Theorem 5.4 for $y_1 = y$, $y_2 = \bar{y}$, $\lambda_1 = \lambda$ and $\lambda_2 = \bar{\lambda}$, we receive that

$$\begin{aligned} \alpha^2 (\lambda - \bar{\lambda}) \int_0^\pi |y(x)|^2 \, d_\alpha(x) &= \int_0^\pi (\bar{y}\, L(y,\alpha) - y L(\bar{y},\alpha)) d_\alpha(x) \\ &= [\bar{y} T_\alpha^\alpha y - y \overline{T_\alpha^\alpha y_2}]|_0^\pi \\ &= 0. \end{aligned} \tag{5.21}$$

Since $a < b$ and $|y(x)|^2$ is positive, then $\int_a^b |y(x)|^2 \, d_\alpha(x) \neq 0$ also, since $\alpha^2 > 0$. Therefore, $\lambda - \bar{\lambda} = 0 \Longrightarrow \lambda = \bar{\lambda}$. So, the eigenvalues must be real and the proof has completed.

Definition 5.6. [14] We suppose that $0 < \alpha \leq 1$ and $y_1(t), y_2(t), \cdots, y_n(t)$ are $(n-1)$ times α-differentiable functions, then the conformable Wronskian of that functions is denoted by $W_\alpha(y_1, y_2, \cdots, y_n)$ and defined as

$$W_\alpha(y_1, y_2, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ T_\alpha^\alpha y_1 & T_\alpha^\alpha y_2 & \cdots & T_\alpha^\alpha y_n \\ \vdots & \vdots & \cdots & \vdots \\ {}^{(n-1)}T_\alpha^\alpha y_1 & {}^{(n-1)}T_\alpha^\alpha y_2 & \cdots & {}^{(n-1)}T_\alpha^\alpha y_n \end{vmatrix}. \tag{5.22}$$

Theorem 5.7. Let $y \in C^{2\alpha}(0,\pi]$, we consider the representation of the solution of the singular Sturm-Liouville eigenvalue problem having modified Coulomb potential (5.1)

$$-T_\alpha T_\alpha y + \alpha^2\left(\frac{A}{x^\alpha} + q(x)\right)y = \alpha^2 \lambda y, \tag{5.23}$$

where the initial conditions are $y(0,\lambda) = 1$ and $T_\alpha^0 y(0,\lambda) = h$ is

$$y(x) = cos(\sqrt{\lambda}x^\alpha) + \frac{h}{\alpha\sqrt{\lambda}} sin(\sqrt{\lambda}x^\alpha)$$
$$+\frac{\alpha}{\sqrt{\lambda}}\int_0^x \left(\frac{A}{t^\alpha} + q(t)\right) y(t)\, sin(\sqrt{\lambda}(x^\alpha - t^\alpha))\, d_\alpha. \tag{5.24}$$

Proof. To solve (5.23) we need to find y_h and y_p.

$$T_\alpha^0 T_\alpha^0 y + \alpha^2 \lambda y = 0. \tag{5.25}$$

We look for the solution of $y = e^{r\frac{x^\alpha}{\alpha}}$, then $T_\alpha y = re^{r\frac{x^\alpha}{\alpha}}$ and $T_\alpha T_\alpha y = r^2 e^{r\frac{x^\alpha}{\alpha}}$. So, in (5.25) we have $(r^2 + \alpha^2\lambda)e^{r\frac{x^\alpha}{\alpha}} = 0 \Rightarrow$. Therefore, we get $r_1 = \alpha\sqrt{\lambda}i$ and $r_2 = -\alpha\sqrt{\lambda}i$. So,

$$y_h = c_1\, cos(\sqrt{\lambda}x^\alpha) + c_2\, sin(\sqrt{\lambda}x^\alpha). \tag{5.26}$$

Let $y_1 = \cos(\sqrt{\lambda}x^\alpha)$ and $y_2 = \sin(\sqrt{\lambda}x^\alpha)$, then

$$y_p = u_1 y_1 + u_2 y_2 \tag{5.27}$$

$$T_\alpha y_p = T_\alpha u_1 y_1 + u_1 T_\alpha y_1 + T_\alpha u_2 y_2 + u_2 T_\alpha y_2 \tag{5.28}$$

$$T_\alpha u_1 y_1 + T_\alpha u_2 y_2 = 0 \tag{5.29}$$

$$T_\alpha T_\alpha y_p = T_\alpha u_1 T_\alpha y_1 + u_1 T_\alpha T_\alpha y_1 + T_\alpha u_2 T_\alpha y_2 + u_2 T_\alpha T_\alpha y_2. \tag{5.30}$$

By putting (5.27) and (5.30) in (5.23), we obtain

$$-T_\alpha u_1 T_\alpha y_1 - u_1 T_\alpha T_\alpha y_1 - T_\alpha u_2 T_\alpha y_2 - u_2 T_\alpha T_\alpha y_2 + \alpha^2\left(\frac{A}{x^\alpha} + q(x)\right)y$$
$$= \alpha^2 \lambda u_1 y_1 + \alpha^2 \lambda u_2 y_2$$

$$\Longrightarrow T_\alpha u_1 T_\alpha y_1 + T_\alpha u_2 T_\alpha y_2 + u_1(T_\alpha T_\alpha y_1 + \alpha^2 \lambda y_1) + u_2(T_\alpha T_\alpha y_2 + \alpha^2 \lambda y_2)$$
$$= \alpha^2\left(\frac{A}{x^\alpha} + q(x)\right)y$$

$$\Longrightarrow T_\alpha u_1 T_\alpha y_1 + T_\alpha u_2 T_\alpha y_2 = \alpha^2\left(\frac{A}{x^\alpha} + q(x)\right)y \tag{5.31}$$

From (5.29) and (5.31) we see that

$$T_\alpha u_1 \cos(\sqrt{\lambda}x^\alpha) + T_\alpha u_2 \sin\left(\sqrt{\lambda}x^\alpha\right) = 0 \tag{5.32}$$

$$-\alpha\sqrt{\lambda}T_\alpha u_1 \sin\left(\sqrt{\lambda}x^\alpha\right) + \alpha\sqrt{\lambda}T_\alpha u_1 \cos(\sqrt{\lambda}x^\alpha) = \alpha^2\left(\frac{A}{x^\alpha} + q(x)\right)y \tag{5.33}$$

using Definition 5.5 to find α-Wronskian of y_1 and y_2 as

$$\begin{aligned} W_\alpha(y_1, y_2) &= y_1 T_\alpha y_2 - y_2 T_\alpha y_1 \\ &= \alpha\sqrt{\lambda}\,(\cos(\sqrt{\lambda}x^\alpha))^2 + \alpha\sqrt{\lambda}(\sin(\sqrt{\lambda}x^\alpha))^2 \\ &= \alpha\sqrt{\lambda}\,. \end{aligned} \tag{5.34}$$

$$T_\alpha u_1 = \frac{\begin{vmatrix} 0 & \sin\left(\sqrt{\lambda}x^\alpha\right) \\ \alpha^2\left(\frac{A}{x^\alpha}+q(x)\right)y & \cos(\sqrt{\lambda}x^\alpha) \end{vmatrix}}{\alpha\sqrt{\lambda}} = -\frac{\alpha}{\sqrt{\lambda}}\left(\frac{A}{x^\alpha} + q(x)\right)y \sin\left(\sqrt{\lambda}x^\alpha\right). \tag{5.35}$$

$$T_\alpha u_2 = \frac{\begin{vmatrix} \cos(\sqrt{\lambda}x^\alpha) & 0 \\ \sin\left(\sqrt{\lambda}x^\alpha\right) & \alpha^2\left(\frac{A}{x^\alpha}+q(x)\right)y \end{vmatrix}}{\alpha\sqrt{\lambda}} = \frac{\alpha}{\sqrt{\lambda}}\left(\frac{A}{x^\alpha} + q(x)\right)y \cos(\sqrt{\lambda}x^\alpha). \tag{5.36}$$

$$u_1 = -\frac{\alpha}{\sqrt{\lambda}}\int_0^x \left(\frac{A}{t^\alpha} + q(t)\right)y(t) \sin\left(\sqrt{\lambda}t^\alpha\right) d_\alpha t. \tag{5.37}$$

$$u_2 = \frac{\alpha}{\sqrt{\lambda}}\int_0^x \left(\frac{A}{t^\alpha} + q(t)\right)y(t) \cos(\sqrt{\lambda}t^\alpha)\, d_\alpha t. \tag{5.38}$$

By putting the value of u_1 and u_2 in (5.27) we get

$$\begin{aligned} y_p &= -\frac{\alpha\cos(\sqrt{\lambda}x^\alpha)}{\sqrt{\lambda}}\int_0^x \left(\frac{A}{t^\alpha} + q(t)\right)y(t)\sin(\sqrt{\lambda}t^\alpha)\, d_\alpha t \\ &\quad + \frac{\alpha\sin(\sqrt{\lambda}x^\alpha)}{\sqrt{\lambda}}\int_0^x \left(\frac{A}{t^\alpha} + q(t)\right)y(t)\cos(\sqrt{\lambda}t^\alpha)\, d_\alpha t \\ &= \frac{\alpha}{\sqrt{\lambda}}\int_0^x \left(\frac{A}{t^\alpha} + q(t)\right)y(t)[\,\sin(\sqrt{\lambda}x^\alpha)\cos(\sqrt{\lambda}t^\alpha) \\ &\quad - \cos(\sqrt{\lambda}x^\alpha)\sin(\sqrt{\lambda}t^\alpha)] d_\alpha t \end{aligned} \tag{5.39}$$

$$= \frac{\alpha}{\sqrt{\lambda}} \int_0^x \left(\frac{A}{t^\alpha} + q(t) \right) y(t) \, sin(\sqrt{\lambda}(x^\alpha - t^\alpha)) \, d_\alpha t.$$

Hence, from (5.26) and (5.39) we see that

$$y(x) = c_1 \, cos(\sqrt{\lambda} x^\alpha) + c_2 \, sin(\sqrt{\lambda} x^\alpha) + \frac{\alpha}{\sqrt{\lambda}} \int_0^x \left(\frac{A}{t^\alpha} + q(t) \right) y(t) \, sin(\sqrt{\lambda}(x^\alpha - t^\alpha)) \, d_\alpha t. \tag{5.40}$$

In boundary conditions, we have $1 = c_1 cos0 + c_2 sin0 \Rightarrow c_1 = 1$ and $h = -\alpha\sqrt{\lambda} c_1 sin0 + \alpha\sqrt{\lambda} c_2 cos0 \Rightarrow c_2 = \frac{h}{\alpha\sqrt{\lambda}}$, then in (5.40), we have

$$y(x) = cos(\sqrt{\lambda} x^\alpha) + \frac{h}{\alpha\sqrt{\lambda}} sin(\sqrt{\lambda} x^\alpha) + \frac{\alpha}{\sqrt{\lambda}} \int_0^x \left(\frac{A}{t^\alpha} + q(t) \right) y(t) \, sin(\sqrt{\lambda}(x^\alpha - t^\alpha)) \, d_\alpha t. \tag{5.41}$$

6. CONCLUSIONS

This research was conducted about the history of fractional calculus. The literature of the researcher about the idea of fractional was explained.

The fractional Taylor power series representation and fractional Laplace transform for some certain functions which they cannot be calculated by old ideas were expanded by using a conformable derivative that discovered by Khalil at al., The conformable generalization of the well-known Sturm-Liouville eigenvalue problems was proposed. The fundamental results of the proposed model properties of a conformable Sturm-Liouville problem with a conformable derivative are very identical with a classical derivative. Also, the eigenvalues of this problem are real and simple. The eigenfunctions corresponding to distinct eigenvalues are orthogonal.

The valuables of the solution of modified power series around a regular singular point $x = 0$ of the conformable eigenvalue problem for singular Sturm- Liouville operator was processed. The equation of conformable Sturm−Liouville problem having modified Coulomb and hydrogen atom potentials around the regular singular point zero was solved. Besides, the obtained solutions were examined. The conformable extension by studying the singular Sturm-Liouville eigenvalue problem having modified Coulomb potential recognized. The eigenfunctions of the singular Sturm-Liouville eigenvalue problem corresponding to distinct eigenvalues are α-orthogonal and that the eigenvalues of this problem are real were proved. Furthermore, the representation of the solution of the singular Sturm-Liouville eigenvalue problem having modified Coulomb potential was discovered.

REFERENCES

[1] Podlubny, I. (1998). *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Elsevier. Academic Press. Vol. 198.

[2] Bas, E., Metin, F. (2013). Fractional singular Sturm-Liouville operator for Coulomb potential, *Advances in Difference Equations.* Vol. 1, 300. https://doi.org/10.1186/1687-1847-2013-300

[3] Wu, G.C., Baleanu, D. (2013). New applications of the variational iteration method-From differential equations to q-Fractional difference equations, *Advances in Difference Equations*. Vol. 21, 1-25. https://doi.org/10.1186/1687-1847-2013-21.

[4] Băleanu, D. , Mustafa, O.G. (2011). On the existence interval for the initial value problem of a fractional differential equation, *Hacettepe Journal of Mathematics and Statistics*. Vol. 40, 581–587.

[5] Cabrera, I.J., Harjani, J., Sadarangani, K.B. (2012). Existence and uniqueness of positive solutions for a singular fractional three-point boundary value problem, *Abstract and Applied Analysis*. Vol. 2012, 1-18, https://doi.org/10.1155/2012/803417.

[6] Bas, E., Acay, B. (2020). The direct spectral problem via local derivative including truncated Mittag-Leffler function, *Applied Mathematics and Computation*. Vol 367, https://doi.org/10.1016/j.amc.2019.124787.

[7] Panakhov, E.S., Bas, E. (2012). Inverse Problem Having Special Singularity Type From Two Spectra, *Tamsui Oxford Journal of Information & Mathematical Sciences (TOJIMS).* Vol. 28(3), 239-258.

[8] Panakhov, E.S., Sat, M. (2013). Reconstruction of potential function for sturm-liouville operator with coulomb potential, *Boundary Value Problems.* Vol. 49, 1–9, https://doi.org/10.1186/1687-2770-2013-49.

[9] Bleanu, D., Mustafa, O.G., Agarwal, R.P. (2011). Asymptotic integration of (1 + alpha) -order fractional differential equations, *Computers and Mathematics with Applications.* Vol. 62, 1492–1500.

[10] Eid, R., Muslih, S.I., Baleanu, D., Rabei, E. (2009). On fractional Schrödinger equation in α-dimensional fractional space, *Nonlinear Analysis: Real World Applications.* Vol. 10, 3, 1299–1304.

[11] Hilfer, R. (2000). Fractional diffusion based on Riemann-Liouville fractional derivatives, *The Journal of Physical Chemistry B*. Vol. 104, 16, 3914–3917.

[12] Unal, E., Gokdogan, A., Celik, E. (2017). Solutions around a regular α singular point of a sequential conformable fractional differential equation, *Kuwait Journal of Science.* Vol. 44,

1, 9–16.

[13] Al-Refai, M., Abdeljawad, T. (2017). Fundamental results of conformable Sturm-Liouville eigenvalue problems, *Complexity*. Vol. 2017, 1-7, https://doi.org/10.1155/2017/3720471.

[14] Gökdoğan, A., Ünal, E., Çelik, E. (2016). Existence and uniqueness theorems for sequential linear conformable fractional differential equations, *Miskolc Mathematical Notes.* Vol. 17,1, 267–279.

[15] Bas, E., Ozarslan, R. (2019). Theory of discrete fractional Sturm–Liouville equations and visual results, *AIMS Mathematics.* Vol. 4, 3, 593–612.

[16] Bas, E., Ozarslan, R. (2017). Sturm-Liouville problem via Coulomb type in difference equations, *Filomat*. Vol. 31,4, 989–998.

[17] Ozarslan, R., Ercan, A., Bas, E. (2019). β− type fractional Sturm-Liouville Coulomb operator and applied results, *Mathematical Methods in the Applied Sciences.* Vol. 42, 6648–6659, https://doi.org/10.1002/mma.5769.

[18] Khalil, R., Abu Hammad, M. (2014). Legendre fractional differential equation and Legender fractional polynomials, *International Journal of Applied Mathematical Research*. Vol. 3, 3, 214–219.

[19] Lakshmikantham, V., Levitan, B.M. and Sargsjan, I.S. (1977). Introduction to spectral theory: Selfadjoint ordinary differential operators, *Bulletin of the American Mathematical Society.* Vol. 83, 188–189.

[20] Gökdoğan, A., Ünal, E., Çelik E. (2015) Conformable fractional Bessel equation and Bessel functions, ArXiv Preprint ArXiv:1506.07382.

[21] Abdeljawad, T. (2015). On conformable fractional calculus, *Journal of Computational and Applied Mathematics,* Elsevier. Vol. 279, 1, 57–66.

[22] Khalil, R., Al Horani, M., Yousef, A., Sababheh, M. (2014) A new definition of fractional derivative, *Journal of Computational and Applied Mathematics.* Vol. 26, 65–70, https://doi.org/10.1016/j.cam.2014.01.002.

[23] Zill, D.G., Cullen, M.R. (2009). *Differential Equations with Boundary-Value Problems*. Academic Press, New York.

[24] Wang, Y., Zhou, J., Li, Y. (2016). Fractional Sobolev's spaces on time scales via conformable fractional calculus and their application to a fractional differential equation on time scales, *Advances in Mathematical Physics.* Vol. 2016, 1-21, https://doi.org/10.1155/2016/9636491.

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1. Erdal Bas and Isam Najemadeen "Singular eigenvalue problem with modified Frobenius method" Cmes2019 Conference Proceeding, vol 2019, No. 84, p. 717-726.