ENTANGLEMENT IN SPIN SYSTEMS AND CAVITY QUANTUM ELECTRODYNAMICS (JAYNES-CUMMINGS MODEL)

by

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ABSTRACT

Entanglement in spin systems and Cavity Quantum Electrodynamics (Jaynes-Cumings Model)

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We have studied the quantum entanglement theory from several aspects. We investigate the properties of twelve entanglement measures for pure states of qutrits by calculating and by comparing the entanglement of the eigenstates of a spin-1 *antiferromagnetic Heisenberg chain*. We find that, among these measures only von Neumann Entropy, negativity, and a lower bound of entanglement of formation (LBE) can catch the degree of entanglement of the states and distinguish them, exactly. We, also, investigate the dynamical properties of the entanglement between two identical atoms and the cavity field in the presence of phase decoherence. The effects of the phase decoherence on the entanglement is obviously displayed. Finally, we also report the results of a study of entanglement dynamics in Jaynes-Cummings Model (JCM) with phase stochastic atom-field interaction parameters. The noise term considered in this work is a phase telegraph noise.

We have found that, the mean dwell time T and the phase jump value a determines the time that the entanglement dies out. Also, the temperature dependence of the entanglement is found to be different in the noisy case compared to the non-noisy JCM.

Keywords: Entanglement, tensor product, partial trace, entanglement measures, spin-1 Heisenberg model, Jaynes-Cummings model.

ÖZET

Spin sistemlerinde dolaşıklık ve Oyuk Kuantum elektrodinamik Jaynes- Cummings modeli

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Bu tezde kuantum dolaşıklık teorisini birkaç açıdan çalıştık. Oniki tane dolaşıklık ölçütünün özelliklerini bir spin-1 Heisenberg modelinin saf qutrit durumlarının dolaşıklıklarını hesaplayarak ve karşılaştırarak araştırdık. Bu ölçütlerin içerisinde sadece von Neumann Entropi'nin negativitenin Schmidt Sayısı'nın, ve Dolaşıklık oluşumunun alt sınırının durumların dolaşıklık derecesini yakalayabildiklerini ve onları tam olarak ayırtedebildiklerini bulduk. Bir de, faz kaybının olduğu durumunda iki tane aynı atomla oyuk alan arasındaki dolaşıklığın dinamik özelliklerini inceledik. Faz kaybının dolaşıklık üzerindeki etkileri açıkça gösterilmiştir. Son olarak, stokastik atom-alan etkileşim parametreli Jaynes-Cummings modelindeki dolaşıklık dinamiğinin çalışmasını bildirdik. Bu çalışmada düşünülen gürültü terimi faz telegraf gürültüdür. Ortalama yaşam süresinin T ve faz atlama değerinin a dolaşıklığın bitme süresini belirlediğini bulduk. Bir de gürültülü durumdaki JCM'deki dolaşıklığın sıcaklık bağımlılığının gürültüsüz

JCM'ne nazaran farklı olduğu bulundu.

Anahtar Kelimeler: Dolaşıklık, dolaşıklık ölçütleri, tensor çarpımı, kısmi iz, spin-1 Heisenberg modeli, Jaynes-Cummings modeli.

To My Family . . .

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CHAPTER 1

INTRODUCTION

Quantum computation and quantum information theory have been very active areas of research for the last two decades. Besides the research for the fundamentals of quantum theory, the aim of these studies is to exploit the unconventional quantum structures to improve and / or develop new techniques in communication and computation to a level that is not possible with the current classical approaches. Along these lines, one can mention the quantum search algorithms which improves the speed of searching in an unstructured database of size N to $\sqrt{N}$, instead of classical N, prime number factoring algorithm of Shore [1], and adiabatic algorithms which are aimed at reducing the NP-complete classical problems to P class. Secure communication is another active area of research that utilizes the quantum information theory.

At the core of quantum mechanical speed-up of the proposed algorithms and security of the communication is a concept that was introduced to Q.M. by Heisenberg and EPR [2], called entanglement. Entanglement is a property of composite systems and its existence in a system signifies the fact that, the system can not be described in terms of the independent constituents. Since entanglement is a resource, like energy [3], it needs to be quantified. There has been a large number of efforts to find measures that would characterize the entanglement of a quantum mechanical state. Since, Q.M. states can come in many varities, such as discrete n-level ($n = 1, 2, 3, ..$) or continuous, pure or mixed states, it is very difficult to have a single measure that would be able to detect and quantify the entanglement in a given state.

Bipartite two-level systems are the most widely studied systems, partly because of the proposed quantum computers are expected to utilize qubits as their buildings blocks. The entanglement properties of bipartite pure as well as mixed states of qubits is well characterized by the so-called entanglement of formation which is calculated by using another measure called concurrence [4, 5].

Bipartite three-level systems can be considered as the generalisation of the qubit case and a single three-level system is called qutrit. There is no universally accepted entanglement measure for qutrit systems.

The first subject investigated in this thesis is a test of all the proposed bipartite qutrit measures for the eigenstates of spin-1 Heisenberg chain. Here, the aim was to measure the performance of entanglement measures that exist in the literature for differentiating between bipartite qutrit states.

The analysis on the entanglement measures covered this study shows that; Of these entanglement measures, von Neumann entropy, negativity, and the lower bound of EOF are very effective entanglement measures. Their results are quite dependable. We can get an idea about the natures of the entangled states by means of these remarkable measures because, they can catch the degree of entanglement very precisely with respect to of maximally entangled and disentangled states very precisely and also they can distinguish the states from each others exactly. But the other entanglement measures can not do all of these.

One of the remarkable models in physics is Jaynes-Cummings Model (JCM) [6]. The JCM describes a single two level atom interacting with a single near -resonant quantized cavity mode of the electromagnetic field. This model plays a crucial role in the atom field interactions. The importance of this model is that, it is fully a quantized model and presents exactly analytical solutions. This model is a very fruitful research field. Because it can be exploited from lots of aspects and it can be extended by including some physical terms of different interactions.

The second subject investigated in this thesis is decoherence and cavity-atom systems. Here, we investigate the dynamical properties of the entanglement between two identical atoms whose states are in the superposition of excited and ground states and cavity which is initially in the vacuum state, in the presence of phase decoherence. The effects of the phase decoherence on the entanglement is clearly displayed. It is shown that, in the resonance case, atom-atom entanglement can survive and finally reaches a stationary case, but in the atom-field case, entanglement decays rapidly. In the non-resonant case, the entanglement in both systems is sensitive to the detuning parameter. The entanglement does not completely dies out in time.

The third subject investigated in this thesis is phase telegraph noise in cavity-atom systems. Here, we investigate the dynamical properties of the entanglement between an atom and the field with noise. The effects of the phase noise on the entanglement is depicted. It is shown that, the mean dwell time T and the phase value a determines the time that the entanglement dies out. Also, the temperature dependence of the entanglement is found to be different in the noisy case compared to the non-noisy JCM.

The plan of this thesis is as follows; in section 2, we give an overview of quantum entanglement theory and some important customary entanglement measures. In Section 3, we introduce our first special physical system called spin-1 Heisenberg model and investigate its entanglement properties. In section 4, we introduce our second system JCM to understand the quantum mechanical description of the interaction of matter with light. In section 5, we introduce an application of JCM on a system which is the interaction of two identical atoms with a single mode cavity field in the presence of phase decoherence to investigate its dynamical properties entanglement coming to existence between the atoms and the cavity field. Finally, in section 6, we investigate the system of an atom coupled to a single mode thermal cavity field in the presence of phase telegraph noise. We properly investigate the dynamics of the entanglement between the atom and the field.

CHAPTER 2

BASIC CONCEPTS OF ENTANGLEMENT RELATED TO QUANTUM MECHANICS

Before start to talk about the entanglement itself, we need to set the stage for the description of systems in Quantum Mechanics. I will present the basic ideas of describing quantum mechanical system, namely the states, density matrices, the meaning of a state being entangled and measuring the entanglement of a given state. This chapter borrows heavily from the textbook B. H. Bransdein and C. J. Joachain *Introduction to quantum mechanics* [7], the review article of J. Myrheim *Measures of entanglement in quantum mechanics* [8], and the lecture notes from Carnagie-Mellon University [9].

2.1 State Vector

A state of a quantum mechanical system can be described by an abstract quantity called *state vector (or ray)* denoted by $|\psi\rangle$. To each state vector, there corresponds a conjugate quantity called bra denoted by $\langle\psi|$. The state vector is independent of any choice of basis. If a quantum system can be described by a single state vector (ray), then such systems are said to be in a pure state, otherwise in a mixed state. There are some specific quantum states such as *qubits* and *qutrits*. A qubit which is a state in a two dimensional Hilbert space, describes spin - 1/2 particles. The most general state of a qubit can be written as

$$|\psi\rangle = a|0\rangle + b|1\rangle \tag{2.1}$$

where a and b are complex numbers that satisfy the normalization condition $|a|^2 + |b|^2 = 1$. A qutrit which is a state in a three dimensional Hilbert space, describes spin-1 particles The most general state of a qutrit can be written as

$$|\psi\rangle = a|0\rangle + b|1\rangle + c|2\rangle \tag{2.2}$$

with again the normalization condition $|a|^2 + |b|^2 + |c|^2 = 1$.

2.2 Density Matrix

Density matrix is another way of describing a given quantum system. It provides a more flexible structure for description of mixed as well as pure states. For a pure state $|\psi\rangle$, the density matrix ρ is simply

$$\rho = |\psi\rangle\langle\psi| \tag{2.3}$$

Consider a system consisting of an ensemble of N-sub-systems $\alpha = 1, 2, ..., N$ and each of these sub-systems is in pure state with a specific probability p_α and so these sub-systems can be described by their own state vectors $|\psi^\alpha\rangle$. In this case, the density matrix of the system is described as;

$$\rho = \sum_\alpha p_\alpha |\psi^\alpha\rangle\langle\psi^\alpha| \tag{2.4}$$

In the above equation, if $\alpha = 1$, the system is in pure state; $\rho^2 = \rho$, otherwise in mixed state; $\rho^2 \neq \rho$.

The density matrix formalism of quantum mechanics is very advantageous, especially to define mixed states.

2.3 Composite Systems

A *composite* system is one involving more than one subsystems. Consider two systems A and B with their own Hilbert spaces H_A and H_B, respectively. The Hilbert space of the composite system comprised by these two distinct physical systems is the tensor product of the two Hilbert spaces for the separate systems, $H_{AB} = H_A \otimes H_B$. This tensor product can be defined in the following way,

Let $|a_i\rangle : i = 1, 2....m$ be an orthonormal basis for the m-dimensional space A, and $|b_j\rangle : j = 1, 2....n$ an orthonormal basis for the n-dimensional basis B , Then the collection of mn elements

$$|a_i\rangle \otimes |b_j\rangle \tag{2.5}$$

forms an orthonormal basis of the tensor product $A \otimes B$. Any state of this composite space can expressed as

$$|\psi\rangle = \sum_{i,j} \alpha_{ij} |a_i\rangle \otimes |b_j\rangle \tag{2.6}$$

where α_{ij} are complex numbers satisfying the normalization condition.

Also, consider two kets $|a\rangle$, $|b\rangle$ again belonging the Hilbert spaces H_A and H_B, respectively. If any element of the space H_{AB} can be written in the direct form of $|\psi\rangle = |a\rangle \otimes |b\rangle$, then the state $|\psi\rangle$ is said to be a product state, otherwise an entangled state.

As an example of the composite systems, Bell states which are the maximal entangled bipartite qubit states, are very common;

$$|\psi^{\mp}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \mp |11\rangle) \tag{2.7}$$

and

$$|\phi^{\mp}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \mp |10\rangle) \tag{2.8}$$

If $|\psi\rangle_{AB}$ is the state of the composite system, the same system can be described by means of the density matrix,

$$\rho_{AB} = |\psi_{AB}\rangle\langle_{AB}\psi| \tag{2.9}$$

In density matrix formalism, a state is product state or disentangled, if it can be written in the form $\rho_{AB} = \sum_i p_i \rho_i^A \otimes \rho_i^B$, otherwise entangled [10].

By using the density matrix of a composite system, the density matrix of one subsystems (the marginal density matrix) can be obtained by taking the partial trace of the composite density matrix over the other sub-system such that;

$$\rho_A = \mathrm{Tr}_B[\rho_{AB}] = \sum_u \langle u_B|\rho_{AB}|u_B\rangle \tag{2.10}$$

and

$$\rho_B = \mathrm{Tr}_A[\rho_{AB}] = \sum_u \langle u_A|\rho_{AB}|u_A\rangle \tag{2.11}$$

2.4 Entanglement Measures

2.4.1 Entropy of Entanglement

Our measure for how a quantum state ρ is, will be the von Neumann entropy, defined as

$$S(\rho) = -\mathrm{Tr}(\rho \log \rho) \tag{2.12}$$

Most easily, it is calculated from nonzero eigenvalues λ_i of ρ as

$$S(\rho) = -\sum_i \lambda_i \log \lambda_i \tag{2.13}$$

von Neumann entropy in fact describes the uncertainty in a quantum state. It is zero for pure states and smaller for mixed states. It is a very special quantity not only to quantify the entanglement of the states, but also its relations to thermodynamics Popescu *et. al.* [11] and also quantum information theory.

2.4.2 Distillable Entanglement and Entanglement Cost

For the pure states, the distillable entanglement sometimes called *entanglement of distillation* E_d is defined as the maximum yield of Bell states that can be obtained, optimised over all possible LOCC protocols.

Entanglement cost E_c is defined as the minimum number of Bell states needed to create a given state by means of LOCC.

It is not easy to calculate E_d and E_c for a given state which is no matter pure or mixed. For any pure state,

$$E_d \leq E_c \tag{2.14}$$

In the asymptotic limit, the result are followings,

$$E_D \equiv E_d^{\infty}(|\psi\rangle) \equiv \lim_{n\to\infty} \frac{E_d(|\psi\rangle^{\otimes n})}{n} \tag{2.15}$$

$$E_C \equiv E_c^{\infty}(|\psi\rangle) \equiv \lim_{n\to\infty} \frac{E_c(|\psi\rangle^{\otimes n})}{n} \tag{2.16}$$

In this case, the distillable entanglement and entanglement cost are equal to the entropy entanglement, $E_D(|\psi\rangle) = E_C(|\psi\rangle) = E_E(|\psi\rangle)$. This means that, if two states $|\psi_1\rangle$ and $|\psi_2\rangle$ have the same entropy, then they can be interconverted with efficiency approaching unity as $n \to \infty$. Otherwise, $|\psi_1\rangle$ can be converted into $|\psi_2\rangle$ with asymptotic yield $E_E(|\psi_1\rangle)/E_E(|\psi_2\rangle)$.

In the case of mixed states, these measures can be defined in the same way as for pure states. The entanglement cost is the minimum number of Bell states needed to produce the state by means of LOCC and the distillable entanglement is the maximum number of Bell states that can be distilled by an optimal LOCC distillation protocol.

For any entanglement measure E which satisfies certain natural conditions for asymptotic measures, the entanglement cost and the distillable entanglement provide upper and lower bounds.

$$E_D \leq E \leq E_C \tag{2.17}$$

2.4.3 Entanglement of formation

This measure is a generalization of the entropy of entanglement to mixed states. Since in asymptotic limit, the entanglement cost in Bell states of preparing a pure state is given by the entropy of entanglement, then one can define the entanglement of formation (EOF) for a pure state as the entropy of entanglement. For a given ensemble of pure states $\varepsilon = (p_i, |\psi_i\rangle)$, the EOF is the average of the entropy of entanglement for states in the ensemble

$$E_f(\rho) = \sum_i p_i E_E(|\psi_i\rangle) \tag{2.18}$$

Since a mixed state is the ensemble of pure states, the entanglement of formation for a mixed state is the entanglement of formation for the "most economic " ensemble.

$$E_f(\rho) = \inf_{\varepsilon} \sum_i p_i E_E(|\psi_i\rangle) \tag{2.19}$$

where the infimum is taken over all ensembles $\varepsilon = (p_i, |\psi_i\rangle)$. The function $E_f(\rho)$ resulting from the above equation is the largest convex function of ρ.

2.4.4 Concurrence and Tangle

Wootters derived an expression for the entanglement of formation for a pair of qubits by introducing the concept concurrence [4, 5]. For a pure state of two qubits, the concurrence $C_2(|\psi\rangle)$ is given by

$$C_2(|\psi\rangle) = |\langle\psi|\widetilde{\psi}\rangle| \tag{2.20}$$

where $|\widetilde{\psi}\rangle = \sigma_y \otimes \sigma_y |\psi^*\rangle$ is the "spin-flip" of $|\psi\rangle$, σ_y is the Pauli matrix, and $|\psi^*\rangle$ is the complex conjugate of $|\psi\rangle$ in the standard basis. The spin flip operation maps the state of each qubit to its corresponding orthogonal state. The relation between the entropy of entanglement and concurrence is given by

$$E_E = \varepsilon(C_2(|\psi\rangle)) \tag{2.21}$$

where the function ε is defined as

$$\varepsilon(C_2(|\psi\rangle)) = h(\frac{1+\sqrt{1-C_2^2}}{2}) \tag{2.22}$$

and

$$h(x) = -x\log_2 x - (1-x)\log_2(1-x) \tag{2.23}$$

is the binary entropy of parameter x.

In the case of mixed states of two qubits, the concurrence is defined as;

$$C_2(\rho) = \inf \sum_i p_i C_2(|\psi_i\rangle) \tag{2.24}$$

For this case, the solution of the concurrence of mixed states is related to the eigenvalues of a non-Hermitian operator $\rho\widetilde{\rho}$ where$\widetilde{\rho} = \sigma_y \otimes \sigma_y \rho^* \sigma_y \otimes \sigma_y$. From these, the concurrence of mixed state of two qubits is given by

$$C_2(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4) \tag{2.25}$$

where λ_i's are the square roots of the eigenvalues of $\rho\widetilde{\rho}$ in decreasing order. The relationship between the entanglement of formation and concurrence is given by

$$E_F(\rho) = \varepsilon(C_2(\rho)) \tag{2.26}$$

The tangle, τ_2, another monotone entanglement measure of the state of two qubits is defined as the square of concurrence

$$\tau_2(\rho) = (C_2(\rho))^2 \tag{2.27}$$

2.4.5 Negativity

A positive map Q takes positive operators to positive operators. But in the case of entanglement, the positive map is not completely positive. So in entanglement case, a quantum state resulting from a positive map can have negative eigenvalues. The positive map in the definition of negativity [12] is the partial transpose operation. Let A and B two sub-systems, then *partial transpose* ρ^{T_A} of ρ with respect to subsystem A is given by

$$\langle i_A, j_B|\rho|l_A, m_B\rangle = \langle l_A, j_B|\rho^{T_A}|i_A, m_B\rangle \tag{2.28}$$

The trace norm of a operator A is defined by

$$||A||_1 = \mathrm{Tr}\sqrt{A^\dagger A} \tag{2.29}$$

is the sum of the absolute eigenvalues of A. Consider the case $A = \rho$, this reduces to the normalization condition $\mathrm{Tr}(\rho) = 1$. Since ρ^{T_A} may have negative eigenvalues, its trace norm reads

$$||\rho^{T_A}||_1 = \sum_i |\lambda_i| = \sum_i \lambda_i^+ + \sum_i |\lambda_i^-| \tag{2.30}$$

From the normalization condition,

$$1 = \sum_i \lambda_i = \sum_i \lambda_i^+ + \sum_i \lambda_i^- \tag{2.31}$$

From equations Equation (2.30)and Equation (2.31), one can obtain that;

$$N(\rho) = \sum_i |\lambda_i^-| = \frac{||\rho^{T_A}||_1 - 1}{2} \tag{2.32}$$

where negativity $N(\rho)$ is defined as the sum of the absolute values of negative eigenvalues of ρ^{T_A} [13].

CHAPTER 3

SPIN-1 HEISENBERG MODEL AND ITS ENTANGLEMENT PROPERTIES

3.1 Spin-1 Heisenberg Model (H.M.)

There have been many studies devoted to understanding entanglement for the different physical systems. One field of these studies is to quantify the magnitude of the entanglement between physical systems. Entanglement for the systems of qubits was fully understood. An effective measure is the concurrence which was proposed by Wootters [4, 5]. This is a very effective entanglement measure because, it is computable, it does not change under local operations and classical communications and it gives the EOF for the pure as well as mixed states of qubits. Wootters defines the concurrence by the σ_y Pauli spin matrix as a spin-flip operator whose action on a qubit density state $\rho = \frac{1}{2}(1 + \overrightarrow{P} \cdot \overrightarrow{\sigma})$ is to flip the spin of the qubit such that;

$$S_2(\rho) = \sigma_y \rho^* \sigma_y = \frac{1}{2}(1 - \overrightarrow{P} \cdot \overrightarrow{\sigma}) \tag{3.1}$$

For qubit states, σ_y more generally S_2 transforms maximally entangled states to themselves. So, the concurrence of these states are 1.

The most important property of S_2 is that, it can be derived from a conjugation Θ which is an anti-unitary operator satisfying $\Theta = \pm 1$ such that;

Let the anti-unitary operator Θ be $\Theta = \sigma_y \cdot C$ such that; $\sigma_y \cdot C = -C \cdot \sigma_y$where C denotes the complex conjugation in the eigenbasis of σ_z $C|\psi\rangle = |\psi^*\rangle$. As can be seen obviously, $\Theta = \pm 1$ and $\Theta^{\dagger} = \pm\Theta$.

Apart from qubit entanglements, also there have been studies devoted to qutrit entanglement. Peres [14] gave a criterion, the partial transposition condition, which determines whether a general state of two qubits or a general state of one qubit and one qutrit is entangled. Then Caves *et. al.* [15] explored the separability of N qutrits.

The first attempt on generalization came from Ulhmann [16]. In his work, he introduced Θ conjugation which is a generalization of Wootters' concurrence to higher dimensions. Then, Rungta *et. al.* [17] proposed a way different from Ulhmann to generalize Wootters's concurrence by means of superoperator called universal inverter. The universal inverter is a natural generalization to higher dimensions of the qubit spin flip.

Not all of these generalization attempts could describe the entangled states of higher dimensions completely. The generator of Wootters' concurrence S_2 commutes with all unitary operators. $C_2(\psi)$ is unchanged by local unitary transformations. S_2 is derived from a conjugation is not a property of its generalizations such as Θ and S_D. Because, except in two dimensions, a conjugation can not commute with all unitaries and therefore can not serve as the basis for measure of entanglement. So, the entanglement measures of these formalisms can not always catch the degree of entanglement of states and can not always distinguish them as will be seen in the thesis. In order to see this, let us now introduce our first model covered in the thesis.

Spin-1 Heisenberg Model is simply an interaction between nearest-neighbour spin-1 particles and on this interaction a magnetic field is applied along z-axis. The model has the Hamiltonian of

$$H(J,M) = J\sum_{i}^{N}\mathbf{S}_i \cdot \mathbf{S}_{i+1} + M\sum_{i}^{N}(S_{iz}) \tag{3.2}$$

where J and M denote the interaction strength and magnetic field, respectively, and $S_i^{\alpha} = 1 \otimes ... \otimes S^{\alpha} \otimes 1.... \otimes 1$ is the spin-1 matrix for the at position i in which **1** is the unit matrix;

It is easy to deduce the form of S_z such that;

$$S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{3.3}$$

with the eigenvectors;

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{3.4}$$

In the basis of S_z, the matrices S_x and S_y can be represented as;

$$S_x = \begin{pmatrix} 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \end{pmatrix}, S_y = \begin{pmatrix} 0 & -i/\sqrt{2} & 0 \\ i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & i/\sqrt{2} & 0 \end{pmatrix} \tag{3.5}$$

Here, we use a two-qutrit version of this model to calculate and to compare various qutrit entanglement measures. Eqn 3.2 for two two-qutrits is a 9×9 matrix and it can be diagonalized analytically.

In our study, we take $J = 1$ and $M = 0$. In this case, the Hamilton of two-qutrit system gets the eigenvalues and the eigenvectors, which are shown in the following table;

Eigenvalues	**Eigenvectors**
-2	$\|\psi\rangle_1 = 1/\sqrt{3}(\|02\rangle - \|11\rangle + \|20\rangle)$
-1	$\|\psi\rangle_2 = 1/\sqrt{2}(-\|02\rangle + \|20\rangle)$
-1	$\|\psi\rangle_3 = 1/\sqrt{6}(\|02\rangle + 2\|11\rangle + \|20\rangle)$
-1	$\|\psi\rangle_4 = (\|22\rangle)$
1	$\|\psi\rangle_5 = 1/\sqrt{2}(-\|12\rangle + \|21\rangle)$
1	$\|\psi\rangle_6 = 1/\sqrt{2}(\|12\rangle + \|21\rangle)$
1	$\|\psi\rangle_7 = 1/\sqrt{2}(-\|01\rangle + \|10\rangle)$
1	$\|\psi\rangle_8 = 1/\sqrt{2}(\|01\rangle + \|10\rangle)$
1	$\|\psi\rangle_9 = (\|00\rangle)$

Table 3.1: The eigenvalues and the eigenvectors of our system.

The eigenvectors of the system are independent of the applied external magnetic field but the eigenvalues are not.

3.2 Qutrit Entanglement Measures Computed for H.M. States

The proposed qutrit entanglement measures can be broadly divided into three categories; those based on density matrices (von Neumann entropy, I-concurrence, Chen *et. al.*'s measure, negativity,...,etc.) [17, 18], those based on state vectors, (concurrence, generalized concurrence,..,etc) [19, 20], and the relative purity [21] which measures the expectation value of Gell-mann matrices for a considered state.

In this section, we introduce the entanglement measures computed for the the states of our system and for the special eigenvector which is the maximally entangled qutrit state such that;

$$|\psi\rangle_{special} = 1/\sqrt{3}(|00\rangle + |11\rangle + |22\rangle) \tag{3.6}$$

For the distance formulations between the states of the system in our study, the special eigenvector is taken to be the reference state. Now let us introduce the entanglement measure formulations.

The first entanglement measure is von Neumann entropy which is a very important quantity not only for quantifying entanglement but also for quantum information theory. von Neumann entropy is defined as;

$$S(\rho_A) = -\mathrm{tr}(\rho_A \log \rho_A) \tag{3.7}$$

where ρ_A is one of the reduced density matrices of the pure composite state matrix ρ_{AB}. More clearly, it can be expressed as

$$S(\rho_A) = -\sum_i \lambda_i \log \lambda_i \tag{3.8}$$

where λ_i is the nonzero eigenvalue of ρ_A, and the logarithm is taken in base-3.

The second entanglement measure in this study was proposed by Chen *et. al.* [18]. They define the entanglement measure P_E by using the reduced elements of the most general matrix for a pair of qutrits as follows; Let

$$\rho_{AB} = \frac{1}{9}(\mathbf{1}\otimes\mathbf{1} + \sqrt{3}\lambda^A.\mathbf{u}\otimes\mathbf{1} + \sqrt{3}\mathbf{1}\otimes\lambda^B.\mathbf{v} + \frac{3}{2}\sum_{i,j=1}^{8}\beta_{ij}\lambda_i^A\otimes\lambda_j^B) \tag{3.9}$$

be the density matrix for two entangled qutrits where $\mathbf{u} = u_1, u_1, ...u_8$, and $\mathbf{v} = v_1, v_1, ...v_8$, are the Bloch vectors for particles A and B, respectively and β_{ij} are real coefficients and also $\lambda_i(i, 1, 2....8)$ are the eight Hermitian matrices which are generators of $SU(3)$;

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.10}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ i & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{3.11}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{3.12}$$

From these, a matrix α of 9×9 can be formed from the $\mathbf{u}$, $\mathbf{v}$, and β_{ij} which are the elements of ρ_{AB} such that;

$$\hat{\alpha} = \begin{pmatrix} 1 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \\ u_1 & \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} & \beta_{15} & \beta_{16} & \beta_{17} & \beta_{18} \\ u_2 & \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} & \beta_{25} & \beta_{26} & \beta_{27} & \beta_{28} \\ u_3 & \beta_{31} & \beta_{32} & \beta_{33} & \beta_{34} & \beta_{35} & \beta_{36} & \beta_{37} & \beta_{38} \\ u_4 & \beta_{41} & \beta_{42} & \beta_{43} & \beta_{44} & \beta_{45} & \beta_{46} & \beta_{47} & \beta_{48} \\ u_5 & \beta_{51} & \beta_{52} & \beta_{53} & \beta_{54} & \beta_{55} & \beta_{56} & \beta_{57} & \beta_{58} \\ u_6 & \beta_{61} & \beta_{62} & \beta_{63} & \beta_{64} & \beta_{65} & \beta_{66} & \beta_{67} & \beta_{68} \\ u_7 & \beta_{71} & \beta_{72} & \beta_{73} & \beta_{74} & \beta_{75} & \beta_{76} & \beta_{77} & \beta_{78} \\ u_8 & \beta_{81} & \beta_{82} & \beta_{83} & \beta_{84} & \beta_{85} & \beta_{86} & \beta_{87} & \beta_{88} \end{pmatrix} \tag{3.13}$$

With this matrix, the entanglement measure of bipartite pure state ρ_{AB} is defined as;

$$P_E = (-\det \alpha)^{1/4} \tag{3.14}$$

Chen *et. al.* also have shown that, P_E is equivalent to Wootters' concurrence for the pure states of bipartite qubits.

Another Wootters' concurrence related qutrit entanglement measure was proposed by Cereceda [19]. Cereceda's concurrence is a straightforward generalization of qubit concurrence to qutrit states. The general state of a bipartite qutrit system can be written as in Eq 3.9. In addition to that, the real coefficients u_i and v_j and β_{ij} can be found as;

$$u_i = \frac{\sqrt{3}}{2}\mathrm{Tr}(\rho_{AB}\lambda_i \otimes \mathbf{1}) \tag{3.15}$$

$$v_j = \frac{\sqrt{3}}{2}\mathrm{Tr}(\rho_{AB}\mathbf{1} \otimes \lambda_i) \tag{3.16}$$

$$\beta_{ij} = \frac{3}{2}\mathrm{Tr}(\rho_{AB}\lambda_j \otimes \lambda_j) \tag{3.17}$$

The point in this entanglement measure comes from;

$$C = \sqrt{1 - |\mathbf{u}|^2} \tag{3.18}$$

This quantity is nominated to be an entanglement measure for the pure state ρ_{AB}. Let the general state of any pure bipartite qutrit system be defined as;

$$C(|\psi\rangle) = \sum_{i,j=1}^{3} \alpha_{ij}|i,j\rangle \tag{3.19}$$

with normalization $\sum_{ij} |\alpha_{ij}|^2 = 1$ By using Eq 3.15 and Eq 3.18, one can obtain the concurrence for a pure state of bipartite qutrit such that;

$$\begin{aligned} C(|\psi\rangle) &= [3(|\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}|^2+|\alpha_{13}\alpha_{21}-\alpha_{11}\alpha_{23}|^2+|\alpha_{12}\alpha_{23}-\alpha_{13}\alpha_{22}|^2 \\ &\quad +|\alpha_{12}\alpha_{31}-\alpha_{11}\alpha_{32}|^2+|\alpha_{11}\alpha_{33}-\alpha_{13}\alpha_{31}|^2+|\alpha_{21}\alpha_{32}-\alpha_{22}\alpha_{31}|^2 \\ &\quad +|\alpha_{23}\alpha_{31}-\alpha_{21}\alpha_{33}|^2+|\alpha_{13}\alpha_{32}-\alpha_{12}\alpha_{33}|^2+|\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32}|^2)]^{1/2} \end{aligned}$$

The forth type of entanglement measure is generalized concurrence that was proposed by Fierro *et.al.* [20]. They extent the Wootters' concurrence to the case of two qutrits for both pure and mixed states in a similar way such that;

$$C_3(\psi) = |\langle\psi|\widetilde{\psi}\rangle| \tag{3.20}$$

where

$$|\psi\rangle = (O_3 \otimes O_3)|\psi^*\rangle \tag{3.21}$$

in which $|\psi^*\rangle$ is the complex conjugate of $|\psi\rangle$ and

$$O_3 = \begin{pmatrix} 0 & -i & i \\ i & 0 & -i \\ -i & i & 0 \end{pmatrix} \tag{3.22}$$

Let the general state be defined as in Eq. 3.19 with the normalization; Then the generalized concurrence Eq. 3.20 can written as;

$$\begin{aligned} C_3(|\psi\rangle) &= |(\alpha_{11}+\alpha_{22}+\alpha_{33})^2+(\alpha_{12}+\alpha_{23}+\alpha_{31})^2 \\ &\quad +(\alpha_{13}+\alpha_{32}+\alpha_{21})^2-(\alpha_{11}+\alpha_{23}+\alpha_{32})^2 \end{aligned}$$

$$-(\alpha_{12}+\alpha_{21}+\alpha_{33})^2-(\alpha_{13}+\alpha_{22}+\alpha_{31})^2|$$

In this formalism, the concurrence of maximally entangled two qunits, n-dimensional states, is given by

$$C_{n,max}=n-1 \tag{3.23}$$

The fifth entanglement measure covered in the study is I-concurrence and it was proposed by Rungta *et. al.* [17]. In this study, Wooters' concurrence of bipartite pure qubit states is generalized to the bipartite pure qudit systems, a $D-$ dimensional quantum system, by using universal inverter S_D such that; Let ρ_{AB} be a pure state of a joint system of arbitrary dimension of $D_1 \times D_2$, then I-concurrence is given by,

$$C(|\psi_{AB}\rangle)=\sqrt{\langle\psi_{AB}|S_{D1}\otimes S_{D2}(|\psi_{AB}\rangle\langle\psi_{AB}|)|\psi_{AB}\rangle} \tag{3.24}$$

where

$$S_D(|\psi_{AB}\rangle\langle\psi_{AB}|)=\nu_D(I-|\psi_{AB}\rangle\langle\psi_{AB}|) \tag{3.25}$$

which acting on the pure state ρ_{AB} maps $|\psi\rangle$ to multiple of the maximally mixed state in the subspace orthogonal to $|\psi\rangle$ and also $\nu_D=1/(D-1)$ is a positive dimension dependent constant. The main point in this measure is to generalize the spin flip operator σ_y to a superoperator S_D called *universal operator*. The universal operator is defined in Hilbert spaces and it enables us to generalize concurrence for joint pure states of two quantum systems to arbitrary dimension. For this purpose, S_D must have foolowing properties;

1.S_D maps Hermitian operators to Hermitian operators.

2.S_D commutes with all unitary operators.

3.$\langle\psi|S_{D_1}\otimes S_{D_1}(|\psi\rangle\langle\psi|)|\psi\rangle$ is non-negative for all joint pure states $|\psi\rangle$ and goes to zero if and only if $|\psi\rangle$ is a product state. By taking into these three properties, the superoperator must have the form of

$$S_D = \nu_D(\mathbf{I} - I) \tag{3.26}$$

From these, I-concurrence can be expressed more clearly,

$$C(|\psi_{AB}\rangle) = \sqrt{2\upsilon_{D1}\upsilon_{D2}(1 - \mathrm{tr}(\rho_A^2)} \tag{3.27}$$

This is the relation of I-concurrence with one of the reduced density matrices of ρ_{AB}.

The sixth qutrit entanglement measure we consider is relative purity which was proposed by Somma *et. al.* [21]. They define relative purity for a generic pure state $|\psi\rangle$, the eigenstate of the system, as

$$P_\eta = \frac{3}{4}\sum_{i=1}^{8}\sum_{\alpha=1}^{2}\langle\lambda_\alpha^i\rangle \tag{3.28}$$

where, λ^i's are Gell-Mann matrices and $\langle\lambda_\alpha^i\rangle$ denotes the expectation value of λ_α^i in the state $|\psi\rangle$. The sum over i is for the eight Gell-mann matrices while α sum is for the position.

The seventh entanglement measure is fidelity proposed by Chen, *et. al.* [22], a strong tool to distinguish two different states of arbitrary dimensions. They give an alternative fidelity measure between two states such that;

Let

$$\rho(\mathbf{n}) = \frac{1}{N}(\mathbf{1} + \sqrt{\frac{N(N-1)}{2}}\overrightarrow{\lambda}.\mathbf{n}) \tag{3.29}$$

be the density matrix of a qunit, the state of dimension of n where $\mathbf{1}$ is the unit matrix of $N \times N$ and $\overrightarrow{\lambda} = (\lambda_1, \lambda_2 .... \lambda_{N^2-1})$ are the generators of the $SU(N)$, and also $\mathbf{n}$ is the (N^2-1) dimensional Bloch vector. In order to define the fidelity; Let $\rho_1 = \rho(\mathbf{u})$ and $\rho_2 = \rho(\mathbf{v})$ be the states of a qunit, which are the reduced matrices of the composite state matrix.

From these, the fidelity is defined as

$$F(\rho_1, \rho_2) = \frac{(1-r)}{2} + \frac{(1+r)}{2}[\mathrm{tr}(\rho_1 \rho_2) + \sqrt{1-\mathrm{tr}(\rho_1^2)}\sqrt{1-\mathrm{tr}(\rho_2^2)}] \tag{3.30}$$

where $r = 1/(N-1)$. For the identical states, the fidelity gives $F(\rho_1, \rho_2) = 1$.

The eighth entanglement measure we consider is negativity which was proposed by Vidal and Werner [12]. It is defined as;

$$N(\rho) = \frac{||\rho^{T_2}||_1 - 1}{2} \tag{3.31}$$

where ρ^{T_2} denotes the partial transpose of ρ with respect to the second particle, and $||\rho^{T_2}||_1$ denotes the trace norm given by

$$||\rho^{T_2}||_1 = \mathrm{tr}\sqrt{(\rho^{T_2})^\dagger \rho^{T_2}} \tag{3.32}$$

From the Eq. 3.31, one can obtain another expression of negativity,

$$N(\rho) = |\sum_j \mu_j| \tag{3.33}$$

So, negativity is the sum of the negative eigenvalues of ρ^{T_2} and μ_j denotes the negative eigenvalues of ρ^{T_2} Zyczkowski *et. al.* [13].

The ninth entanglement measure taken into account is Schmidt number which is the number of nonzero eigenvalues of the reduced density matrix ρ_A (or ρ_B).

Schmidt number is a useful and an effective way to determine the entanglement of a bipartite pure state of arbitrary dimension.

The next entanglement measure we deal with was proposed by Chen *et.al* [23]. In their study, they give a lower bound $||R(\rho)||$ for the entanglement of formation for arbitrary bipartite mixed states by using the convex hull construction of a certain function such that;

A pure quantum state of $m \otimes n$ can always be written in the standard Schmidt form;

$$|\psi\rangle = \sum_i \sqrt{\mu_i}|a_i b_i\rangle \tag{3.34}$$

where $\sqrt{\mu_i}(i = 1, 2, ...m)$ are the Schmidt coefficients, and $|a_i\rangle$ and $|b_i\rangle$ are the orthonormal basis for the subsystems, respectively.

Then, for the pure state of Eq. 3.34,

$$E(\rho) \geq R(\rho) \tag{3.35}$$

From these, they define this bound as;

$$||R(\rho)|| = ||\rho^{T_A}|| \tag{3.36}$$

where ρ^{T_A} is the reduced density matrix of the first sub-system which is invariant under local unitary transformations. From Eq 3.36, one can express this bound as;

$$||R(\rho)|| = (\sum_i \sqrt{\mu_i})^2 = \lambda \tag{3.37}$$

where λ varies from 1 to m.

The eleventh entanglement measure we deal with was proposed by Rastegin [24]. In this study, it is defined a sort of distance called *sine distance* between the states of pure and also mixed such that; Let $\delta(x, y)\epsilon[0, \frac{\pi}{2}]$ be the angle between pure states $|x\rangle$ and $|y\rangle$ defined as;

$$\delta(x, y) = \arccos(|\langle x|y\rangle|) \tag{3.38}$$

It is obvious that, the states $|x\rangle$ and $|y\rangle$ are identical if and only if $\delta(x, y) = 0$ Then, this entanglement measure is defined as;

$$d(x, y) = \sin\delta(x, y) \tag{3.39}$$

For the identical states, the sine distance gives $d(x, y) = 0$

The final entanglement measure we consider was proposed by Schulman *et.al* [25]. The formulation of this sort of entanglement measure is as follows; Let ρ be a one of the marginal density matrices of a pure composite state density matrix. Then this measure is defined as;

$$J = 1 - \lambda \tag{3.40}$$

where λ is the maximum eigenvalue of this sub-matrix. In this formalism, the degree of maximal entangled states of dimensions of n is given as

$$J = 1 - \frac{1}{n} \tag{3.41}$$

The results of all of these entanglement measures are normalized except $R(\rho)$ and Schmidt number because of their own peculiar formulations. So, the results of these entanglement measure on our system naturally varies from zero to one. These entanglement measures give the value one for the maximally entangles states and give the value of zero for the disentangled (or product) states except purity which gives the contrary results. For the distance formulations, the fidelity gives the value of one for the identical states whereas sine distance gives the value of zero for the identical states.

3.3 Results

When we apply these entanglement measures on our specific system, we get the results as shown in the following table,

	$\|\psi\rangle_1$	$\|\psi\rangle_2$	$\|\psi\rangle_3$	$\|\psi\rangle_4$	$\|\psi\rangle_5$	$\|\psi\rangle_6$	$\|\psi\rangle_7$	$\|\psi\rangle_8$	$\|\psi\rangle_9$	$\|\psi\rangle_{special}$
von Neumann Ent.	1	0,63	0,79	0	0,63	0,63	0,63	0,63	0	1
P_E	1	0	0,63	0	0	0	0	0	0	1
Concurrence	1	0,87	0,87	0	0,87	0,87	0,87	0,87	0	1
Gen. concurrence	0,67	1	1,67	0	1	1	1	1	0	2
I-Concurrence	1	0,87	0,87	0	0,87	0,87	0,87	0,87	0	1
P_η	0	0,25	0,25	1	0,25	0,25	0,25	0,25	1	0
Fidelity	1	0,93	0,93	0,5	0,93	0,93	0,93	0,93	0,5	1
N	1	0,5	0,83	0	0,5	0,5	0,5	0,5	0	1
Sch.Num.	3	2	3	1	2	2	2	2	1	3
$\boldsymbol{R}(\rho)$	3	2	2,67	1	2	2	2	2	1	3
sine distance	0.94	1	0.88	0.82	1	1	1	1	0.82	0
free ent.	1	0,75	0,50	0	0,75	0,75	0,75	0,75	0	1

Table 3.2: The results of the entanglement measures on our system.

3.4 The Analysis of The Results of Entanglement Measures on the Model

Entanglement measures $E(\rho)$ are very useful physical tools to quantify the entangled states. An entanglement measure must satisfy those two expectations; First, it must catch the entanglement of the states precisely and give reasonable values for the degree of entanglement of the states with respect to maximally entangled states and disentangled states whose results are certain. Second, it must also distinguish the states with different entanglement degrees. In our analysis, we will take into these two important factors.

Examining the entangled states of our model shows that; the states $|\psi_1\rangle$ and $|\psi_{special}\rangle$ are maximally entangled states. So, the entanglement measures in our study have to give the maximum or minimum value depending on their natures. And the states $|\psi_4\rangle$ and $|\psi_9\rangle$ are the disentangled states. Therefore, the entanglement measures in our study have to give the minimum or maximum value depending again on their natures. These states are indispensable states to analyse the entanglement measures. And also, $|\psi_3\rangle$ is a close state to the maximally entangled state. So, the entanglement measures must give a value between minimum and middle or between middle and maximum depending on their natures, again. In addition to these approaches, we know, the formulations of $R(\rho)$ and Schmidt number are related to the eigenvalues of the states of the sub-systems. So, their results for the disentangled states are not zero but one. Therefore, by means of these insights, we can properly examine the results of these entanglement measures in our study.

The results of von Neumann entropy on our system satisfy all of the expectations already mentioned above. Its results are very reasonable. So, it gives an idea us about the nature of the states. So, one more time, we can understand that, von Neumann entropy is a very remarkable entanglement measure.

The results of the second entanglement measure P_E satisfies the rules for the maximally entangled and disentangled states. But for the rest of states it gives the value of zero. So, it can not catch the degree of entanglement precisely and also it can not distinguish them. In fact, this entanglement measure was formulated for the qubits. So, it seems that, its generalization to the qutrits is not healthy.

The third measure concurrence obviously gives the expected results for the states $|\psi_1\rangle$ and $|\psi_{special}\rangle$. It gives an idea about the entanglement degrees of the states. But Its result for the state $|\psi_3\rangle$ is the same as the rest ones. So, it can not distinguish $|\psi_3\rangle$ from these states.

Generalized concurrence provides all expected results except the state $|\psi_1\rangle$. The problem arises from the algebraic signs in the coefficients of the state. So, we think, the formulation of this entanglement measure is not complete.

The results of relative purity on the states give an sight about their entanglement degrees. It depicts the maximally entangled and the disentangled states, exactly. But it can not distinguish the state $|\psi_3\rangle$ from these states exactly.

The seventh measure fidelity gives the distance of the states to the special eigenvector. It gives an idea about the distances of the states very well, except the state $|\psi_3\rangle$. Fidelity can not describe the distance of the state $|\psi_3\rangle$ to the state $|\psi_{special}\rangle$, well.

Negativity, as can be seen from the results, gives very reasonable values for the states. Its results give an dependable idea about the degree of entanglement of the states, and also it can distinguish these states exactly. So, it is like von Neumann entropy, a remarkable entanglement measure.

Schmidt number only gives the answer to whether a given state is entangled or disentangled, not to the degree of entanglement of states and to their distinguishabilities from some other states. So, the results it gives satisfy the expected answers. It says, the states $|\psi_4\rangle$ and $|\psi_9\rangle$ are the disentangled states and the rest of states are entangled.

The tenth entanglement measure, a lower bound for the EOF $R(\rho)$, gives very reasonable results for the states of the our model. It can describe the entanglement of all of the states very precisely. So, we can get a powerful idea about the entanglement degrees of these states. Additionally, it can distinguish the the states very exactly. Its results match what we expect roughly. So, in addition to von Neumann entropy and negativity, it is a remarkable entanglement measure, too.

Finally, the free entanglement describes the natures of our states very well. It satisfies the expected results for the states with the maximally entanglement and disentanglement. It also can distinguish the states because of their different entanglement degrees exactly. But, it can not catch the degree of entanglement of the states exactly.

Summarizing all of these analysis on the entanglement measures covered this study shows that; Of these entanglement measures, von Neumann entropy, negativity, and the lower bound of EOF are very effective entanglement measures. Their results are quite dependable. We can get an idea about the natures of the entangled states by means of these remarkable measures because, they can catch the degree of entanglement very precisely with respect to of maximally entangled and disentangled states very precisely and also they can distinguish the states from each others exactly. But the other entanglement measures can not do all of these. This is because of their formalisms which are the generalizations of spin flip operator of Wootters' concurrence. So, they are not complete entanglement measures. Another point is lower bound of EOF, the $R(\rho)$. Its results are very reasonable by comparing with definite measures, von Neumann entropy and negativity . Also, its formulation is different from the formulation of Wootters' concurrence. So, its formulation can be a basis for new formulations of entanglement measures proposed to quantify the qutrit or higher dimensional states entanglements.

CHAPTER 4

THE JAYNES CUMMINGS MODEL (JCM)

In this chapter, we will introduce the Jaynes-Cummings Model with its solutions in detail. This chapter borrows heavily from the review article of Shore and Knight [26] and lecture notes of Han Pu [27].

4.1 Model History

The JCM [6] describes a single two level atom interacting with a single near - resonant quantized cavity mode of the electromagnetic field. This model plays a crucial role in the atom field interactions. The importance of this model is that, it is fully a quantized model and presents exactly analytical solutions.

The level structure of a real atom is infinite. The reason behind why a two-level atom is a good approximation lies on two factors: First; Resonance excitation, Second; Selection rules. The dipole selection rules dictates that, only certain magnetic sub-levels are excited. So, the field causes transitions between a small number of discrete states. So one can consider the system only as a two-state and their energies are separated by $\hbar\omega_0$, the atomic transition energy. Under resonance condition, many levels lying far away from the resonance can be easily ignored. Another assumption by this model is that; it ignores some terms coming from interaction term called *Rotating Wave Approximation*(RWA). Because those terms violate the conservation of energy. But, this is not the case for appearing terms in the interaction term. Because each photon creation accompanies an atomic de-excitation, and each photon annihilation accompanies atomic excitation. This model was first used to examine the classical aspects of spontaneous emission. Then it was discovered that, the JCM atomic population histories presented direct evidence for the discreteness of photons. It was also recognized that, the statistical properties of the JCM interaction are not found in classical fields.

This model is a very fruitful research field. Because it can be exploited from lots of aspects and it can be extended by including some physical terms of different interactions.

The original JCM Hamiltonian is expressed in the form of

$$H = \hbar\omega\hat{a}^{\dagger}a + E_1\hat{S}_{11} + E_2\hat{S}_{22} + \frac{\hbar}{2}\Omega_1(\hat{a}^{\dagger}\hat{S}_{12} + \hat{a}\hat{S}_{21}) \tag{4.1}$$

where E_k is the energy of atomic state ψ_k, ω is the field frequency the atom-field coupling constant Ω_1 is the vacuum (or single-photon) Rabi frequency, $|\Omega_1|^2 = 4d^2\omega/\hbar V\epsilon_0$ in which V is the cavity volume and d is the atomic transition moment, and $\hat{S}_{jk}$ is a transition operator acting on atomic states, defined by the expressions

$$\hat{S}_{jk}|\psi_n\rangle = \delta_{kn}|\psi_j\rangle \tag{4.2}$$

$$\hat{S}_{jk}S_{nm} = \delta_{kn}S_{jm} \tag{4.3}$$

In the case of two states, one can re-express these atomic transition operators in terms of Pauli spin matrices such that;

$$\sigma_x = \hat{S}_{12} + \hat{S}_{21} \tag{4.4}$$

$$\sigma_y = i(\hat{S}_{12} - \hat{S}_{21}) \tag{4.5}$$

$$\sigma_2 = \hat{S}_{22} - \hat{S}_{11} \tag{4.6}$$

And also in the common used notation, $\sigma^\dagger = \hat{S}_{21}$ and $\sigma = \hat{S}_{12}$. The photon creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ which satisfy the commutation relation,

$$[\hat{a}, \hat{a}^\dagger] = 1 \tag{4.7}$$

act on photon states (coherent states) such that;

$$\hat{a}^\dagger \hat{a}|n\rangle = n|n\rangle \tag{4.8}$$

$$\hat{a}^\dagger|n\rangle = (n+1)^{1/2}|n+1\rangle \tag{4.9}$$

$$\hat{a}|n\rangle = (n)^{1/2}|n-1\rangle \tag{4.10}$$

where the states $|n\rangle$ are called *Fock states*. The some missing physical cases in this model are cavity loss, multiple cavity modes, atomic sublevel degeneracy and atomic polarizability.

4.2 The Solutions of the Model

In the model, since each photon creation accompanies an atomic de-excitation, and each photon annihilation accompanies atomic excitation, in addition of the conservation of energy, there are other some conserved physical quantities which are the atomic probability;

$$\langle \hat{S_{11}} \rangle + \langle \hat{S_{22}} \rangle = 1 \tag{4.11}$$

and also atomic excitation;

$$\langle \hat{a^\dagger a} \rangle + \langle \hat{S_{22}} \rangle = constant \tag{4.12}$$

Therefore the vision of the problem is that; an infinite set of uncoupled two-state Schrödinger equations, each pair identified by the number of photons that are present when the atom is in the lowest-energy state. We can write the atom - field state vector for specified photon number as a combination of two basis states;

$$\psi(n,t) = \exp(-in\omega t - iE_1 t/\hbar)[C_1(n,t)\phi_1(n) + C_2(n,t)\phi_2(n)] \tag{4.13}$$

where $\phi_k(n)$ is the atom-field product state;

$$\phi_1(n) = |n, \psi_1\rangle \tag{4.14}$$

$$\phi_2(n) = |n-1, \psi_2\rangle \tag{4.15}$$

From these, the 2×2 Hamiltonian matrix of one such pair has the form of

$$H(n) = \frac{\hbar}{2} \begin{pmatrix} 0 & \Omega(n) \\ \Omega(n) & 2\Delta \end{pmatrix} \tag{4.16}$$

$$\psi(n,t) = \exp(in\omega t - iE_1 t/\hbar) \begin{pmatrix} C_1(n,t) \\ C_1(n,t) \end{pmatrix} \tag{4.17}$$

where Δ is the cavity atom detuning parameter, n is the number of photons, and $\Omega(n)$ is the Rabi frequency.

$$\hbar\Delta = E_2 - E_1 - \hbar\omega \tag{4.18}$$

and

$$\Omega(n) = n^{1/2}\Omega_1 \tag{4.19}$$

The solution in Eqn. (4.17) is the solution of a two-state atom in a monochromatic field.

We can make the problem more specific such that; since in the Eq 4.1, $E_2 - E_1 = \hbar\omega_0$, we can take the ground state energy as $E_1 = 0$ and the excited state energy as $E_2 = \hbar\omega_0$ and also $\frac{\Omega}{2} = g$. In this case our Hamilton in the Eq 4.1 becomes;

$$H = \hbar\omega_0\hat{S_{11}} + \hbar\omega\hat{a}^\dagger a + \hbar g(\hat{a}^\dagger\hat{S}_{12} + \hat{a}\hat{S}_{21}) \tag{4.20}$$

In this case, the sub-matrix in the basis $|n-1,\psi_2\rangle$ and $|n,\psi_1\rangle$, can be easily obtained as;

$$\hbar\begin{pmatrix} (n-1)\omega + \omega_0 & \sqrt{n}g \\ \sqrt{n}g & n\omega \end{pmatrix} = \hbar(n\omega - \frac{\Delta}{2})\mathbf{1} + \frac{\hbar}{2}\begin{pmatrix} -\Delta & -R_n \\ -R_n & \Delta \end{pmatrix} \tag{4.21}$$

where $\Delta = \omega - \omega_0$ is the detuning parameter and $R_n = -2\sqrt{n}g$ is the Rabi frequency of the nth block. The eigenvalues of the system can be found as;

$$E_{n\pm} = \hbar(n\omega - \frac{\Delta}{2}) \pm \frac{\hbar}{2}\sqrt{\Delta^2 + R_n^2} \tag{4.22}$$

with the corresponding eigenstates;

$$|\phi_n^+\rangle = -\cos\frac{\theta_n}{2}|n-1,\psi_2\rangle + \sin\frac{\theta_n}{2}|n,\psi_1\rangle \tag{4.23}$$

and

$$|\phi_n^-\rangle = \sin\frac{\theta_n}{2}|n-1,\psi_2\rangle + \cos\frac{\theta_n}{2}|n,\psi_1\rangle \tag{4.24}$$

where

$$\cos\frac{\theta_n}{2} = \sqrt{\frac{\Omega_n - \Delta}{2\Omega_n}}, \sin\frac{\theta_n}{2} = \sqrt{\frac{\Omega_n + \Delta}{2\Omega_n}} \tag{4.25}$$

with

$$\Omega_n = \sqrt{\Delta^2 + R_n^2} \tag{4.26}$$

The eigenstates of the JCM are $|\theta^{\pm}\rangle$ are called *dressed states*. The mixing angle θ_n can be defined as $\theta_n = \tan^{-1}(-R_n/\Delta)$ The coefficients $\cos\frac{\theta_n}{2}$ and $\sin\frac{\theta_n}{2}$ are the special solutions of the $C(n)$.

Now, consider an atom is initially unexcited while the field is in a coherent state $|\alpha\rangle$ with exactly n photons such that; $|\psi(0)\rangle = |\alpha\rangle \otimes |\psi(1)\rangle$ Therefore, one can write this initial state as;

$$|\psi(0)\rangle = |\alpha\rangle \otimes |\psi_1\rangle = \exp(-|\alpha|^2/2)\sum_n \frac{\alpha^n}{\sqrt{n!}}|n,\psi_1\rangle = \sum_n \frac{\alpha^n}{\sqrt{n!}}(\sin\frac{\theta_n}{2}|\phi_n^+\rangle + \cos\frac{\theta_n}{2}|\phi_n^-\rangle) \tag{4.27}$$

Then the time evolution of the state is given by;

$$|\psi(\mathrm{t})\rangle = \exp(-|\alpha|^2/2)\sum_n \frac{\alpha^n}{\sqrt{n!}}(\sin\frac{\theta_n}{2}|\phi_n^+\rangle(\mathrm{t}) + \cos\frac{\theta_n}{2}|\phi_n^+\rangle(\mathrm{t})) \tag{4.28}$$

where $|\phi_n^{\pm}\rangle(\mathrm{t}) = \exp(-iE_{n\pm}\mathrm{t}/\hbar)|\phi_n^{\pm}\rangle$

The population of inversion at time t is given as;

$$\omega(\mathrm{t}) = P_2(t) - P_1(t) \tag{4.29}$$

where $P_2(t)$ and $P_1(t)$ are the probabilities that atom is in the excited state and the ground state at time t such that;

$$P_2(\mathrm{t}) = |\langle n, \psi_1|\psi(\mathrm{t})\rangle|^2 = \exp(-|\alpha|^2)\sum_{n=1}^{\infty}\frac{|\alpha|^{2n}}{n!}\sin^2\theta_n\sin^2\frac{\Omega_n\mathrm{t}}{2} = \sum_{n=1}^{\infty}p_\alpha(n)\sin^2\theta_n\sin^2\frac{\Omega_n\mathrm{t}}{2} \tag{4.30}$$

where $p_\alpha(n) = |\langle n|\alpha\rangle|^2 = \exp(-|\alpha|^2)|\alpha|^{2n}/n!$ is the probability that the coherent state has exactly n photons. Similarly,

$$P_1(\mathrm{t}) = \sum_{n=0}^{\infty}p_\alpha(n)(1-\sin^2\theta_n\sin^2\frac{\Omega_n\mathrm{t}}{2}) = 1 - P_1(\mathrm{t}) \tag{4.31}$$

From these, one can obtain the atomic inversion $\omega((\mathrm{t}))$ such that;

$$\omega((\mathrm{t})) = 2P_2(\mathrm{t})) - 1 = 2\sum_{n=1}^{\infty}p_\alpha(n)\sin^2\theta_n\sin^2\frac{\Omega_n\mathrm{t}}{2} - 1 \tag{4.32}$$

In the same way, one can find the solution of the atomic inversion, when the atom is initially in the ground state.

CHAPTER 5

ENTANGLEMENT DYNAMICS IN A TWO ATOMS ONE CAVITY MODE SYSTEM IN THE PRESENCE OF PHASE DECOHERENCE

5.1 Introduction

In this section, we study the dynamics of entanglement in a revised Jaynes-Cummings Model, sometimes called Tavis-Cummings Model [28]. Here, we have a single mode cavity and has two identical atoms. The cavity is imperfect, so we have decoherence. The system is investigated by a Liouville type of dynamical equation with a solution method due to Milburn [29].

The main aim here is to understand how entanglement between the atoms as well as the atom and the field changes depending on the decoherence as well as the initial state of the system. A very similar problem was solved by Li *et.al* [30] who considered the initial system as a vacuum cavity and one of the atom in excited state, the other one in the ground state.

In this study, we have investigated the entanglement dynamics for a thermal initial atomic distribution.

For this situation we consider that, two identical atoms whose states are in the superposition of excited and ground states are trapped inside cavity which is initially in the vacuum state.

$$\rho_{A1,2}(0) = \lambda|g\rangle\langle g| + (1-\lambda)|e\rangle\langle e| \tag{5.1}$$

$$\rho_F(0) = |0\rangle\langle 0| \tag{5.2}$$

So, the initial state which describes the entire field-atoms system is

$$\rho(0) = |0\rangle\langle 0| \otimes (\lambda^2 |gg\rangle\langle gg| + (1-\lambda)^2 |ee\rangle\langle ee| + \lambda(1-\lambda)|ge\rangle\langle ge| + \lambda(1-\lambda)|eg\rangle\langle eg|) \tag{5.3}$$

The Hamiltonian describing the system can be written as ($\hbar = 1$)

$$H = \frac{\omega_0}{2}\sigma_z^{(1)} + \frac{\omega_0}{2}\sigma_z^{(2)} + \omega a^\dagger a + g(a\sigma_+^{(1)} + a^\dagger \sigma_-^{(1)}) + g(a\sigma_+^{(2)} + a^\dagger \sigma_-^{(2)}) \tag{5.4}$$

where $\sigma_z^{(1,2)}$ and $\sigma_\pm^{(1,2)}$ are atomic operators such that;

$$\sigma_z|e\rangle = |e\rangle \tag{5.5}$$

$$\sigma_z|g\rangle = -|g\rangle \tag{5.6}$$

and

$$\sigma_+|g\rangle = |e\rangle \tag{5.7}$$

$$\sigma_-|e\rangle = |g\rangle \tag{5.8}$$

also, ω_0 is atomic transition frequency, g is the coupling constant, and $a(a^\dagger)$ is the annihilation (creation) operator of cavity field with frequency ω. In this study, we investigate the entanglement between atoms and cavity in the presence of the pure phase decoherence only. In quantum mechanics, decoherence is the process by which quantum systems in complex environments exhibit classical behaviour. It occurs when a system interacts with its environment in such a way that different portions of its wave function can no longer interfere each other. So there occurs a loss of phase coherence between these portions of wave function of the system.

In this situation, the master equation governing the time evolution for the system under the Markovian approximation is given by [31]

$$\frac{d\rho}{dt} = -i[H,\rho] - \frac{\gamma}{2}[H,[H,\rho]] \tag{5.9}$$

where γ is the phase decoherence rate. The formal solution of the master equation Eq 5.9 is expressed as [32]

$$\rho(t) = \sum_{k=0}^{\infty} \frac{(\gamma t)^k}{k!} M^k(t)\rho(0)M^{\dagger k}(t) \tag{5.10}$$

where $\rho(0)$ is the density operator of the initial atom-field system and $M^k(t)$ is defined by

$$M^k(t) = H^k \exp(-iHt)\exp(-\frac{\gamma t}{2}H^2) \tag{5.11}$$

Substituting $\rho(0)$ into Eq 5.10 gives the time evolution of $\rho(t)$ as;

$$\begin{aligned}
\rho(t) \;=\; & \frac{\lambda^2}{2}|0gg\rangle\langle 0gg| \\
& + \frac{\lambda(1-\lambda)}{4}(1+\frac{\Delta^2}{\Omega^2}+(1-\frac{\Delta^2}{\Omega^2})\cos\Omega t\exp(-\frac{\gamma t}{2}\Omega^2))|0B_+\rangle\langle 0B_+| \\
& + 2\frac{\lambda(1-\lambda)g^2}{\Omega^2}(1-\cos\Omega t\exp(-\frac{\gamma t}{2}\Omega^2))|1gg\rangle\langle 1gg| + \frac{\lambda(1-\lambda)}{2}|0B_-\rangle\langle 0B_-| \\
& + \frac{\sqrt{2}g\lambda(1-\lambda)}{\Omega}(\frac{\Delta}{\Omega}(1-\cos\Omega t\exp(-\frac{\gamma t}{2}\Omega^2)) + i\sin\Omega t\exp(-\frac{\gamma t}{2}\Omega^2))|0B_+\rangle\langle 1gg| \\
& + \frac{\sqrt{2}g(1-\lambda)^2}{\Omega}(\frac{\Delta}{\Omega}(1-\cos\Omega t\exp(-\frac{\gamma t}{2}\Omega^2)) + i\sin\Omega t\exp(-\frac{\gamma t}{2}\Omega^2))|0ee\rangle\langle 1B_+| \\
& + \frac{(1-\lambda)^2 g^2}{\Omega^2}(1-\cos\Omega t\exp(-\frac{\gamma t}{2}\Omega^2))|1B_+\rangle\langle 1B_+| \\
& + \frac{(1-\lambda)^2}{4}(1+\frac{\Delta^2}{\Omega^2}+(1-\frac{\Delta^2}{\Omega^2})\cos\Omega t\exp(-\frac{\gamma t}{2}\Omega^2))|0ee\rangle\langle 0ee| + H.c.
\end{aligned}$$

where $\Delta = \omega_0 - \omega$ is the detuning parameter between the atoms and the cavity field, $\Omega = (\Delta^2 + 8g^2)^{1/2}$, and $|B_\pm\rangle = \frac{1}{\sqrt{2}}(|eg\rangle \pm |ge\rangle)$ are the Bell states. By tracing over the degree of the freedom of the cavity field and the second atom, we obtain the reduced density matrices $\rho_{s1,s2}(t)$ describing the atom-atom and field-atom subsystems as;

$$\begin{aligned}
\rho_{s1}(t) \;=\;& \frac{1}{2}(\lambda^2 + \frac{4g^2\lambda(1-\lambda)}{\Omega^2}(1 - \exp(-\frac{\gamma t}{2}) \cos\Omega t))|gg\rangle\langle gg| \\
&+\frac{1}{4}(\frac{(1-\lambda)}{4\Omega^2}(8g^2(1-\lambda)(1-\exp(-\frac{\gamma t}{2}\Omega^2)) + \lambda(\Delta^2 + 3\Omega^2) \\
&+(\Omega^2 - \Delta^2)\cos\Omega t)))(|ge\rangle\langle ge| + |eg\rangle\langle eg|) \\
&+\frac{(1-\lambda)(8g^2(1-\lambda) + \lambda(\Omega^2 - \Delta^2))}{4\Omega^2}(1 - \exp(-\frac{\gamma t}{2}\Omega^2)\cos\Omega t)|ge\rangle\langle eg| \\
&+\frac{(1-\lambda)^2}{4}(1 + \frac{\Delta^2}{\Omega^2} + (1 - \frac{\Delta^2}{\Omega^2})\cos\Omega t \exp(-\frac{\gamma t}{2}\Omega^2)|ee\rangle\langle ee| + H.c.
\end{aligned}$$

$$\begin{aligned}
\rho_{s2}(t) \;=\;& \frac{1}{2}(\frac{\exp(-\frac{\gamma t}{2}\Omega^2)\lambda(1-\lambda)}{4\Omega^2}(\exp(-\frac{\gamma t}{2}\Omega^2)(\Delta^2 \\
&+(3+\lambda)\Omega^2) + (\Omega^2 - \Delta^2)\cos\Omega t)|0g\rangle\langle 0g| \\
&+\frac{1}{2}(\frac{(\lambda-1)}{4\Omega^2}(((\lambda-2)\Delta^2 - (2+\lambda)\Omega^2) \\
&+(\lambda-2)(\Omega^2-\Delta^2)\cos\Omega t)\exp(-\frac{\gamma t}{2}\Omega^2)))|0e\rangle\langle 0e| \\
&+\frac{g(\lambda-1)}{\Omega^2}(-\Delta + \exp(-\frac{\gamma t}{2}\Omega^2)(\Delta\cos\Omega t - i\Omega\sin\Omega t))|0e\rangle\langle 1g| \\
&+\frac{1}{2}(\frac{g^2(1-\lambda^2)}{\Omega^2}(1 - \cos\Omega t\exp(-\frac{\gamma t}{2}\Omega^2)))|1g\rangle\langle 1g| \\
&+\frac{1}{2}(\frac{g^2(1-\lambda)^2}{\Omega^2}(1 - \cos\Omega t\exp(-\frac{\gamma t}{2}\Omega^2))|1e\rangle\langle 1e| + H.c.
\end{aligned}$$

5.2 Entanglement between atoms and between cavity and atom

In this study, in order to quantify the degree of entanglement between subsystems, we take into account the concurrence *et. al.* [4, 5] defined as Eq 2.25 in chapter 2 to calculate the entanglement between the systems of atom and atom ρ_{s1} and atom and cavity ρ_{s2} such that;

$$C(\rho_{s1,s2}) = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\} \tag{5.12}$$

where the $\lambda_i (i = 1, 2, 2, 4)$ are the square roots of the eigenvalues in decreasing order of the operator

$$R_{s1,s2} = \rho_{s1,s2}(\sigma_y \otimes \sigma_y)\rho^*_{s1,s2}(\sigma_y \otimes \sigma_y) \tag{5.13}$$

5.3 Results and Discussion

The explicit expression of the concurrence for the systems 1 and 2 can obtained by the equations Eq 5.13 and Eq 5.12.

In Fig. 5.1-6, we plot the concurrence of system 1 as the function of time and the parameter λ for decoherence values of $\gamma = 0.1$ and $\gamma = 0.2$ and three different values of Δ. In Figs, it is shown that, the concurrence never goes to zero for particular values of the parameter λ. This means that, the phase decoherence does not completely destroy the entanglement. As times goes on, entanglement reaches a constant value. So, we get a stationary entangled state of two atoms. Also, the increase in the magnitude of Δ suppresses the entanglement.

In Fig. 5.7-9, we plot the concurrence of system 1 as the function of time and the phase decoherence rate γ for the parameter value of $\lambda = 0.1$ and three

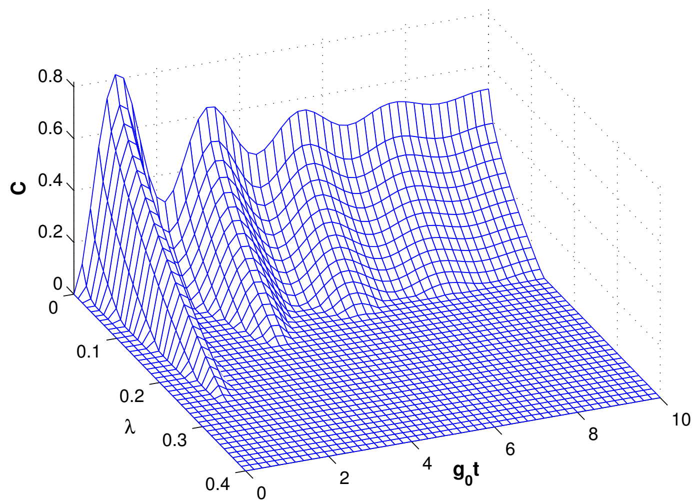

Figure 5.1: The concurrence of the atoms as a function of time g_0t and λ, where $\Delta = 0g$, $\gamma = 0.1$

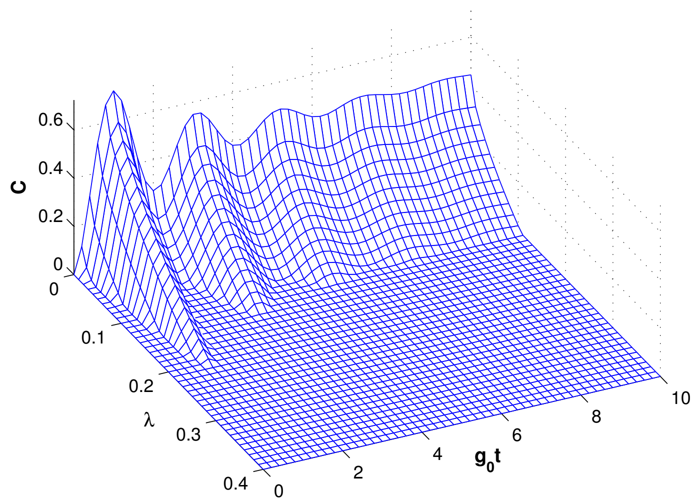

Figure 5.2: The concurrence of the atoms as a function of time g_0t and λ, where $\Delta = g$, $\gamma = 0.1$

different values of Δ. In Figs, it is shown that, the concurrence never goes to zero for $t > 0$ and $\gamma \neq 0$. This means that, the phase decoherence does not completely destroy the entanglement. And also when $\gamma = 0$, we get the dynamical behaviour of ordinary JCM. As times goes on, entanglement reaches a constant value. So, we

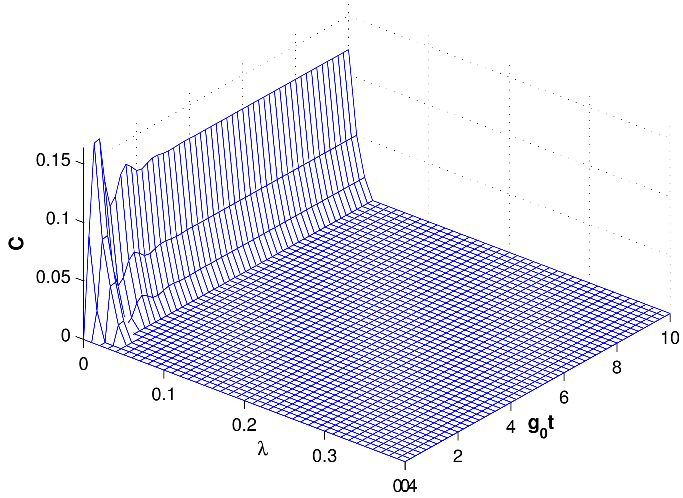

Figure 5.3: The concurrence of the atoms as a function of time $g_0 t$ and λ, where $\Delta = 5g$, $\gamma = 0.1$

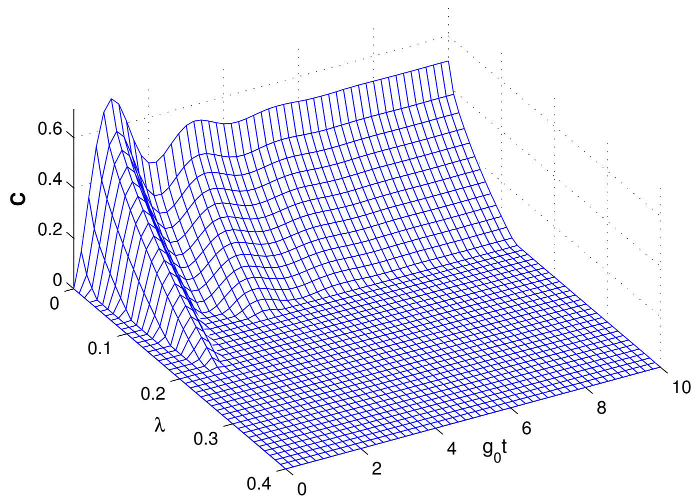

Figure 5.4: The concurrence of the atoms as a function of time $g_0 t$ and λ, where $\Delta = 0g$, $\gamma = 0.2$

get a stationary entangled state of two atoms. Also, the increase in the magnitude of Δ suppresses the entanglement.

In Fig. 5.10-15, we plot the concurrence of system 2 again as the function time and the parameter λ for again decoherence values of $\gamma = 0.1$ and $\gamma = 0.2$

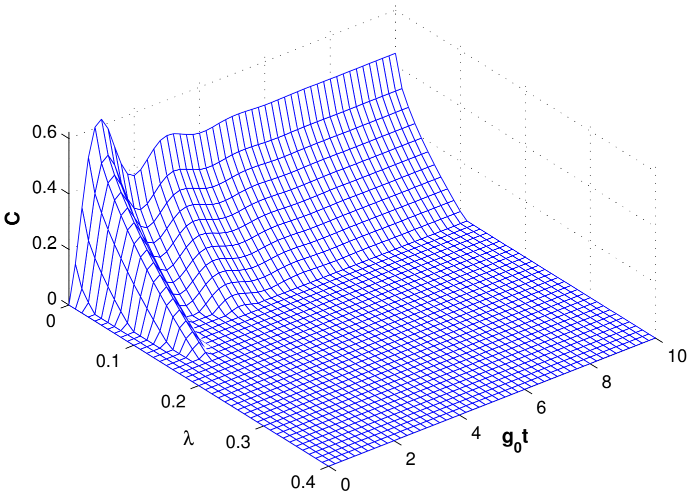

Figure 5.5: The concurrence of the atoms as a function of time g_0t and λ, where $\Delta = g$, $\gamma = 0.2$

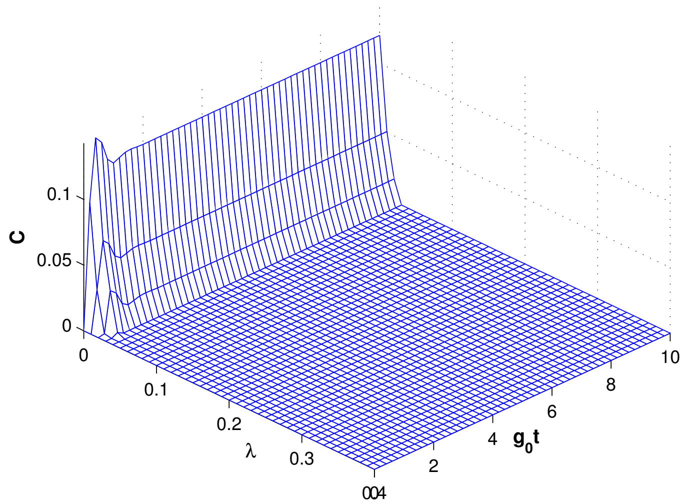

Figure 5.6: The concurrence of the atoms as a function of time g_0t and λ, where $\Delta = 5g$, $\gamma = 0.2$

and three different values of Δ. The increase in the magnitude of Δ feeds the lifetime of the entanglement between atom and the field.

In Fig. 5.16-18, we plot the concurrence of system 2 again as the function time and the parameter γ for again the parameter value of $\lambda = 0.1$ and three

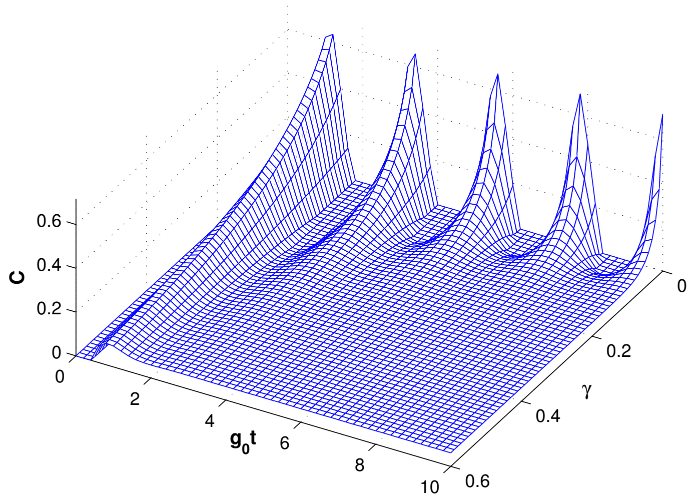

Figure 5.7: The concurrence of the atoms as a function of time $g_0 t$ and γ, where $\Delta = 0g$, $\lambda = 0.1$

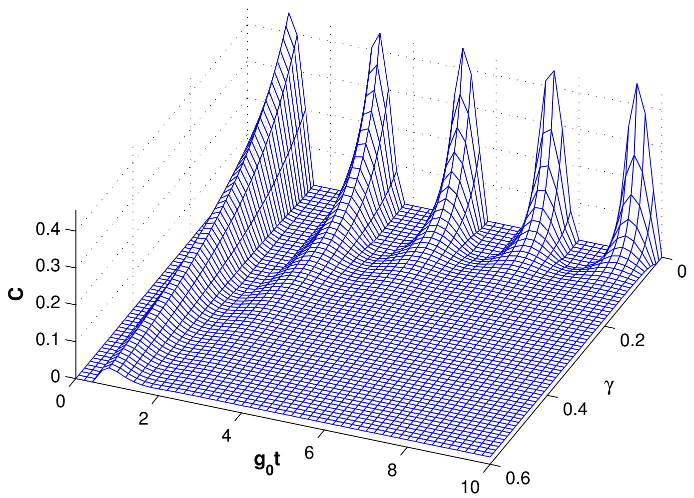

Figure 5.8: The concurrence of the atoms as a function of time $g_0 t$ and γ, where $\Delta = g$, $\lambda = 0.1$

different values of Δ. The increase in the magnitude of Δ feeds the lifetime of the entanglement between atom and the field.

By figures, we can say that, the dynamical behaviours of the entanglements of the both atom-atom and atom-field systems exhibit the similar properties.

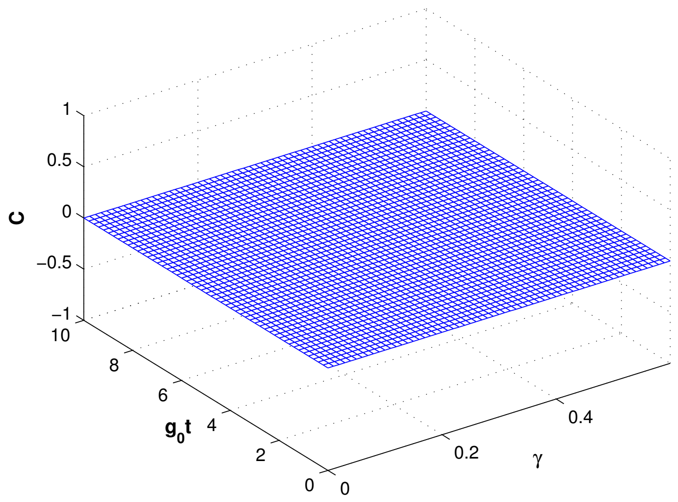

Figure 5.9: The concurrence of the atoms as a function of time g_0t and γ, where $\Delta = 5g$, $\lambda = 0.1$

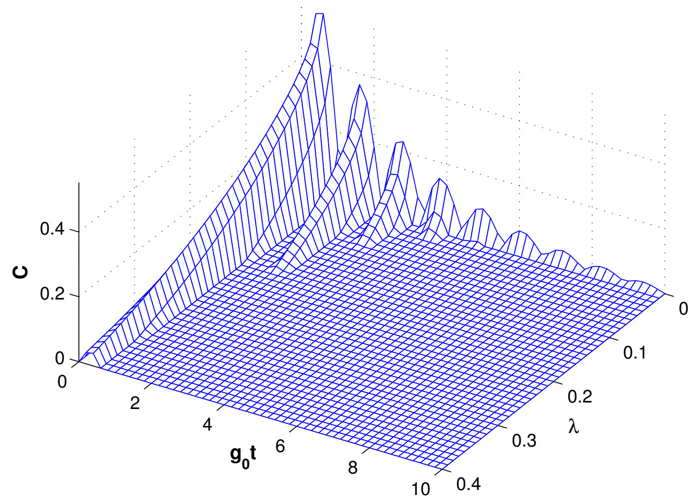

Figure 5.10: The concurrence of the atom and the field as a function of time g_0t and λ, where $\Delta = 0g$, $\gamma = 0.1$

5.4 Conclusion

In this study, we investigate the dynamical properties of the entanglement between two identical atoms and the field in the presence of phase decoherence. The

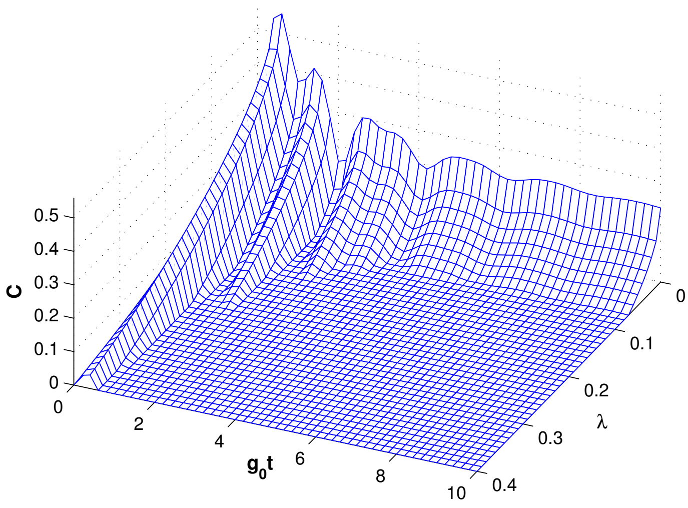

Figure 5.11: The concurrence of the atom and the field as a function of time g_0t and λ, where $\Delta = g$, $\gamma = 0.1$

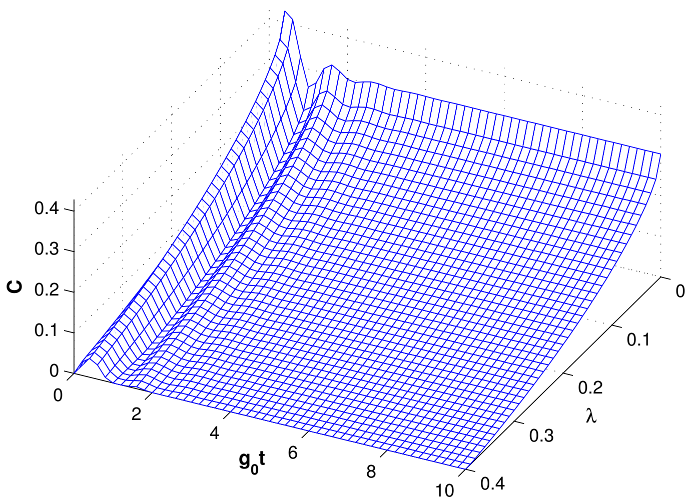

Figure 5.12: The concurrence of the atom and the field as a function of time g_0t and λ, where $\Delta = 5g$, $\gamma = 0.1$

effects of the phase decoherence on the entanglement is obviously displayed by the means of the figures. It is shown that, in the resonance case, atom-atom entanglement can survive and finally reaches a stationary case, but in the atom-field case, entanglement decays rapidly. In the non-resonant case, the entanglement

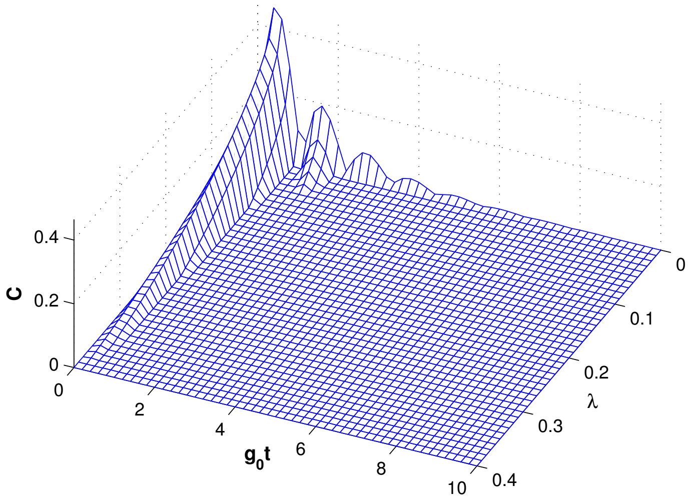

Figure 5.13: The concurrence of the atom and the field as a function of time g_0t and λ, where $\Delta = 0g$, $\gamma = 0.2$

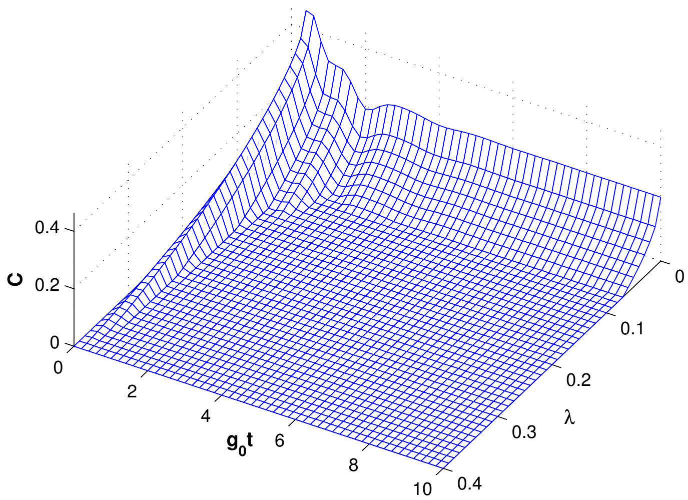

Figure 5.14: The concurrence of the atom and the field as a function of time g_0t and λ, where $\Delta = g$, $\gamma = 0.2$

in both systems is sensitive to the detuning parameter. The entanglement does not completely dies out in time for $\gamma \neq 0$ and for the particular values of the parameter λ.

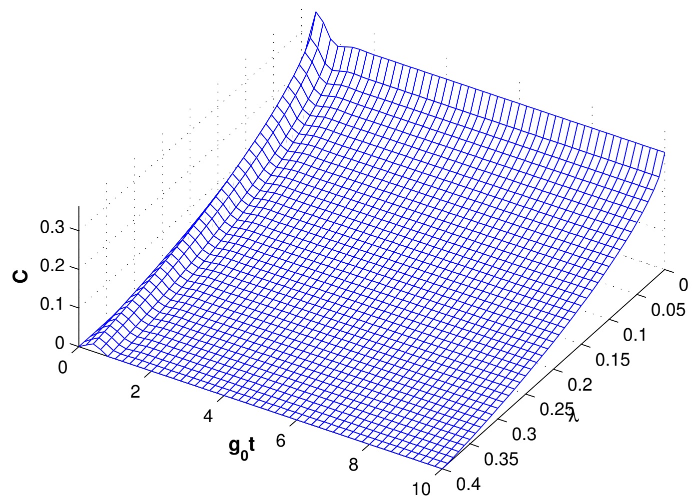

Figure 5.15: The concurrence of the atom and the field as a function of time g_0t and λ, where $\Delta = 5g$, $\gamma = 0.2$

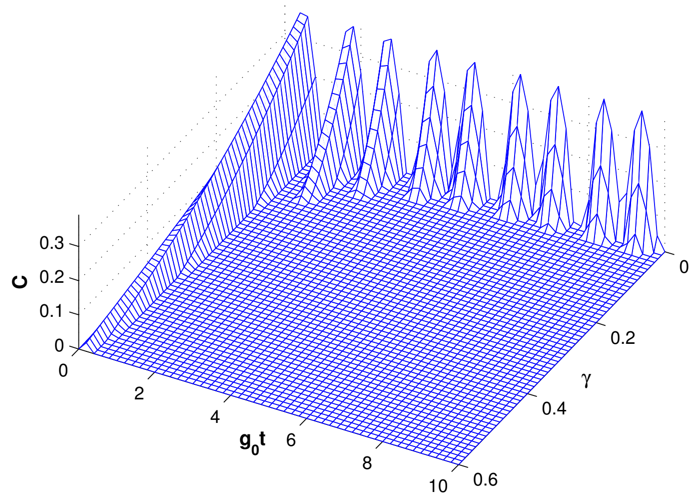

Figure 5.16: The concurrence of the atom and the field as a function of time g_0t and γ, where $\Delta = 0g$, $\lambda = 0.1$

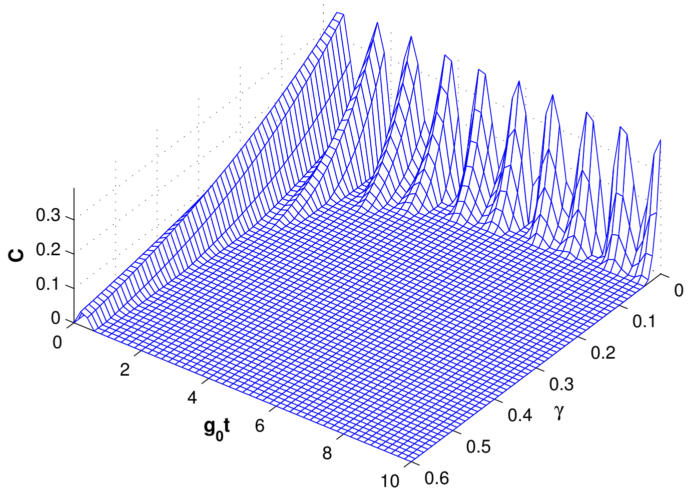

Figure 5.17: The concurrence of the atom and the field as a function of time g_0t and γ, where $\Delta = g$, $\lambda = 0.1$

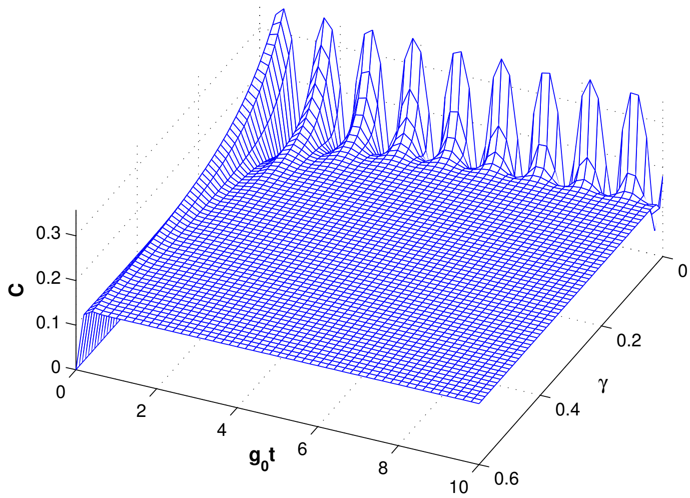

Figure 5.18: The concurrence of the atom and the field as a function of time g_0t and γ, where $\Delta = 5g$, $\lambda = 0.1$

CHAPTER 6

ENTANGLEMENT DYNAMICS IN JAYNES-CUMMINGS MODEL WITH PHASE TELEGRAPH NOISE

6.1 Introduction

JCM is the simplest quantum mechanical model that describes the interaction of a two-level atom with a single mode cavity field [6]. Although this model was proposed for this simple setting, there has been many generalizations of it by taking the atom or the atoms as multi-levels and the cavity as multi-modes, and also different physical situations such as cavity loss ...etc. [33]-[57]. Also, the stochastic versions of the model were investigated in 80s and 90s [58]-[63]. The entanglement in these models were investigated by the numerous groups. In Sainz *et. al.* [64] s' studies, JCM embedded in an atomic environment and the field is in Fock states, they found that; the environment modifies the entanglement dynamics. In Scheel *et. al.* [65] s' studies, the field is the thermal field and a two-level atom is in mixed state, they report that, depending on the initial properties of the field mode, entanglement can show a nonzero behaviour. Bose *et. al.* [66] found that, entanglement exists at arbitrarily high temperatures for JCM. In Khemmani *et. al.* [67] s' studies, they look at the negativity and they prove that, the field is always entangled with the qubits. Kim *et. al.* [68] looked at the interaction of the thermal field with two qubits and they found that, the chaotic field was able to entangle two atoms. Li *et. al.* [69] showed that, the coherent atoms can create entanglement between the modes of two thermal fields. One important finding from these studies is that, the existence of entanglement in the system as time evolves depends strongly on the statistics of the cavity field as well as the initial state of the atoms.

Another important physical situation is the stochasticity. The stochasticity in JCM was first introduced by Burshtein *et. al.* [70] to describe the noisy atom-field interactions. The simplest model for the stochasticity is the two- state random telegraph. In this paper, we take into this model which is described in the characterization of the atom-field coupling coefficient to investigate the the noisy interaction of a two-level atom with a cavity field. We present the explicit expression of the general solution of the problem and give the exact algebraic expressions of all elements of the average density matrix describing the atom-field system. By using these elements, we reduce the general solutions into a specific system in which we consider the field as thermal and the atom as in the excited state. We properly investigate the properties of the entanglement coming to existence between them. We also investigate the purities of the atom and the field to display the characteristic changes in their states. It is shown that, the states evolve to fully mixed states as the time goes on. Finally, we investigate the nonlocality of the system. It is found that, as the more the atom and the field violate Bell inequalities, the larger entanglement happens between them, or vice versa.

The organization of the study is as follows; In section II, we briefly present the formulation of the problem. We firstly introduce our physical model in which the stochasticity is defined and then give the exact algebraic expressions of the general solution. We close this section by introducing the entanglement criteria. In section III, we give the results and some discussions. We reduce the general solutions obtained in the last section into a specific system and discuss the properties of its entanglement dynamics. We also investigate the purities of the atom and the field and their nonlocality. Finally, in section IV, we present our conclusions.

6.2 The Formulation of the Problem

6.2.1 The Model

We consider the interaction of a two-level atom described by the spin-1/2 operators $S_{\pm}, S_z$ with a single mode of quantized radiation field described by the annihilation and the creation operators $a, a^{\dagger}$, respectively. For the sake of simplicity, we assume that, the field is in resonance with the atomic transition frequency ω_0. In this case, the Hamiltonian of the system under the rotating wave approximation takes the form of

$$H = \omega_0 S_z + \omega_0 a^{\dagger} a + (g^*(t) S_+ a + g(t) S_- a^{\dagger}) \tag{6.1}$$

where the coupling coefficient between the atom and the field is time dependent. Since we consider the problem in the case of noise, the nature of $g(t)$ is completely stochastic. The model to describe noise is the two-state telegraph. For this, the coupling coefficient is described as

$$g(t) = g_0 e^{-i\phi(t)} \tag{6.2}$$

where g_0 is is a positive real constant amplitude and $\phi(t)$ is a stochastic variable. Since, we consider the random-phase telegraph, so $\phi(t)$ fluctuates between two states denoted as a and $-a$. These fluctuations are completely random and they obey the Poisson process.

6.2.2 The Exact Solution of the Problem

We can deduce the analytical expressions of the diagonal and off-diagonal elements of the state by using the master equation introduced by Burshtein.

$$\frac{\partial}{\partial t}V_\alpha(t) = -iM_\alpha V_\alpha(t) - \frac{1}{T}\sum_\beta[\delta_{\alpha\beta} - \mathrm{f}(\alpha|\beta)]V_\beta(t) \tag{6.3}$$

where the function $\mathrm{f}(\alpha|\beta)$ is the conditional probability that in the interval $t, t+dt$ this change takes the telegraph from the state β to α, $\hat{V}$ is the vector form of the components of atomic state operator, and also M is effective Liouville operator constituted by the coefficients of the solutions of Liouville-Neumann equation in interaction picture ($\hbar = 1$) such that;

$$i\partial\rho/\partial t = [H_I, \rho] \tag{6.4}$$

From the Eq.6.4 one can obtain the diagonal and the off-diagonal elements of the state ρ in the basis $|\psi_n^+\rangle = |n, e\rangle$ and $|\psi_n^-\rangle = |n+1, g\rangle$; are

$$\frac{d}{dt}\rho_{mn}^{\pm\pm}(t) = i[\alpha_n e^{\mp i\phi}\rho_{mn}^{\pm\mp}(t) - \alpha_m e^{\pm i\phi}\rho_{mn}^{\mp\pm}(t)] \tag{6.5}$$

$$\frac{d}{dt}\rho_{mn}^{\pm\mp}(t) = ie^{\pm i\phi}[\alpha_n\rho_{mn}^{\pm\pm}(t) - \alpha_m\rho_{mn}^{\mp\pm}(t)] \tag{6.6}$$

where $\rho_{mn}^{\pm\mp} = \langle\psi_m^\pm|\rho(t)|\psi_n^\mp\rangle$ and $\alpha_m = g_0\sqrt{m+1}$.

And the operator M can be constructed by the equation

$$\dot{\hat{V}}^k(t) = -iM^{kl}\hat{V}^l(t) \tag{6.7}$$

By using these, one can obtain the Burshtein equation as;

$$\frac{d}{dt}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = -i \begin{pmatrix} M_1 - i(1/T) & i(1/T) \\ i(1/T) & M_2 - i(1/T) \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \tag{6.8}$$

where $V_{1,2}$ is the atomic state component of the state $a, -a$ and $M_{1,2} = M(a, -a)$.

$$V_1 = \begin{pmatrix} \rho_{mn}^{++}(a) \\ \rho_{mn}^{--}(a) \\ \rho_{mn}^{-+}(a) \\ \rho_{mn}^{+-}(a) \end{pmatrix} \tag{6.9}$$

$$V_2 = \begin{pmatrix} \rho_{mn}^{++}(-a) \\ \rho_{mn}^{--}(-a) \\ \rho_{mn}^{-+}(-a) \\ \rho_{mn}^{+-}(-a) \end{pmatrix} \tag{6.10}$$

$$M_1 = \begin{pmatrix} 0 & 0 & \alpha_m \exp(ia) & -\alpha_n \exp(-ia) \\ 0 & 0 & -\alpha_n \exp(ia) & \alpha_m \exp(-ia) \\ \alpha_m \exp(-ia) & -\alpha_n \exp(-ia) & 0 & 0 \\ -\alpha_n \exp(ia) & \alpha_m \exp(ia) & 0 & 0 \end{pmatrix} \tag{6.11}$$

$$M_2 = \begin{pmatrix} 0 & 0 & \alpha_m \exp(-ia) & -\alpha_n \exp(ia) \\ 0 & 0 & -\alpha_n \exp(-ia) & \alpha_m \exp(ia) \\ \alpha_m \exp(ia) & -\alpha_n \exp(ia) & 0 & 0 \\ -\alpha_n \exp(-ia) & \alpha_m \exp(-ia) & 0 & 0 \end{pmatrix} \tag{6.12}$$

Eq. 6.8 can be solved exactly by Laplace transformation technique,

$$\begin{pmatrix} V_1(s) \\ V_2(s) \end{pmatrix} = (1s + iM)^{-1} \begin{pmatrix} V_1(0) \\ V_1(0) \end{pmatrix} \tag{6.13}$$

The stochastic expectation value of $\hat{V}$ is given by;

$$\langle \hat{V} \rangle(t) = 1/2(V_a(t) + V_{-a}(t)) \tag{6.14}$$

To make the algebraic results more compact and simpler, let us introduce the quantities

$$X^{\pm}_{mn}(t) = \frac{1}{2}(\langle \rho^{++}_{mn}(t) \rangle \pm \langle \rho^{--}_{mn}(t) \rangle) \tag{6.15}$$

and

$$Y^{\pm}_{mn}(t) = \frac{1}{2}(\langle \rho^{+-}_{mn}(t) \rangle \pm \langle \rho^{+-}_{mn}(t) \rangle) \tag{6.16}$$

From these, one can obtain the analytical expressions of $X^{\pm\pm}_{mn}(s)$ and $X^{\pm\mp}_{mn}(s)$ as;

$$X^{\pm}_{mn}(s) = \frac{(s^3 + (4/T)s^2 + (4/T^2 + (\Omega^{\mp}_{mn})^2)s + (2/T)(\Omega^{\mp}_{mn})^2 \sin^2 a)X^{\pm}_{mn}(0)}{s^4 + (4/T)s^3 + 2(2/T^2 + \Gamma_{mn})s^2 + (4/T\Upsilon^{\pm}_{mn})s + (\Omega^{\pm}_{mn})^2(4/T^2 \cos^2 a + (\Omega^{\mp}_{mn})^2)} \tag{6.17}$$

and

$$Y^{\pm}_{mn}(s) = \frac{-i\Omega^{\pm}_{mn}(s^2 + (4/T)s + (4/T^2 + (\Omega^{\mp}_{mn})^2)) \cos a X^{\pm}_{mn}(0)}{s^4 + (4/T)s^3 + 2(2/T^2 + \Gamma_{mn})s^2 + (4/T\Upsilon^{\pm}_{mn})s + (\Omega^{\pm}_{mn})^2(4/T^2 \cos^2 a + (\Omega^{\mp}_{mn})^2)} \tag{6.18}$$

where $\Gamma_{mn} = \alpha^2_m + \alpha^2_n$, $\Upsilon^{\pm}_{mn} = \Gamma_{mn} \mp 2\alpha_m\alpha_n \cos^2 a$ and $\Omega^{\pm}_{mn} = (\alpha_m \pm \alpha_n)$.

Here, we have assumed that $\rho_{mn}^{\pm\mp}(0) = 0$

The inverse Laplace transformation of the Eq. 6.17 and Eq. 6.18 yields the general solutions of the problem.

$$X_{mn}^{\pm}(t) = X_{mn}^{\pm}(0) \sum_{j=1}^{4} \frac{\lambda_j((\lambda_j + 2/T\lambda_j)^2 + (\Omega_{mn}^{\mp})^2) + 2/T)(\Omega_{mn}^{\mp})^2 \sin^2 a}{\prod_{k\neq j}(\lambda_j - \lambda_k)} \exp(\lambda_j t) \tag{6.19}$$

$$Y_{mn}^{\pm}(t) = X_{mn}^{\pm}(0) \sum_{j=1}^{4} \frac{-i\Omega_{mn}^{\pm} \cos a((\lambda_j + 2/T\lambda_j)^2 + (\Omega_{mn}^{\mp})^2)}{\prod_{k\neq j}(\lambda_j - \lambda_k)} \exp(\lambda_j t) \tag{6.20}$$

where λ_j are the roots of the equation

$$\lambda^4 + (4/T)\lambda^3 + 2(2/T^2 + \Gamma_{mn})\lambda^2 + (4/T\Upsilon_{mn}^{\pm})\lambda + (\Omega_{mn}^{\pm})^2(4/T^2 \cos^2 a + (\Omega_{mn}^{\mp})^2) = 0 \tag{6.21}$$

6.2.3 Entanglement criteria

The dimensions of the system is $2 \otimes \infty$. In fact, there is no known entanglement measure to quantify such systems. For this, we take into account two entanglement measures. First one is the lower bound of concurrence (LBC) for which the entire state of the atom field system is projected onto $2 \otimes 2$ systems by the means of the projection operator

$$\Pi_n = (|g\rangle\langle g| + |e\rangle\langle e|) \otimes (|n\rangle\langle n| + |n+1\rangle\langle n+1|) \tag{6.22}$$

The resulting density operator is

$$\rho_n(t) = \frac{1}{T_n(t)} \Pi_n \rho(t) \Pi_n \tag{6.23}$$

where $T_n(t) = \mathrm{Tr}(\Pi_n \rho(t) \Pi_n)$.

The resulting states $\rho_n(t)$ are the $2 \otimes 2$ systems. For these systems, the degree of entanglement can be quantified by Wootters' concurrence [4, 5]

$$C_n(t) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} \tag{6.24}$$

where λs are the square roots of the eigenvalues of the matrix

$$R_n(t) = \rho_n(t)(\sigma_y \otimes \sigma_y)\rho_n(t)^*(\sigma_y \otimes \sigma_y) \tag{6.25}$$

The lower bound of concurrence(LBC) is obtained by taking the average of $C_n(t)$ over all possible values of n [71] such that;

$$C(t) = \frac{\sum_{n=0,2,4} T_n(t) C_n(t)}{\sum_{n=0,2,4} T_n(t)} \tag{6.26}$$

The second measure is the mutual entropy $S(A:F)$ of the system which is defined as $S(A:F) = S(A) + S(F) - S(AF)$. For this, we take the dimension of the system n_{upper} such that, the probability distribution of the field reaches approximately unity, $\sum_{n=0}^{n_{upper}} P_n \cong 1$ [72].

To calculate the mutual entropy and the purities , we take the dimension of the system n_{upper} as until probability distribution of the field reaches to approximately unity such that; $\sum_{n=0}^{n_{upper}} P_n \cong 1$ [72].

6.3 Results and Discussion

We can now use the general solutions given by the equations Eq. 6.19 and Eq. 6.20 to investigate the entanglement dynamics of a particular system. For this, we consider the field as thermal and the atom as in the excited state. So, our initial state becomes

$$\rho(0) = \sum_n P_n |n, e\rangle\langle n, e| \tag{6.27}$$

where $|n\rangle$ is the Fock state, $P_n = \frac{\bar{n}^n}{(\bar{n}+1)^n}$ is the weight function called Planck distribution in which $\bar{n}$ is the average number of photons in the cavity given by $\bar{n} = \frac{1}{\exp(\beta)-1}$ in which β is proportional to the inverse of the temperature of the cavity. It is clear that, $X^{\pm}_{nn}(0) = P_n/2$ and $Y^{\pm}_{nn}(0) = 0$.

The complete solution of the average density matrix of the state is the sum of 2×2 sub-block forms;

$$\begin{aligned} \langle\rho(t)\rangle &= \sum_n (\langle\rho^{++}_{nn}(t)\rangle |n, e\rangle\langle n, e| + \langle\rho^{--}_{nn}(t)\rangle |n+1, g\rangle\langle n+1, g| + \\ &\quad \langle\rho^{+-}_{nn}(t)\rangle |n, e\rangle\langle n+1, g| + \langle\rho^{-+}_{nn}(t)\rangle |n+1, g\rangle\langle n, e|) \end{aligned}$$

It is so clear to apply LBC on our system such that; From the equation Eq. 6.23, one can find that;

$$\langle\rho_n(t)\rangle = \frac{1}{T_n} \begin{pmatrix} x & 0 & 0 & y \\ 0 & z & 0 & 0 \\ 0 & 0 & t & 0 \\ y^* & 0 & 0 & w \end{pmatrix} \tag{6.28}$$

where $x = \langle\rho^{++}_{nn}(t)\rangle$, $y = \langle\rho^{+-}_{nn}(t)\rangle$, $z = \langle\rho^{++}_{n+1n+1}(t)\rangle$, $t = \langle\rho^{--}_{n-1n-1}(t)\rangle$, $w =$

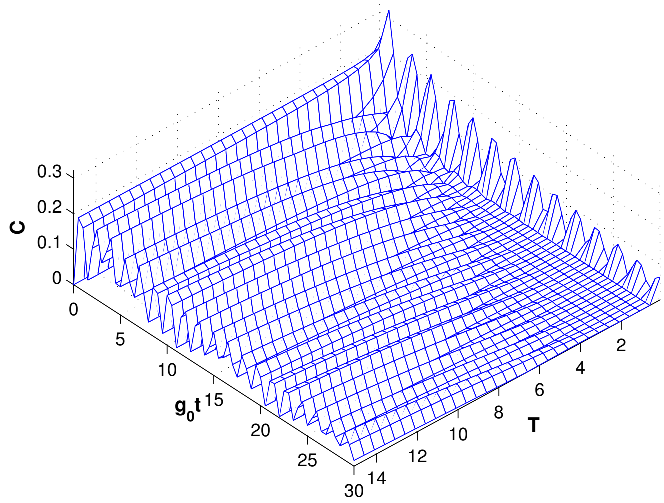

Figure 6.1: The lower bound of concurrence as the function of the mean dwell time T and time g_0t: $a = \pi/4$ and $\bar{n} = 1$

$\langle \rho_{nn}^{--}(t) \rangle$ and also $T_n = (x + z + t + w)_n$. From the Eq. 6.24, C_n can be obtained as ;

$$C_n(t) = \frac{2}{T_n}(|y| - \sqrt{zt}), \quad if \ \sqrt{xw} \geq |y| \tag{6.29}$$

$$C_n(t) = \frac{2}{T_n}(\sqrt{xw} - \sqrt{zt}), \ \ if \ \sqrt{xw} \leq |y| \tag{6.30}$$

The effects of the phase telegraph on the entanglement dynamics of the atom-field interaction are depicted in Figs. 6.1-13.

In Figs. 6. 1-3 and Figs. 6. 6-8, we plot LBC (C) and the mutual entropy ($S(A:F)$) as the function of time g_0t and the mean dwell time T for the particular values of phase value a and the mean photon number in the cavity $\bar{n}$. It is shown that, there happens a decay in the value of the concurrence in time. As the value of mean dwell time T increases, the value of entanglement increases and reaches a stationary value. Because, as T increases, the effect of the noise on the

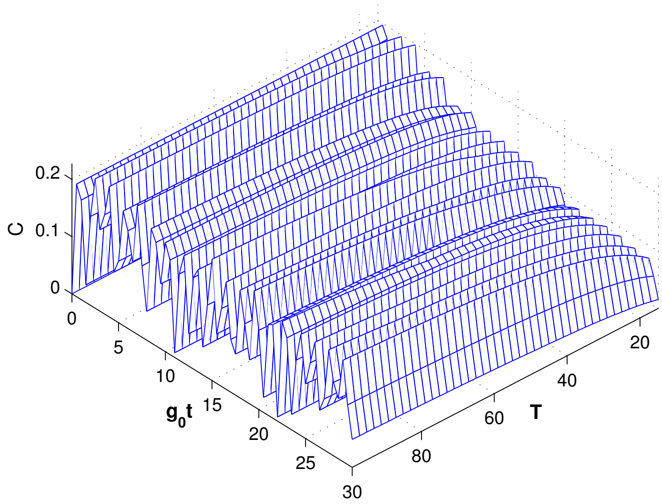

Figure 6.2: The lower bound of concurrence as the function of the mean dwell time T and time g_0t: $a = \pi/4$ and $\bar{n} = 1$

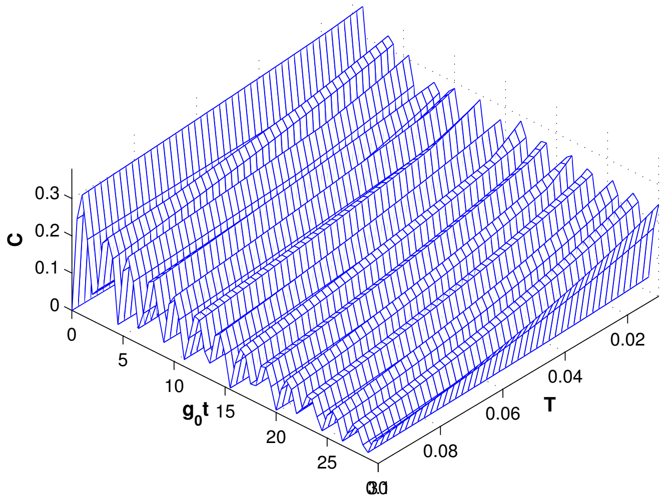

Figure 6.3: The lower bound of concurrence as the function of the mean dwell time T and time g_0t: $a = \pi/4$ and $\bar{n} = 1$

interaction weakens. In this case, we get the entanglement dynamics of ordinary JCM. For the small values the mean dwell time, the decrease in the mean dwell time causes the increase in the value of entanglement. Because, the changes in the phase value are happening very fast. And, the system can not follow these

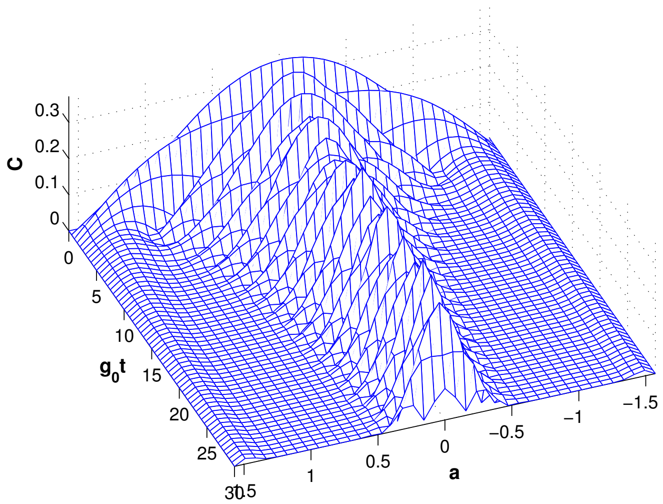

Figure 6.4: The lower bound of concurrence as the function of the phase value a and time g_0t: $T = 1$ and $\bar{n} = 1$

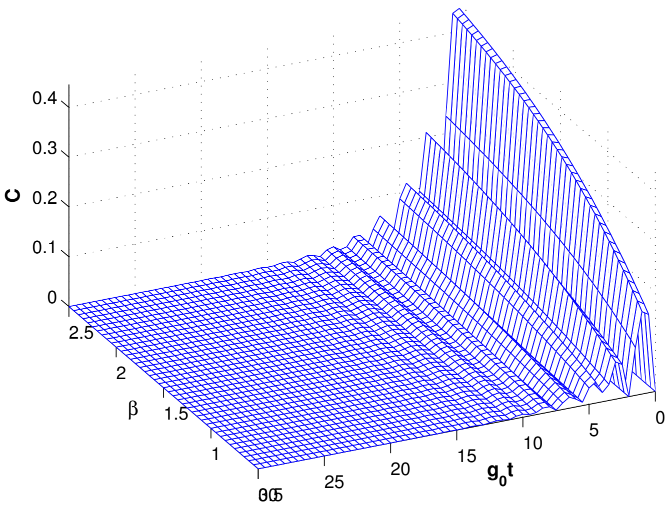

Figure 6.5: The lower bound of concurrence as the function of β and time g_0t: $T = 1$ and $a = \pi/4$

changes. So, the system only feels the average of the changes.

In Figs. 6.4 and 6.9, C and $S(A : F)$ are plotted as the function of time g_0t and the phase value a for the particular values of the mean dwell time T and the mean photon number $\bar{n}$. Obviously, as the value of a increases, there happens a

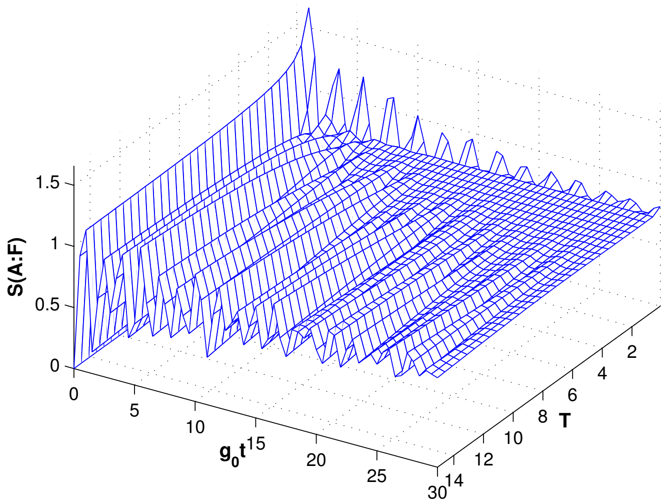

Figure 6.6: The mutual entropy as the function of the mean dwell time T and time g_0t: $a = \pi/4$ and $\bar{n} = 0.1$

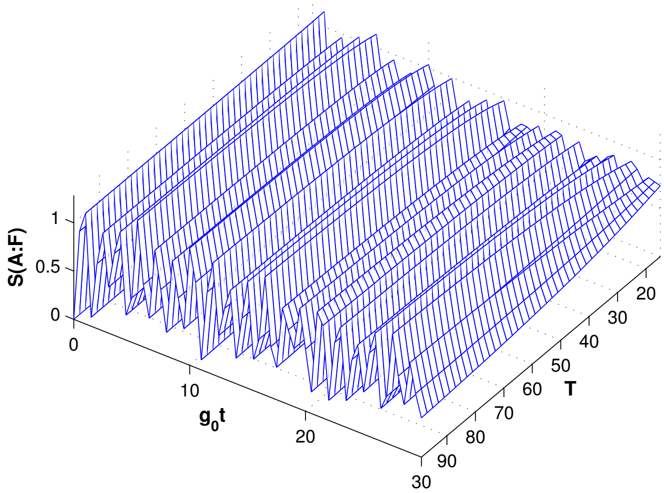

Figure 6.7: The mutual entropy as the function of the mean dwell time T and time g_0t: $a = \pi/4$ and $\bar{n} = 0.1$

decay in the value of concurrence in time. For $a = 0$, we get the entanglement dynamics of the ordinary JCM, as expected.

In Figs. 6.5 and 6.10, we plot C and $S(A : F)$ as the function of time g_0t and the temperature parameter β for the particular values of the mean dwell time

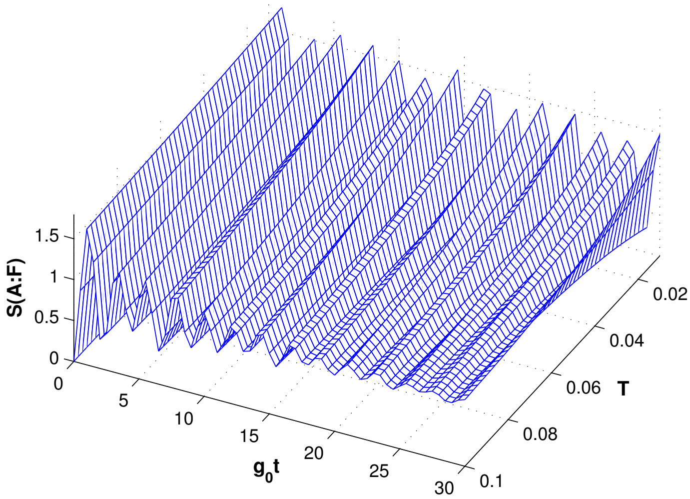

Figure 6.8: The mutual entropy as the function of the mean dwell time T and time g_0t: $a = \pi/4$ and $\bar{n} = 0.1$

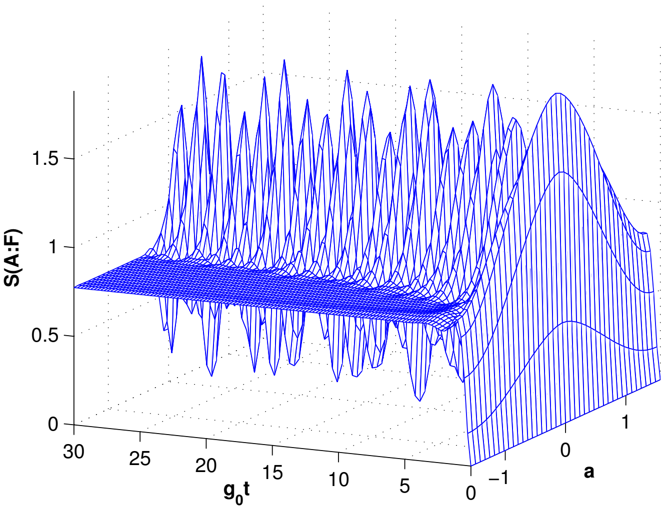

Figure 6.9: The mutual entropy as the function of the phase value a and time g_0t: $T = 1$ and $\bar{n} = 0.1$

and the phase value. The increase in the value of β equivalently, the decrease in the temperature of the cavity, causes the increase in the value of entanglement. Also, entanglement goes to zero as the time passes.

We also investigate the purities of the atom and the field. The time evolution

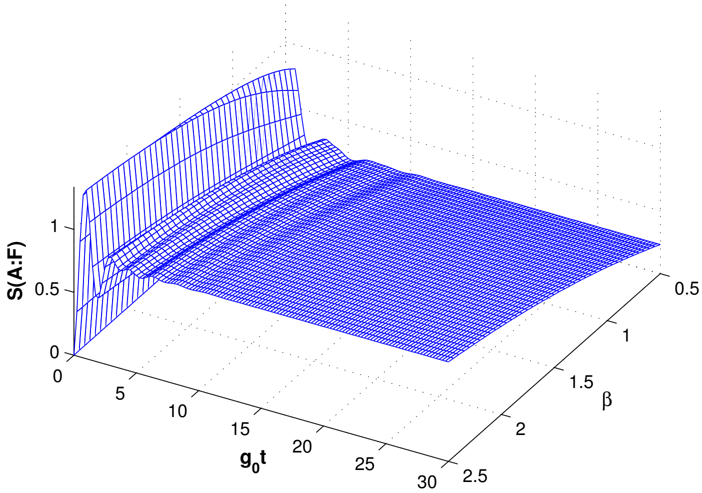

Figure 6.10: The mutual entropy as the function of β and time g_0t: $T = 1$ and $a = \pi/4$

of the purities of the atom and the field depict us the changes in the characters of their states. The purities are calculated as in Eq. **??**. The density matrices of the atom and the field states are in diagonal form

$$\rho_A(t) = \begin{pmatrix} \sum_n \rho_{nn}^{++}(t) & 0 \\ 0 & \sum_n \rho_{nn}^{--}(t) \end{pmatrix} \tag{6.31}$$

and

$$\rho_F^{nn}(t) = \rho_{nn}^{++}(t) + \rho_{n-1n-1}^{--}(t) \tag{6.32}$$

The Figs. 6.11 and 6.12 obviously show that, the states of the atom and the field evolves to fully mixed states as the time goes on.

6.4 The Nonlocality in the System

In this section, we investigate the nonlocality of atom-field system. The nonlocality can be characterized by CHSH inequality [73, 74] which is the maximal

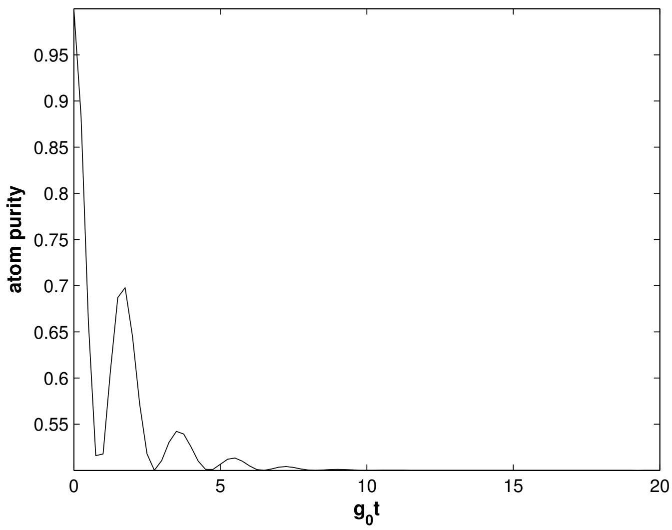

Figure 6.11: The atom purity as a function of time g_0t: $\bar{n} = 0.1$, $T = 1$ and $a = \pi/4$

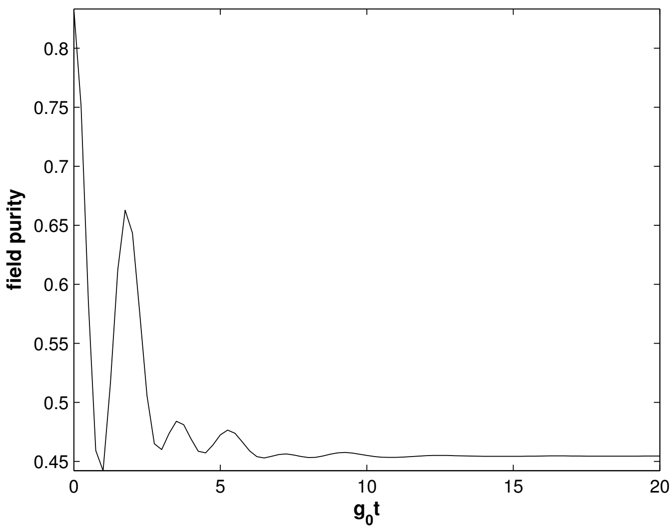

Figure 6.12: The field purity as a function of time g_0t: $\bar{n} = 0.1$, $T = 1$ and $a = \pi/4$

violation of Bell's inequalities.

The CHSH operator is defined as;

$$\hat{B} = \overrightarrow{a} \cdot \overrightarrow{\sigma} \otimes (\overrightarrow{b} + \overrightarrow{b'}) \cdot \overrightarrow{\sigma} + \overrightarrow{a'} \cdot \overrightarrow{\sigma} \otimes (\overrightarrow{b} - \overrightarrow{b'}) \cdot \overrightarrow{\sigma} \tag{6.33}$$

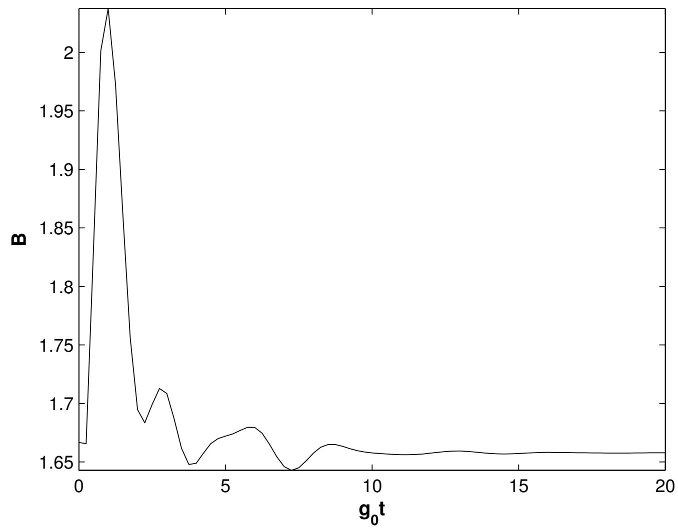

Figure 6.13: The maximal violation of Bell's inequality B as the function of time g_0t

where $\overrightarrow{a}, \overrightarrow{a'}, \overrightarrow{b}, \overrightarrow{b'}$ are unit vectors. CHSH inequality is described as $|\langle \hat{B} \rangle| \leq 2$. The maximal amount of Bell's violation is defined as [75]

$$B = 2\sqrt{\lambda + \tilde{\lambda}} \tag{6.34}$$

where λ and $\tilde{\lambda}$ are the two largest eigenvalues of $T_\rho^\dagger T_\rho$. The elements of the matrix T_ρ being a 3×3 are constructed by $(T_\rho)_{mn} = \mathrm{Tr}(\rho\sigma_m \otimes \sigma_n)$,$(m,n) = 1,2,3$ where $\sigma_{m,n}$ are the Pauli spin matrices. B indicates the Bell violation if $B > 2$ and the maximal violation if $B = 2\sqrt{2}$. Here, we adopt the same method used for calculating the lower bound of concurrence such that;

$$B(t) = \frac{\sum_{n=0,2,4} T_n(t) B_n(t)}{\sum_{n=0,2,4} T_n(t)} \tag{6.35}$$

where $B_n(t)$ is calculated on the reduced matrix given by Eq 6.28.

The figure 6.13 shows the time evolution the amount of maximal violation of Bell's inequality B. Its value increases at the beginning of the evolution $B > 2$

and finally reaches a stable value $B < 2$. This means that, the atom-field entanglement survives until a certain time and then dies out. This result is completely compatible with the results of LBC, the mutual entropy and the purities. So, one can say that, the more violation of Bell inequalities, the larger entanglement, or vice versa.

6.5 Conclusion

In this study, we investigate the system of a single two-level atom interacting with a single mode of optical cavity with the phase telegraph noise and calculate the entanglement between the atom and the cavity field. The effects of the relevant parameters of the noise on the system are obviously displayed. We show that, the mean dwell time T and the phase value a determines the time that the entanglement dies out. Also, the temperature dependence of the entanglement is found to be different in the noisy case compared to the non-noisy JCM. We also investigate the changes in the characters of the states of the atom and the field by calculating their purities. It is shown that, the states of the atom and the field evolve to fully mixed states. Finally, the nonlocality of the atom and the field is investigated. We depict that, the more violation of the maximal of Bell inequalities, the larger entanglement, or vice versa.

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