STABILIZATION AND STRONG STABILIZATION OF LTI SYSTEMS BY STRUCTURED TIME-DELAY CONTROLLERS

PhD Dissertation

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Elektrik-Elektronik Mühendisliği Anabilim Dalı
Danışman: Prof. Dr. Altuğ İFTAR

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FINAL APPROVAL FOR THESIS

This thesis titled **"Stabilization and Strong Stabilization of LTI Systems by Structured Time-delay Controllers"** has been prepared and submitted by **S. Mert ÖZER** in partial fullfillment of the requirements in "Eskişehir Technical University Directive on Graduate Education and Examination" for the Degree of Doctor of Philosophy (PhD) in Department of Electrical and Electronics Engineering has been examined and approved on 18/08/2022.

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ABSTRACT

Stabilization and Strong Stabilization of LTI Systems by Structured Time-delay Controllers

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Department of Electrical and Electronics Engineering
Programme in Control and Command Systems
Eskişehir Technical University, Institute of Graduate Programs, August 2022

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The main objective of this thesis is to develop controller design methods for stabilization and strong stabilization of linear time-invariant multi-input multi-output systems. In particular, the open-loop systems to be considered are allowed to be retarded or neutral time-delay systems which may have an arbitrary number of discrete time delays in the input/output channels or in the dynamics of the system itself. In order to stabilize such systems, an eigenvalue optimization-based controller design method is proposed in which the stabilization problem is formulated as the minimization of spectral abscissa function over the controller parameters. In this context, not only conventional finite-dimensional controllers but also time-delay controllers with an arbitrary number of artificial time delays are aimed to be designed. Furthermore, the controllers to be designed are allowed to be structured in an arbitrary form. A strong stabilization method is also developed as an extension of the stabilization method. More precisely, the centralized and/or decentralized strong stabilization problem is formulated by adding the controller stability as a constraint to the present stabilization problem. Finally, a new MATLAB-based software, which implements the analysis and controller design methods proposed in this thesis, is developed.

Keywords: Stabilization, strong stabilization, time-delay systems, time-delay controllers, optimization.

ÖZET

Doğrusal Zamanla Değişmeyen Sistemlerin Yapılandırılmış Zaman Gecikmeli Denetleyiciler ile Kararlılaştırılması ve Kuvvetli Kararlılaştırılması

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Bu tezin temel amacı, doğrusal zamanla değişmeyen çok girdili çok çıktılı sistemlerin kararlılaştırılması ve kuvvetli kararlılaştırılması için denetleyici tasarım yöntemleri geliştirmektir. Özellikle, ele alınacak olan açık döngü sistemlerin, girdi/çıktı kanallarında veya sistemin kendi dinamiklerinde keyfi sayıda ayrık zaman gecikmesine sahip olabilen geri-tipli veya tarafsız zaman gecikmeli sistemler olmasına izin verilmiştir. Bu tür sistemleri kararlılaştırmak için, kararlılaştırma probleminin denetleyici parametreleri üzerinden spektral apsis fonksiyonunun minimizasyonu olarak formüle edildiği özdeğer optimizasyonu tabanlı bir denetleyici tasarım yöntemi önerilmiştir. Bu kapsamda sadece geleneksel sonlu boyutlu denetleyiciler değil, isteğe bağlı sayıda yapay zaman gecikmelerine sahip zaman gecikmeli denetleyicilerin de tasarlanması amaçlanmıştır. Ayrıca, tasarlanacak olan hem sonlu boyutlu hem de zaman gecikmeli denetleyicilerin keyfi bir biçimde yapılandırılmasına izin verilmiştir. Kararlılaştırma yönteminin bir uzantısı olarak bir kuvvetli kararlılaştırma yöntemi de geliştirilmiştir. Daha kesin olarak, merkezi ve/veya merkezi olmayan kuvvetli kararlılaştırma problemi, mevcut kararlılaştırma problemine bir kısıt olarak denetleyici kararlılığı eklenerek formüle edilmiştir. Son olarak, bu tezde önerilen analiz ve denetleyici tasarım yöntemlerini uygulayan yeni bir MATLAB tabanlı yazılım geliştirilmiştir.

Anahtar Sözcükler: Kararlılaştırma, kuvvetli kararlılaştırma, zaman gecikmeli sistemler, zaman gecikmeli denetleyiciler, optimizasyon.

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S. Mert ÖZER
18/08/2022

18/08/2022

STATEMENT OF COMPLIANCE WITH ETHICAL PRINCIPLES AND RULES

I hereby truthfully declare that this thesis is an original work prepared by me; that I have behaved in accordance with the scientific ethical principles and rules throughout the stages of preparation, data collection, analysis, and presentation of my work; that I have cited the sources of all the data and information that could be obtained within the scope of this study, and included these sources in the references section; and that this study has been scanned for plagiarism with "scientific plagiarism detection program" used by Eskişehir Technical University, and that "it does not have any plagiarism" whatsoever. I also declare that, if a case contrary to my declaration is detected in my work at any time, I hereby express my consent to all the ethical and legal consequences that are involved.

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ABBREVIATIONS

CFM	: Centralized Fixed Mode
DFM	: Decentralized Fixed Mode
DDE	: Delay-Difference Equation
DDAE	: Delay-Differential-Algebraic Equation
FM	: Fixed Mode
LTI	: Linear Time-Invariant
MIMO	: Multi-Input Multi-Output
PD	: Proportional Derivative
PI	: Proportional Integral
PID	: Proportional Integral Derivative
PIP	: Parity Interlacing Property
PIR	: Proportional Integral Retarded
PR	: Proportional Retarded
SISO	: Single-Input Single-Output
TDC	: Time-Delay Controller
TDS	: Time-Delay System
TFM	: Transfer Function Matrix

NOTATION

$\mathbb{R}$	: The set of real numbers
$\mathbb{C}$	: The set of complex numbers
$\mathcal{H}^{\infty}$	: Hardy space of bounded and analytic functions
$\mathcal{H}^{\infty}_{\mathbb{R}}$	: Hardy space of bounded and analytic real functions
$M(\mathcal{R})$	: The set of all matrices whose entries are in a ring $\mathcal{R}$
$\mathbb{N}$	: The set of natural numbers with zero
$\mathbb{R}^n$	: The space of n-dimensional real vectors
$\mathbb{C}^n$	: The space of n-dimensional complex vectors
$\mathbb{R}^{m\times q}$	: The space of $m \times q$-dimensional real matrices
$\mathbb{C}^{m\times q}$	: The space of $m \times q$-dimensional complex matrices
Re(s)	: Real part of $s \in \mathbb{C}$
Im(s)	: Imaginary part of $s \in \mathbb{C}$
$\mathbb{C}^e$	: $\mathbb{C} \cup \{\infty\}$ (extended complex plane)
$\mathbb{C}^-_{\epsilon}$	: $\{s \in \mathbb{C} \mid \mathrm{Re}(s) < \epsilon\}$ for $\epsilon \in \mathbb{R}$
$\mathbb{C}^+_{\epsilon}$	: $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq \epsilon\}$ for $\epsilon \in \mathbb{R}$
$\mathbb{C}^{+e}_{\epsilon}$	: $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq \epsilon\} \cup \{\infty\}$ for $\epsilon \in \mathbb{R}$
$\mathbb{C}_0$	: $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$ (open right-half plane)
$\mathbb{C}^+_0$	: $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 0\}$ (closed right-half plane)
$\mathbb{C}^{+e}_0$	: $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 0\} \cup \{\infty\}$ (extended right-half plane)
$\mathbb{R}^{+e}_0$	: $\{s \in \mathbb{R} \mid s \geq 0\} \cup \{\infty\}$
j	: Imaginary unit, $\mathrm{j}^2 = -1$
$\mathrm{nrank}(T(s))$	: Normal rank of a transfer function matrix $T(s)$ where $s \in \mathbb{C}$
$f(x; y)$	: A function with a variable x and a parameter y
∇f	: Gradient of a function f
I	: Identity matrix of appropriate dimensions
I_n	: $n \times n$ identity matrix
0	: Zero matrix of appropriate dimensions
0_n	: $n \times n$ zero matrix
$0_{m\times q}$	: $m \times q$ zero matrix
$\rho(\Gamma)$	: Spectral radius of a matrix Γ

$\mathrm{rank}(\Gamma)$: Rank of a matrix Γ

$\det(\Gamma)$: Determinant of a matrix Γ

Γ^T : Transpose of a matrix or vector Γ

Γ^* : Complex-conjugate transpose of a matrix or vector Γ

$\Gamma^\perp$: Perpendicular complement of a matrix or vector Γ

Γ^{-1} : Inverse of a full-rank matrix Γ

$\mathrm{bdiag}(\cdots)$: A block diagonal matrix with blocks $\cdots$ on its main diagonal

1. INTRODUCTION

In Section 1.1, the existing literature about the content of this thesis is overviewed. Then, the objectives of the thesis and the motivation are presented in Section 1.2. In Section 1.3, the outline of the subsequent chapters is given.

1.1. Literature Overview

In feedback control systems, acquiring information, creating control signals, and executing decisions do not occur instantaneously [1]. Because of that, there always exist certain time delays. Furthermore, time delays may also occur in the system dynamics due to communication and transportation lags (e.g., see [2] and references therein). In many cases, it is not possible to completely eliminate time delays. In other words, the presence of time delays is inevitable. It is well-known that neglecting these delays may lead to poor performance, lack of robustness, or even cause a destabilizing effect in feedback control systems. Therefore, time delays must be taken into account in the controller design. In particular, such systems are described by *delay-differential* or *delay-differential-algebraic equations* (DDAEs) and called *time-delay systems*. In general, it is more difficult to analyze and stabilize time-delay systems, compared to the delay-free systems, since their states can not be represented by finitely many state variables, thus, they are infinite-dimensional; therefore, an LTI time-delay system has infinitely many modes (characteristic roots), in general [3]. A system described by a delay-differential equation which does not involve the delayed version of the derivative of the state vector is said to be a *retarded* time-delay system. On the other hand, if the delay-differential equation involves the delayed versions of the derivative of the state vector or if the system is described by DDAEs, then the system may become a *neutral* time-delay system. It is more difficult to deal with neutral systems compared to retarded systems, because, although an LTI retarded system always has finitely many modes in any right-half complex plane (as in finite-dimensional systems), neutral systems may exhibit infinitely many modes tending to infinity along vertical axes at the right-hand side of a real number on the complex plane [4]. Furthermore, even if an LTI neutral system has finitely many unstable modes, this situation can dramatically change when an infinitesimal perturbation in the time delays occurs [3].

In feedback control, stability is the primary expectation when it comes to controller design. Even though there exist a rich literature and various controller design methods for finite-dimensional systems, the stabilization of time-delay systems is still an open problem [5]. For time-delay systems, the analytical controller design methods can only be derived for relatively simple systems such as generally low-dimensional and/or having a limited number of time delays (see [3] and the references therein). In this context, many different controller design methods for stabilization of time-delay systems have been proposed in the literature (e.g., see [3], [5], and references therein) so far.

Although stability of an overall closed-loop system is the primary expectation in feedback control, in most of the practical cases, there are some other requirements such as tracking performance, disturbance rejection and robustness [6]. Even though closed-loop stability can be achieved by an unstable controller, the unstable controllers may have certain disadvantages in some practical applications. For instance, system integrity may be a motivation to use stable controllers. In the feedback control, when a sensor or an actuator fails or is deliberately turned off during start-up or shutdown, hence the feedback loop opens; overall stability can not be maintained if the controller is unstable. A stable controller, however, guarantees overall stability under such a fault if the plant is also stable [7]. On the other hand, an unstable controller introduces additional right-half plane zeros, which reduce the tracking ability and disturbance rejection of the closed-loop system and makes it more sensitive to numerical errors and nonlinear effects [8]. Such effects may in fact cause an unstable behavior in practical implementations [9]. Because of these reasons, stable controller design, which is also referred to as strong stabilization, became an important problem [8]. Although strong stabilization is important, not all plants are strongly stabilizable. It is well-known that a stabilizable finite-dimensional LTI system is strongly stabilizable if and only if it satisfies the parity interlacing property (PIP); a system is said to satisfy the PIP if the number of poles (counted according to their McMillan degrees) between any pair of blocking zeros on the extended positive real-axis is even [10]. Strong stabilization problem is first studied for finite-dimensional systems (e.g., [8, 11–18]), then, for infinite-dimensional systems in single-input single-output (SISO) case [19–21] and in multi-input multi-output (MIMO) case (see [22] and references therein). In particular, a number of analytical methods to solve the strong stabilization problem for both finite-dimensional (e.g., [12, 17]) and time-delay (e.g., [9, 19, 21, 23]) systems have been proposed. Most of these analytical methods developed to solve the strong stabilization problem rely on the construction of a unit function that interpolates the value of the characteristic function of the system at the blocking zeros of the system located in the extended right-half plane. Due to the interpolation-based design approach, the analytical methods are proposed only for systems with a restricted number of unstable poles and/or blocking zeros (see [24] and references therein). Furthermore, as reported in many studies, those analytical methods developed to solve the strong stabilization problem usually lead to high-dimensional controllers. Moreover, the existence of infinitely many unstable poles and/or blocking zeros makes the interpolation phase of those analytical methods even more complicated [23].

In recent years, the eigenvalue-based methods are used for stabilization of LTI time-delay systems, effectively. First, the so-called *continuous pole placement* algorithm was presented in [25] for retarded time-delay systems, then, extended to the neutral case in [26]. Then, the *eigenvalue optimization*-based controller design method, which was originally introduced for robust stabilization of finite-dimensional systems in [27], was adapted to retarded time-delay systems in [28], and to neutral time-delay systems in [29]. In addition to these papers, this method has been successfully used in many studies [30–43]. The underlying idea of the eigenvalue optimization-based stabilization is minimizing the real part of the rightmost mode, i.e., *spectral abscissa* of the closed-loop system. From this perspective, this strategy not only leads to a stable closed-loop system but also to a stable closed-loop system with a maximum exponential decay rate. We note that, achieving the maximum exponential decay rate is quite desirable in some applications such as load frequency control [44], fast consensus in networked control systems [45], synchronization in network systems [46], multi-agent systems [47], oscillatory systems [48], etc.

Even though the closed-loop stability is not guaranteed by using the eigenvalue optimization-based method, it can, in principle, be achieved through minimizing spectral abscissa as a function of sufficiently large number of controller parameters. Therefore, it is advised to increase the dimension of the controller to be designed until the stability is achieved. Thus, the success of this method highly depends on the degree of freedom of the controller to be used. If the stability can not be achieved, then, the degree of freedom of the controller to be designed must be increased, gradually, as it is suggested in [49], by increasing the number of differential order of a delay-free dynamic controller. However, for complex systems, e.g., time-delay systems, such a strategy may cause very high-dimensional delay-free controllers which are quite undesirable for various reasons, such as cost, reliability, robustness, etc. [50]. In [51], it has been shown that rather than using such delay-free controllers with high differential degree, simpler time-delay controllers (i.e., *delayed-feedback*) may be employed, since it is known that time-delay controllers, which introduce artificial time delays, can have a stabilizing effect on certain systems (see [1, 52, 53], and the references therein). For instance, [48] has shown that an oscillatory system can be stabilized using a simple delayed-feedback instead of using a derivative feedback or an observer. Furthermore, time-delay controllers have several advantages in practical implementations [54]. It has been shown that time-delay controllers can improve the exponential decay rate [55], noise attenuation [56], and stability margins [57]. However, the analytical design of such controllers is quite difficult even for low-order systems [58, 59]. Besides their difficulties, most of the analytic methods are dedicated to determine the stability maps over the controller parameters, e.g., [60]. On the other hand, iterative methods, such as continuous pole placement [25] or eigenvalue optimization [29], can be used to design stabilizing time-delay controllers for high-dimensional or even time-delay systems [55, 61]. The optimization-based stabilization method has been successfully adapted to the design of time-delay controllers in [55]. In that study, the designed controllers had no differential parts, i.e., no integrators in their implementation. Even so, by benchmark examples, it has been shown that such time-delay controllers may be more effective than delay-free controllers with differential parts, i.e., finite-dimensional dynamic controllers. However, for some systems, the stabilizing controller may need to have differential parts also. In this case, the controller must be described by delay-differential equations. So, in [51], controller type considered in [55] has been extended to a more general form, which may have an arbitrary number of time delays and differential degree, in order to achieve the stabilization of complex systems.

The growing size and complexity of real-world industrial processes are raising the importance of modeling and control of large-scale systems. For many large-scale systems, it may be very costly, even impossible, to collect all the information in a centralized place, process it there, and dispatch the control commands from there [62]. Practical examples

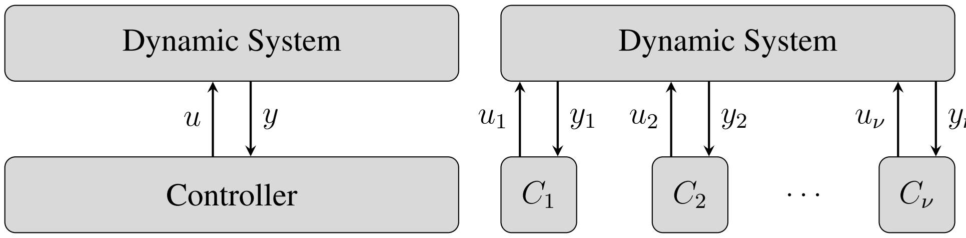

Figure 1.1: *Centralized (left) and decentralized (right) feedback control.*

of large-scale systems are electric power distribution systems, transportation and traffic systems, water transmission systems, inventory management systems, and the global economic system (see [63–65] and references therein). The idea of information flow structure usually describes the fundamental difference between feedback control of small-scale and large-scale systems. Thus, unlike the small-scale systems, the overall plant is not controlled by a single controller but by several independent local controllers each of which can operate only on certain input-output channels of the system as illustrated in Figure 1.1. This control strategy is known as decentralized control, and those independent local controllers all together represent the overall decentralized controller [66]. The decentralized controllers have several advantages over the centralized ones. First, they can reduce communication requirements. Thus, employing a decentralized control strategy is more economical in real-life control systems. Second, decentralized controllers are inherently robust to uncertainties in the interconnections [63]. Therefore, they operate in industrial processes more efficiently. Third, the overall control system can be more reliable against both interconnection and controller failures [67]. However, due to the information flow constraints, well-known feedback control methodologies, which have been established for the centralized case, can not be applied directly to the decentralized case. To this end, many studies have been conducted to analyze and design decentralized feedback control systems particularly.

Considering the existence of multiple input-output channels and the high complexity of large-scale systems, the existence of time delays is inevitable for such systems. Therefore, it is crucial to take time delays into account in the feedback control design of large-scale systems. There are various studies on this topic, e.g., [38,67–82]. Even though there exist many controller design methods for time-delay systems, only a few of them has been adapted to decentralized case in recent years [64]. There might be two main reasons for this circumstance. The obvious reason is that the high complexity of time-delay systems makes the decentralized control of large-scale systems even more complicated. As a second reason, although the importance of time delays in a feedback control system has been realized long ago, most enlightening studies have been carried out in the last three decades [5]. Furthermore, recent developments in computational tools facilitated the control of time-delay systems. Therefore, it is necessary to fill this gap by developing a decentralized controller design method for time-delay systems, which can be thought of as an extension of existing centralized controller design methods to the decentralized case in the presence of time delays.

In the literature, there are several decentralized controller design methods developed for finite-dimensional systems in the last five decades (see e.g., [66, 83–88]). Essentially, there are two main design approaches, which are known as *sequential design* and *simultaneous design*. In the simultaneous design, the overall decentralized controller is described as a block diagonal controller. Then, such a controller can be designed in a single step as in the design of a centralized controller [37]. On the other hand, in the sequential design, the local controllers are designed one-by-one [33]. The main advantage of using the sequential design approach is to be able to split the controller design problem into subproblems by considering a single input/output channel each time. So, the sequential design approach can be preferred because of the fact that a large-scale system generally has an excessive number of input/output channels which would dramatically increase the complexity of the controller design problem [49]. However, when a sequential design is considered, the question of the most advantageous design order arises. Emprical evidence have shown that the result directly depends on the design order of the local controllers when they are designed sequentially. Even though there exists recent studies on finding the most effective design order, it is a higly complicated problem. The only solution existing right now is the trial-and-error method which brings a computational burden as the number of local controllers to be designed increases. So, in the light of above disscusion, the computational cost of these two design stategies can not be a reason to chose one of them. Other than these two main approaches, an alternative method has been proposed in [83] that a decentralized controller is designed by making the closed-loop system stabilizable and detectable, i.e., free of unstable fixed-modes, from a single channel by applying decentralized static output feedback controllers to the remaining channels. One can conclude that this method is disadvantageous, especially in the case of time-delay systems, because, unlike the sequential or simultaneous design, the overall complexity of a decentralized controller is not distributed among the local controllers, which causes to end up with a single highly-complex local controller responsible for the stability of the overall closed-loop system.

Both centralized and decentralized stabilizability (i.e., stabilizability under centralized and decentralized feedback, respectively) can be determined by the existence of *fixed modes*. Basically, a mode of an LTI system is called fixed if its location on the complex plane does not change under any LTI static output feedback controller applied to the system [89]. As far as the centralized control is concerned, a fixed mode of a system is named as a *centralized fixed mode* (CFM), which corresponds to an uncontrollable and/or unobservable mode of the system. We note that CFMs are characterized as spectrally uncontrollable and/or spectrally unobservable modes of an LTI time-delay system [2]. It has been shown in [90] that an LTI retarded time-delay system can be stabilized by a proper LTI dynamic finite-dimensional controller if and only if the system does not have any unstable CFMs. A *decentralized fixed mode* (DFM), on the other hand, can be thought of as a fixed mode under decentralized control. It has been shown in [89] that an LTI finite-dimensional system can be stabilized by a proper LTI finite-dimensional dynamic decentralized controller if and only if it does not have any unstable DFMs. Similar to this result, it has been shown in [83] that a strongly connected LTI finite-dimensional system can be stabilized by a proper LTI finite-dimensional decentralized controller if and only if certain system matrices belonging to the complementary subsystems are all complete. Then, in [91], it has been shown that the completness condition of [83] is equal to the absence of unstable DFMs. Furthermore, it has been shown that the strong connectedness assumption of [83] can be removed when a proper LTI finite-dimensional decentralized dynamic controller is applied [85, 92]. Then, the result of [89] was extended to an LTI retarded time-delay system with commensurate time delays in [93]. The same result was extended to retarded time-delay systems with incommensurate time delays in [71] and then to neutral time-delay systems in [77]. In this concept, the most general centralized and decentralized stabilizability conditions of possibly neutral time-delay systems (without restricting the type of time delays as commensurate or incommensurate) have been established in [80]. It has been shown in [80] that such a system can be stabilized by a centralized (decentralized) LTI finite-dimensional and/or time-delay controller if and only if it does not have any unstable CFMs (DFMs). Note that the above result is valid only under the assumption that the system has finitely many unstable modes. As it is mentioned above, since retarded systems always have finitely many unstable modes, this assumption is necessary only for a neutral time-delay system. Considering the result of [94], which basically states that a proper LTI controller can not stabilize a time-delay system with infinitely many unstable modes, this assumption is necessary as long as a proper LTI controller is to be designed.

Even though the centralized strong stabilization problem has been well studied in the past four decades, the literature is not that rich for the decentralized counterpart of the strong stabilization problem. As far as the author's knowledge, the decentralized strong stabilization problem is studied explicitly only in [95–98]. In [98], the decentralized blocking zeros (i.e., blocking zeros that occur under decentralized feedback) are characterized for finite-dimensional systems. The decentralized blocking zeros are defined as the common blocking zeros of various complementary TFMs, and the TFMs appear in the main diagonal of the TFM of the system. Then, it has been shown that PIP (where decentralized blocking zeros are considered particularly instead of centralized blocking zeros) is a necessary condition for decentralized strong stabilization of finite-dimensional systems, and also a sufficient condition under some assumptions. A stable controller design method, which is based on the sequential design of stable local controllers, is also proposed in [98] for finite-dimensional systems. However, to the author's best knowledge, there is no result on either the decentralized strong stabilizability or the decentralized strong stabilization as far as the time-delay systems are concerned.

In recent years, a number of Matlab-based software, which use the eigenvalue optimization-based controller design method, have been developed. First, HIFOO has been developed to design finite-dimensional stabilizing controllers for finite-dimensional systems [27]. Even though the main objective was to design a stabilizing controller, the capabilities of HIFOO have been extended to $\mathcal{H}^\infty$ optimization [21], strong stabilization [99], and multiobjective optimization [100] over the years. It has been shown through benchmark examples that the design method leads to the design of low-dimensional (relative to the dimension of the system to be handled) controllers [21]. Then, TDS_STABIL [29] and TDS_HIOPT [101] have been developed to design finite-dimensional stabilizing and $\mathcal{H}^\infty$ controllers, respectively, for time-delay systems described by delay-differential-algebraic equations (DDAEs). For such systems, then, DCD_TDS has been developed to design finite-dimensional decentralized stabilizing controllers [35]. Note that being able to design low-dimensional controllers is more crucial for time-delay systems since these systems are infinite-dimensional; hence, it is quite possible to end up with high-dimensional controllers using conventional design methods. We note that, for the optimization phase, all these software use a Matlab-based solver named HANSO, which is specialized for nonsmooth and nonconvex unconstrained optimization [102].

1.2. Motivation and Objectives

The first objective of this thesis is to develop an eigenvalue optimization-based stabilization method for LTI MIMO time-delay systems. The systems to be considered are allowed to be so-called complex in which no restriction has been made on system dimension, input-output number, and the number of time delays. In this respect, multi-agent large-scale systems are particularly considered. The controllers to be designed, on the other hand, are allowed to be LTI output feedback controllers with an arbitrary number of time delays and differential degree. Thus, it is possible to design complex controllers with a high degree of freedom which is in general needed as the complexity of the system increases. As it is mentioned in the previous section, time-delay controllers have several advantages over delay-free ones. As far as the design and implementation of a controller are concerned, it is possible to add an extra degree of freedom by increasing the number of time delays of the controller instead of increasing the dimension of the state vector, i.e., the number of integration units necessary to implement the controller. Despite the ability of complex controller design, we always aim to achieve the closed-loop stability by using the simplest (e.g., low-dimensional in the finite-dimensional controller case) yet the most effective (i.e., in terms of the achieved minimal spectral abscissa value of the corresponding closed-loop system) controller. To this end, the eigenvalue optimization-based controller design strategy is specially chosen, because it has been shown in many studies that this strategy leads to the simplest yet the most effective controller because of the fact that the solution of the eigenvalue optimization problem corresponds to the point where the maximum exponential decay rate is achieved. Furthermore, it is well-known that the existing analytical methods are generally restricted to relatively simple systems and/or may be quite conservative (e.g., methods using Lyapunov-Krasovskii functionals), in the sense that stabilizing controllers can not be found even if they exist. Furthermore, using analytical methods, designing complex controllers such as time-delay controllers is mathematically intense.

The second objective of the thesis is to design structured time-delay controllers. In this way, it is possible to design time-delay controllers which introduce different time delays to each I/O channel of a MIMO system. Furthermore, it is also possible to design well-known servocompensators such as PI/PD/PID, which are the far most popular controllers used in practical control systems. In this respect, one can also design PR/PIR controllers, which can be thought of as delayed versions of PD/PID controllers. As discussed in the previous section, these time-delay servocompensators may be more advantageous in practical applications. On top of those, the structured controller design leads to the design of decentralized controllers where the overall controller is composed of several local controllers, each of which is connected to a certain part of a large-scale system.

The third objective of the thesis is to extend the stabilization methods to strong stabilization. In other words, it is aimed to design stable controllers to achieve closed-loop stability. Because stable controllers are preferable in practical control systems for many reasons such as robustness, tracking performance, etc., as discussed in the previous section. However, as mentioned in the previous section, not all systems are strongly stabilizable. Because of that, in this thesis, certain analysis methods are also aimed to be developed to investigate the strong stabilizability of a given LTI MIMO time-delay system. Here, the decentralized strong stabilizability problem is especially considered since the existing literature lacks results in this manner even though the literature is quite rich for the centralized counterpart. On the other hand, even though there exist decentralized strong stabilization methods developed for finite-dimensional systems, those methods can not be applied to time-delay systems directly. Furthermore, as it is mentioned in the previous section, most of the proposed strong stabilization methods lead to unnecessarily complex controllers even for finite-dimensional systems. Therefore, the development of new centralized and decentralized strong stabilization methods, which can be used for both finite-dimensional and time-delay systems, is also aimed.

Last but not least, the fourth objective of the thesis is to develop a Matlab-based software that implements the proposed analysis and controller design methods. The main reason for this objective lies behind the fact that the proposed methods require algorithms that consist of many symbolic and numeric operations. Because of that, a Matlab-based software, which can also be integrated with the necessary tools such as mode/zero computation and optimization solver, is aimed to be developed. Furthermore, the developed software will be publicly available so that any researcher or control engineer can use the proposed analysis and design methods effectively. Because of this motivation, special attention is given to developing not only effective but also a user-friendly software.

1.3. Outline

A brief outline of the thesis is given in this section. A detailed outline of subsequent chapters is given at the beginning of each chapter. In Chapter 2, the open-loop system descriptions are introduced along with the necessary background on the spectral properties and stabilizability conditions of LTI time-delay systems. Furthermore, the centralized and decentralized structured time-delay controllers, which are considered in the following chapters, are also introduced in Chapter 2. In Chapter 3, the design method of centralized stabilizing and strongly stabilizing time-delay controllers are presented, along with the technical details of the optimization-based design phase. At the end of this chapter, the proposed methods are illustrated through various examples. In Chapter 4, first, the decentralized strong stabilizability conditions of LTI time-delay systems are investigated; then, the controller design methods presented in Chapter 3 are extended to the decentralized case. Then, as in Chapter 3, the proposed methods are illustrated through various examples at the end of Chapter 4. Finally, the concluding remarks of the thesis are given in Chapter 5 along with the discussion of possible research objectives which can be considered in the future.

2. SYSTEMS, CONTROLLERS, AND STABILIZABILITY

In Section 2.1, the spectral properties of an LTI (retarded or neutral) time-delay system with an arbitrary number of discrete time delays are outlined. Those properties are the necessary background for the rest of the thesis since the autonomous part of all the open-loop systems to be controlled, and time-delay controllers to be designed are in the form of (2.1) given below. In Section 2.2, two different descriptions of LTI MIMO interconnected time-delay systems, which are considered as the open-loop systems (i.e., plants) in this thesis, are introduced. In Section 2.3, both centralized and decentralized LTI output feedback controllers are introduced along with various structures, each of which can be used for different aspects. In Section 2.4, centralized and decentralized stabilizability conditions of the open-loop systems, introduced in Section 2.2, are given.

2.1. Spectral Properties of Time-delay Systems

Let us consider an LTI time-delay system described by DDAEs of the form:

$$E\dot{x}(t) = \sum_{i=0}^{\sigma} A_i x(t - h_i) \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector at time t. Here, $h_i > 0$, $i = 1, \ldots, \sigma$, where σ indicates the number of distinct time delays involved in (2.1), are the time delays. In particular, $h_0 := 0$ is used for notational convenience, thus $i = 0$ corresponds to the delay-free part of the system. The matrices E and A_i, $i = 0, \ldots, \sigma$, are constant real matrices. In the case $\operatorname{rank}(E) < n$, let the columns of U (respectively V) be a minimal basis for the left (respectively right) null space of E. Then, $U \in \mathbb{R}^{n\times\tilde{n}}$ and $V \in \mathbb{R}^{n\times\tilde{n}}$, where

$$\tilde{n} := n - \operatorname{rank}(E), \tag{2.2}$$

are such that

$$U^T E = 0 \quad \text{and} \quad EV = 0\ . \tag{2.3}$$

In order to ensure the solvability of (2.1), it is assumed that either $\operatorname{rank}(E) = n$ or $U^T A_0 V$ is nonsingular [29], [103].

Definition 1. For any given $\epsilon \in \mathbb{R}$, the set of ϵ-modes of (2.1) is defined as

$$\Omega_\epsilon = \left\{ s \in \mathbb{C}_\epsilon^+ \mid \det\left(\phi(s)\right) = 0 \right\} \tag{2.4}$$

where

$$\phi(s) := sE - A(s). \tag{2.5}$$

is the characteristic matrix of the system where

$$A(s) := \sum_{i=0}^{\sigma} A_i e^{-sh_i}. \tag{2.6}$$

Remark 1. By Definition 1, for any $s_0 \in \Omega_\epsilon$, it follows that

$$\mathrm{rank}(A(s_0) - s_0 E) < n. \tag{2.7}$$

Definition 2. For any given $\epsilon \in \mathbb{R}$, $s_0 \in \Omega_\epsilon$ is said to be *simple*, if

$$\frac{d}{ds}(\det(\phi(s)))|_{s=s_0} \neq 0$$

and is said to be *multiple*, otherwise.

Definition 3. For any given $\mu \in \mathbb{R}$, where μ is the *relative stability boundary*, (2.1) is said to be *μ-stable* if there exist a $\xi > 0$, such that $\Omega_{\mu-\xi} = \emptyset$.

Note that, for $\mu \leq 0$, μ-stability is equivalent to the exponential stability with a decay rate less than μ [3]. The μ-stability condition can also be expressed in terms of the *spectral abscissa* of the system, which is defined as

$$c := \sup \left\{ \mathrm{Re}(s) \mid \det\left(\phi(s)\right) = 0 \right\} \ . \tag{2.8}$$

Hence, (2.1) is *μ-stable* if and only if $c < \mu$.

When $\tilde{n} = 0$, where $\tilde{n}$ is the rank deficiency of E as defined in (2.2), (2.1) has only delay-differential equations; hence it describes a retarded system. For retarded systems, Ω_ϵ is a finite set for any $\epsilon \in \mathbb{R}$ [3]. In other words, even though the time-delay system has infinitely many modes, there are always finitely many of them on the right-hand side of any vertical line of the complex plane. Consequently, for any given $\mu \in \mathbb{R}$, a retarded system has always finitely many μ-modes.

On the other hand, in the case $1 \leq \tilde{n} < n$, i.e., when E is a rank deficient matrix, (2.1) may also have delay-difference equations. Let the columns of $U^{\perp} \in \mathbb{R}^{n\times(n-\tilde{n})}$ and $V^{\perp} \in \mathbb{R}^{n\times(n-\tilde{n})}$ be a basis for the orthogonal complement of U and V, respectively, so that

$$\bar{U} := [U^{\perp}\ U] \ \text{ and } \ \bar{V} := [V^{\perp}\ V] \tag{2.9}$$

are full-rank matrices. Also let x_1 and x_2 be defined such that

$$x = \bar{V}[x_1^T\ x_2^T]^T = V^{\perp}x_1 + Vx_2 \tag{2.10}$$

where $x_1 \in \mathbb{R}^{n-\tilde{n}}$ and $x_2 \in \mathbb{R}^{\tilde{n}}$. Now, (2.1) can be described as coupled delay-differential and delay-difference equations, via multiplying (2.1) by $\bar{U}^T$ from the left and substituting (2.10), which yields

$$\begin{aligned} U^{\perp T}EV^{\perp}\dot{x}_1(t) &= \sum_{i=0}^{\sigma}\Big(U^{\perp T}A_iV^{\perp}x_1(t-h_i) + U^{\perp T}A_iVx_2(t-h_i)\Big) \\ 0 &= \sum_{i=0}^{\sigma}\big(U^TA_iV^{\perp}x_1(t-h_i) + U^TA_iVx_2(t-h_i)\big)\,. \end{aligned} \tag{2.11}$$

Clearly (2.11), accordingly (2.1), describes a neutral system unless $U^TA_iV = 0$, $i = 1, \ldots, \sigma$.

Note that neutral systems are generally described in the following form:

$$\dot{\tilde{x}}(t) + \sum_{i=1}^{\sigma}\big(\tilde{E}_i\dot{\tilde{x}}(t-h_i)\big) = \sum_{i=0}^{\sigma}\big(\tilde{A}_i\tilde{x}(t-h_i)\big) \tag{2.12}$$

where the delayed version(s) of state derivatives are explicitly given. In this case, (2.12) can be brought into the form of (2.1) by defining the state vector as

$$x(t) := [\bar{x}(t)^T\ \tilde{x}(t)^T]^T\ , \tag{2.13}$$

where

$$\bar{x}(t) := \tilde{x}(t) + \sum_{i=1}^{\sigma} \left(\tilde{E}_i \tilde{x}(t-h_i)\right) . \tag{2.14}$$

Accordingly, the system matrices in (2.1) become

$$E = \begin{bmatrix} I_n & 0 \\ 0 & 0_n \end{bmatrix} , \; A_0 = \begin{bmatrix} 0 & \tilde{A}_0 \\ -I_n & I_n \end{bmatrix} , \tag{2.15}$$

and, for $i = 1, \ldots, \sigma$,

$$A_i = \begin{bmatrix} 0_n & \tilde{A}_i \\ 0 & \tilde{E}_i \end{bmatrix} . \tag{2.16}$$

When (2.1) describes a neutral system, the associated delay-difference equations can be described as

$$\sum_{i=0}^{\sigma} \hat{A}_i \hat{x}(t-h_i) = 0 , \tag{2.17}$$

where $\hat{A}_i := U^T A_i V$, $i = 0, \ldots, \sigma$, and U and V are as in (2.3). Here, $\hat{x}(\cdot) \in \mathbb{R}^{\tilde{n}}$ is a dummy state vector. The spectral absicssa of (2.17) can be defined as

$$c_D := \sup \left\{ \mathrm{Re}(s) \mid \det\left(\phi_D(s)\right) = 0 \right\} , \tag{2.18}$$

where

$$\phi_D(s) := \sum_{i=0}^{\sigma} \hat{A}_i e^{-sh_i} \tag{2.19}$$

is the characteristic matrix of (2.17). Then, (2.17) is μ-stable if and only if $c_D < \mu$ [29].

Unlike the retarded systems, neutral systems may have infinitely many modes on the right-hand side of some vertical line on the complex plane. However, Ω_ϵ is a finite set for all $\epsilon > c_D$ [3]. Consequently, for any given $\mu > c_D$, a neutral system has always finitely many μ-modes. For this reason, $c_D < \mu$ is a necessary condition for the μ-stability of (2.1) [4].

However, even though (2.18) is continuous in the elements of the system matrices, it is not continuous in the time delays. As a consequence of this, the high-frequency modes of (2.17), accordingly c_D, may be highly sensitive to infinitesimal perturbations in the time delays [4]. In other words, Ω_ϵ may become an infinite set when the nominal values of time delays change [104]. For this reason, the concept of *strong stability* has been introduced in [4]. In this thesis, however, since we reserve the term strong stability for the closed-loop stability achieved by a stable controller, we will call this property *insensitive stability* as in [49] and [51].

Definition 4. (2.1) is said to be *insensitively μ-stable* if it is μ-stable for nominal values of the time delays and remains μ-stable when infinitesimal perturbations occur in the time delays.

In this regard, the system (2.1) is insensitively μ-stable if and only if it is μ-stable, i.e., $c < \mu$, and the real part of the modes of (2.17) remains less than μ under all infinitesimal perturbations in the time delays [29]. A necessary condition for the insensitive μ-stability of (2.1) is the insensitive μ-stability of the associated DDE (2.17). Considering the hypersensitivity of c_D, we need to calculate the least upper bound of the real part of the modes of (2.17) under all infinitesimal perturbations in the time delays. We will call this bound as the *insensitive bound* (this was called *safe upper bound* in [29]) of (2.1).

Definition 5. The insensitive bound, indicated by C_D, is the least upper bound of the real part of the modes of (2.17) under all infinitesimal perturbations in the time delays.

The insensitive bound, C_D, is equal to the unique root of

$$g(\zeta) - 1 = 0, \tag{2.20}$$

where, for $\sigma = 1$,

$$g(\zeta) := \rho\left(\hat{A}_0^{-1}\hat{A}_1 e^{-\zeta h_1}\right), \tag{2.21}$$

and, for $\sigma \geq 2$,

$$g(\zeta) := \max_{\theta\, \in\, [0, 2\pi]^{\sigma-1}} \rho\left(\hat{A}_0^{-1}\left[\hat{A}_1 e^{-\zeta h_1} + \sum_{k=2}^{\sigma} \hat{A}_k e^{-\zeta h_k} e^{\mathrm{j}\theta_k}\right]\right), \tag{2.22}$$

where $\theta := \{\theta_2, \ldots, \theta_\sigma\}$ [29].

Note that the set of ϵ-modes is finite for all $\epsilon > C_D$ and remains finite even when infinitesimal perturbations in the time delays occur. Furthermore, if the set of ϵ-modes is a compact set that consists of finitely many elements, then the *supremum* in (2.8) reduces to a *maximum*. As a result, for all $\epsilon > C_D$, the system in the form of (2.1) is μ-stable if and only if the real part of the rightmost ϵ-mode is strictly less than $\mu \in \mathbb{R}$. Clearly, the above result is also valid for finite-dimensional systems, i.e., systems in the form of (2.1) with $\sigma = 0$, since it is well-known that the set of ϵ-modes is a finite compact set for all finite-dimensional systems.

Although C_D gives direct information about the location of the rightmost roots of (2.19), the computation of C_D is quite expensive. Taking this situation into consideration, we use the following measure, which is computationally less demanding than C_D, to determine the insensitive μ-stability. The associated DDE (2.17) is insensitively μ-stable if and only if $\gamma_\mu < 1$, where, for $\sigma = 1$,

$$\gamma_\mu := \rho\left(\hat{A}_0^{-1}\hat{A}_1 e^{-\mu h_1}\right) \tag{2.23}$$

and, for $\sigma \geq 2$,

$$\gamma_\mu := \max_{\theta \in [0, 2\pi]^{\sigma-1}} \rho\left(\hat{A}_0^{-1}\left[\hat{A}_1 e^{-\mu h_1} + \sum_{k=2}^{\sigma} \hat{A}_k e^{-\mu h_k} e^{\mathrm{j}\theta_k}\right]\right), \tag{2.24}$$

where $\theta := \{\theta_2, \ldots, \theta_\sigma\}$ [49]. We refer the reader to [26] for the case $\mu = 0$; the general case follows by the transformation $s \to s - \mu$. Notice that, for $\mu = 0$, (2.23) and (2.24) are independent of time delays. Therefore, the system (2.1) is insensitively 0-stable if and only if it is 0-stable [104].

The insensitive μ-stability condition for (2.1), then, can be expressed in terms of c and either γ_μ or C_D. As a result, (2.1) is insensitively μ-stable if and only if $c < \mu$ and $C_D < \mu$, while the latter condition can also be expressed as $\gamma_\mu < 1$.

2.2. System Descriptions

In this section, we consider the single-agent and multi-agent systems separately. In Subsection 2.2.1, the description of LTI MIMO single-agent time-delay systems is given by (2.25). It is shown that neutral time-delay systems with direct feed-through terms given by (2.28) can be compactly represented as (2.25). Then, in Subsection 2.2.2, the description of LTI MIMO multi-agent time-delay system is given by (2.31). It is shown that a multi-agent system described as (2.31) can be represented as (2.25) by collecting the agents.

2.2.1. Single-agent Systems

We consider an LTI MIMO time-delay system described by DDAEs of the form:

$$\begin{aligned} E\dot{x}(t) &= \sum_{i=0}^{\sigma} \Big(A_i x(t-h_i) + B_i u(t-h_i) \Big) \\ y(t) &= \sum_{i=0}^{\sigma} C_i x(t-h_i) \ , \end{aligned} \tag{2.25}$$

where $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^q$ are, respectively, the input and the output vectors at time t. The autonomous part of (2.25) is in the form of (2.1). In addition to the autonomous part of the system, the matrices $B_i \in \mathbb{R}^{n \times m}$ and $C_i \in \mathbb{R}^{q \times n}$, $i = 0, \ldots, \sigma$, are constant real matrices, where σ indicates the number of distinct time delays. The system may have an arbitrary number of time delays both in the autonomous part and the I/O channels of the system. We note that, although we represent the time delays compactly here, the state, the input, and the output time delays may indeed be different, in which case certain matrices in (2.25) would be zero. Since the autonomous part of (2.25) is in the form of (2.1), (2.25) can describe both retarded and neutral systems. Respectively, the spectral properties given in the previous section are also valid for (2.25).

If $\tilde{n} = 0$, where $\tilde{n}$ is the rank deficiency of E as defined in (2.2), (2.25) has only delay-differential equations; hence it describes a retarded system. Furthermore, (2.25) does not have any direct feed-through term in this case. However, if $1 \leq \tilde{n} < n$, i.e., when E is a rank deficient matrix, (2.25) describes a neutral system unless $U^T A_i V = 0$, $i = 1, \ldots, \sigma$. In this case, the state equations of (2.25) may become coupled delay-differential and delay-difference equations. The existence of delay-difference equations may cause (2.25) to have direct feed-through terms. In order to reveal this circumstance, we take (2.25) into the form of coupled delay-differential and delay-difference equations. Let $\bar{U}$ and $\bar{V}$ be as described in (2.9). Also let x_1 and x_2 be as in (2.10). We multiply the state equations of (2.25) by $\bar{U}^T$ from the left, then substitute (2.10) into both the state and output equations of (2.25), which yields

$$
\begin{aligned}
U^{\perp^T} E V^{\perp} \dot{x}_1(t) &= \sum_{i=0}^{\sigma} \left(U^{\perp^T} A_i V^{\perp} x_1(t-h_i) + U^{\perp^T} A_i V x_2(t-h_i) + U^{\perp^T} B_i u(t-h_i) \right) \\
0 &= \sum_{i=0}^{\sigma} \left(U^T A_i V^{\perp} x_1(t-h_i) + U^T A_i V x_2(t-h_i) + U^T B_i u(t-h_i) \right) \\
y(t) &= \sum_{i=0}^{\sigma} \left(C_i V^{\perp} x_1(t-h_i) + C_i V x_2(t-h_i) \right)
\end{aligned} \tag{2.26}
$$

Then, we solve the delay-difference parts of the state equations for $x_2(t)$, which yields

$$
x_2(t) = \mathcal{W}^{-1} \left(\sum_{i=0}^{\sigma} \left(U^T A_i V^{\perp} x_1(t-h_i) + U^T B_i u(t-h_i) \right) + \sum_{i=1}^{\sigma} U^T A_i V x_2(t-h_i) \right)
$$

where $\mathcal{W} := U^T A_0 V$ has already been assumed to be invertible to ensure the solvability of the system. Then, we substitute $x_2(t)$ into the output equations in (2.26), which yields

$$
\begin{aligned}
y(t) = \sum_{i=0}^{\sigma} \left(C_i V^{\perp} x_1(t-h_i) \right) &+ \sum_{i=0}^{\sigma} \sum_{j=0}^{\sigma} \left(C_i V \mathcal{W}^{-1} U^T A_j V^{\perp} x_1(t-h_i-h_j) \right. \\
&\left. + C_i V \mathcal{W}^{-1} U^T B_j u(t-h_i-h_j) \right) + \sum_{i=0}^{\sigma} \sum_{j=1}^{\sigma} \left(C_i V \mathcal{W}^{-1} U^T A_j V x_2(t-h_i-h_j) \right).
\end{aligned}
$$

As a result, (2.26), accordingly (2.25), does not have any direct feed-through term if and only if

$$
C_i V \mathcal{W}^{-1} U^T B_j = 0, \tag{2.27}
$$

for all $i, j = 0, \ldots, \sigma$, when $1 \leq \tilde{n} < n$.

In order to illustrate the generality of the system description (2.25), let us consider the following system description:

$$
\begin{aligned}
\tilde{E} \dot{\tilde{x}}(t) &= \sum_{i=0}^{\sigma} \left(\tilde{A}_i \tilde{x}(t-h_i) + \tilde{B}_i u(t-h_i) \right) \\
y(t) &= \sum_{i=0}^{\sigma} \left(\tilde{C}_i \tilde{x}(t-h_i) + \tilde{D}_i u(t-h_i) \right),
\end{aligned} \tag{2.28}
$$

where the direct feed-through terms are explicitly given. Now, (2.28) can be brought into the the form of (2.25) by defining the state vector as

$$x(t) := [\tilde{x}(t)^T \; w(t)^T]^T \; , \tag{2.29}$$

where

$$w(t) := \sum_{i=0}^{\sigma} \left(\tilde{D}_i u(t-h_i)\right) \; . \tag{2.30}$$

Accordingly, the system matrices in (2.1) become

$$E = \begin{bmatrix} \tilde{E} & 0 \\ 0 & 0_q \end{bmatrix} , \; A_0 = \begin{bmatrix} \tilde{A}_0 & 0 \\ 0 & -I_q \end{bmatrix} , \; B_0 = \begin{bmatrix} \tilde{B}_0 \\ \tilde{D}_0 \end{bmatrix} , C_0 = \begin{bmatrix} \tilde{C}_0 & I_q \end{bmatrix} ,$$

and, for $i = 1, \ldots, \sigma$,

$$A_i = \begin{bmatrix} \tilde{A}_i & 0 \\ 0 & 0_q \end{bmatrix} , \; B_i = \begin{bmatrix} \tilde{B}_i \\ \tilde{D}_i \end{bmatrix} , \; C_i = \begin{bmatrix} \tilde{C}_i & 0_q \end{bmatrix} .$$

2.2.2. Multi-agent Systems

In the concept of large-scale systems, where especially decentralized control is considered, open-loop systems are generally described in the I/O oriented form since it gives more information about the structure of the inputs and outputs of the system. Therefore, for such systems, we consider the description of

$$\begin{aligned} E\dot{x}(t) &= \sum_{i=0}^{\sigma} \Big(A_i x(t-h_i) + \sum_{k=1}^{\nu} B_{i,k} u_k(t-h_i) \Big) \\ y_k(t) &= \sum_{i=0}^{\sigma} C_{i,k} x(t-h_i) \; , \; k = 1, \ldots, \nu \; , \end{aligned} \tag{2.31}$$

where $u_k(t) \in \mathbb{R}^{m_k}$ and $y_k(t) \in \mathbb{R}^{q_k}$ are, respectively, the input and output vectors at time t. Here, ν is the number of local control stations each of which is associated with the corresponding I/O channel which consists of a certain number of inputs and outputs [62]. That is, k^{th} local control station has access only to y_k and determines only u_k. As in (2.25), the autonomous part of (2.31) is in the form of (2.1). Thus, (2.31) can describe both retarded and neutral systems. In addition to the autonomous part of the system, the matrices $B_{i,k} \in \mathbb{R}^{n \times m_k}$ and $C_{i,k} \in \mathbb{R}^{q_k \times n}$, $i = 0, \ldots, \sigma$, $k = 1, \ldots, \nu$, are constant real matrices, where σ indicates the number of distinct time delays. The system may have an arbitrary number of time delays both in the autonomous part and the I/O channels of each local control station.

Note that a large-scale system given in the form of (2.31) can be brought into the form of (2.25) by defining

$$u(t) := [u_1^T(t) \ \cdots \ u_\nu^T(t)]^T \in \mathbb{R}^m, \tag{2.32}$$

where $m := \sum_{k=1}^{\nu} m_k$, and,

$$y(t) := [y_1^T(t) \ \cdots \ y_\nu^T(t)]^T \in \mathbb{R}^q, \tag{2.33}$$

where $q := \sum_{k=1}^{\nu} q_k$. Thus, for $i = 0, \ldots, \sigma$, B_i and C_i can be composed as

$$B_i := [B_{i,1} \ \cdots \ B_{i,\nu}]$$

and

$$C_i := [C_{i,1}^T \ \cdots \ C_{i,\nu}^T]^T.$$

Likewise, a large-scale system described in the form of (2.25) can be brought into the form of (2.31) by decomposing B_i and C_i, $i = 0, \ldots, \sigma$, into submatrices (regarding the associated time delays) whose dimensions are compatiable with the corresponding input and output vectors u_k and y_k, $k = 1, \ldots, \nu$.

We should note that (2.31) does not have any direct feed-through term if either $\operatorname{rank}(E) = n$ or

$$C_{i,k} V \mathcal{W}^{-1} U^T B_{j,l} = 0, \tag{2.34}$$

for all $k, l = 1, \ldots, \nu$ and $i, j = 0, \ldots, \sigma$, where $\mathcal{W} = U^T A_0 V$, and, U and V are as defined in (2.3).

2.3. Controller Descriptions and Structures

We consider LTI output feedback controllers described in the form:

$$\begin{aligned} \dot{z}(t) &= \sum_{i=0}^{\hat{\sigma}} \left(F_i z(t-\hat{h}_i) + G_i y(t-\hat{h}_i) \right) \\ u(t) &= \sum_{i=0}^{\hat{\sigma}} \left(H_i z(t-\hat{h}_i) + K_i y(t-\hat{h}_i) \right) . \end{aligned} \tag{2.35}$$

Here, $z(t) \in \mathbb{R}^l$ is the state vector at time t where $l \in \mathbb{N}$ is the differential degree of the controller. For $i = 0, \dots, \hat{\sigma}$, $F_i \in \mathbb{R}^{l\times l}$, $G_i \in \mathbb{R}^{l\times q}$, $H_i \in \mathbb{R}^{m\times l}$, and $K_i \in \mathbb{R}^{m\times q}$ are real-valued matrices, and $\hat{h}_i > 0$, are the time delays of the controller (introduced intentionally), where $\hat{\sigma}$ indicates the number of distinct time delays. In particular, we let $\hat{h}_0 := 0$, thus, $i = 0$ corresponds to the delay-free part of the controller. The autonomous part of (2.35) is in the form of (2.1) with $E = I_l$, which follows that all the spectral properties introduced for (2.1) are also valid for (2.35).

Note that (2.35) describes a broad class of LTI output feedback controllers. For instance, when $\hat{\sigma} = 0$, (2.35) describes a finite-dimensional controller, otherwise, (2.35) describes a time-delay controller which may have an arbitrary number of distinct time delays both in the I/O channels and dynamics of the controller itself.

Note that (2.35) may describe a centralized or decentralized controller depending on the structures of the controller matrices. These two types of controllers are separately considered in the next two subsections.

2.3.1. Centralized Controllers

In a feedback control system, if there exists a single agent (i.e., control station) which can access all the I/O channels, then such a control structure is said to be *centralized control*. In centralized control, the complete set of output signals is collected in a centralized place, and the complete set of input signals is dispatched from there. In other words, the information flow of a centralized controller is not constrained. In this subsection, some of the centralized LTI output feedback controller classes are introduced. Each controller class is determined by the existence of a differential part and time delay in (2.35). Even though a centralized controller is not constrained by means of information flow, the controller matrices can be intentionally structured to obtain different types of centralized controllers. Some of the essential types of centralized controllers which are obtained by using proper structuring will be introduced under the associated controller classes.

Centralized controller classes:

- $\mathbf{K}^{\mathbf{c}}_{\mathbf{fs}}$: = the class of all centralized delay-free static controllers in the form of (2.35) with $l = 0$ and $\hat{\sigma} = 0$. Thus, a controller in this class can be described as

$$u(t) = K_0 y(t), \tag{2.36}$$

  which has mq number of parameters to be determined (i.e., *free parameters*) and requires same number of gains to be implemented.

- $\mathbf{K}^{\mathbf{c}}_{\mathbf{fd}}$: = the class of all centralized delay-free dynamic controllers in the form of (2.35) with $l > 0$ and $\hat{\sigma} = 0$. Thus, a controller in this class can be described as

$$\begin{aligned} \dot{z}(t) &= F_0 z(t) + G_0 y(t) \\ u(t) &= H_0 z(t) + K_0 y(t) \end{aligned} \tag{2.37}$$

  which has $l^2 + lq + lm + mq$ number of free parameters when none of the controller matrices are structured. However, the input-output relation of such a controller is unique only up to a similarity transformation. In this regard, it is possible to reduce the number of free parameters of the controller by structuring the controller matrices in a suitable canonical form without changing the effective complexity, i.e., the degree of freedom, of the controller [49]. In these forms, the controller matrices can be structured depending on l, m, and q. If $q \leq m$, then the *multivariable controllable canonical form* [105] can be used, in which case F_0 and G_0 matrices have a certain number of fixed parameters while H_0 and K_0 matrices have free parameters only. If the dimension of the controller, $l \geq q$, then F_0 and G_0 matrices take the following forms:

$$F_0 = \begin{bmatrix} 0_{(l-q)\times q} & & I_{l-q} \\ f^0_{1,1} & \cdots & f^0_{1,l} \\ \vdots & & \vdots \\ f^0_{q,1} & \cdots & f^0_{q,l} \end{bmatrix}, G_0 = \begin{bmatrix} 0_{(l-q)\times q} \\ I_q \end{bmatrix},$$

otherwise

$$F_0 = \begin{bmatrix} f^0_{1,1} & \cdots & f^0_{1,l} \\ \vdots & & \vdots \\ f^0_{l,1} & \cdots & f^0_{l,l} \end{bmatrix}, G_0 = \begin{bmatrix} & g^0_{1,1} & \cdots & g^0_{1,q-l} \\ I_l & \vdots & & \vdots \\ & g^0_{l,1} & \cdots & g^0_{l,q-l} \end{bmatrix}.$$

If $q > m$, then the *multivariable observable canonical form* [105] can be used, in which case F_0 and H_0 matrices have a certain number of fixed parameters while G_0 and K_0 matrices have free parameters only. If $l \geq m$, then the controller matrices take the following forms:

$$F_0 = \begin{bmatrix} 0_{m\times(l-m)} & f^0_{1,1} & \cdots & f^0_{1,m} \\ & \vdots & & \vdots \\ I_{l-m} & f^0_{l,1} & \cdots & f^0_{l,m} \end{bmatrix}, H_0 = \begin{bmatrix} 0_{m\times(l-m)} & I_m \end{bmatrix},$$

otherwise

$$F_0 = \begin{bmatrix} f^0_{1,1} & \cdots & f^0_{1,l} \\ \vdots & & \vdots \\ f^0_{l,1} & \cdots & f^0_{l,l} \end{bmatrix}, H_0 = \begin{bmatrix} & I_l & \\ h^0_{1,1} & \cdots & h^0_{1,l} \\ \vdots & & \vdots \\ h^0_{m-l,1} & \cdots & h^0_{m-l,l} \end{bmatrix}.$$

Note that, in any of these forms, the number of free parameters is equal to $lq + lm + qm$, which means that the number of free parameters is simply reduced by l^2 compared to the case when the controller matrices are not structured in a canonical form.

Besides the canonical forms, the controller matrices of (2.37) can also be structured to obtain the well-known servocompansators. For instance, consider a SISO controller described in the Laplace domain as

$$u(s) = k_p y(s) + \frac{k_i}{s} y(s) + \frac{k_d s}{T_f s + 1} y(s) \tag{2.38}$$

where $u(s)$ and $y(s)$ are the Laplace transform of the input and output, respectively, and, k_p, k_i, and k_d are the proportional, integral, and derivative gains of the controller, respectively. Such a controller is known as *proportional-integral-derivative* (PID)[1]. Furthermore, it is also known as *proportional-integral* (PI) when $k_d = 0$, and *proportional-derivative* (PD) when $k_i = 0$. Considering (2.37), a PID controller can be designed by structuring the controller matrices as

$$\begin{gathered} F_0 = \begin{bmatrix} 0 & 1 \\ 0 & -1/T_f \end{bmatrix}, \ G_0 = \begin{bmatrix} 0 \\ 1/T_f \end{bmatrix}, \\ H_0 = \begin{bmatrix} k_i & T_f k_i - k_d/T_f \end{bmatrix}, \ K_0 = k_p + k_d/T_f, \end{gathered} \tag{2.39}$$

where k_p, k_i, and k_d are the free parameters of the controller. Therefore, a PID controller has three free parameters and requires three gains along with two integrators to be implemented. In addition, T_f, i.e., the time-constant of the filter, may be fixed or can also be left as a free parameter (see [107] for the design of a derivative filter). However, the time-constant must be chosen as a non-negative number if it is determined to be fixed; otherwise, it must be guaranteed that $T_f > 0$ during the controller design phase (see Subsection 4.5.2 for an example).

- $\mathbf{K}_{\mathbf{ts}}^{\mathbf{c}}$: = the class of all centralized time-delay controllers in the form of (2.35) with $\hat{\sigma} > 0$ and $l = 0$. Thus, a controller in this class can be described as

$$u(t) = \sum_{i=0}^{\hat{\sigma}} K_i y(t - \hat{h}_i) \tag{2.40}$$

which has $mq(\hat{\sigma}+1)+\hat{\sigma}$ number of free parameters and requires $mq(\hat{\sigma}+1)$ number of gains along with $\hat{\sigma}$ number of delay elements to be implemented. Obviously, (2.40) is only a delayed version of (2.36). The controller in the form of (2.40) introduces $\hat{\sigma}$ number of artificial time delays to the I/O channels of a system. In

[1] A controller with a pure derivative term (with $k_d \neq 0$, $T_f = 0$), i.e., without a low-pass filter, is intentionally excluded for many practical reasons [106].

the case of multi-input and/or multi-output, the default option is to introduce the same time delay to each I/O channel. However, it is also possible to introduce different time delays to each I/O channel by choosing appropriate $\hat{\sigma}$ (depending on the number of input and/or output) and structuring K_i's properly. For instance, if, besides a delay-free feedback, a different single time delay is to be introduced from each output to each input of a two-input two-output system, (2.40) takes the form

$$\begin{aligned} u(t) = K_0 y(t) &+ \begin{bmatrix} k^1_{1,1} & 0 \\ 0 & 0 \end{bmatrix} y(t-\hat{h}_1) + \begin{bmatrix} 0 & k^2_{1,2} \\ 0 & 0 \end{bmatrix} y(t-\hat{h}_2) \\ &+ \begin{bmatrix} 0 & 0 \\ k^3_{2,1} & 0 \end{bmatrix} y(t-\hat{h}_3) + \begin{bmatrix} 0 & 0 \\ 0 & k^4_{2,2} \end{bmatrix} y(t-\hat{h}_4), \end{aligned} \tag{2.41}$$

where all the elements of K_0, each $k^i_{j,k}$, and each $\hat{h}_i$ is a free parameter. Here, we should also note that a controller in the form of (2.40) with $\hat{\sigma} = 1$ is known as *proportional-retarded* (PR) controller which can be thought as an approximation of a PD controller with a pure derivative term, i.e., a controller whose description in the Laplace domain is (2.38) when $k_i = T_f = 0$ [108].

- $\mathbf{K^c_{td}}$: = the class of all centralized time-delay controllers in the form of (2.35) with $\hat{\sigma} > 0$ and $l > 0$. Thus, a controller in this class can exactly be described as (2.35). As a particular case, the dynamics of a controller may be delay-free, yet, the controller can introduce an arbitrary number of artificial time delays to the I/O channels of a system. Such a controller can be described as

$$\begin{aligned} \dot{z}(t) &= F_0 z(t) + \sum_{i=0}^{\hat{\sigma}} \left(G_i y(t-\hat{h}_i) \right) \\ u(t) &= \sum_{i=0}^{\hat{\sigma}} \left(H_i z(t-\hat{h}_i) + K_i y(t-\hat{h}_i) \right) , \end{aligned} \tag{2.42}$$

which is in the form of (2.35) with, for $i = 1, \dots, \hat{\sigma}$, $F_i = 0$. Furthermore, the controller matrices of (2.35) can also be structured to obtain a servocompensator known as *proportional-integral-retarded* (PIR) which can be described in the Laplace domain as

$$u(s) = k_p y(s) + \frac{k_i}{s} y(s) + k_r e^{-h_r s} y(s) \tag{2.43}$$

where k_r and h_r are the retarded gain and time delay of the controller, respectively. Considering (2.35), a PIR controller can be designed by letting $l = 1$ and $\hat{\sigma} = 1$ and structuring the controller matrices as

$$\begin{aligned} &F_0 = F_1 = 0,\ G_0 = 1,\ G_1 = 0, \\ &H_0 = k_i,\ H_1 = 0,\ K_0 = k_p,\ K_1 = k_r \end{aligned} \tag{2.44}$$

where k_p, k_i, and k_r are free parameters. The time delay $\hat{h}_1 = h_r$ is also a free parameter. Therefore, a PIR controller has four free parameters, and, requires a single integrator, a single delay element, and three gains to be implemented.

2.3.2. Decentralized Controllers

In a feedback control system, if there exist $\nu > 1$ agents, each of which can access a specific I/O channel, then such a control structure is said to be *decentralized control*. In decentralized control, there exist ν *local control stations*. Unlike centralized control, only a subset of output signals is collected in a local control station, and a subset of input signals is dispatched from there. The controller applied from a local control station is said to be a *local controller*. So, the information flow of a decentralized controller, which consists of ν local controllers, is constrained. Therefore, the decentralized controller is naturally in a structured form. The local controllers are described as

$$\begin{aligned} \dot{z}_k(t) &= \sum_{i=0}^{\hat{\sigma}_k} \Big(F_{i,k} z_k(t - \hat{h}_{i,k}) + G_{i,k} y_k(t - \hat{h}_{i,k}) \Big) \\ u_k(t) &= \sum_{i=0}^{\hat{\sigma}_k} \Big(H_{i,k} z_k(t - \hat{h}_{i,k}) + K_{i,k} y_k(t - \hat{h}_{i,k}) \Big)\ , \end{aligned} \tag{2.45}$$

where $z_k(t) \in \mathbb{R}^{l_k}$, $k = 1, \ldots, \nu$, is the state vector of the k^{th} local controller at time t. Thus, the k^{th} local controller given above can access only $\{u_k, y_k\}$ pairs of the system given in (2.31). Here, $\hat{h}_{i,k} > 0$, $i = 1, \ldots, \hat{\sigma}_k$, $k = 1, \ldots, \nu$, are the time delays where $\hat{\sigma}_k$ indicates the number of distinct time delays of the k^{th} local controller. In particular, we let $\hat{h}_{0,k} := 0$, $k = 1, \ldots, \nu$, for notational convenience, hence $i = 0$ corresponds to the delay-free part of the k^{th} local controller. Furthermore, $F_{i,k}$, $G_{i,k}$, $H_{i,k}$, and $K_{i,k}$, $i = 0, \ldots, \hat{\sigma}_k$, $k = 1, \ldots, \nu$ are real-valued matrices of appropriate dimensions.

If we let

$$z(t) := \begin{bmatrix} z_1^T(t) & \cdots & z_\nu^T(t) \end{bmatrix}^T \in \mathbb{R}^l, \tag{2.46}$$

where $l := \sum_{k=1}^{\nu} l_k$,

$$u(t) := \begin{bmatrix} u_1^T(t) & \cdots & u_\nu^T(t) \end{bmatrix}^T \in \mathbb{R}^m, \quad y(t) := \begin{bmatrix} y_1^T(t) & \cdots & y_\nu^T(t) \end{bmatrix}^T \in \mathbb{R}^q, \tag{2.47}$$

and,

$$\{\hat{h}_1, \ldots, \hat{h}_{\hat{\sigma}}\} = \{\hat{h}_{1,1}, \ldots, \hat{h}_{\hat{\sigma}_1,1}\} \cup \cdots \cup \{\hat{h}_{1,\nu} \ldots, \hat{h}_{\hat{\sigma}_\nu,\nu}\}, \tag{2.48}$$

then the overall decentralized controller[2] can be compactly represented in the form of (2.35). Due to the structural constraint of the overall decentralized controller, the controller matrices in (2.35) must be structured in the form

$$\begin{aligned} F_i &:= \text{bdiag}(\hat{F}_{i,1}, ..., \hat{F}_{i,\nu}), \;\; G_i := \text{bdiag}(\hat{G}_{i,1}, ..., \hat{G}_{i,\nu}), \\ H_i &:= \text{bdiag}(\hat{H}_{i,1}, ..., \hat{H}_{i,\nu}), \;\; K_i := \text{bdiag}(\hat{K}_{i,1}, ..., \hat{K}_{i,\nu}), \end{aligned} \tag{2.49}$$

where

$$\hat{M}_{i,k} := \begin{cases} M_{j,k}, & \text{if } \hat{h}_i = \hat{h}_{j,k} \\ 0, & \text{otherwise} \end{cases},$$

where M stands for F, G, H, or K, $i = 0, \ldots, \hat{\sigma}$, $k = 1, \ldots, \nu$. As a result, the overall decentralized controller is actually a block diagonal controller where the main diagonal of each controller matrix given by (2.49) consists of the associated controller matrices of the corresponding local controller.

In the remainder of this subsection, some of the essential decentralized LTI output feedback controller classes are introduced. Each decentralized controller class can be thought as a counterpart of the corresponding centralized controller class introduced in the previous subsection. Therefore, as in the centralized case, each decentralized controller class is determined by the existence of a differential part and time delay in (2.45).

Decentralized controller classes:

- $\mathbf{K}_{\mathbf{fs}}^{\mathbf{d}}$: = the class of all decentralized delay-free static controllers where, for $k =$

[2]Here, we especially use the term "overall" because of the fact that a controller which consists of, say, s number of local controllers, where $1 < s < \nu$, is also a decentralized controller. However, the overall decentralized controller is a controller that consists of ν number of local controllers, i.e., the complete set of all local controllers.

$1, \ldots, \nu$, each local controller is in the form of (2.45) with $l_k = 0$ and $\hat{\sigma}_k = 0$. Thus, a controller in this class can be described as

$$u(t) = \mathrm{bdiag}(K_{0,1}, \ldots, K_{0,\nu})y(t). \tag{2.50}$$

where $u(t)$ and $y(t)$ are as in (2.47).

- $\mathbf{K^d_{fd}}$: = the class of all decentralized delay-free dynamic controllers where, for $k = 1, \ldots, \nu$, the local controllers is in the form of (2.45), at least one with $l_k > 0$, but all of them with $\hat{\sigma}_k = 0$. Thus, a controller in this class can be described as

$$\begin{aligned} \dot{z}(t) &= \mathrm{bdiag}(F_{0,1}, \ldots, F_{0,\nu})z(t) + \mathrm{bdiag}(G_{0,1}, \ldots, G_{0,\nu})y(t) \\ u(t) &= \mathrm{bdiag}(H_{0,1}, \ldots, H_{0,\nu})z(t) + \mathrm{bdiag}(K_{0,1}, \ldots, K_{0,\nu})y(t). \end{aligned} \tag{2.51}$$

  where $z(t)$, $u(t)$, and $y(t)$ are as in (2.46)-(2.47).

- $\mathbf{K^d_{ts}}$: = the class of all decentralized time-delay controllers where, for $k = 1, \ldots, \nu$, the local controllers is in the form of (2.45), at least one with $\hat{\sigma}_k > 0$, but all of them with $l_k = 0$. Thus, a controller in this class can be described as

$$u(t) = \sum_{i=0}^{\hat{\sigma}} K_i y(t - \hat{h}_i) \tag{2.52}$$

  where $u(t)$, $y(t)$, $\hat{h}_i$, $i = 1, ..., \hat{\sigma}$, and $\hat{\sigma}$ are as in (2.47)-(2.48). Furthermore, K_i, $i = 0, \ldots, \hat{\sigma}$, are as in (2.49).

- $\mathbf{K^d_{td}}$: = the class of all decentralized time-delay controllers where, for $k = 1, \ldots, \nu$, the local controllers is in the form of (2.45), at least one with $l_k > 0$ and at least one with $\hat{\sigma}_k > 0$. Thus, a controller in this class can be described as

$$\begin{aligned} \dot{z}(t) &= \sum_{i=0}^{\hat{\sigma}} \Big(F_i z(t - \hat{h}_i) + G_i y(t - \hat{h}_i) \Big) \\ u(t) &= \sum_{i=0}^{\hat{\sigma}} \Big(H_i z(t - \hat{h}_i) + K_i y(t - \hat{h}_i) \Big) \ , \end{aligned} \tag{2.53}$$

  where $z(t)$, $u(t)$, $y(t)$, $\hat{h}_i$, $i = 1, ..., \hat{\sigma}$, and $\hat{\sigma}$ are as in (2.46)-(2.48). Furthermore,

F_i, G_i, H_i, and K_i, $i = 0, \ldots, \hat{\sigma}$, are as in (2.49).

Note that, since each local controller described by (2.45) is in the form of (2.35), the local controllers can be structured individually as introduced in the previous subsection. For instance, it is possible to reduce the free parameters of a decentralized controller in the class of $\mathbf{K_{fd}^{d}}$ by using a proper multivariable canonical form. Likewise, the decentralized versions of the centralized sercocompensators such as PID and PIR (and their sub-versions without an integrator such as PD/PR) can be obtained when the classes $\mathbf{K_{fd}^{d}}$ and $\mathbf{K_{td}^{d}}$, respectively, are concerned.

2.4. Stabilizability

In this section, we consider the stabilizability of LTI MIMO time-delay systems. To be consistent, we consider a system described as (2.25) when centralized stabilizability (i.e., stabilizability by means of a centralized controller) is concerned (see Subsection 2.4.1), and, a system described as (2.31) when decentralized stabilizability (i.e., stabilizability by means of a decentralized controller) is concerned (see Subsection 2.4.2).

It has been shown in [94] that it is not possible to assign infinitely many unstable modes of an LTI time-delay system to the stable region in the complex plane by using an LTI proper controller. According to this result, to be able to design a μ-stabilizing controller in any class of centralized/decentralized controllers introduced in Subsection 2.3.1-2.3.2, it must be ensured that the open-loop system has finitely many μ-modes, for $\mu < 0$. Therefore, $C_D < \mu$ is a necessary condition for stabilizability of (2.25) and (2.31) since only then it is guaranteed that the open-loop system has finitely many μ-modes (see Section 2.1). To this end, we make the following assumption before going into the centralized and decentralized stabilizability conditions of (2.25) and (2.31), respectively.

Assumption 1. (2.25) *and* (2.31) *satisfy* $C_D < \mu$, *where* $\mu \in \mathbb{R}$ *is the relative stability boundary.*

In the following two subsections, the centralized stabilizability of (2.25) and the decentralized stabilizability of (2.31) are considered, respectively. Before going into them, we make the following definitions:

Definition 6. If there exists a centralized controller, say $\mathcal{K}$, which makes (2.25) μ-stable, i.e., the closed-loop system of $\mathcal{K}$ and (2.25) is μ-stable, then (2.25) is said to be *centralized μ-stabilizable*, and, the controller $\mathcal{K}$ is said to be a *centralized μ-stabilizing* controller.

Definition 7. If there exists a decentralized controller, say $\mathcal{K}$, which makes (2.31) μ-stable, i.e., the closed-loop system of $\mathcal{K}$ and (2.31) is μ-stable, then (2.31) is said to be *decentralized μ-stabilizable*, and, the controller $\mathcal{K}$ is said to be a *decentralized μ-stabilizing* controller.

2.4.1. Centralized Stabilizability

In this section, we consider the stabilizability of an open-loop system descibed as (2.25) by using a centralized controller in any class introduced in Subsection 2.3.1. Let us introduce the following definition:

Definition 8. For a given $\mu \in \mathbb{R}$, the set of *μ-centralized fixed modes* (μ-CFMs) of (2.25) with respect to the class of controllers $\mathbf{K}$, where $\mathbf{K}$ can be any class introduced in Subsection 2.3.1, is defined as

$$\Lambda_\mu^c = \{s \in \mathbb{C}_\mu^+ \mid \det(\bar{\phi}(s;\mathcal{K})) = 0,\ \forall \mathcal{K} \in \mathbf{K}\} \tag{2.54}$$

where $\bar{\phi}(s;\mathcal{K})$ is the characteristic matrix of the closed-loop system obtained by applying controller $\mathcal{K}$ to (2.25).

Notice that the centralized controller classes introduced in Subsection 2.3.1 have the following relations

$$\mathbf{K}_{\mathbf{fs}}^{\mathbf{c}} \subset \mathbf{K}_{\mathbf{fd}}^{\mathbf{c}} \subset \mathbf{K}_{\mathbf{td}}^{\mathbf{c}}, \tag{2.55}$$

and

$$\mathbf{K}_{\mathbf{fs}}^{\mathbf{c}} \subset \mathbf{K}_{\mathbf{ts}}^{\mathbf{c}} \subset \mathbf{K}_{\mathbf{td}}^{\mathbf{c}}. \tag{2.56}$$

Remark 2. Λ_μ^c is exactly same for all the centralized controller classes given above (see [90] for retarded time-delay systems under controllers in the class of $\mathbf{K}_{\mathbf{fd}}^{\mathbf{c}}$, and [80] for time-delay systems described as (2.31) under controllers in the class of $\mathbf{K}_{\mathbf{td}}^{\mathbf{c}}$).

Furthermore, under Assumption 1, $\Lambda_\mu^c = \emptyset$ is a necessary condition for the existence of a μ-stabilizing centralized controller in the classes of $\mathbf{K}_{\mathbf{fs}}^{\mathbf{c}}$ or $\mathbf{K}_{\mathbf{ts}}^{\mathbf{c}}$. On the other hand, again under Assumption 1, $\Lambda_\mu^c = \emptyset$ is both a necessary and sufficient condition for the existence of a μ-stabilizing centralized controller in the classes of $\mathbf{K}^{\mathbf{c}}_{\mathbf{fd}}$ and $\mathbf{K}^{\mathbf{c}}_{\mathbf{td}}$ [80]. As a result, in order to determine the centralized μ-stabilizability of an LTI time-delay system described as (2.25), each μ-mode of the system must be checked whether it is a μ-CFM or not. The rank conditions given in the following lemma can be used for this purpose [109].

Lemma 1. $s_o \in \mathbb{C}^+_\mu$ *is a* μ*-CFM of* (2.25) *if and only if*

$$\operatorname{rank}\begin{bmatrix} s_0E - A(s_0) & B(s_0) \end{bmatrix} < n \tag{2.57}$$

or

$$\operatorname{rank}\begin{bmatrix} s_0E - A(s_0) \\ C(s_0) \end{bmatrix} < n \tag{2.58}$$

where

$$A(s) := \sum_{i=0}^{\sigma} A_i e^{-sh_i}, \ B(s) := \sum_{i=0}^{\sigma} B_i e^{-sh_i}, \ \text{and}\ C(s) := \sum_{i=0}^{\sigma} C_i e^{-sh_i}. \tag{2.59}$$

According to Lemma 1, (2.25) is both *spectrally stabilizable* and *spectrally detectable* if and only if

$$\operatorname{rank}\begin{bmatrix} s_0E - A(s_0) & B(s_0) \end{bmatrix} = n,$$

and

$$\operatorname{rank}\begin{bmatrix} s_0E - A(s_0) \\ C(s_0) \end{bmatrix} = n$$

for all $s_o \in \mathbb{C}^+_\mu$ [2]. These two rank conditions can be thought of as a generalization of Popov-Belevitch-Hautus (PBH) stabilizability and detectability tests, respectively, to time-delay systems [110]. Note that, since

$$\operatorname{rank}\begin{bmatrix} s_0E - A(s_0) & B(s_0) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} s_0E - A(s_0) \\ C(s_0) \end{bmatrix} = n, \tag{2.60}$$

unless $\operatorname{rank}(s_oE - A(s_o)) < n$, any μ-CFM of (2.25) is a μ-mode of (2.25), thus

$$\Lambda^c_\mu \subset \Omega_\mu\ . \tag{2.61}$$

From this follows, Λ^c_μ is also a finite set under Assumption 1. Thus, once the μ-modes of

(2.25) are determined (e.g., by the method of [111]), the μ-CFMs can be determined by checking the rank conditions (2.57) and (2.58).

2.4.2. Decentralized Stabilizability

In this section, we consider the stabilizability of an open-loop system descibed as (2.31) by using a decentralized controller in any class introduced in Subsection 2.3.2. Let us introduce the following definition:

Definition 9. For a given $\mu \in \mathbb{R}$, the set of *μ-decentralized fixed modes* (μ-DFMs) of (2.31) with respect to the class of controllers $\mathbf{K}$, where $\mathbf{K}$ can be any class introduced in Subsection 2.3.2, is defined as

$$\Lambda_{\mu}^{d} = \{s \in \mathbb{C}_{\mu}^{+} \mid \det(\bar{\phi}(s;\mathcal{K})) = 0, \ \forall \mathcal{K} \in \mathbf{K}\} \tag{2.62}$$

where $\bar{\phi}(s;\mathcal{K})$ is the characteristic matrix of the closed-loop system obtained by applying controller $\mathcal{K}$ to (2.31).

Notice that the decentralized controller classes introduced in Subsection 2.3.2 have the following relations

$$\mathbf{K}_{\mathbf{fs}}^{\mathbf{d}} \subset \mathbf{K}_{\mathbf{fd}}^{\mathbf{d}} \subset \mathbf{K}_{\mathbf{td}}^{\mathbf{d}}, \tag{2.63}$$

and

$$\mathbf{K}_{\mathbf{fs}}^{\mathbf{d}} \subset \mathbf{K}_{\mathbf{ts}}^{\mathbf{d}} \subset \mathbf{K}_{\mathbf{td}}^{\mathbf{d}}. \tag{2.64}$$

Remark 3. Λ_{μ}^{d} is exactly same for all the decentralized controller classes given above (see [89] for finite-dimensional systems under controllers in the class of $\mathbf{K}_{\mathbf{fd}}^{\mathbf{d}}$, [71] for retarded time-delay systems under controllers in the class of $\mathbf{K}_{\mathbf{fd}}^{\mathbf{d}}$, and [80] for time-delay systems described as (2.31) under controllers in the class of $\mathbf{K}_{\mathbf{td}}^{\mathbf{d}}$).

Furthermore, under Assumption 1, $\Lambda_{\mu}^{d} = \emptyset$ is a necessary condition for the existence of a μ-stabilizing decentralized controller in the classes of $\mathbf{K}_{\mathbf{fs}}^{\mathbf{d}}$ or $\mathbf{K}_{\mathbf{ts}}^{\mathbf{d}}$. On the other hand, again under Assumption 1, $\Lambda_{\mu}^{d} = \emptyset$ is both a necessary and sufficient condition for the existence of a μ-stabilizing decentralized controller in the classes of $\mathbf{K}_{\mathbf{fd}}^{\mathbf{d}}$ and $\mathbf{K}_{\mathbf{td}}^{\mathbf{d}}$ [80]. As a result, in order to determine the decentralized μ-stabilizability of an LTI time-delay system described as (2.31), each μ-mode of the system must be checked wheter it is a

μ-DFM or not. The rank condition given in the following lemma can be used for this purpose [109].

Lemma 2. *$s_o \in \mathbb{C}_\mu^+$ is a μ-DFM of* (2.31) *if and only if there exist $k \in \{0, \dots, \nu\}$ and $\{j_1, \dots, j_k\} \subset \{1, \dots, \nu\}$, where $\{j_1, \dots, j_k\}$ are distinct, such that*

$$\operatorname{rank} \begin{bmatrix} A(s_0) - s_0 E & B_{j_1}(s_0) & \dots & B_{j_k}(s_0) \\ C_{j_{k+1}}(s_0) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_{j_\nu}(s_0) & 0 & \dots & 0 \end{bmatrix} < n \tag{2.65}$$

where $\{j_{k+1}, \dots, j_\nu\} := \{1, \dots, \nu\} \backslash \{j_1, \dots, j_k\}$,

$$A(s) := \sum_{i=0}^{\sigma} A_i e^{-sh_i}, \; B_j(s) := \sum_{i=0}^{\sigma} B_{i,j} e^{-sh_i}, \text{ and } C_j(s) := \sum_{i=0}^{\sigma} C_{i,j} e^{-sh_i}. \tag{2.66}$$

Proof. See [80]. □

Note that, as in the centralized case, any μ-DFM of (2.31) is a μ-mode of (2.31) [80], thus

$$\Lambda_\mu^d \subset \Omega_\mu \, . \tag{2.67}$$

From this follows, Λ_μ^d is also a finite set under Assumption 1. Thus, once the μ-modes of (2.31) are determined (e.g., by the method of [111]), the μ-CFMs can be determined by checking the rank conditions (2.65) [80].

2.4.3. Numerical Procedure for Stabilizability

As a result of the previous subsections, the centralized and decentralized μ-stabilizability can be determined by the existence of μ-CFMs and μ-DFMs, respectively. In this context, once all the μ-modes of the system are calculated, a μ-mode can be checked whether it is a μ-CFM or μ-DFM by the rank conditions given in Lemma 1 and 2, respectively. However, such a procedure is disadvantageous for at least two reasons.

According to the rank test-based procedure, for the determination of centralized μ-stabilizability, it is necessary to check the rank of at most two matrices (i.e., (2.57) and (2.58)) for each μ-mode. For the determination of decentralized μ-stabilizability, on the other hand, it is necessary to check the rank of up to 2^ν matrices for each μ-mode. Thus, even for finite-dimensional systems, where the number of μ-modes can not be greater than n, the number of rank computations dramatically increases as ν increases. For time-delay systems, on the other hand, the number of rank computations can be extremely high since the number of μ-modes of a time-delay system may be much greater than n. Furthermore, as n increases, the rank computation problem of even a single matrix may become computationally intense and/or numerically ill-conditioned[3] [112].

As an alternative to the rank test-based procedure, we use a *numerical procedure* to determine the centralized and decentralized μ-stabilizability. The main idea of the numerical procedure is based on the definition of the sets of μ-CFMs and μ-DFMs given in Definition 8 and 9, respectively [113]. In this procedure, we apply an arbitrary controller, whose gains are randomly chosen, to the open-loop system. If all the μ-modes of the open-loop system are moved, then the system is said to be μ-stabilizable. However, if there exists any μ-mode of the open-loop system whose location on the complex plane is not changed, then such a μ-mode can be determined as a fixed mode. This procedure relies on the fact that the location of a non-fixed mode on the complex plane is changed by *almost all* controllers whose gains are randomly chosen [66]. Furthermore, relying on Remark 2 and 3, the controller to be applied in this procedure can be a delay-free static output feedback controller to ease the process.

Such a numerical procedure can be applied for both centralized and decentralized stabilizability. The main differences between these two cases are the open-loop systems to be considered and the controllers to be applied. In this respect, the decentralized counterparts are indicated inside parentheses, when necessary, in the test given below.

Centralized (Decentralized) μ-stabilizability test:

0) Choose an $\epsilon \in \mathbb{R}$ such that $C_D < \epsilon < \mu$ for mode calculation. Also, choose a small positive $\delta \in \mathbb{R}$, e.g., $\delta = 10^{-6}$, $a < \mu$, and the number of random controllers, $n_r > 0$, to be applied sequentially.

1) Calculate the set of ϵ-modes of (2.25) ((2.31)). Let $\mathcal{S} := \Omega_\mu$ denote the set of all μ-modes of the open-loop system.

[3] An ill-conditioned matrix is close to being singular. In other words, the columns of a matrix tend to be linearly dependent as n increases, which generally causes an error in numeric computations.

2) Randomly choose $L \in \mathbb{R}^{m\times q}$ ($L_k \in \mathbb{R}^{m_k\times q_k}$, $k = 1,\ldots,\nu$).

3) If $\operatorname{rank}(E) = n$ or (2.27) is satisfied, then apply the controller (2.36) with $K_0 = L$ ((2.50) with $K_{0,k} = L_k$, $k = 1,\ldots,\nu$); otherwise, apply the controller (2.37) with

$$F_0 = aI_q,\ G_0 = I_q,\ H_0 = L,\ K_0 = 0_{m\times q}$$

if $q \leq m$,

$$F_0 = aI_m,\ G_0 = L,\ H_0 = I_m,\ K_0 = 0_{m\times q}$$

otherwise ((2.51) with

$$F_{0,k} = aI_{q_k},\ G_{0,k} = I_{q_k},\ H_{0,k} = L_k,\ K_{0,k} = 0_{m_k\times q_k}$$

if $q_k \leq m_k$,

$$F_{0,k} = aI_{m_k},\ G_{0,k} = L_k,\ H_{0,k} = I_{m_k},\ K_{0,k} = 0_{m_k\times q_k}$$

otherwise, for $k = 1,\ldots,\nu$).

4) Calculate the ϵ-modes of the closed-loop system. Let $\bar{\mathcal{S}}$ denote the set of all μ-modes of the closed-loop system.

5) Let $\hat{\mathcal{S}} = \emptyset$.

6) For all $\bar{s} \in \bar{\mathcal{S}}$ do the following: if $|\bar{s} - s| \leq \delta$ for some $s \in \mathcal{S}$ then let $\hat{\mathcal{S}} = \hat{\mathcal{S}} \cup \{s\}$.

7) Let $\mathcal{S} = \hat{\mathcal{S}}$ and $n_r = n_r - 1$.

8) If $\mathcal{S} = \emptyset$, go to step **10**, otherwise continue with the next step.

9) If $n_r > 0$, go to step **2**, otherwise continue with the next step.

10) If $\mathcal{S} = \emptyset$, then the open-loop system does not have any μ-CFM (μ-DFM). Therefore, the open-loop system is centralized (decentralized) μ-stabilizable. If $S \neq \emptyset$, then, with probability 1, $\mathcal{S}$ is the set of μ-CFMs (μ-DFMs), hence the open-loop system is not centralized (decentralized) μ-stabilizable.

We should note that a mode can be determined as nonfixed if the condition in step **6** is violated, i.e., the location of the mode changes considerably. However, a mode may not be fixed even though the condition in step **6** is satisfied, i.e., the location of the mode does not change, or the present change is not satisfactory. Therefore, using multiple random controllers with different gains would increase the accuracy of the result; thus, one should choose $n_r > 1$ [112].

Note that, in step **3**, the applied controller is chosen as a finite-dimensional static controller if $\mathrm{rank}(E) = n$ or (2.27), for all $i, j = 0, \ldots, \sigma$, ((2.34), for all $k, l = 1, \ldots, \nu$ and $i, j = 0, \ldots, \sigma$, in decentralized case) is satisfied, which means that the open-loop system does not have any direct feed-through terms. However, if $\mathrm{rank}(E) < n$ and (2.27), for at least one combination of $i, j = 0, \ldots, \sigma$, ((2.34), for at least one combination of $k, l = 1, \ldots, \nu$ and $i, j = 0, \ldots, \sigma$, in decentralized case) is not satisfied, then the controller is chosen as a finite-dimensional dynamic controller without any direct feed-through terms. Due to the controller structures, the closed-loop system is guaranteed to be well-posed in both cases. Furthermore, it is also guaranteed that $\bar{C}_D = C_D$, where $\bar{C}_D$ is the insensitive bound of the closed-loop system. Since $\epsilon > C_D$, all the ϵ-modes (thus all the μ-modes) of the closed-loop system can be calculated. For details of these two arguments and the construction of the closed-loop system, we refer to Section 3.1 and 4.1 for centralized and decentralized cases, respectively.

3. CENTRALIZED STABILIZATION AND STRONG STABILIZATION

The main objective of this chapter is to propose eigenvalue optimization-based stabilization and strong stabilization methods for LTI MIMO time-delay systems, described in (2.25), using centralized LTI output feedback controllers introduced in Subsection 2.3.1. In Section 3.1, first, the closed-loop system is obtained, then, the stabilization problem is formulated. In Section 3.2, first, the strong μ-stabilizability test is introduced, then, the strong stabilization problem is formulated. In Section 3.3, the technical details of the controller design phase are discussed. Finally, in Section 3.4, the controller design methods proposed in this chapter, along with the associated analysis methods given in the previous chapter, are illustrated through various examples.

3.1. Stabilization

We consider a MIMO LTI time-delay system described by DDAEs described as (2.25). According to Subsection 2.4.1, given that $C_D < \mu$, (2.25) is centralized μ-stablizable by a centralized controller in the form of (2.35) if and only if it does not have any μ-CFMs. For this reason, we make the following assumption in addition to Assumption 1.

Assumption 2. (2.25) *does not have any μ-CFMs.*

Under Assumptions 1 and 2, all the μ-modes of (2.25) can be assigned to the left-hand side of $\mu \in \mathbb{R}$. To describe the closed-loop system of (2.25) and the controller $\mathcal{K}$ described as (2.35), let

$$\{\bar{h}_1, \dots, \bar{h}_{\bar{\sigma}}\} := \{h_1, \dots, h_\sigma\} \cup \{\hat{h}_1, \dots, \hat{h}_{\hat{\sigma}}\}\ , \tag{3.1}$$

where $\bar{\sigma} = \sigma + \hat{\sigma}$. Then, the closed-loop system can be described in the form of (2.1) as

$$\mathcal{E}\dot{\eta}(t) = \mathcal{A}_0\eta(t) + \sum_{i=1}^{\bar{\sigma}} \mathcal{A}_i\eta(t - \bar{h}_i) \tag{3.2}$$

where $\eta(t) := \begin{bmatrix} x(t)^T & y(t)^T & z(t)^T & u(t)^T \end{bmatrix}^T \in \mathbb{R}^{\bar{n}}$, $\bar{n} := n + m + q + l$,

$$\mathcal{E} := \begin{bmatrix} E & 0 & 0 & 0 \\ 0 & 0_q & 0 & 0 \\ 0 & 0 & I_l & 0 \\ 0 & 0 & 0 & 0_m \end{bmatrix}, \quad \mathcal{A}_0 := \begin{bmatrix} A_0 & 0 & 0 & B_0 \\ C_0 & -I_q & 0 & 0 \\ 0 & G_0 & F_0 & 0 \\ 0 & K_0 & H_0 & -I_m \end{bmatrix},$$

and, for $i = 1, \ldots, \bar{\sigma}$,

$$\mathcal{A}_i := \begin{bmatrix} \bar{A}_i & 0 & 0 & \bar{B}_i \\ \bar{C}_i & 0_q & 0 & 0 \\ 0 & \bar{G}_i & \bar{F}_i & 0 \\ 0 & \bar{K}_i & \bar{H}_i & 0_m \end{bmatrix},$$

where

$$\bar{A}_i := \begin{cases} A_j, & \text{if } \bar{h}_i = h_j \\ 0_n, & \text{otherwise} \end{cases}, \qquad \bar{B}_i := \begin{cases} B_j, & \text{if } \bar{h}_i = h_j \\ 0_{n\times m}, & \text{otherwise} \end{cases},$$

$$\bar{C}_i := \begin{cases} C_j, & \text{if } \bar{h}_i = h_j \\ 0_{q\times n}, & \text{otherwise} \end{cases},$$

$$\bar{F}_i := \begin{cases} F_j, & \text{if } \bar{h}_i = \hat{h}_j \\ 0_l, & \text{otherwise} \end{cases}, \qquad \bar{G}_i := \begin{cases} G_j, & \text{if } \bar{h}_i = \hat{h}_j \\ 0_{l\times q}, & \text{otherwise} \end{cases},$$

$$\bar{H}_i := \begin{cases} H_j, & \text{if } \bar{h}_i = \hat{h}_j \\ 0_{m\times l}, & \text{otherwise} \end{cases}, \qquad \bar{K}_i := \begin{cases} K_j, & \text{if } \bar{h}_i = \hat{h}_j \\ 0_{m\times q}, & \text{otherwise} \end{cases}.$$

Let $\tilde{\bar{n}}$ denote the rank deficiency of $\mathcal{E}$, thus

$$\tilde{\bar{n}} := \bar{n} - \operatorname{rank}(\mathcal{E}) = \tilde{n} + m + q,$$

where $\tilde{n}$ is the rank deficiency of E as defined in (2.2). Let $\mathcal{U} \in \mathbb{R}^{\bar{n}\times\tilde{\bar{n}}}$ and $\mathcal{V} \in \mathbb{R}^{\bar{n}\times\tilde{\bar{n}}}$, be such that

$$\mathcal{U} = \mathcal{V} := \begin{bmatrix} 0_{n\times q} & 0 \\ I_q & 0 \\ 0_{l\times q} & 0 \\ 0 & I_m \end{bmatrix}, \tag{3.3}$$

if $\operatorname{rank}(E) = n$,

$$\mathcal{U} := \begin{bmatrix} U & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0_{l\times q} & 0 \\ 0 & 0 & I_m \end{bmatrix}, \quad \mathcal{V} := \begin{bmatrix} V & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0_{l\times q} & 0 \\ 0 & 0 & I_m \end{bmatrix}, \tag{3.4}$$

where U and V are as in (2.3), otherwise. Thus, the columns of $\mathcal{U}$ and $\mathcal{V}$ form a minimal basis for the left and right null spaces of $\mathcal{E}$, respectively. Also, let the columns of $\mathcal{U}^\perp \in \mathbb{R}^{\bar{n}\times(\bar{n}-\tilde{\bar{n}})}$ and $\mathcal{V}^\perp \in \mathbb{R}^{\bar{n}\times(\bar{n}-\tilde{\bar{n}})}$ be a basis for the orthogonal complement of $\mathcal{U}$ and $\mathcal{V}$, respectively, in which case

$$\bar{\mathcal{U}} := [\mathcal{U}^\perp \, \mathcal{U}] \text{ and } \bar{\mathcal{V}} := [\mathcal{V}^\perp \, \mathcal{V}] \tag{3.5}$$

are full-rank matrices. Also let η_1 and η_2 be defined such that

$$\eta = \bar{\mathcal{V}}[\eta_1^T \; \eta_2^T]^T = \mathcal{V}^\perp \eta_1 + \mathcal{V}\eta_2 \tag{3.6}$$

where $\eta_1 \in \mathbb{R}^{\bar{n}-\tilde{\bar{n}}}$ and $\eta_2 \in \mathbb{R}^{\tilde{\bar{n}}}$. Now, (3.2) can be described as coupled delay-differential and delay-difference equations, via multiplying (3.2) by $\bar{\mathcal{U}}^T$ from the left and substituting (3.6), which yields

$$\begin{aligned} \mathcal{U}^{\perp T}\mathcal{E}\mathcal{V}^\perp \dot{\eta}_1(t) &= \sum_{i=0}^{\bar{\sigma}} \left(\mathcal{U}^{\perp T}\mathcal{A}_i\mathcal{V}^\perp \eta_1(t-\bar{h}_i) + \mathcal{U}^{\perp T}\mathcal{A}_i\mathcal{V}\eta_2(t-\bar{h}_i) \right) \\ 0 &= \sum_{i=0}^{\bar{\sigma}} \left(\mathcal{U}^T\mathcal{A}_i\mathcal{V}^\perp \eta_1(t-\bar{h}_i) + \mathcal{U}^T\mathcal{A}_i\mathcal{V}\eta_2(t-\bar{h}_i) \right). \end{aligned} \tag{3.7}$$

Therefore the associated delay-difference equation of (3.2) is

$$\mathcal{H}_0\eta_2(t) + \sum_{i=1}^{\bar{\sigma}} \mathcal{H}_i\eta_2(t-\bar{h}_i) = 0 \ , \tag{3.8}$$

where $\mathcal{H}_i := \mathcal{U}^T\mathcal{A}_i\mathcal{V}$, $i = 0, \ldots, \bar{\sigma}$. Then, if $\mathrm{rank}(E) = n$, we have

$$\mathcal{H}_0 = \begin{bmatrix} -I_q & 0 \\ K_0 & -I_m \end{bmatrix}, \tag{3.9}$$

and, for $i = 1, \ldots, \bar{\sigma}$,

$$\mathcal{H}_i = \begin{bmatrix} 0_q & 0 \\ \bar{K}_i & 0_m \end{bmatrix}, \tag{3.10}$$

otherwise,

$$\mathcal{H}_0 = \begin{bmatrix} U^T A_0 V & 0 & U^T B_0 \\ C_0 V & -I_q & 0 \\ 0 & K_0 & -I_m \end{bmatrix}, \tag{3.11}$$

and, for $i = 1, \ldots, \bar{\sigma}$,

$$\mathcal{H}_i = \begin{bmatrix} U^T \bar{A}_i V & 0 & U^T \bar{B}_i \\ \bar{C}_i V & 0_q & 0 \\ 0 & \bar{K}_i & 0_m \end{bmatrix}. \tag{3.12}$$

Notice that, by (3.9) and (3.10), the closed-loop system can not be neutral if $\mathrm{rank}(E) = n$. However, if $\mathrm{rank}(E) < n$, the closed-loop system might be neutral.

The closed-loop system (3.2) is well-posed if and only if the characteristic matrix of (3.2) defined as $\bar{\phi}(s;\mathcal{K}) := s\mathcal{E} - \mathcal{A}(s)$, where

$$\mathcal{A}(s) := \sum_{i=0}^{\bar{\sigma}} \mathcal{A}_i e^{-s\bar{h}_i}, \tag{3.13}$$

is invertible and its inverse is proper [6]. From this follows, the closed-loop system (3.2) is well-posed if and only if

$$\lim_{Re(s)\to+\infty} \det\left(\bar{\phi}(s;\mathcal{K})\right) \neq 0. \tag{3.14}$$

If $\mathrm{rank}(E) < n$, then, using (3.5) and (3.6), (3.14) can be explicitly given as

$$\lim_{Re(s)\to+\infty} \det\left(\begin{bmatrix} {U^{\perp}}^T A_0 V^{\perp} - s({U^{\perp}}^T E V^{\perp}) & {U^{\perp}}^T A_0 V & 0 & 0 & {U^{\perp}}^T B_0 \\ U^T A_0 V^{\perp} & U^T A_0 V & 0 & 0 & U^T B_0 \\ C_0 V^{\perp} & C_0 V & -I_q & 0 & 0 \\ 0 & 0 & G_0 & F_0 - sI_l & 0 \\ 0 & 0 & K_0 & H_0 & -I_m \end{bmatrix}\right) \neq 0 \tag{3.15}$$

since

$$\lim_{Re(s)\to+\infty} \mathcal{A}(s) = \mathcal{A}_0.$$

By rearanging the rows and colums of $\bar{\phi}(s;\mathcal{K})$, (3.15) becomes

$$\lim_{Re(s)\to+\infty} \det\left(\begin{bmatrix} {U^{\perp}}^T A_0 V^{\perp} - s({U^{\perp}}^T E V^{\perp}) & 0 & {U^{\perp}}^T A_0 V & 0 & {U^{\perp}}^T B_0 \\ 0 & F_0 - sI_l & 0 & G_0 & 0 \\ U^T A_0 V^{\perp} & 0 & U^T A_0 V & 0 & U^T B_0 \\ C_0 V^{\perp} & 0 & C_0 V & -I_q & 0 \\ 0 & H_0 & 0 & K_0 & -I_m \end{bmatrix}\right) \neq 0 \tag{3.16}$$

Then, (3.16) can be written as

$$\lim_{Re(s)\to+\infty} \left(\det(\bar{\phi}^{11})\det(\bar{\phi}^{22} - \bar{\phi}^{21}(\bar{\phi}^{11})^{-1}\bar{\phi}^{12}\right) \neq 0, \tag{3.17}$$

where

$$\bar{\phi}^{11} = \begin{bmatrix} {U^{\perp}}^T A_0 V^{\perp} - s({U^{\perp}}^T E V^{\perp}) & 0 \\ 0 & F_0 - sI_l \end{bmatrix}, \ \bar{\phi}^{12} = \begin{bmatrix} {U^{\perp}}^T A_0 V & 0 & {U^{\perp}}^T B_0 \\ 0 & G_0 & 0 \end{bmatrix},$$

$$\bar{\phi}^{21} = \begin{bmatrix} U^T A_0 V^{\perp} & 0 \\ C_0 V^{\perp} & 0 \\ 0 & H_0 \end{bmatrix}, \ \bar{\phi}^{22} = \begin{bmatrix} U^T A_0 V & 0 & U^T B_0 \\ C_0 V & -I_q & 0 \\ 0 & K_0 & -I_m \end{bmatrix}.$$

Then, using

$$\lim_{Re(s)\to+\infty} \left(\det(\bar{\phi}^{11})\right) \neq 0 \tag{3.18}$$

and

$$\lim_{Re(s)\to+\infty}\left((\bar{\phi}^{11})^{-1}\right)=0, \tag{3.19}$$

(3.17) reduces to

$$\lim_{Re(s)\to+\infty}\left(\det(\bar{\phi}^{22})\right)=\det(\bar{\phi}^{22})\neq 0. \tag{3.20}$$

Recalling that $U^T A_0 V$ is assumed to be nonsingular, i.e., $\det(U^T A_0 V)\neq 0$, (3.20) further reduces to

$$\det\left(I+K_0C_0V(U^TA_0V)^{-1}U^TB_0\right)\neq 0. \tag{3.21}$$

As a result, to gurantee the well-posedness of the closed-loop system (3.2), the controller (2.35) must satisfy (3.21). Note that, if $\mathrm{rank}(E)=n$, then the closed-loop system is always well-posed.

The main objective is to make (3.2) μ-stable by minimizing the spectral abscissa as a function of the free parameters of the controller in the form of (2.35) which satisfy (3.14). Considering (2.35), let $p\in\mathbb{R}^N$ be the vector of free parameters in the controller matrices where N is the number of free parameters involved. Furthermore, let $\hat{h}\in\mathbb{R}^{\hat{\sigma}}$ be the vector of time delays of the controller described as

$$\hat{h}:=[\hat{h}_1,\ldots,\hat{h}_{\hat{\sigma}}]. \tag{3.22}$$

So, the overall vector of parameters can be constructed as

$$\bar{p}:=[p\;\hat{h}]. \tag{3.23}$$

In order to find a controller, which μ-stabilizes (3.2), we solve the following optimization problem:

$$\min_{\bar{p}} c(\bar{p}) \tag{3.24}$$

where $c(\bar{p})$ indicates the spectral abscissa of the closed-loop system (3.2) with the parameter vector $\bar{p}$. If a local minimizer, $\bar{p}^+$, of (3.24) satisfies $c(\bar{p}^+)<\mu$, then, the controller corresponding to $\bar{p}^+$ μ-stabilizes the system (2.25) for a given $\mu\in\mathbb{R}$. If a time-delay controller, i.e., (2.35) with $\hat{\sigma}>0$, is aimed, then the constraint

$$\{\hat{h}_1,\ldots,\hat{h}_{\hat{\sigma}}\}\geq 0 \tag{3.25}$$

must be imposed to make sure that the designed controller will be causal. However, if a delay-free controller, i.e., (2.35) with $\hat{\sigma} = 0$, is aimed, then (3.25) should be omitted.

By definition, the spectral absicssa, $c(\bar{p})$ for a particular $\bar{p}$, can be determined as the real part of the rightmost closed-loop ϵ-mode (the ϵ-mode with largest real part), where, to achieve μ-stability, ϵ must be chosen such that $\epsilon < \mu$. Therefore, the determination of $c(\bar{p})$ requires that the set of closed-loop ϵ-modes be finite. To this end, it must be guranteed that $\bar{C}_D(\bar{p}) < \epsilon$, where $\bar{C}_D(\bar{p})$ indicates the insentive bound of the closed-loop system (3.2) with the parameter vector $\bar{p}$, during the course of the optimization. Considering this circumstance, we will introduce two different cases regarding the relation between C_D and $\bar{C}_D(\bar{p})$. Before getting into the cases, recall that the system (2.25) does not have any direct feed-through term if either $\mathrm{rank}(E) = n$ or (2.27) is satisfied for all $i, j = 0, \ldots, \sigma$. On the other hand, the controller (2.35) does not have any direct feed-through term if $K_i = 0$, $i = 0, \ldots, \hat{\sigma}$. Furthermore, the controller (2.35) can not be neutral.

Case 1: If either the system or the controller does not have any direct feed-through terms, then $\bar{C}_D(\bar{p}) = C_D$. As a result, it is guranteed that $\bar{C}_D(\bar{p}) < \epsilon$ for all $\bar{p}$ since $C_D < \epsilon$.

Case 2: If a controller (2.35) with some direct feed-through terms is to be designed and the open-loop system (2.25) has some direct feed-through terms, then for some $\bar{p}$ we might have $\bar{C}_D(\bar{p}) > C_D$. [1] Then, the constraint of $\bar{C}_D(\bar{p}) \leq \tilde{\mu}$ must be imposed, where $\tilde{\mu} \in \mathbb{R}$. In this case, chosing $\tilde{\mu}$ and ϵ such that $C_D < \tilde{\mu} < \epsilon$ gurantees that the set of closed-loop ϵ-modes be finite. However, as it is discussed in Section 2.1, calculating the insenstive bound is computationally expensive. On the other hand, $\bar{C}_D(\bar{p}) \leq \tilde{\mu}$ can be expressed as $\gamma_{\tilde{\mu}}(\bar{p}) \leq 1$, where, for $\bar{\sigma} = 1$,

$$\gamma_{\tilde{\mu}}(\bar{p}) := \rho\left(\mathcal{H}_0^{-1}\mathcal{H}_1 e^{-\tilde{\mu}\bar{h}_1}\right) \tag{3.26}$$

and, for $\bar{\sigma} \geq 2$,

$$\gamma_{\tilde{\mu}}(\bar{p}) := \max_{\theta \in [0, 2\pi]^{\bar{\sigma}-1}} \rho\left(\mathcal{H}_0^{-1}\left[\mathcal{H}_1 e^{-\tilde{\mu}\bar{h}_1} + \sum_{k=2}^{\bar{\sigma}} \mathcal{H}_k e^{-\tilde{\mu}\bar{h}_k} e^{\mathrm{j}\theta_k}\right]\right), \tag{3.27}$$

[1] $\bar{C}_D(\bar{p})$ can not be less than C_D since a controller in the form of (2.35) can not reduce $\bar{C}_D(\bar{p})$ any further than C_D [29].

where $\theta := \{\theta_2, \ldots, \theta_{\bar{\sigma}}\}$. Therefore, rather than $\bar{C}_D(\bar{p}) \leq \tilde{\mu}$, the constraint

$$\gamma_{\tilde{\mu}}(\bar{p}) \leq 1 \tag{3.28}$$

can be imposed to (3.24). As a result, under the constraint (3.28), it is guranteed that $\bar{C}_D(\bar{p}) < \epsilon$ for all $\bar{p}$ since $\tilde{\mu} < \epsilon$.

Note that, in *Case 1*, the closed-loop system is guaranteed to be well-posed during the course of the optimization. On the other hand, in *Case 2*, (3.21) may be violated for some $\bar{p}$, which causes $\bar{C}_D(\bar{p}) = \infty$, i.e., $\gamma_{\tilde{\mu}}(\bar{p}) > 1$. However, by imposing (3.28), the closed-loop system is guaranteed to be well-posed during the course of the optimization. Furthermore, in both of these cases, if the controller corresponding to $\bar{p}^+$ μ-stabilizes the system (2.25) then the system (2.25) is guaranteed to be insensitively μ-stable, since $\tilde{\mu} < \mu$.

3.2. Strong Stabilization

Although the solution to the optimization problem (3.24) leads to the design of a stabilizing controller, the designed controller may not be stable. However, (3.24) can be modified in such a way that the controller stability can be defined as a constraint. In this case, such an optimization problem is solvable only if (2.25) is strongly stabilizable, i.e, (2.25) can be stabilized by a stable controller. As mentioned in Section 1.1, however, under Assumption 1, a stabilizable system of the form (2.25) is strongly stabilizable if and only if it satisfies PIP. Although, PIP is normally defined with reference to extended positive real-axis [10], since here we aim μ-stability, for some (normally non-positive) real μ, we define it with reference to *extended μ-positive real-axis*, which is defined as all real numbers, including infinity, greater than or equal to μ:

Definition 10. A system is said to satisfy PIP if the number of real modes (counted according to their McMillan degrees) between any pair of real blocking zeros on the extended μ-positive real-axis is even.

In the concept of strong stabilization, we have two main questions that must be answered:

Q1: Does there exist a μ-stable μ-stabilizing controller, in the form of (2.35), for (2.25)?

and

- **Q2**: If a μ-stable μ-stabilizing controller for (2.25) does not exist, what is the minimum possible number of μ-modes, counted with multiplicities, that any μ-stabilizing controller must-have?

In fact, **Q1** is directly related to the solvability of the strong μ-stabilization problem. On the other hand, the solution of **Q2** yields a solution to **Q1**, therefore it can be thought of as a generalization of the strong μ-stabilization problem. So, we need to have a test that gives the answer to **Q2** for (2.25).

First, let us define the *blocking zeros* of (2.25).

Definition 11. $z \in \mathbb{C}$ is said to be a *blocking zero* of (2.25), if $T(z) = 0$, where $T(s) := C(s)\,(sE - A(s))^{-1}\,B(s)$ is the transfer function matrix (TFM) of (2.25), where $A(s)$, $B(s)$, and $C(s)$ are as defined in Lemma 1.

The following test answers **Q2**:

Strong μ-stabilizability test:

- **A**) Compute the ϵ-modes of the system for some ϵ satisfying $C_D < \epsilon < \mu$. Form an interval $\mathcal{I} \subset \mathbb{R}$ with the boundaries (ϵ, ς), where ς is the rightmost real ϵ-mode of (2.25) (if there exists no such mode, then the system satisfies PIP, let $l^{\min} = 0$ and exit). Find all $z \in \mathcal{I}$ satisfying $T(z) = 0$, where $T(s)$ is the TFM of (2.25), to find the blocking zeros of (2.25) on $\mathcal{I}$.

- **B**) Determine the pairs of consecutive real blocking zeros $\{\kappa_i^-, \kappa_i^+\}$ inside $\mathcal{I}$ with an odd number (counting multiplicities) of real modes between them, for $i = 1, \ldots, l^+$, where l^+ is the number of such pairs (if there are no such pairs, let $l^+ = 0$). Also, let κ^+ indicate the rightmost blocking zero in $\mathcal{I}$ and check whether there are an odd number of real modes to the right of κ^+. If so, let $l^{\min} = l^+ + 1$; otherwise, let $l^{\min} = l^+$.

- **C**) If $l^{\min} = 0$, then (2.25) is strongly μ-stabilizable. Otherwise, the dimension of the μ-stabilizing controller of (2.25) must be at least $l^{\min}$. Furthermore, such a controller must include at least one real mode (or any odd number, counting multiplicities) in each interval (κ_i^-, κ_i^+), $i = 1, \ldots, l^+$, plus at least one real mode (or any odd number, counting multiplicities) to the right of κ^+ (but to the left of next real blocking zero if any - if blocking zeros outside $\mathcal{I}$ are not known, these modes must be placed between κ^+ and ς) if there are an odd number of real system modes to the right of κ^+.

The interval $\mathcal{I}$ can be chosen as a finite interval with upper limit ς, since whether there are any real blocking zeros to the right of ς or not does not affect the end result. Thus, the roots of $T(z) = 0$ in $\mathcal{I}$ can be computed using any numerical algorithm since $\mathcal{I}$ is a finite interval. In particular, *quasi-polynomial mapping based root-finder* (QPmR) algorithm of [114] can be used.

If (2.25) is strongly μ-stabilizable, i.e., $l^{\min} = 0$, then a strongly μ-stabilizing controller in the form of (2.35) can be designed by adding the constraint of controller stability to (3.24). Let $\hat{c}(\bar{p})$ denote the spectral abscissa, i.e., the real part of the rightmost mode, of the controller which corresponds to $\bar{p}$. Suppose that we want all the controller modes to have real parts less than $\hat{\mu}$ for some $\hat{\mu} \leq \mu$. Then we impose an additional constriant

$$\hat{c}(\bar{p}) \leq \hat{\mu}. \tag{3.29}$$

to the stabilization problem (3.24). If a local minimizer, $\bar{p}^+$, satisfies $c(\bar{p}^+) < \mu$, then, the controller corresponding to $\bar{p}^+$ strongly μ-stabilizes (2.25) for the given $\mu \in \mathbb{R}$. Note, however, that, constraint (3.29) is needed only when $l > 0$. If $l = 0$, then the controller reduces to the form (2.40), which is $\hat{\mu}$-stable for any $\hat{\mu} \in \mathbb{R}$. Hence, when $l = 0$, constraint (3.29) must be omitted.

3.3. Details of Centralized Controller Design

This section presents the technical details of the analysis and controller design methods presented in Section 3.1 and 3.2. First, the properties of the object function (3.24) and the possible constraints, i.e., (3.25), (3.28), and (3.29), are discussed in Subsection 3.3.1. In Subsection 3.3.2, the initialization of the free parameters of the controller to be designed is given. Then, in Subsection 3.3.3, the computation of the function values, i.e., c, $\hat{c}$, C_D, and $\gamma_{\hat{\mu}}$ is given. Finally, in Subsection 3.3.4, the computation of the gradient vectors are given along with the derivation of the associated analytical formulas.

3.3.1. Optimization Phase

It is known that the spectral abscissa, $c(\bar{p})$, is both nonsmooth and nonconvex function of $\bar{p}$ [28]. The main reason for the nonsmoothness is that by definition $c(\bar{p})$ may be determined by more than one mode whose real parts are precisely equal. Furthermore, as it is reported in many studies (e.g., in [27] and [28] and references therein) that the local minima of $c(\bar{p})$ is likely to be a nonsmooth point because of the fact that, as $c(\bar{p})$ is minimized, more than one modes are assigned to the left as far as possible in the complex plane. Furthermore, there might be more than one such local minima because of the nonconvexity of the spectral abscissa function. Obviously, these properties also hold for $\hat{c}(\bar{p})$. Furthermore, $\gamma_{\hat{\mu}}(\bar{p})$ has the same properties since it is a maximum eigenvalue function as defined in (3.27) [29].

However, by utilizing a proper random initialization, for almost every $\bar{p}$, gradients of $c(\bar{p})$, $\hat{c}(\bar{p})$, and $\gamma_{\hat{\mu}}(\bar{p})$ exist [29]. For this reason, the optimization problem (3.24) can be solved by utilizing a gradient-based algorithm. Relying on that, we utilize the specialized algorithm of [115] and its associated Matlab-based solver named as GRANSO [116]. The algorithm of [115] is based on a quasi-Newton Hessian approximation BFGS and an exact penalty function whose parameters are tuned using a steering strategy. Here, the Matlab-based software DrStabilization is integrated with GRANSO to solve (3.24) for designing controllers of the form (2.35) with arbitrary l and $\hat{\sigma}$.

GRANSO requires a feasible initial point, i.e., the initial controller in our case, the function values, i.e., $c(\bar{p})$, $\hat{c}(\bar{p})$, and $\gamma_{\hat{\mu}}(\bar{p})$, and the corresponding gradient vectors, i.e., $\nabla c(\bar{p})$, $\nabla \hat{c}(\bar{p})$, and $\nabla \gamma_{\hat{\mu}}(\bar{p})$, whenever the function is differentiable.

3.3.2. Initilization of the Controller Paramaters

The optimization algorithm of GRANSO requires a feasible initial parameter vector which corresponds to a feasible initial controller in our case. Considering the nonconvexity of (3.24), the initial parameter vector must be determined randomly as much as possible. Furthermore, due to the nonconvexity of (3.24), the optimization problem must be solved starting from a number of different initial parameter vectors, and the best result (in terms of the minimum spectral abscissa value) must be chosen (in Section 3.4, we solve each example starting from 20 different randomly chosen initial points and take the best result). However, before the initialization of the controller parameters, the type (time-delay or delay-free) and the complexity (l and $\hat{\sigma}$ in (2.35)) of the controller must be determined in advance. Recall that, the strong μ-stabilizability test given in Section 3.2 determines l^{min}, i.e., the least possible differential degree that any μ-stabilizing controller must have. In this regard, in the choice of the controller complexity, the differential degree must be chosen such that $l \geq l^{min}$.

After the determination of the complexity of the controller, the default option is to let all the controller parameters (i.e., all the elements of the controller matrices and all the time delays of the controller) be free for the optimization phase. However, as given in Subsection 2.3.1, the controller can be structured in a prescribed form by fixing some of the terms to a predetermined number while letting the rest of them be free. So, the structure of the controller must be determined before the initilization.

Once the complexity and the structure of the controller are determined, in order to find a feasible initial controller, the initialization of the controller parameters must be carried out according to the existing constraints imposed to (3.24). However, regardless of the constraint, we apply a general rule: to avoid having a high-gain controller, the free parameters of the controller matrices (i.e., the elements of the vector p) are initialized randomly according to a normal distribution centered at zero (the standard deviation for each parameter must be chosen according to the problem at hand; in the examples in Section 3.4, we take all standard deviations as unity).

If the constraint (3.25) is imposed, thus in the case of a time-delay controller design, the time delays of the controller (i.e., the elements of the vector $\hat{h}$) must be initialized as randomly chosen non-negative numbers (DrStabilization randomly initialize the time delays according to a normal distribution with mean zero and standard deviation unity and then take the absolute value).

If the constraint (3.29) is imposed, in order to ensure that the initial controller is a feasible point for the optimization, the initial controller must be $\hat{\mu}$-stable. To find such an initial controller, the parameters of the F_0 matrix (in case $l > 0$; if $l = 0$ the controller is inherently stable) are initialized such that the real part of the rightmost eigenvalue of F_0 is less than $\hat{\mu}$; and (in case $l > 0$ and $\hat{\sigma} > 0$) matrices F_i, $i = 1, \ldots, \hat{\sigma}$, are initialized as zero matrices. The parameters of the other controller matrices (i.e., G_i, H_i, K_i, $i = 0, \ldots, \hat{\sigma}$) are initialized randomly according to a normal distribution centered at zero. Also, the time delays of the controller are initialized as randomly chosen non-negative numbers as mentioned before.

If the constraint (3.28) is imposed, the initial controller must satisfy $\gamma_{\tilde{\mu}}(\bar{p}) \leq 1$ which corresponds to $\bar{C}_D(\bar{p}) \leq \tilde{\mu}$. Here, we should keep in mind that $\gamma_{\tilde{\mu}}(\bar{p})$ (and $\bar{C}_D(\bar{p})$) depends only a subset of $\bar{p}$, which will be denoted as $\bar{p}^K$ from now on. More precisely, $\bar{p}^K$ consists of the free parameters of K_i, $i = 0, \ldots, \hat{\sigma}$ and the associated time delays introduced to the direct feed-through of the controller (see Section 3.1). Therefore, as long as the constraint (3.28) is concerned, we only consider $\bar{p}^K$. In order to find such an initial controller, there are two different ways to follow. The first one is the *bisection-based initilization*. In this approach, we set all the time delays in $\bar{p}^K$ (if any) to zero and randomly choose the rest of the controller parameters in $\bar{p}^K$. Then, we check $\gamma_{\tilde{\mu}}(\bar{p}^K) \leq 1$. If it is not satisfied, then we divide all the non-zero parameters of $\bar{p}^K$ by two until $\gamma_{\tilde{\mu}}(\bar{p}^K) \leq 1$ is satisfied. This approach relies on the fact that $\gamma_{\tilde{\mu}}(\bar{p}^K) \leq 1$ can be satisfied when all the controller parameters are set to zero. Accordingly, a feasible initial controller can be found when the controller parameters become sufficiently small. However, we should note that the controller would not be entirely random since the time delays are set to zero. The alternative approach is the *optimization-based initilization*. In this approach we randomly choose all the elements of $\bar{p}^K$ including time delays (which are forced to be non-negative as mentioned before). Then, starting with the corresponding parameter vector, say $\bar{p}_0$, we solve

$$\min_{\bar{p}^K} \gamma_{\tilde{\mu}}(\bar{p}) \tag{3.30}$$

where $C_D < \tilde{\mu} < \epsilon$. If a local minimizer, $\bar{p}_0^{K+}$, of (3.30) satisfies $\gamma_{\tilde{\mu}}(\bar{p}) \leq 1$, then, the controller corresponding to $\bar{p}_0^{K+}$ can be a feasible initial controller for the main optimization problem (3.24). However, if there exist time delays in $\bar{p}^K$, then the constraint (3.25) must be imposed to (3.30) to preserve the causality of the controller. Compared to the first approach, all the controller parameters can be chosen randomly in this approach. However, randomly chosen different initial controllers may converge to the same (or almost the same) local minima of (3.30) which leads to ending up with the same (or almost the same) local minimizer. This situation can be a problem because we need multiple randomly chosen initial controllers for the main optimization problem (3.24). However, this problem can be solved by stopping the optimization algorithm as soon as the $\gamma_{\tilde{\mu}}(\bar{p}) \leq 1$

is achieved. By this way, the randomness of the initial controller can be preserved.

Notice that, the constraint (3.29) only depends on F_i, $i = 0, \ldots, \hat{\sigma}$ and the associated time delays. On the other hand, (3.28) only depends on K_i, $i = 0, \ldots, \hat{\sigma}$ and the associated time delays. Therefore, if both (3.28) and (3.29) are imposed, then, first a fesaible initial controller can be found considering (3.29); then, by saving the intialized F_i, $i = 0, \ldots, \hat{\sigma}$ and the associated time delays, a feasible initial controller can be found considering (3.28).

Finally, we should stress that a feasible initial controller may not be found because of improper structuring. That means, if the determined structure of the controller makes the satisfaction of the constraints impossible for all $\bar{p}$, then none of the proposed approaches leads to a feasible initial controller. Therefore, the controller must be structured by taking the imposed constraints into account.

3.3.3. Computation of Function Values

For the optimization phase, the values of $c(\bar{p})$, $\hat{c}(\bar{p})$, and $\gamma_{\hat{\mu}}(\bar{p})$ must be computed for a particular $\bar{p}$.

The evaluation of $\gamma_{\hat{\mu}}(\bar{p})$ can be handled by applying a predictor-corrector approach, as it was done in [29]. In the prediction step, Dekker-Brents method can be used, and in the correction step, Newton's method can be used, where the evaluations are approximated by restricting each θ_i to a grid.

Note that if (3.2) is a retarded system, then there always exist a finite number of ϵ-modes for all $\epsilon \in \mathbb{R}$. On the other hand, if (3.2) is a neutral system, then (3.2) might have infinitely many modes. However, it is guranteed that, there always exist finitely many ϵ-modes for any $\epsilon > \tilde{\mu}$ (see Section 3.1). Thus, we can always compute the value of $c(\bar{p})$ by computing the ϵ-modes of (3.2), for $\epsilon > \tilde{\mu}$ by the *spectral discretization approach* of [29], which extends the approach of [111] to systems whose autonomous part can be described by DDAEs. For a system with finite-dimensional dynamics (when $\bar{\sigma} = 0$ or $\mathcal{A}_1 = \ldots = \mathcal{A}_{\bar{\sigma}} = 0$), however, to find all the modes (which, counting multiplicities, there exist exactly $\operatorname{rank}(\mathcal{E})$ of them), we simply calculate the roots of $\det(s\mathcal{E} - \mathcal{A}_0)$. Once the set of ϵ-modes are computed, the spectral abscissa can be determined as the real part of the rightmost ϵ-mode (the ϵ-mode with largest real part). On the other hand, in the computation of the value of $\hat{c}(\bar{p})$, we follow the same procedure as in $c(\bar{p})$.

3.3.4. Computation of Gradient Vectors

Although the gradient vectors can be computed by finite differences, it is more reliable to use exact gradients. Therefore, the analytic formulas of $\nabla c(\bar{p})$, $\nabla \hat{c}(\bar{p})$, and $\nabla \gamma_{\bar{\mu}}(\bar{p})$ must be derived, explicitly. Note that, while $\nabla c(\bar{p})$ is essential for the controller design, $\nabla \hat{c}(\bar{p})$ and $\nabla \gamma_{\bar{\mu}}(\bar{p})$ are necessary only when (3.29) and (3.28) are imposed, respectively. In addition to these, when (3.25) is imposed, the corresponding gradient vector must also be calculated, which is relatively straightforward.

3.3.4.1. *Computation of* $\nabla c(\bar{p})$

Let λ be a simple root of

$$\det(\bar{\phi}(\lambda; \bar{p})) = 0 \tag{3.31}$$

at some $\bar{p}$, which is the overall vector of free parameters of the controller. If λ is a multiple root of (3.31), then it can be separated into simple roots by perturbing the controller parameters, as long as it is not a fixed mode of (2.25); however, by Assumption 2, there are no μ-fixed modes of (2.25). So, we assume that λ is a simple root of (3.31) without loss of any generality. Furthermore, if there are more than one such mode with the same real part, then, any one of them can be used to determine the spectral abscissa [28]. Then, there exist non-zero vectors $u, v \in \mathbb{C}^{\bar{n}}$ such that

$$u^* \bar{\phi}(\lambda; \bar{p}) = 0 \tag{3.32a}$$

$$\bar{\phi}(\lambda; \bar{p}) v = 0 \tag{3.32b}$$

Note that, since $\bar{\phi}$ depends on $\bar{p}$, λ depends on $\bar{p}$ as well. Accordingly, u and v depend on $\bar{p}$ too. Let $\hat{h}$ be fixed and take the derivative of (3.32b) with respect to p_j, $j = 1, \ldots, N$, and multiply both sides from left by u^* to obtain

$$u^* \Big(\frac{\partial \bar{\phi}}{\partial p_j} + \frac{\partial \bar{\phi}}{\partial \lambda} \frac{\partial \lambda}{\partial p_j} \Big) v + u^* \bar{\phi}(\lambda; \bar{p}) \Big(\frac{\partial v}{\partial p_j} + \frac{\partial v}{\partial \lambda} \frac{\partial \lambda}{\partial p_j} \Big) = 0, \tag{3.33}$$

then, use (3.32a) and rearrange the terms to obtain

$$\frac{\partial \lambda}{\partial p_j} = -\frac{u^* \frac{\partial \bar{\phi}}{\partial p_j} v}{u^* \frac{\partial \bar{\phi}}{\partial \lambda} v}. \tag{3.34}$$

Thus, if λ is the rightmost mode of (3.2), which determines the spectral abscissa, then,

$$\frac{\partial c}{\partial p_j} = -\mathrm{Re}\left(\frac{u^* \frac{\partial \bar{\phi}}{\partial p_j} v}{u^* \frac{\partial \bar{\phi}}{\partial \lambda} v} \right), \tag{3.35}$$

where

$$\frac{\partial \bar{\phi}}{\partial p_j} = -\sum_{i=0}^{\bar{\sigma}} \frac{\partial \mathcal{A}_i}{\partial p_j} e^{-\lambda \bar{h}_i} , \tag{3.36}$$

and

$$\frac{\partial \bar{\phi}}{\partial \lambda} = \mathcal{E} + \sum_{i=1}^{\bar{\sigma}} \bar{h}_i \mathcal{A}_i e^{-\lambda \bar{h}_i}. \tag{3.37}$$

Then, the gradient of $c(\bar{p})$ with respect to p is obtained as

$$\nabla_p c(\bar{p})) = \left[\tfrac{\partial c}{\partial p_1} \dots \tfrac{\partial c}{\partial p_N} \right]. \tag{3.38}$$

Next, let p be fixed and take the derivative of (3.32b) with respect to $\hat{h}_k$, $k = 1, \dots, \hat{\sigma}$, and multiply both sides from left by u^* to obtain

$$u^* \Big(\frac{\partial \bar{\phi}}{\partial \hat{h}_k} + \frac{\partial \bar{\phi}}{\partial \lambda} \frac{\partial \lambda}{\partial \hat{h}_k} \Big) v + u^* \phi(\lambda; \bar{p}) \Big(\frac{\partial v}{\partial \hat{h}_k} + \frac{\partial v}{\partial \lambda} \frac{\partial \lambda}{\partial \hat{h}_k} \Big) = 0, \tag{3.39}$$

then, use (3.32a) and rearrange the terms to obtain

$$\frac{\partial \lambda}{\partial \hat{h}_k} = -\frac{u^* \frac{\partial \bar{\phi}}{\partial \hat{h}_k} v}{u^* \frac{\partial \bar{\phi}}{\partial \lambda} v}. \tag{3.40}$$

Thus, if λ is the rightmost mode of (3.2), which determines the spectral abscissa, then,

$$\frac{\partial c}{\partial \hat{h}_k} = -\mathrm{Re}\left(\frac{u^* \frac{\partial \bar{\phi}}{\partial \hat{h}_k} v}{u^* \frac{\partial \bar{\phi}}{\partial \lambda} v} \right), \tag{3.41}$$

where

$$\frac{\partial \bar{\phi}}{\partial \hat{h}_k} = \lambda \mathcal{A}_i e^{-\lambda \bar{h}_i} \tag{3.42}$$

where i is such that $\hat{h}_k = \bar{h}_i$, and $\frac{\partial \bar{\phi}}{\partial \lambda}$ is as given by (3.37). Thus, the gradient of $c(\bar{p})$ with respect to $\hat{h}$ is obtained as

$$\nabla_{\hat{h}} c(\bar{p}) = \left[\frac{\partial c}{\partial \hat{h}_1} \dots \frac{\partial c}{\partial \hat{h}_{\hat{\sigma}}} \right]. \tag{3.43}$$

As a result, the gradient vector of $c(\bar{p})$ can be constructed as

$$\nabla c(\bar{p}) = \left[\nabla_p c(\bar{p}) \quad \nabla_{\hat{h}} c(\bar{p}) \right]. \tag{3.44}$$

3.3.4.2. *Computation of* $\nabla \hat{c}(\bar{p})$

If the strong stabilization problem is considered, then, in addition to $\nabla c(\bar{p})$, $\nabla \hat{c}(\bar{p})$ must also be derived when $l > 0$.[2] To derive $\nabla \hat{c}(\bar{p})$ when $l > 0$, let λ be a simple root of

$$\det(\hat{\phi}(\lambda; \bar{p})) = 0 \tag{3.45}$$

at some $\bar{p}$, where

$$\hat{\phi}(\lambda; \bar{p}) := \lambda I - \sum_{i=0}^{\hat{\sigma}} F_i e^{-\lambda \hat{h}_i} \tag{3.46}$$

is the characteristic matrix of (2.35). Then, there exist non-zero vectors $\hat{u}, \hat{v} \in \mathbb{C}^l$ such that

$$\hat{u}^* \hat{\phi}(\lambda; \bar{p}) = 0 \tag{3.47a}$$

$$\hat{\phi}(\lambda; \bar{p}) \hat{v} = 0 \tag{3.47b}$$

Then, proceeding as before, the gradient vector of $\hat{c}(\bar{p})$ can be constructed as

$$\nabla \hat{c}(\bar{p}) = \left[\nabla_p \hat{c}(\bar{p}) \quad \nabla_{\hat{h}} \hat{c}(\bar{p}) \right] = \left[\frac{\partial \hat{c}}{\partial p_1} \quad \dots \quad \frac{\partial \hat{c}}{\partial p_N} \quad \frac{\partial \hat{c}}{\partial \hat{h}_1} \quad \dots \quad \frac{\partial \hat{c}}{\partial \hat{h}_{\hat{\sigma}}} \right], \tag{3.48}$$

where, for $j = 1, \dots, N$,

$$\frac{\partial \hat{c}}{\partial p_j} = -\mathrm{Re}\left(\frac{\hat{u}^* \frac{\partial \hat{\phi}}{\partial p_j} \hat{v}}{\hat{u}^* \frac{\partial \hat{\phi}}{\partial \lambda} \hat{v}} \right), \tag{3.49}$$

[2]Recall that, when $l = 0$, the controller reduces to the form (2.40), which is $\hat{\mu}$-stable for any $\hat{\mu} \in \mathbb{R}$; in which case, constraint (3.29) must be omitted; thus $\nabla \hat{c}(\bar{p})$ is not needed.

and, for $k = 1, \ldots, \hat{\sigma}$,

$$\frac{\partial \hat{c}}{\partial \hat{h}_k} = -\mathrm{Re}\left(\frac{\hat{u}^* \frac{\partial \hat{\phi}}{\partial \hat{h}_k} \hat{v}}{\hat{u}^* \frac{\partial \hat{\phi}}{\partial \lambda} \hat{v}} \right), \tag{3.50}$$

where

$$\frac{\partial \hat{\phi}}{\partial p_j} = -\sum_{i=0}^{\hat{\sigma}} \frac{\partial F_i}{\partial p_j} e^{-\lambda \hat{h}_i} , \tag{3.51}$$

$$\frac{\partial \hat{\phi}}{\partial \hat{h}_k} = \lambda F_k e^{-\lambda \hat{h}_k} , \tag{3.52}$$

and

$$\frac{\partial \hat{\phi}}{\partial \lambda} = I + \sum_{i=1}^{\hat{\sigma}} \hat{h}_i F_i e^{-\lambda \hat{h}_i} , \tag{3.53}$$

where λ is the rightmost mode of (2.35), which determines the spectral abscissa of the controller.

3.3.4.3. ***Computation of*** $\nabla \gamma_{\tilde{\mu}}(\bar{p})$

If the constraint (3.28) is imposed to (3.24), then, in addition to $\nabla c(\bar{p})$ and $\nabla \hat{c}(\bar{p})$, $\nabla \gamma_{\tilde{\mu}}(\bar{p})$ must also be derived. Let $\mathcal{H}_i$, $i = 0, \ldots, \bar{\sigma}$, be the matrices in (3.8) and let

$$\Psi(\bar{p}) := (\mathcal{H}_0)^{-1} \Psi_0 \tag{3.54}$$

where

$$\Psi_0 := \mathcal{H}_1 e^{-\tilde{\mu} \bar{h}_1} \tag{3.55}$$

if $\bar{\sigma} = 1$, and, if $\bar{\sigma} \geq 2$,

$$\Psi_0 := \Psi_1(\theta^+) \tag{3.56}$$

where

$$\Psi_1(\theta) := \mathcal{H}_1 e^{-\tilde{\mu} \bar{h}_1} + \sum_{i=2}^{\bar{\sigma}} \mathcal{H}_i e^{-\tilde{\mu} \bar{h}_i} e^{j\theta_i} \tag{3.57}$$

where $\theta := [\theta_2, \ldots, \theta_{\bar{\sigma}}] \in [0, 2\pi]^{\bar{\sigma}-1}$ and $\theta^+ \in [0, 2\pi]^{\bar{\sigma}-1}$ is such that

$$\rho\left((\mathcal{H}_0)^{-1}\,\Psi_1(\theta^+)\right) \geq \rho\left((\mathcal{H}_0)^{-1}\,\Psi_1(\theta)\right), \quad \forall\theta \in [0, 2\pi]^{\bar{\sigma}-1}.$$

Then, let λ be an eigenvalue of $\Psi(\bar{p})$ such that $|\lambda| = \rho(\Psi(\bar{p}))$. Thus, λ is the largest (in magnitude) eigenvalue of $\Psi(\bar{p})$ and

$$\gamma_{\tilde{\mu}} = |\lambda|. \tag{3.58}$$

Note that, if there are more than one such eigenvalue which corresponds to $\rho(\Psi(\bar{p}))$, then, any one of them can be used to determine $\rho(\Psi(\bar{p}))$. Then, there exist non-zero vectors $\tilde{u}, \tilde{v} \in \mathbb{C}^{\tilde{\bar{n}}}$ such that

$$\tilde{u}^*\Psi(\bar{p}) = \lambda\tilde{u}^* \tag{3.59a}$$

$$\Psi(\bar{p})\tilde{v} = \lambda\tilde{v} \tag{3.59b}$$

and $\tilde{u}^*\tilde{v} = 1$. Note that, since Ψ depends on $\bar{p}$, λ depends on $\bar{p}$ as well. Accordingly, $\tilde{u}$ and $\tilde{v}$ depend on $\bar{p}$ too. Let $\hat{h}$ be fixed and take the derivative of (3.59b) with respect to p_j, $j = 1, \ldots, N$, and multiply both sides from left by $\tilde{u}^*$ to obtain

$$\tilde{u}^*\frac{\partial\Psi}{\partial p_j}\tilde{v} + \tilde{u}^*\Psi\Big(\frac{\partial\tilde{v}}{\partial p_j} + \frac{\partial\tilde{v}}{\partial\lambda}\frac{\partial\lambda}{\partial p_j}\Big) = \tilde{u}^*\frac{\partial\lambda}{\partial p_j}\tilde{v} + \tilde{u}^*\lambda\Big(\frac{\partial\tilde{v}}{\partial p_j} + \frac{\partial\tilde{v}}{\partial\lambda}\frac{\partial\lambda}{\partial p_j}\Big), \tag{3.60}$$

then, use (3.59a) to obtain

$$\tilde{u}^*\frac{\partial\Psi}{\partial p_j}\tilde{v} = \tilde{u}^*\frac{\partial\lambda}{\partial p_j}\tilde{v} \tag{3.61}$$

which further reduces to

$$\tilde{u}^*\frac{\partial\Psi}{\partial p_j}\tilde{v} = \frac{\partial\lambda}{\partial p_j} \tag{3.62}$$

since $\frac{\partial\lambda}{\partial p_j}$ is scalar and $\tilde{u}^*\tilde{v} = 1$. Now, using $|\lambda| = (\lambda\bar{\lambda})^{1/2}$, where $\bar{\lambda}$ indicates the complex conjugate of λ, we take the derivative of both sides of (3.58) with respect to p_j, $j = 1, \ldots, N$ to obtain

$$\frac{\partial\gamma_{\tilde{\mu}}}{\partial p_j} = \Big(\frac{\partial\lambda}{\partial p_j}\bar{\lambda} + \lambda\frac{\partial\bar{\lambda}}{\partial p_j}\Big)\frac{1}{2(\lambda\bar{\lambda})^{1/2}}\,, \tag{3.63}$$

Then, using

$$\left(\frac{\partial\lambda}{\partial p_j}\bar{\lambda}+\lambda\frac{\partial\bar{\lambda}}{\partial p_j}\right)=2\mathrm{Re}\left(\frac{\partial\lambda}{\partial p_j}\bar{\lambda}\right)$$

and (3.62), we obtain

$$\frac{\partial\gamma_{\tilde{\mu}}}{\partial p_j}=\frac{1}{|\lambda|}\mathrm{Re}\left(\bar{\lambda}\tilde{u}^*\frac{\partial\Psi}{\partial p_j}\tilde{v}\right) \tag{3.64}$$

where

$$\frac{\partial\Psi}{\partial p_j}=\frac{\partial(\mathcal{H}_0)^{-1}}{\partial p_j}\Psi_0+(\mathcal{H}_0)^{-1}\frac{\partial\Psi_0}{\partial p_j} \tag{3.65}$$

which can be derived by taking the derivative of (3.54) with respect to p_j, $j=1,\ldots,N$. Keep in mind that both $\mathcal{H}_0$ and Ψ_0 might depend on p_j, $j=1,\ldots,N$. In particular, if $\mathcal{H}_0$ depends on p_j, $j=1,\ldots,N$, then

$$\frac{\partial(\mathcal{H}_0)^{-1}}{\partial p_j}=-(\mathcal{H}_0)^{-1}\frac{\partial(\mathcal{H}_0)}{\partial p_j}(\mathcal{H}_0)^{-1}. \tag{3.66}$$

Then, the gradient of $\gamma_{\tilde{\mu}}(\bar{p})$ with respect to p is obtained as

$$\nabla_p\gamma_{\tilde{\mu}}(\bar{p}))=\left[\tfrac{\partial\gamma_{\tilde{\mu}}}{\partial p_1}\ldots\tfrac{\partial\gamma_{\tilde{\mu}}}{\partial p_N}\right]. \tag{3.67}$$

Now, let p be fixed and take the derivative of (3.59b) with respect to $\hat{h}_k$, $k=1,\ldots,\hat{\sigma}$. By proceeding as for p, we obtain

$$\frac{\partial\gamma_{\tilde{\mu}}}{\partial\hat{h}_k}=\frac{1}{|\lambda|}\mathrm{Re}\left(\bar{\lambda}\tilde{u}^*\frac{\partial\Psi}{\partial\hat{h}_k}\tilde{v}\right) \tag{3.68}$$

where

$$\frac{\partial\Psi}{\partial\hat{h}_k}=(\mathcal{H}_0)^{-1}\frac{\partial\Psi_0}{\partial\hat{h}_k} \tag{3.69}$$

which can be derived by taking the derivative of (3.54) with respect to $\hat{h}_k$, $k=1,\ldots,\hat{\sigma}$. Keep in mind that $\mathcal{H}_0$ does not depend on $\hat{h}_k$, $k=1,\ldots,\hat{\sigma}$, but Ψ_0 does. Therefore,

$$\frac{\partial\Psi_0}{\partial\hat{h}_k}=-\tilde{\mu}\mathcal{H}_ie^{-\tilde{\mu}\bar{h}_i}e^{j\theta_i^+} \tag{3.70}$$

where i is such that $\bar{h}_i=\hat{h}_k$ and $\theta_i^+:=0$ if $i=1$. Then, the gradient of $\gamma_{\tilde{\mu}}(\bar{p})$ with respect to $\hat{h}$ is obtained as

$$\nabla_{\hat{h}}\gamma_{\tilde{\mu}}(\bar{p}))=\left[\tfrac{\partial\gamma_{\tilde{\mu}}}{\partial\hat{h}_1}\ldots\tfrac{\partial\gamma_{\tilde{\mu}}}{\partial\hat{h}_{\hat{\sigma}}}\right]. \tag{3.71}$$

As a result, the gradient vector of $\gamma_{\tilde{\mu}}(\bar{p})(\bar{p})$ can be constructed as

$$\nabla\gamma_{\tilde{\mu}}(\bar{p}) = \begin{bmatrix}\nabla_p\gamma_{\tilde{\mu}}(\bar{p}) & \nabla_{\hat{h}}\gamma_{\tilde{\mu}}(\bar{p})\end{bmatrix}. \tag{3.72}$$

3.4. Examples

In this section, the analysis and controller design methods are illustrated through various control problems. Each subsection begins with an introduction of the given open-loop system, then continues with the open-loop system analysis, controller design, and closed-loop system analysis phases, respectively. The main objective is not only to stabilize (or strongly stabilize) the given system but also to achieve the minimum spectral abscissa value, i.e. maximum exponential decay rate, with a controller whose complexity is as low as possible. However, even if a stabilizing controller is found, we also investigate the performances of more complex controllers by gradually increasing the controller complexity. In addition to the achieved spectral abscissa value, specific performance measures and/or time-domain responses are also obtained to make fair comparisons when needed. In particular, the examples presented in Subsection 3.4.1, 3.4.2, and 3.4.3 are considered as a benchmark. The examples presented in Subsection 3.4.4 and 3.4.5, on the other hand, are dedicated to showing the effectiveness of the design method in more complex systems. Note that all the analysis and controller design phases are performed by using DrStabilization v1.0. We should also note that some parts of this section have also been presented in [55], [51] and [117].

3.4.1. Two-mass-spring System

We consider an oscillatory mechanical system that consists of two masses attached to each other by a spring, as illustrated in Figure 3.1. It is a simple yet challenging benchmark control problem introduced in [118] for the first time. The external force acts on mass 1 while the position of mass 2 is measured. In order to describe the dynamics of the system in the state-space form, we let $x := \begin{bmatrix}x_1 & x_2 & x_3 & x_4\end{bmatrix}^T$ be the state vector where x_1 and x_2 are the displacements, and, x_3 and x_4 are the velocities of mass 1 and mass 2,

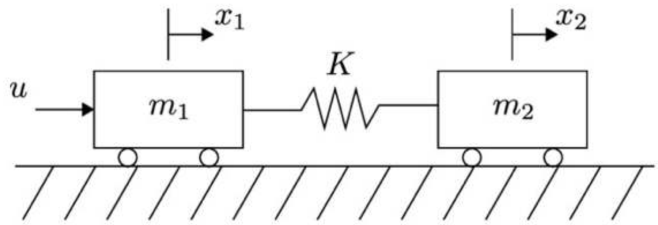

Figure 3.1: *Two-mass-spring system.*

respectively. Then,

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -K/m_1 & K/m_1 & 0 & 0 \\ K/m_2 & -K/m_2 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1/m_1 \\ 0 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} x(t) \tag{3.73}$$

where $m_1 = m_2 = 1$ and $K = 1$. For this particular system, the analytic solution to the achievable maximum exponential decay rate, i.e., the achievable minimum spectral abscissa value, problem is considered in [119]. It has been shown that to be able to 0-stabilize the system, a delay-free controller whose differential order is at least 2 is needed. Then, in [27] using HIFOO, a 2^{nd} order delay-free controller is designed. The achieved minimum spectral abscissa value is given in Table 3.1 which is also consistent with the result of [119]. In [55] on the other hand, stabilization by a delayed output feedback controller of the form (2.40) with a single time-delay (equivalently a time-delay controller of the form (2.35) with $l = 0$ and $\hat{\sigma} = 1$) was considered. By minimizing the spectral abscissa, the controller

$$u(t) = -0.5391y(t) + 0.3471y(t - 3.0693) \tag{3.74}$$

was obtained. Here, we increase the complexity of the time-delay controller by increasing its differential order l in order to achieve better performance in the sense of minimal spectral abscissa value. We let each controller to have four time-delays, one (say $\hat{h}_1$) associated with the F matrix (thus we let $F_2 = F_3 = F_4 = 0$), one (say $\hat{h}_2$) associated with the G matrix (thus we let $G_1 = G_3 = G_4 = 0$), one (say $\hat{h}_3$) associated with the H

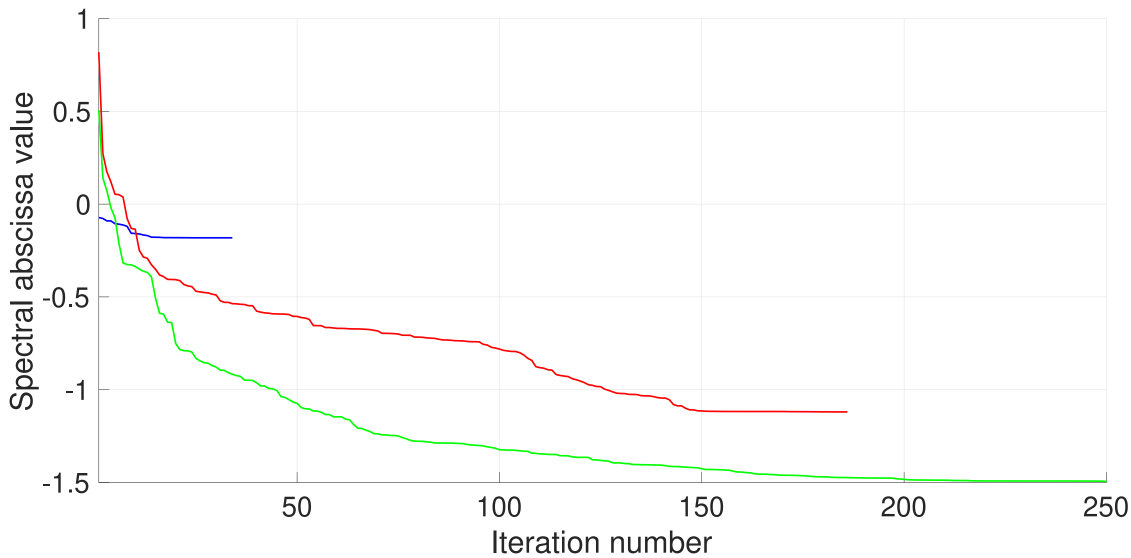

Figure 3.2: *The course of optimization during the design of controllers* (3.74) *(blue),* (3.75) *(red), and* (3.76) *(green).*

matrix (thus we let $H_1 = H_2 = H_4 = 0$), and one (say $\hat{h}_4$) associated with the K matrix (thus we let $K_1 = K_2 = K_3 = 0$). By choosing $l = 1$, we obtain the following controller:

$$\begin{aligned} \dot{z}(t) =& -2.8090z(t) + 1.4096y(t) - 0.7340z(t-0.8577) + 0.1505y(t-0.3404) \\ u(t) =& 1.2211z(t) + 0.0887y(t) + 2.0852z(t-1.9280) - 1.6307y(t-0.9497) \end{aligned} \tag{3.75}$$

and by choosing $l = 2$, we obtain the following controller:

$$\begin{aligned} \dot{z}(t) =& \begin{bmatrix} -2.0934 & -0.9186 \\ -0.9180 & -0.5602 \end{bmatrix} z(t) + \begin{bmatrix} 0.5979 \\ 1.4804 \end{bmatrix} y(t) \\ &+ \begin{bmatrix} -1.1351 & 1.7956 \\ -0.1061 & -1.2533 \end{bmatrix} z(t-0.4242) + \begin{bmatrix} 0.4585 \\ -0.4138 \end{bmatrix} y(t-1.0244) \\ u(t) =& \begin{bmatrix} -0.9221 & 0.4633 \end{bmatrix} z(t) - 0.4891y(t) + \begin{bmatrix} 1.1305 & -1.1145 \end{bmatrix} z(t-1.4136) \\ &+ 0.5041y(t-0.3472). \end{aligned} \tag{3.76}$$

The course of optimization during the design of each controller is shown in Figure 3.2. As shown there, as the complexity of the controller is increased, more iterations are needed to design the controller. However, the spectral abscissa value is significantly improved. The closed-loop modes under each controller are shown in Figure 3.3. The closed-loop spectral abscissa value achieved by each controller is shown in Table 3.1

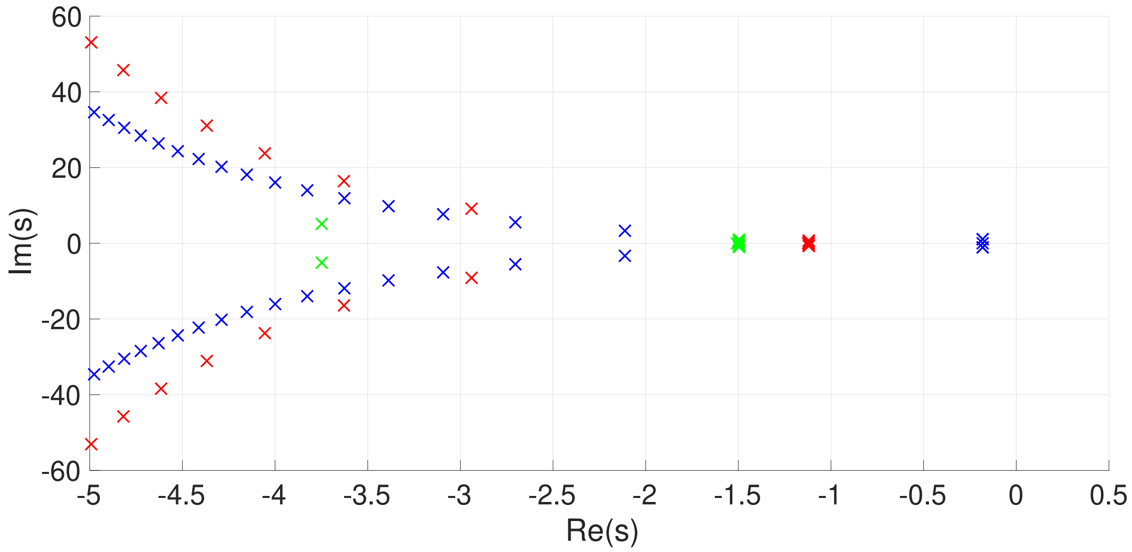

Figure 3.3: *The closed-loop* -5*-modes of the system* (3.73) *under the controllers* (3.74) *(blue),* (3.75) *(red), and* (3.76) *(green).*

Table 3.1: *The minimum spectral abscissa values achieved by various controllers for the example in Subsection 3.4.1. The improvements on the spectral abscissa values are calculated against the delay-free controller.*

Controller	**Min. Spec. Abs.**	**Imp.**
Delay-free, $l = 2$, [27]	-0.7706	-
Delayed, $l = 0$, (3.74)	-0.1816	-
Delayed, $l = 1$, (3.75)	-1.1199	45%
Delayed, $l = 2$, (3.76)	-1.4963	94%

together with the spectral abscissa value achieved by the 2^{nd} order controller designed in [27]. Although the minimum order of a stabilizing delay-free controller is 2, here we could stabilize the system by a time-delay controller with 0 differential order using only one time-delay. However, as shown in Table 3.1, by increasing the differential order of the controller it is possible to further decrease the spectral abscissa value and obtain a significant improvement over that achieved by the 2^{nd} order delay-free controller.

We also examine the time-domain performance of the four controllers. The closed-loop step responses under each controller is shown in Figure 3.4. Although the delay-free controller of [27] causes a significant undershoot, the time-delay controllers with $l = 0$ and $l = 2$ do not cause any undershoot, and the time-delay controller with $l = 1$ causes a much smaller undershoot. Furthermore, the maximum overshoot under each time-delay controller is less than the maximum overshoot of the delay-free controller of [27] and

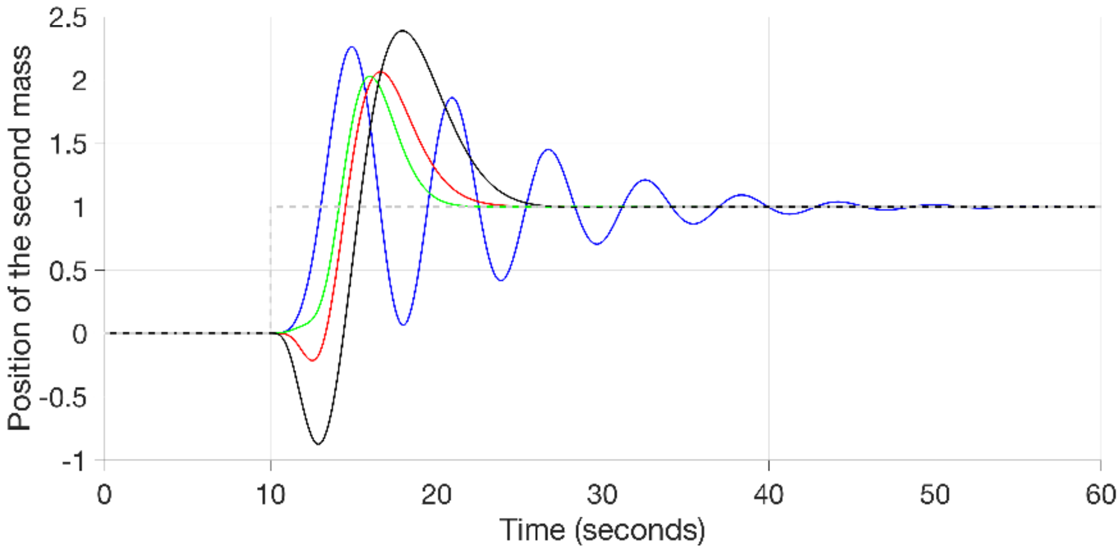

Figure 3.4: *Closed-loop step responses under the delay-free controller of [27] (black) and the time-delay controllers described by* (3.74) *(blue),* (3.75) *(red), and* (3.76) *(green).*

decreases as the differential order of the time-delay controller increases. Finally, the response under the time-delay controllers with $l = 1$ and $l = 2$ settles much faster than the one under the delay-free controller of [27], as also suggested by the spectral abscissa values given in Table 3.1.

3.4.2. Acrobot Robotic System

We consider the strong stabilization problem of an underactuated robot called Acrobot which is a two-link planar robot moving in a vertical plane. The system description of Acrobot is given in [120], where the associated transfer function of the system linearized at the upright equilibrium point is obtained as

$$P(s) = \frac{-1.3545(s-1.281)(s+1.281)}{(s-6.101)(s-2.24)(s+6.101)(s+2.24)} \tag{3.77}$$

Clearly, the system has two stable, $s_1 = -2.24$ and $s_2 = -6.101$, and two unstable, $s_1 = 2.24$ and $s_2 = 6.101$, modes. Furthermore, the system has two real finite blocking zeros, $z_1 = -1.281$ and $z_2 = 1.281$. Since $z_1 < 0$ and there are an even number (two) of modes to the right of z_2, as far as 0-stability is concerned, the system satisfies PIP. So, if we apply the strong μ-stabilizability test, for $\mu = 0$, it results $l^{\min} = 0$. Therefore the system is strongly 0-stabilizable. In [121], a stable 2^{nd} order delay-free controller is designed, and, the achieved spectral absicssa value is reported as -1.2806. On the other hand, in [122],

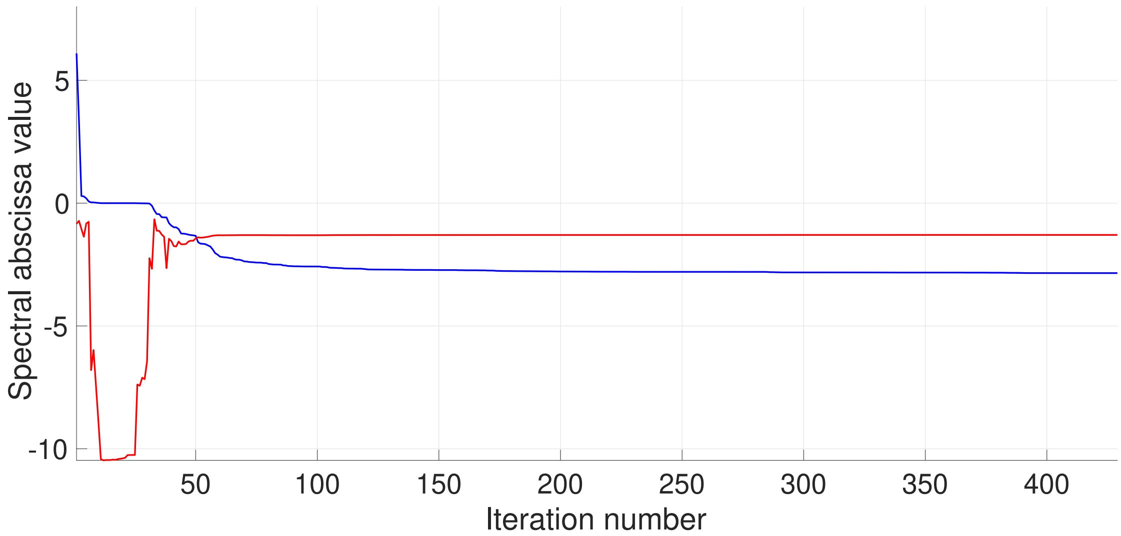

Figure 3.5: *Spectral abscissa values of the closed-loop system (blue) and of the controller (red) during the design of* (3.78).

a stable 3^{rd} order delay-free controller is designed, and, the achieved spectral abscissa value is calculated as -1.2726. Here, we aim to design $\hat{\mu}$-stable controllers for $\hat{\mu} = 0$.

First, we design a 0-stable 2^{nd} order delay-free controller in the form of (2.35) with $l = 2$ and $\hat{\sigma} = 0$. The designed controller is

$$\begin{aligned} \dot{z}(t) &= \begin{bmatrix} -15.2610 & 0.6496 \\ 11.3650 & -1.8223 \end{bmatrix} z(t) + \begin{bmatrix} -19.9390 \\ 24.7037 \end{bmatrix} y(t) \\ u(t) &= \begin{bmatrix} 49.3085 & 1.7497 \end{bmatrix} z(t) + 106.7103 y(t). \end{aligned} \tag{3.78}$$

The minimization of the closed-loop spectral abscissa value is shown in Figure 3.5, together with the spectral abscissa of the controller. The closed-loop modes are shown in Figure 3.6. The achieved spectral abscissa value is -2.8472 which is less than that achieved by both [121] and [122].

Next, we design a time-delay controller of the form (2.40) with a single time-delay. The designed controller is

$$u(t) = 64.1787 y(t) + 4.3205 y(t - 1.6695) \tag{3.79}$$

which is inherently $\hat{\mu}$-stable for any $\hat{\mu}$. The minimization of the closed-loop spectral abscissa value is shown in Figure 3.7 and the closed-loop modes are shown in Figure 3.6.

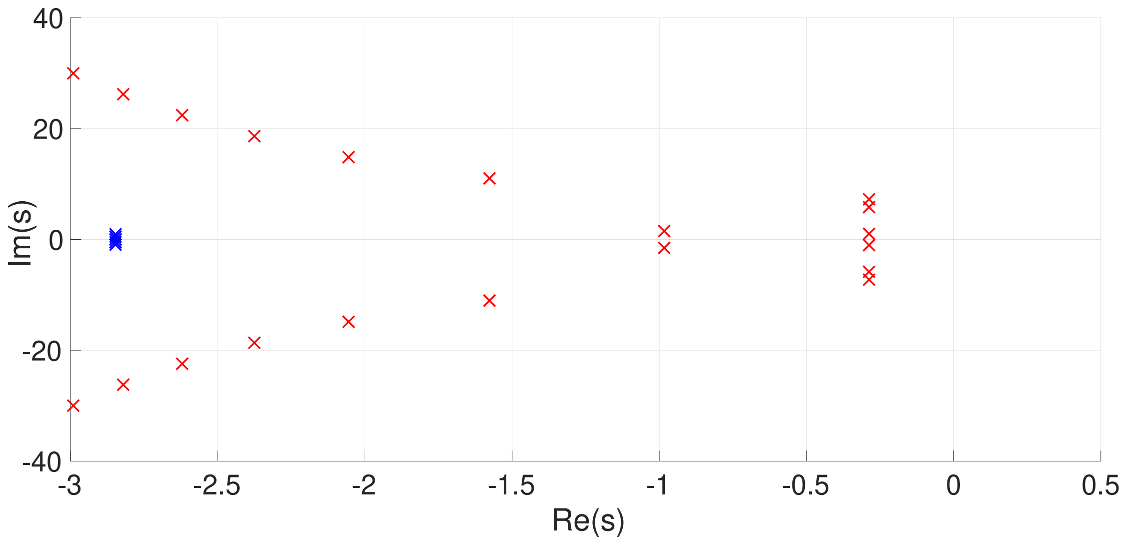

Figure 3.6: *The closed-loop modes of* (3.77) *under the controllers* (3.78) *(blue) and* (3.79) *(red).*

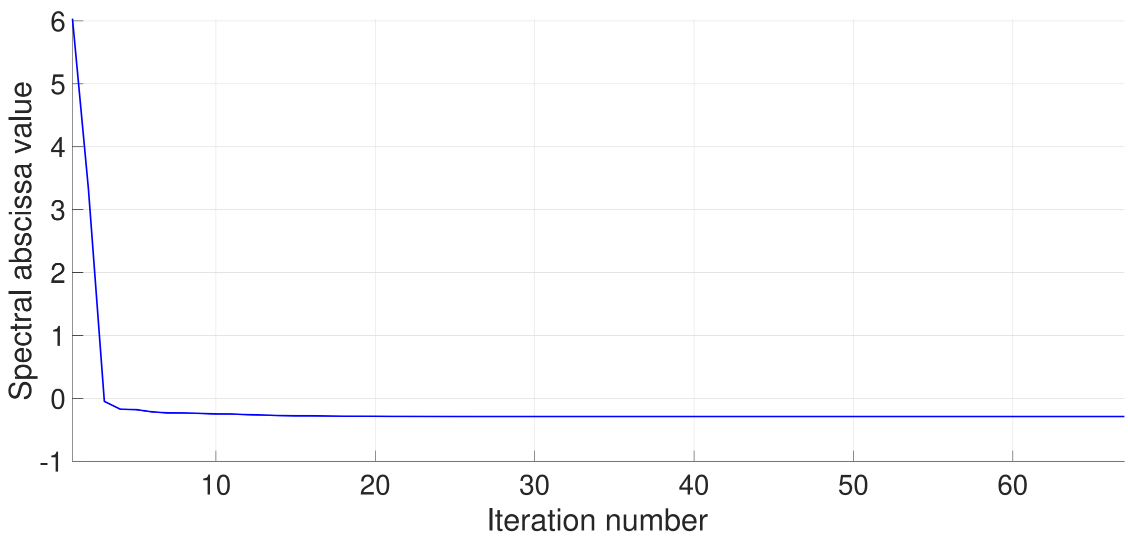

Figure 3.7: *The course of optimization during the design of the controller* (3.79).

The achieved spectral abscissa value is -0.2874. Even though this value is greater than that achieved by (3.78), (3.79) requires only two gains and one delay element, where as (3.78) requires two integrators and nine gains (the number of gains, however, could be reduced to five if a canonical form was used - see [49]).

3.4.3. Input-delayed System

We consider the input-delayed system described as

$$\dot{x}(t) = A_0 x(t) + B_1 u(t-5) \tag{3.80}$$

Table 3.2: *The minimum spectral abscissa values achieved by various controllers for the example in Subsection 3.4.3. The improvements on the spectral abscissa values are calculated against the 0-order delay-free controller of [29].*

Controller	**Min. Spec. Abs.**	**Imp.**
Delay-free, $l = 0$, [29]	-0.1489	-
Delay-free, $l = 1$, [29]	-0.2293	54%
Delay-free, $l = 2$, [29]	-0.2682	80%
Delay-free, $l = 3$, [29]	-0.4575	207%
Delayed, $l = 0$, (3.81)	-0.2492	67%
Delayed, $l = 1$, (3.82)	-0.3449	232%
Delayed, $l = 2$, (3.83)	-0.5171	347%

where

$$A_0 = \begin{bmatrix} -0.08 & -0.03 & 0.2 \\ 0.2 & -0.04 & -0.005 \\ -0.06 & 0.2 & -0.07 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.1 \\ -0.2 \\ 0.1 \end{bmatrix},$$

which was considered in [29], where it was assumed that the whole state vector is measurable. Thus, here too, we let $y(t) = x(t)$. The open-loop system modes are $s_1 = 0.1081$ and $s_{1,2} = -0.1490 \pm$ j0.2015, thus the system is unstable. In [29], using TDS_STABIL, delay-free controllers in the form of (2.37) were designed. The achieved spectral abscissa values of [29] are given in Table 3.2. Here, on the other hand, we design controllers of the form (2.35), for $l = 0$, 1, and 2. Not that, for the 0^{th} order controller, we consider the same time delay in all the output channels. For the 1^{st} and 2^{nd} order controllers, on the other hand, as in Subsection 3.4.1, we consider four time delays, each associated with a different controller matrix. The designed controllers are as follows:

$$\begin{aligned} u(t) = &\begin{bmatrix} 1.4567 & 3.4589 & 1.3211 \end{bmatrix} y(t) \\ &+ \begin{bmatrix} -0.9218 & -2.7055 & -0.6651 \end{bmatrix} y(t - 0.2917) \end{aligned} \tag{3.81}$$

for $l = 0$;

$$
\begin{aligned}
\dot{z}(t) =& -1.3606z(t) + \begin{bmatrix} -0.2784 & 1.3153 & 0.1485 \end{bmatrix} y(t) - 1.0789z(t - 2.7422) \\
& + \begin{bmatrix} 1.0194 & -0.6851 & -1.5088 \end{bmatrix} y(t - 0.9014) \\
u(t) =& 0.4263z(t) + \begin{bmatrix} 1.4360 & 0.4853 & 0.5869 \end{bmatrix} y(t) - 0.6845z(t - 2.2906) \\
& + \begin{bmatrix} -0.6712 & 0.8023 & -0.1375 \end{bmatrix} y(t - 0.0446)
\end{aligned}
\tag{3.82}
$$

for $l = 1$; and

$$
\begin{aligned}
\dot{z}(t) =& \begin{bmatrix} -2.2475 & -1.3285 \\ 2.5137 & 0.6410 \end{bmatrix} z(t) + \begin{bmatrix} 0.2049 & -0.0311 & -0.2119 \\ 0.7912 & 1.7789 & 1.5988 \end{bmatrix} y(t) \\
& + \begin{bmatrix} -0.0741 & -1.5393 \\ 0.3514 & 1.3903 \end{bmatrix} z(t - 1.9559) \\
& + \begin{bmatrix} -0.6628 & 0.2405 & -1.4212 \\ 0.6263 & -1.0781 & 0.8367 \end{bmatrix} y(t - 0.1578) \\
u(t) =& \begin{bmatrix} 0.0927 & 1.7344 \end{bmatrix} z(t) + \begin{bmatrix} -0.4296 & -0.3518 & -0.3189 \end{bmatrix} y(t) \\
& + \begin{bmatrix} -0.2418 & -0.6569 \end{bmatrix} z(t - 1.3349) \\
& + \begin{bmatrix} 0.4174 & 0.4872 & 0.6127 \end{bmatrix} y(t - 0.0427)
\end{aligned}
\tag{3.83}
$$

for $l = 2$.

The course of optimization during the design of each controller is shown in Figure 3.8. As in the previous example, as the complexity of the controller is increased, more iterations are needed to design the controller. However, the spectral abscissa value is improved significantly. The closed-loop modes under each controller is shown in Figure 3.9 and the spectral abscissa values achieved are shown in Table 3.2. As shown there, comparing to the delay-free controllers of [29], the time-delay controllers designed here made significant improvements in the spectral abscissa values.

Next, we design a controller of the form (2.40), where different time delays are introduced in each output channel. Such a controller can be described as

$$
u(t) = K_0 y(t) + K_1 y(t - \hat{h}_1) + K_2 y(t - \hat{h}_2) + K_3 y(t - \hat{h}_3) \tag{3.84}
$$

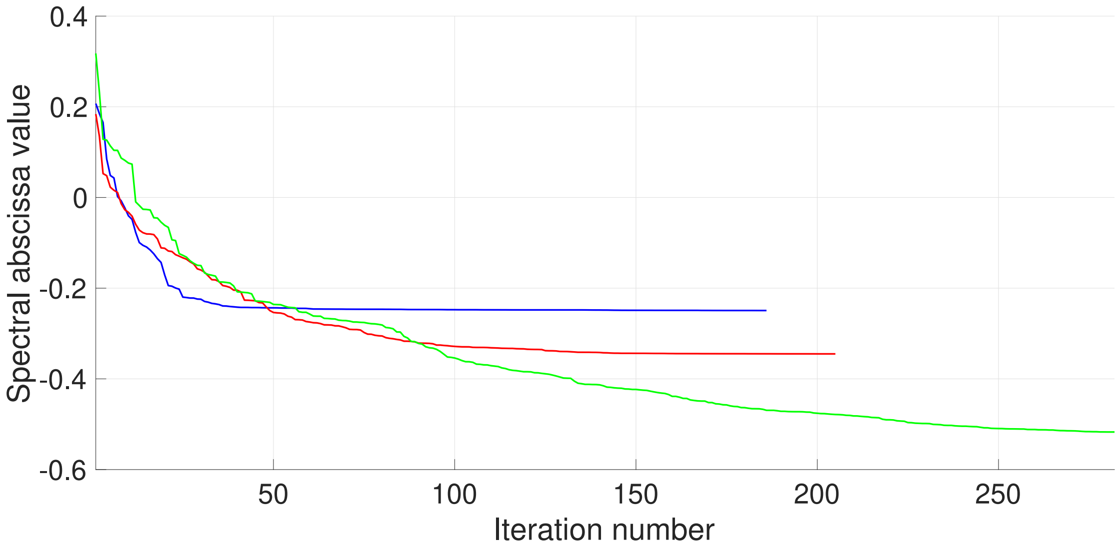

Figure 3.8: *The course of optimization during the design of controllers* (3.81) *(blue),* (3.82) *(red), and* (3.83) *(green).*

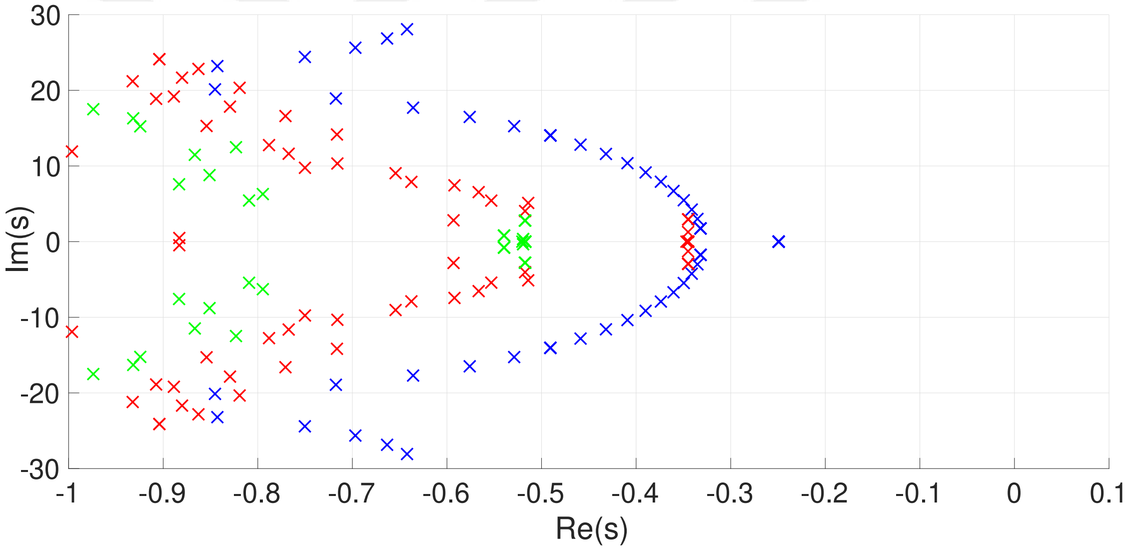

Figure 3.9: *The closed-loop* -1*-modes of the system* (3.80) *under the controllers* (3.81) *(blue),* (3.82) *(red), and* (3.83) *(green).*

where all the elements of K_0 are free, however, the rest of them are structured as (using a similar notation with (2.41)) $K_1 := \begin{bmatrix} k_1^1 & 0 & 0 \end{bmatrix}$, $K_2 := \begin{bmatrix} 0 & k_2^2 & 0 \end{bmatrix}$, and $K_3 := \begin{bmatrix} 0 & 0 & k_3^3 \end{bmatrix}$ where k_1^1, k_2^2, and k_3^3 are free parameters while the rest of them are fixed to zero. The controller is designed by minimizing the spectral abscissa function as shown

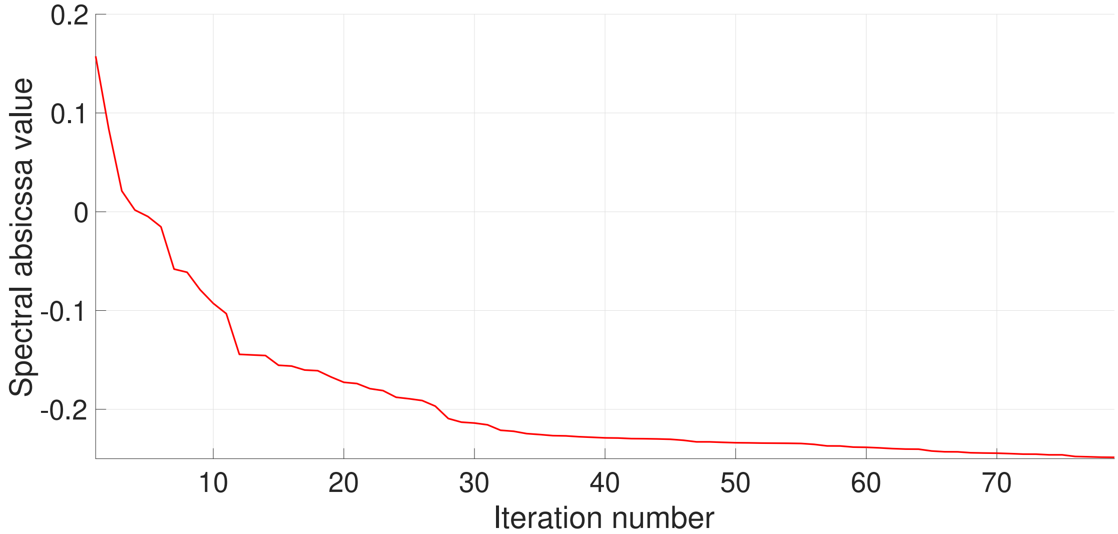

Figure 3.10: *Minimization of the spectral absicssa value during the design process of* (3.85).

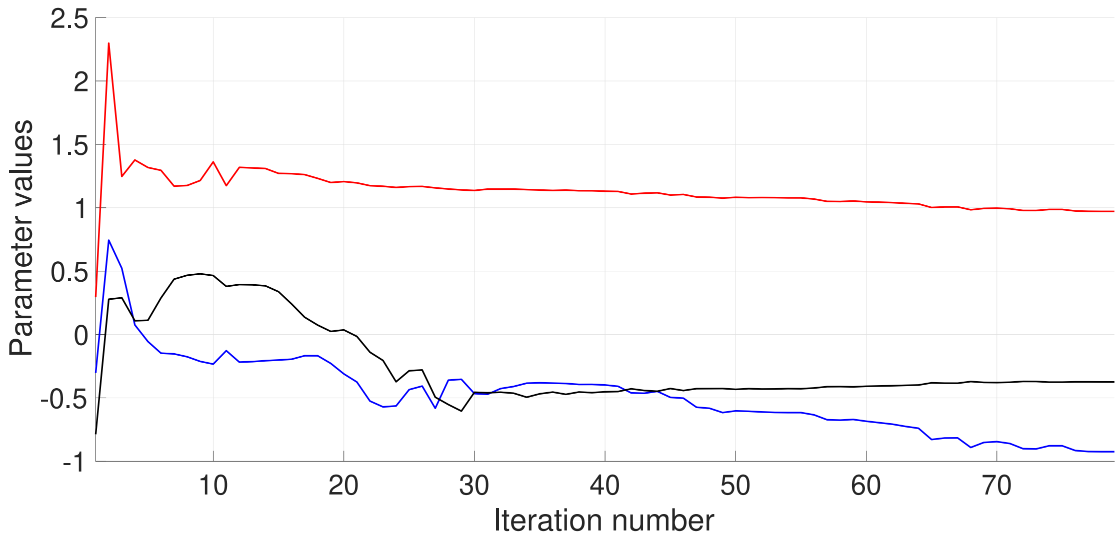

Figure 3.11: *Changes of parameters K_1^0 (blue line), K_2^0 (red line), and K_3^0 (black line) of* (3.85), *during the course of the optimization algorithm.*

in Figure 3.10 which yields

$$\begin{aligned} u(t) = &\begin{bmatrix} -0.9243 & 0.9709 & -0.3744 \end{bmatrix} y(t) \\ &+ 1.5059 y_1(t - 0.0015) + 0.1562 y_2(t - 2.9420) \\ &+ 0.9352 y_3(t - 1.8246). \end{aligned} \tag{3.85}$$

where the free parameters changed during the optimization process as shown in Figure 3.11, Figure 3.12, and Figure 3.13. By using the QPmR tool [114], we calculate all the -0.5-modes of the closed-loop system which are shown in Figure 3.14. The achieved

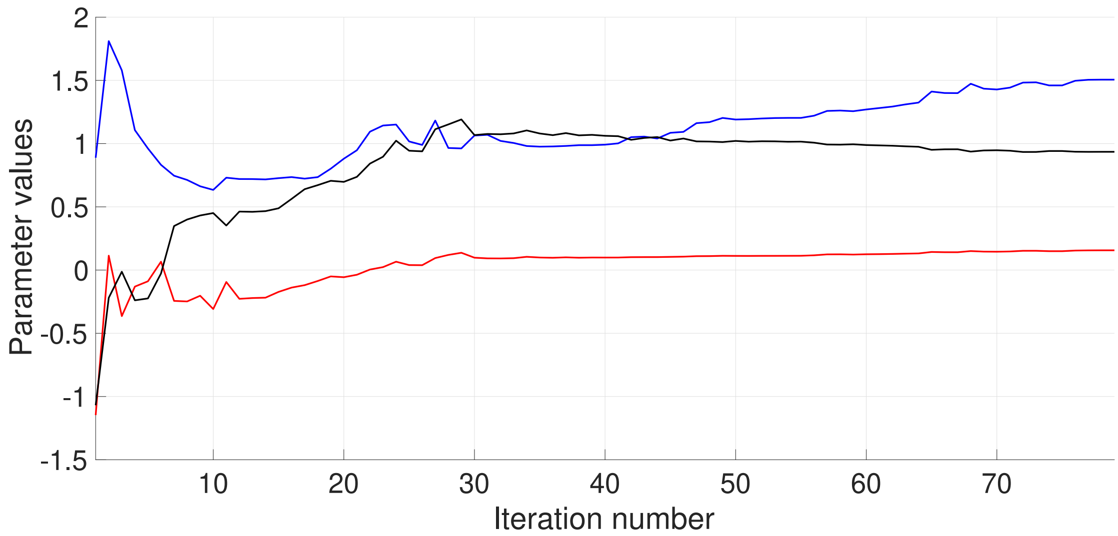

Figure 3.12: *Changes of parameters K_1^1 (blue line), K_2^2 (red line), and K_3^3 (black line) of* (3.85), *during the course of the optimization algorithm.*

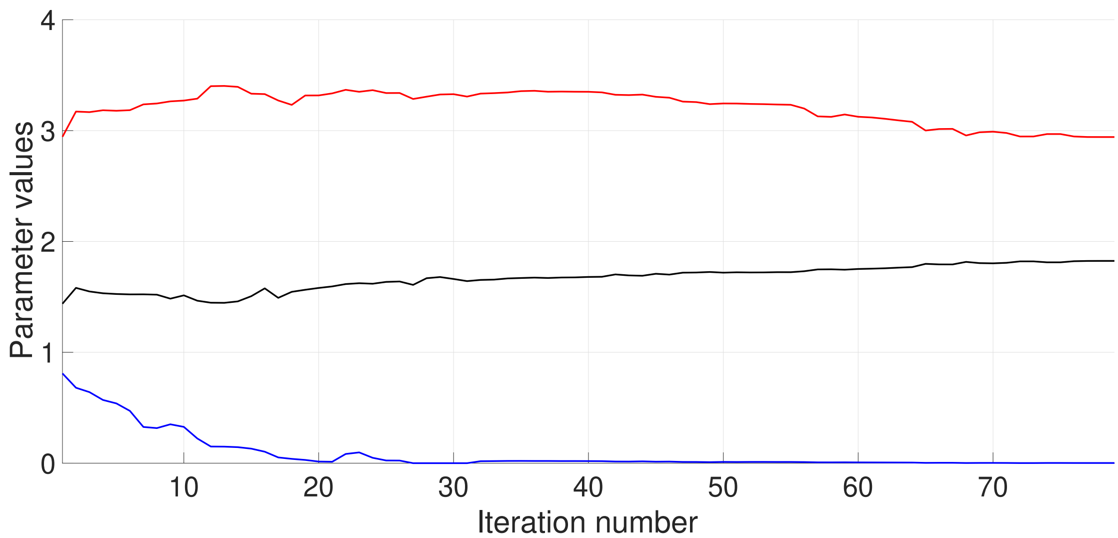

Figure 3.13: *Changes of parameters $\hat{h}_1$ (blue line), $\hat{h}_2$ (red line), and $\hat{h}_3$ (black line) of* (3.85), *during the course of the optimization algorithm.*

spectral abscissa value is -0.2486 which is almost equal to the one achieved by (3.81) where same time delay introduced in each output channel. Thus, we conclude that (3.81) is more favorable since even though both (3.81) and (3.85) require 6 gains to be implemented, (3.81) requires only delay element along with the gains while (3.85) requires three delay elements along with the gains. Furthermore, (3.81) has 7 free parameters to be determined while (3.85) has 9. Thus, (3.81) is also favorable than (3.85) as long as the controller design is concerned.

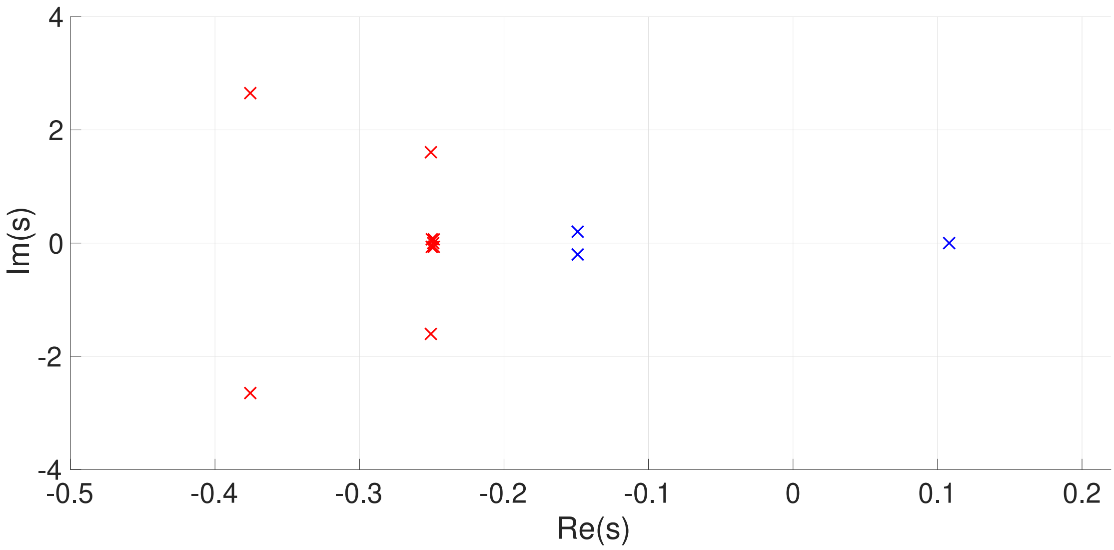

Figure 3.14: *The open-loop system modes (blue) of* (3.80) *and the closed-loop system modes associated with the controller described by* (3.85) *(red).*

3.4.4. Neutral Type System

We consider the LTI neutral time-delay system, described as

$$\dot{\tilde{x}}(t) + \tilde{E}_2\dot{\tilde{x}}(t-0.7) + \tilde{E}_3\dot{\tilde{x}}(t-1.7) = \tilde{A}_0\tilde{x}(t) + \tilde{B}_1 u(t-0.5) \tag{3.86}$$

where

$$\tilde{E}_2 = \begin{bmatrix} 0 & -0.2 & 0.4 \\ 0.5 & -0.3 & 0 \\ -0.2 & -0.7 & 0 \end{bmatrix}, \quad \tilde{E}_3 = \begin{bmatrix} 0.3 & 0.1 & 0 \\ 0 & -0.2 & 0 \\ -0.1 & 0 & -0.4 \end{bmatrix},$$

$$\tilde{A}_0 = \begin{bmatrix} -4.8 & 4.7 & 3 \\ 0.1 & 1.4 & -0.4 \\ 0.7 & 3.1 & -1.5 \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} 0.3 & 0.7 & 0.1 \end{bmatrix}^T,$$

which was considered in [26], where it was assumed that the whole vector $\tilde{x}(t)$ is measurable. Accordingly, a finite-dimensional state vector feedback controller was desgined. Therefore, we let $y(t) = \tilde{x}(t)$. The system can be put into the form (2.25) by defining $x(t) := \begin{bmatrix} w(t)^T & \tilde{x}(t)^T \end{bmatrix}^T$, where

$$w(t) := \tilde{x}(t) + \tilde{E}_2\tilde{x}(t-0.7) + \tilde{E}_3\tilde{x}(t-1.7)\ ,$$

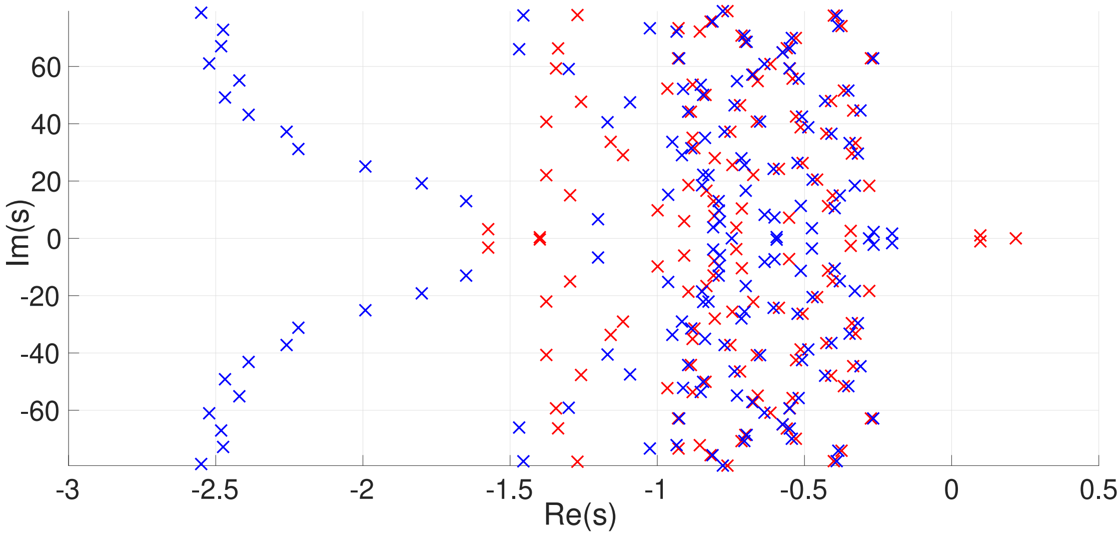

Figure 3.15: *Open-loop modes (red) of* (3.86) *and the closed-loop modes (blue) of* (3.86) *under the controller* (3.87).

$h_0 = 0$, $h_1 = 0.5$, $h_2 = 0.7$, $h_3 = 1.7$,

$$E = \begin{bmatrix} I_3 & 0 \\ 0 & 0_3 \end{bmatrix}, \ A_0 = \begin{bmatrix} 0 & \tilde{A}_0 \\ -I_3 & I_3 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0_3 & 0 \\ 0 & \tilde{E}_2 \end{bmatrix}, \ A_3 = \begin{bmatrix} 0_3 & 0 \\ 0 & \tilde{E}_3 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} \tilde{B}_1^T & 0_{1\times 3} \end{bmatrix}^T, C_0 = \begin{bmatrix} 0_3 & I_3 \end{bmatrix},$$

and all other matrices as zero.

We calculate $C_D = -0.2751$. Thus, we can calculate Ω_ϵ by the method of [29], for any $\epsilon > -0.2751$. We choose $\epsilon = -0.25$ and determine $\Omega_{-0.25} = \{0.2180, 0.0976 \pm 1.0396j\}$. Hence, the system is not stable. By using the QPmR tool [114], we calculate all the -3-modes of (3.86) with imaginary parts between -80 and 80 and show them in Figure 3.15.

As stated in Section 3.1, C_D can not be reduced by any controller in the form of (2.35). Therefore, the spectral abscissa can be minimized only up to -0.2751. Note that, in [26], the continuous pole placement algorithm of [25] was used to design a 0-stabilizing controller. In this respect, the design process has been stopped as soon as $c < 0$ is achieved. Here, we aim μ-stability for $\mu = -0.2$ which is close enough to C_D. We apply the proposed optimization procedure to design a controller of the form (2.40) with a single time delay. The minimization of the spectral abscissa is shown in Figure 3.16. We stop the iterations as soon as μ-stability is achieved. The designed controller is

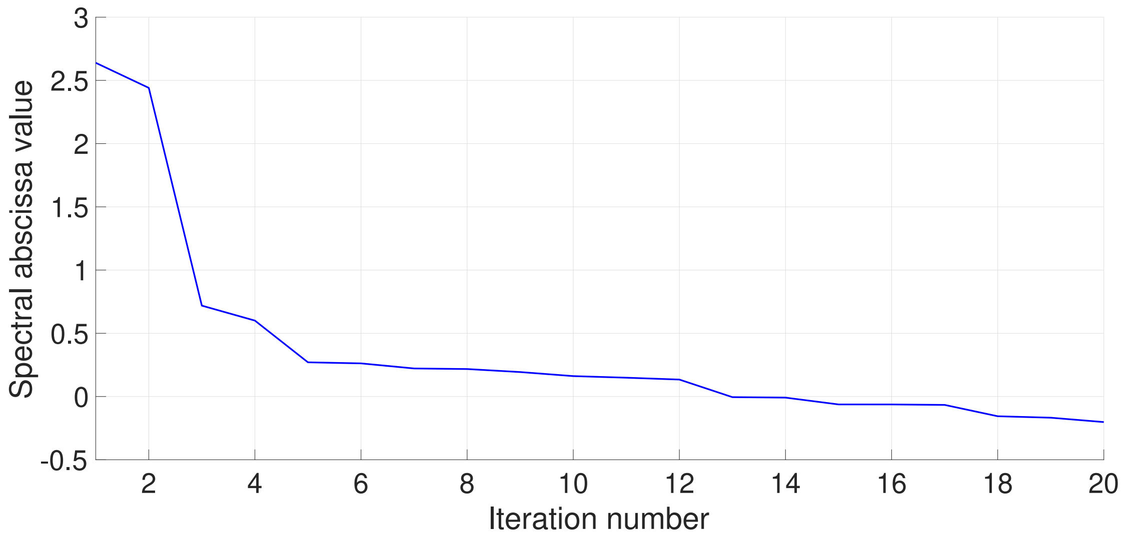

Figure 3.16: *The course of optimization during the design of the controller* (3.87).

$$\begin{aligned} u(t) = &\begin{bmatrix} -2.1150 & 0.1124 & 1.5747 \end{bmatrix} y(t) \\ &+ \begin{bmatrix} -0.4309 & 0.8969 & 0.2282 \end{bmatrix} y(t - 2.2582). \end{aligned} \tag{3.87}$$

The closed-loop -3-modes with imaginary parts between -80 and 80 (calculated using QPmR) are shown in Figure 3.15.

3.4.5. Multiple Delay System

This example is borrowed from [61], where the continuous pole placement-based controller design method is used. The system is a LTI retarded time-delay system, which has delays in both the states and the input:

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_2 x(t - h_2) x(t) + B_1 u(t - h_1) \\ y(t) &= C_0 x(t) \ , \end{aligned} \tag{3.88}$$

where $h_1 = 0.7$, $h_2 = 0.8$,

$$A_0 = \begin{bmatrix} -2.5 & 0.5 \\ 0 & -3.5 \end{bmatrix}, \ A_2 = \begin{bmatrix} 3 & -1 \\ 0 & 5 \end{bmatrix}, \ B_1 = \begin{bmatrix} -3.5 \\ b_2 \end{bmatrix}, \ \text{and} \ C_0 = \begin{bmatrix} c_1 & 2.25 \end{bmatrix}.$$

-2-modes of the system are shown in Figure 3.17. Because of the existence of two unstable modes, $s_1 = 0.3324$ and $s_2 = 0.1534$, the system is unstable. We consider two

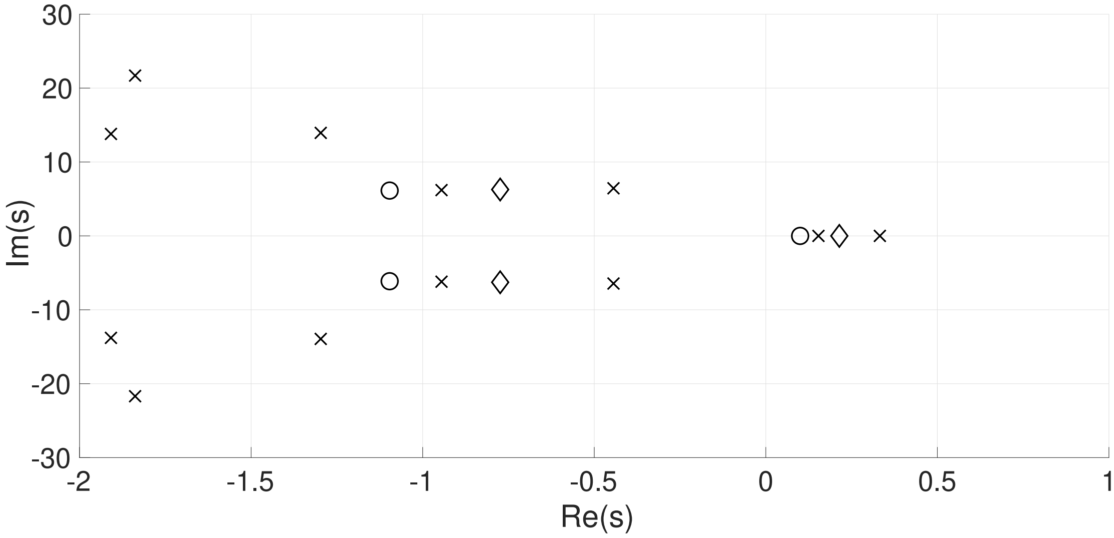

Figure 3.17: *Open-loop modes (x) and blocking zeros of* (3.88) *with* $b_2 = 3$*,* $c_1 = 0.5$ *(o), and with* $b_2 = 10$ *and* $c_1 = 2.5$ *(◇).*

sets of values for b_2 and c_1.

Scenario 1: As in [61], we let $b_2 = 3$ and $c_1 = 0.5$. In this case, the system has a real blocking zero at $z_1 = 0.1$ (see Figure 3.17). We apply the strong μ-stabilizability test for $\mu = 0$. Because of the existence of an even number of real modes to the right of z_1, the system satisfies PIP. Therefore, the system is strongly μ-stabilizable. Accordingly, the minimum possible order of the controller is determined as $l^{\min} = 0$ which is consistent with the result of [61] where the system is stabilized by an 0-order delayed controller. Here, we particularly focus on designing -0.5-stable controllers. First, we design a -0.5-stable 1^{st} order delay-free controller. The designed controller is

$$\begin{aligned} \dot{z}(t) &= -2.1498z(t) - 26.0069y(t) \\ u(t) &= 0.0863z(t) + 0.5779y(t) \ . \end{aligned} \tag{3.89}$$

The minimization of the spectral abscissa value is shown in Figure 3.18 together with the spectral abscissa of the controller. The resulting closed-loop system modes are shown in Figure 3.19. The achieved closed-loop spectral abscissa value is -0.1817, which is comparable with the one (-0.1825) obtained in [61]. Furthermore, the spectral abscissa value of the controller is -2.1498, which is well below the required -0.5.

Next, we design a -0.5-stable 1^{st} order time-delay controller. The designed con-

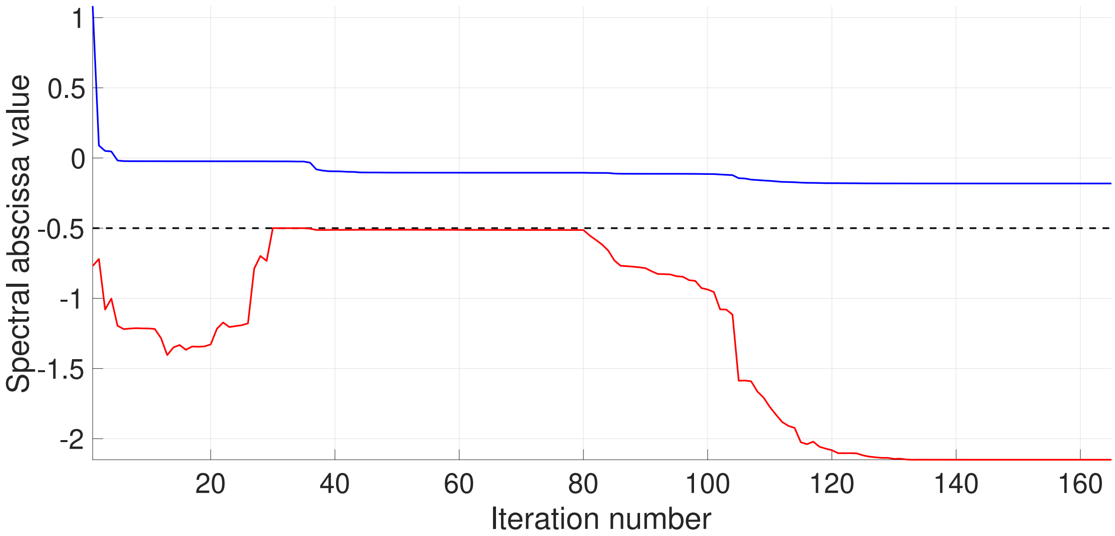

Figure 3.18: *Spectral abscissa values of the closed-loop system (blue) and of the controller (red) during the design of* (3.89).

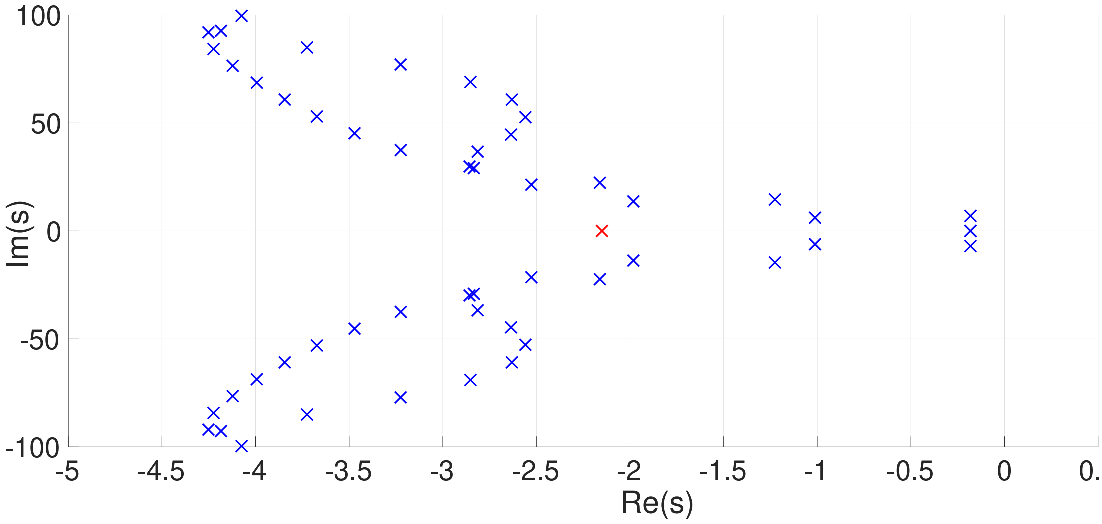

Figure 3.19: *Closed-loop* -5*-modes (blue) of* (3.88) *(Scenario 1) under the controller* (3.89) *and the controller mode (red).*

troller is

$$\begin{aligned}\dot{z}(t) =& -1.6651z(t) + 1.3208y(t) - 1.6670z(t-0.5301) - 0.3210y(t-0.8723)\\ u(t) =& 2.3699z(t) + 0.3812y(t) - 2.5378z(t-0.5559) - 0.7800y(t-0.2578).\end{aligned} \tag{3.90}$$

The minimization of the spectral abscissa value is shown in Figure 3.20 together with the spectral abscissa of the controller. The resulting closed-loop system modes are shown in Figure 3.21 together with those of the controller. The achieved closed-loop

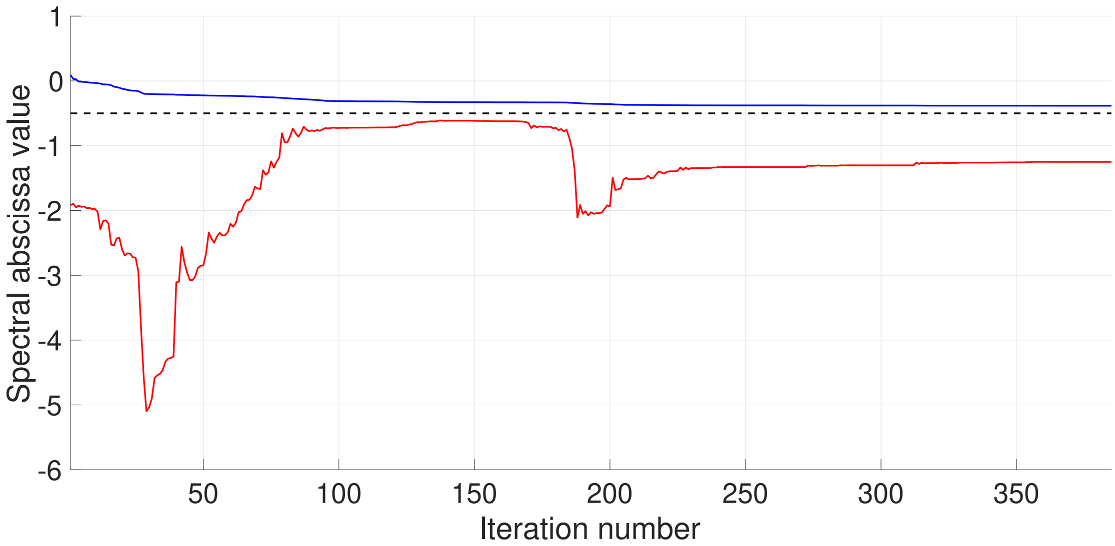

Figure 3.20: *Spectral abscissa values of the closed-loop system (blue) and of the controller (red) during the design of* (3.90).

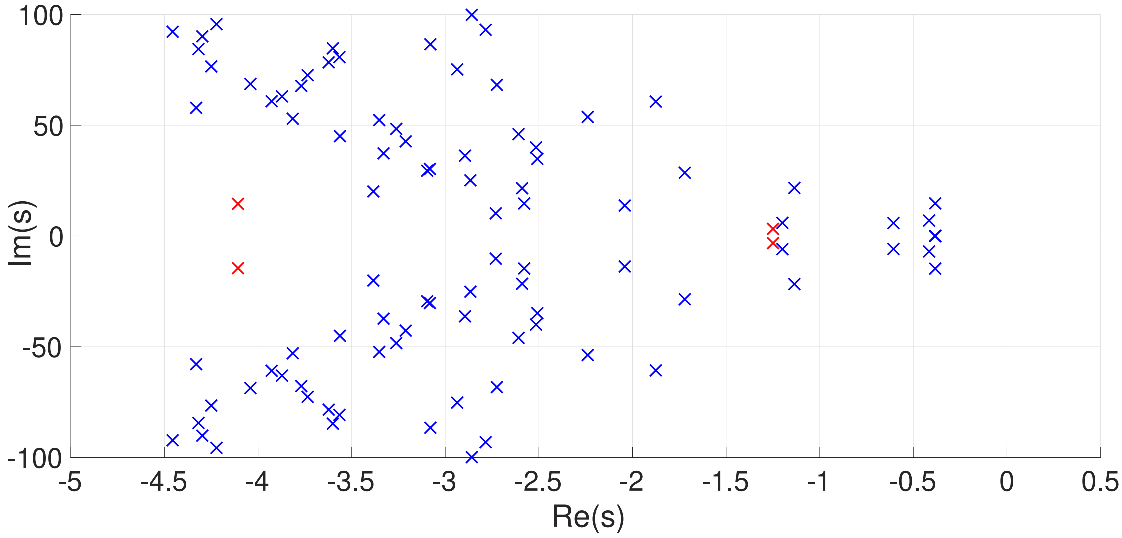

Figure 3.21: *Closed-loop* -5*-modes (blue) of* (3.88) *(Scenario 1) under the controller* (3.90) *and those of the controller (red).*

spectral abscissa value is -0.3833, which shows an improvement of 217% compared to the one achieved by the delay-free controller (3.89). Furthermore, the spectral abscissa value of the controller is -1.25, which is also well below the required -0.5.

Scenario 2: We let $b_2 = 10$ and $c_1 = 2.5$. In this case, the system has a real blocking zero at $z_1 = 0.2143$ (see Figure 3.17). Unlike Scenario 1, there exists an odd number (one) of real modes to the right of z_1. Therefore, the strong μ-stabilizability test yields $l^{min} = 1$ for $\mu = 0$. Thus, the system is not strongly μ-stabilizable for any $\mu \leq 0$ since the PIP is not satisfied. However, the system can be stabilized by an unstable controller that has at

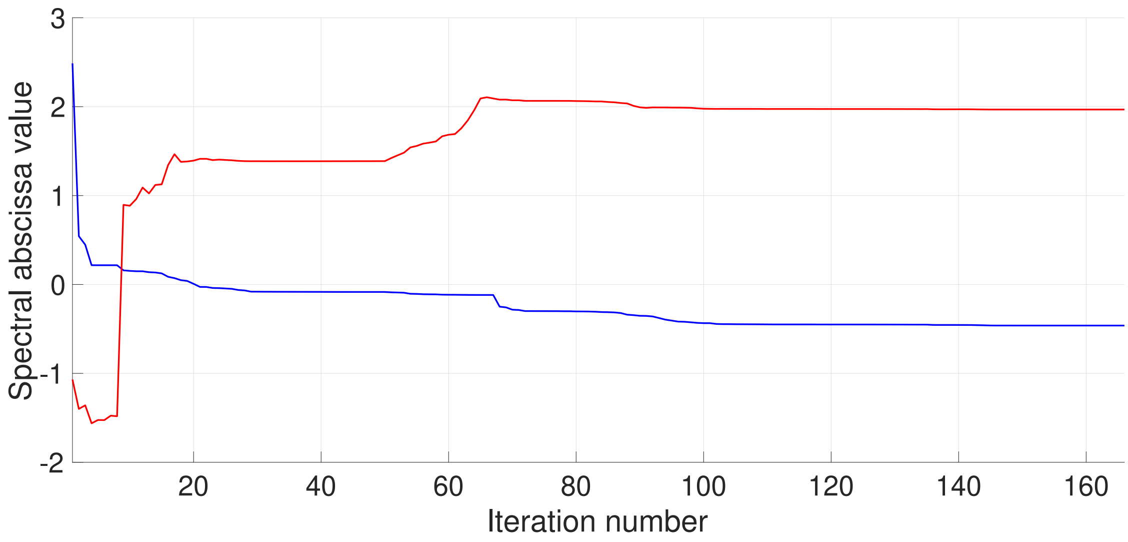

Figure 3.22: *Spectral abscissa values of the closed-loop system (blue) and of the controller (red) during the design of* (3.91).

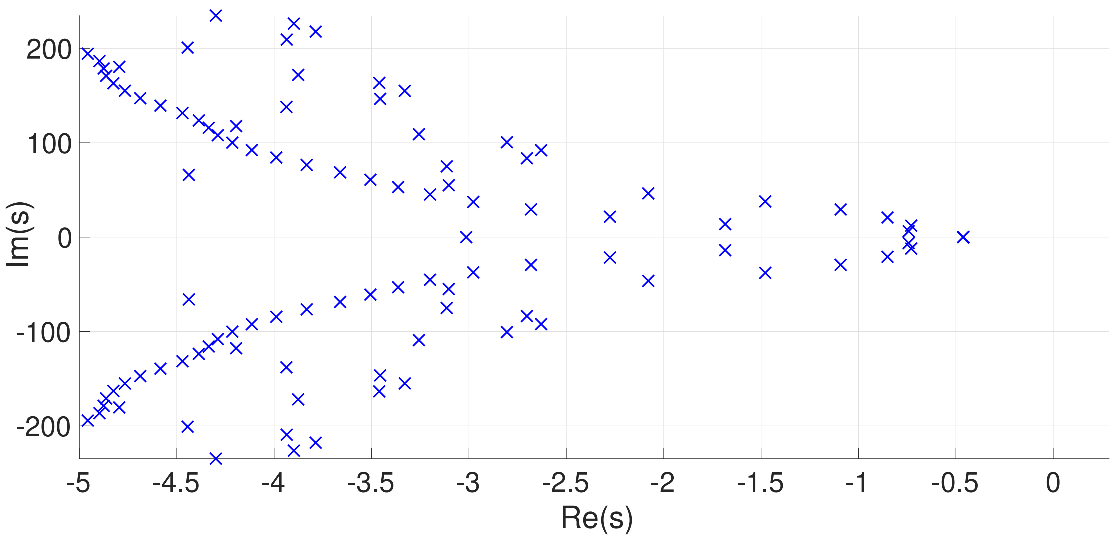

Figure 3.23: *Closed-loop* -5*-modes (blue) of* (3.88) *(Scenario 2) under the controller* (3.91).

least one real unstable controller mode located to the right of z_1. We design a 1^{st} order delay-free controller. The designed controller is

$$\begin{aligned} \dot{z}(t) &= 1.9678z(t) + 0.4593y(t) \\ u(t) &= -2.4336z(t) - 0.6404y(t) \end{aligned} \tag{3.91}$$

which has one unstable mode (at 1.9678) to the right of z_1, as required. The minimization of the spectral abscissa value is shown in Figure 3.22 together with the spectral abscissa of the controller. The resulting closed-loop system modes are shown in Figure 3.23. The achieved closed-loop spectral abscissa value is -0.4621.

4. DECENTRALIZED STABILIZATION AND STRONG STABILIZATION

The main objective of this chapter is to propose eigenvalue optimization-based stabilization and strong stabilization methods for LTI MIMO time-delay systems, described in (2.31), using decentralized LTI output feedback controllers introduced in Subsection 2.3.2. In Section 4.1, first, the closed-loop system is obtained, then, the stabilization problem is formulated. In Section 4.2, the decentralized strong stabilizability conditions of LTI time-delay systems are investigated, and the decentralized strong μ-stabilizability test is introduced (see Remark 6). In Section 4.3, the decentralized strong stabilization problem is formulated. We should note that Section 4.1 and 4.3 can be thought of as special cases of Section 3.1 and 3.2, where a decentralized controller is aimed to be designed. Furthermore, the decentralized controllers are designed using a simultenous design approach. Therefore, the controller design methods have only slight modifications. Because of that, to avoid redundancy, the reader is referred to the related parts of Section 3.1 and 3.2 whenever necessary. In Section 4.4, the technical details of the decentralized controller design phase are discussed. Finally, in Section 4.5, the decentralized controller design methods proposed in this chapter, along with the associated analysis methods given in Chapter 2, are illustrated through examples.

4.1. Decentralized Stabilization

In this section, we consider an LTI time-delay system described as (2.31). According to Subsection 2.4.2, given that $C_D < \mu$, (2.31) is decentralized μ-stablizable for a decentralized controller whose k^{th} local controller is in the form of (2.45) if and only if it does not have any μ-DFMs. For this reason, we make the following assumption in addition to Assumption 1.

Assumption 3. (2.31) *does not have any μ-DFMs.*

Under Assumptions 1 and 3, all the μ-modes of (2.31) can be assigned to the left-hand side of $\mu \in \mathbb{R}$. Before the construction of the overall closed-loop system, we first take (2.31) into the form of (2.25) as described in Subsection 2.2.2. Furthermore, since the set of all local controllers (2.45) of the overall decentralized controller will be designed simultaneously, we construct the overall decentralized controller in the form of (2.35) as described in Subsection 2.3.2. Recall that each local controller is allowed to have different time delays (see (4.21)-(4.22) for example). Therefore, the set of time delays of the overall decentralized controller will be the union of the set of time delays of each local controller.

From now on, the closed-loop system and decentralized controller refer to the overall closed-loop system and overall decentralized controller, respectively. Then we apply the same procedure described in Section 3.1 to obtain the corresponding closed-loop system. First, we let

$$\{\bar{h}_1, \ldots, \bar{h}_{\bar{\sigma}}\} = \{h_1, \ldots, h_\sigma\} \cup \{\hat{h}_1, \ldots, \hat{h}_{\hat{\sigma}}\}, \tag{4.1}$$

where $\{\hat{h}_1, \ldots, \hat{h}_{\hat{\sigma}}\}$ are as in (2.48), and $\bar{h}_0 := 0$. Then, the closed-loop system can be described by DDAEs as

$$\mathcal{E}\dot{\eta}(t) = \mathcal{A}_0\eta(t) + \sum_{i=1}^{\bar{\sigma}} \mathcal{A}_i\eta(t - \bar{h}_i) \tag{4.2}$$

where $\eta(t) := \begin{bmatrix} x(t)^T & y(t)^T & z(t)^T & u(t)^T \end{bmatrix}^T \in \mathbb{R}^{\bar{n}}$, where $z(t)$, $u(t)$, and $y(t)$ are as in (2.46)-(2.47), $\bar{n} := n + m + q + l$,

$$\mathcal{E} := \begin{bmatrix} E & 0 & 0 & 0 \\ 0 & 0_q & 0 & 0 \\ 0 & 0 & I_l & 0 \\ 0 & 0 & 0 & 0_m \end{bmatrix}, \quad \mathcal{A}_0 := \begin{bmatrix} A_0 & 0 & 0 & B_0 \\ C_0 & -I_q & 0 & 0_{q\times m} \\ 0 & G_0 & F_0 & 0 \\ 0 & K_0 & H_0 & -I_m \end{bmatrix},$$

and, for $i = 1, \ldots, \bar{\sigma}$,

$$\mathcal{A}_i := \begin{bmatrix} \bar{A}_i & 0 & 0 & \bar{B}_i \\ \bar{C}_i & 0_q & 0 & 0_{q\times m} \\ 0 & \bar{G}_i & \bar{F}_i & 0 \\ 0 & \bar{K}_i & \bar{H}_i & 0_m \end{bmatrix},$$

where

$$\bar{A}_i := \begin{cases} A_j, & \text{if } \bar{h}_i = h_j \\ 0_n, & \text{otherwise} \end{cases}, \quad \bar{B}_i := \begin{cases} B_j, & \text{if } \bar{h}_i = h_j \\ 0_{n\times m}, & \text{otherwise} \end{cases},$$

$$\bar{C}_i := \begin{cases} C_j, & \text{if } \bar{h}_i = h_j \\ 0_{q\times n}, & \text{otherwise} \end{cases},$$

$$B_j := \left[\begin{array}{ccc} B_{j,1} & \cdots & B_{j,\nu} \end{array}\right], \quad C_j := \left[\begin{array}{ccc} C_{j,1}^T & \cdots & C_{j,\nu}^T \end{array}\right]^T,$$

and,

$$\bar{F}_i := \begin{cases} F_j, & \text{if } \bar{h}_i = \hat{h}_j \\ 0_l, & \text{otherwise} \end{cases}, \qquad \bar{G}_i := \begin{cases} G_j, & \text{if } \bar{h}_i = \hat{h}_j \\ 0_{l\times q}, & \text{otherwise} \end{cases},$$
$$\bar{H}_i := \begin{cases} H_j, & \text{if } \bar{h}_i = \hat{h}_j \\ 0_{m\times l}, & \text{otherwise} \end{cases}, \qquad \bar{K}_i := \begin{cases} K_j, & \text{if } \bar{h}_i = \hat{h}_j \\ 0_{m\times q}, & \text{otherwise} \end{cases}.$$

Note that, since the open-loop system is in the form of (2.25) and the controller is in the form (2.35), the closed-loop system (4.2) is in the form of (3.2). The only difference is that the matrices of the decentralized controller are block diagonal matrices as in (2.49). Therefore, the associated delay-difference of (4.2) can be obtained as for (3.2). Accordingly, the associated delay-difference of (4.2) is in the form of (3.8). Thus, following from Section 3.1, the closed-loop system (4.2) can not be neutral if $\mathrm{rank}(E) = n$; however, if $\mathrm{rank}(E) < n$, the closed-loop system (4.2) might be neutral. We also note that the closed-loop system (4.2) is well-posed if and only if (3.14) is satisfied. In particular, if $\mathrm{rank}(E) < n$, then (3.21) must be satisfied to gurantee the well-posedness of the closed-loop system (4.2). However, if $\mathrm{rank}(E) = n$, then the closed-loop system (4.2) is always well-posed.

The main objective is to make (4.2) μ-stable by minimizing the spectral abscissa as a function of the free parameters of the decentralized controller. Considering (2.45), let $p \in \mathbb{R}^N$ be the vector of free parameters in the controller matrices where N is the number of free parameters involved. Furthermore, let $\hat{h} \in \mathbb{R}^{\hat{\sigma}}$ be the vector of free parameters described as

$$\hat{h} := [\hat{h}_1, \ldots, \hat{h}_{\hat{\sigma}}], \tag{4.3}$$

thus, $\hat{h}$ consists of the time delays of the decentralized controller. So, the vector of parameters can be constructed as

$$\bar{p} := [p\ \hat{h}]. \tag{4.4}$$

In order to design a decentralized controller, which μ-stabilizes (3.2), we solve the optimization problem:

$$\min_{\bar{p}} c(\bar{p}) \tag{4.5}$$

where $c(\bar{p})$ indicates the spectral abscissa of the closed-loop system (4.2) with the parameter vector $\bar{p}$. If a local minimizer, $\bar{p}^+$, of (4.5) satisfies $c(\bar{p}^+) < \mu$, then, the decentralized controller corresponding to $\bar{p}^+$ μ-stabilizes the system (2.31) for a given $\mu \in \mathbb{R}$. If a decentralized time-delay controller, i.e., (2.45) with $\hat{\sigma}_k > 0$ for at least one k, is aimed, then the constraint

$$\{\hat{h}_1, \ldots, \hat{h}_{\hat{\sigma}}\} \geq 0 \tag{4.6}$$

must be imposed to make sure that the designed decentralized controller will be causal. However, if a decentralized delay-free controller, i.e., (2.45) with $\hat{\sigma}_k = 0$ for $k = 1, \ldots, \nu$, is aimed, then (4.6) should be omitted.

In order to determine $c(\bar{p})$ for a particular $\bar{p}$, it must be guaranteed that $\bar{C}_D(\bar{p}) < \epsilon < \mu$, where $\bar{C}_D$ indicates the insensitive bound of the closed-loop system (4.2), during the course of the optimization. According to this, two different cases, introduced in Section 3.1, can also be applied to decentralized controller design. However, careful attention must be paid that the open-loop system (2.31) does not have any direct feed-through term if either $\operatorname{rank}(E) = n$ or (2.34) is satisfied for all $k, l = 1, \ldots, \nu$ and $i, j = 0, \ldots, \sigma$. Furthermore, the decentralized controller does not have any direct feed-through term if $K_{i,k} = 0$ for all $i = 0, \ldots, \hat{\sigma}_k$, $k = 1, \ldots, \nu$. Moreover, the decentralized controller (2.45) can not be neutral since none of the local controllers are neutral.

As in Section 3.1, in *Case 1*, it is guaranteed that $\bar{C}_D(\bar{p}) < \epsilon$ for all $\bar{p}$. However, in *Case 2*, the constraint

$$\gamma_{\tilde{\mu}}(\bar{p}) \leq 1 \tag{4.7}$$

must be imposed to (4.5), where $\gamma_{\tilde{\mu}}(\bar{p})$ is as described in (3.26) and (3.27). As a result, under the constraint (4.7), it is guranteed that $\bar{C}_D(\bar{p}) < \epsilon$ for all $\bar{p}$. We note that, in both of these cases, if the decentralized controller corresponding to $\bar{p}^+$ μ-stabilizes the system (2.31), then the system (2.31) is guaranteed to be insensitively μ-stable, since $\tilde{\mu} < \mu$.

4.2. Decentralized Strong Stabilizability of Time-delay Systems

In this section, we try to find answers to the questions **Q1** and **Q2** given in Section 3.2 in the case of decentralized control. For **Q2**, we investigate the minimum possible number of unstable modes, counting with multiplicities, that any decentralized stabilizing controller must have. As in the centralized case, the answer to **Q2** yields the answer to **Q1** so that we determine whether there exists a stable decentralized controller or not. Furthermore, according to the answer to **Q2**, the strong stabilizability test given in Section 3.2 can be extended to the decentralized case. As it has been mentioned in Section 1.1, these two questions have been solved in [98] for finite-dimensional systems, i.e., systems in the form of (2.31) with $\sigma = 0$. Here, we extend the results of [98] to time-delay systems, i.e., systems in the form of (2.31) with $\sigma \neq 0$, by following the line of [98], and using the results of [22] which considers the centralized strong stabilizability problem for a broad class of LTI MIMO systems including time-delay systems. In the remainder of this section, we extensively use bicoprime factorization for the system and left/right coprime factorization for the controller. We note that there are several advantages of using both of these two types of factorization approaches (see [123] and [124]).

Let us denote the TFM of (2.31) as $\mathcal{Z} = [\mathcal{Z}_{i,j}]$, where $\mathcal{Z}_{i,j}$, $i, j = 1, \ldots, \nu$, are the $q_i \times m_j$ dimensional TFMs of each subsystem associated with $\mathcal{Z}$. Also, let us denote the $m_k \times q_k$ dimensional TFM of the k^{th} local controller (2.45) as $\mathcal{C}_k$, then denote the TFM of the overall decentralized controller as $\mathcal{C} = \operatorname{diag}\{\mathcal{C}_1, \ldots, \mathcal{C}_\nu\}$. The decentralized strong stabilization problem can be defined as designing a decentralized stable controller $\mathcal{C}$, which is composed of stable local controllers $\mathcal{C}_k$, $k = 1, \ldots, \nu$, for $\mathcal{Z}$ such that the overall closed-loop system of $(\mathcal{Z}, \mathcal{C})$ is stable.

It is known that the ordered triple (P, Q, R) is bicoprime in $\mathcal{H}_{\mathbb{R}}^{\infty}$ if the pairs (P, Q) and (Q, R) are right and left coprime over $\mathcal{H}_{\mathbb{R}}^{\infty}$, respectively [124]. Furhermore the ordered triple (P, Q, R) is a bicoprime factorization of $\mathcal{Z}$ over $\mathcal{H}_{\mathbb{R}}^{\infty}$ if (P, Q, R) is bicoprime in $\mathcal{H}_{\mathbb{R}}^{\infty}$, Q is a square matrix with $\det(Q(\infty)) \neq 0$, and $\mathcal{Z} = PQ^{-1}R$, where P, Q and R are $q \times \xi$, $\xi \times \xi$ and $\xi \times m$ dimensional matrices in $\mathcal{H}_{\mathbb{R}}^{\infty}$, respectively. Here, ξ must be such that

$$\operatorname{nrank}(\mathcal{Z}) \leq \xi$$

where $\operatorname{nrank}(\mathcal{Z})$ is the normal rank of $\mathcal{Z}$ [124]. Note that, for $\mathcal{Z}$, one can always find these pairs (P_r, Q_r) and (Q_l, R_l) which are right and left coprime in $\mathcal{H}_{\mathbb{R}}^{\infty}$, respectively [125]. Then a bicoprime factorization can be found via right or leftcoprime factorization. For example, once a right coprime factorization of $\mathcal{Z} = P_r Q_r^{-1}$ over $\mathcal{H}_{\mathbb{R}}^{\infty}$ is found, then the ordered triple (P, Q, R) can be determined as a bicoprime factorization of $\mathcal{Z}$ over $\mathcal{H}_{\mathbb{R}}^{\infty}$ by choosing $P = P_r$, $Q = Q_r$, and $R = I_m$. A dual procedure can also be applied by using the left coprime factorization. Furhermore a bicoprime factorization can be directly derived from (2.31) if and only if (2.31) is both spectrally stabilizable and spectrally detectable [124].

A decentralized controller $\mathcal{C}$, whose each local controller is in the form of (2.45), has always a *right coprime factorization* $\mathcal{C} = P_c Q_c^{-1}$ over $\mathcal{H}_{\mathbb{R}}^{\infty}$ such that P_c and Q_c are right coprime over $\mathcal{H}_{\mathbb{R}}^{\infty}$ where P_c and Q_c are $m \times q$ and $q \times q$ dimensional matrices in $\mathcal{H}_{\mathbb{R}}^{\infty}$ [125].

Lemma 3. *Let $\mathcal{Z} = PQ^{-1}R$ be a bicoprime factorization over $\mathcal{H}_{\mathbb{R}}^{\infty}$. Also let $\mathcal{C} = P_c Q_c^{-1}$ be a right coprime factorization over $\mathcal{H}_{\mathbb{R}}^{\infty}$. Then, the closed-loop system of $(\mathcal{Z}, \mathcal{C})$ is stable if and only if*

$$\begin{bmatrix} Q & RP_c \\ -P & Q_c \end{bmatrix} \tag{4.8}$$

is invertible in $\mathcal{H}_{\mathbb{R}}^{\infty}$ (see Theorem 11 in [124]).

According to Lemma 3, if $\mathcal{C} \in M(\mathcal{H}_{\mathbb{R}}^{\infty})$, i.e., Q_c is invertible in $\mathcal{H}_{\mathbb{R}}^{\infty}$, then (4.8) can be written as

$$\begin{bmatrix} I & RP_cQ_c^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} Q + RP_cQ_c^{-1}P & 0 \\ 0 & Q_c \end{bmatrix} \begin{bmatrix} I & 0 \\ -Q_c^{-1}P & I \end{bmatrix}$$

by using the Shur complement decomposition. Accordingly, the closed-loop system of $(\mathcal{Z}, \mathcal{C})$ is stable if and only if $Q + RP_cQ_c^{-1}P$ is invertible in $\mathcal{H}_{\mathbb{R}}^{\infty}$, where $\mathcal{C} \in M(\mathcal{H}_{\mathbb{R}}^{\infty})$.

Now we state the following two results which give the characterization of centralized blocking zeros through the framework of factorization in $\mathcal{H}_{\mathbb{R}}^{\infty}$.

Lemma 4. *Let $\mathcal{Z} = PQ^{-1}R$ be a factorization over $\mathcal{H}_{\mathbb{R}}^{\infty}$. For any $z_0 \in \mathbb{C}^{+e}$ satisfying $\mathcal{Z}(z_0) = 0$, one has*

$$\operatorname{rank} \begin{bmatrix} Q & R \\ -P & 0 \end{bmatrix} (z_0) \leq \xi \tag{4.9}$$

where strict equaliy is achieved if either (P, Q) is right coprime or (Q, R) left coprime over $\mathcal{H}_{\mathbb{R}}^{\infty}$ [126].

Lemma 5. *Let $\mathcal{Z} = PQ^{-1}R$ be a bicoprime factorization over $\mathcal{H}_{\mathbb{R}}^{\infty}$. For any $z_0 \in \mathbb{C}^{+e}$*

$$\operatorname{rank} \begin{bmatrix} Q & R \\ -P & 0 \end{bmatrix} (z_0) = \xi \tag{4.10}$$

if and only if $\mathcal{Z}(z_0) = 0$ [126].

Now let us introduce the notion of *decentralized blocking zeros* which will play a central role in the remainder of this section.

Definition 12. $z_0 \in \mathbb{C}^e$ is a decentralized blocking zero of $\mathcal{Z} = [\mathcal{Z}_{i,j}]$, $i, j = 1, \ldots, \nu$, if, when evaluated at z_0, all the main block diagonal entries and the entries below the main diagonal blocks of the system TFM become zero after a suitable symmetric permutation of the block rows and columns. In other words, z_0 is a decentralized blocking zero of $\mathcal{Z} = [\mathcal{Z}_{i,j}]$ if for some permutation $\{i_1, \ldots, i_\nu\}$, $\mathcal{Z}_{i_k, i_l} = 0$ where $k = 1, \ldots \nu$ and $l = 1, \ldots, k$. Thus, the set of decentralized blocking zeros can be desribed as

$$\Lambda_{\mathcal{Z}} = \{z_0 \in \mathbb{C}^e \mid there\ exists\ a\ permutation\ \{i_1, \ldots, i_\nu\}\ of\ \{1, \ldots, \nu\}\ such\ that$$

$$\begin{bmatrix} \mathcal{Z}_{i_1,i_1} & 0 & 0 & \ldots & 0 \\ \mathcal{Z}_{i_2,i_1} & \mathcal{Z}_{i_2,i_2} & 0 & \ldots & 0 \\ \mathcal{Z}_{i_3,i_1} & \mathcal{Z}_{i_3,i_2} & \mathcal{Z}_{i_3,i_3} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \mathcal{Z}_{i_\nu,i_1} & \mathcal{Z}_{i_\nu,i_2} & \mathcal{Z}_{i_\nu,i_3} & \ldots & \mathcal{Z}_{i_\nu,i_\nu} \end{bmatrix} (z_0) = 0\}.$$

For instance, when $\nu = 2$, the set of decentralized blocking zeros of $\mathcal{Z}$ can be defined as

$$\Lambda_{\mathcal{Z}} = \{z_0 \in \mathbb{C}^e \mid \begin{bmatrix} \mathcal{Z}_{1,1} & 0 \\ \mathcal{Z}_{2,1} & \mathcal{Z}_{2,2} \end{bmatrix} (z_0) = 0\ or\ \begin{bmatrix} \mathcal{Z}_{1,1} & \mathcal{Z}_{1,2} \\ 0 & \mathcal{Z}_{2,2} \end{bmatrix} (z_0) = 0\}$$

which can also be expressed as

$$\Lambda_{\mathcal{Z}} = \{the\ intersection\ of\ blocking\ zeros\ of\ \mathcal{Z}_{1,1}\ and\ \mathcal{Z}_{2,2}\} \cap$$

$$\{the\ union\ of\ blocking\ zeros\ of\ \mathcal{Z}_{1,2}\ and\ \mathcal{Z}_{2,1}\}$$

Some other equivalent descriptions of the decentralized blocking zeros can be found in [98]. We note that, for a single-channel system, i.e, when $\nu = 1$, the decentralized blocking zeros are equal to the centralized blocking zeros. By definition, while a decentralized blocking zero may not be a centralized blocking zero, any centralized blocking zero is a decentralized blocking zero. Consequently, the set of decentralized blocking zeros can be a much larger set than the set of centralized blocking zeros. Let us denote the decentralized unstable blocking zeros of the system as

$$\Lambda_{\mathcal{Z}}^{+} = \Lambda_{\mathcal{Z}} \cap \mathbb{C}_0^{+e}.$$

Lemma 6. *Let $\mathcal{Z}$ be a TFM of* (2.31). *If* (2.31) *does not have any unstable DFMs, then the set of modes of $\mathcal{Z}$ and $\Lambda_{\mathcal{Z}}^{+}$ are disjoint [126].*

Before going into the main theorem, first, we should investigate how a local controller applied to any channel of $\mathcal{Z}$ affects the set of unstable decentralized blocking zeros.

Lemma 7. *Let $\mathcal{Z}$ have no unstable decetralized fixed modes, and, $\mathcal{Z} = PQ^{-1}R$ is a bicoprime factorization over $\mathcal{H}_{\mathbb{R}}^{\infty}$. Let $\mathcal{C}_\nu = P_{c\nu}Q_{c\nu}^{-1}$ be the local controller applied to the ν^{th} channel of $\mathcal{Z}$, where $(P_{c\nu}, Q_{c\nu})$ is a pair of right coprime matrices over $\mathcal{H}_{\mathbb{R}}^{\infty}$ with appropriate dimensions. Let the resulting $\nu - 1$ channel system has a bicoprime factorization over $\mathcal{H}_{\mathbb{R}}^{\infty}$*

$$\mathcal{Z}^{\nu-1} := \begin{bmatrix} P_1 & 0 \\ \vdots & \vdots \\ P_{\nu-1} & 0 \end{bmatrix} \begin{bmatrix} Q & R_\nu P_{c\nu} \\ -P_\nu & Q_{c\nu} \end{bmatrix}^{-1} \begin{bmatrix} R_1 & \cdots & R_{\nu-1} \\ 0 & \cdots & 0 \end{bmatrix}, \tag{4.11}$$

and, $\mathcal{Z}^{\nu-1}$ has no unstable DFMs. Then $\Lambda_{\mathcal{Z}}^{+} \subset \Lambda_{\mathcal{Z}^{\nu-1}}^{+}$ [126].

Lemma 7 states that the set of unstable decentralized blocking zeros of the resulting system with $\nu - 1$ channels includes the set of unstable decentralized blocking zeros of the given system with ν channels. Note that this result is crucial for the proof of the following theorem.

Theorem 1. *Let Assumptions 1 and 3 are made for $\mathcal{Z}$. Let $z_1, z_2, \ldots, z_t$ denote the elements of $\Lambda_{\mathcal{Z}} \cap \mathbb{R}_0^{+e}$ arranged in an ascending order. Also, let η_i denote the number of $\mathbb{R}_0^{+e}$ modes of $\mathcal{Z}$ counted with multiplicities in the interval (z_i, z_{i+1}), $i \in 1, 2, \ldots, t-1$. Define η to be the number of odd integers in the set $\eta_1, \ldots, \eta_{t-1}$. Then, every decentralized stabilizing controller $\mathcal{C} = \operatorname{diag}\{\mathcal{C}_1, \ldots, \mathcal{C}_\nu\}$ has at least η modes in $\mathbb{R}_0^{+e}$ (counting multiplicities).*

Proof. Let $\mathcal{Z} = PQ^{-1}R$ be a bicoprime factorization over $\mathcal{H}_{\mathbb{R}}^{\infty}$ where $P = [P_1^T \ldots P_\nu^T]^T$ and $R = [R_1 \ldots R_\nu]$. The proof will be given by induction. Let $\nu = 2$. By Definition 12, for a two-channel system, the set of decentralized unstable real blocking zeros can be defined as

$$\begin{aligned} \Lambda_{\mathcal{Z}}^{+} = \{z_0 \in \mathbb{R}^{+e} \mid & \begin{bmatrix} \mathcal{Z}_{1,1}^T & \mathcal{Z}_{2,1}^T \end{bmatrix}(z_0) = 0 \text{ and } \mathcal{Z}_{2,2}(z_0) = 0\} \\ \cup\{z_0 \in \mathbb{R}^{+e} \mid & \begin{bmatrix} \mathcal{Z}_{1,1} & \mathcal{Z}_{1,2} \end{bmatrix}(z_0) = 0 \text{ and } \mathcal{Z}_{2,2}(z_0) = 0\}. \end{aligned} \tag{4.12}$$

Also, if $z_0 \in \Lambda_{\mathcal{Z}}^{+}$ satisfies $[\mathcal{Z}_{11}^T \ \mathcal{Z}_{21}^T]^T(z_0) = 0$, then by Lemma 4, we have

$$\operatorname{rank} \begin{bmatrix} Q & R_1 \\ -P_1 & 0 \\ -P_2 & 0 \end{bmatrix}(z_0) = \xi \tag{4.13}$$

since (P, Q) is right coprime. If $z_0 \in \Lambda_{\mathcal{Z}}^{+}$ satisfies $[\mathcal{Z}_{11} \ \mathcal{Z}_{12}](z_0) = 0$, then by Lemma 4, we have

$$\operatorname{rank} \begin{bmatrix} Q & R_1 & R_2 \\ -P_1 & 0 & 0 \end{bmatrix}(z_0) = \xi \tag{4.14}$$

since (Q, R) is left coprime.

Assume that $\mathcal{C} = \operatorname{diag}\{\mathcal{C}_1, \mathcal{C}_2\}$ is a decentralized stabilizing controller whose local controllers $\mathcal{C}_1$ and $\mathcal{C}_2$ have n_1 and n_2 unstable modes, respectively, counted with multiplicities. Let $\mathcal{C}_2 = P_{c2}Q_{c2}^{-1}$ be a right coprime factorization over $\mathcal{H}_{\mathbb{R}}^{\infty}$. Then, the resulting system, which is obtained by applying $\mathcal{C}_2$, can be obtained as

$$\mathcal{Z}^1 := \begin{bmatrix} P_1 & 0 \end{bmatrix} \begin{bmatrix} Q & R_2P_{c2} \\ -P_2 & Q_{c2} \end{bmatrix}^{-1} \begin{bmatrix} R_1 \\ 0 \end{bmatrix} \tag{4.15}$$

which is a bicoprime factorization over $\mathcal{H}_{\mathbb{R}}^{\infty}$ [85]. Note that the closed-loop system of $(\mathcal{Z}^1, \mathcal{C}_1)$ is stable since $\mathcal{C} = \text{diag}\{\mathcal{C}_1, \mathcal{C}_2\}$ is assumed to be a decentralized stabilizing controller. By the bicoprimeness of (4.15), for any $z_0 \in \mathbb{R}_0^{+e}$ for which (4.13) or (4.14) holds, i.e., z_0 is an unstable real blocking zero, we have

$$\text{rank}\begin{bmatrix} Q & R_2P_{c2} & R_1 \\ -P_2 & Q_{c2} & 0 \\ -P_1 & 0 & 0 \end{bmatrix}(z_0) = \xi + m_2. \tag{4.16}$$

Since (4.15) is a bicoprime factorization over $\mathcal{H}_{\mathbb{R}}^{\infty}$, we can apply Lemma 5 for $\mathcal{Z}^1$. Accordingly, every $z_0 \in \Lambda_{\mathcal{Z}}^{+}$ is an unstable real blocking zero of $\mathcal{Z}^1$. By Lemma 3 of [22], $\mathcal{C}_1$ stabilizes $\mathcal{Z}^1$ only if the number of sign changes of

$$\det\left(\begin{bmatrix} Q & R_2P_{c2} \\ -P_2 & Q_{c2} \end{bmatrix} \right) \tag{4.17}$$

at the blocking zeros of $\mathcal{Z}^1$ is not be greater than n_1. Since, by Lemma 7, the sequence $z_1, z_2, \ldots, z_t$ is a subset of the blocking zeros of $\mathcal{Z}^1$, the sign changes of (4.17) at these sequence can not be greater than n_1. On the other hand, for any $z_0 \in \Lambda_{\mathcal{Z}}^{+}$, it holds that $\mathcal{Z}_{22}(z_0) = 0$ by (4.12). Therefore, the number of sign changes of (4.17) and that of $\det(Q)\det(Q_{c2})$ in the sequence $z_1, z_2, \ldots, z_t$ are equal. It follows that the number of sign changes of $\det(Q)$ in this sequence equals η. Then, $\det(Q)\det(Q_{c2})$ has at least $\eta - n_2$ sign changes in the sequence $z_1, z_2, \ldots, z_t$. In other words, for $\mathcal{C}_1$ to stabilize $\mathcal{Z}^1$, it must hold that $\eta - n_2 \leq n_1$. This proves Theorem 1 for $\nu = 2$.

Now, assume that the statement in Theorem 1 holds for $\nu - 1$ for any $\nu - 1 \geq 2$. We need to establish the statement for ν. Let $\mathcal{C}$ be a decentralized stabilizing controller whose local controller $\mathcal{C}_k$ has n_k unstable modes, counted with multiplicities, for $k = 1, \ldots, \nu$. Let $\mathcal{C}_\nu = P_{c\nu}Q_{c\nu}^{-1}$ be a right coprime factorization over $\mathcal{H}_{\mathbb{R}}^{\infty}$. Consider the closed-loop system $\mathcal{Z}^{\nu-1}$ given in (4.11) which is a bicoprime factorization over $\mathcal{H}_{\mathbb{R}}^{\infty}$ [85]. Let $\Lambda_{\mathcal{Z}^{\nu-1}}^{+}$ be the set of unstable real decentralized blocking zeros of $\mathcal{Z}^{\nu-1}$. By Lemma 7, we have $\Lambda_{\mathcal{Z}}^{+} \subset \Lambda_{\mathcal{Z}^{\nu-1}}^{+}$. By Lemma 6, $\Lambda_{\mathcal{Z}^{\nu-1}}^{+}$ and the set of mode of $\mathcal{Z}^{\nu-1}$ are disjoint. Let $\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_{\bar{t}}$ denote the elements of $\Lambda_{\mathcal{Z}^{\nu-1}}^{+}$ arranged in an ascending order. Also, let $\bar{\eta}_i$ denote the number of $\mathbb{R}_0^{+e}$ modes of $\mathcal{Z}^{\nu-1}$, counted with multiplicities, in the interval

$(\bar{z}_i, \bar{z}_{i+1})$, $i \in 1, 2, \ldots, \bar{t} - 1$. By hypotesis of induction, the number of odd integers in the set $\bar{\eta}_1, \ldots, \bar{\eta}_{\bar{t}-1}$ is less than or equal to $n_1 + n_2 + \ldots + n_{\nu-1}$. In this case, the number of sign changes of

$$\det\Big(\begin{bmatrix} Q & R_\nu P_{c\nu} \\ -P_\nu & Q_{c\nu} \end{bmatrix} \Big) \tag{4.18}$$

in the sequence $z_1, z_2, \ldots, z_t$ is not greater than $n_1+n_2+\ldots+n_{\nu-1}$. Also, in this sequence (4.18) and $\det(Q)\det(Q_{c\nu})$ takes the same sign as every decentralized blocking zero of $\mathcal{Z}$, say z_0, satisfy $\mathcal{Z}_{\nu\nu}(z_0) = 0$ by Definition 12. The number of sign changes of $\det(Q)$ in $z_1, z_2, \ldots, z_t$ is no less than $\eta - \eta_\nu$. So, $\eta - \eta_\nu \leq n_1 + n_2 + \ldots + n_{\nu-1}$. Recall that the number of unstable modes of C is equal to $n_1 + n_2 + \ldots + n_\nu$. Thus, the statement has been proved. □

Remark 4. The proof of Theorem 1 is the modified version of the proof given for the necessity part of Theorem 2 in [98] in which a very similar statement has been made. Although the proof of [98] has been conducted for fractional system matrices, we use the same strategy to extend to time-delay systems using the results of [22] instead of [8]. Furthermore, for the bicoprime factorization of time-delay systems, we refer to [124] when necessary.

Corollary 1. *Under Assumptions 1 and 3,* (2.31) *is decentralized strongly stabilizable only if* $\eta = 0$.

Theorem 1 and Corollary 1 can be thought of as the generalizations of the part (i) of Theorem 2 and Corollary 1 given by [98], respectively, to time-delay systems. Furthermore, Corollary 1 can also be thought of as the generalization of the necessity part of Theorem 1 given by [22] to decentralized strong stabilizability.

Remark 5. In Theorem 1 and Corollary 1, we consider the strong 0-stabilizability for the sake of ease. However, Theorem 1 and Corollary 1 can be generalized to decentralized strong μ-stabilizability, for any $\mu \in \mathbb{R}$, by considering decentralized μ-blocking zeros.

Remark 6. According to Theorem 1 and Remark 5, the strong μ-stabilizability test given in Section 3.2 can be used to determine the least possible number of unstable modes that any decentralized μ-stabilizing controller must have by considering decentralized blocking zeros instead of centralized blocking zeros.

4.3. Decentralized Strong Stabilization

If (2.31) is decentralized strongly μ-stabilizable, then a decentralized strongly μ-stabilizing controller in the form of (2.45) can be designed by adding the constraint of controller stability to (4.5). Let $\hat{c}(\bar{p})$ denote the spectral abscissa, i.e., the real part of the rightmost mode, of the decentralized controller which corresponds to $\bar{p}$. Then, the decentralized controller is μ-stable if and only if $\hat{c}(\bar{p}) < \mu$. Since the decentralized controller is a block diagonal controller, μ-stability of the decentralized controller implies that each local controller is also μ-stable. Suppose that we want all the modes of the decentralized controller to have real parts less than $\hat{\mu}$ for some $\hat{\mu} \leq \mu$. Then we impose an additional constriant

$$\hat{c}(\bar{p}) \leq \hat{\mu}. \tag{4.19}$$

to the stabilization problem (4.5). If a local minimizer, $\bar{p}^+$, satisfies $c(\bar{p}^+) < \mu$, then, the decentralized controller corresponding to $\bar{p}^+$ decentralized strongly μ-stabilizes (2.31) for the given $\mu \in \mathbb{R}$. Note, however, that, constraint (4.19) is needed only when $l > 0$. If $l = 0$, then the controller reduces to the form (2.52), which is $\hat{\mu}$-stable for any $\hat{\mu} \in \mathbb{R}$. Hence, when $l = 0$, constraint (4.19) must be omitted.

4.4. Details of Decentralized Controller Design

As it is mentioned before, the decentralized controller design methods introduced in Section 4.1 and 4.3 are the slightly modified versions of the centralized controller design methods introduced in Section 3.1 and 3.2. More precisely, the formulations of decentralized stabilization and strong stabilization problems are conducted as for centralized stabilization and strong stabilization. Because of that, the characteristics of the optimization problems are the same. Therefore, details of the optimization phase, computation of function values, and computation of the gradient vectors, directly follows from Subsection 3.3.1, 3.3.3, and 3.3.4, respectively. Furthermore, the initialization phase proposed in Subsection 3.3.2 can also be applied for the decentralized controller design. However, we should remind that, unlike a centralized controller, a decentralized controller is structured in a diagonal form by its nature. Therefore, the initialization phase proposed in Subsection 3.3.2 must be applied after the decentralized controller (2.45) is taken into the form

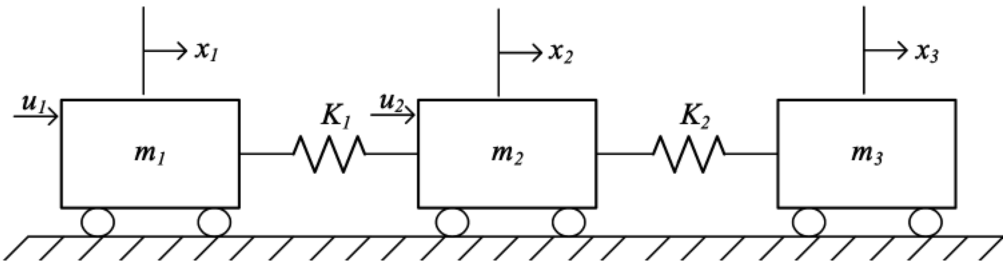

Figure 4.1: Three-mass-two-spring system.

of (2.35).

4.5. Examples

In this section, the analysis and controller design methods are illustrated through control problems. As in Section 3.4, the following subsections begin with an introduction of the given open-loop system, then continues with the open-loop system analysis, controller design, and closed-loop system analysis phases, respectively. Again as in Section 3.4, the main objective is not only to stabilize (or strongly stabilize) the given system but also to achieve the minimum spectral abscissa value, i.e. maximum exponential decay rate, with a controller whose complexity is as low as possible. In addition to the achieved spectral abscissa value, specific performance measures and/or time-domain responses are also obtained to make fair comparisons when needed. Particularly, in Subsection 4.5.2, we design PI and PID controllers. In addition to the optimal (in the sense of minimal spectral abscissa) PI and PID controllers, we also focus on the sub-optimal PI controller, whose results were presented in [49]. Then, we compare the time-domain responses of the sub-optimal PI controller with the results of some other PI controllers proposed in the literature. Note that, as in Subsection 3.4, all the analysis and controller design phases are performed by using DrStabilization v1.0.

4.5.1. Three-mass-two-spring System

We consider a mechanical system illustrated in Figure 4.1. The external forces act on the first (m_1) and second masses (m_2) while the position of second (m_2) and third masses (m_3) are measured along with the velocity of first (m_1) and second masses (m_2). Let $x := \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{bmatrix}^T$ be the state vector where x_1, x_2, and x_3 are the positions, and, x_4, x_5, and x_6 are the velocities of m_1, m_2, and m_3, respectively. Then, the system can be described in the state-space form as

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -K_1/m_1 & K_1/m_1 & 0 & 0 & 0 & 0 \\ K_1/m_2 & -(K_1+K_2)/m_2 & K_2/m_2 & 0 & 0 & 0 \\ 0 & K_2/m_3 & -K_2/m_3 & 0 & 0 & 0 \end{bmatrix} x(t) \\ &+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_1 \\ 0 \\ 0 \end{bmatrix} u_1(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/m_2 \\ 0 \end{bmatrix} u_2(t) \\ y_1(t) &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} x(t) \\ y_2(t) &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} x(t) \ , \end{aligned} \tag{4.20}$$

where $m_1 = m_2 = m_3 = 1$ and $K_1 = K_2 = 1$. It is a highly oscillatory system because of the existence of multiple modes on the imaginary axis. It is calculated that the system has six modes, $s_{1,2} = 0$, $s_{3,4} = 0 \pm i$, and $s_{5,6} = 0 \pm 1.7321i$, on the imaginary axis. Since none of these modes are decentralized fixed mode, the system is decentralized stabilizable. Furthermore, the system has no decentralized blocking zero, thus, it is also decentralized strong stabilizable.

We first design a decentralized controller in the form of (2.45) with $l_1 = l_2 = 0$, $\hat{\sigma}_1 = 1$, and $\hat{\sigma}_2 = 1$ to μ-stabilize the system for $\mu = 0$. Thus, the decentralized controller consists of two local time-delay controllers. The controller design is performed by using the optimization-based method proposed in Section 4.1. The first and second local controllers, which corresponds to $\{u_1, y_1\}$ and $\{u_2, y_2\}$ channels, respectively, can

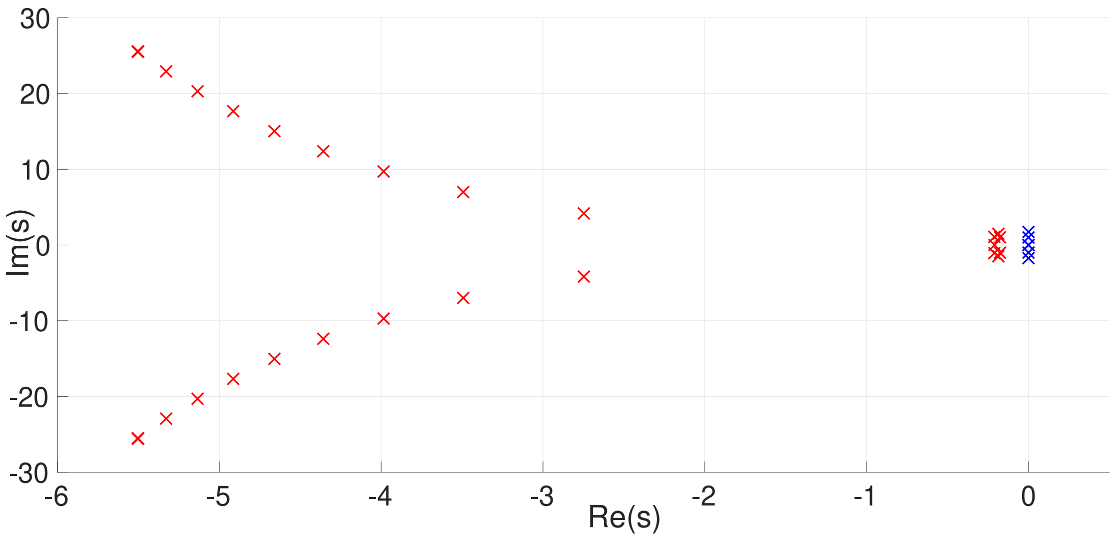

Figure 4.2: *The open-loop system modes (blue), and, the closed-loop system modes associated with the decentralized controller described by* (4.21)-(4.22) *(red).*

be represented in the form of (2.45) as

$$u_1(t) = \begin{bmatrix} -1.1441 & 0.0185 \end{bmatrix} y_1(t) + \begin{bmatrix} 0.7281 & -0.0956 \end{bmatrix} y_1(t - 2.4261) \tag{4.21}$$

$$u_2(t) = \begin{bmatrix} -0.2368 & 0.2693 \end{bmatrix} y_2(t) + \begin{bmatrix} 0.4117 & -0.3545 \end{bmatrix} y_2(t - 0.0773) \tag{4.22}$$

The modes of the associated closed-loop system are shown in Figure 4.2. The spectral abscissa value is calculated as -0.1760. As a result, the closed-loop system is 0-stable.

Then, we design a decentralized stable controller in the form of (2.45) with $l_1 = l_2 = 2$, $\hat{\sigma}_1 = 0$, and $\hat{\sigma}_2 = 0$. Thus, the decentralized controller consists of two local 2^{nd} order stable delay-free controllers. The controller design is performed by using the optimization-based method proposed in Section 4.3. In particular, the upper bound on the exponential decay rate of the decentralized controller is determined as $\hat{\mu} = 0$. The first and second local controllers, which corresponds to $\{u_1, y_1\}$ and $\{u_2, y_2\}$ channels, respectively, can be compactly represented in the form of (2.45) as

$$\begin{aligned} \dot{z}(t) &= \begin{bmatrix} F_{0,1} & 0 \\ 0 & F_{0,2} \end{bmatrix} z(t) + \begin{bmatrix} G_{0,1} & 0 \\ 0 & G_{0,2} \end{bmatrix} y(t) \\ u(t) &= \begin{bmatrix} H_{0,1} & 0 \\ 0 & H_{0,2} \end{bmatrix} z(t) + \begin{bmatrix} K_{0,1} & 0 \\ 0 & K_{0,2} \end{bmatrix} y(t) \end{aligned} \tag{4.23}$$

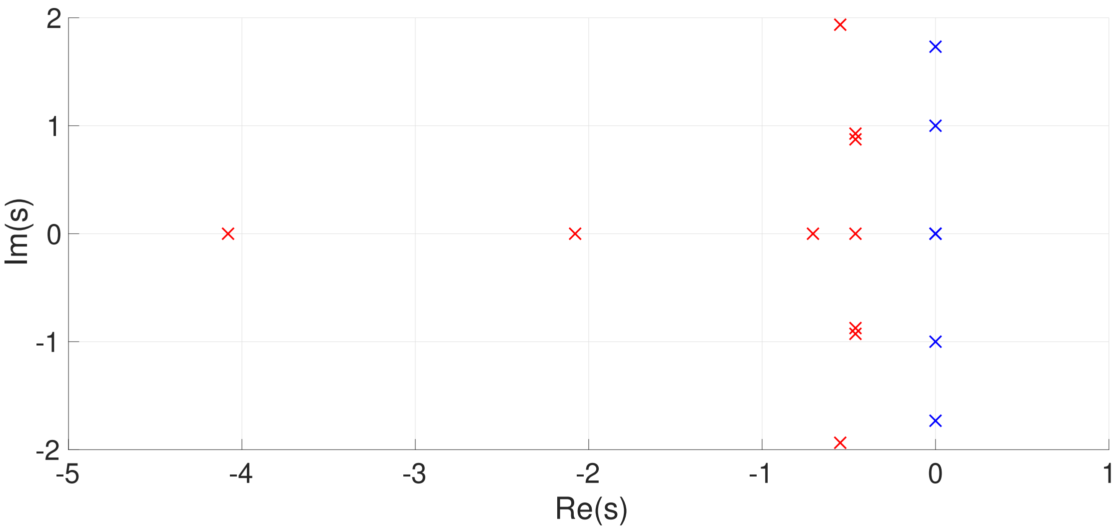

Figure 4.3: *The open-loop system modes (blue), and, the closed-loop system modes associated with the controller described by* (4.23) *(red).*

where

$$F_{0,1} = \begin{bmatrix} -1.7666 & 0.3121 \\ -0.7763 & -1.4377 \end{bmatrix} \quad G_{0,1} = \begin{bmatrix} 1.1903 & -1.2515 \\ -0.6037 & -2.4309 \end{bmatrix}$$

$$H_{0,1} = \begin{bmatrix} 1.1291 & 1.1060 \end{bmatrix} \quad K_{0,1} = \begin{bmatrix} 0.6037 & -0.1729 \end{bmatrix},$$

and,

$$F_{0,2} = \begin{bmatrix} -2.0186 & 0.1087 \\ -1.0854 & -4.2495 \end{bmatrix} \quad G_{0,1} = \begin{bmatrix} -0.2932 & -0.4040 \\ -2.0318 & -0.7652 \end{bmatrix}$$

$$H_{0,1} = \begin{bmatrix} 0.4510 & 0.8460 \end{bmatrix} \quad K_{0,1} = \begin{bmatrix} 0.0758 & -0.6267 \end{bmatrix},$$

Note that, the modes of the first local controller are $\lambda_{1,2} = -1.6022 \pm 0.4639i$, and, the modes of the second local controller are $\lambda_3 = -2.0728$ and $\lambda_4 = -4.1952$. Thus, the decentralized controller is 0-stable. The modes of the associated closed-loop system are shown in Figure 4.3. The spectral abscissa value is calculated as -0.4616. As a result, the closed-loop system is strongly 0-stable.

4.5.2. Wood-Berry Distillation Column System

In this example, we consider the pilot-scale distillation column, introduced in [127], which is used to separate a methanol-water mixture. The system is described in the

Laplace domain as

$$\begin{bmatrix} X_D(s) \\ \\ \\ X_B(s) \end{bmatrix} = \begin{bmatrix} \frac{12.8e^{-s}}{16.7s+1} & \frac{-18.9e^{-3s}}{21s+1} \\ \\ \\ \frac{6.6e^{-7s}}{10.9s+1} & \frac{-19.4e^{-3s}}{14.4s+1} \end{bmatrix} \begin{bmatrix} R(s) \\ \\ \\ S(s) \end{bmatrix} \tag{4.24}$$

where the outputs X_D and X_B are the mole fractions of methanol on overhead and bottom, respectively. Furthermore, the inputs R and S are the reflux flow rate and the steam flow rate, respectively. The system has strong interactions among its I/O channels where there exist significant time delays. The system with the parameters given in (4.24) has been used as a benchmark example by many studies for verification of the decentralized PI controller design methods. Among all these studies, some of them have chosen to decouple the TFM of the system through a decoupler matrix (e.g., [128, 129]). However, it is also possible to design decentralized PI controllers without using any decoupler. Therefore, in order to make a fair comparison, we will compare our results to the results obtained by [130], [131], and [132], which do not use any decoupler either.

To apply our controller design approach, we first obtain a minimal realization of (4.24) in the form of (2.31) as

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -0.0599 & 0 & 0 & 0 \\ 0 & -0.0476 & 0 & 0 \\ 0 & 0 & -0.0917 & 0 \\ 0 & 0 & 0 & -0.0694 \end{bmatrix} x(t) \\ &\quad + \begin{bmatrix} 0.7665 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1(t-1) + \begin{bmatrix} 0 \\ 0 \\ 0.6055 \\ 0 \end{bmatrix} u_1(t-7) + \begin{bmatrix} 0 \\ -0.9 \\ 0 \\ -1.3472 \end{bmatrix} u_2(t-3) \\ y_1(t) &= \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} x(t) \\ y_2(t) &= \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} x(t). \end{aligned}$$

Clearly, the system has four modes, $s_1 = -0.0599$, $s_2 = -0.0476$, $s_3 = -0.0917$, and $s_4 = -0.0694$. We calculated that none of these modes are decentralized fixed mode.

Table 4.1: *Decentralized PI controller parameters and the corresponding minimum spectral abscissa values.*

Controller	Parameters				Min. Spec. Abs.
	k_{p1}	k_{i1}	k_{p2}	k_{i2}	
Optimal PI	-0.7467	-0.1303	0.1085	0.0310	-0.0395
Sub-optimal PI	-0.5921	-0.0571	0.0679	0.0187	-0.0388
PI of [130]	-0.375	-0.045	0.075	0.003	-0.0183
PI of [131]	-0.9320	-0.0558	0.1240	0.0086	-0.0348
PI of [132]	-0.7757	-0.0886	0.0288	0.0141	-0.0388

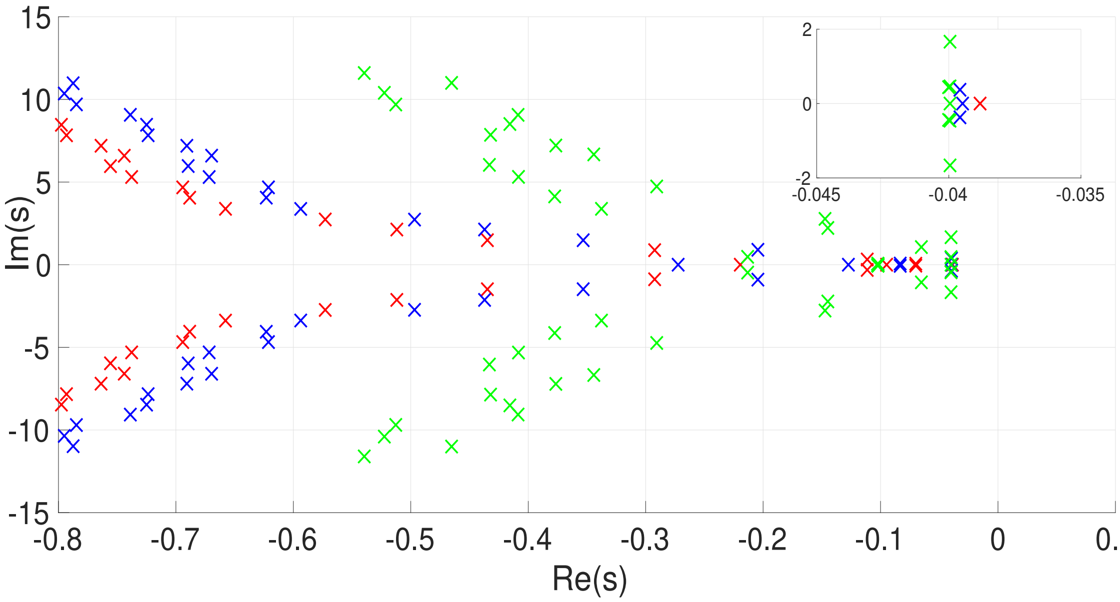

Figure 4.4: -0.8-*modes of the closed-loop system with PI (blue), sub-optimal PI (red), and PID (green) controller.*

Therefore, we aim 0-stability. Furthermore, the system has no decentralized blocking zeros.

First, we design a decentralized PI controller in the form of (2.45) where $z_k \in \mathbb{R}^1$, $k = 1, 2$, is the state of the k^{th} local controller. Furthermore, the matrices of the k^{th} local controller can be structured as

$$F_k = 0, \ G_k = 1, \ H_k = k_{ik}, \ K_k = k_{pk} \tag{4.25}$$

where the proportional gain k_{pk} and the integral gain k_{ik} are considered as the only free parameters of the controller. The gains of the designed PI controller are given in Table 4.1. The modes of the associated closed-loop system are shown in Figure 4.4. The achieved spectral abscissa value is calculated as -0.0395.

We should note that the designed PI controller is so-called optimal in the sense of

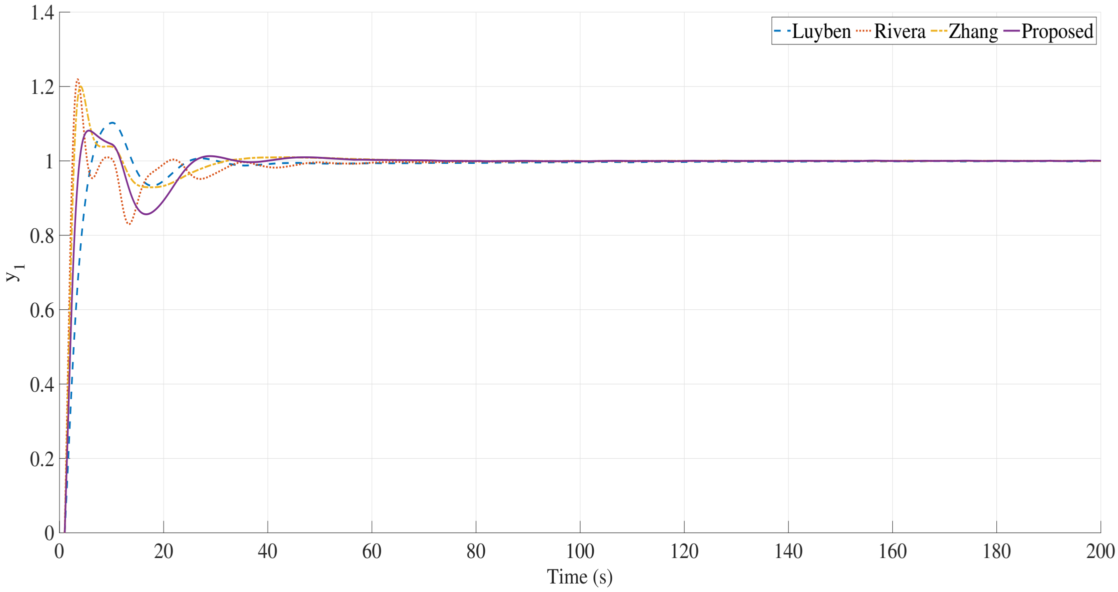

Figure 4.5: *Response of the first output to a unit step reference for the first output.*

minimal spectral abscissa. However, achieving the minimal spectral abscissa value does not imply the best time-domain response in general. Therefore, we also present a sub-optimal PI controller, which is found by searching for the one which achieves the best time-domain performance in the optimization iterations. The gains of the sub-optimal PI controller are given in Table 4.1, together with three other PI controllers obtained in the literature. The modes of the associated closed-loop system are shown in Figure 4.4. The achieved spectral abscissa value of the sub-optimal controller is -0.0388.

Then, we investigate the time-domain performance of the sub-optimal controller. In this regard, the unit step responses are shown in Figures 4.5-4.8, both for the sub-optimal controller and three other controllers obtained in the literature. Among all the controllers considered, the sub-optimal controller has the minimum average integral-absolute-error (IAE) and the minimum average settling time as given in Table 4.2. [1]

Finally, we design a decentralized PID controller in the form of (2.45) where $z_k \in \mathbb{R}^2$, $k = 1, 2$, is the state of the k^{th} local controller. Furthermore, the matrices of the k^{th} local controller can be structured as (2.39). We should note that, in the design phase, the time-constant of each low-pass filter, T_{fk}, $k = 1, 2$, is determined as a free parameter.

[1] We note that y_i-r_j indicates response of output y_i to a unit step reference r_j for output j. The $IAE := \int_0^T |y_i(t) - r_i(t)|dt$ is calculated for a simulation time of $300s$ and sampling time of $0.001s$. The settling time is calculated as the time it takes for the output $y_i(t)$, $i = 1, 2$, to reach and remain between $y^f + 0.02$ and $y^f - 0.02$ where $y^f = 1$ for y_1-r_1 and y_2-r_2, and, $y^f = 0$ for y_1-r_2 and y_2-r_1.

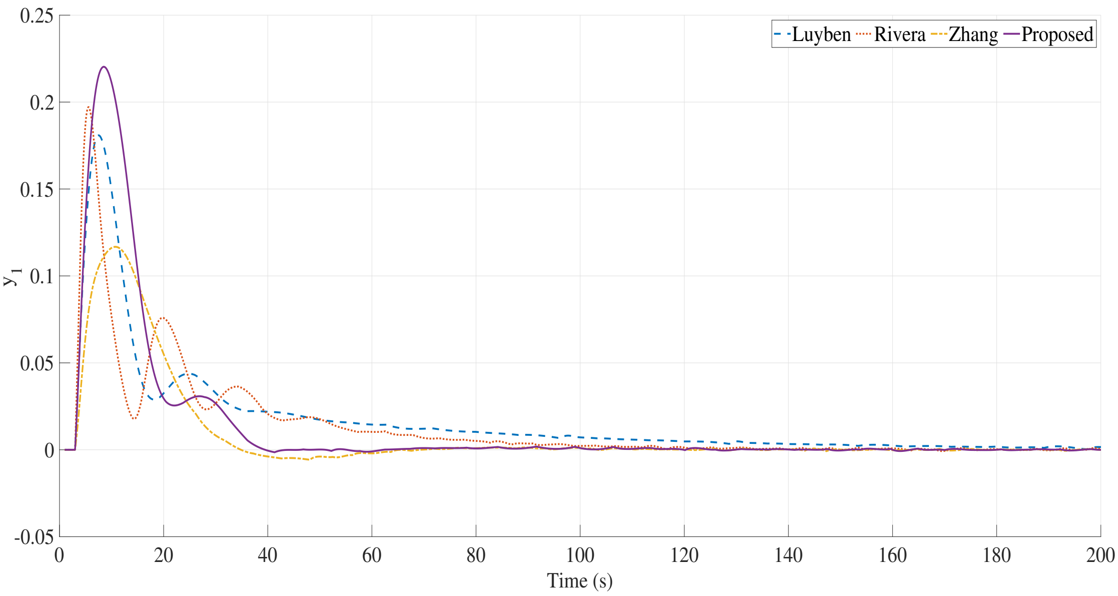

Figure 4.6: *Response of the first output to a unit step reference for the second output.*

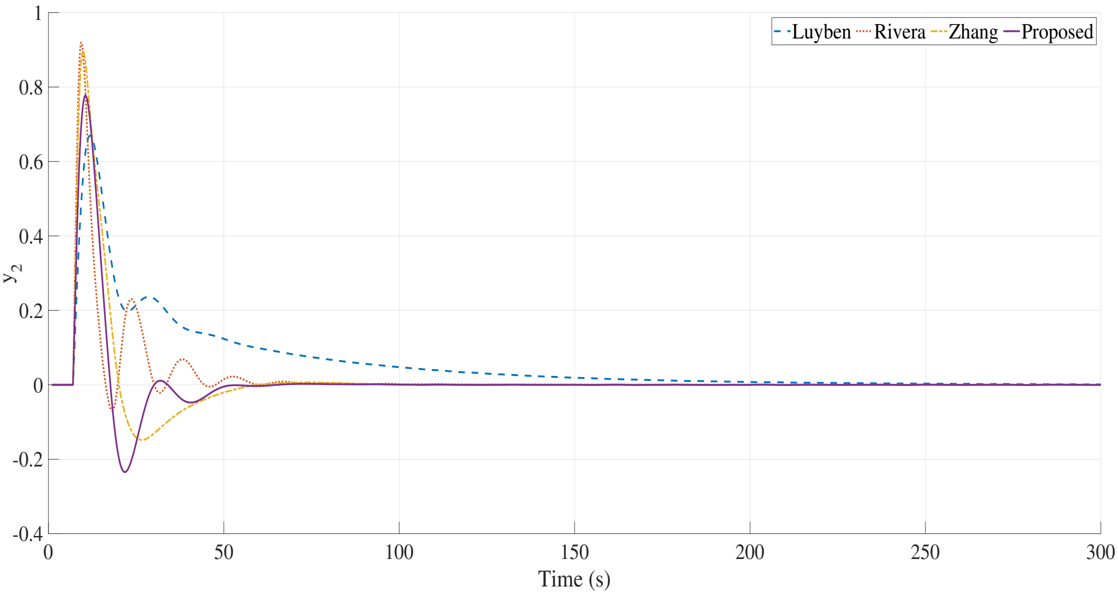

Figure 4.7: *Response of the second output to a unit step reference for the first output.*

Table 4.2: *The time-domain performances (integral absolute error and settling time) achieved by decentralized PI controllers.*

	IAE					Settling Time (s)				
Controller	y_1-r_1	y_1-r_2	y_2-r_1	y_2-r_2	Avg.	y_1-r_1	y_1-r_2	y_2-r_1	y_2-r_2	Avg.
Sub-opt. PI	3.05	2.73	7.17	7.88	5.21	24.36	31.91	46.54	39.40	35.55
PI of [130]	3.57	3.39	17.74	33.40	14.53	22.76	45.36	147.21	184.36	99.92
PI of [131]	2.62	2.77	6.64	11.06	5.77	31.82	40.29	54.06	63.64	47.45
PI of [132]	2.64	1.96	8.93	10.51	6.01	27.38	26.33	50.20	48.13	38.01

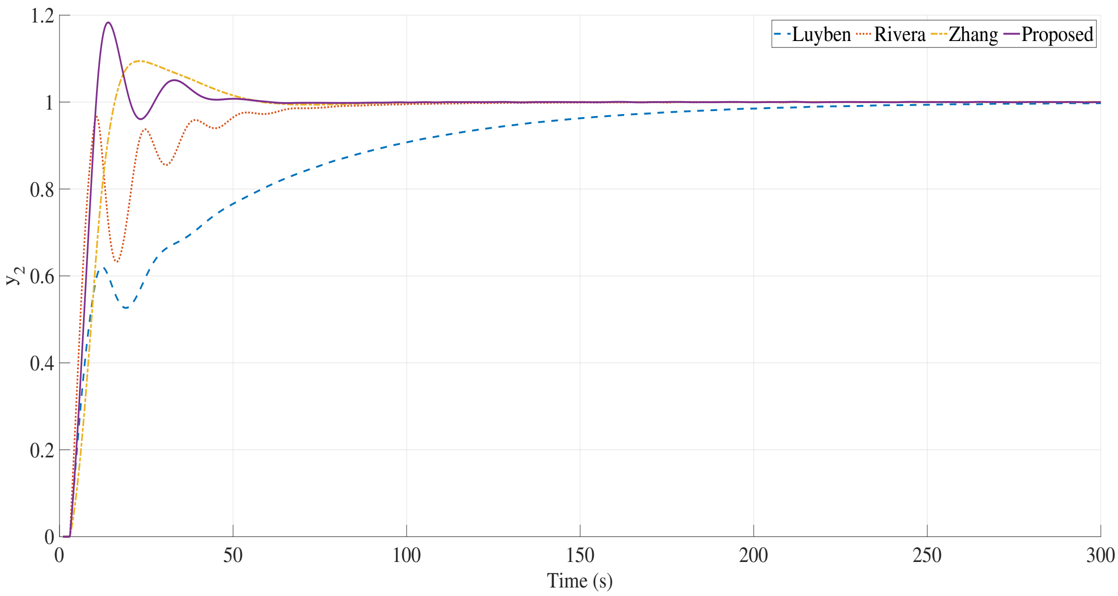

Figure 4.8: *Response of the second output to a unit step reference for the second output.*

Therefore, the initial values of T_{fk}, $k = 1, 2$, are chosen as non-negative random numbers, and the constraint

$$\{T_{f1}, T_{f2}\} \geq 0.1$$

is imposed to the optimization problem (3.24). As a result, the gains of the decentralized PID controller decribed as (2.39) are found as $k_{p1} = -0.5188$, $k_{i1} = -0.5888$, $k_{d1} = -1.6599$, $k_{p2} = 0.0504$, $k_{i2} = 0.1462$, and $k_{d2} = 0.7120$. Additionally, the time-constants of the low-pass filters are found as $T_{f1} = 1.1087$ and $T_{f2} = 0.9624$. The modes of the associated closed-loop system are shown in Figure 4.4. The achieved spectral abscissa value of the optimal PID controller is calculated as -0.04, which is slightly better than the spectral abscissa value that the optimal PI controller achieves.

5. CONCLUSION AND FUTURE RESEARCH

In Section 5.1, the concluding remarks of the thesis are given. Then, in Section 5.2, regarding the present contributions of this thesis, the possible future research objectives are discussed.

5.1. Conclusion

In this thesis, stabilization and strong stabilization of LTI MIMO systems have been considered. The open-loop systems are allowed to be retarded or neutral time-delay systems with discrete time delays. In this respect, no restriction has been made neither in the dimension nor the number of time delays of the system. In order to stabilize (or strongly stabilize) such systems, LTI time-delay controllers have been considered. The controllers are allowed to have an arbitrary number of time delays and differential degrees. Thus, the degree-of-freedom of the controller to be designed can be determined arbitrarily through the choice of differential degree and the number of time delays. Furthermore, the controllers to be designed can be structured in an arbitrary form. It has been shown that by using proper structuring, one can design various types of servocompesators. Furthermore, it is also possible to structure the matrices of a finite-dimensional controller in a suitable canonical form. As far as the time-delay controllers are concerned, different time delays can be introduced to each I/O channel of a multi-input and/or multi-output system. On top of these, proper structuring also leads to the design of decentralized controllers, which are composed of several local controllers, each of which can access only a specific part of the system. After the detailed descriptions of the open-loop systems to be considered and the controllers to be designed, the stabilizability properties of the open-loop systems by means of both centralized and decentralized controllers have been outlined. In this regard, a stabilizability test that can be used to determine whether a centralized (or a decentralized) stabilizing controller exists has been introduced.

Since centralized and decentralized stabilization (and strong stabilization) are two different concepts, these two subjects have been considered separately. First, the centralized stabilization and strong stabilization of LTI systems have been considered. In the controller design phase, the eigenvalue optimization-based stabilization method has been used. In this method, the stabilization problem has been formulated as the minimization of the spectral abscissa of the corresponding closed-loop system. The strong stabilization problem, on the other hand, has been formulated as the minimization of the spectral abscissa of the closed-loop system under the constraint of controller stability, which has been expressed in terms of the spectral abscissa of the controller. Although stable controllers are preferred over unstable ones because of various practical reasons, not all stabilizable systems are strongly stabilizable. It is well-known that a centralized stabilizing stable controller exists if and only if the stabilizable open-loop system satisfies the PIP condition. In this respect, a centralized strong stabilizability test has been introduced to determine whether a stabilizable system is also strongly stabilizable. The strong stabilizability test determines the least number of unstable real modes that any stabilizing controller must-have. As a natural consequence, if such a number is zero, then the system is strongly stabilizable. On the other hand, if it is not zero, i.e., the controller must have a certain number of real unstable modes, then, regarding this number, one can determine the least possible differential degree of a stabilizing controller before starting the controller design phase. It has been shown through various examples that the proposed design method is quite effective both in the stabilization and strong stabilization of LTI systems. It is worth noting that time-delay controllers have several advantages over conventional finite-dimensional ones. Empirical results have shown that one can achieve better results in terms of minimum spectral abscissa by using a relatively simple time-delay controller instead of using high-dimensional finite-dimensional controllers. It has also been stated that simple time-delay controllers are not only more powerful than finite-dimensional controllers but also more cost-effective as far as the controller design and implementation are concerned.

The decentralized stabilization and strong stabilization of LTI systems have been considered as an extension of their centralized counterparts. As in the centralized case, the eigenvalue optimization-based controller design method has been used to design decentralized controllers. As mentioned before, one can design decentralized controllers by structuring the controller matrices in an appropriate form. In this respect, the centralized stabilization problem has been modified in such a way that the local controllers of an overall decentralized controller are designed simultaneously by minimizing the spectral abscissa of the corresponding overall closed-loop system. In the decentralized strong stabilization problem, on the other hand, the stability of each local controller has been added as a constraint. Even though the decentralized strong stabilizability conditions exist for LTI finite-dimensional systems in the literature, these results had never been verified for LTI time-delay systems. For this reason, the existing necessary condition of decentralized strong stabilizability has been extended to LTI time-delay systems. It has been shown that the decentralized strong stabilization problem is solvable only if the system is decentralized stabilizable and satisfies PIP. We should note that the centralized stable controller design method proposed in this thesis can be thought of as an alternative contribution to the existing literature. However, the decentralized stable controller design method proposed here is a novel contribution since there exists no controller design method developed in this manner as far as the author's knowledge. It has been shown through various examples that the proposed design method is quite effective both in decentralized stabilization and decentralized strong stabilization of LTI systems.

As a final contribution to this thesis, a new Matlab-based software named DrStabilization has been developed. The main objective of this software is to utilize the analysis and design methods introduced in this thesis. In the analysis phase, one can determine the centralized/decentralized stabilizability and strong stabilizability of LTI time-delay systems. Then, in the design phase, centralized/decentralized structured time-delay controllers can be designed to stabilize or strongly stabilize LTI time-delay systems. As a final note, DrStabilization v1.0 has been used along with the necessary tools, e.g., QPmR and GRANSO, in the analysis and design phases of all examples presented in this thesis.

5.2. Future Research

As it is discussed above, both in centralized and decentralized cases, an effective controller design method has been developed in this thesis. In the light of the results of this thesis and the existing literature, there are at least three main research subjects that deserve attention in the near future. First, even though it has been known for a long time that the time-delay controllers can be used in the stabilization of LTI systems effectively, because of the difficulties in analytic design methods, the time-delay controllers do not get the attention they deserve. It is our belief that using the proposed controller design method, the use of time-delay controllers can be widespread. Second, as it is stressed in Chapter 2, the stabilization (and strong stabilization) methods proposed in this thesis are based on the minimization of the spectral abscissa function. Thus, the local minimum of such an optimization problem leads to the maximum (or very close to the maximum) exponential decay rate of the closed-loop system. Even though it might be required for certain applications, it is well-known that as the spectral abscissa approaches its local minima, the sensitivity of the rightmost modes increases, which causes a lack of robustness. Generally, there must be a balance between the time-domain performance (such as tracking and disturbance rejection ability) and robustness. In order to establish such a balance, the controller design method must be modified in such a way once the closed-loop stability is achieved, $\mathcal{H}^{\infty}$ norm of a related transfer function of the closed-loop must be minimized (for finite-dimensional controllers, see [27] for finite-dimensional systems, and [101] for time-delay systems). Third, as it is stressed in Subsection 2.3.2, each local controller of a decentralized controller can access only a certain part of a system. More precisely, k^{th} local controller, described as (2.45), is connected to $\{u_k, y_k\}$ input/output pairs of a system described as (2.31). However, instead of a strict decentralized controller, a decentralized overlapping controller can be designed where k^{th} local controller can access a certain part of the system in addition to $\{u_k, y_k\}$ input/output pairs (see [133] for finite-dimensional systems under the control of finite-dimensional decentralized overlapping controllers). By this way, it is possible to increase the degree-of-freedom of a decentralized controller by paying the communication network cost as a price. Such a decentralized overlapping controller can be designed using the controller design methods proposed in this thesis almost directly by using proper structuring. Finally, DrStabilization can be modified to be more efficient and user-friendly in the near future. Furthermore, DrStabilization can also be extended in line with the possible research objectives given above.

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Curriculum Vitae

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EDUCATION

Doctor of Philosophy, Electrical and Electronics Engineering, August 2022

Eskişehir Technical University, Eskişehir, Turkey

Thesis - "Stabilization and Strong Stabilization of LTI Systems by Structured Time-delay Controllers"

Master of Science, Electrical and Electronics Engineering September 2016

Anadolu University, Eskişehir, Turkey

Thesis - "Stabilization of Time-Delay Systems by Nonconvex Optimization Techniques"

European Embedded Control Institute (EECI)

International Graduate School on Control Independent Graduate Modules

- Module M5 - Decentralized and Distributed Control
  Instructors : Professor Giancarlo Ferrari-Trecate and Professor Marcello Farina
  Paris-Saclay, 2015
- Module M8 - Introduction to Nonlinear Systems Analysis and Nonlinear Feedback Control
  Instructor : Professor Hassan K. Khalil
  Yildiz Technical University, 2015
- Module M15 - Advances in Feedback Design for MIMO Nonlinear Systems
  Instructor : Professor Alberto Isidori
  Yildiz Technical University, 2015
- Module M16 - Time-delay Systems: Lyapunov Functional and Matrices
  Instructor : Professor Vladimir Kharitonov

St.Petersburg State University, 2015

- Module M12 - Model Predictive Control
  Instructor : Professor Jan Maciejowski
  Paris-Saclay, 2014

Bachelor of Science, Electrical and Electronics Engineering August 2013
Anadolu University, Eskişehir, Turkey
GPA 3.04

PROFESSIONAL EXPERIENCE

Eskişehir Technical University June 2018 - ongoing
Eskişehir, Turkey

- Serving as a research/teaching assistant in the department of electrical and electronics engineering.
- As a research assistant, employed in several research projects which are listed below.
- As a teaching assistant, participated in various undergraduate courses and their laboratory sections. Also, consulted to dissertation projects of senior students.
- Served as an instructor of *Fundementals of Control Laboratory* and *Advanced Control Systems Laboratory*.

Anadolu University March 2014 - June 2018
Eskişehir, Turkey

- Served as a research/teaching assistant in the department of electrical and electronics engineering.

Turkish Aircraft Industries Corporation (TAI) June-July 2012
Ankara, Turkey

- Attended to an engineering intern program.

ASELSAN A.Ş. June-July 2010

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RESEARCH INTERESTS

- Stabilization of complex dynamical systems
- Multi-agent systems and decentralized control
- Stability and stabilization of time-delay systems
- Time-delay controller design
- Optimization-based controller design methods
- Software development

RESEARCH PROJECTS

1) "Robust Controller Design for Time-delay Systems," supported by the Scientific Research Projects Commission of Eskişehir Technical University, 2021-2022.
2) "Decentralized Control of Time-delay Systems," supported by the Scientific Research Projects Commission of Eskişehir Technical University under grant number of 20DRP067, 2020-2022.
3) "Control of Time-delay Systems," supported by the Scientific Research Projects Commission of Anadolu University, 2018-2020.
4) "Decentralized Time-Delay Controller Design for Decentralized Time-delay Systems," supported by the Scientific and Technical Research Council of Turkey (TUBITAK) under grant number of 115E379, 2015-2018.
5) "Control of Time-delay Systems," supported by the Scientific Research Projects Commission of Anadolu University, 2017-2018.
6) "Decentralized Control of Infinite-dimensional Systems," supported by the Scientific Research Projects Commission of Anadolu University, 2016-2017.
7) "Stabilization of Time-delay Systems by Nonconvex Optimization Techniques," supported by the Scientific Research Projects Commission of Anadolu University under grant number of 1410F418, 2015-2016.

8) “Decentralized Control of Systems with Commensurate and Uncommensurate Time-delay Systems,” supported by the Scientific and Technical Research Council of Turkey (TUBITAK) under grant number of 115E153, 2012-2015.

PUBLICATIONS

Journal Papers

1) “Delay-based Stabilization and Strong Stabilization of LTI Systems by Nonsmooth Constrained Optimization”, S.M. Özer and A. İftar, International Journal of Control, 2022.

   Abstact: Stabilization and strong stabilization (i.e., the problem of stabilizing by a stable controller) of linear time-invariant (LTI) multi-input multi-output systems are considered. The considered systems may be finite-dimensional or time-delay systems, which may have multiple discrete time-delays. The controllers to be designed may also be either finite-dimensional or time-delay LTI controllers. In the controller design phase, a constrained non-smooth optimization-based design method is employed. The effectiveness of the proposed method is illustrated through various benchmark examples which are solved by using the recently developed Matlab-based software called DrStabilization. The examples also show that, in many cases, time-delay controllers may have certain advantages over conventional finite-dimensional controllers.

2) “Eigenvalue-Optimisation-based Centralised and Decentralised Stabilisation of Time-delay Systems”, S.M. Özer and A. İftar, International Journal of Control, 2021.

   Abstact: Both centralised and decentralised stabilisation of multi-input multi-output linear time-invariant (LTI) systems with discrete time-delays are considered. No restriction has been made on the number and type of the time-delays. The main objective is to design LTI finite-dimensional static/dynamic stabilising centralised/decentralised output feedback controllers. To design decentralised controllers, a sequential approach is used in order to ease the design process. For this, the decentralised pole assignment algorithm developed for decentralised stabilisation of finite-dimensional systems is extended to the time-delay case. In this approach, decentralised controllers are designed sequentially, where a centralised design approach is used in each step. As the centralised design approach, the eigenvalue optimisation-based stabilisation method is utilised, where the effect of small perturbations in the time-delays is also considered. Regarding the nonconvexity of the optimisation problem, an initialisation procedure, which determines the minimum possible dimension of the stabilising controller and the initial controller parameters, is also developed. Furthermore, in order to lessen the computational burden of the optimisation phase, the controller matrices are structured in a suitable canonical form. As a particular case of dynamic output feedback, the decentralised proportional-integral (PI) control is also considered. The proposed controller design methods are verified through numerical and practical examples.

Conference Papers

1) **(Invited Session)** "DrStabilization: A Matlab-based Software to Design Structured Time-delay Controllers", S.M. Özer and A. İftar, in Preprints of the 17th IFAC Workshop on Time-delay Systems, Montreal, Canada, 2022.

   Abstact: A new Matlab-based software called DrStabilization is introduced. The main purpose of the software is to stabilize multi-input multi-output linear time-invariant systems which may have an arbitrary number of discrete time-delays. The stabilization problem is formulated as the minimization of spectral abscissa function over the controller parameters. By modifying this problem, it is also possible to consider strong stability, i.e., stability achieved by a stable controller. The software allows designing structured dynamic output feedback controllers. By structuring the controller matrices, one can design decentralized controllers and/or well-known servocompansators such as PI/PID/PIR. Furthermore, the software also allows designing time-delay controllers in which case an arbitrary number of artificial time-delays can be introduced both to the I/O channels and the dynamics of the controller itself. The utilization of the software is illustrated through examples.

2) "Eigenvalue-Optimization-based Stabilization by Delayed Output Feedback", S.M. Özer and A. İftar, in Preceedings of the 29th Mediterranean Conference on Control and Automation, Bari, Italy, 2021.

Abstact: Eigenvalue-optimization-based stabilization of multi-input multi-output linear time-invariant systems is considered. The system is allowed to be a retarded time-delay system with discrete time-delays. No restriction has been made on the number of time-delays. Instead of conventional static or dynamic delay-free output feedback, delayed output feedback is considered in the controller part. Furthermore, in the case of multi-input and/or multi-output, the controller is allowed to introduce a different time-delay in each input-output channel. Benchmark examples illustrate the effectiveness of the proposed method.

3) "Performance Analysis of a DC-Motor Control System with Time-delay: Smith Predictor vs Optimization-based Controller Design", S.M. Özer, S. Yıldız, and A. İftar, in Preceedings of the 6th International Conference on Control Engineering and Information Technology, İstanbul, Turkey, 2018.

   Abstact: Smith predictor-based and optimization-based controller design for a DC-motor control system with a time-delay in the feedback loop is considered. It is shown that, compared to the Smith predictor-based controller, the optimization-based controller can preserve stability against larger variations in the system parameters and the time-delay. Furthermore, by simulations and actual experiments, it is verified that the optimization-based controller outperforms the Smith predictor-based controller when the system parameters and, especially, the time-delay varies.

4) "Optimization-based PV/PI Design for a DC-Motor System with Delayed Feedback", S.M. Özer, S. Yıldız, and A. İftar, in Preceedings of the 26th Mediterranean Conference on Control and Automation, Zadar, Croatia, 2018.

   Abstact: Proportional velocity (PV) and proportional integral (PI) controllers are designed to regulate the angular position and angular velocity, respectively, of a DC motor system with a pointwise time-delay in the feedback loop. Because of the time-delay, the system is described by delay-differential equations which have infinitely many modes that can not be assigned by using classical pole placement methods. The proposed method is based on minimizing the real part of the rightmost closed-loop mode, i.e., spectral abscissa, as a function of the controller parameters. The effectiveness of the method is illustrated by simulations and real-time experiments.

5) "Simultaneous Decentralized Controller Design for Time-Delay Systems", S.M. Özer

and A. İftar, in Preprints of the 20th World Congress of the International Federation of Automatic Control, Toulouse, France, 2017.

Abstact: Decentralized stabilization problem of linear time-invariant time-delay systems is considered. A non-smooth optimization based fixed-order controller design method is used to simultaneously design decentralized dynamic output feedback controllers for such systems. Furthermore, the MATLAB based software DCD-TDS is extended to realize the proposed method. An example design using this software is also presented.

6) "Observer-Based Stabilization of Time-Delay Systems: An Application to Diesel Engine", S.M. Özer and A. İftar, in Preceedings of the Asian Control Conference, Gold Coast, Australia, 2017.

Abstact: Observer-based stabilization of a general class of linear time-invariant dynamic systems with time-delays is considered. A recently developed non-smooth optimization based method is used to design the observer-based controller. The proposed stabilization method is then applied to a linear model of a diesel engine which involves time-delay.

7) "A Software to Design Decentralized Controllers for Time-delay Systems", S.M. Özer, G. Gülmez, and A. İftar, in Proceedings of the IEEE Conference on Computer Aided Control System Design, Buenos Aires, Argentina, 2016.

Abstact: A new MATLAB based software package which, besides analysis, can be used to design centralized or decentralized controllers to stabilize time-delay systems is introduced.

8) "Controller design for neutral time-delay systems by nonsmooth optimization", S.M. Özer and A. İftar, in Preprints of the 13th IFAC Workshop on Time-delay Systems, Istanbul, Turkey, 2016.

Abstact: Strong stabilization of linear time-invariant neutral time-delay systems is considered. A controller design algorithm based on a nonsmooth optimization approach is proposed. An initialization procedure for the non-convex optimization problem is also presented. Finally, the proposed approach is demonstrated by an example.

9) "Decentralized controller design for time-delay systems by optimization", S.M. Özer and A. İftar, in Preprints of the 12th IFAC Workshop on Time-delay Systems, Ann Arbor, MI, USA, 2015.

Abstact: Decentralized controller design problem for linear time-invariant retarded time-delay systems is considered. Algorithms, based on nonsmooth optimization and decentralized pole assignment, are developed to design decentralized stabilizing controllers for such systems. A MATLAB based software is being developed to realize these algorithms. An example design using this software is also presented.

10) "Design of a GUI for a Software to Design Decentralized Controllers for Time-Delay System", G. Gülmez, S.M. Özer, and A. İftar, in TOK Otomatik Kontrol Ulusal Toplantısı Bildiriler Kitabı, Eskişehir, Turkey, 2016 (Turkish).
11) "A software to design decentralized controllers for time-delay systems", S.M. Özer and A. İftar, in TOK Otomatik Kontrol Ulusal Toplantısı Bildiriler Kitabı, Denizli, Turkey, 2015 (Turkish).
12) "Zaman Gecikmeli Sistemler için Merkezi Olmayan Denetleyici Tasarımı", S.M. Özer, H.E. Erol, and A. İftar, In TOK Otomatik Kontrol Ulusal Toplantısı Bildiriler Kitabı, Kocaeli, Turkey, 2014 (Turkish).
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COMPUTER PROGRAMS

- *DrStabilization (Delayed Robust Stabilization)*

  DrStabilization is a MATLAB-based software package. It has been created for the analysis and stabilization of LTI time-delay systems. In addition to stability, strong stability (i.e., stability achieved with a stable controller) can also be aimed. The classes of the controllers to be designed are both finite-dimensional and time-delay structured output feedback controllers. The controller design algorithm is based on eigenvalue optimization. In particular, DrStabilization is used with a Matlab-based optimization package GRANSO which utilizes BFGS algorithms to solve nonsmooth and nonconvex constrained optimization problems. The software package is publicly available both in `https://github.com/smertozer` and in `https://mathworks.com/matlabcentral/profile/authors/21374970`.
- *DCD-TDS (Decentralized and Centralized Controller Design for Time-delay Systems)*

DCD-TDS is a MATLAB-based software package. It has been created for the analysis and stabilization of LTI time-delay systems. The class of the controllers to be designed is finite-dimensional dynamic output feedback controllers. The controller design algorithm is based on eigenvalue optimization. In particular, DCD-TDS is used with a Matlab-based optimization package HANSO which utilizes BFGS algorithms to solve nonsmooth and nonconvex unconstrained optimization problems.

LANGUAGE SKILLS

- Turkish (mother tongue)
- English (IELTS Academic; Speaking: 7.5, Listening: 6.0, Reading: 5.5, Writing: 5.5, Overall: 6.0, YOKDİL Exam; 84.5/100)