A Method for Estimating Stock-Out Based Substitution Rates by Using Point-of-Sale Data

by

Omer Cem Ozturk

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This is to certify that I have examined this copy of a master's thesis by

Omer Cem Ozturk

and have found that it is complete and satisfactory in all respects,

and that any and all revisions required by the final

examining committee have been made.

Committee Members:

Selcuk Karabati, Ph. D. (Advisor)

Baris Tan, Ph. D. (Advisor)

Fikri Karaesmen, Ph. D.

Ozden Gur Ali, Ph. D.

Serdar Sayman, Ph. D.

Date: 18.07.2006

ABSTRACT

Empirical studies in retailing suggest that stock-out rates are considerably large for many product categories. Stock-outs result in demand spill-over, or substitution, among items within a product category. Product assortment and inventory management decisions can be improved when the substitution rates are known. In this thesis, we study the problem of estimating product substitution rates using Point-of-Sale (POS) data only. We propose a practical approach that clusters POS intervals into states where each state corresponds to a specific substitution scenario. We then consolidate available POS data for each state, and estimate the substitution rates using the consolidated information. We first analyze the properties of the estimation method under a deterministic setting. We then provide an extensive computational analysis of the proposed substitution rate estimation method. A comparison with another estimation method from the literature and a simulation-based case study show that the estimation method we propose performs quite well with limited information.

ÖZETÇE

Perakendecilik alanında yapılan ampirik çalışmalar stok tükenmesi oranlarının çoğu ürün kategorisi için önemli ölçüde yüksek olduğunu göstermektedir. Ürünlerin stoklarının tükenmesi bu ürünlere gelen taleplerin aynı kategorideki diğer ürünlere kaymasına, yani ürün ikamesine neden olmaktadır. Kategorideki ürün çeşitliliği ve envanter yönetimi kararları ürün ikame oranları dikkate alınarak iyileştirilebilir. Bu tezde ele alınan problem sadece ürünlere ait satış noktası verilerini kullanarak ürünler arası ikame oranlarının tahmin edilmesidir. Önerdiğimiz yöntem ilk olarak satış noktası aralıklarını çeşitli ikame senaryolarına karşılık gelecek şekilde gruplamaktadır. Daha sonra her bir duruma ait satış noktası verileri konsolide edilmekte ve elde edilen bu konsolide veriler de ikame oranlarının tahmin edilmesinde kullanılmaktadır. Öncelikle yöntemimizin deterministik durumdaki akışkan modeli incelenmekte, daha sonra kapsamlı bir sayısal analizi sunulmaktadır. Yazında sunulan bir başka yöntemle yapılan karşılaştırma ve benzetim tabanlı bir vaka çalışması önerdiğimiz yöntemin kısıtlı veriyle oldukça iyi sonuçlar verdiğini göstermektedir.

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NOMENCLATURE

i, j	product indices
N	number of products in a category
L_i	order-up-to level of product i
λ_i	real demand for product i per unit time
$\hat{\lambda}_i$	estimated demand for product i per unit time
δ	probability of substitution
α_{ij}	probability of substituting product j for product i
$\hat{\alpha}_{ij}$	estimated probability of substituting product j for product i
p	POS interval index
τ	POS interval length
ρ_h	seasonality factor of h^{th} hour
H	operating time in a week
P_d	seasonality factor of d^{th} day
n	period index
$A_i[n]$	indicator variable for customer arrival to product i in period n
Δ	length of a period
$S_i[n]$	indicator variable for sales of product i in period n
$I_i[n]$	indicator variable for inventory status of product i in period n
$\underline{I}[n]$	inventory status of the category in period n
$\delta_{i,\underline{I}_0,n}$	indicator variable for inventory status of product i in period n given $\underline{I}_0$

$\delta_{j,i,\underline{I}_0,n}$	indicator variable for joint inventory status of product i and j in period n given $\underline{I}_0$
$\aleph_{\underline{I}_0}$	total number of periods where the inventory indicator state is $\underline{I}_0$
$s_{i,\underline{I}_0}$	expected value for the number of units sold of product i per unit time given the inventory indicator state is $\underline{I}_0$
$\hat{s}_{i,\underline{I}_0}$	approximation with finite sum for $s_{i,\underline{I}_0}$
$\varepsilon_{i,\underline{I}_0,\mathrm{T}}$	approximation error term due to using a finite sum
$s_i^{\,p}$	observed demand for product i in POS interval p
$\hat{s}_{i,\underline{I}^{pos}}$	average observed demand for product i in the POS intervals with inventory state vector $\underline{I}_0^{\,pos}$
r	error index in the optimization problem
w_r	weight of r^{th} error
$\aleph_{\underline{I}_0^{pos}}^{\;pos}$	total number of POS intervals where the inventory indicator state is $\underline{I}_0^{\,pos}$
$\alpha^f{}_{ij}$	deterministic fraction of substituting product j for product i
δ^f	total substitution fraction in the fluid model
T	replenishment cycle length in the fluid model
$I_i(t)$	inventory level of product i at time t
λ_i^{pos}	real demand for product i per POS interval $i \in A,B,C$
$\lambda_i^{pos'}$	demand for product i per POS interval when product A is out-of-stock $i \in B,C$
$\lambda_C^{pos''}$	demand for product C per POS interval when both product A and B are out-of-stock

st_i	stock-out time of product i in the fluid model
π	number of POS intervals per hour
$c_{\underline{I}_0^{pos}}$	number of POS intervals with inventory state vector $\underline{I}_0^{pos}$ per replenishment cycle in the fluid model
L'_B	inventory level of product B when product A stocks out in the fluid model
L'_C	inventory level of product C when product A stocks out in the fluid model
L''_C	inventory level of product C when product B stocks out in the fluid model
$L_{c_{\underline{I}_0^{pos}}}$	lower bound on $c_{\underline{I}_0^{pos}}$
$U_{c_{\underline{I}_0^{pos}}}$	upper bound on $c_{\underline{I}_0^{pos}}$
$\lambda^h{}_i$	hourly real demand for product i
$\lambda^w{}_i$	weekly real demand for product i
ECR	efficient consumer response
GMA	grocery manufacturers of America
DSD	direct store delivery
POS	point-of-sale
MLE	maximum likelihood estimation
SKU	stock keeping unit
BLUE	best linear unbiased estimator
EM	expectation-maximization
MNL	multinomial logit
SSBE	state space based estimation
MAPE	mean absolute percentage error
MAD	mean absolute deviation
MAX	maximum absolute deviation

Chapter 1

INTRODUCTION

Despite the adoption of initiatives such as category management and Efficient Consumer Response (ECR), empirical studies show that the measured stock-out rates in retailing are quite high and increase substantially during store promotions (Gruen et al. [1], Andersen Consulting [2], Mason and Wilkinson [3]). Empirical studies measure stock out rate in two different ways. Most of the studies cited in a report by Grocery Manufacturers of America (GMA) [1] and also a study by Andersen Consulting [2] define stock-out rate as percentage of SKUs that are out-of-stock in the retailer at a particular moment in time. Based on this definition, the average stock-out rate of the studies cited in GMA report was 8.3 percent for all products. The study of Andersen Consulting [2] also gives an average out-of stock rate of 8.2 percent. Alternatively, stock-out can also be measured as the proportion of time the consumer does not find a particular SKU. The number of empirical studies using this definition of stock-out rate is limited.

Stock-out situations affect consumers' decisions including whether, how much, and what to buy (Campo et al. [4]). Consumer response types can be grouped into different types: buy item at another store, delay purchase, substitute - same brand, substitute – different brand, and do not purchase the item (Gruen et al. [1]). Consumer responses can also be grouped as substitution actions and non-substitution actions. Recent studies reported that substitution actions, which differ from each other with respect to category, brand, variety, and size variations, comprised around 40 percent of consumer responses to out-of-stocks.

In this thesis, we consider a retailer that stocks a certain number of products in a category. If customers cannot find their first-choice product on the shelf, the demand for that product will either spill over to another product according to a probabilistic substitution structure or it will be lost. The substitution rate between product A and B is defined as the fraction (or probability) of customers who arrive demanding product A but purchase product B since product A is not available. In our setting, if demand substitution occurs and the second-choice product is not available either, the demand is lost.

We develop a method to estimate stock-out based substitution rates among products in a category by using Point-of-Sale (POS) data only. In stock-out based substitution, the sale opportunity is lost if consumers' first-choice product is carried in the retailer but temporarily unavailable due to a stock out. On the other hand, in assortment based substitution, consumers switch to a substitute if their favorite product is not carried in the store. The models presented in this thesis work with the POS data, and therefore the demand observed through the POS data reflect the demands of items included in the pre-specified assortment.

To estimate the stock-out based substitution patterns in this setting, we propose an approach that clusters POS intervals into states where each state corresponds to a specific substitution scenario. We then consolidate available POS data for each state, and estimate the substitution rates using the consolidated information. We analyze the dynamics and the performance of the method by using a deterministic flow model. We present the comparison of our method with Maximum Likelihood Estimation (MLE) method proposed in the literature in the case where initial stocking levels and replenishment lengths are known. Furthermore, we provide a case study of our proposed substitution rate estimation method. The results suggest that the estimation method we propose performs quite well with limited information.

Out-of-stocks and subsequent responses of consumers (especially product substitution) have a significant impact on both retailers and manufacturers. For example, for manufacturers, product substitution may result in a partial loss due to a substitution to a less costly item (in the case of substitution to the same brand) or, more unfavorably, results in a shift within the category to competitors' items (in the case of substitution to a different brand). When it comes to retailers, product substitution affects order quantities and expected profits of retailers (Rajaram and Tang [5]). Substitution rates can be used to determine the optimal assortment of products and the optimal inventory levels of products included in the assortment. Consequently, retailers should take into account the substitutability of products when making assortment and inventory decisions about a category.

The main contribution of this thesis is the development of a method that can be used under very general conditions to estimate product substitution rates in a category by using only POS data. More precisely, our method does not assume any arrival, substitution, or inventory control structure.

The outline of the thesis is as follows. In Chapter 2, previous work related to the estimation of product substitution rates is reviewed. In Chapter 3, we describe our model and the proposed method of estimation. In Chapter 4, the analysis of our method as a fluid model is presented. In Chapter 5, we provide the computational results regarding the performance of our method. Finally, we present the results of a comprehensive simulation-based case study in Chapter 6 and concluding remarks in Chapter 7.

Chapter 2

LITERATURE REVIEW

2.1 Consumer Response to Stock-Outs

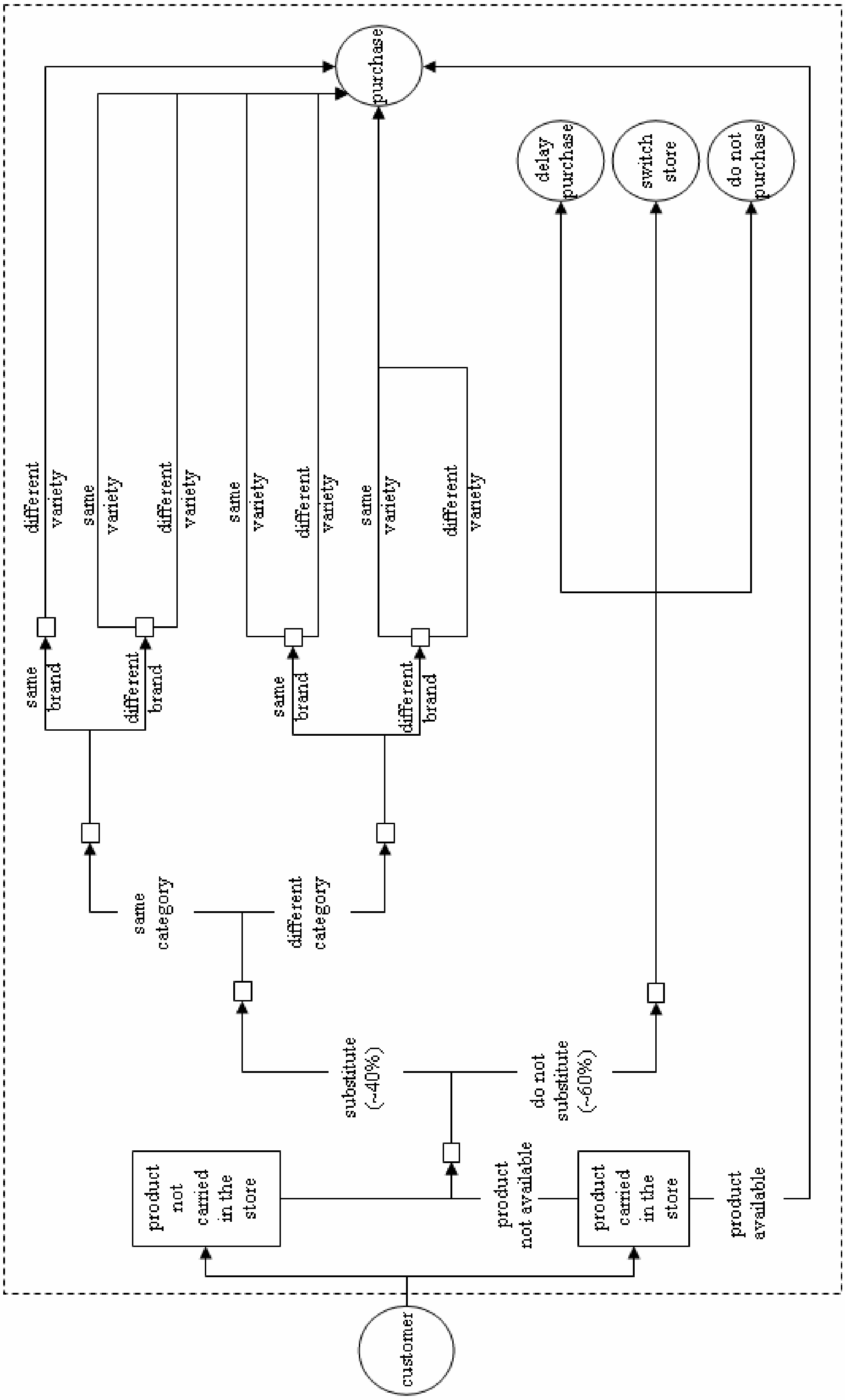
2.1.1 Consumer Response Types

In the literature, over 10 different ways consumers respond to an out-of-stock situation are reported. In the taxonomy of Colacchio et al. [7], consumer responses are primarily divided into two categories: substitution actions and non-substitution actions. Moreover, substitution actions differ from each other with respect to category, brand, variety, and size variations. As for non-substitution actions, a consumer can switch the store, delay her purchase, arrange a special trip only for that product, or simply decide not to buy that item.

In practice, researchers focus on five main responses that can be observed during an out-of-stock period (Gruen et al. [1]):

- Buy item at another store (i.e. store switch),
- Delay purchase (i.e. buy item later at the same store),
- Substitute – same brand (for a different type or size),
- Substitute – different brand (brand switch),
- Do not purchase the item (i.e. lost sale).

Figure 2.1 depicts possible responses of consumers in the case where they cannot find their favorite product on the shelf.



Source: Author's own drawing based on Gruen et al. [1] and Colacchio et al. [7].

Figure 2.1: Consumer responses to product unavailability situations

A study by Gruen et al. [1] provides comprehensive statistics about average consumer responses by different categories and regions. The results reveal that, on the average, 32 percent of the consumers choose "buy item at another store," 17 percent "delay purchase," 20 percent "substitute-same brand," 20 percent "substitute-different brand," and 11 percent "do not purchase the item" option. The results indicate that consumer responses differ considerably by category. Especially, paper towels and toilet tissue categories exhibit high levels of substitution which amount to 50 percent total substitution (i.e. sum of substitution within or among brands). On the other hand, cosmetics and coffee exhibit low levels of substitution. When it comes to the average responses by region, we observe that European consumers tend to substitute more when their favorite product is out-of-stock with a total substitution percentage of 48 percent. Whereas, U.S. consumers prefer to switch brands less; in other words, European consumers are approximately 50 percent more likely to buy another brand in the case of a stock-out. On the other hand, U.S. consumers are more likely to choose "substitution-same brand" option (e.g. substitute to a different package size).

Table 2.1 provides total substitution rates reported in the major consumer response studies; these values demonstrate that stock-out-based substitution rates are considerably high. This is parallel to the findings cited in Urban et al. [8], Emmelhainz and Stock [9] and Food Marketing Institute [10].

2.1.2 Effects of Consumer Responses on Retailers and Manufacturers

Consumer responses, especially product substitution, in out-of-stock situations have substantial effects on retailers and manufacturers. Facing a stock-out, consumer need to decide whether, how much and what to buy. Campo et al. [4] examines this issue in a one store setting by using two categories. Their results suggest that stock-outs may change each of the previously mentioned consumer decisions. Also the results reveal that overall losses foreseen for the retailer are lower than is often believed, whereas manufacturer losses may be significant. In a similar study, Rajaram and Tang [5] discuss the impact of product substitution on order quantities and expected profits of the retailer.

Study	**Total Substitution (%)**
A.C. Nielsen, 1962, U.S.	58
Schary and Christopher, 1979, UK	22
Coca-Cola Retailing Research Council, 1996, U.S.	60
National Association of Convenience Stores, 1998, U.S.	71
Data Ventures top 2000 SKUs, 1999, U.S.	30
Campo et al., 2000, Belgium	55
GMA DSD, 2002, U.S.	60
Gruen et al., 2002, U.S. data (11 categories)	40
Gruen et al., 2002, Worldwide data (8 categories)	45

Source: Gruen et al. [1].

Table 2.1: Total substitution percentages in consumer response studies

First, they analyze the extended news-vendor model for two products analytically. Then, they provide numerical results for the *n* product case. The result of this study is four-fold. First, it is observed that demand substitution between products always leads to higher expected profits for the retailer than that obtained in no-substitution case. Second, the retailer should give more importance to designing a substitutable assortment. Third, the value of planning for substitutable demand becomes more significant when the degree of substitutability between products is low, demand variation of products is high, there exist disparate levels of demand uncertainty of products in the assortment and product demand is positively correlated. Finally, the highest gains in expected profit from the base case (i.e.

no substitution) are obtained when individual product demand is highly negatively correlated and there is a high level of demand uncertainty.

Colacchio et al. [7] discuss the implications of temporary stock-outs for the retailer. They classify the effects as negative, potentially negative and non-negative. They suggest that the decisions like not to buy -including store substitution- and switch to less costly goods have negative effects on the retailer's outcome; on the other hand, decisions like delaying purchase or planning a special trip are classified as potentially negative. Then substitution to a more expensive or an equivalent product is evaluated as a non-negative class.

Gruen et al. [1] examine the effects of out-of-stock situations on both sides. Actually, they classify losses into two parts: direct and indirect losses. This study suggests that consumer responses such as "buy item at another store" and "do not purchase the item" cause direct losses to retailers, whereas "substitute-different brand" causes the biggest loss to manufacturers. In addition, "delay purchase" and "substitute-same brand" adversely affect the financial flows and result in partial losses for both parts.

Besides the direct losses, the retailer and the manufacturer may incur indirect losses. In fact, these losses are mostly due to decreased consumer satisfaction. For example, if the consumer chooses "buy item at another store" option she will have a chance to try a different store, which may eventually lead to a permanent store switching. Moreover, responses to out-of-stock situations cause problems in the supply chain because sales figures do not reflect the actual demand.

In sum, when faced with a stock-out a consumer should decide to buy another item or switch the store or delay purchase or do not buy the item at all. This decision has a considerable effect on retailers' and manufacturers' performance (e.g. expected profits) both in a direct way and in an indirect way. Consequently, both parties should analyze this decision process carefully, which entails the estimation the substitution rates.

2.2 Estimation of Demand from Sales Information

The estimation of demand from sales information is similar to the problem of estimation of censored and truncated distributions, which has been widely analyzed in the econometrics literature. The occurrences of stock-outs and consumer substitution affect the observed sales of a product, which in turn affect the estimation of demand. The observed sales of an item no longer reflect its core consumer demand –demand when all other items in the category are available. In other words, observed sales of goods that become unavailable exhibit a data stream that is right-censored. On the other hand, sales figures of goods that are available –when the others are not- will probably be higher than their core demands as a result of substitution.

A number of studies (Shaw [11], Grogger and Carson [12], and Mullahy [13]) examine estimators from count data from truncated samples. To be more precise, Shaw [11] suggests Poisson and normal regression models for the analysis of truncated samples of count data whereas Grogger and Carson [12] present a class of maximum likelihood estimators for count data from truncated samples. They exemplify estimators for Poisson and negative binomial distributions. Mullahy [13] provides an application of count regression model in a study of daily beverage consumption. However, in order for these methods for estimating demands from sales information perform accurately, the truncated or censored parts should be small. Greene [14] and Allison [15] provide detailed information about censored and truncated distributions and the estimation of their parameters.

In the literature, authors mainly try to estimate demand rates from sales information in two different settings: estimation in the single product case and estimation in the multi-product case. Nahmias [16] considers the former setting in an inventory system with lost sales. He presumes that demands form a sequence of IID normal random variables and compares three estimators for the mean and standard deviation of the demand distribution through extensive simulations. These three estimators are a maximum likelihood estimator, a best linear unbiased estimator (BLUE), and a new estimator proposed in the study. Agrawal and Smith [17] propose a parameter estimation methodology for the case of single product with negative binomial distribution and unobserved lost sales.

The estimation of demand distribution parameters in the case of multiple partially substitutable products is analyzed by Anupindi et al. [6] (a detailed explanation of the method is presented in Appendix A). In this paper, they present a method for estimating consumer demand by assuming a Poisson arrival process for all the products and solving for the arrival rates and substitution probabilities through maximizing the likelihood function by means of the expectation-maximization (EM) method. They also assume that products are replenished with some periodicity. First, they present a model of customer arrivals and choice between products. Then, maximum likelihood estimates (MLEs) of demand parameters are obtained for both perpetual (including inventory data at restocking points with POS data) and periodic (including inventory data at restocking points) inventory tracking systems. On the other hand, Campo et al. [4] present an application of the multinomial logit (MNL) estimation together with availability data and quantify the effect of out-of-stock situations in category sales (In fact, MNL is extensively employed as a choice model in the marketing literature. We do not elucidate this framework in detail since it is not directly related to our work. For further explanation see, for example, Greene [14]). Kok and Fisher [18] provide a new methodology for assortment-based demand estimation under an assumed relatively simple substitution structure and use the estimations for the stock-out based substitution. Our estimation procedure is different from all these methods in that it does not assume any arrival or substitution structure.

2.3 Customer Arrival and Choice Models

In retail settings with substitution, observed sales are a ramification of two processes: customer arrival process and customer choice process. Researchers have made different assumptions regarding these processes in the literature.

When it comes to consumer arrival process, Agrawal and Smith [17] provide results – found by using sales data from a major retailing chain- indicating that the negative binomial distribution fits considerably better than the Poisson or the normal distribution. As mentioned earlier, Nahmias [16] presumes that demands constitute a sequence of IID normal random variables. This can be seen as a residual demand process after eliminating seasonality and promotional nonstationarities. Anupindi et al. [6] assume that the customer arrivals to the retailer constitute a Poisson Process. The authors support their arrival process choice with two arguments: (1) the population generating the arrivals can be assumed to be large enough that arrivals between successive time intervals are independent and (2) customers make their shopping at the retailer such that at most one arrival occurs at any given time instant. Campo et al. [4] define a purchase incidence and quantity model; that is, the probability that a customer will realize a category purchase on a given purchase occasion is modeled by a logistic function and the probability that a certain quantity will be purchased is modeled as a truncated Poisson process.

The other important process affecting observed sales is the consumer choice process. Different choice processes and different restrictions on them are presented in the marketing literature. For example, Noonan [19] presumes that the demand is composed of two stages. The first choice demand is satisfied with the inventory at hand in the first stage; then, the remaining demand is directed to the other products in deterministic proportions in the second stage (an approach also utilized by McGillivray and Silver [20], Parlar and Goyal [21], Parlar [22], Ernst and Kouvelis [23], Rajaram and Tang [5], and Netessine and Rudi

[24]). Moreover, the choice process is restricted such that there is only one substitution attempt. A similar approach is also used by Netessine and Rudi [24]. Bell and Fitzsimons [25] employ a random utility model of consumer choice. On the other hand, Mahajan and Ryzin [26] assume that the customer's choice process is based on a simple utility maximization mechanism. According to that mechanism, each customer assigns a certain utility to purchasing a certain item and to the no-purchase option. Then, based on the inventory level and the utility vector (composed of utilities assigned to each variant and the no-purchase option), customer makes her decision so as to maximize her utility. Thus, the final decision would be either to purchase an item or not to buy an item depending on the utility vector. Kok and Fisher [18] model the consumer choice with MNL framework, which is also considered as a special case by Mahajan and Ryzin [26].

Anupindi et al. [6] describe their choice process for two products as follows: when both variants are available, the consumer chooses variants with respect to certain probabilities (i.e. p_A for product A and p_B for product B); then the probability of "not buying an item" is the remaining proportion (i.e. $1-p_A-p_B$). When one of the products is not available from a certain point in time until replenishment, the authors assume that the other product is chosen by a greater probability then its initial value because the two products are partially substitutable. In other words, some customers substitute the available product for the out-of-stock one. Moreover, it is assumed in the study that the demand cannot be backlogged and the consumer choice probabilities are independent for successive arrivals. For the multi-item case they use one-stage substitution restriction, which is also used by Noonan [19] and Agrawal and Smith [17].

In sum, there is a sizable marketing literature dealing with consumer choice among variant alternatives. Along with the examples mentioned above, we can refer the interested readers to studies by Guadagni and Little [27], Bultez and Naert [28], Shugan [29], Jain et al. [30], Gupta [31], Chiang [32], and, Fader and Hardie [33].

2.4 Assortment Optimization and Inventory Control under Substitution

The inventory management and assortment optimization when core demand and substitution rates are known have been studied in the literature. The inventory models with demand substitution in the two product case are presented in McGillivray and Silver [20], Parlar and Goyal [21], Parlar [22], Pasternack and Drezner [34], Moinzadeh and Ingene [35], Drezner et al. [36], Narayanan and Raman [37], and Rajaram and Tang [5]. The above mentioned studies consider a single period problem setting with deterministic substitution patterns. These models may not be suitable for the retail setting where the substitution patterns are probabilistic.

In inventory models with multiple items, some studies assume that substitution is realized in deterministic proportions; works by Pentico [38], Bitran and Dasu [39], Klein et al. [40], Chand et al. [41], Hsu and Bassok [42], Bassok et al. [43], Noonan [19], and Netessine and Rudi [24] fall into that group. On the other hand, van Ryzin and Mahajan [44], Smith and Agrawal [45], and Mahajan and van Ryzin [26] consider optimal inventory levels of more than two items when substitution pattern is assumed to be probabilistic.

Along with the issue of determining optimal stocking levels of items for a fixed assortment, another important issue is choosing the optimal assortment. In that field, van Ryzin and Mahajan [44] examine the structural properties of the optimal assortments in a stochastic single period problem. Cachon et al. [46] study different models of assortment planning in the presence of consumer search (i.e., consumers may search other retailers before buying the item). Smith and Agrawal [45] try to find an optimal inventory policy with multi-period base-stock inventory models. On the other hand, Rajaram [47] provides a mean-variance analysis of assortment planning without substitution.

Chapter 3

MODEL DESCRIPTION AND ESTIMATION METHOD

Many studies about the retailing industry indicate that temporary stock-outs occur frequently in most of the supermarkets all over the world (see Chapter 1). During these temporary periods, if a customer cannot find her first-choice variant she might switch to an available variant in the same category. Therefore, if we take into account these substitution effects, the dynamics of the consumer demand process alter significantly. In this chapter, we will attempt to answer the question whether the demand and substitution rates of multiple partially substitutable products can be estimated when only POS data is available.

3.1 Model Description

We consider a retailer that stocks and sells N products in a category. Demand rates for these N products are $\lambda_1, \lambda_2, ..., \lambda_N$, respectively. We first consider the case with time-homogenous demand rates. Later we consider the case where demand rates change according to store-specific daily seasonality factors. In this case, we first deseasonalize the time series with the store-specific seasonality factors and then use the method outlined below directly. If customers cannot find their first-choice product on the shelf, the demand for that product will either spill over to another product according to a probabilistic substitution structure or will be lost. With probability $1-\delta$ customers do not substitute. The study by GMA [1] report that δ is approximately 40 percent. Given that a customer decides to substitute, the probability of substituting product j for product i is α_{ij}. The retailer has POS data including information on the number of units sold of each item in each POS interval p, which has a length of τ (hours).

3.2 Estimation Method

In our method, we define a set of sub-states and a related state structure to form the basis for estimation. First sub-states are labeled by taking into account the corresponding amount of sales of each product. Then, states are identified for different combinations of sub-states. After the identification of states, our method estimates demand rates of each product. Finally, an optimization is carried out by using the estimated demand rates and realized demands in POS intervals to estimate substitution rates among products in the category. The procedure is elucidated in more detail with examples for the three product case in the subsequent part.

We first explain our method: State Space Based Estimation (SSBE). Then, we provide a detailed description of the implementation of SSBE. For ease of explanation, we initially elucidate the case with complete information, i.e., the case with known initial stocking levels and replenishment cycle lengths. Finally, we explain SSBE with POS data only.

3.2.1 State Space Based Estimation (SSBE)

Consider a discrete time stochastic process where a customer demanding product i arrives with rate λ_i. Then, we can observe the state of the system by a number of indicator variables: A_i, S_i, and I_i.

So, first let

$$A_i[n]=\begin{cases}1 & \text{if customer demanding product } i \text{ arrives in period } n\\ 0 & \text{otherwise.}\end{cases}$$

Then, we will have:

$$\lambda_i=\lim_{T\to\infty}\frac{\sum_{n=1}^{T}A_i[n]}{T}. \qquad (3.1)$$

Moreover, let

$$S_i[n]=\begin{cases}1 & \text{if product } i \text{ is sold in period } n \text{ (immediately following the arrival of a customer)}\\ 0 & \text{otherwise}\end{cases}$$

$$I_i[n]=\begin{cases}1 & \text{if product } i \text{ is available in period } n \text{ (just after the customer arrives and before the sales is completed)}\\ 0 & \text{otherwise}\end{cases}$$

Then, the probability that product i is sold in period n can be written as:

$$P[S_i[n]=1]=P[A_i[n]=1,I_i[n]=1]+\sum_{j\neq i}\alpha_{ji}P[A_j[n]=1,I_j[n]=0,I_i[n]=1]. \qquad (3.2)$$

Let $\underline{I}[n]=\left[I_1[n],I_2[n],\ldots,I_N[n]\right]$ be the inventory indicator state vector for period n. If we condition Equation 3.2 on a particular realization of the inventory indicator state vector (i.e. $\underline{I}[n]=\underline{I}_0$), we obtain:

$$\begin{aligned}P[S_i[n]=1\,|\,\underline{I}[n]=\underline{I}_0]&=P[A_i[n]=1,I_i[n]=1\,|\,\underline{I}[n]=\underline{I}_0]\\&+\sum_{j\neq i}\alpha_{ji}P[A_j[n]=1,I_j[n]=0,I_i[n]=1\,|\,\underline{I}[n]=\underline{I}_0].\end{aligned} \qquad (3.3)$$

Since the arrival process is independent of the inventory status, Equation 3.3 becomes:

$$P[S_i[n]=1 \mid \underline{I}[n]=\underline{I}_0] = P[A_i[n]=1]P[I_i[n]=1 \mid \underline{I}[n]=\underline{I}_0] + \sum_{j \neq i} \alpha_{ji} P[A_j[n]=1]P[I_j[n]=0, I_i[n]=1 \mid \underline{I}[n]=\underline{I}_0]. \quad (3.4)$$

Let two indicator variables $\delta_{i,\underline{I}_0,n}$ and $\delta'_{j,i,\underline{I}_0,n}$ be defined as follows:

$$\delta_{i,\underline{I}_0,n} = \begin{cases} 1 & \text{if } I_i[n]=1 \text{ in } \underline{I}_0 \\ 0 & \text{otherwise} \end{cases} \qquad \delta'_{j,i,\underline{I}_0,n} = \begin{cases} 1 & \text{if } I_i[n]=1 \text{ and } I_j[n]=0 \text{ in } \underline{I}_0 \\ 0 & \text{otherwise.} \end{cases}$$

Then, Equation 3.4 can be expressed as:

$$P[S_i[n]=1 \mid \underline{I}[n]=\underline{I}_0] = P[A_i[n]=1]\delta_{i,\underline{I}_0,n} + \sum_{j \neq i} \alpha_{ji} P[A_j[n]=1]\delta'_{j,i,\underline{I}_0,n}. \quad (3.5)$$

Note that if $\delta_{i,\underline{I}_0,n}=0$ then $\delta'_{j,i,\underline{I}_0,n}=0$. In addition, if $\delta_{i,\underline{I}_0,n}=1$ then $\delta'_{j,i,\underline{I}_0,n}=0$ or 1 depending on $I_j[n]$. Therefore, we have $\delta'_{j,i,\underline{I}_0,n}=\delta_{i,\underline{I}_0,n}(1-\delta_{j,\underline{I}_0,n})$. Then, Equation 3.5 can be rewritten as:

$$P[S_i[n]=1 \mid \underline{I}[n]=\underline{I}_0] = \delta_{i,\underline{I}_0,n}[P[A_i[n]=1] + \sum_{j \neq i} \alpha_{ji} P[A_j[n]=1](1-\delta_{j,\underline{I}_0,n})] \quad (3.6)$$

By using Equation 3.6 we can write:

$$\lim_{T\to\infty}\frac{\sum_{n=1}^{T}P[S_i[n]=1\,|\,\underline{I}[n]=\underline{I}_0]}{T}=\delta_{i,\underline{I}_0,n}[\lim_{T\to\infty}\frac{\sum_{n=1}^{T}P[A_i[n]=1]}{T}$$
$$+\sum_{j\neq i}\alpha_{ji}\lim_{T\to\infty}\frac{\sum_{n=1}^{T}P[A_j[n]=1]}{T}(1-\delta_{j,\underline{I}_0,n})] \qquad (3.7)$$

Therefore, through the use of Equation 3.1 we can rewrite Equation 3.7 as follows:

$$\lim_{T\to\infty}\frac{\sum_{n=1}^{T}P[S_i[n]=1\,|\,\underline{I}[n]=\underline{I}_0]}{T}=\delta_{i,\underline{I}_0,n}[\lambda_i+\sum_{j\neq i}\alpha_{ji}\lambda_j(1-\delta_{j,\underline{I}_0,n})] \qquad (3.8)$$

Moreover, the expected sales of product i in period n given the inventory indicator state $\underline{I}_0$ is:

$$E[S_i[n]\,|\,\underline{I}[n]=\underline{I}_0]=P[S_i[n]=1\,|\,\underline{I}[n]=\underline{I}_0]. \qquad (3.9)$$

Thus, the expected sales of product i per unit time given the inventory indicator state $\underline{I}_0$, denoted by $s_{i,\underline{I}_0}$, can be expressed by using Equations 3.8 and 3.9 as follows:

$$s_{i,\underline{I}_0}=\lim_{T\to\infty}\frac{\sum_{n=1}^{T}E[S_i[n]\,|\,\underline{I}[n]=\underline{I}_0]}{T}=\delta_{i,\underline{I}_0,n}[\lambda_i+\sum_{j\neq i}\alpha_{ji}\lambda_j(1-\delta_{j,\underline{I}_0,n})]. \qquad (3.10)$$

Also note that, for all n such that $\underline{I}[n]=\underline{I}_0$, we have $\delta_{i,\underline{I}_0,n}=\delta_{i,\underline{I}_0}$. Thus, Equation 3.10 becomes:

$$s_{i,\underline{I}_0} = \delta_{i,\underline{I}_0}[\lambda_i + \sum_{j \neq i} \alpha_{ji} \lambda_j (1 - \delta_{j,\underline{I}_0})]. \quad (3.11)$$

In Equation 3.11, it is clear that if $\delta_{i,\underline{I}_0} = 0$ then $s_{i,\underline{I}_0} = 0$. On the other hand, if we choose $\underline{I}_0'$ such that $\delta_{i,\underline{I}_0'} = 1$, from Equation 3.11 we will have:

$$s_{i,\underline{I}_0'} = \lambda_i + \sum_{j \neq i} \alpha_{ji} \lambda_j (1 - \delta_{j,\underline{I}_0'}). \quad (3.12)$$

Consider a POS interval, which consists of a certain number of periods. Then, all of those periods have the same inventory indicator state as the POS interval they belong. Therefore, an approximation for $s_{i,\underline{I}_0'}$ with a finite sum, denoted by $\hat{s}_{i,\underline{I}_0'}$, can be obtained by taking the average of sales observed in the POS intervals with the inventory indicator state equal to $\underline{I}_0'$.

With this approximation, Equation 3.12 becomes:

$$\hat{s}_{i,\underline{I}_0'} = \lambda_i + \sum_{j \neq i} \alpha_{ji} \lambda_j (1 - \delta_{j,\underline{I}_0',n}) + \varepsilon_{i,\underline{I}_0',\mathrm{T}} \quad (3.13)$$

where $\varepsilon_{i,\underline{I}_0',\mathrm{T}}$ is the error due to estimating the average sales rate by using a finite number of sales periods.

So, when we choose the intervals in which all of the products are available we have $\delta_{i,\underline{I}_0'} = 1$ for each i. Therefore from Equation 3.13, $\hat{s}_{i,(1,1,...,1)} = \lambda_i + \varepsilon_{i,(1,1,...,1),\mathrm{T}}$; in other words, we have N equations to estimate λ_i s. Likewise, we have $(2^{N-1})N$ equations for the other product availability states. Furthermore, when $\underline{I}_0' = (0,0,...,0)$ we have: $0 = 0$ (N equations). So, we have at most $(2^{N-1} - 1)N$ equations and $N^2 - N$ unknowns (substitution parameters) and we solve the equations to minimize the estimation errors. We call this method State-Space Based Estimation (SSBE) method.

3.2.2 SSBE with Complete Information

In this section, initial stocking levels of products are assumed to be available as an input to the estimation procedure. For explanation simplicity, suppose that we have a category with three items, labeled A, B, and C. In this case, we try to estimate three demand parameters ($\lambda_A, \lambda_B, \lambda_C$) along with six substitution parameters ($\alpha_{AB}, \alpha_{AC}, \alpha_{BA}$, $\alpha_{BC}, \alpha_{CA}, \alpha_{CB}$).

Consider the POS intervals throughout which all of the three products are available. In those POS intervals the observed demands of products are generated only from the core demands of products. Therefore, demand parameters can be estimated by taking averages of demands for each product in such POS intervals. On the other hand, in estimating the substitution rates, we should focus on situations in which at least one of the products is unavailable. For example, the effect of α_{AB} can be observed in the POS intervals where product A is out-of-stock and product B is available. Clearly, the effects of the other substitution rates can be examined from the corresponding combinations of availability of the products in the category. In fact, SSBE provides a systematic way, which requires the definition of a set of sub-states and related state structure, to capture these situations, calculating appropriate statistics, and finally finding estimates for the parameters.

Before describing the sub-state and state labeling procedure, it would be useful to see an example of POS data providing the basic information needed in the estimation process. Table 3.1 presents a weekly POS data (with a POS interval length of one hour) of a category in a retailer operating 10 hours/day, 7 days/week. We will later use this table for illustration. The POS data table, initial stocking levels of each product, and the replenishment intervals, are the inputs of the estimation procedure. So, we define the state of each POS interval by using these input data.

An overview of the five steps of SSBE with complete information is presented in Figure 3.1.

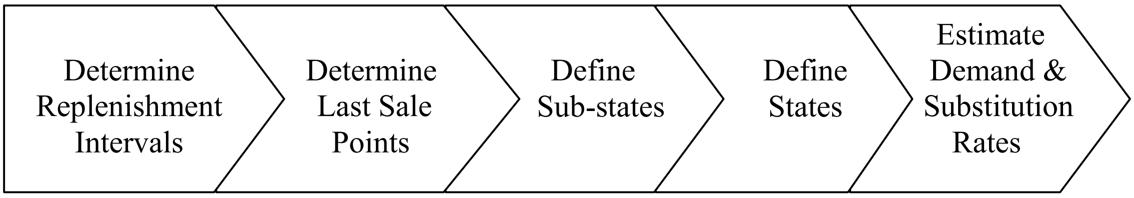


Figure 3.1: SSBE with complete information

The following part provides detailed explanations of the 5 steps.

Step 1: Determine Replenishment Intervals

In this step, we determine the POS intervals in which replenishments occur by using the periodicity of replenishment and the length of a POS interval. For example, consider the example of POS data given in Table 3.1. Suppose that the orders are given once a week and the length of POS interval is equal to one hour. Assuming that the retailer gives an order at the beginning of the first interval (and receives it instantly as lead time is zero), the first replenishment interval is the first POS interval. The second replenishment occurs 70 hours later, which is just the beginning of 71^{st} POS interval, and so on.

POS No	Number of Sales		
	Product A	Product B	Product C
1	2	0	6
2	0	1	7
3	2	1	2
4	0	1	2
5	1	0	3
6	2	2	0
7	4	3	2
8	0	1	5
9	1	0	4
10	3	2	5
11	4	1	4
12	5	1	6
13	1	1	4
14	0	1	4
15	0	2	2
16	2	1	2
17	5	2	3
18	2	1	2
19	3	1	3
20	2	2	1
21	3	3	3
22	2	2	2
23	2	1	2
24	0	1	1
25	2	1	1
26	1	2	1
27	0	1	0
28	1	1	2
29	3	2	1
30	3	2	2
31	2	3	3
32	2	0	3
33	0	3	1
34	3	0	2
35	2	1	3
36	1	3	2
37	1	1	5
38	4	1	0
39	1	1	3
40	2	2	5
41	1	0	0
42	1	3	3
43	4	1	1
44	6	2	1
45	3	0	4
46	3	2	4
47	4	1	7
48	3	0	1
49	1	2	6
50	3	2	2
51	1	2	7
52	3	0	4
53	6	0	3
54	2	0	2
55	3	0	5
56	2	0	3
57	3	0	3
58	3	0	4
59	3	0	2
60	3	0	2
61	3	0	4
62	1	0	2
63	0	0	4
64	0	0	5
65	0	0	6
66	0	0	5
67	0	0	3
68	0	0	0
69	0	0	0
70	0	0	0

Table 3.1: An example of POS data

Step 2: Determine Last Sale Points

After specifying the replenishment intervals, our method determines the last sale points of each product for each replenishment cycle. This is equivalent to determining the last point where we encounter a non-zero sale in the cycle. So, for our example the last sale points for products are calculated as the 62nd, 51st and 67th intervals.

Step 3: Define Sub-States

Before explaining how the method labels the POS intervals, we should make clear what the sub-states mean and how they are represented. Actually, we define three different sub-states as described in Table 3.2.

Sub-State	**Description**
✓	Product is on the shelf during the POS interval.
O	Product has been depleted completely within the POS interval.
✕	Product is unavailable from the beginning of the POS interval.

Table 3.2: Descriptions of sub-states

In order to label sub-states, we first determine the amount of sale of a product in a cycle. When this is equal to its initial stock number, then the last sale point for this product is the transition interval. Again, consider the example above. We know that the initial stock levels for product A, product B, and product C are 136, 70, and 202 respectively (these values are calculated by the method explained earlier for a 95 percent service level). These levels are equal to the number of units sold of each product, which can be calculated by using the sales figures in Table 3.1. In other words, all three products stocked out in the cycle.

Since we have already determined the last sale POSs for the cycle, we know that this point corresponds exactly to the sub-state O, which means that the last unit of product is sold within this interval. Moreover, all of the POS intervals before the last sale point in the same cycle can be represented by ✓; because even if we see a zero sale case before the last sale point, we are sure that the product was on the shelf but no demand was realized. Finally, the product is not available from the beginning of the POS interval for all of the remaining POS intervals, and we then represent these with $\times$ (See Table 3.3).

Step 4: Define States

Every POS interval is described by a state, which is related to each of the sub-states of products in that POS interval. In fact, the states used by the method correspond to the inventory indicator states defined for POS intervals in section 3.4.1. We consolidate available POS data for each state and through the use of flow equations explained in section 3.2.1 we define an optimization problem and eventually solve it for substitution rates.

Since we have three different sub-states for each product, we may encounter at most 3^N different states, where N denotes the number of items in the category. Therefore, for the three product case we have a total of 27 different states. However, we will ignore the states with transition sub-states (i.e. sub-states representing the intervals in which the product is depleted completely). As a result, we will only use the states given in Table 3.4 to estimate the demand and substitution rates in the final step of SSBE.

POS No	Number of Sales			POS No	Number of Sales		
	Product A	Product B	Product C		Product A	Product B	Product C
1	2 ✓	0 ✓	6 ✓	36	1 ✓	3 ✓	2 ✓
2	0 ✓	1 ✓	7 ✓	37	1 ✓	1 ✓	5 ✓
3	2 ✓	1 ✓	2 ✓	38	4 ✓	1 ✓	0 ✓
4	0 ✓	1 ✓	2 ✓	39	1 ✓	1 ✓	3 ✓
5	1 ✓	0 ✓	3 ✓	40	2 ✓	2 ✓	5 ✓
6	2 ✓	2 ✓	0 ✓	41	1 ✓	0 ✓	0 ✓
7	4 ✓	3 ✓	2 ✓	42	1 ✓	3 ✓	3 ✓
8	0 ✓	1 ✓	5 ✓	43	4 ✓	1 ✓	1 ✓
9	1 ✓	0 ✓	4 ✓	44	6 ✓	2 ✓	1 ✓
10	3 ✓	2 ✓	5 ✓	45	3 ✓	0 ✓	4 ✓
11	4 ✓	1 ✓	4 ✓	46	3 ✓	2 ✓	4 ✓
12	5 ✓	1 ✓	6 ✓	47	4 ✓	1 ✓	7 ✓
13	1 ✓	1 ✓	4 ✓	48	3 ✓	0 ✓	1 ✓
14	0 ✓	1 ✓	4 ✓	49	1 ✓	2 ✓	6 ✓
15	0 ✓	2 ✓	2 ✓	50	3 ✓	2 ✓	2 ✓
16	2 ✓	1 ✓	2 ✓	51	1 ✓	2 ○	7 ✓
17	5 ✓	2 ✓	3 ✓	52	3 ✓	0 ✗	4 ✓
18	2 ✓	1 ✓	2 ✓	53	6 ✓	0 ✗	3 ✓
19	3 ✓	1 ✓	3 ✓	54	2 ✓	0 ✗	2 ✓
20	2 ✓	2 ✓	1 ✓	55	3 ✓	0 ✗	5 ✓
21	3 ✓	3 ✓	3 ✓	56	2 ✓	0 ✗	3 ✓
22	2 ✓	2 ✓	2 ✓	57	3 ✓	0 ✗	3 ✓
23	2 ✓	1 ✓	2 ✓	58	3 ✓	0 ✗	4 ✓
24	0 ✓	1 ✓	1 ✓	59	3 ✓	0 ✗	2 ✓
25	2 ✓	1 ✓	1 ✓	60	3 ✓	0 ✗	2 ✓
26	1 ✓	2 ✓	1 ✓	61	3 ✓	0 ✗	4 ✓
27	0 ✓	1 ✓	0 ✓	62	1 ○	0 ✗	2 ✓
28	1 ✓	1 ✓	2 ✓	63	0 ✗	0 ✗	4 ✓
29	3 ✓	2 ✓	1 ✓	64	0 ✗	0 ✗	5 ✓
30	3 ✓	2 ✓	2 ✓	65	0 ✗	0 ✗	6 ✓
31	2 ✓	3 ✓	3 ✓	66	0 ✗	0 ✗	5 ✓
32	2 ✓	0 ✓	3 ✓	67	0 ✗	0 ✗	3 ○
33	0 ✓	3 ✓	1 ✓	68	0 ✗	0 ✗	0 ✗
34	3 ✓	0 ✓	2 ✓	69	0 ✗	0 ✗	0 ✗
35	2 ✓	1 ✓	3 ✓	70	0 ✗	0 ✗	0 ✗

Table 3.3: An example of sub-state labeling

State	Sub-State of Product A	Sub-State of Product B	Sub-State of Product C
(1,1,1)	✓	✓	✓
(1,1,0)	✓	✓	✗
(1,0,1)	✓	✗	✓
(1,0,0)	✓	✗	✗
(0,1,1)	✗	✓	✓
(0,1,0)	✗	✓	✗
(0,0,1)	✗	✗	✓

Table 3.4: States used in the estimation procedure

Note that the state where all sub-states are ✗ is not included in the table, because in that case POS intervals will not provide any information about demand and substitution rates (no substitution can occur in that state). Now it is very easy to label the states in our example by using Tables 3.3 and 3.4.

Step 5: Estimate Demand and Substitution Rates

In this step, we will first estimate demand rates λ_A, λ_B, and λ_C, and then substitution rates α_{AB}, α_{AC}, α_{BA}, α_{BC}, α_{CA}, and α_{CB} given state labels of POS intervals and the amount of sales in each POS interval. To estimate demand rates, we use only the POS intervals with the inventory indicator state vector (1,1,1). The reason for this is clear: in these POS intervals, as it is known that all products are available on the shelf, we only observe real demands for each product. That is to say, there exists no demand resulting from the spillover effect. Furthermore, the demand rates can be estimated quiet accurately; for, especially under conditions where the service level is high, we observe enough POS intervals with the inventory indicator state vector (1,1,1) .

The second stage in this step is the estimation of substitution rates. In estimating the elements of substitution matrix, we can only use the states defined in Table 3.4 except (1,1,1) where no substitution can take place.

We expect that Equation 3.13 holds for each combination of product and state. Therefore, we have at most $(2^{N-1}-1)N$ equations and N^2-N unknowns (substitution parameters). However, by using only the states where only one product is unavailable, we will have N^2-N equations through which we can calculate N^2-N unknown substitution rates. Therefore, our method can estimate substitution rates when the number of products in a category is large.

In addition, to increase precision of the estimation method through the use of the states where more than one product is not available, we can solve the following optimization problem to find the estimates of the substitution probabilities, namely $\hat{\alpha}_{ij}$, that minimize the sum of squared errors:

$$min \quad \sum_{r=1}^{9} {e_r}^2$$

subject to

$$\begin{aligned}
\hat{s}_{A,(1,1,0)} &= \hat{\alpha}_{CA}\hat{\lambda}_C + \hat{\lambda}_A + e_1 \\
\hat{s}_{A,(1,0,1)} &= \hat{\alpha}_{BA}\hat{\lambda}_B + \hat{\lambda}_A + e_2 \\
\hat{s}_{A,(1,0,0)} &= \hat{\alpha}_{BA}\hat{\lambda}_B + \hat{\alpha}_{CA}\hat{\lambda}_C + \hat{\lambda}_A + e_3 \\
\hat{s}_{B,(1,1,0)} &= \hat{\alpha}_{CB}\hat{\lambda}_C + \hat{\lambda}_B + e_4 \\
\hat{s}_{B,(0,1,1)} &= \hat{\alpha}_{AB}\hat{\lambda}_A + \hat{\lambda}_B + e_5 \\
\hat{s}_{B,(0,1,0)} &= \hat{\alpha}_{AB}\hat{\lambda}_A + \hat{\alpha}_{CB}\hat{\lambda}_C + \hat{\lambda}_B + e_6 \\
\hat{s}_{C,(1,0,1)} &= \hat{\alpha}_{BC}\hat{\lambda}_B + \hat{\lambda}_C + e_7 \\
\hat{s}_{C,(0,1,1)} &= \hat{\alpha}_{AC}\hat{\lambda}_A + \hat{\lambda}_C + e_8 \\
\hat{s}_{C,(0,0,1)} &= \hat{\alpha}_{AC}\hat{\lambda}_A + \hat{\alpha}_{BC}\hat{\lambda}_B + \hat{\lambda}_C + e_9
\end{aligned}$$

where $0 \le \hat{\alpha}_{ij} \le 1$, $\sum_j \hat{\alpha}_{ij} \le 1 \quad \forall i, j \ne i \in A, B, C$ and $r = 1,...,9$. In this optimization problem $\hat{\lambda}_i$ denotes the estimate for demand rate of product i obtained in the first stage of Step 5, and $\hat{s}_{i,\underline{I}_0^{pos}}$ denotes the average observed demand for product i in the POS intervals with inventory state vector $\underline{I}_0^{pos}$.

For example, the first constraint states that in the POS intervals with state vector (1,1,0), where only product C is out-of stock, the average effective demand of product A is made up of its average real demand for a POS interval and α_{CA} times the average real demand of product C. The explanation of the third constraint is also very similar; but this time as both product B and product C are out-of-stock the average effective demand of product A will include α_{BA} times demands of product B and α_{CA} times demands of product C respectively. All of the other equations follow the same reasoning.

In order to further improve the estimation method by taking the number of observed states into account, the above optimization method can be modified by assigning weights to the error terms depending on the number of observations.

The weights must affect only the equations used in the estimation of the same probabilities. Note that the equations in the optimization problem can be grouped into three. First three equations are for the estimation of $\hat{\alpha}_{CA}$ and $\hat{\alpha}_{BA}$. Next three equations are used in the estimation of $\hat{\alpha}_{CB}$ and $\hat{\alpha}_{AB}$. And the final three equations estimate $\hat{\alpha}_{BC}$ and $\hat{\alpha}_{AC}$. So we can assign weights within each group according to the numbers of observations of states and solve three separate optimization problems given below where w_r denotes the weight of r^{th} error.

$$min \quad \sum_{r=1}^{3} w_r e_r^{\;2}$$

$$subject\ to$$

$$\begin{aligned}
\hat{s}_{A,(1,1,0)} &= \hat{\alpha}_{CA}\hat{\lambda}_C + \hat{\lambda}_A + e_1 \\
\hat{s}_{A,(1,0,1)} &= \hat{\alpha}_{BA}\hat{\lambda}_B + \hat{\lambda}_A + e_2 \\
\hat{s}_{A,(1,0,0)} &= \hat{\alpha}_{BA}\hat{\lambda}_B + \hat{\alpha}_{CA}\hat{\lambda}_C + \hat{\lambda}_A + e_3
\end{aligned}$$

where $0 \le \hat{\alpha}_{ij} \le 1$, $\forall i, j \ne i \in A, B, C$, $r = 1,2,3$,

$w_1 = \dfrac{\aleph_{(1,1,0)}^{\;pos}}{\aleph_{(1,1,0)}^{\;pos} + \aleph_{(1,0,1)}^{\;pos} + \aleph_{(1,0,0)}^{\;pos}}$, $w_2 = \dfrac{\aleph_{(1,0,1)}^{\;pos}}{\aleph_{(1,1,0)}^{\;pos} + \aleph_{(1,0,1)}^{\;pos} + \aleph_{(1,0,0)}^{\;pos}}$, and

$w_3 = \dfrac{\aleph_{(1,0,0)}^{\;pos}}{\aleph_{(1,1,0)}^{\;pos} + \aleph_{(1,0,1)}^{\;pos} + \aleph_{(1,0,0)}^{\;pos}}$ where $\aleph_{\underline{I}^{pos}}^{\;pos}$ is the total number of POS intervals with inventory state vector $\underline{I}_0^{\;pos}$.

$$min \quad \sum_{r=4}^{6} w_r e_r^{\;2}$$

$$subject\ to$$

$$\begin{aligned}
\hat{s}_{B,(1,1,0)} &= \hat{\alpha}_{CB}\hat{\lambda}_C + \hat{\lambda}_B + e_4 \\
\hat{s}_{B,(0,1,1)} &= \hat{\alpha}_{AB}\hat{\lambda}_A + \hat{\lambda}_B + e_5 \\
\hat{s}_{B,(0,1,0)} &= \hat{\alpha}_{AB}\hat{\lambda}_A + \hat{\alpha}_{CB}\hat{\lambda}_C + \hat{\lambda}_B + e_6
\end{aligned}$$

where $0 \le \hat{\alpha}_{ij} \le 1, \forall i, j \ne i \in A,B,C$, $r = 4,5,6$,

$$w_4 = \frac{\aleph_{(1,1,0)}{}^{pos}}{\aleph_{(1,1,0)}{}^{pos} + \aleph_{(0,1,1)}{}^{pos} + \aleph_{(0,1,0)}{}^{pos}}, w_5 = \frac{\aleph_{(0,1,1)}{}^{pos}}{\aleph_{(1,1,0)}{}^{pos} + \aleph_{(0,1,1)}{}^{pos} + \aleph_{(0,1,0)}{}^{pos}}, \text{ and}$$

$$w_6 = \frac{\aleph_{(0,1,0)}{}^{pos}}{\aleph_{(1,1,0)}{}^{pos} + \aleph_{(0,1,1)}{}^{pos} + \aleph_{(0,1,0)}{}^{pos}}.$$

$$min \quad \sum_{r=7}^{9} w_r e_r{}^2$$

$$subject\ to$$

$$\begin{aligned} \hat{s}_{C,(1,0,1)} &= \hat{\alpha}_{BC}\hat{\lambda}_B + \hat{\lambda}_C + e_7 \\ \hat{s}_{C,(0,1,1)} &= \hat{\alpha}_{AC}\hat{\lambda}_A + \hat{\lambda}_C + e_8 \\ \hat{s}_{C,(0,0,1)} &= \hat{\alpha}_{AC}\hat{\lambda}_A + \hat{\alpha}_{BC}\hat{\lambda}_B + \hat{\lambda}_C + e_9 \end{aligned}$$

where $0 \le \hat{\alpha}_{ij} \le 1, \forall i, j \ne i \in A,B,C$, $r = 7,8,9$,

$$w_7 = \frac{\aleph_{(1,0,1)}{}^{pos}}{\aleph_{(1,0,1)}{}^{pos} + \aleph_{(0,1,1)}{}^{pos} + \aleph_{(0,0,1)}{}^{pos}}, w_8 = \frac{\aleph_{(0,1,1)}{}^{pos}}{\aleph_{(1,0,1)}{}^{pos} + \aleph_{(0,1,1)}{}^{pos} + \aleph_{(0,0,1)}{}^{pos}}, \text{ and}$$

$$w_9 = \frac{\aleph_{(0,0,1)}{}^{pos}}{\aleph_{(1,0,1)}{}^{pos} + \aleph_{(0,1,1)}{}^{pos} + \aleph_{(0,0,1)}{}^{pos}}.$$

If the estimations from these separate optimization problems do not violate $\sum_j \hat{\alpha}_{ij} \le 1$ these estimations are also feasible; so we use them. If this constraint is violated (in our computational experiments it is never violated), we can simply use the non-weighted optimization for the estimation of substitution parameters.

Note that, for N products, we will have N separate optimization problems with $2^{N-1} - 1$ equality constraints and $N - 1$ decision variables. Therefore, the optimization problems are tractable for the estimations for a category with a moderate number of products. On the other hand, it is also important to note that when the number of products is large, the substitution rates among them will be too small to be taken into account. Thus, a certain level of aggregation of products will be appropriate in practical situations (as we will present in section 6.2).

3.2.3 SSBE with POS Data Only

So far we assumed that we had information on the initial stocking levels of products and the replenishment intervals besides POS data. However, gathering such data seems very difficult in a real setting. As a result, in order for a method to be used practically by a retailer it should only employ POS data. In this section, we present the SSBE method for the case where neither initial stocking levels nor replenishment intervals are known. Therefore, we try to estimate the states corresponding to each POS interval and then use these estimated states to estimate substitution rates.

In this setting we will not be able to determine the POS intervals in which replenishments occur. Moreover, we cannot know exactly which POS intervals correspond to out-of-stock intervals. Thus, we need to label POS intervals carefully. One way to do that consists of calculating the probability of observing a zero sale due to no arrival in a POS interval for each product and thus calculating the probability of observing consecutive zero sales groups resulting from no arrivals. Then, according to the probability of observing consecutive zero sales groups resulting from no arrivals we decide whether a consecutive zero sales group corresponds to a stock-out situation or not. Apparently, the most plausible out-of-stock intervals are the ones with “large” values of consecutive zeros. A “large” value means that it is clearly different from the majority of the rest of the values.

So after determining these outliers, our method treats them as stock-out intervals and labels the sub-states accordingly. Subsequent steps of SSBE, including the labeling of the states and the estimation of demand and substitution rates, are the same as those applied in the case where initial stocking levels and replenishment cycle lengths are known. Adaptations made to the steps of SSBE are as follows.

Note that we need to know the form of the distribution of demands to determine which of the consecutive zero-sales groups will be treated as outliers. Therefore, we first specify the forms of the distribution of demands and approximate their parameters through the POS intervals with positive sales. Then, we use the probability of observing a zero sale along with the predetermined threshold values (upper and lower) in order to classify zero sales groups. More precisely, if the probability of observing a zero sales group due to no arrival is less than the lower threshold, this zero sales group is labeled as a stock-out group and the POS interval just before this group is labeled as a transition interval. On the other hand, if the probability of observing a zero sales group due to no arrival is greater than the upper threshold, this zero sales group is labeled as an in-stock group. The rest of the zero sales groups are not labeled and left out of the estimation procedure.

In fact, the procedure explained in the above paragraph is related to the test of the hypothesis whether a sample consisting of zeros is coming from the specified demand distribution. In addition, the threshold values are related to the significance of the hypothesis test. To avoid mis-labeling of substitution intervals (i.e., labeling non-substitution intervals as substitution intervals) at the expense of non-labeling, we determine a very small lower threshold value (e.g. $1x10^{-5}$). That is, a zero group must be significantly different from the specified demand distribution to be labeled as a stock-out group. The upper threshold is not as critical as the lower one, because when the service level is high we observe many in-stock intervals to accurately estimate the core demands. For example, we choose the upper threshold as 0.001.

Then, just as we proceed in Step 4 and Step 5 of the SSBE with complete information, POS intervals are labeled according to the corresponding sub-state labels. Table 3.5b presents a sample state labeling, where the critical consecutive zero sales numbers (i.e., the numbers of consecutive zeros for which the method cannot label the corresponding state) are 5, 3, 3 for products A, B, and C, respectively, for the data provided in Table 3.5a. Consequently, demand and substitution rates are estimated by solving an optimization problem using input parameters calculated from different states.

We can summarize SSBE with POS data only as follows:

Step 1: Determine the form of the distribution of demands

Step 2: Approximate distribution parameters by using positive sales

Step 3: Define sub-states by specifying threshold values

Step 4: Define states

Step 5: Estimate demand and substitution rates

POS No	Number of Sales		
	Product A	Product B	Product C
1	0	0	4
2	0	0	3
3	1	1	0
4	0	1	0
5	0	0	2
6	0	0	1
7	2	1	1
8	0	4	3
9	0	1	0
10	0	0	3
11	2	3	1
12	0	1	1
13	0	1	1
14	0	0	4
15	1	1	3
16	0	0	4
17	0	1	1
18	0	0	0
1	0	1	4
20	0	0	4
21	0	1	1
22	0	1	2
23	0	0	1
24	1	0	0
25	0	2	0
26	3	0	3
27	0	0	2
28	0	1	4
29	0	1	1
30	1	1	0
31	0	1	0
32	0	0	2
33	0	0	1
34	0	1	5
35	0	2	2
36	0	3	0
37	1	0	1
38	0	0	0
39	0	2	0
40	1	0	3
41	0	0	2
42	0	0	1
43	0	0	2
44	0	2	1
45	1	1	1
46	1	1	1
47	0	0	1
48	1	3	3
49	0	2	1
50	0	0	3
51	0	0	4
52	0	0	0
53	0	1	0
54	0	1	0
55	0	0	3
56	1	1	1
57	0	0	2
58	0	0	2
59	1	1	0
60	0	0	1
61	0	1	1
62	0	2	1
63	1	2	1
64	0	2	2
65	0	1	2
66	1	0	2
67	0	1	2
68	0	2	1
69	0	0	1
70	0	0	2

Table 3.5a: Consecutive zeros in an example POS data

POS No	Number of Sales			State	POS No	Number of Sales			State
	Product A	Product B	Product C			Product A	Product B	Product C	
1	0 ✓	0 ✓	4 ✓	(1,1,1)	36	1 ✓	3 ✓	0 ✓	(1,1,1)
2	0 ✓	0 ✓	3 ✓	(1,1,1)	37	1 ✓	0 ✓	1 ✓	(1,1,1)
3	1 ✓	1 ✓	0 ✓	(1,1,1)	38	0 ✓	0 ✓	0 ✓	(1,1,1)
4	0 ✓	1 ✓	0 ✓	(1,1,1)	39	0 ✓	2 ○	0 ✓	-
5	0 ✓	0 ✓	2 ✓	(1,1,1)	40	1 ✓	0 ✗	3 ✓	(1,0,1)
6	0 ✓	0 ✓	1 ✓	(1,1,1)	41	0 ✓	0 ✗	2 ✓	(1,0,1)
7	2 ✓	1 ✓	1 ✓	(1,1,1)	42	0 ✓	0 ✗	1 ✓	(1,0,1)
8	0 ✓	4 ✓	3 ✓	(1,1,1)	43	0 ✓	0 ✗	2 ✓	(1,0,1)
9	0 ✓	1 ✓	0 ✓	(1,1,1)	44	0 ✓	2 ✓	1 ✓	(1,1,1)
10	0 ✓	0 ✓	3 ✓	(1,1,1)	45	1 ✓	1 ✓	1 ✓	(1,1,1)
11	2 ✓	3 ✓	1 ✓	(1,1,1)	46	1 ✓	1 ✓	1 ✓	(1,1,1)
12	0 ✓	1 ✓	1 ✓	(1,1,1)	47	0 ✓	0 ✓	1 ✓	(1,1,1)
13	0 ✓	1 ✓	1 ✓	(1,1,1)	48	1 ○	3 ✓	3 ✓	-
14	0 ✓	0 ✓	4 ✓	(1,1,1)	49	0 ✗	2 ✓	1 ✓	(0,1,1)
15	1 ○	1 ✓	3 ✓	-	50	0 ✗	0	3 ✓	-
16	0 ✗	0 ✓	4 ✓	(0,1,1)	51	0 ✗	0	4 ✓	-
17	0 ✗	1 ✓	1 ✓	(0,1,1)	52	0 ✗	0	0	-
18	0 ✗	0 ✓	0 ✓	(0,1,1)	53	0 ✗	1 ✓	0	-
19	0 ✗	1 ✓	4 ✓	(0,1,1)	54	0 ✗	1 ✓	0	-
20	0 ✗	0 ✓	4 ✓	(0,1,1)	55	0 ✗	0 ✓	3 ✓	(0,1,1)
21	0 ✗	1 ✓	1 ✓	(0,1,1)	56	1 ✓	1 ✓	1 ✓	(1,1,1)
22	0 ✗	1 ✓	2 ✓	(0,1,1)	57	0 ✓	0 ✓	2 ✓	(1,1,1)
23	0 ✗	0 ✓	1 ✓	(0,1,1)	58	0 ✓	0 ✓	2 ✓	(1,1,1)
24	1 ✓	0 ✓	0 ✓	(1,1,1)	59	1 ✓	1 ✓	0 ✓	(1,1,1)
25	0 ✓	2 ✓	0 ✓	(1,1,1)	60	0 ✓	0 ✓	1 ✓	(1,1,1)
26	3 ✓	0 ✓	3 ✓	(1,1,1)	61	0 ✓	1 ✓	1 ✓	(1,1,1)
27	0 ✓	0 ✓	2 ✓	(1,1,1)	62	0 ✓	2 ✓	1 ✓	(1,1,1)
28	0 ✓	1 ✓	4 ✓	(1,1,1)	63	1 ✓	2 ✓	1 ✓	(1,1,1)
29	0 ✓	1 ✓	1 ✓	(1,1,1)	64	0 ✓	2 ✓	2 ✓	(1,1,1)
30	1 ✓	1 ✓	0 ✓	(1,1,1)	65	0 ✓	1 ✓	2 ✓	(1,1,1)
31	0	1 ✓	0 ✓	-	66	1 ✓	0 ✓	2 ✓	(1,1,1)
32	0	0 ✓	2 ✓	-	67	0 ✓	1 ✓	2 ✓	(1,1,1)
33	0	0 ✓	1 ✓	-	68	0 ✓	2 ✓	1 ✓	(1,1,1)
34	0	1 ✓	5 ✓	-	69	0 ✓	0 ✓	1 ✓	(1,1,1)
35	0	2 ✓	2 ✓	-	70	0 ✓	0 ✓	2 ✓	(1,1,1)

Table 3.5b: Corresponding states

Chapter 4

ANALYSIS OF THE ESTIMATION METHOD IN A DETERMINISTIC SETTING

4.1 Introduction

Before examining the results of the comprehensive simulation study, it would be more insightful to analyze the estimation method under a deterministic setting. This part consists of calculating the substitution rates and the number of observed states through POS data in the case where demand rates are deterministic and substitution occurs in deterministic proportions. The deterministic fraction $\alpha^f{}_{ij} \in [0,1]$, where $\sum_{j=1}^{N} \alpha^f{}_{ij} \leq 1$, of the demand of product i that cannot be satisfied will spillover to product j as a substitute. These fractions are determined as follows based on the Logit model as done in Kök and Fisher [18], Netessine and Rudi [24], and Smith and Agrawal [45]:

$$\alpha^f{}_{ij} = \delta^f \frac{\lambda_j}{\sum_{l \in N \setminus \{i\}} \lambda_l}$$ where δ^f is the total substitution fraction.

We will analyze the three product case and assume that replenishments occur periodically for each product (e.g. each item is replenished at the beginning of the week). Without loss of generality, it is presumed that the POS interval length τ is a divisor of the replenishment cycle T and during these replenishments stock levels of products are raised to the predetermined levels L_A, L_B, and L_C. Moreover, while finding formulas for substitution rates and number of observed states we assume that product A is depleted first, product B second, and product C third. Figure 4.1 provides a representation of the fluid model in the deterministic case, where $I_i(t)$ is the inventory level of product i at time t.

4.2 Analysis

This part consists of the calculation of substitution rates, calculation of exact times of stock-outs, and calculation of the number of observed states. These calculations will provide the basis for further discussion on the effect of system parameters on the quality of SSBE.

4.2.1 Substitution Rates

Under the current assumptions, we can obtain substitution rates by only analyzing one replenishment cycle; because the system will repeat itself in every replenishment cycle. We will consider the situation in which product A and product B are depleted in the middle of a POS interval. Let λ_i^{pos} denote the real demand for product i per POS interval $i \in A, B, C$, $\lambda_i^{pos'}$ denote the demand for product i per POS interval when product A is out-of-stock $i \in B, C$, and $\lambda_C^{pos''}$ denote the demand for product C per POS interval when both products A and B are out-of-stock. We can observe the rates $\lambda_A^{pos}, \lambda_B^{pos}$, $\lambda_B^{pos'}, \lambda_C^{pos}, \lambda_C^{pos'}$, $\lambda_C^{pos''}$ in the POS data table if we have at least one complete POS interval other than stock-out POS intervals. More precisely, we have: $\lambda_A^{pos} = s_A{}^P$, $\lambda_B^{pos} = s_B{}^P$, and $\lambda_C^{pos} = s_C{}^P$ for

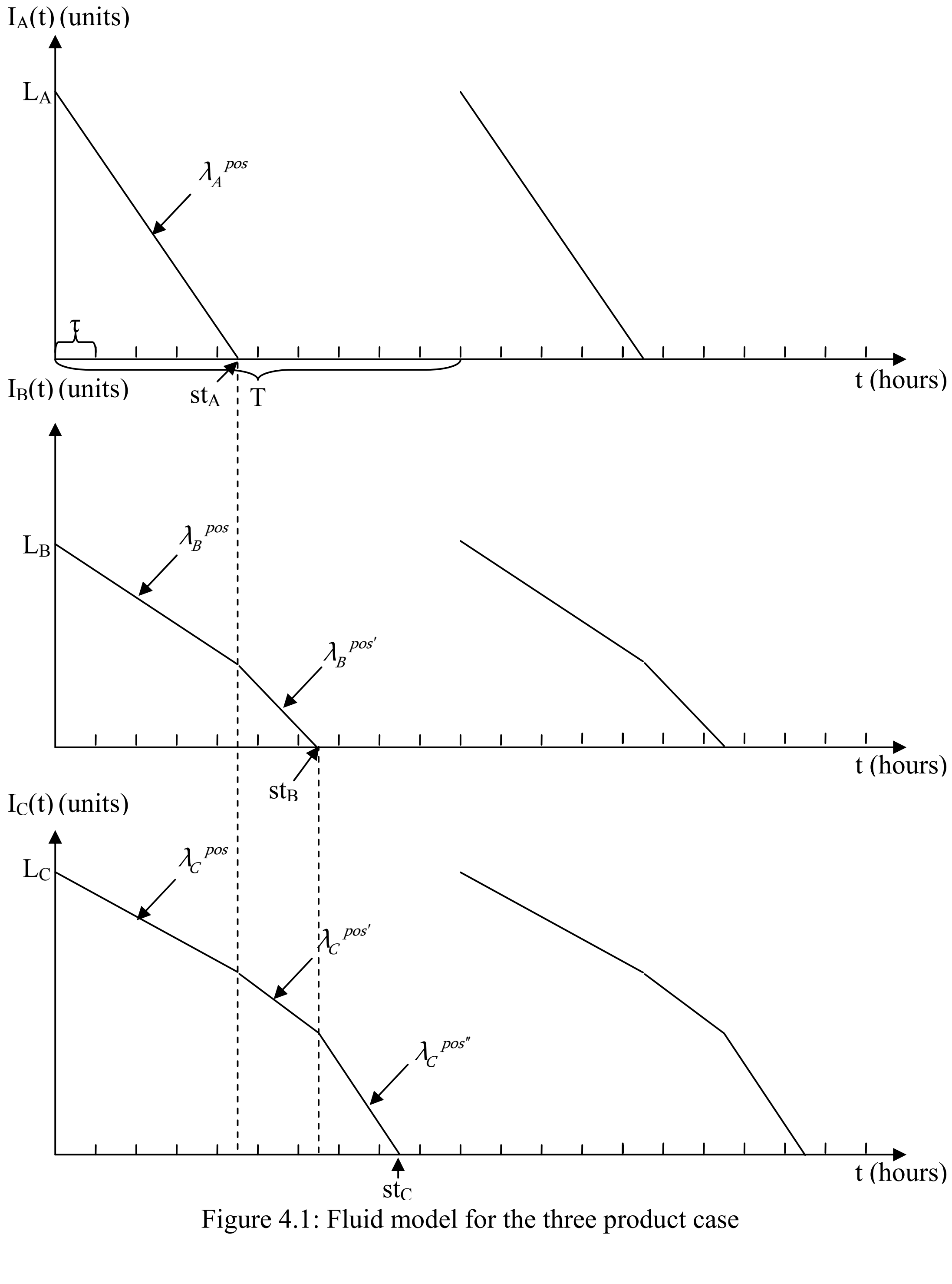


Figure 4.1: Fluid model for the three product case

any p such that $\underline{I}^{pos}[p]=(1,1,1)$; $\lambda_B^{pos'}=s_B^{p'}$ and $\lambda_C^{pos'}=s_C^{p'}$ for any p' such that $\underline{I}^{pos}[p']=(0,1,1)$; and $\lambda_C^{pos''}=s_C^{p''}$ for any p'' such that $\underline{I}^{pos}[p'']=(0,0,1)$, where s_i^p denotes the observed demand for product i in POS interval p. So by assuming this and thus using the observed demand rates we are able to calculate the substitution rates easily.

First, as $\lambda_B^{pos'}$ is composed of the core demand rate of product B and the spillover demand from only product A we can calculate $\alpha^f{}_{AB}$ as follows:

$$\lambda_B^{pos'}=\lambda_B^{pos}+\alpha^f{}_{AB}\lambda_A^{pos}\Rightarrow\alpha^f{}_{AB}=\frac{s_B^{p'}-s_B^{p}}{s_A^{p}} \tag{4.1}$$

for any p such that $\underline{I}^{pos}[p]=(1,1,1)$ and p' such that $\underline{I}^{pos}[p']=(0,1,1)$.

Similarly, we calculate $\alpha^f{}_{AC}$ and $\alpha^f{}_{BC}$ as follows:

$$\lambda_C^{pos'}=\lambda_C^{pos}+\alpha^f{}_{AC}\lambda_A^{pos}\Rightarrow\alpha^f{}_{AC}=\frac{s_C^{p'}-s_C^{p}}{s_A^{p}} \tag{4.2}$$

$$\lambda_C^{pos''}=\lambda_C^{pos'}+\alpha^f{}_{BC}\lambda_B^{pos}\Rightarrow\alpha^f{}_{BC}=\frac{s_C^{p''}-s_C^{p'}}{s_B^{p}} \tag{4.3}$$

for any p such that $\underline{I}^{pos}[p]=(1,1,1)$, p' such that $\underline{I}^{pos}[p']=(0,1,1)$, and p'' such that $\underline{I}^{pos}[p'']=(0,0,1)$.

However, in this setting, it is interesting to realize that although everything is deterministic we cannot calculate the values of $\alpha^f{}_{BA}$, $\alpha^f{}_{CA}$, and $\alpha^f{}_{CB}$. This may give an idea of the difficulty of estimating product substitution rates in a non-deterministic setting, which we provide in Chapter 5.

4.2.2 Numbers of Observed States

Since we know the initial stocking levels of the three products and we are able to capture core and effective demand rates from the POS data, the calculation of the exact stock-out time of each of the product, st_i, is trivial. Consider the stock-out time for product A. It can be found by the following formula:

$$st_A = \frac{L_A}{\lambda_A^{pos}\pi} \tag{4.4}$$

where π is the number of POS intervals per hour. For the exact stock-out time of product B we need to use the following:

$$st_B = st_A + \frac{L_B - \lambda_B^{pos}\pi st_A}{\lambda_B^{pos'}\pi}. \tag{4.5}$$

Finally, for product C we have the following equation:

$$st_C = st_B + \frac{L_C - \lambda_C^{pos}\pi st_A - \lambda_C^{pos'}\pi(st_B - st_A)}{\lambda_C^{pos''}\pi}. \tag{4.6}$$

Given initial stocking levels, replenishment length, and POS data we can also calculate the numbers of observed states per cycle, namely $c_{L_0^{pos}}$. Under the current assumptions, we will only observe state vectors (1,1,1), (0,1,1), and (0,0,1) (we may observe the state where all of the products are out-of-stock but these states are not used during the estimation). Figure 4.2 shows the sub-states corresponding to the deterministic case for three products.

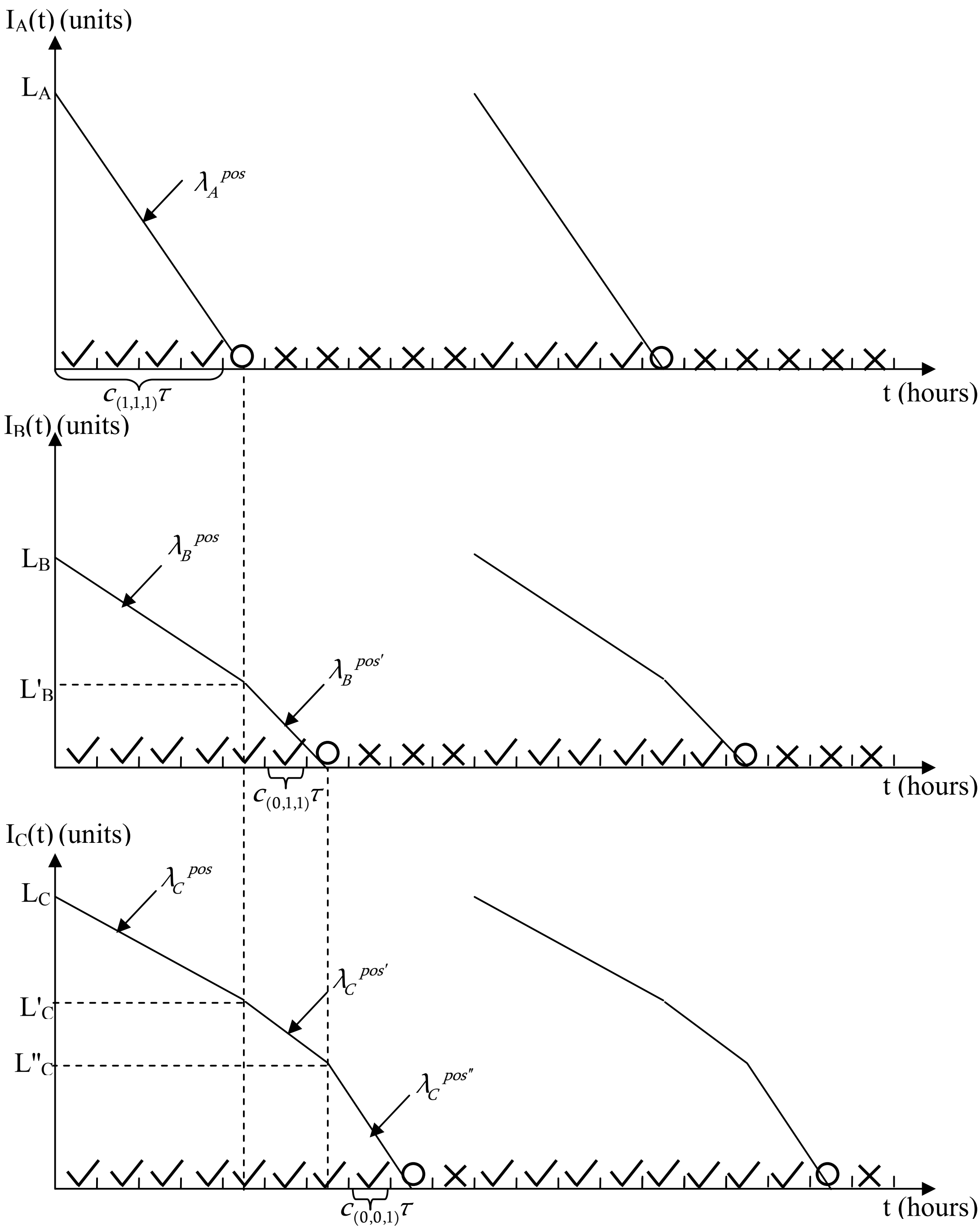


Figure 4.2: Sub-states for the fluid model for the three products case

Numbers of POS intervals corresponding to each of these states in one cycle are denoted by $c_{(1,1,1)}$, $c_{(0,1,1)}$, and $c_{(0,0,1)}$ respectively, and calculated as follows:

$$c_{(1,1,1)} = \left\lfloor \frac{L_A}{\lambda_A^{pos}} \right\rfloor \tag{4.7}$$

$$c_{(0,1,1)} = \left\lfloor \frac{L'_B - \lambda_B^{pos'}\pi\left[\left(c_{(1,1,1)}+1\right)\tau - st_A\right]}{\lambda_B^{pos'}} \right\rfloor \tag{4.8}$$

$$c_{(0,0,1)} = \left\lfloor \frac{L''_C - \lambda_C^{pos''}\pi\left[\left(c_{(1,1,1)}+c_{(0,1,1)}+2\right)\tau - st_B\right]}{\lambda_C^{pos''}} \right\rfloor \tag{4.9}$$

where $L'_B = L_B - \lambda_B^{pos}\pi st_A$, $L'_C = L_C - \lambda_C^{pos}\pi st_A$, and $L''_C = L'_C - \lambda_C^{pos'}\pi(st_B - st_A)$. Since these numbers correspond to one replenishment cycle, total numbers of observed states can be found by multiplying these values by the number of total replenishments.

4.3 The Effect of Problem Parameters on the Numbers of Observed States

States are the fundamental structures for the estimation of both demand and substitution rates. Recall that we use the POS intervals with state vector (1,1,1) in estimating the core demand rates of products; whereas we exploit the rest of the states for the estimation of substitution probabilities. Therefore, examining the effect of problem parameters such as the substitution fraction and the length of the POS intervals on the numbers of observed states would be interesting in that one might expect better estimation performance as the numbers of observed states increase. The proofs of the lemmas and propositions provided in this section are given in Appendix A.

4.3.1 The Effect of Substitution Fraction (δ)

In this section, we examine the effect of substitution fracion.

4.3.1.1 The Effect on the Number of States Where All Products are Available ($c_{(1,1,1)}$)

Since $c_{(1,1,1)}$ depends only on L_A and λ_A, which are independent of δ, substitution fraction has no effect on $c_{(1,1,1)}$. This is logical because $c_{(1,1,1)}$ is the number of states in which all three products are available, i.e. no substitution occurs.

4.3.1.2 The Effect on the Number of States Where Only One Product is Unavailable ($c_{(0,1,1)}$)

One of the important states with regard to the substitution rate estimation is the state in which one of the products is not available.

Proposition 4.1. $c_{(0,1,1)}(\delta)$ is nonincreasing in δ.

Proposition 4.1 states that the number of states where only one product is unavailable is nonincreasing in the substitution fraction. This is expected because as the substitution fraction increases $\lambda_B^{pos'}$ increases, i.e., the slope of the line becomes steeper. As a result, we will have fewer intervals to observe the substitution from the stock-out product.

4.3.1.3 The Effect on the Number of States Where Two Products are Unavailable ($c_{(0,0,1)}$)

The number of states where two of the products are unavailable is also crucial for the substitution rate estimation.

Lemma 4.1. Let $c'_{(0,0,1)}(\delta) = \dfrac{L''_C(\delta)}{\lambda_C^{pos''}(\delta)} - [c_{(1,1,1)} + c_{(0,1,1)}(\delta) + 2] + \pi st_B(\delta)$. (4.10)

That is, $c_{(0,0,1)}(\delta) = \left\lfloor c_{(0,0,1)}{}'(\delta) \right\rfloor$. There exist a lower bound $L_{c'_{(0,0,1)}(\delta)}$ and an upper bound $U_{c'_{(0,0,1)}(\delta)}$ on $c'_{(0,0,1)}(\delta)$ that are decreasing in δ and have a difference of 2.

Proposition 4.2. There exist a lower bound $L_{c_{(0,0,1)}(\delta)}$ and an upper bound $U_{c_{(0,0,1)}(\delta)}$ on $c_{(0,0,1)}(\delta)$ that are nonincreasing in δ and have a difference of 2.

Proposition 4.2 shows that the number of states where two of the products are not available is tightly bounded by two functions, which are nonincreasing in the substitution fraction. That is, we will observe fewer (except for temporary small jumps) intervals where there is a potential substitution from the stock-out products when the substitution fraction goes up.

4.3.1.4 The Effect on the Total Number of Substitution States ($c_{(0,1,1)} + c_{(0,0,1)}$)

The sum of the number of states where only one product is unavailable and the number of states where two of the products are unavailable gives the total number of substitution states, which reflects the total extent of data available for substitution rate estimation.

Proposition 4.3. $c_{(0,1,1)}(\delta)+c_{(0,0,1)}(\delta)$ is nonincreasing in δ.

Proposition 4.3 states that the total number of substitution states is nonincreasing in the substitution fraction. In other words, our estimation procedure will be able to use fewer intervals for estimation when the substitution fraction increases.

4.3.2 The Effect of POS Interval Length (τ)

In this section, we analyze the effect of POS interval length, which is of importance for the quality of estimation. Throughout this section we assume that hourly demands of products are fixed: $\lambda_i^{pos}\pi = \lambda^h{}_i$ is constant for all $i \in A,B,C$.

4.3.2.1 The Effect on the Number of States Where All Products are Available ($c_{(1,1,1)}$)

First, we examine the effect of POS interval length on the number of states where all products are available.

Proposition 4.4. $c_{(1,1,1)}(\tau)$ is nonincreasing in τ.

Proposition 4.4 shows that the number of states where all products are available is nonincreasing in the POS interval length. In fact, an increase in the POS interval length does not cause any loss of data. However, the precision of data, which is used for estimation, decreases as the POS interval length gets larger.

4.3.2.2 The Effect on the Number of States Where Only One Product is Unavailable ($c_{(0,1,1)}$)

Second, we analyze the effect of POS interval length on the number of states where only one product is out-of-stock.

Lemma 4.2. Let $c_{(0,1,1)}(\tau) = \lfloor c'_{(0,1,1)}(\tau) \rfloor$. There exist a lower bound $L_{c'_{(0,1,1)}(\tau)}$ and an upper bound $U_{c'_{(0,1,1)}(\tau)}$ on $c'_{(0,1,1)}(\tau)$ that are decreasing in τ and have a difference of 1.

Proposition 4.5. There exist a lower bound $L_{c_{(0,1,1)}(\tau)}$ and an upper bound $U_{c_{(0,1,1)}(\tau)}$ on $c_{(0,1,1)}(\tau)$ that are nonincreasing in τ and have a difference of 1.

Proposition 4.5 shows that the number of states where only one of the products is not available is tightly bounded by two functions, which are nonincreasing in the POS interval length. In other words, we will have fewer (except for temporary small jumps) intervals where there is a potential substitution from the stock-out product as the POS interval length increases.

4.3.2.3 The Effect on the Number of States Where Two Products are Unavailable ($c_{(0,0,1)}$)

Third, we try to understand the effect of POS interval length on the number of states where two of the products are out-of-stock.

Lemma 4.3. Let $c_{(0,0,1)}(\tau)=\lfloor c'_{(0,0,1)}(\tau)\rfloor$. There exist a lower bound $L_{c'_{(0,0,1)}(\tau)}$ and an upper bound $U_{c'_{(0,0,1)}(\tau)}$ on $c'_{(0,0,1)}(\tau)$ that are decreasing in τ and have a difference of 1.

Proposition 4.6. There exist a lower bound $L_{c_{(0,0,1)}(\tau)}$ and an upper bound $U_{c_{(0,0,1)}(\tau)}$ on $c_{(0,0,1)}(\tau)$ that are nonincreasing in τ and have a difference of 1.

Proposition 4.6 states that the number of states where two of the products are unavailable is tightly bounded by two functions, which are nonincreasing in the POS interval length. That is, we will be able to use fewer (except for temporary small jumps) intervals where there is a potential substitution from the two stock-out products if the POS interval length increases.

4.3.2.4 The Effect on the Total Number of Substitution States ($c_{(0,1,1)}+c_{(0,0,1)}$)

Finally, it is crucial to examine the effect of POS interval length on the total number of substitution states.

Lemma 4.4. Let $c_{(0,1,1)}(\tau)+c_{(0,0,1)}(\tau)=\lfloor c'_{(0,1,1)}(\tau)+c'_{(0,0,1)}(\tau)\rfloor$. There exist a lower bound $L_{c'_{(0,1,1)}(\tau)+c'_{(0,0,1)}(\tau)}$ and an upper bound $U_{c'_{(0,1,1)}(\tau)+c'_{(0,0,1)}(\tau)}$ on $c'_{(0,1,1)}(\tau)+c'_{(0,0,1)}(\tau)$ that are decreasing in τ and have a difference of 1.

Proposition 4.7. There exist a lower bound $L_{c_{(0,1,1)}(\tau)+c_{(0,0,1)}(\tau)}$ and an upper bound $U_{c_{(0,1,1)}(\tau)+c_{(0,0,1)}(\tau)}$ on $c_{(0,1,1)}(\tau)+c_{(0,0,1)}(\tau)$ that are nonincreasing in τ and have a difference of 1.

Proposition 4.7 shows that the total number of substitution states is tightly bounded by two functions, which are nonincreasing in the POS interval length. In other words, the precision of the data including potential substitutions decreases as the POS interval length gets larger.

4.3.3 Discussion

In section 4.3.2, we have analyzed the effect of problem parameters such as the substitution proportion and the POS interval length on the numbers of observed states. More precisely, we prove that the number of states where only one product is unavailable, the number of states where two products are unavailable and the total number of substitution states are either nonincreasing or bounded by nonincreasing functions in the substitution proportion. Thus, we may expect that the performance of our estimation method decreases (or at least does not increase) as the substitution proportion increases because of the reduction in the observed number of substitution states.

Moreover, we prove that the number of states where only one product is unavailable, the number of states where two products are unavailable and the total number of substitution states are bounded by nonincreasing functions in the POS interval length. Therefore, we may also expect that the performance of our estimation method increases (or at least does not decrease) as we decrease the POS interval length, i.e. obtain more detailed sales data. The findings of the computational study presented in the next chapter are parallel to this expectation.

Chapter 5

COMPUTATIONAL RESULTS

5.1 Problem Generation

Problems used in the comparison are generated according to predefined customer arrival and choice processes and replenishment policy for a category consisting of three products. This section describes the details of this problem generation process.

5.1.1 Customer Arrival and Choice Process

The customer arrival process is presumed to be a Poisson process. This assumption is valid in a retail setting; because the population generating the arrivals can be presumed large enough that arrivals between successive time intervals are independent and customers make their shopping such that there is at most one arrival at any given time instant [6]. The arrival rate for a particular item i is assumed to be Poisson with rate λ_i. Upon arrival, a customer may buy one unit of one of the available items or choose the no-purchase option.

As for the choice process, we put a restriction only allowing for one substitution attempt, which is used by Smith and Agrawal [45] and Anupindi et al. [6]. Thus, if a customer cannot find her favorite product on the shelf the demand for that product may spill over to another product in the category. Nevertheless, if the second-choice product is not available then the demand is lost.

We employ a probabilistic model of substitution (see Netessine and Rudi [24], Smith and Agrawal [45], and Kok and Fisher [18]), which is mostly used in the related literature. In this model, every customer has a first-choice product in a certain category that consists of N products. If the first-choice product is not available due to a temporary stock-out, the second favorite product is chosen according to the substitution probability matrix (α_{ij}). As mentioned earlier, we do not allow more substitution attempts in line with the substitution structure used in Anupindi et al. [6] and Smith and Agrawal [45].

The substitution probability matrix is determined via a simple structure based on market shares of the products in a category. We assume that the probability of substitution is δ, which reflects the intensity of substitution in the category. Then, the probability of substituting product j for i is calculated as follows:

$$\alpha_{ij} = \delta \frac{\lambda_j}{\sum_{l \in N \setminus \{i\}} \lambda_l}. \tag{5.1}$$

For example, consider the three product case. That is, a retailer stocks three different products A, B, and C in a category. The corresponding demand arrivals for these items are Poisson with rates λ_A, λ_B and λ_C respectively. Suppose that a customer arrival occurs for product A. If product A is on the shelf at that time, the customer buys a unit of product A. If this is not the case, ($1-\delta$) fraction of the demand for product A will result in lost sale, and the remaining fraction of demand will be directed to product B and product C in proportion to their demand (i.e. λ_B and λ_C). Specifically, corresponding substitution probabilities are:

$$\alpha_{AC} = \delta \frac{\lambda_C}{\lambda_B + \lambda_C} \tag{5.2}$$

$$\alpha_{AB} = \delta \frac{\lambda_B}{\lambda_B + \lambda_C}. \tag{5.3}$$

The other substitution parameters in this case, namely α_{BA}, α_{BC}, α_{CA}, and α_{CB}, can be determined similarly.

5.1.2 Inventory Control System

A fixed review period order-up-to level (R,S) type inventory control system is employed. According to this system, every R units of time (i.e. at every review point) an order is given to raise inventory position to a predetermined level S. The lead time is assumed to be zero.

Specifically, we assume that replenishments occur on a weekly basis. The order-up-to level, in case of a periodic review system with lost sales and normal demand, is determined for a given desired service level β as follows (Silver et al. [48]):

Step 1: Select the safety factor z_β which satisfies $G_u(z_\beta) = \dfrac{(1-\beta)\mu R}{\beta\sigma\sqrt{R+L}}$

where μ is the mean and σ is the standard deviation of demand. In addition,

$$G_u(z_\beta) = \int_z^\infty (x-z)\frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx .$$

Here, we define the service level β as the fraction of demand satisfied directly from the shelf. Moreover, since demand is Poisson normal approximation to Poisson can be used in calculating initial stock levels. That is, we take $\mu = \lambda$ and $\sigma = \sqrt{\lambda}$.

Step 2: Reorder level $S = \mu(R+L) + z_\beta \sigma \sqrt{R+L}$ (if not integer, increase to the next higher integer).

Since in our setting we allow substitution among products, the service level used in calculating reorder levels will not probably be realized. However, we compare the performances of the methods for the same demand realizations; thus, this will not pose any problem in that regard.

5.2 Real vs. Realized Rates of Product Substitution

While evaluating the performance of an estimation method for product substitution rates we can compare the real values of substitution rates with the estimated ones. In fact, the real values of substitution rates are the values in the customers' mind. However, due to various reasons these real values may not be reflected in the sales information of the products. For example, think about the following extreme case. Consider again a retailer that stocks three items in a category. If the retailer stocks one of the items such that it never stocks out, then whatever the real substitution rates from this product are we cannot observe these rates through the sales information. Even consider a more extreme case in which the retailer stocks every item in the category such that none of them stocks out. Then it is obvious that no substitution will occur, and thus estimation methods will seem to perform very poorly.

Taking into account this situation, it will also be useful to evaluate the performance of an estimation method according to the realized substitution rates. Actually, these rates will reflect the substitution rates which are resulted from the substitution rates on the customers' mind and the dynamics of the considered system (e.g. inventory control policy). Therefore, using realized substitution rates along with the real ones can provide us with more insight on the performance of an estimation method in some cases.

5.3 Performance Measures

To compare performance of different estimation methods we employ three common estimation error measures: mean absolute percentage error, mean absolute deviation, and maximum absolute deviation. The formulas of the corresponding error metrics for N products are as follows:

Mean Absolute Percentage Error: $$MAPE = \frac{\sum_i \sum_{j \neq i} \frac{\left|\hat{\alpha}_{ij} - \alpha_{ij}\right|}{\alpha_{ij}} * 100}{N^2 - N}$$

Mean Absolute Deviation: $$MAD = \frac{\sum_i \sum_{j \neq i} \left|\hat{\alpha}_{ij} - \alpha_{ij}\right|}{N^2 - N}$$

Maximum Absolute Deviation: $$MAX = \max_{\substack{(i,j)\\ i \neq j}} \left|\hat{\alpha}_{ij} - \alpha_{ij}\right|$$

where α_{ij} denotes the probability of substituting product j for product i and $\hat{\alpha}_{ij}$ denotes its estimated value.

5.4 Comparison of SSBE with Maximum Likelihood Estimation (MLE)

It would be useful to compare SSBE with another method in order to get an insight on its performance. One existing method in the literature (Anupidi et al. [6]) uses a MLE procedure to obtain estimates of both demand and substitution parameters (see Appendix A). MLE with perpetual system data requires a detailed transaction data for each sale. More precisely, data corresponding to the time of a stock-out, the identity of the product that runs out, and the sales of other products up to that time are known for each replenishment cycle. Therefore, we can compare SSBE with known initial stocking levels and replenishment lengths with MLE with perpetual system data. In this section, a detailed comparison of these two methods is presented for different parameter sets (using 100 replications for each parameter set) in terms of MAPE, MAD, and MAX.

5.4.1 MAPE

All of the problem parameters, including market shares, simulation length, probability of substitution, and POS interval length affect the performance of the methods in terms of MAPE. The computational results for MAPE are obtained with respect to both real and realized substitution rates, and since they were found to be very similar in the experiments the following results are valid for both of them. Figures 5.1 to 5.3 provide detailed information about MAPE values for different combinations of demands ([3.3,3.3,3.3], [2.5,2.5,5], and [1,3,6]), POS interval lengths (0.01, 0.1, and 0.25 hours), probability of substitutions (0.2, 0.4, and 0.6) and simulation lengths (3 and 6 months) with respect to the real substitution matrix (see Appendix B for detailed error values with respect to both real and realized substitution matrices) obtained in the experiments.

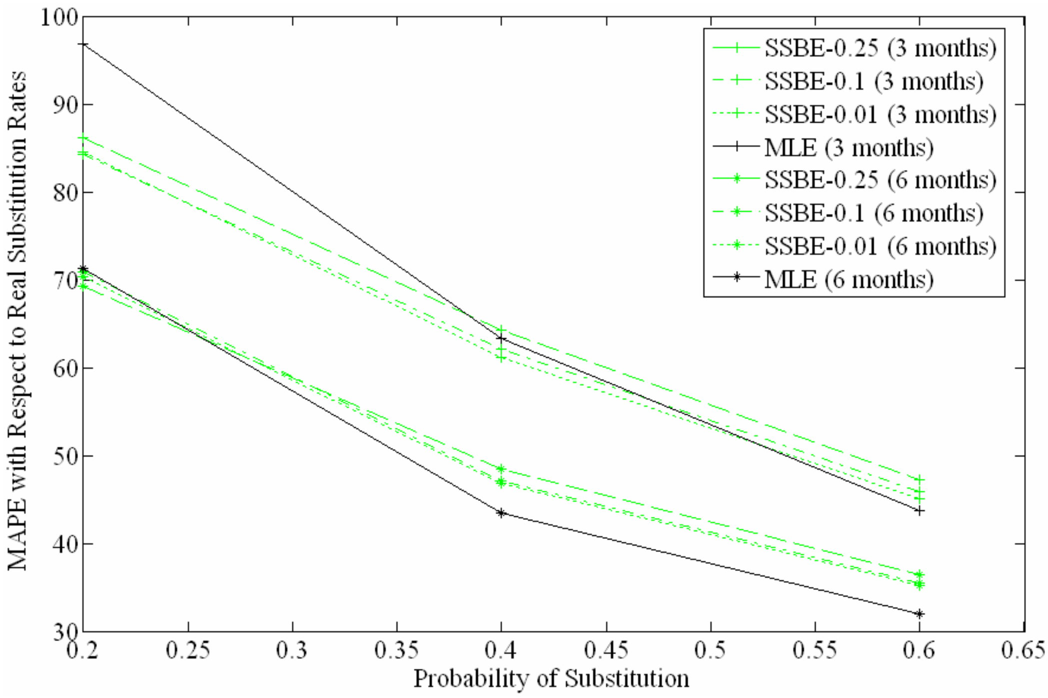


Figure 5.1: MLE vs. SSBE – MAPE with respect to real substitution matrix for demand combination [3.3, 3.3, 3.3]

First, we observe that as the probability of substitution increases, MAPE values for the two methods decrease significantly. When we compare the two methods in terms of probability of substitution, the results reveal that SSBE performs better than MLE when the probability of substitution is 20 percent and 40 percent and performs worse when it is 60 percent under the simulation length of 3 months. On the other hand, when the simulation length is 6 months, SSBE performs better than MLE when the probability of substitution is 20 percent and performs worse when it is 40 percent and 60 percent. This result is due to the fact that the number of observed states used by SSBE decreases as the substitution probability increases (as we discussed for the fluid model in Chapter 4). After all, the two methods behave similarly with respect to this parameter and generate error values which are more or less the same when the probability of substitution is around the average value reported in the literature (i.e., 40 percent).

When it comes to the comparison for different types of market shares, that is different demand rate combinations, we can say that MAPE values increase for both methods as market shares of products become less homogeneous. Moreover, the performance of SSBE improves compared to that of MLE as the difference of the demand rates increases.

As for the analysis of the experiments in terms of POS interval length, first note that MAPE values for MLE do not change with respect to the different values of POS interval length, because MLE does not use POS intervals as input. Instead, it takes the exact times of stock-outs, the identities of the products that run out, and the sales of other products up to these run-out times for each of the replenishment cycles as input values. So we can only analyze the effect of POS interval length on SSBE. In this regard, the most important result is that MAPE values change only slightly when we increase the POS interval length from 0.01 hour to 0.25 hour. This is equivalent to say that there will be practically no difference in terms of estimation performance if we use POS data tables with a precision of 15 minutes in lieu of those with a precision of 36 seconds. Moreover, we observe that in most of the cases error values tend to decrease with the increasing precision of sales data (as discussed in section 4.3).

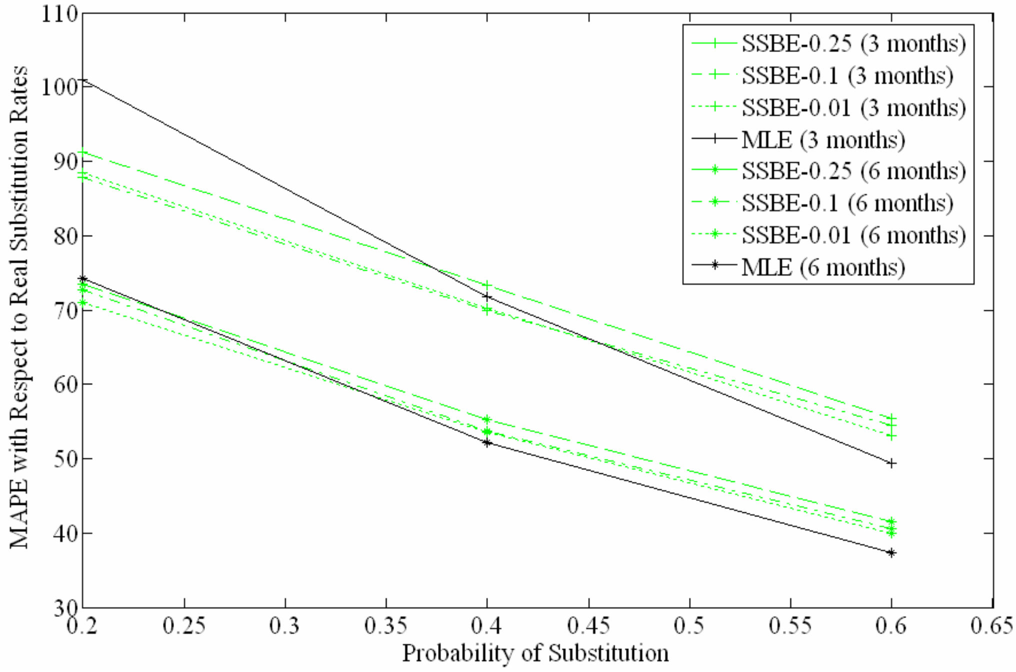


Figure 5.2: MLE vs. SSBE – MAPE with respect to real substitution matrix for demand combination [2.5, 2.5, 5]

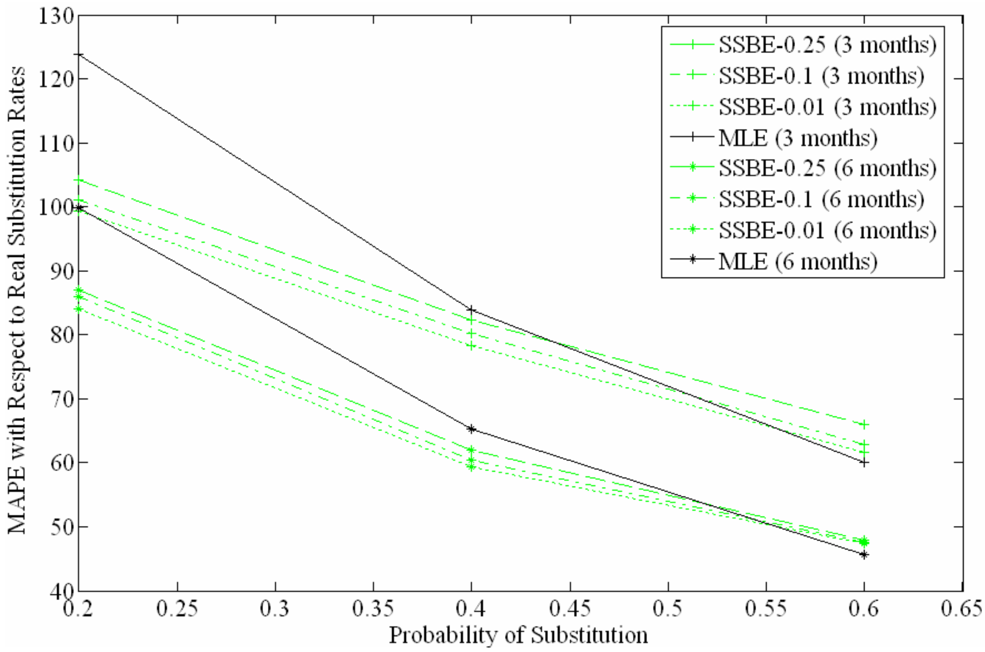


Figure 5.3: MLE vs. SSBE – MAPE with respect to real substitution matrix for demand combination [1, 3, 6]

Finally, when we consider the impact of simulation length on the estimation performances of the two methods, it is clear that as the simulation length increases MAPE values decrease substantially as a result of using more data in the estimation process.

5.4.2 MAD

MAD values show similar patterns to those of MAPE values with respect to the POS interval length and simulation length parameters, while it represents a dissimilar pattern with respect to the probability of substitution parameter. First, as the probability of substitution goes up MAD values also go up. Recall that MAPE values go down when the probability of substitution increases. In fact, the results are not contradictory because when we increase the probability of substitution from 0.2 to 0.6, i.e. the substitution probabilities are tripled, MAD values are not even doubled. So even if the absolute deviations increase to some extent, the percentage errors decrease. Still, SSBE provides better results than MLE when the probability of substitution is low and the two methods behave in the same manner in terms of the probability of substitution. Second, mean absolute errors are decreasing in demand rate homogeneity. Third, POS interval length has no impact on MLE and a slight impact on SSBE. Finally, MAD values tend to decrease when the simulation length increases. Figures 5.4-5.6 present error values corresponding to MAD criterion for different parameter combinations.

5.4.3 MAX

Another important metric for estimation performance is the maximum absolute deviation showing the largest distance between a substitution probability and its estimate. MAX values also change with different combinations of parameters. When we analyze the effect of the probability of substitution, we first notice that MAX values do not present a clear pattern for the simulation of three months. On the other hand, for the six-month simulation they are increasing in the probability of substitution. However, unlike the case with MAPE and MAD, SSBE performs better when the probability of substitution is around 40 percent and the simulation length is 6 months. The rest of the results are parallel to those mentioned in MAPE and MAD sections. Figures 5.7-5.9 show corresponding MAX values for each of the parameter sets.

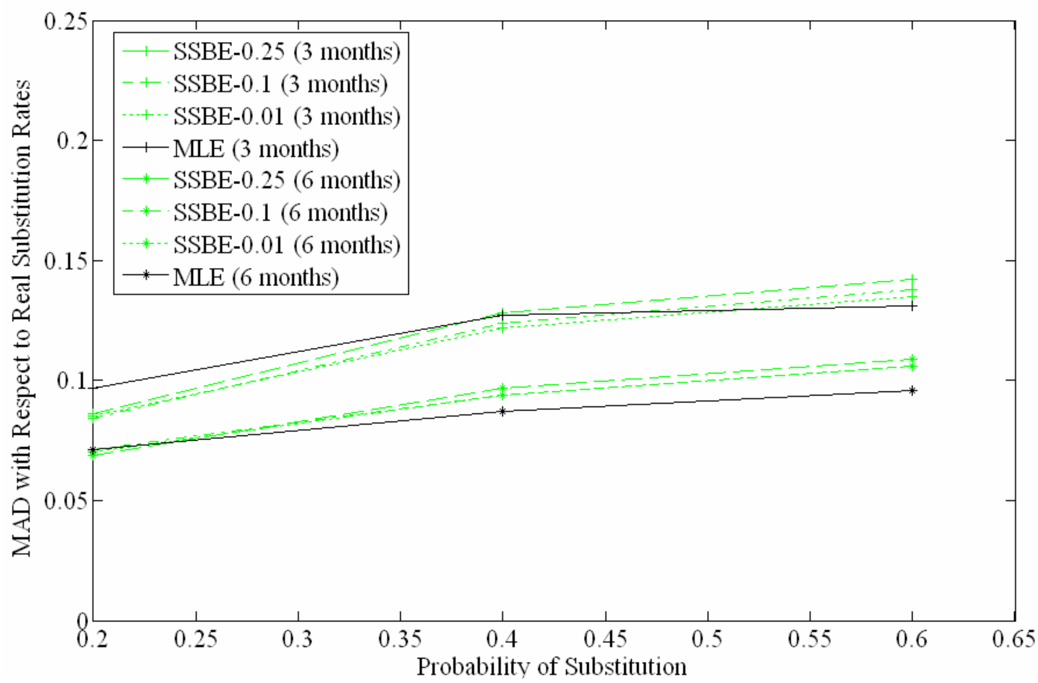


Figure 5.4: MLE vs. SSBE – MAD with respect to real substitution matrix for demand combination [3.3, 3.3, 3.3]

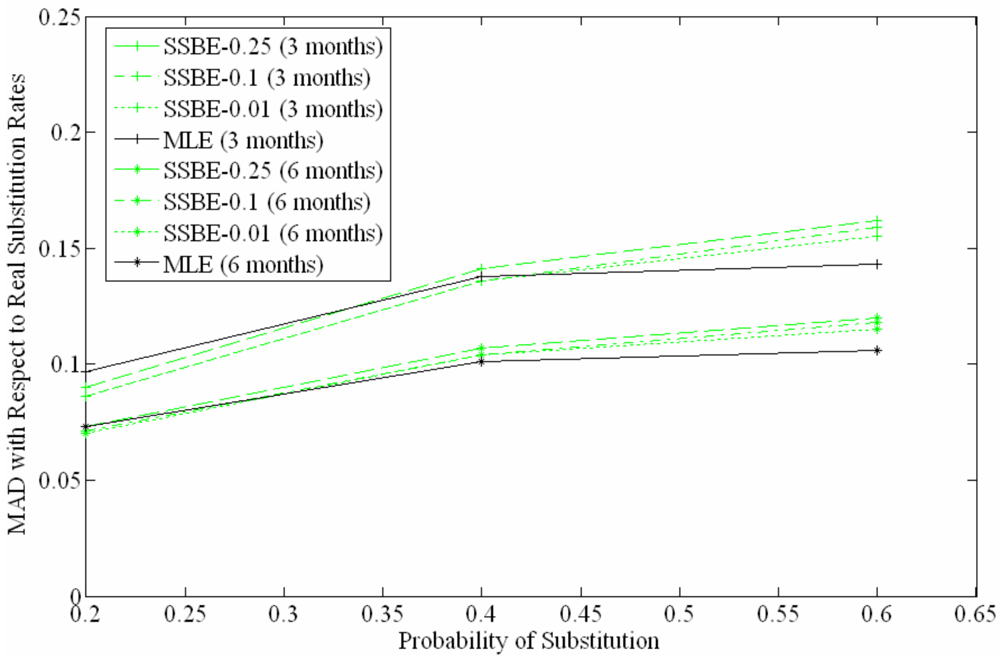


Figure 5.5: MLE vs. SSBE – MAD with respect to real substitution matrix for demand combination [2.5, 2.5, 5]

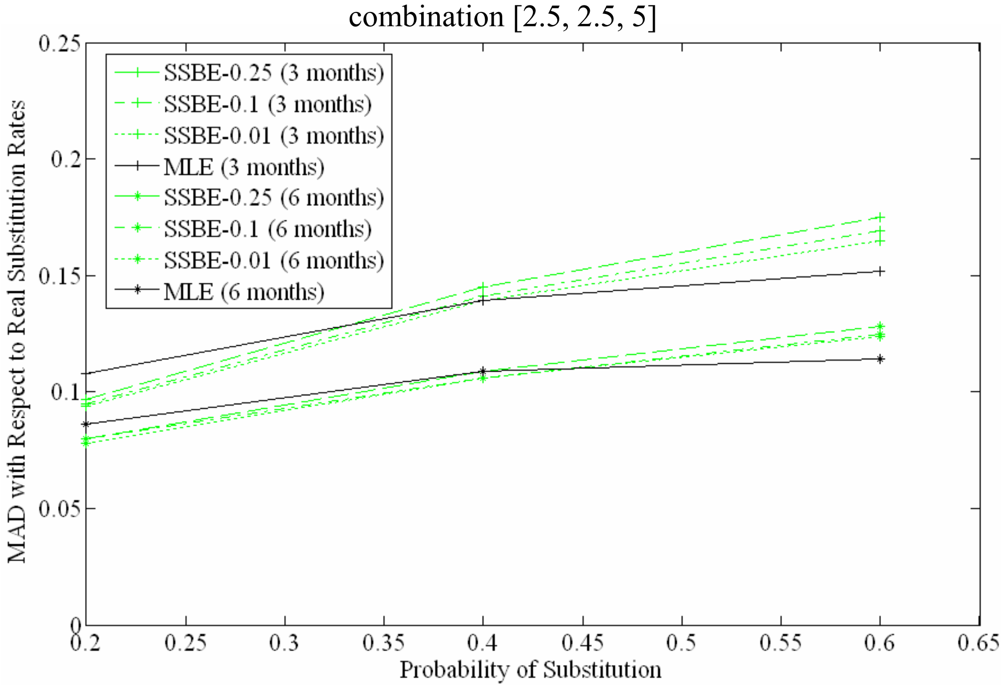


Figure 5.6: MLE vs. SSBE – MAD with respect to real substitution matrix for demand combination [1, 3, 6]

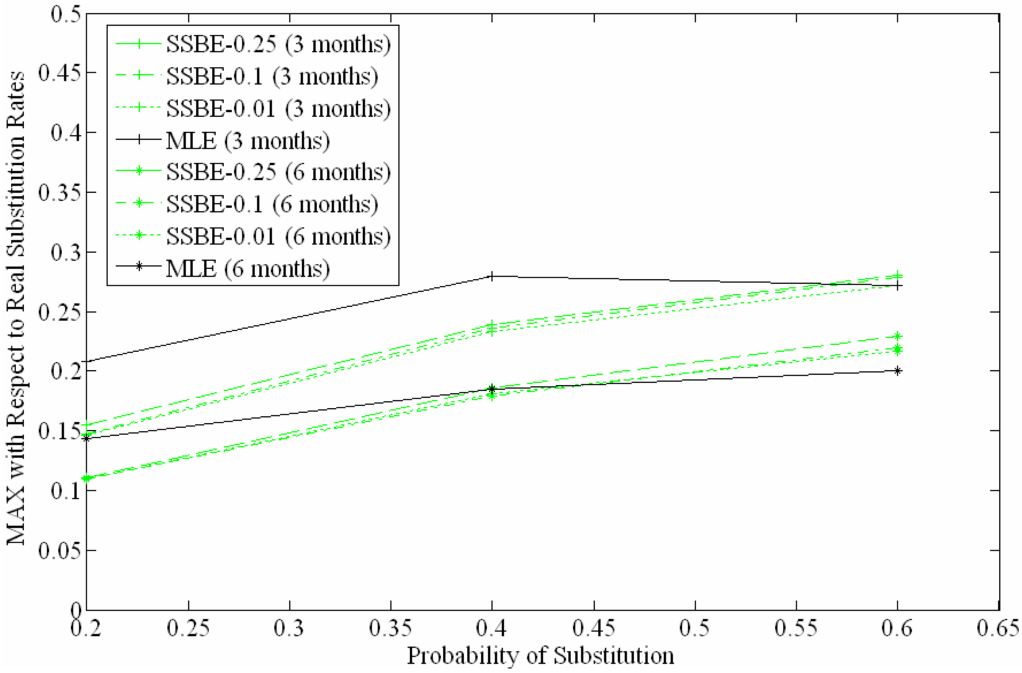


Figure 5.7: MLE vs. SSBE – MAX with Respect to real substitution matrix for demand combination [3.3, 3.3, 3.3]

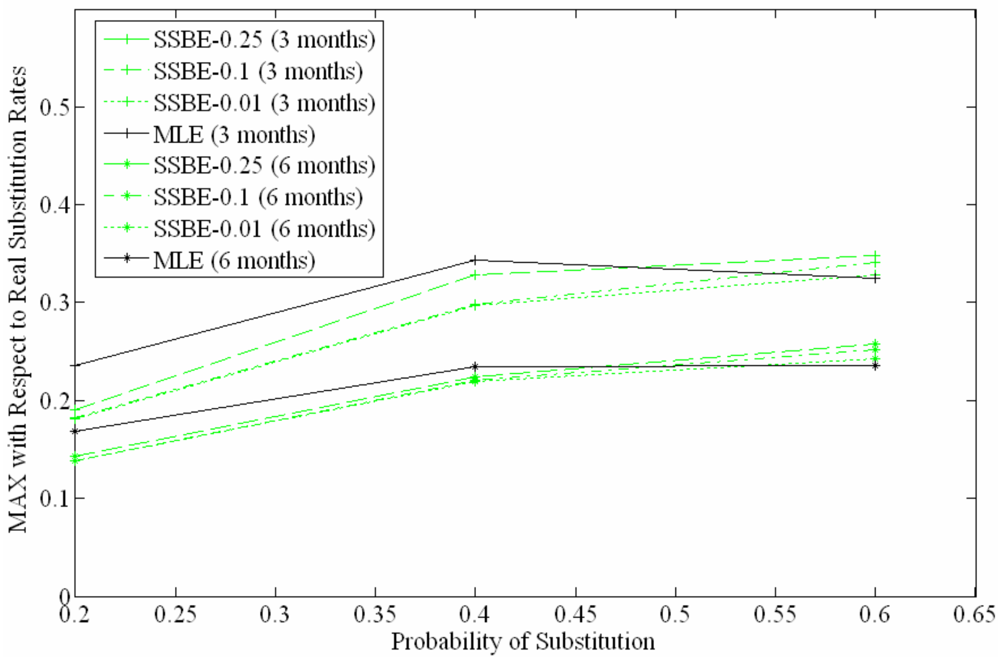


Figure 5.8: MLE vs. SSBE – MAX with respect to real substitution matrix for demand combination [2.5., 2.5, 5]

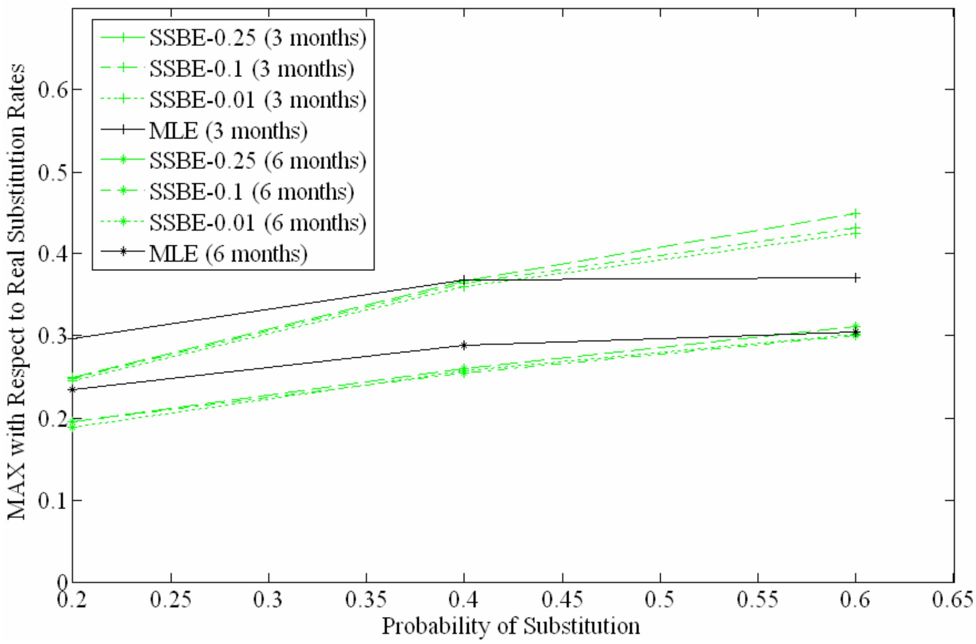


Figure 5.9: MLE vs. SSBE – MAX with respect to real substitution matrix for demand combination [1, 3, 6]

5.5 Summary of the Computational Results

In this chapter, we analyzed the performances of the two estimation methods, SSBE and MLE, for different combinations of problem parameters by using criteria including MAPE, MAD, and MAX. According to the results, MAD and MAX values are increasing in the probability of substitution, whereas MAPE values are decreasing. The results also suggest that the error values are generally decreasing in demand rate homogeneity and in the simulation length for both of the methods. Another important result reveals that the POS interval length has a very slight impact on the error values generated by SSBE, which is very favorable for the practical use of the method.

Along with the effect of problem parameters on the two estimation methods, we compared the performances of the two methods. This comparison suggests that when the probability of substitution is medium they perform closely. For a small probability of substitution SSBE performs better, whereas MLE generates smaller mean error values for a large probability of substitution. But after all, SSBE and MLE exhibit similar behavior to changes in problem parameters and yield more or less the same error values in most of the cases.

Chapter 6

CASE STUDY

6.1 Introduction

In Chapter 5, we provided the results of the computational study in a setting with a category of three products. We compared the estimation performance of our method to that of MLE to obtain insights on the quality of our estimation. In this chapter, we test SSBE in the case where we only have POS data by comparing it to the case where initial stocking levels and replenishment cycle lengths are known.

When we generated problem data for the comparison with MLE we assumed that the lead time was zero and replenishments were made periodically (once a week). However, in a real retail setting there is a replenishment lead time different than zero. Items can be replenished several times in a week and replenishment days may differ among the items in a category.

Furthermore, we presumed that the POS data did not include seasonality effects. However, in a real life setting, demand in supermarkets varies both over the day and over the week. For example, van Donselaar et al. [49] found that 54 percent of the demand at the retailer occurred on Fridays and Saturdays (for another study about consumer habit in shopping hours see East et al. [50]). Therefore, we assume that the seasonality factors (determined by a forecasting method) for a product category for different hours of the day and different days of the week are given. Moreover, we assume that these seasonality factors are valid for all of the products in the category. By using these seasonality factors we obtain deseasonalized POS data ready to be used in our proposed estimation procedure.

Therefore, in this chapter, we present a case study for evaluating the quality of estimation of SSBE with POS data only in a more complex setting by relaxing the assumptions on the replenishment lead times and policy along with the seasonality of demand. We analyze the case with five products instead of three and provide the necessary adaptations made to the input data used by the method.

6.2 Problem Generation and Adjustments Made to the Input Data and Step 3 of SSBE

In this case we have a category of five products: A, B, C, D, and E. Similar to the problem generation in the computational results part, the customer arrival processes for these five products are assumed to be Poisson Process. Hourly demand rates for the products are 18, 15, 6, 3, and 3 respectively. The POS interval length is 15 minutes. During the choice process, again, we only allow for one substitution attempt. We employ the probabilistic substitution structure in which the total substitution probability δ (set to 40 percent) is shared among products proportionate to their market shares (see 5.1.1 for details). We assume that the hourly seasonality factors (ρ_h $h = 1,2,...,H$, where H is the operating time in a week), given in Table 6.1, are available to the retailer (these factors can also be estimated by using standard forecasting techniques).

The replenishment lead time is 1 day for all products. Orders for products C, D, and E are released once a week (on Wednesday mornings). Order-up-to levels for these products are calculated by using a fixed review period inventory control with lost sales. On the other hand, the retailer orders product A and product B twice per week (on Wednesday and Friday mornings); first order is for the peak demand days and the other is for the rest of the week. Replenishments are made according to an adaptation to the fixed review period order-up-to level type inventory control system with lost sales (see Appendix C for details).

	Mon	**Tu**	**Wed**	**Th**	**Fr**	**Sa**	**Su**
9-10 a.m	1.18	1.73	1.18	2.49	2.01	2.56	0.28
10-11 a.m.	1.25	1.18	1.18	2.81	2.35	2.01	0.42
11 a.m.-12 noon	0.9	0.62	1.04	2.14	2.21	1.11	0.28
12 noon-1 p.m.	0.28	0.21	0.76	0.83	1.11	1.25	0.14
1 p.m.-2 p.m	0.35	0.42	0.48	1.11	1.59	0.9	0.14
2 p.m-3 p.m.	0.62	0.83	1.04	1.25	1.94	0.97	0.14
3 p.m.-4 p.m.	0.21	0.76	0.35	0.83	1.73	0.97	0.14
4 p.m.-5 p.m.	0.21	0.28	0.48	1.25	1.31	0.35	0.07
5 p.m.-6 p.m.	0.28	0.28	0.69	1.66	2.49	0.48	0
6 p.m.-7 p.m.	0.42	0.48	1.18	2.35	3.25	0.21	0

Adapted from East et al. [49].

Table 6.1: Hourly seasonality factors in problem generation

Recall that we estimated demand and substitution rates for three products in Chapter 5. So, we obtained three estimates for the demand rates and six estimates for the substitution probabilities. However, in the case of five products we have to estimate five demand parameters and twenty substitution probabilities. And we need to solve 5 separate optimization problems each including 15 ($(2^{N-1}-1)$) equality constraints.

Nevertheless, we can make an aggregation for both decreasing the size of the optimization problems and obtaining more insightful (i.e. not negligible) substitution rates. In this case study we prefer to consider three products as one aggregate product and proceed with the estimation procedure as if we had a category with three products (including an aggregate product). More precisely, we choose the combination of products C, D, and E; and denote the aggregate product by C-D-E. Therefore, we try to estimate six substitution rates among products A, B, and C-D-E.

By taking into account the seasonality effects and the aggregation, we must make the necessary adjustments to the input data and Step 3 of the method in order for SSBE to be able to work properly. So, first we deseasonalize POS data by using hourly seasonality factors given in Table 6.1. When it comes to SSBE, we apply the first two steps for each of the five products similar to the case with three products. However, since we consider three products as one product we define the sub-state of the aggregate product as follows. First we define the sub-states of the three products (C, D, and E) separately. Then if the sub-states of these three products in a particular POS interval are $\checkmark$ then that of the aggregate product (C-D-E) is labeled as $\checkmark$. If the sub-states of these three products in a particular POS interval are $\times$ then that of the aggregate product is labeled as $\times$ (This modification can easily be generalized for the case of N products.). Finally, SSBE proceed with Step 4 and Step 5 without any modification.

6.3 Results

In this section, the results of the case study of five products are presented. More precisely, Figures 6.1-6.6 show the quality of the estimation for six substitution parameters with respect to different simulation lengths for a particular random seed. We first present the results for a particular random seed because when we apply our method to a real problem we will have POS data corresponding to a particular realization of demand. Figures 6.7-6.9 provide the quality of the estimation in terms of performance measures. Then, we provide results that are obtained through hundred different seeds showing the estimation performance (mean errors and confidence intervals) in Figures 6.10-6.15. In these figures SSBE–CI shows the estimation of SSBE with complete information (i.e., while the replenishment points and amounts for every product are known), whereas SSBE–POS presents the estimation of SSBE with POS data only.

The results suggest that SSBE performs accurately in the case where the category consists of more than three products and demand seasonality is present. Moreover, the estimations of SSBE with POS data only are very close to those of SSBE with known replenishment points and amounts. It is also apparent that estimation performance improves in general as the simulation length increases, because when the simulation length goes up we observe more states, especially the states where substitution occur.

Note that these results are obtained in the setting where the total category demand is 45 units/hour and the POS interval length is 15 minutes. Since these parameters are only scaling parameters, presenting the results for different values they may assume will not provide additional insight.

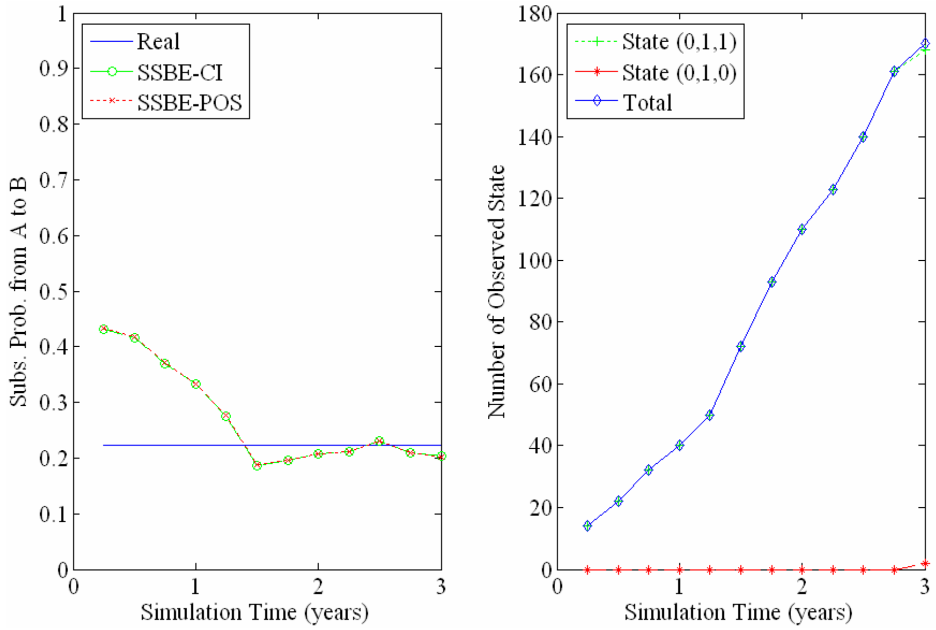


Figure 6.1: Estimation performance for the substitution rate from A to B

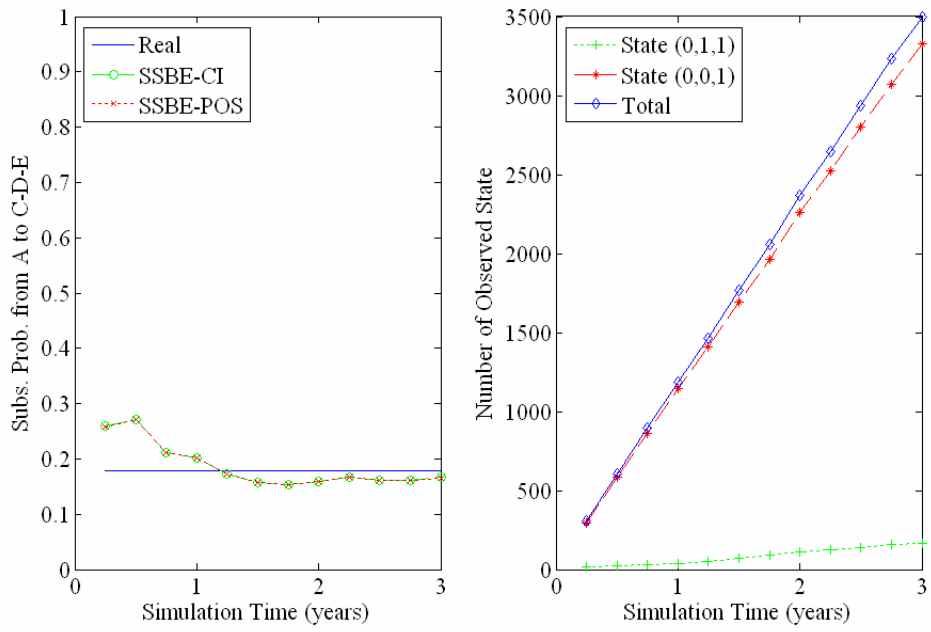


Figure 6.2: Estimation performance for the substitution rate from A to C-D-E

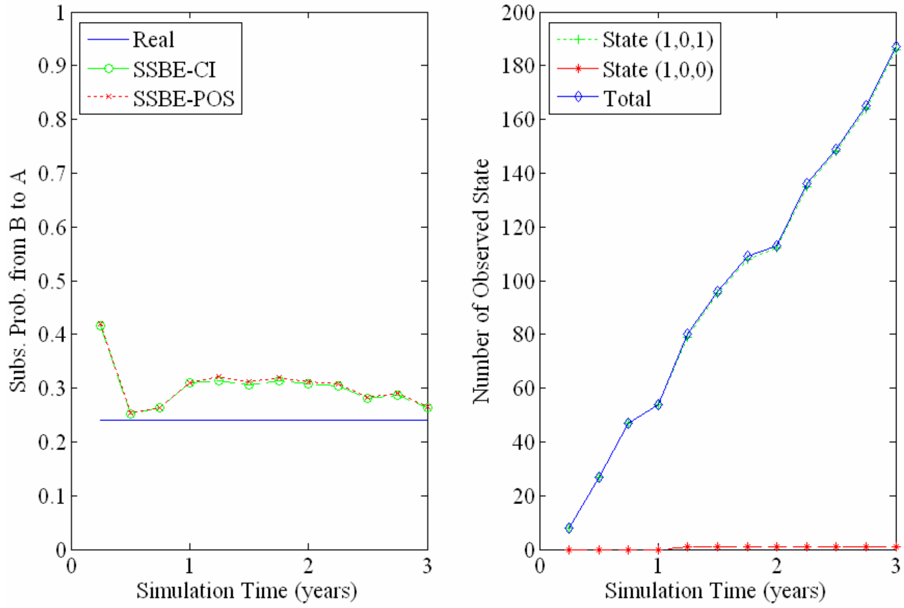


Figure 6.3: Estimation performance for the substitution rate from B to A

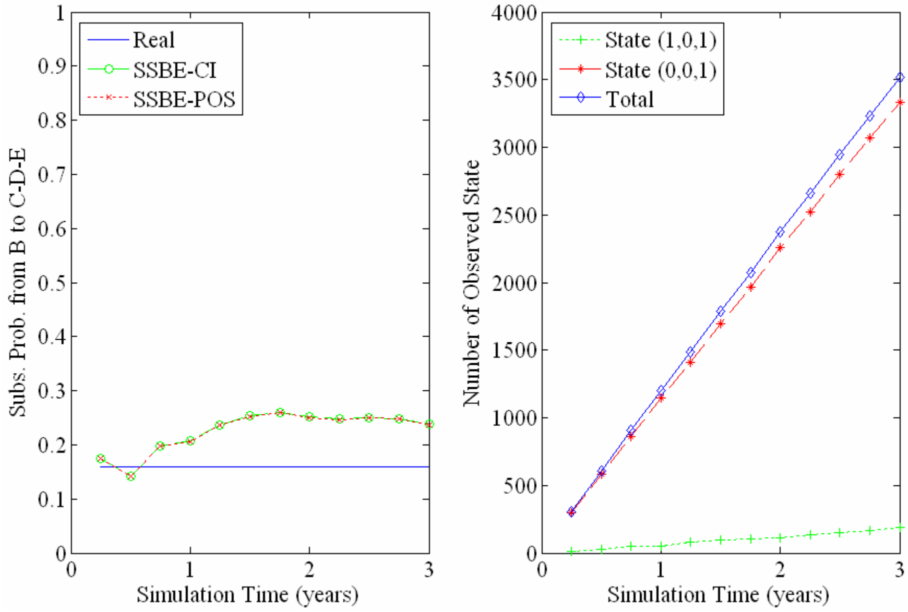


Figure 6.4: Estimation performance for the substitution rate from B to C-D-E

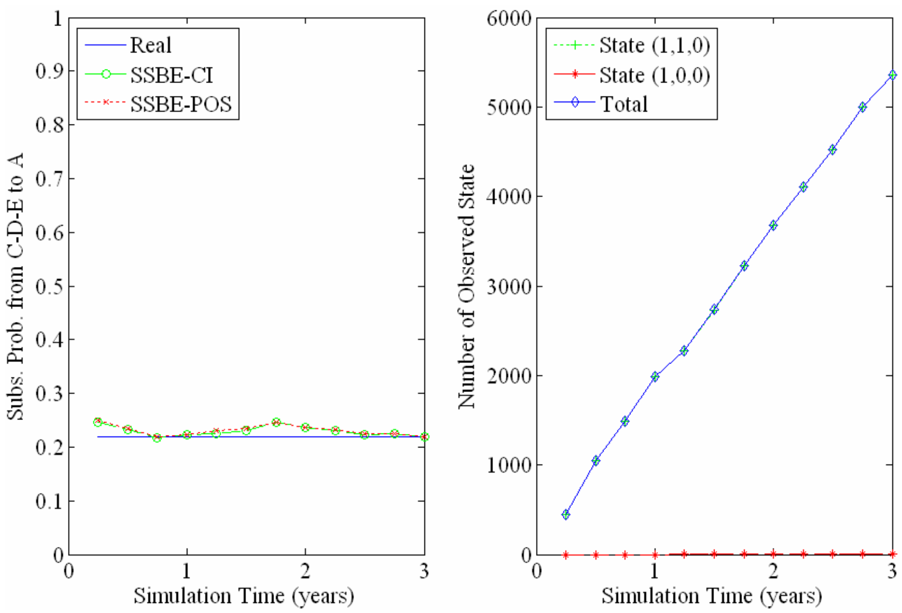


Figure 6.5: Estimation performance for the substitution rate from C-D-E to A

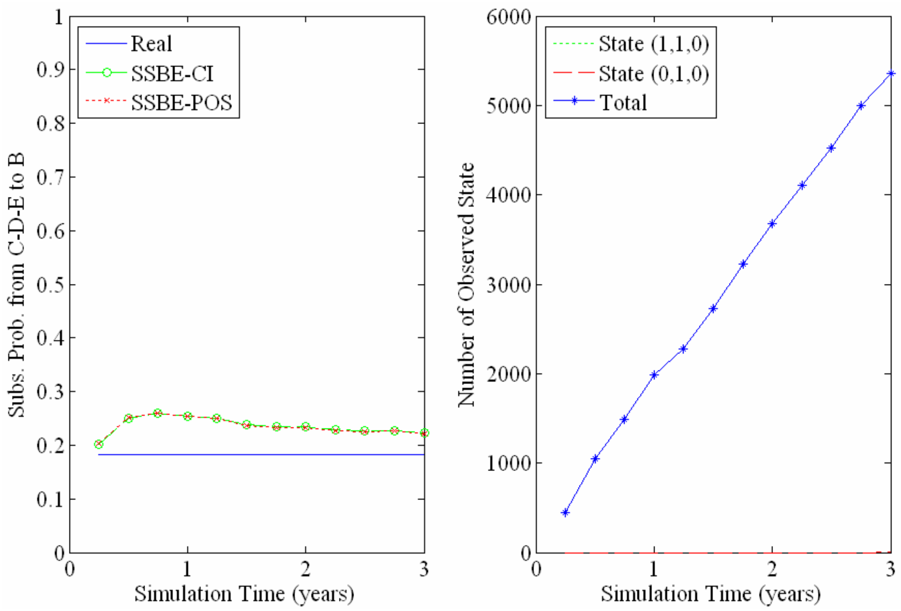


Figure 6.6: Estimation performance for the substitution rate from C-D-E to B

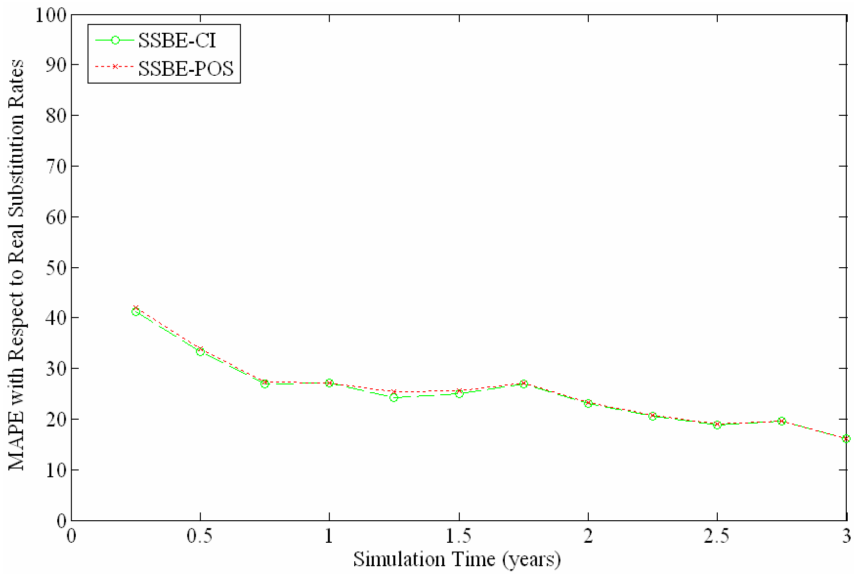


Figure 6.7: MAPE with respect to real substitution rates

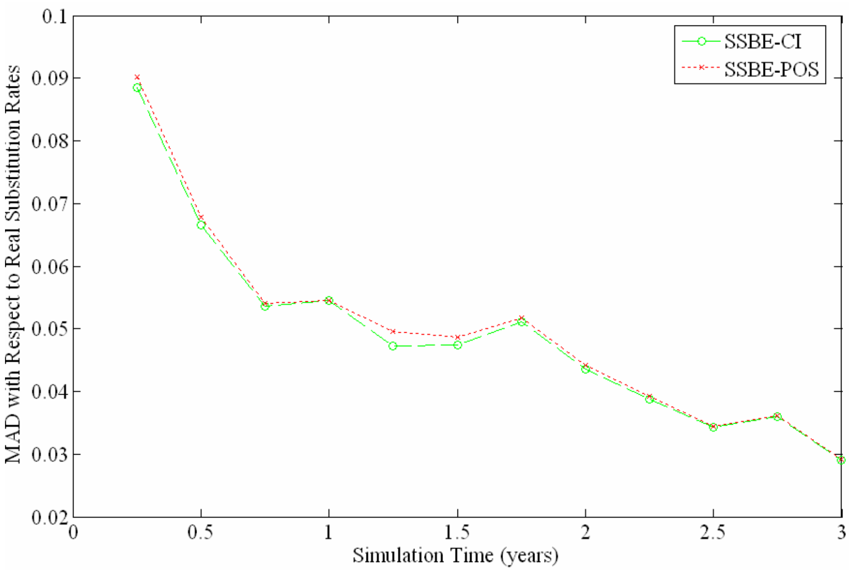


Figure 6.8: MAD with respect to real substitution rates

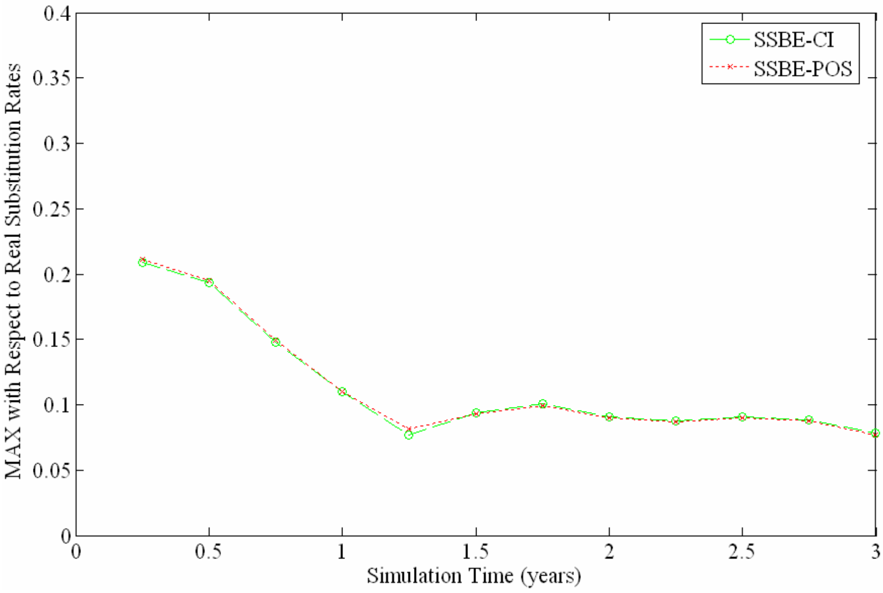


Figure 6.9: MAX with respect to real substitution rates

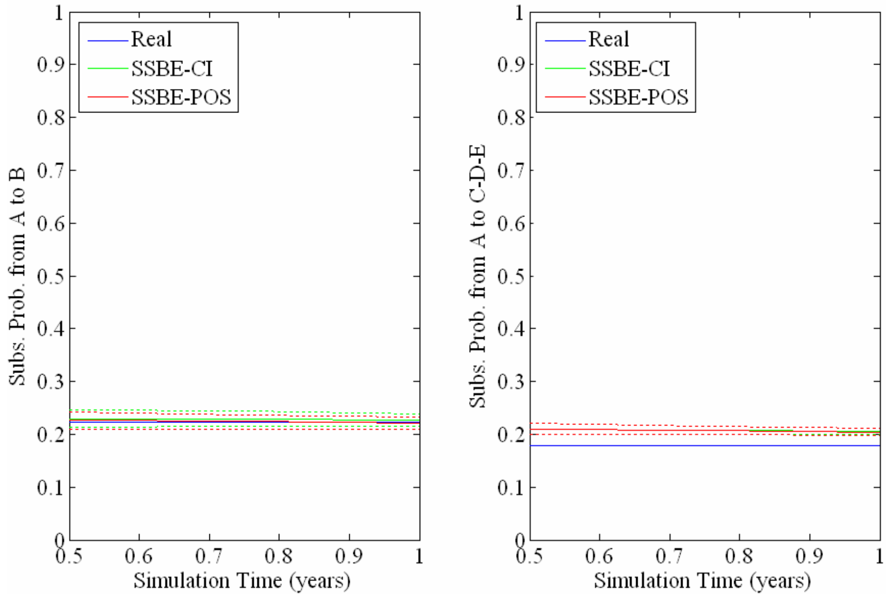


Figure 6.10: Estimation performance (average for 100 seeds) for the substitution rate from A

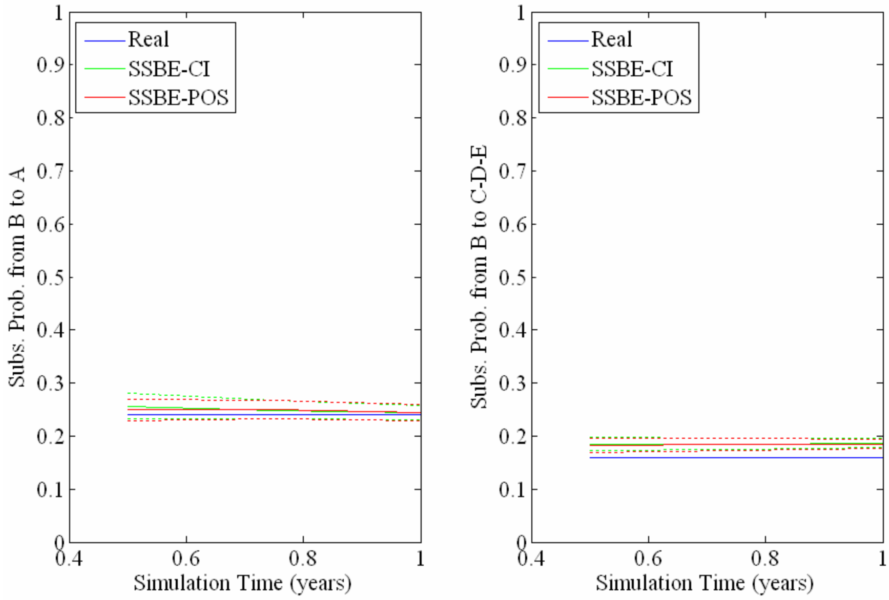


Figure 6.11: Estimation performance (average for 100 seeds) for the substitution rate from B

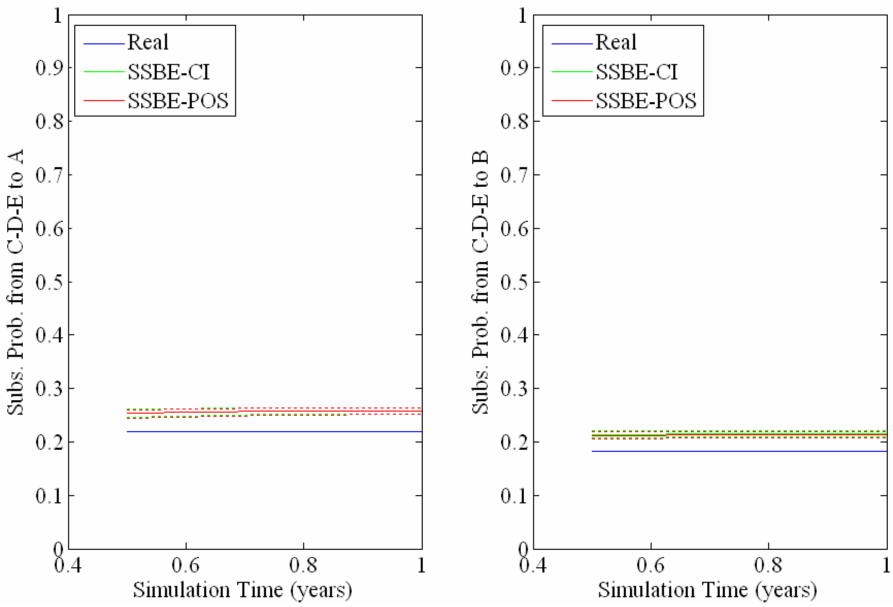


Figure 6.12: Estimation performance (average for 100 seeds) for the substitution rate from C-D-E

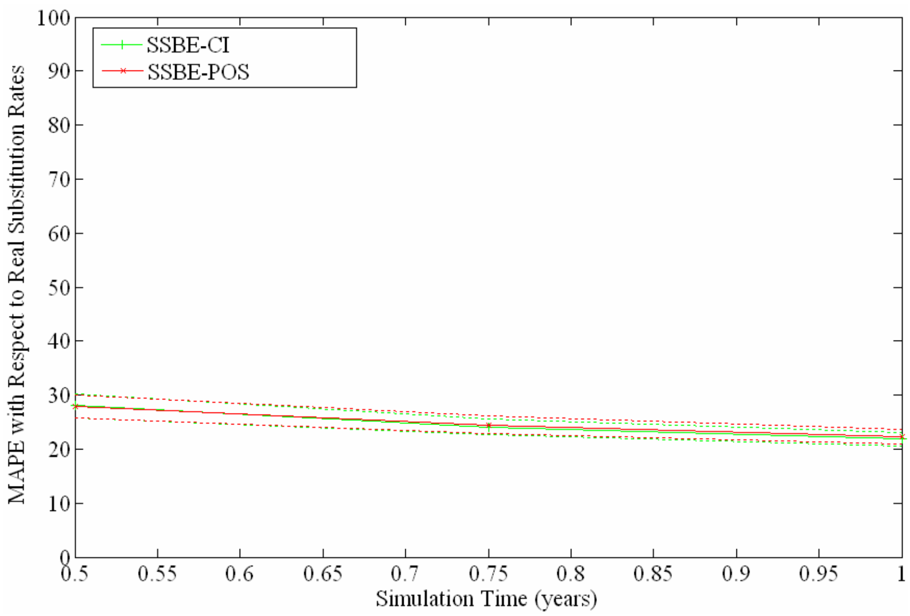


Figure 6.13: MAPE (average for 100 seeds) with respect to real substitution rates

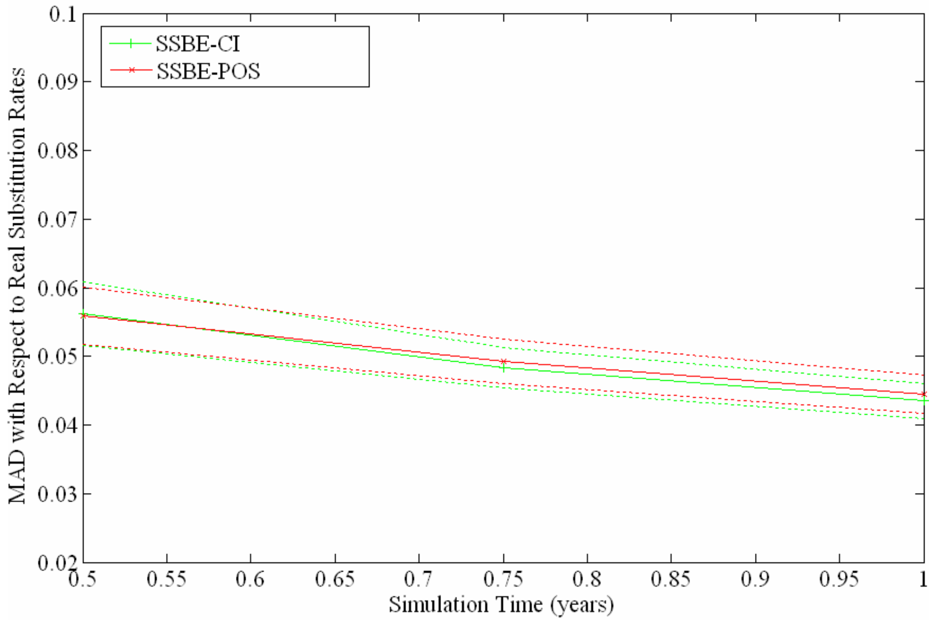


Figure 6.14: MAD (average for 100 seeds) with respect to real substitution rates

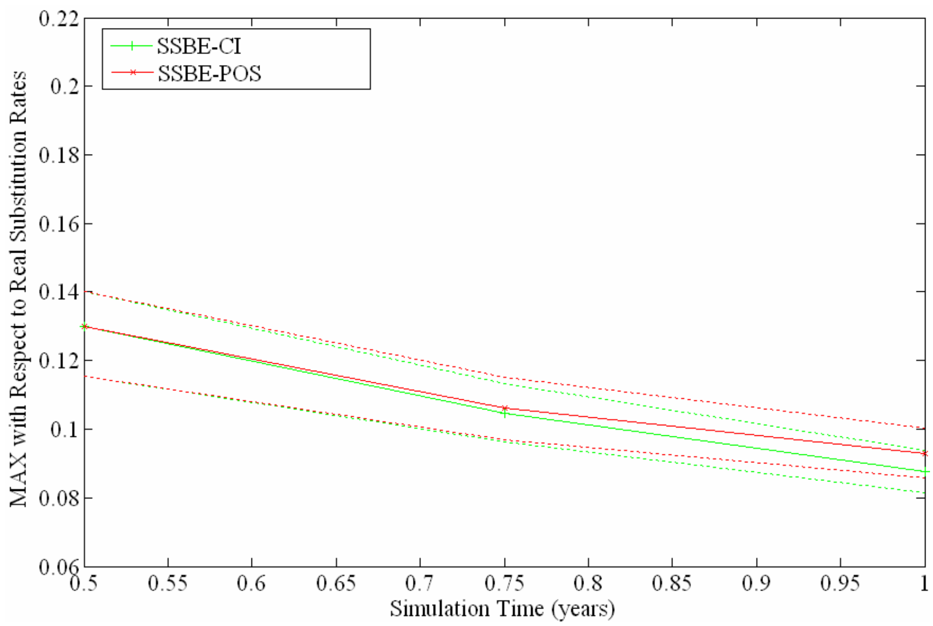


Figure 6.15: MAX (average for 100 seeds) with respect to real substitution rates

Chapter 7

CONCLUSION

Retailers have long been trying to reduce the rates of products unavailability by implementing novel approaches including category management and ECR. However, recent reports on worldwide stock-out rates suggest that the average product unavailability on a typical day, which is reported to be 8.2 percent, has stayed the same over the last ten years. Moreover these rates are reported to be considerably higher during promotion days.

The temporary stock-outs affect the performance of retailers as well as manufacturers. Therefore, both parties need to recognize the responses of consumers in the case of stock-outs. Practitioners generally divide these responses into different groups: buy item at another store, delay purchase, substitute to same brand, substitute to different brand, and do not purchase the item. In this thesis, we focus on substitution response of consumers involving 40 percent of consumer responses to out-of-stocks.

Stock-out based product substitution affect retailers and manufacturers both directly and indirectly. On one hand, it alters the optimal order quantities and expected profit of retailers. On the other hand, retailers may incur indirect losses such as a decrease in the consumer satisfaction level. Therefore, retailers should adapt their inventory control and product assortment decisions to the product substitution possibility. Similarly, manufacturers may experience negative effects as a result of a substitution to a competitor's product or a cheaper product.

Substitution rates are primarily used as inputs to the product assortment and inventory control decisions. In this thesis, we study the estimation of stock-out based substitution rates among products in a category. Our main research question is: can we estimate the demand and substitution rates of multiple partially substitutable products when only POS data is available?

We consider a retailer that stocks a certain number of products in a category. If consumers cannot find their favorite item the demand for that item either spills over to another item according to a probabilistic substitution structure or it is lost. We allow only one substitution attempt. The retailer has POS data including information on the number of units sold of each item for each POS interval and on the seasonality factors.

Our estimation method is based on a set of sub-states and a related state structure. First sub-states are defined given the corresponding amount of sales of each product. Subsequently, states are labeled for different combinations of sub-states. After the identification of states, our method finds estimates for demand rates of each product. Finally, separate optimization problems are solved by using the estimated demand rates and realized demands in POS intervals to estimate product substitution probabilities.

We provide a detailed description of the implementation of SSBE. For explanation simplicity, we first explain the case with complete information. Subsequently, we give details of SSBE with POS data only.

Then we analyze our method for the fluid model under deterministic case. More precisely, first we derive formulae for the calculations of the substitution proportions, the exact times of stock-outs, and the number of observed states. These calculations are then used to examine the effect of system parameters on the numbers of observed states, which may affect the estimation performance of SSBE. We show that both the number of states where only one product is unavailable and the total number of substitution states are nondecreasing in the substitution probability. Furthermore, we prove that the number of states where only one product is unavailable, the number of states where two products are unavailable and the total number of substitution states are bounded by nonincreasing functions in the POS interval length.

After analyzing our method for the fluid model, we compare SSBE with MLE in order to test its performance. To be more precise, we carry out a comparison between SSBE with known initial stocking levels and replenishment lengths with MLE with perpetual system data for different parameter sets in terms of MAPE, MAD, and MAX. The results suggest that SSBE generally performs better than MLE when the probability of substitution is low and performs worse when it is high. For a medium probability of substitution their performances are close. Moreover, we observe that the two methods behave similarly to the changes in parameter values (except the POS interval length, which does not affect MLE estimations).

Finally, we provide a simulation-based case study to determine the estimation performance of SSBE with POS data only in a more complicated setting through relaxing the assumptions on the replenishment lead times and policy along with the seasonality of demand. We examine a category with five products and explain the required adjustments made to the input data used by SSBE.

The results reveal that SSBE performs accurately in the case where the category consists of more than three products and demand seasonality is present. In addition, the estimations obtained by SSBE with POS data only are very close to those of SSBE with known replenishment points and amounts. We also observe that the estimation performance improves in general as the simulation length increases, i.e. the numbers of observed states go up.

This thesis contributes to the existing literature regarding inventory control and assortment optimization by providing a practical and accurate method that can be applied under very general conditions to estimate product substitution rates in a category by using only POS data.

A possible extension to this thesis is to test the performance of SSBE under different demand distribution assumptions, e.g. negative binomial. A further extension is to identify the best strategy in manipulating service levels of different products in order to estimate substitution probabilities as soon and accurately as possible by minimizing the negative effects resulting from the strategy. Another promising future direction is to determine the optimum inventory control policy under the described setting.

APPENDIX A

Proofs of the Lemmas and Propositions

Proof of Proposition 4.1. From Equation 4.8 we know that

$$c_{(0,1,1)}(\delta)=\left\lfloor \frac{L_B'-\pi\left[\left(c_1+1\right)\tau-st_A\right]\lambda_B^{pos'}(\delta)}{\lambda_B^{pos'}(\delta)}\right\rfloor.$$

Let $c_{(0,1,1)}{}'(\delta)=\dfrac{L_B'-\pi\left[\left(c_{(1,1,1)}+1\right)\tau-st_A\right]\lambda_B^{pos'}(\delta)}{\lambda_B^{pos'}(\delta)}$. That is, $c_{(0,1,1)}(\delta)=\left\lfloor c_{(0,1,1)}{}'(\delta)\right\rfloor$. To show that $c_{(0,1,1)}(\delta)$ is nonincreasing in δ, we first show that $c_{(0,1,1)}{}'(\delta)$ is decreasing in δ. Since we assume that product B is depleted after product A, we know that L_B' is positive. Moreover, let $I=\pi\left[\left(c_{(1,1,1)}+1\right)\tau-st_A\right]$. Then we have:

$$\begin{aligned} I &= c_{(1,1,1)}+1-st_A\pi \\ &= \frac{L_A}{\lambda_A^{pos}}-\varepsilon+1-\frac{L_A}{\lambda_A^{pos}} \qquad \text{with } 0<\varepsilon<1 \\ &= 1-\varepsilon. \end{aligned}$$

So $I>0$ and we can write: $c_{(0,1,1)}{}'(\delta)=\dfrac{L_B'-I\lambda_B^{pos'}(\delta)}{\lambda_B^{pos'}(\delta)}=\dfrac{L_B'-I\left(\lambda_B^{pos}+\dfrac{\lambda_A^{pos}\lambda_B^{pos}}{\lambda_A^{pos}+\lambda_B^{pos}}\delta\right)}{\lambda_B^{pos}+\dfrac{\lambda_A^{pos}\lambda_B^{pos}}{\lambda_A^{pos}+\lambda_B^{pos}}\delta}$

by using Equation 4.1.

Since all of the constants in $c_{(0,1,1)}'(\delta)$ are positive, the nominator is decreasing in δ and the denominator is increasing in δ. Therefore, $c_{(0,1,1)}'(\delta)$ is decreasing in δ.

To prove that $c_{(0,1,1)}(\delta)$ is nonincreasing in δ, we have to show that if $\delta_1 < \delta_2$ then $c_{(0,1,1)}(\delta_1) \geq c_{(0,1,1)}(\delta_2)$. Recall that we consider the case in which all products are depleted in the middle of a POS interval. So we should prove the above statement for the case in which $c_{(0,1,1)}'(\delta_1)$ and $c_{(0,1,1)}'(\delta_2)$ are not integers; and thus we have two different cases.

Case 1: There exists at least one integer d between $c_{(0,1,1)}'(\delta_1)$ and $c_{(0,1,1)}'(\delta_2)$.

Take $\delta_1, \delta_2 \in (0,1]$ such that $\delta_1 < \delta_2$. We know that $c_{(0,1,1)}'(\delta_1) > c_{(0,1,1)}'(\delta_2)$ because $c_{(0,1,1)}'(\delta)$ is decreasing in δ. So we have $c_{(0,1,1)}'(\delta_1) > d > c_{(0,1,1)}'(\delta_2)$. By the definition of the floor function we know that $c_{(0,1,1)}(\delta_1) = c'_{(0,1,1)}(\delta_1) - \varepsilon_1 \geq d$ with $0 < \varepsilon_1 < 1$, and similarly $c_{(0,1,1)}(\delta_2) = c'_{(0,1,1)}(\delta_2) - \varepsilon_2$ with $0 < \varepsilon_2 < 1$.

Since $d > c_{(0,1,1)}'(\delta_2)$, $d > c_{(0,1,1)}'(\delta_2) - \varepsilon_2 = c_{(0,1,1)}(\delta_2)$ is also true. In sum we have the following inequalities: $c_{(0,1,1)}(\delta_1) \geq d > c_{(0,1,1)}(\delta_2)$. In other words, $c_{(0,1,1)}(\delta)$ is decreasing in δ.

Case 2: There does not exist any integer between $c_{(0,1,1)}'(\delta_1)$ and $c_{(0,1,1)}'(\delta_2)$.

Take $\delta_1, \delta_2 \in (0,1]$ such that $\delta_1 < \delta_2$. Then we have $d < c'_{(0,1,1)}(\delta_2) < c'_{(0,1,1)}(\delta_2) < d+1$ with d the largest integer less than $c_{(0,1,1)}'(\delta_2)$. By using the definition of the floor function we obtain $c_{(0,1,1)}(\delta_1) = c_{(0,1,1)}(\delta_2)$.

Thus, by combining these two cases we know that for any $\delta_1, \delta_2 \in (0,1]$ if $\delta_1 < \delta_2$ then $c_{(0,1,1)}(\delta_1) \geq c_{(0,1,1)}(\delta_2)$. That is, $c_{(0,1,1)}(\delta)$ is nonincreasing in δ.

Proof of Lemma 4.1. Since $c_{(1,1,1)} < c'_{(1,1,1)}$ and $c_{(0,1,1)}(\delta) < c'_{(0,1,1)}(\delta)$ we can define a lower bound on $c'_{(0,0,1)}(\tau)$ as such:

$$
\begin{aligned}
L_{c'_{(0,0,1)}(\tau)} &= \frac{L''_C(\delta)}{\lambda_C^{pos''}(\delta)} - [c'_{(1,1,1)} + c'_{(0,1,1)}(\delta) + 2] + \pi st_B(\delta) \\
&= \frac{L''_C(\delta)}{\lambda_C^{pos''}(\delta)} - [\pi st_A + \frac{L'_B - \pi\left[\left(c_{(1,1,1)}+1\right)\tau - st_A\right]\lambda_B^{pos'}(\delta)}{\lambda_B^{pos'}(\delta)} + 2] + \frac{L'_B}{\lambda_B^{pos'}(\delta)} + \pi st_A \\
&= \frac{L''_C(\delta)}{\lambda_C^{pos''}(\delta)} + \pi\left[\left(c_{(1,1,1)}+1\right)\tau - st_A\right] - 2 \\
&= \frac{L''_C(\delta)}{\lambda_C^{pos''}(\delta)} + 1 - \varepsilon - 2 \\
&= \frac{L''_C(\delta)}{\lambda_C^{pos''}(\delta)} - 1 - \varepsilon
\end{aligned}
$$

with $0 < \varepsilon < 1$.

First consider $L''_C(\delta)$:

$$
\begin{aligned}
L''_C(\delta) &= L'_C - \lambda_C^{pos'}(\delta)\pi\left[st_B(\delta) - st_A\right] \\
&= L'_C - \lambda_C^{pos'}(\delta)\pi \frac{L'_B}{\lambda_B^{pos'}(\delta)\pi} \\
&= L'_C - L'_B \frac{\lambda_C^{pos'}(\delta)}{\lambda_B^{pos'}(\delta)}.
\end{aligned}
$$

Since $\lambda_B^{pos'}(\delta)$ and $\lambda_C^{pos'}(\delta)$ are increasing in δ, L'_B and L'_C are positive constants $L''_C(\delta)$ is decreasing in δ.

Second consider $\lambda_C^{pos''}(\delta)$. From Equation 4.3 we know that $\lambda_C^{pos''}(\delta)=\lambda_C^{pos}$ $+\lambda_A^{pos}\alpha_{AC}(\delta)+\lambda_B^{pos}\alpha_{BC}(\delta)$. Since $\alpha_{AC}(\delta)$ and $\alpha_{BC}(\delta)$ are increasing in δ by definition and λ_A^{pos}, λ_B^{pos}, and λ_C^{pos} are positive constants $\lambda_C^{pos''}(\delta)$ is increasing in δ.

In sum, $\dfrac{L_C''(\delta)}{\lambda_C^{pos''}(\delta)}$ is decreasing δ; thus $L_{c'_{(0,0,1)}(\delta)}$ is decreasing in δ.

Similarly, as $c'_{(1,1,1)}-1<c_{(1,1,1)}$ and $c'_{(0,1,1)}(\delta)-1<c_{(0,1,1)}(\delta)$ we can find an upper bound on $c'_{(0,0,1)}(\delta)$ as such:

$$\begin{aligned}U_{c'_{(0,0,1)}(\tau)}&=\frac{L_C''(\delta)}{\lambda_C^{pos''}(\delta)}-[c'_{(1,1,1)}-1+c'_{(0,1,1)}(\delta)-1+2]+\pi st_B(\delta)\\&=\frac{L_C''(\delta)}{\lambda_C^{pos''}(\delta)}+1-\varepsilon\end{aligned}$$

with $0<\varepsilon<1$. As a result, $U_{c'_{(0,0,1)}(\delta)}$ is decreasing in δ and $U_{c'_{(0,0,1)}(\delta)}=L_{c'_{(0,0,1)}(\delta)}+2$.

Proof of Proposition 4.2. From Lemma 4.1 we know that $L_{c'_{(0,0,1)}(\delta)}<c'_{(0,0,1)}(\delta)<U_{c'_{(0,0,1)}(\delta)}$. Then we have $\left\lfloor L_{c'_{(0,0,1)}(\delta)}\right\rfloor\le\left\lfloor c'_{(0,0,1)}(\delta)\right\rfloor\le\left\lfloor U_{c'_{(0,0,1)}(\delta)}\right\rfloor\Leftrightarrow\left\lfloor L_{c'_{(0,0,1)}(\delta)}\right\rfloor\le c_{(0,0,1)}(\delta)\le\left\lfloor U_{c'_{(0,0,1)}(\delta)}\right\rfloor$. So the bounds on $c_{(0,0,1)}(\delta)$ are $L_{c_{(0,0,1)}(\delta)}=\left\lfloor L_{c'_{(0,0,1)}(\delta)}\right\rfloor$ and $U_{c_{(0,0,1)}(\delta)}=\left\lfloor U_{c'_{(0,0,1)}(\delta)}\right\rfloor$; and they are nonincreasing in δ because of the floor function. Moreover, $U_{c_{(0,0,1)}(\delta)}=L_{c_{(0,0,1)}(\delta)}+2$.

Proof of Proposition 4.3. Since $c_{(0,1,1)}(\delta)$ is an integer, we have:

$$c_{(0,1,1)}(\delta)+c_{(0,0,1)}(\delta)=c_{(0,1,1)}(\delta)+\left\lfloor c'_{(0,0,1)}(\delta)\right\rfloor=\left\lfloor c_{(0,1,1)}(\delta)+c'_{(0,0,1)}(\delta)\right\rfloor.$$

From Equation 4.10 we obtain:

$$c_{(0,1,1)}(\delta)+c_{(0,0,1)}(\delta)=\left\lfloor \frac{L_C''(\delta)}{\lambda_C^{pos''}(\delta)}-[c_{(1,1,1)}+2]+\pi st_B(\delta)\right\rfloor . \qquad \text{(A.1)}$$

We know from the previous section that $\frac{L_C''(\delta)}{\lambda_C^{pos''}(\delta)}$ is decreasing δ. If we consider $st_B(\delta)$ from Equation 4.5 we know that $st_B(\delta)=st_A+\frac{L_B'}{\pi\lambda_B^{pos'}(\delta)}$. Since $\lambda_B^{pos'}(\delta)$ is increasing in δ, $st_B(\delta)$ is decreasing in δ.

Turning to the function inside the floor function in Equation 4.11, it is decreasing in δ; because $\frac{L_C''(\delta)}{\lambda_C^{pos''}(\delta)}$ and $\pi st_B(\delta)$ are decreasing in δ. Thus by taking into account the floor function we can conclude that $c_{(0,1,1)}(\delta)+c_{(0,0,1)}(\delta)$ is nonincreasing in δ.

Proof of Proposition 4.4. Let $c_{(1,1,1)}(\tau)=\lfloor c'_{(1,1,1)}(\tau)\rfloor$. Then we have $c'_{(1,1,1)}(\tau)=\frac{L_A}{\lambda_A^{pos}(\tau)}$. Note that L_A is independent of τ, because we assume that hourly demands of products are constant. Furthermore, as $\lambda_A^{pos}\pi$ is fixed λ_A^{pos} is increasing in τ (recall that $\pi=\frac{1}{\tau}$). Thus, $c'_{(1,1,1)}(\tau)$ is decreasing in τ; and $c_{(1,1,1)}(\tau)$ is nonincreasing in τ.

Proof of Lemma 4.2. By manipulating Equation 4.8 we obtain:

$$c'_{(0,1,1)}(\tau)=\frac{L_B'}{\lambda_B^{pos'}(\tau)}-c_{(1,1,1)}(\tau)+\pi(\tau)st_A-1 . \qquad \text{(A.2)}$$

Since $c_{(1,1,1)}(\tau)<c'_{(1,1,1)}(\tau)$ we can define a lower bound on $c'_{(0,1,1)}(\tau)$ as such:

$$
\begin{aligned}
L_{c'_{(0,1,1)}(\tau)} &= \frac{L'_B}{\lambda_B{}^{pos'}(\tau)} - c'_{(1,1,1)}(\tau) + \pi(\tau)st_A - 1 \\
&= \frac{L'_B}{\lambda_B{}^{pos'}(\tau)} - \frac{L_A}{\lambda_A{}^{pos}(\tau)} + \pi(\tau)st_A - 1 \\
&= \frac{L'_B}{\lambda_B{}^{pos'}(\tau)} - \pi(\tau)st_A + \pi(\tau)st_A - 1
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{L'_B}{\lambda_B{}^{pos'}(\tau)} - 1 \\
&= \frac{L'_B}{(\lambda^h{}_B + \alpha_{AB}\lambda^h{}_A)\tau} - 1.
\end{aligned}
$$

Therefore, $L_{c'_{(0,1,1)}(\tau)}$ is decreasing in τ.

Similarly, since $c'_{(1,1,1)}(\tau) - 1 < c_{(1,1,1)}(\tau)$ we can define an upper bound on $c'_{(0,1,1)}(\tau)$ as such:

$$
\begin{aligned}
U_{c'_{(0,1,1)}(\tau)} &= \frac{L'_B}{\lambda_B{}^{pos'}(\tau)} - (c'_{(1,1,1)}(\tau) - 1) + \pi(\tau)st_A - 1 \\
&= \frac{L'_B}{(\lambda^h{}_B + \alpha_{AB}\lambda^h{}_A)\tau}.
\end{aligned}
$$

As a result, $U_{c'_{(0,1,1)}(\tau)}$ is decreasing in τ and $U_{c'_{(0,1,1)}(\tau)} = L_{c'_{(0,1,1)}(\tau)} + 1$.

Proof of Proposition 4.5. From Lemma 4.1 we know that $L_{c'_{(0,1,1)}(\tau)} < c'_{(0,1,1)}(\tau) < U_{c'_{(0,1,1)}(\tau)}$. Then we have $\lfloor L_{c'_{(0,1,1)}(\tau)} \rfloor \le \lfloor c'_{(0,1,1)}(\tau) \rfloor \le \lfloor U_{c'_{(0,1,1)}(\tau)} \rfloor \Leftrightarrow \lfloor L_{c'_{(0,1,1)}(\tau)} \rfloor \le c_{(0,1,1)}(\tau) \le \lfloor U_{c'_{(0,1,1)}(\tau)} \rfloor$. So the bounds on $c_{(0,1,1)}(\tau)$ are $L_{c_{(0,1,1)}(\tau)} = \lfloor L_{c'_{(0,1,1)}(\tau)} \rfloor$ and $U_{c_{(0,1,1)}(\tau)} = \lfloor U_{c'_{(0,1,1)}(\tau)} \rfloor$; and they are nonincreasing in τ because of the floor function. Furthermore, $U_{c_{(0,1,1)}(\tau)} = L_{c_{(0,1,1)}(\tau)} + 1$.

Proof of Lemma 4.3. By manipulating Equation 4.9 we obtain: $c'_{(0,0,1)}(\tau) = \frac{L''_C}{\lambda_C^{pos''}(\tau)}$ $-c_{(1,1,1)}(\tau) - c_{(0,1,1)}(\tau) + \pi(\tau)st_B - 2$. Since $c_{(0,1,1)}(\tau) < c'_{(0,1,1)}(\tau)$ we can define a lower bound on $c'_{(0,0,1)}(\tau)$ as such:

$$
\begin{aligned}
L_{c'_{(0,0,1)}(\tau)} &= \frac{L''_C}{\lambda_C^{pos''}(\tau)} - c_{(1,1,1)}(\tau) - c'_{(0,1,1)}(\tau) + \pi(\tau)st_B - 2 \\
&= \frac{L''_C}{\lambda_C^{pos''}(\tau)} - c_{(1,1,1)}(\tau) - \left[\frac{L'_B}{\lambda_B^{pos'}(\tau)} - c_{(1,1,1)}(\tau) + \pi(\tau)st_A - 1 \right] + \pi(\tau)st_B - 2 \\
&= \frac{L''_C}{(\lambda^h{}_C + \alpha_{AC}\lambda^h{}_A + \alpha_{BC}\lambda^h{}_B)\tau} - \frac{L'_B}{\lambda_B^{pos'}(\tau)} + \pi(\tau)\left[st_B - st_A\right] - 1 \\
&= \frac{L''_C}{(\lambda^h{}_C + \alpha_{AC}\lambda^h{}_A + \alpha_{BC}\lambda^h{}_B)\tau} - \pi(\tau)\left[st_B - st_A\right] + \pi(\tau)\left[st_B - st_A\right] - 1 \\
&= \frac{L''_C}{(\lambda^h{}_C + \alpha_{AC}\lambda^h{}_A + \alpha_{BC}\lambda^h{}_B)\tau} - 1
\end{aligned}
$$

by using Equations 4.5 and A.2. Apparently, $L_{c'_{(0,0,1)}(\tau)}$ is decreasing in τ.

Similarly, as $c'_{(0,1,1)}(\tau) - 1 < c_{(0,1,1)}(\tau)$ we can find an upper bound on $c'_7(\tau)$ as such:

$$
\begin{aligned}
U_{c'_{(0,0,1)}(\tau)} &= \frac{L''_C}{\lambda_C^{pos''}(\tau)} - c_{(1,1,1)}(\tau) - [c'_{(0,1,1)}(\tau) - 1] + \pi(\tau)st_B - 2 \\
&= \frac{L''_C}{(\lambda^h{}_C + \alpha_{AC}\lambda^h{}_A + \alpha_{BC}\lambda^h{}_B)\tau}.
\end{aligned}
$$

As a result, $U_{c'_{(0,0,1)}(\tau)}$ is decreasing in τ and $U_{c'_{(0,0,1)}(\tau)} = L_{c'_{(0,0,1)}(\tau)} + 1$.

Proof of Proposition 4.6. From Lemma 4.3 we know that $L_{c'_{(0,0,1)}(\tau)} < c'_{(0,0,1)}(\tau) < U_{c'_{(0,0,1)}(\tau)}$. Then we have $\lfloor L_{c'_{(0,0,1)}(\tau)} \rfloor \le \lfloor c'_{(0,0,1)}(\tau) \rfloor \le \lfloor U_{c'_{(0,0,1)}(\tau)} \rfloor \Leftrightarrow \lfloor L_{c'_{(0,0,1)}(\tau)} \rfloor \le c_{(0,0,1)}(\tau) \le \lfloor U_{c'_{(0,0,1)}(\tau)} \rfloor$. So the bounds on $c_{(0,0,1)}(\tau)$ are $L_{c_{(0,0,1)}(\tau)} = \lfloor L_{c'_{(0,0,1)}(\tau)} \rfloor$ and $U_{c_{(0,0,1)}(\tau)} = \lfloor U_{c'_{(0,0,1)}(\tau)} \rfloor$; and they are nonincreasing in τ because of the floor function. In addition, $U_{c_{(0,0,1)}(\tau)} = L_{c_{(0,0,1)}(\tau)} + 1$.

Proof of Lemma 4.4. From Equation 4.10 we obtain:

$$c'_{(0,1,1)}(\tau) + c'_{(0,0,1)}(\tau) = \frac{L''_C}{\lambda_C^{pos''}(\tau)} - c_{(1,1,1)}(\tau) + \pi(\tau) st_B - 2.$$

Since $c_{(1,1,1)}(\tau) < c'_{(1,1,1)}(\tau)$ we can find a lower bound on $c'_{(0,1,1)}(\tau) + c'_{(0,0,1)}(\tau)$ as such:

$$\begin{aligned}
L_{c'_{(0,1,1)}(\tau)+c'_{(0,0,1)}(\tau)} &= \frac{L''_C}{\lambda_C^{pos''}(\tau)} - c'_{(1,1,1)}(\tau) + \pi(\tau) st_B - 2 \\
&= \frac{L''_C}{\lambda_C^{pos''}(\tau)} - \frac{L_A}{\lambda_A^{pos}(\tau)} + \pi(\tau) st_B - 2 \\
&= \frac{L''_C}{\lambda_C^{pos''}(\tau)} - \pi(\tau) st_A + \pi(\tau) st_B - 2 \\
&= \frac{L''_C}{(\lambda^h{}_C + \alpha_{AC}\lambda^h{}_A + \alpha_{BC}\lambda^h{}_B)\tau} + \frac{st_B - st_A}{\tau} - 2.
\end{aligned}$$

Since all of the constants in the first term are positive and $st_B > st_A$, $L_{c'_{(0,1,1)}(\tau)+c'_{(0,0,1)}(\tau)}$ is decreasing in τ.

Similarly, as $c'_{(1,1,1)}(\tau)-1 < c_{(1,1,1)}(\tau)$ we can find an upper bound on $c'_{(0,1,1)}(\tau)+c'_{(0,0,1)}(\tau)$ as such:

$$\begin{aligned}U_{c'_{(0,1,1)}(\tau)+c'_{(0,0,1)}(\tau)} &= \frac{L''_C}{\lambda_C{}^{pos''}(\tau)} - [c'_{(1,1,1)}(\tau)-1] + \pi(\tau)st_B - 2 \\ &= \frac{L''_C}{(\lambda^h{}_C + \alpha_{AC}\lambda^h{}_A + \alpha_{BC}\lambda^h{}_B)\tau} + \frac{st_B - st_A}{\tau} - 1.\end{aligned}$$

Therefore, $U_{c'_{(0,1,1)}(\tau)+c'_{(0,0,1)}(\tau)}$ is decreasing in τ and $U_{c'_{(0,1,1)}(\tau)+c'_{(0,0,1)}(\tau)} = L_{c'_{(0,1,1)}(\tau)+c'_{(0,0,1)}(\tau)} + 1$.

Proof of Proposition 4.7. From Lemma 4.3 we know that $L_{c'_{(0,1,1)}(\tau)+c'_{(0,0,1)}(\tau)} < c'_{(0,1,1)}(\tau)+c'_{(0,0,1)}(\tau) < U_{c'_{(0,1,1)}(\tau)+c'_{(0,0,1)}(\tau)}$. Then we have:

$$\begin{aligned}&\left\lfloor L_{c'_{(0,1,1)}(\tau)+c'_{(0,0,1)}(\tau)} \right\rfloor \le \left\lfloor c'_{(0,1,1)}(\tau)+c'_{(0,0,1)}(\tau) \right\rfloor \le \left\lfloor U_{c'_{(0,1,1)}(\tau)+c'_{(0,0,1)}(\tau)} \right\rfloor \\ \Leftrightarrow &\left\lfloor L_{c'_{(0,1,1)}(\tau)+c'_{(0,0,1)}(\tau)} \right\rfloor \le c_{(0,1,1)}(\tau)+c_{(0,0,1)}(\tau) \le \left\lfloor U_{c'_{(0,1,1)}(\tau)+c'_{(0,0,1)}(\tau)} \right\rfloor.\end{aligned}$$

So the bounds on $c_{(0,1,1)}(\tau)+c_{(0,0,1)}(\tau)$ are $L_{c_{(0,1,1)}(\tau)+c_{(0,0,1)}(\tau)} = \left\lfloor L_{c'_{(0,1,1)}(\tau)+c'_{(0,0,1)}(\tau)} \right\rfloor$ and $U_{c_{(0,1,1)}(\tau)+c_{(0,0,1)}(\tau)} = \left\lfloor U_{c'_{(0,1,1)}(\tau)+c'_{(0,0,1)}(\tau)} \right\rfloor$; and they are nonincreasing in τ because of the floor function. Moreover, $U_{c_{(0,1,1)}(\tau)+c_{(0,0,1)}(\tau)} = L_{c_{(0,1,1)}(\tau)+c_{(0,0,1)}(\tau)} + 1$.

APPENDIX B

Estimation of Substitution Parameters by MLE

First of all, it is assumed that a detailed transaction data for each sale is available. Therefore, data corresponding to the time of a stock-out, the identity of the product that runs out, and the sales of other products up to that time are at hand. Let the N products be indexed by n_k. Let λ_{n_k} be the arrival rate for n_k when all products are in stock and $\lambda_{n_k \tilde{n}_i}$ be the arrival rate for n_k when n_i is not in stock. The arrival rate for product n_k when two products n_i and n_j are not in stock is denoted by $\lambda_{n_k \tilde{n}_i \tilde{n}_j}$.

By using the transaction data the following parameters are identified:

- Total (cumulative) time when all products are in stock (W),
- Total time when all products but n_i are in stock ($W_{\tilde{n}_i}$),
- Total time when all products but n_i and n_j are in stock ($W_{\tilde{n}_i \tilde{n}_j}$),
- Sales of product n_k in time interval $W(Z(n_k))$,
- Sales of product n_k in time interval $W_{\tilde{n}_i}(Z_{\tilde{n}_i}(n_k))$,
- Sales of product n_k in time interval $W_{\tilde{n}_i \tilde{n}_j}(Z_{\tilde{n}_i \tilde{n}_j}(n_k))$.

Then the following optimization problem is solved and demand rates are obtained. The substitution probabilities are calculated by $(\lambda_{n_k \tilde{n}_i} - \lambda_{n_k}) / \lambda_{\tilde{n}_i}$, representing the proportion of brand n_i's demand that spills-over to brand n_k when n_i is unavailable.

$$\mathit{max} \quad \log L = \sum_{k=1}^{N}\left[Z(n_k)\log\lambda_{n_k} - W\lambda_{n_k}\right] + \sum_{k=1}^{N}\sum_{\substack{i=1\\ i\neq k}}^{N}\left[Z_{\tilde{n}_i}(n_k)\log\lambda_{n_k\tilde{n}_i} - W_{\tilde{n}_i}\lambda_{n_k\tilde{n}_i}\right]$$

$$+\sum_{k=1}^{N}\sum_{\substack{i=1\\ i\neq k}}^{N}\sum_{j=i+1}^{N}\left[Z_{\tilde{n}_i\tilde{n}_j}(n_k)\log\lambda_{n_k\tilde{n}_i\tilde{n}_j} - W_{\tilde{n}_i\tilde{n}_j}\lambda_{n_k\tilde{n}_i\tilde{n}_j}\right]$$

subject to

$$\lambda_{n_k\tilde{n}_i\tilde{n}_j} = \lambda_{n_k} + (\lambda_{n_k\tilde{n}_i} - \lambda_{n_k}) + (\lambda_{n_k\tilde{n}_j} - \lambda_{n_k}) \text{ for all } i \neq j \neq k$$

$$\sum_{\substack{k=1\\ k\neq i}}^{N}\left[\lambda_{n_k\tilde{n}_i} - \lambda_{n_k}\right] \leq \lambda_{n_i} \text{ for all } i = 1,\ldots,N$$

$$\lambda_{n_k\tilde{n}_i} \geq \lambda_{n_k} \text{ for all } i \neq k \text{ and } i,k = 1,\ldots N$$

APPENDIX C

Detailed Comparison of MLE with SSBE

In section 5.4 graphs were provided for error values using 100 replications for each parameter set. In this section, we provide detailed figures for the sample means and confidence intervals on means for different error measures using 100 replications for each parameter set unless otherwise stated (we run 250 and 500 replications for some of the parameters sets to obtain more significant results). Also, p-values, resulting from the associated paired-t tests of the hypothesis that two samples come from distributions with equal means, are given.

We generated the problem data through a simulation coded with MATLAB 7.0 and solved the optimization problems by GAMS IDE 2.0.30.1 using CPLEX as the quadratic programming solver.

Simulation Length: 3 months							
Prob. of Subs.	Demand (units/hour)	POS Interval (hour)	MAPE - MLE		MAPE - SSBE		p
0.2	[3.3,3.3,3.3]	0.01	96.870	[90.350 ,103.390]	84.506	[80.149 ,88.862]	0.000
		0.1	96.870	[90.350 ,103.390]	84.296	[79.430 ,89.161]	0.000
		0.25	96.870	[90.350 ,103.390]	86.138	[80.815 ,91.460]	0.000
	[2.5,2.5,5]	0.01	100.986	[92.867 ,109.104]	88.448	[83.313 ,93.582]	0.000
		0.1	100.986	[92.867 ,109.104]	87.763	[82.243 ,93.283]	0.000
		0.25	100.986	[92.867 ,109.104]	91.179	[84.462 ,97.896]	0.000
	[1,3,6]	0.01	123.897	[114.074 ,133.720]	99.367	[91.022 ,107.712]	0.000
		0.1	123.897	[114.074 ,133.720]	101.067	[92.426 ,109.707]	0.000
		0.25	123.897	[114.074 ,133.720]	104.182	[95.798 ,112.566]	0.000
0.4	[3.3,3.3,3.3]	0.01	64.163	[61.692 ,66.633]	59.453	[57.467 ,61.439]	0.000*
		0.1	64.163	[61.692 ,66.633]	60.197	[58.115 ,62.279]	0.000*
		0.25	64.163	[61.692 ,66.633]	61.560	[59.450 ,63.670]	0.018*
	[2.5,2.5,5]	0.01	72.710	[69.815 ,75.605]	66.194	[63.817 ,68.572]	0.000*
		0.1	72.710	[69.815 ,75.605]	66.951	[64.520 ,69.381]	0.000*
		0.25	72.710	[69.815 ,75.605]	68.626	[66.029 ,71.224]	0.000*
	[1,3,6]	0.01	82.071	[79.411 ,84.732]	77.193	[74.199 ,80.186]	0.000*
		0.1	82.071	[79.411 ,84.732]	78.399	[75.298 ,81.500]	0.003*
		0.25	82.071	[79.411 ,84.732]	80.262	[77.005 ,83.519]	0.170*
0.6	[3.3,3.3,3.3]	0.01	46.172	[44.929 ,47.415]	47.983	[46.755 ,49.212]	0.000
		0.1	46.492	[44.674 ,48.311]	49.099	[47.297 ,50.901]	0.000*
		0.25	46.492	[44.674 ,48.311]	50.281	[48.400 ,52.162]	0.000*
	[2.5,2.5,5]	0.01	50.966	[49.001 ,52.930]	53.292	[51.344 ,55.240]	0.002*
		0.1	50.966	[49.001 ,52.930]	54.012	[52.052 ,55.971]	0.000*
		0.25	50.966	[49.001 ,52.930]	54.361	[52.369 ,56.353]	0.000*
	[1,3,6]	0.01	61.360	[58.581 ,64.138]	64.886	[61.697 ,68.076]	0.013*
		0.1	61.360	[58.581 ,64.138]	64.702	[61.535 ,67.869]	0.018*
		0.25	60.126	[55.374 ,64.878]	65.909	[60.435 ,71.384]	0.005

***** Found by 250 replications
** Found by 500 replications

Table C.1: MLE vs. SSBE - 3-Month MAPE with respect to real substitution matrix

Simulation Length: 6 months							
Prob. of Subs.	Demand (units/hour)	POS Interval (hour)	MAPE - MLE		MAPE - SSBE		p
0.2	[3.3,3.3,3.3]	0.01	72.111	[69.255 ,74.968]	69.612	[67.681 ,71.543]	0.042*
		0.1	72.111	[69.255 ,74.968]	69.843	[67.811 ,71.876]	0.067*
		0.25	71.640	[69.633 ,73.647]	70.636	[69.215 ,72.058]	0.266**
	[2.5,2.5,5]	0.01	78.843	[75.567 ,82.120]	70.434	[68.237 ,72.630]	0.000*
		0.1	78.843	[75.567 ,82.120]	71.171	[68.932 ,73.410]	0.000*
		0.25	78.843	[75.567 ,82.120]	71.839	[69.607 ,74.071]	0.000*
	[1,3,6]	0.01	99.821	[92.315 ,107.327]	84.054	[79.023 ,89.085]	0.000
		0.1	99.821	[92.315 ,107.327]	85.923	[80.706 ,91.140]	0.000
		0.25	99.821	[92.315 ,107.327]	86.913	[81.449 ,92.377]	0.000
0.4	[3.3,3.3,3.3]	0.01	44.608	[42.760 ,46.455]	46.357	[44.724 ,47.990]	0.014*
		0.1	44.608	[42.760 ,46.455]	46.987	[45.354 ,48.619]	0.002*
		0.25	43.814	[42.509 ,45.119]	48.729	[47.568 ,49.890]	0.000**
	[2.5,2.5,5]	0.01	48.335	[46.934 ,49.737]	50.150	[49.015 ,51.284]	0.001**
		0.1	48.335	[46.934 ,49.737]	50.910	[49.744 ,52.075]	0.000**
		0.25	48.335	[46.934 ,49.737]	51.722	[50.539 ,52.906]	0.000**
	[1,3,6]	0.01	59.816	[57.953 ,61.680]	59.241	[57.682 ,60.800]	0.415**
		0.1	59.816	[57.953 ,61.680]	59.405	[57.858 ,60.952]	0.562**
		0.25	65.208	[60.304 ,70.112]	61.973	[58.552 ,65.393]	0.092
0.6	[3.3,3.3,3.3]	0.01	31.979	[30.166 ,33.792]	35.236	[33.248 ,37.223]	0.000
		0.1	31.979	[30.166 ,33.792]	35.455	[33.425 ,37.486]	0.000
		0.25	31.979	[30.166 ,33.792]	36.389	[34.287 ,38.492]	0.000
	[2.5,2.5,5]	0.01	36.457	[34.958 ,37.955]	38.544	[37.191 ,39.897]	0.000*
		0.1	37.312	[34.724 ,39.900]	40.591	[38.334 ,42.847]	0.001
		0.25	37.312	[34.724 ,39.900]	41.439	[39.084 ,43.794]	0.000
	[1,3,6]	0.01	44.657	[43.277 ,46.037]	47.922	[46.700 ,49.144]	0.000**
		0.1	44.657	[43.277 ,46.037]	48.288	[47.023 ,49.552]	0.000**
		0.25	44.657	[43.277 ,46.037]	49.061	[47.767 ,50.355]	0.000**

***** Found by 250 replications
** Found by 500 replications

Table C.2: MLE vs. SSBE - 6-Month MAPE with respect to real substitution matrix

Simulation Length: 3 months							
Prob. of Subs.	Demand (units/hour)	POS Interval (hour)	MAD - MLE		MAD - SSBE		p
0.2	[3.3,3.3,3.3]	0.01	0.097	[0.090 ,0.103]	0.085	[0.080 ,0.089]	0.000
		0.1	0.097	[0.090 ,0.103]	0.084	[0.079 ,0.089]	0.000
		0.25	0.097	[0.090 ,0.103]	0.086	[0.081 ,0.091]	0.000
	[2.5,2.5,5]	0.01	0.097	[0.090 ,0.104]	0.086	[0.082 ,0.091]	0.000
		0.1	0.097	[0.090 ,0.104]	0.086	[0.081 ,0.091]	0.000
		0.25	0.097	[0.090 ,0.104]	0.090	[0.084 ,0.096]	0.001
	[1,3,6]	0.01	0.108	[0.100 ,0.117]	0.094	[0.085 ,0.103]	0.000
		0.1	0.108	[0.100 ,0.117]	0.095	[0.086 ,0.105]	0.000
		0.25	0.108	[0.100 ,0.117]	0.097	[0.088 ,0.105]	0.000
0.4	[3.3,3.3,3.3]	0.01	0.128	[0.123 ,0.133]	0.119	[0.115 ,0.123]	0.000*
		0.1	0.128	[0.123 ,0.133]	0.120	[0.116 ,0.125]	0.000*
		0.25	0.128	[0.123 ,0.133]	0.123	[0.119 ,0.127]	0.018*
	[2.5,2.5,5]	0.01	0.139	[0.134 ,0.144]	0.129	[0.125 ,0.134]	0.000*
		0.1	0.139	[0.134 ,0.144]	0.131	[0.126 ,0.135]	0.000*
		0.25	0.139	[0.134 ,0.144]	0.134	[0.129 ,0.138]	0.004*
	[1,3,6]	0.01	0.140	[0.136 ,0.144]	0.139	[0.133 ,0.144]	0.555*
		0.1	0.140	[0.136 ,0.144]	0.141	[0.135 ,0.147]	0.568*
		0.25	0.140	[0.136 ,0.144]	0.145	[0.139 ,0.151]	0.044*
0.6	[3.3,3.3,3.3]	0.01	0.139	[0.135 ,0.142]	0.144	[0.140 ,0.148]	0.000
		0.1	0.139	[0.134 ,0.145]	0.147	[0.142 ,0.153]	0.000*
		0.25	0.139	[0.134 ,0.145]	0.151	[0.145 ,0.156]	0.000*
	[2.5,2.5,5]	0.01	0.144	[0.139 ,0.149]	0.154	[0.148 ,0.160]	0.000*
		0.1	0.144	[0.139 ,0.149]	0.156	[0.150 ,0.162]	0.000*
		0.25	0.144	[0.139 ,0.149]	0.157	[0.151 ,0.163]	0.000*
	[1,3,6]	0.01	0.154	[0.147 ,0.160]	0.170	[0.162 ,0.179]	0.000*
		0.1	0.154	[0.147 ,0.160]	0.170	[0.161 ,0.178]	0.000*
		0.25	0.152	[0.141 ,0.162]	0.175	[0.161 ,0.188]	0.000

***** Found by 250 replications
** Found by 500 replications

Table C.3: MLE vs. SSBE - 3-Month MAD with respect to real substitution matrix

Simulation Length: 6 months							
Prob. of Subs.	Demand (units/hour)	POS Interval (hour)	MAD - MLE		MAD - SSBE		P
0.2	[3.3,3.3,3.3]	0.01	0.072	[0.069 ,0.075]	0.070	[0.068 ,0.072]	0.042*
		0.1	0.072	[0.069 ,0.075]	0.070	[0.068 ,0.072]	0.067*
		0.25	0.072	[0.070 ,0.074]	0.071	[0.069 ,0.072]	0.266**
	[2.5,2.5,5]	0.01	0.077	[0.074 ,0.081]	0.070	[0.068 ,0.072]	0.000*
		0.1	0.077	[0.074 ,0.081]	0.070	[0.068 ,0.073]	0.000*
		0.25	0.077	[0.074 ,0.081]	0.071	[0.069 ,0.073]	0.000*
	[1,3,6]	0.01	0.086	[0.080 ,0.092]	0.078	[0.073 ,0.083]	0.000
		0.1	0.086	[0.080 ,0.092]	0.080	[0.075 ,0.085]	0.007
		0.25	0.086	[0.080 ,0.092]	0.080	[0.075 ,0.086]	0.028
0.4	[3.3,3.3,3.3]	0.01	0.089	[0.086 ,0.093]	0.093	[0.089 ,0.096]	0.014*
		0.1	0.089	[0.086 ,0.093]	0.094	[0.091 ,0.097]	0.002*
		0.25	0.088	[0.085 ,0.090]	0.097	[0.095 ,0.100]	0.000**
	[2.5,2.5,5]	0.01	0.093	[0.090 ,0.096]	0.098	[0.096 ,0.100]	0.000**
		0.1	0.093	[0.090 ,0.096]	0.099	[0.097 ,0.102]	0.000**
		0.25	0.093	[0.090 ,0.096]	0.101	[0.099 ,0.104]	0.000**
	[1,3,6]	0.01	0.103	[0.100 ,0.106]	0.105	[0.102 ,0.108]	0.018**
		0.1	0.103	[0.100 ,0.106]	0.106	[0.103 ,0.109]	0.010**
		0.25	0.109	[0.101 ,0.116]	0.109	[0.103 ,0.115]	0.933
0.6	[3.3,3.3,3.3]	0.01	0.096	[0.090 ,0.101]	0.106	[0.100 ,0.112]	0.000
		0.1	0.096	[0.090 ,0.101]	0.106	[0.100 ,0.112]	0.000
		0.25	0.096	[0.090 ,0.101]	0.109	[0.103 ,0.115]	0.000
	[2.5,2.5,5]	0.01	0.103	[0.099 ,0.107]	0.110	[0.106 ,0.114]	0.000*
		0.1	0.106	[0.099 ,0.113]	0.118	[0.111 ,0.124]	0.000
		0.25	0.106	[0.099 ,0.113]	0.120	[0.113 ,0.127]	0.000
	[1,3,6]	0.01	0.113	[0.110 ,0.116]	0.125	[0.121 ,0.129]	0.000**
		0.1	0.113	[0.110 ,0.116]	0.126	[0.122 ,0.130]	0.000**
		0.25	0.113	[0.110 ,0.116]	0.128	[0.124 ,0.131]	0.000**

***** Found by 250 replications
** Found by 500 replications

Table C.4: MLE vs. SSBE - 6-Month MAD with respect to real substitution matrix

Simulation Length: 3 months							
Prob. of Subs.	Demand (units/hour)	POS Interval (hour)	MAX - MLE		MAX - SSBE		P
0.2	[3.3,3.3,3.3]	0.01	0.208	[0.188 ,0.228]	0.146	[0.132 ,0.160]	0.000
		0.1	0.208	[0.188 ,0.228]	0.147	[0.132 ,0.162]	0.000
		0.25	0.208	[0.188 ,0.228]	0.155	[0.136 ,0.174]	0.000
	[2.5,2.5,5]	0.01	0.236	[0.208 ,0.263]	0.181	[0.160 ,0.201]	0.000
		0.1	0.236	[0.208 ,0.263]	0.183	[0.161 ,0.205]	0.000
		0.25	0.236	[0.208 ,0.263]	0.191	[0.165 ,0.217]	0.000
	[1,3,6]	0.01	0.296	[0.257 ,0.334]	0.245	[0.203 ,0.286]	0.000
		0.1	0.296	[0.257 ,0.334]	0.248	[0.204 ,0.292]	0.001
		0.25	0.296	[0.257 ,0.334]	0.249	[0.207 ,0.290]	0.000
0.4	[3.3,3.3,3.3]	0.01	0.271	[0.257 ,0.285]	0.219	[0.210 ,0.228]	0.000*
		0.1	0.271	[0.257 ,0.285]	0.223	[0.214 ,0.233]	0.000*
		0.25	0.271	[0.257 ,0.285]	0.229	[0.218 ,0.240]	0.000*
	[2.5,2.5,5]	0.01	0.321	[0.305 ,0.336]	0.272	[0.261 ,0.283]	0.000*
		0.1	0.321	[0.305 ,0.336]	0.275	[0.263 ,0.287]	0.000*
		0.25	0.321	[0.305 ,0.336]	0.282	[0.269 ,0.295]	0.000*
	[1,3,6]	0.01	0.370	[0.354 ,0.387]	0.359	[0.337 ,0.381]	0.110*
		0.1	0.370	[0.354 ,0.387]	0.363	[0.340 ,0.386]	0.359*
		0.25	0.370	[0.354 ,0.387]	0.371	[0.347 ,0.396]	0.913*
0.6	[3.3,3.3,3.3]	0.01	0.289	[0.280 ,0.298]	0.289	[0.280 ,0.298]	0.964
		0.1	0.285	[0.273 ,0.297]	0.294	[0.283 ,0.305]	0.062*
		0.25	0.285	[0.273 ,0.297]	0.300	[0.287 ,0.313]	0.010*
	[2.5,2.5,5]	0.01	0.321	[0.307 ,0.335]	0.333	[0.314 ,0.351]	0.111*
		0.1	0.321	[0.307 ,0.335]	0.335	[0.317 ,0.353]	0.046*
		0.25	0.321	[0.307 ,0.335]	0.344	[0.323 ,0.365]	0.010*
	[1,3,6]	0.01	0.379	[0.361 ,0.397]	0.446	[0.409 ,0.482]	0.000*
		0.1	0.379	[0.361 ,0.397]	0.449	[0.412 ,0.487]	0.000*
		0.25	0.371	[0.342 ,0.401]	0.449	[0.391 ,0.506]	0.002

***** Found by 250 replications
** Found by 500 replications

Table C.5: MLE vs. SSBE - 3-Month MAX with respect to real substitution matrix

Simulation Length: 6 months							
Prob. of Subs.	Demand (units/hour)	POS Interval (hour)	MAX - MLE		MAX - SSBE		P
0.2	[3.3,3.3,3.3]	0.01	0.148	[0.141 ,0.156]	0.114	[0.109 ,0.118]	0.000*
		0.1	0.148	[0.141 ,0.156]	0.114	[0.110 ,0.118]	0.000*
		0.25	0.144	[0.139 ,0.150]	0.114	[0.111 ,0.117]	0.000**
	[2.5,2.5,5]	0.01	0.180	[0.168 ,0.192]	0.143	[0.135 ,0.151]	0.000*
		0.1	0.180	[0.168 ,0.192]	0.144	[0.136 ,0.151]	0.000*
		0.25	0.180	[0.168 ,0.192]	0.143	[0.136 ,0.150]	0.000*
	[1,3,6]	0.01	0.235	[0.209 ,0.260]	0.188	[0.169 ,0.206]	0.000
		0.1	0.235	[0.209 ,0.260]	0.195	[0.174 ,0.216]	0.000
		0.25	0.235	[0.209 ,0.260]	0.195	[0.174 ,0.216]	0.000
0.4	[3.3,3.3,3.3]	0.01	0.188	[0.179 ,0.197]	0.181	[0.174 ,0.187]	0.012*
		0.1	0.188	[0.179 ,0.197]	0.183	[0.176 ,0.189]	0.073*
		0.25	0.181	[0.176 ,0.187]	0.181	[0.177 ,0.185]	0.925**
	[2.5,2.5,5]	0.01	0.207	[0.199 ,0.215]	0.207	[0.201 ,0.212]	0.810**
		0.1	0.207	[0.199 ,0.215]	0.208	[0.202 ,0.214]	0.849**
		0.25	0.207	[0.199 ,0.215]	0.212	[0.206 ,0.219]	0.142**
	[1,3,6]	0.01	0.272	[0.261 ,0.284]	0.258	[0.247 ,0.268]	0.000**
		0.1	0.272	[0.261 ,0.284]	0.259	[0.248 ,0.269]	0.000**
		0.25	0.289	[0.260 ,0.318]	0.260	[0.239 ,0.281]	0.007
0.6	[3.3,3.3,3.3]	0.01	0.200	[0.186 ,0.214]	0.217	[0.205 ,0.228]	0.003
		0.1	0.200	[0.186 ,0.214]	0.220	[0.209 ,0.232]	0.001
		0.25	0.200	[0.186 ,0.214]	0.229	[0.217 ,0.242]	0.000
	[2.5,2.5,5]	0.01	0.231	[0.220 ,0.242]	0.242	[0.231 ,0.253]	0.004*
		0.1	0.236	[0.218 ,0.254]	0.252	[0.234 ,0.269]	0.051
		0.25	0.236	[0.218 ,0.254]	0.258	[0.240 ,0.276]	0.008
	[1,3,6]	0.01	0.295	[0.283 ,0.306]	0.310	[0.298 ,0.322]	0.000**
		0.1	0.295	[0.283 ,0.306]	0.314	[0.301 ,0.326]	0.000**
		0.25	0.295	[0.283 ,0.306]	0.315	[0.302 ,0.327]	0.000**

***** Found by 250 replications
** Found by 500 replications

Table C.6: MLE vs. SSBE - 6-Month MAX with respect to real substitution matrix

Simulation Length: 3 months							
Prob. of Subs.	Demand (units/hour)	POS Interval (hour)	MAPE - MLE		MAPE - SSBE		p
0.2	[3.3,3.3,3.3]	0.01	96.966	[90.603 ,103.329]	84.930	[80.566 ,89.294]	0.000
		0.1	96.966	[90.603 ,103.329]	84.676	[79.856 ,89.497]	0.000
		0.25	96.966	[90.603 ,103.329]	86.223	[80.771 ,91.676]	0.000
	[2.5,2.5,5]	0.01	101.638	[93.478 ,109.798]	89.011	[83.853 ,94.168]	0.000
		0.1	101.638	[93.478 ,109.798]	88.388	[82.802 ,93.974]	0.000
		0.25	101.638	[93.478 ,109.798]	91.913	[85.152 ,98.675]	0.000
	[1,3,6]	0.01	126.735	[115.629 ,137.841]	101.230	[91.890 ,110.570]	0.000
		0.1	126.735	[115.629 ,137.841]	102.544	[93.002 ,112.086]	0.000
		0.25	126.735	[115.629 ,137.841]	105.813	[96.421 ,115.205]	0.000
0.4	[3.3,3.3,3.3]	0.01	64.261	[61.784 ,66.738]	59.452	[57.436 ,61.468]	0.000*
		0.1	64.261	[61.784 ,66.738]	60.249	[58.130 ,62.368]	0.000*
		0.25	64.261	[61.784 ,66.738]	61.524	[59.343 ,63.706]	0.013*
	[2.5,2.5,5]	0.01	72.729	[69.692 ,75.766]	66.012	[63.532 ,68.492]	0.000*
		0.1	72.729	[69.692 ,75.766]	66.782	[64.225 ,69.339]	0.000*
		0.25	72.729	[69.692 ,75.766]	68.727	[66.023 ,71.431]	0.000*
	[1,3,6]	0.01	82.962	[80.222 ,85.701]	77.781	[74.747 ,80.815]	0.000*
		0.1	82.962	[80.222 ,85.701]	78.894	[75.777 ,82.011]	0.001*
		0.25	82.962	[80.222 ,85.701]	80.749	[77.467 ,84.031]	0.092*
0.6	[3.3,3.3,3.3]	0.01	46.096	[44.854 ,47.338]	47.859	[46.639 ,49.079]	0.000
		0.1	46.153	[44.323 ,47.983]	48.803	[47.015 ,50.591]	0.000*
		0.25	46.153	[44.323 ,47.983]	50.016	[48.115 ,51.916]	0.000*
	[2.5,2.5,5]	0.01	51.055	[49.041 ,53.070]	53.268	[51.284 ,55.252]	0.004*
		0.1	51.055	[49.041 ,53.070]	53.804	[51.813 ,55.796]	0.001*
		0.25	51.055	[49.041 ,53.070]	54.267	[52.235 ,56.299]	0.000*
	[1,3,6]	0.01	61.208	[58.432 ,63.983]	64.694	[61.303 ,68.085]	0.019*
		0.1	61.208	[58.432 ,63.983]	64.624	[61.225 ,68.023]	0.021*
		0.25	60.519	[55.849 ,65.190]	65.998	[60.497 ,71.499]	0.010

* Found by 250 replications
** Found by 500 replications

Table C.7: MLE vs. SSBE - 3-Month MAPE with respect to realized substitution matrix

Simulation Length: 6 months							
Prob. of Subs.	Demand (units/hour)	POS Interval (hour)	MAPE - MLE		MAPE - SSBE		p
0.2	[3.3,3.3,3.3]	0.01	72.514	[69.574 ,75.454]	69.911	[67.962 ,71.860]	0.037*
		0.1	72.514	[69.574 ,75.454]	70.108	[68.036 ,72.179]	0.056*
		0.25	71.688	[69.676 ,73.700]	70.685	[69.288 ,72.081]	0.274**
	[2.5,2.5,5]	0.01	79.051	[75.776 ,82.327]	70.426	[68.216 ,72.635]	0.000*
		0.1	79.051	[75.776 ,82.327]	71.174	[68.911 ,73.436]	0.000*
		0.25	79.051	[75.776 ,82.327]	72.054	[69.853 ,74.254]	0.000*
	[1,3,6]	0.01	99.797	[92.008 ,107.585]	84.449	[78.889 ,90.009]	0.000
		0.1	99.797	[92.008 ,107.585]	86.301	[80.642 ,91.959]	0.000
		0.25	99.797	[92.008 ,107.585]	87.203	[81.493 ,92.913]	0.000
0.4	[3.3,3.3,3.3]	0.01	44.572	[42.735 ,46.410]	46.225	[44.599 ,47.852]	0.022*
		0.1	44.572	[42.735 ,46.410]	46.833	[45.204 ,48.463]	0.003*
		0.25	43.746	[42.435 ,45.057]	48.530	[47.373 ,49.687]	0.000**
	[2.5,2.5,5]	0.01	48.031	[46.630 ,49.432]	50.044	[48.890 ,51.198]	0.000**
		0.1	48.031	[46.630 ,49.432]	50.824	[49.644 ,52.004]	0.000**
		0.25	48.031	[46.630 ,49.432]	51.612	[50.416 ,52.808]	0.000**
	[1,3,6]	0.01	59.827	[57.911 ,61.744]	59.289	[57.704 ,60.874]	0.452**
		0.1	59.827	[57.911 ,61.744]	59.447	[57.887 ,61.006]	0.599**
		0.25	65.840	[60.990 ,70.689]	62.187	[58.789 ,65.586]	0.064
0.6	[3.3,3.3,3.3]	0.01	31.874	[30.102 ,33.645]	35.098	[33.093 ,37.102]	0.000
		0.1	31.874	[30.102 ,33.645]	35.400	[33.353 ,37.448]	0.000
		0.25	31.874	[30.102 ,33.645]	36.318	[34.199 ,38.438]	0.000
	[2.5,2.5,5]	0.01	36.490	[34.991 ,37.989]	38.528	[37.185 ,39.871]	0.000*
		0.1	37.362	[34.728 ,39.996]	40.612	[38.363 ,42.862]	0.001
		0.25	37.362	[34.728 ,39.996]	41.293	[38.977 ,43.608]	0.000
	[1,3,6]	0.01	44.753	[43.367 ,46.139]	47.939	[46.743 ,49.135]	0.000**
		0.1	44.753	[43.367 ,46.139]	48.317	[47.069 ,49.564]	0.000**
		0.25	44.753	[43.367 ,46.139]	49.047	[47.763 ,50.332]	0.000**

***** Found by 250 replications
** Found by 500 replications

Table C.8: MLE vs. SSBE - 6-Month MAPE with respect to realized substitution matrix

Simulation Length: 3 months							
Prob. of Subs.	Demand (units/hour)	POS Interval (hour)	MAD - MLE		MAD - SSBE		P
0.2	[3.3,3.3,3.3]	0.01	0.097	[0.090 ,0.103]	0.085	[0.080 ,0.089]	0.000
		0.1	0.097	[0.090 ,0.103]	0.084	[0.080 ,0.089]	0.000
		0.25	0.097	[0.090 ,0.103]	0.086	[0.081 ,0.091]	0.000
	[2.5,2.5,5]	0.01	0.097	[0.089 ,0.104]	0.086	[0.081 ,0.090]	0.000
		0.1	0.097	[0.089 ,0.104]	0.085	[0.080 ,0.090]	0.000
		0.25	0.097	[0.089 ,0.104]	0.089	[0.083 ,0.095]	0.001
	[1,3,6]	0.01	0.108	[0.099 ,0.116]	0.094	[0.085 ,0.102]	0.000
		0.1	0.108	[0.099 ,0.116]	0.095	[0.085 ,0.104]	0.000
		0.25	0.108	[0.099 ,0.116]	0.096	[0.088 ,0.105]	0.000
0.4	[3.3,3.3,3.3]	0.01	0.128	[0.123 ,0.133]	0.118	[0.114 ,0.122]	0.000*
		0.1	0.128	[0.123 ,0.133]	0.120	[0.116 ,0.124]	0.000*
		0.25	0.128	[0.123 ,0.133]	0.122	[0.118 ,0.127]	0.011*
	[2.5,2.5,5]	0.01	0.138	[0.133 ,0.143]	0.128	[0.124 ,0.132]	0.000*
		0.1	0.138	[0.133 ,0.143]	0.129	[0.125 ,0.134]	0.000*
		0.25	0.138	[0.133 ,0.143]	0.133	[0.128 ,0.138]	0.004*
	[1,3,6]	0.01	0.140	[0.136 ,0.144]	0.139	[0.134 ,0.144]	0.616*
		0.1	0.140	[0.136 ,0.144]	0.141	[0.136 ,0.147]	0.562*
		0.25	0.140	[0.136 ,0.144]	0.145	[0.139 ,0.151]	0.042*
0.6	[3.3,3.3,3.3]	0.01	0.138	[0.134 ,0.141]	0.143	[0.139 ,0.147]	0.000
		0.1	0.138	[0.133 ,0.143]	0.146	[0.141 ,0.151]	0.000*
		0.25	0.138	[0.133 ,0.143]	0.150	[0.144 ,0.155]	0.000*
	[2.5,2.5,5]	0.01	0.143	[0.138 ,0.148]	0.153	[0.147 ,0.158]	0.000*
		0.1	0.143	[0.138 ,0.148]	0.154	[0.149 ,0.160]	0.000*
		0.25	0.143	[0.138 ,0.148]	0.156	[0.150 ,0.162]	0.000*
	[1,3,6]	0.01	0.152	[0.146 ,0.158]	0.169	[0.160 ,0.177]	0.000*
		0.1	0.152	[0.146 ,0.158]	0.168	[0.160 ,0.177]	0.000*
		0.25	0.151	[0.141 ,0.161]	0.175	[0.161 ,0.188]	0.000

***** Found by 250 replications
** Found by 500 replications

Table C.9: MLE vs. SSBE - 3-Month MAD with respect to realized substitution matrix

Simulation Length: 6 months							
Prob. of Subs.	Demand (units/hour)	POS Interval (hour)	MAD - MLE		MAD - SSBE		P
0.2	[3.3,3.3,3.3]	0.01	0.072	[0.069 ,0.075]	0.069	[0.068 ,0.071]	0.046*
		0.1	0.072	[0.069 ,0.075]	0.070	[0.068 ,0.072]	0.068*
		0.25	0.071	[0.069 ,0.073]	0.070	[0.069 ,0.072]	0.334**
	[2.5,2.5,5]	0.01	0.077	[0.074 ,0.080]	0.069	[0.067 ,0.072]	0.000*
		0.1	0.077	[0.074 ,0.080]	0.070	[0.068 ,0.072]	0.000*
		0.25	0.077	[0.074 ,0.080]	0.071	[0.069 ,0.073]	0.000*
	[1,3,6]	0.01	0.085	[0.079 ,0.091]	0.077	[0.073 ,0.082]	0.001
		0.1	0.085	[0.079 ,0.091]	0.079	[0.074 ,0.084]	0.012
		0.25	0.085	[0.079 ,0.091]	0.080	[0.075 ,0.085]	0.039
0.4	[3.3,3.3,3.3]	0.01	0.089	[0.085 ,0.093]	0.092	[0.089 ,0.096]	0.023*
		0.1	0.089	[0.085 ,0.093]	0.093	[0.090 ,0.097]	0.004*
		0.25	0.087	[0.085 ,0.090]	0.097	[0.095 ,0.099]	0.000**
	[2.5,2.5,5]	0.01	0.092	[0.089 ,0.095]	0.097	[0.095 ,0.099]	0.000**
		0.1	0.092	[0.089 ,0.095]	0.099	[0.096 ,0.101]	0.000**
		0.25	0.092	[0.089 ,0.095]	0.101	[0.098 ,0.103]	0.000**
	[1,3,6]	0.01	0.102	[0.099 ,0.105]	0.105	[0.102 ,0.108]	0.012**
		0.1	0.102	[0.099 ,0.105]	0.105	[0.102 ,0.108]	0.006**
		0.25	0.109	[0.101 ,0.117]	0.110	[0.103 ,0.116]	0.866
0.6	[3.3,3.3,3.3]	0.01	0.095	[0.090 ,0.101]	0.105	[0.099 ,0.111]	0.000
		0.1	0.095	[0.090 ,0.101]	0.106	[0.100 ,0.112]	0.000
		0.25	0.095	[0.090 ,0.101]	0.109	[0.102 ,0.115]	0.000
	[2.5,2.5,5]	0.01	0.103	[0.098 ,0.107]	0.110	[0.106 ,0.114]	0.000*
		0.1	0.106	[0.099 ,0.113]	0.117	[0.111 ,0.124]	0.000
		0.25	0.106	[0.099 ,0.113]	0.119	[0.112 ,0.126]	0.000
	[1,3,6]	0.01	0.113	[0.109 ,0.116]	0.125	[0.121 ,0.128]	0.000**
		0.1	0.113	[0.109 ,0.116]	0.126	[0.122 ,0.130]	0.000**
		0.25	0.113	[0.109 ,0.116]	0.127	[0.124 ,0.131]	0.000**

***** Found by 250 replications
** Found by 500 replications

Table C.10: MLE vs. SSBE - 6-Month MAD with respect to realized substitution matrix

Simulation Length: 3 months							
Prob. of Subs.	Demand (units/hour)	POS Interval (hour)	MAX - MLE		MAX - SSBE		P
0.2	[3.3,3.3,3.3]	0.01	0.211	[0.191 ,0.231]	0.150	[0.136 ,0.164]	0.000
		0.1	0.211	[0.191 ,0.231]	0.151	[0.137 ,0.166]	0.000
		0.25	0.211	[0.191 ,0.231]	0.159	[0.141 ,0.178]	0.000
	[2.5,2.5,5]	0.01	0.233	[0.206 ,0.261]	0.180	[0.160 ,0.200]	0.000
		0.1	0.233	[0.206 ,0.261]	0.183	[0.161 ,0.205]	0.000
		0.25	0.233	[0.206 ,0.261]	0.191	[0.165 ,0.217]	0.000
	[1,3,6]	0.01	0.295	[0.257 ,0.333]	0.244	[0.204 ,0.285]	0.000
		0.1	0.295	[0.257 ,0.333]	0.246	[0.202 ,0.290]	0.001
		0.25	0.295	[0.257 ,0.333]	0.247	[0.206 ,0.288]	0.000
0.4	[3.3,3.3,3.3]	0.01	0.270	[0.256 ,0.284]	0.222	[0.212 ,0.231]	0.000*
		0.1	0.270	[0.256 ,0.284]	0.226	[0.216 ,0.235]	0.000*
		0.25	0.270	[0.256 ,0.284]	0.231	[0.219 ,0.242]	0.000*
	[2.5,2.5,5]	0.01	0.319	[0.304 ,0.334]	0.271	[0.260 ,0.282]	0.000*
		0.1	0.319	[0.304 ,0.334]	0.274	[0.262 ,0.286]	0.000*
		0.25	0.319	[0.304 ,0.334]	0.281	[0.268 ,0.294]	0.000*
	[1,3,6]	0.01	0.371	[0.355 ,0.388]	0.360	[0.338 ,0.382]	0.121*
		0.1	0.371	[0.355 ,0.388]	0.365	[0.342 ,0.387]	0.408*
		0.25	0.371	[0.355 ,0.388]	0.373	[0.349 ,0.397]	0.843*
0.6	[3.3,3.3,3.3]	0.01	0.287	[0.278 ,0.296]	0.289	[0.279 ,0.298]	0.681
		0.1	0.283	[0.271 ,0.295]	0.292	[0.281 ,0.303]	0.051*
		0.25	0.283	[0.271 ,0.295]	0.299	[0.286 ,0.313]	0.004*
	[2.5,2.5,5]	0.01	0.319	[0.305 ,0.333]	0.333	[0.314 ,0.351]	0.063*
		0.1	0.319	[0.305 ,0.333]	0.334	[0.316 ,0.353]	0.032*
		0.25	0.319	[0.305 ,0.333]	0.344	[0.323 ,0.365]	0.005*
	[1,3,6]	0.01	0.377	[0.360 ,0.395]	0.443	[0.406 ,0.479]	0.000*
		0.1	0.377	[0.360 ,0.395]	0.446	[0.408 ,0.483]	0.000*
		0.25	0.370	[0.341 ,0.398]	0.446	[0.389 ,0.503]	0.002

***** Found by 250 replications
** Found by 500 replications

Table C.11: MLE vs. SSBE - 3-Month MAX with respect to realized substitution matrix

Simulation Length: 6 months							
Prob. of Subs.	Demand (units/hour)	POS Interval (hour)	MAX - MLE		MAX - SSBE		P
0.2	[3.3,3.3,3.3]	0.01	0.149	[0.141 ,0.157]	0.117	[0.113 ,0.121]	0.000*
		0.1	0.149	[0.141 ,0.157]	0.117	[0.113 ,0.121]	0.000*
		0.25	0.144	[0.139 ,0.150]	0.116	[0.113 ,0.120]	0.000**
	[2.5,2.5,5]	0.01	0.180	[0.168 ,0.192]	0.144	[0.136 ,0.152]	0.000*
		0.1	0.180	[0.168 ,0.192]	0.145	[0.137 ,0.152]	0.000*
		0.25	0.180	[0.168 ,0.192]	0.144	[0.137 ,0.151]	0.000*
	[1,3,6]	0.01	0.232	[0.206 ,0.258]	0.187	[0.167 ,0.206]	0.000
		0.1	0.232	[0.206 ,0.258]	0.194	[0.172 ,0.215]	0.000
		0.25	0.232	[0.206 ,0.258]	0.194	[0.172 ,0.215]	0.000
0.4	[3.3,3.3,3.3]	0.01	0.188	[0.179 ,0.197]	0.180	[0.173 ,0.187]	0.014*
		0.1	0.188	[0.179 ,0.197]	0.182	[0.175 ,0.189]	0.072*
		0.25	0.179	[0.174 ,0.185]	0.181	[0.178 ,0.185]	0.391**
	[2.5,2.5,5]	0.01	0.206	[0.198 ,0.214]	0.206	[0.200 ,0.212]	0.879**
		0.1	0.206	[0.198 ,0.214]	0.207	[0.201 ,0.213]	0.771**
		0.25	0.206	[0.198 ,0.214]	0.211	[0.205 ,0.218]	0.140**
	[1,3,6]	0.01	0.272	[0.261 ,0.284]	0.258	[0.247 ,0.268]	0.000**
		0.1	0.272	[0.261 ,0.284]	0.258	[0.248 ,0.269]	0.000**
		0.25	0.292	[0.263 ,0.320]	0.262	[0.241 ,0.283]	0.005
0.6	[3.3,3.3,3.3]	0.01	0.199	[0.185 ,0.212]	0.217	[0.205 ,0.229]	0.002
		0.1	0.199	[0.185 ,0.212]	0.220	[0.209 ,0.232]	0.000
		0.25	0.199	[0.185 ,0.212]	0.230	[0.217 ,0.242]	0.000
	[2.5,2.5,5]	0.01	0.229	[0.218 ,0.240]	0.242	[0.231 ,0.253]	0.002*
		0.1	0.236	[0.218 ,0.254]	0.253	[0.235 ,0.270]	0.038
		0.25	0.236	[0.218 ,0.254]	0.259	[0.241 ,0.277]	0.006
	[1,3,6]	0.01	0.295	[0.283 ,0.307]	0.310	[0.298 ,0.322]	0.001**
		0.1	0.295	[0.283 ,0.307]	0.313	[0.301 ,0.326]	0.000**
		0.25	0.295	[0.283 ,0.307]	0.314	[0.301 ,0.327]	0.000**

***** Found by 250 replications
** Found by 500 replications

Table C.12: MLE vs. SSBE - 6-Month MAX with respect to realized substitution matrix

APPENDIX D

An Adaptation to the Fixed Review Period Order-Up-To Level Inventory Control System for the Products Replenished Twice a Week

Recall that the retailer orders product A and product B twice per week (on Wednesday and Friday mornings); and the replenishment lead time is 1 day. Moreover, daily seasonality factors (P_d) can be calculated by using hourly seasonality factors. Figure C.1 depicts this setting.

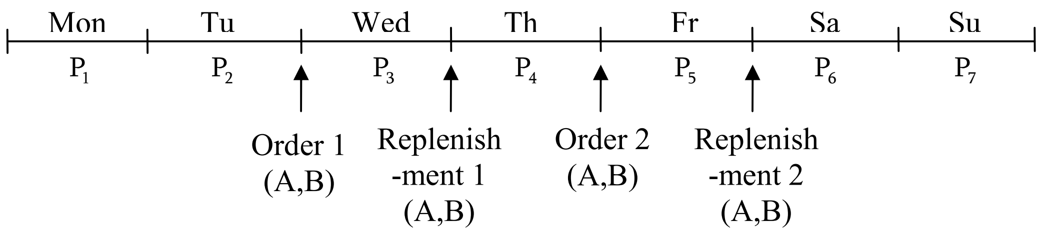


Figure D.1: Replenishments for products A and B

In that case we have to determine two order-up-to levels for each of the orders for products A and B. So we follow the steps of the fixed review period order-up-to level inventory control system for each order of a product with appropriate demand and review period lengths:

Step 1: Select the safety factor z_β which satisfies $G_u(z_\beta) = \dfrac{(1-\beta)\mu R}{\beta\sigma\sqrt{(R+L)}}$

where μ is the mean and σ is the standard deviation of demand.

Step 2: Reorder level $S = \mu(R+L) + z_\beta \sigma \sqrt{R+L}$ (if not integer, increase to the next higher integer).

In these two steps we need values for R, μ, and σ ($L=1$). For the first order-up-to level of product $i \in A, B$ (which is used for the orders on Wednesday mornings) we take $R=2$ days (Wednesday and Thursday), $\mu_i = \lambda^w{}_i \dfrac{(\mathrm{P}_3 + \mathrm{P}_4)}{2}$ units/day where $\lambda^w{}_i$ is the weekly demand rate of product i, and $\sigma_i = \sqrt{\lambda^w{}_i \dfrac{(\mathrm{P}_3 + \mathrm{P}_4 + \mathrm{P}_5)}{3}}$ units/day.

Similarly, for the second order-up-to level of product i (which is used for the orders on Friday mornings) we take $R=5$ days (Friday-Tuesday), $\mu_i = \lambda^w{}_i \dfrac{(\mathrm{P}_1 + \mathrm{P}_2 + \mathrm{P}_5 + \mathrm{P}_6 + \mathrm{P}_7)}{5}$, and $\sigma_i = \sqrt{\lambda^w{}_i \dfrac{(\mathrm{P}_1 + \mathrm{P}_2 + \mathrm{P}_3 + \mathrm{P}_5 + \mathrm{P}_6 + \mathrm{P}_7)}{6}}$.

As for the order-up-to levels for product $i \in C, D, E$, we take $R=7$ days, $\mu_i = \lambda^w{}_i \dfrac{(\mathrm{P}_1 + \mathrm{P}_2 + 2\mathrm{P}_3 + \mathrm{P}_4 + \mathrm{P}_5 + \mathrm{P}_6 + \mathrm{P}_7)}{8}$, and $\sigma_i = \sqrt{\lambda^w{}_i \dfrac{(\mathrm{P}_1 + \mathrm{P}_2 + 2\mathrm{P}_3 + \mathrm{P}_4 + \mathrm{P}_5 + \mathrm{P}_6 + \mathrm{P}_7)}{8}}$.

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VITA

Ömer Cem Öztürk was born in Ankara, on July 3, 1981. He graduated from Ankara Atatürk Anatolian High School in 1999. He received his B.S. degree in Industrial Engineering from Galatasaray University, Istanbul, in 2004. In September 2004, he joined the Industrial Engineering Department of Koç University, Istanbul, as a teaching and research assistant.