WEAK SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS AND ONE-SIDED STICKY BROWNIAN MOTION

by

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Koç University
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ABSTRACT

One-sided sticky Brownian motion is a slowly reflecting Brownian motion because it spends positive amount of time at zero before reflection. The primary purpose of this thesis is to analyze the stochastic differential equation or the equivalent system which has one-sided sticky Brownian motion as a weak solution. The work of H. J. Engelbert and G. Peskir entitled "Stochastic Differential Equations for Sticky Brownian Motion" published in *Stochastics* in 2012 is taken into consideration. They study the stochastic differential equation for one-sided sticky Brownian motion by first analyzing the stochastic differential system for two-sided sticky Brownian motion and then using these results for the one-sided system. In this thesis, one-sided version is analyzed without the use of the results for the two-sided version, but with the help of the relation between one-sided and two-sided sticky Brownian motion. The proofs for two-sided sticky Brownian motion given in [Engelbert and Peskir, 2012] paper are followed. We prove the stochastic differential equation, or the equivalent system, admits a weak solution but does not have a strong solution.

ÖZETÇE

Tek taraflı yapışkan Brown hareketi yansımadan önce sıfırda pozitif miktar zaman geçirdiği için yavaş yansıyan Brown hareketidir. Bu tezin birincil amacı tek taraflı yapışkan Brown hareketinin zayıf çözümü olduğu stokastik diferansiyel denklemi ve denkleme eşdeğer olan sistemi anlamaktır. Özellikle, H. J. Engelbert ve G. Peskir'in "Stochastic Differential Equations for Sticky Brownian Motion" isimli 2012'de *Stochastics*'de yayınlanmış makaleleri dikkate alınmıştır. H. J. Engelbert ve G. Peskir tek taraflı yapışkan Brown hareketi için olan stokastik diferansiyel denklemi, çift taraflı yapışkan Brown hareketinin stokastik diferansiyel denklemini analiz ettikten sonra çalışmışlardır ve çift taraflı yapışkan Brown hareketinin stokastik diferansiyel denklemi için buldukları sonuçları kullanmışlardır. Bu tezde tek taraflı yapışkan Brown hareketi çift taraflı yapışkan Brown hareketi için bulunan sonuçlar kullanılmadan analiz edildi. Tek taraflı yapışkan Brown hareketi ile çift taraflı yapışkan Brown hareketi arasındaki bağlantı kullanıldı. [Engelbert and Peskir, 2012]'de çift taraflı yapışkan Brown hareketi için yapılan ispatlar takip edildi. Stokastik diferansiyel denklem ve ona eşdeğer olan sistemin zayıf çözümü olduğu ama güçlü çözümünün olmadığı ispat edildi.

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Chapter 1

INTRODUCTION

Solvability of stochastic differential equations is essential in both theory and application. There are some stochastic differential equations which have only weak solutions. Our main interest in this thesis is the stochastic differential equation (SDE):

$$dX_t = a1_{\{X_t=0\}}dt + 1_{\{X_t>0\}}dB_t, \tag{1.0.1}$$

where $X_0 = x \in \mathbb{R}^+,\ a \in (0, \infty)$ is a given constant and B is a standard Brownian motion.

Sticky reflecting (one-sided sticky) Brownian motion $\{X_t; t \geq 0\}$ is the weak solution of this stochastic differential equation defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is similar to reflected Brownian motion away from 0 but satisfies [Howitt, 2007, p.14]

$$\mathbb{P}\Big(\int_0^t 1_{\{X_s=0\}}ds > 0\Big) > 0.$$

Since it spends positive amount of time at zero, X is called sticky. Chitashvili [Chitashvili, 1997] proved that this stochastic differential equation has no strong solution.

Engelbert and Peskir [Engelbert and Peskir, 2012] are interested in Equation (1.0.1) and also the stochastic differential system for two-sided sticky Brownian motion which is:

$$d\tilde{X}_t = 1_{\{\tilde{X}_t \neq 0\}}d\tilde{B}_t \tag{1.0.2}$$

$$1_{\{\tilde{X}_t=0\}}dt = \frac{1}{a}d\ell_t^0(\tilde{X}), \tag{1.0.3}$$

where $\tilde{X}_0 = \tilde{x} \in \mathbb{R},\ a \in (0, \infty)$ is a given constant, $\tilde{B}$ is a standard Brownian motion and $\ell^0(\tilde{X})$ is the local time of $\tilde{X}$ at 0. Their no strong solution proof is different from Chitashvili's because they used proofs of two-sided sticky Brownian motion for one-sided sticky Brownian motion.

Bass [Bass, 2014] also studied stochastic differential equations with sticky point from a more general perspective. Both Engelbert-Peskir and Bass are interested in Equation

(1.0.2), but instead of Equation (1.0.3) Bass used a side condition

$$m(dx) = dx + \gamma\delta_0(dx), \tag{1.0.4}$$

where δ_0 is point of mass at 0, m is a speed measure. Stochastic differential equation examined in his paper (1.0.2)+ (1.0.4) is equivalent to the system (1.0.2)+ (1.0.3).

There are some other researchers who studied sticky Brownian motion from other aspects. Harrison and Lemoine [Harrison and Lemonie, 1981] examined as the limit of storage processes. Amir [Amir, 1991] examined as the strong limit of a sequence of random walks. Hajri, Caglar and Arnaudon [Hajri et al., 2017] examined application of stochastic flows to the sticky Brownian equation.

In this thesis, we first give some preliminary definitions and theorems used in our analysis of solutions of Equation (1.0.1). Then, we show that the SDE (1.0.1) has a weak solution and study some of its basic properties. Finally, we prove that the SDE has no strong solution. For this, we follow the same lines of proof for two-sided version given in Engelbert and Peskir [Engelbert and Peskir, 2012].

Chapter 2

PRELIMINARIES

In this chapter, some essential definitions and theorems which are used in the next chapter are introduced. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}_t\}$ be a filtration on it.

2.1 Martingales

We can define a martingale as in [Karatzas and Shreve, 1998, p.11].

Definition 2.1.1. *A real-valued stochastic process $\{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$, where X_t is integrable for every $0 \leq t < \infty$ and adapted to $\mathcal{F}_t$, is said to be sub-martingale if $\mathbb{E}(X_t|\mathcal{F}_s) \geq X_s$ super-martingale if $\mathbb{E}(X_t|\mathcal{F}_s) \leq X_s$ martingale if $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$ whenever $s < t$.*

Definition 2.1.2. *Let X be a square integrable martingale ($\mathbb{E}X_t^2 < \infty$), quadratic variation of X is a process $\langle M, M\rangle$ such that unique, adapted, natural increasing process for which $\langle M, M\rangle_0 = 0$ almost surely and $M^2 - \langle M, M\rangle$ is a martingale.*

Definition 2.1.3. *A real-valued stochastic process $\{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is said to be local martingale if there exists a nondecreasing sequence of stopping times $\{\mathcal{T}_n; 0 \leq n\}$ of $\mathcal{F}_t$ with $\lim\limits_{n\to\infty} \mathcal{T}_n = \infty$ a.s. such that $\{X_{t\wedge\mathcal{T}_n}, \mathcal{F}_t; 0 \leq t < \infty\}$ is a martingale.*

Every martingale is a local martingale but converse is not true.

Definition 2.1.4. *A real-valued stochastic process $\{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is said to be continuous semi-martingale if it can be written as $X = M + A$ where $M = \{M_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a continuous local martingale and $A = \{A_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a continuous adapted process of finite variation.*

Although definition of the Brownian motion will be stated in the next section, we can characterize standard Brownian motion by martingales [Çınlar, 2010, p.231].

Proposition 2.1.5. *Let X be a continuous martingale adapted to $\mathcal{F}$ with $X_0 = 0$. If $\{X_t^2 - t, \mathcal{F}_t; 0 \leq t < \infty\}$ is also a martingale adapted to $\mathcal{F}_t$, then X is a standard Brownian motion with respect to $\mathcal{F}$.*

2.2 Brownian Motion

Brownian motion can be considered as a model of a particle moving randomly in d-dimensional space such as in a fluid. In this thesis, linear (d=1) Brownian motion is taken into the consideration.

2.2.1 Brownian Motion on $\mathbb{R}$

We can define a Brownian motion as in Mörters and Peres [Mörters and Peres, 2010, p.7].

Definition 2.2.1. *A real-valued stochastic process* $\{B(t); t \geq 0\}$ *is called Brownian Motion starting* $\mathbf{x} \in \mathbb{R}$ *if*

- $B(0) = x$,
- $B(t_n) - B(t_{n-1}), B(t_{n-1}) - B(t_{n-2}), ..., B(t_2) - B(t_1)$ *are independent random variables, for* $0 \leq t_1 \leq t_2 \leq ... \leq t_n$,
- $B(t+h) - B(t)$ *are normally distributed with expectation* 0 *and variance* h *for all* $t \geq 0$ *and* $h > 0$,
- *the function* $t \to B(t)$ *is continuous almost surely.*

Brownian motion is sometimes called as Wiener Process to honor Norbert Wiener and if $x = 0$, called as standard Brownian Motion.

From the definition, one can see that Brownian motion is a stochastic process for which almost all sample paths are continuous, and their increments are independent, normally distributed. After these properties, a theorem which helps to determine whether a stochastic process is a Brownian motion or not is needed. Levy's characterization theorem [Karatzas and Shreve, 1998, p.157] shows that one can reach the answer using the quadratic variation of the stochastic process.

Theorem 2.2.1. *(Levy's Characterization Theorem) Let* $\{X_t = (X_t^{(1)}, X_t^{(2)}, ..., X_t^{(d)}), \mathcal{F}_t; 0 \leq t < \infty\}$ *a continuous, adapted, real-valued stochastic process satisfying the following conditions:*

- *the process* $M_t^{(i)} := X_t^{(i)} - X_0^{(i)}$ *is a continuous local martingale adapted to* $\{\mathcal{F}_t\}$ *for* $1 \leq i \leq d$ *and* $0 \leq t < \infty$,
- *the cross variations are* $\langle M^{(i)}, M^{(j)} \rangle = \delta_{ij} t$ *for* $1 \leq i, j \leq d$.

Then, $\{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ *is a Brownian motion.*

If Levy's characterization theorem is not applicable to a local martingale, one may use theorem of Dambis, Dubins, Schwarz [Karatzas and Shreve, 1998, p.174] which roughly tells that continuous local martingales with unbounded quadratic variations are time-changed Brownian motions.

Theorem 2.2.2. *(Dambis, Dubins, Schwarz Theorem) Let* $M := \{M_t, \mathcal{F}_t; 0 \leq t < \infty\} \in \mathcal{M}^{c,loc}$ *satisfying* $\lim_{t\to\infty} \langle M, M\rangle_t = \infty$ *almost surely in probability. Define the stopping time* $T(s) := inf\ \{t \geq 0; \langle M, M\rangle_t > s\}$, *for each* $0 \leq s < \infty$. *Then, the time-changed process*

$$B_s := M_{T(s)} \text{ with the filtration } \mathscr{G}_s := \mathscr{F}_{T(s)}, \text{ for each } 0 \leq s,$$

is a standard one dimensional Brownian motion. In particular, the filtration $\{\mathscr{G}_s\}$ *satisfy the usual conditions and we have almost surely in probability:*

$$M_t = B_{\langle M,M\rangle_t}, \quad 0 \leq t.$$

We can also define *a reflected Brownian motion* as the absolute value of a Brownian motion, ($|B|$). By the Skorokhod representation theorem [Karatzas and Shreve, 1998, p.210], we can say

$$|B_t| \sim B_t - \big(0 \wedge \inf_{0\leq s\leq t} B_s\big), \tag{2.2.1}$$

where we use symbol '$\sim$' in the meaning of "has the same distribution with".

Our stochastic process, one-sided sticky Brownian motion, can be considered as slowly reflecting Brownian motion. Therefore, we are also interested in the the amount of time the process spent around zero. Local time helps us to measure this time. It behaves as a clock which advances when and only when particle is at origin [Çınlar, 2010, p.409]. We can define *local time of Brownian motion* B at origin as in [Revuz and Yor, 2005, p.227]

$$\ell_t^0(B) = \lim_{\epsilon\to\infty} \frac{1}{2\epsilon} \int_0^t 1_{\{-\epsilon\leq B_s\leq\epsilon\}} ds, \quad \text{for } t \geq 0. \tag{2.2.2}$$

If X is a continuous semi-martingale then its local time at a is

$$\ell_t^a(X) = \lim_{\epsilon\to\infty} \frac{1}{\epsilon} \int_0^t 1_{\{a\leq X_s\leq a+\epsilon\}} d\langle X, X\rangle_s, \quad \text{for } t \geq 0 \text{ and every } a. \tag{2.2.3}$$

If X is a continuous local martingale then its local time at a is

$$\ell_t^a(X) = \lim_{\epsilon\to\infty} \frac{1}{2\epsilon} \int_0^t 1_{\{a-\epsilon\leq X_s\leq a+\epsilon\}} d\langle X, X\rangle_s, \quad \text{for } t \geq 0 \text{ and every } a.$$

Stochastic calculus provides us with a generalized rule for differentiation with respect to stochastic processes of unbounded variation, such as Brownian motion. See Karatzas ans Shreve [Karatzas and Shreve, 1998, p.128] for an account of stochastic integration. The fundamental theorem of stochastic calculus is known as Itô's formula, which necessitates the existence of second derivative.

Theorem 2.2.3. *(Itô's formula for Brownian motion) If $f : \mathbb{R} \to \mathbb{R}$ is the continuous, twice differentiable function, then for the Brownian motion B*

$$f(B(t)) = f(B(0)) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds.$$

The following theorem is known as Tanaka formula [Revuz and Yor, 2005, Thm.1.2]. In this thesis, Tanaka formula will be applied many times.

Theorem 2.2.4. *(Tanaka formula) For any $a \in \mathbb{R}$, there exists an increasing, continuous process ℓ_t^a such that,*

$$\begin{aligned} |X_t - a| &= |X_0 - a| + \int_0^t sgn(X_s - a)dX_s + \ell_t^a, \\ (X_t - a)^+ &= (X_0 - a)^+ + \int_0^t 1_{\{X_s > a\}} dX_s + \frac{1}{2}\ell_t^a, \\ (X_t - a)^- &= (X_0 - a)^- - \int_0^t 1_{\{X_s \le a\}} dX_s + \frac{1}{2}\ell_t^a. \end{aligned}$$

In particular, $|X_t - a|$, $(X_t - a)^+$ and $(X_t - a)^-$ are semi-martingales.

2.3 Stochastic Differential Equations

A differential equation of the form

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t \tag{2.3.1}$$

is called a stochastic differential equation (SDE) in which μ is called $(d \times 1)$ dimensional drift vector, σ is called $(d \times r)$ dimensional dispersion matrix and B is an r-dimensional Brownian motion, X_t is a stochastic process with values in $\mathbb{R}^d$. The integral version of the above equation can also be written. The integrals should be understood as stochastic integration [Karatzas and Shreve, 1998].

2.3.1 Strong Solutions of SDE's

It is essential to know whether an SDE has a strong solution or not, for many problems because strong solutions are known as soon as B is known. In [Karatzas and Shreve, 1998, p.285] a strong solution is defined as follows.

Definition 2.3.1. *A strong solution of an SDE is a process* $X = \{X_t;\ 0 \leq t \leq \infty\}$ *with continuous sample paths on the given probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and with respect to the fixed Brownian motion* B *and initial condition* ζ *satisfying*

- X *is adapted to the natural filtration* $\{\mathcal{F}_t\}$ *of the given Brownian motion* B,
- $X_0 = \zeta$ *almost surely,*
- $\int_0^t \big(|\mu(s, X_s)| + \sigma^2(s, X_s)\big)ds < \infty$ *almost surely for every* $0 \leq t < \infty$,
- $X_s = X_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s$ *holds almost surely.*

Strong solutions are functions of the given Brownian motion in the SDE. After determining the existence of a strong solution for an SDE, we consider its uniqueness.

Definition 2.3.2. *Let* μ *and* σ *be given. Assume* B *is an r-dimensional Brownian motion on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$. *If* X *and* $\tilde{X}$ *are two strong solutions of an SDE adapted to the natural filtration of* B *and satisfy* $\mathbb{P}\big\{X = \tilde{X}; 0 \leq t < \infty\big\} = 1$, *then we say that strong uniqueness holds for the pair* (μ, σ).

Theorem 2.3.1. *If* (μ, σ) *satisfies that for every integer* $n \geq 1$, *there exists a constant* $K_n > 0$ *such that for every* $t \geq 0$, $||x|| \leq n$, $||y|| \leq n$ *such that*

$$||\mu(t,x) - \mu(t,y)|| + ||\sigma(t,x) - \sigma(t,x)|| \leq K_n||x - y||$$

then strong uniqueness holds for an SDE.

Theorem 2.3.1 and its proof are given in [Karatzas and Shreve, 1998, p.287].

2.3.2 Weak Solutions of SDE's

Weak solutions of stochastic differential equations are useful in both theory and applications.

Definition 2.3.3. *A weak solution of a stochastic differential equations* (2.3.1) *is a pair* (X, B) *of adapted process on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *filtered by* $\{\mathcal{F}_t\}$ *satisfying following properties:*

- $X = \{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ *is a continuous, adapted,* $\mathbb{R}^d$*-valued process and* $B = \{B_t, \mathcal{F}_t; 0 \leq t < \infty\}$ *is an r-dimensional standard Brownian motion,*
- $\int_0^t \big(|\mu(s, X_s)| + \sigma^2(s, X_s)\big)ds < \infty$ *almost surely for every* $0 \leq t < \infty$,
- $X_s = X_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s$ *holds almost surely.*

Remark 2.3.4. *Existence of a weak solution does not imply existence of a strong solution but clearly, existence of a strong solution implies existence of a weak solution.* $\big($*Existence of a weak solution* $\not\Leftarrow$ *Existence of a strong solution.*$\big)$

Example. *Tanaka's Equation*

$$X_t = x + \int_0^t sgn(X_s) dB_s \tag{2.3.2}$$

where $0 \leq t < \infty$, $x \in \mathbb{R}$ *and* B *is a standard Brownian Motion and*

$$sgn(x) = \begin{cases} 1 & x \geq 0, \\ -1 & x < 0. \end{cases}$$

Equation (2.3.2), known as Tanaka's Equation, does not have strong solution but admits a weak solution. First, the quadratic variation of X is,

$$\big\langle X, X \big\rangle_t = \int_0^t \mathrm{sgn}^2(X_s) d\big\langle B, B\big\rangle_s = t. \tag{2.3.3}$$

By Levy's characterization (Theorem 2.2.1), X is a Brownian motion. We can define B as

$$\begin{aligned} dX_t &= \mathrm{sgn}(X_t) dB_t \\ \mathrm{sgn}(X_t) dX_t &= dB_t \\ x + \int_0^t \mathrm{sgn}(X_s) dX_s &= B_t \end{aligned}$$

which is also a Brownian motion, similar to X. (X, B) satisfying Tanaka Equation is a weak solution. To show it does not admit strong solution, we can use contradiction method. Assume that X on $(\Omega, \mathcal{F}, \mathbb{P})$ with given B satisfies Equation. Then, X is a function of B which means $\mathcal{F}_t^X \subseteq \mathcal{F}_t^B$ for every $t \geq 0$.

We can apply Tanaka formula (Theorem 2.2.4) to X_t for $a = 0$ with $X_0 = x$ as,

$$|X_t| = |x| + \int_0^t \mathrm{sgn}(X_s) dB_s + \ell_t^0(X),$$

Then, we have

$$\int_0^t \mathrm{sgn}(X_s) dX_s = |X_t| - \ell_t^0(X) - |x|$$

where $\ell_t^0(X) = \lim_{\epsilon \to \infty} \frac{1}{2\epsilon} \int_0^t 1_{\{-\epsilon \leq X_s \leq \epsilon\}} d\big\langle X, X\big\rangle_s$ for $t \in [0, \infty)$. It leads us

$$\begin{aligned} B_t = \int_0^t \mathrm{sgn}(X_s) dX_s &= |X_t| - \ell_t^0(X) - |x| \\ &= |X_t| - \lim_{\epsilon \to \infty} \frac{1}{2\epsilon} \int_0^t 1_{\{|X_s| \leq \epsilon\}} ds - |x|, \quad \text{for } t \geq 0. \end{aligned}$$

with this equation we have $\mathcal{F}_t^B \subseteq \mathcal{F}_t^{|X|}$. Regarding our assumption, we get $\mathcal{F}_t^X \subseteq \mathcal{F}_t^B \subseteq \mathcal{F}_t^{|X|}$ which contradicts with the fact that $\mathcal{F}_t^{|X|} \subseteq \mathcal{F}_t^X$. Hence, Tanaka's Equation does not have a strong solution.

After determining the existence of a weak solution for an SDE, we consider its uniqueness.

Definition 2.3.5. *Path-wise uniqueness holds for an SDE if for any two weak solutions (X, B) and $(\tilde{X}, \tilde{B})$ defined on the same filtered probability space with common Brownian motion $B = \tilde{B}$ and common initial value $\mathbb{P}\big\{X_0 = \tilde{X}_0\big\} = 1$, we have $\mathbb{P}\big\{X_t = \tilde{X}_t;\ \text{for all } 0 \leq t < \infty\big\} = 1$.*

Definition 2.3.6. *Uniqueness-in-law holds for an SDE if for any two weak solutions (X, B) and $(\tilde{X}, \tilde{B})$, not necessarily defined on the same probability space, with possibly different Brownian motions $B, \tilde{B}$ but with the same initial distribution $X_0 \sim \tilde{X}_0$, then $X \sim \tilde{X}$.*

Definition 2.3.7. *Joint uniqueness-in-law holds for an SDE if for any two weak solutions (X, B) and $(\tilde{X}, \tilde{B})$, not necessarily defined on the same probability space, with possibly different Brownian motions $B, \tilde{B}$ but with the same initial distribution $X_0 \sim \tilde{X}_0$, then $(X, B) \sim (\tilde{X}, \tilde{B})$.*

Remark 2.3.8. *For a large class of stochastic differential equations uniqueness-in-law and joint uniqueness-in-law are the same [Cherny, 2001].*

Remark 2.3.9. *Uniqueness-in-law does not imply existence of path-wise uniqueness but clearly existence of path-wise uniqueness implies uniqueness-in-law. $\big($Uniqueness-in-law $\not\leftarrow$ Path-wise uniqueness.$\big)$*

Example. *Weak solution of a Tanaka's equation is unique-in-law but equation does not have path-wise uniqueness.*

Assume that $(X, B), (\Omega, \mathcal{F}, \mathbb{P}), \{\mathcal{F}_t\}$ be a weak solution for Equation (2.3.2). We know that X is a Brownian motion by Equation (2.3.3). Therefore, uniqueness-in-law holds. By the way, if $(X, B), (\Omega, \mathcal{F}, \mathbb{P}), \{\mathcal{F}_t\}$ is a weak solution, then $(-X, B)$, $(\Omega, \mathcal{F}, \mathbb{P}), \{\mathcal{F}_t\}$ is also a weak solution. Then, path-wise uniqueness fails.

The following theorem relates weak and strong solutions to uniqueness-in-law and path-wise uniqueness.

Theorem 2.3.2. *For a solution of an SDE, strong existence and uniqueness-in-law hold if and only if weak existence and path-wise uniqueness hold.*
(Strong existence + Uniqueness-in-law $\iff$ Weak existence + Path-wise uniqueness.)

Sufficiency in this theorem is proven by Yamada and Watanabe [Yamada and Watanabe, 1971] and neccessity is proven by Cherny [Cherny, 2001].

Remark 2.3.10. *We have shown that Tanaka's equation does not have a strong solution but admits a weak solution. When we construct unique-in-law weak solution, we will have already proven that Equation is not path-wise unique in view of Theorem 2.3.2.*

Chapter 3

ONE-SIDED STICKY BROWNIAN MOTION

Consider a stochastic process X which defined as a strong Markov process on $[0,\infty)$ and behaves like a Brownian motion when it is away from zero, reflects when it hits zero. Different from reflecting Brownian motion, it spends positive amount of time at zero. It can be obtained by reflected Brownian motion $|B|$ by time change [Engelbert and Peskir, 2012], [Warren, 1997]. They have the same sample paths but one-sided sticky Brownian motion spends more time at zero [Engelbert and Peskir, 2012, p.2] in the sense that

$$\mathbb{P}\big(\int_0^t 1_{\{X_s=0\}}ds > 0\big) > 0.$$

Then, one can say one-sided sticky Brownian motion is a slowly reflecting Brownian motion. We call this process one-sided sticky Brownian motion.

In this chapter, one-sided sticky Brownian motion is defined as a weak solution of an SDE. Then, some properties of the weak solution are examined and it is proven that this SDE does not admit a strong solution.

All processes we examined in this chapter are continuous and taking values in the space [Engelbert and Peskir, 2012, p.8]

$$\mathbb{S} = \{x : [0,\infty) \to \mathbb{R}; x \text{ is continuous}\}$$

equipped with the topology of uniform convergence on compacts which is induced by the metric $d(x,y) = \sum_{n=1}^{\infty} 1/2^N \big(1 \wedge \sup_{0\le t\le N} |x(t)-y(t)|\big)$.

3.1 SDE for One-Sided Sticky Brownian Motion

The stochastic differential equation for one-sided sticky Brownian motion is constructed by Itô-McKean [Itô and McKean, 1963]. The SDE for one-sided sticky Brownian motion is given by

$$dX_t = a1_{\{X_t=0\}}dt + 1_{\{X_t>0\}}dB_t \tag{3.1.1}$$

or equivalently the system

$$dX_t = \frac{1}{2}d\ell_t^0(X) + 1_{\{X_t>0\}}dB_t \tag{3.1.2}$$

$$1_{\{X_t=0\}}dt = \frac{1}{2a}d\ell_t^0(X) \tag{3.1.3}$$

where $X_0 = x \in \mathbb{R}^+,\ a \in (0,\infty)$ is a given constant, B is a standard Brownian motion and $\ell^0(X)$ is the local time of X at 0.

We are interested in one-sided sticky Brownian motion. However, two-sided sticky Brownian motion is also referred to. SDE for the latter process is also constructed by Itô-McKean [Itô and McKean, 1963] as

$$d\tilde{X}_t = 1_{\{\tilde{X}_t\neq 0\}}d\tilde{B}_t \tag{3.1.4}$$

$$1_{\{\tilde{X}_t=0\}}dt = \frac{1}{a}d\ell_t^0(\tilde{X}) \tag{3.1.5}$$

where $\tilde{X}_0 = \tilde{x} \in \mathbb{R},\ a \in (0,\infty)$ is a given constant, $\tilde{B}$ is a standard Brownian motion and $\ell^0(\tilde{X})$ is the local time of $\tilde{X}$ at 0.

First of all, we need to be convinced of the equivalence of Equation (3.1.1) and System (3.1.2)+(3.1.3).

Theorem 3.1.1. *[Engelbert and Peskir, 2012] Equation* (3.1.1) *and System* (3.1.2)+ (3.1.3) *are equivalent.*

Proof. By (3.1.3), we have

$$d\ell_t^0(X) = 2a1_{\{X_t=0\}}dt. \tag{3.1.6}$$

When we put Equation (3.1.6) into (3.1.2), we have Equation (3.1.1). Therefore, system (3.1.2)+ (3.1.3) implies Equation (3.1.1).

For the other side, we will show $X^- = \max\{-X,0\} = 0$ because we need to be convinced that X is non-negative. Apply Tanaka formula to X for $a=0$:

$$\begin{aligned}
X_t^- &= X_0^- - \int_0^t 1_{\{X<0\}}dX_s + \frac{1}{2}\ell_t^0(X)\\
&= 0 - \int_0^t 1_{\{X<0\}}dX_s + \frac{1}{2}\ell_t^0(X)\\
&= -\int_0^t 1_{\{X<0\}}(a1_{\{X_t=0\}}ds + 1_{\{X_t>0\}}dB_s) + \frac{1}{2}\ell_t^0(X)\\
&= -\int_0^t a(1_{\{X<0\}}1_{\{X_t=0\}})ds - \int_0^t (1_{\{X<0\}}1_{\{X_t>0\}})dB_s + \frac{1}{2}\ell_t^0(X)\\
&= -0 - 0 + \frac{1}{2}\ell_t^0(X) = \frac{1}{2}\ell_t^0(X).
\end{aligned}$$

where in the second equality we use $X_0 = x \in \mathbb{R}^+$ and in the third equality we substitute X with Equation (3.1.1). Since X is a continuous semi-martingale, by Equation (2.2.3) we have

$$\begin{aligned}
\ell_t^{0-}(X) &= \lim_{\epsilon \to \infty} \frac{1}{\epsilon} \int_0^t 1_{\{-\epsilon \le X_s \le 0\}} d < X, X >_s, \quad \text{for } 0 \le s \le t, \\
&= \lim_{\epsilon \to \infty} \frac{1}{\epsilon} \int_0^t 1_{\{-\epsilon \le X_s \le 0\}} 1_{\{X_s > 0\}} ds \\
&= 0.
\end{aligned}$$

Therefore, we have

$$X_t^- = 0 \tag{3.1.7}$$

which means X is non-negative, $X = X^+$.

Now, we will show that every solution of Equation (3.1.1) is also solution for system (3.1.2)+(3.1.3) using non-negativeness of X. Let X be a solution for Equation (3.1.1). We will again apply Tanaka formula to X for $a = 0$:

$$\begin{aligned}
X_t^+ &= X_0^+ + \int_0^t 1_{\{X_s > 0\}} dX_s + \frac{1}{2} \ell_t^0(X) \\
X_t &= x + \int_0^t 1_{\{X_s > 0\}} dX_s + \frac{1}{2} \ell_t^0(X) \\
&= x + \int_0^t 1_{\{X_s > 0\}} (a 1_{\{X_s = 0\}} ds + 1_{\{X_s > 0\}} dB_s) + \frac{1}{2} \ell_t^0(X) \\
&= x + \int_0^t a (1_{\{X_s > 0\}} 1_{\{X_s = 0\}}) ds + \int_0^t (1_{\{X_s > 0\}} 1_{\{X_s > 0\}}) dB_s + \frac{1}{2} \ell_t^0(X) \\
&= x + \int_0^t 1_{\{X_s > 0\}} dB_s + \frac{1}{2} \ell_t^0(X)
\end{aligned}$$

where in the second equality we use non-negativeness of X and $X_0 = x \in \mathbb{R}^+$, in the third equality we substitute X with Equation (3.1.1). It is equal to

$$dX_t = 1_{\{X_t > 0\}} dB_t + \frac{1}{2} d\ell_t^0(X)$$

by differentiation which is Equation (3.1.2). When we compare this with Equation (3.1.1), we have

$$\frac{1}{2} d\ell_t^0(X) = a 1_{\{X_t = 0\}} dt$$

which is Equation (3.1.3). Hence, every solution of Equation (3.1.1) is also a solution for System (3.1.2) + (3.1.3). $\square$

Equation (3.1.1) and System (3.1.2)+(3.1.3) are equivalent. Suitable one can be used. In the following, "the SDE" refers to both System (3.1.2)+(3.1.3) and Equation (3.1.1).

As we have discussed in the previous chapter, X needs to be adapted to the natural filtration of B in order to be a strong solution of the SDE. We will prove the SDE has no strong solution. Nonetheless, we will also prove the SDE has a weak solution. Therefore, we can find a standard Brownian motion B and a stochastic process X which is not adapted to natural filtration of B but (X, B) satisfies the SDE.

3.2 Weak Solutions of the SDE

In this part, we adapt the proof of weak existence [Engelbert and Peskir, 2012, Thm.1] for two-sided sticky Brownian motion to our case of one-sided equation (3.1.1).

Theorem 3.2.1. *The stochastic differential equation/system for one-sided sticky Brownian motion (the SDE) has a weak solution.*

Proof. We will construct a one-sided sticky Brownian motion using the reflecting Brownian motion. Therefore, suppose we have defined one dimensional standard Brownian motion $\hat{B}$ on a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ and define

$$\hat{W}_t := |x + \hat{B}_t| \quad \text{where } x \in \mathbb{R} \text{ given and fixed.} \tag{3.2.1}$$

We will use technique of time change, then define

$$A_t := t + \frac{1}{2a}\ell_t^0(\hat{W}) \tag{3.2.2}$$

where A_t is continuous, strictly increasing, and $\lim_{t\to\infty} A_t = \infty$. The proper inverse of A_t is well-defined and is denoted by

$$T_t := A_t^{-1}, \quad \text{for all } t \geq 0. \tag{3.2.3}$$

T_t is also continuous, strictly increasing and $\lim_{t\to\infty} T_t = \infty$. Now, define the time-changed of (3.2.1) as

$$X_t := \hat{W}_{T_t} = |x + \hat{B}_{T_t}|. \tag{3.2.4}$$

To explore $|x + \hat{B}_{T_t}|$, first focus on $x + \hat{B}_{T_t}$ as follows

$$\begin{aligned} x + \hat{B}_{T_t} &= \int_0^{T_t} 1_{\{x+\hat{B}_s \neq 0\}} d\hat{B}_s = \int_0^{T_t} 1_{\{|x+\hat{B}_s| \neq 0\}} d\hat{B}_s = \int_0^{T_t} 1_{\{W_s \neq 0\}} d\hat{B}_s \\ &= \int_0^t 1_{\{W_{T_s} \neq 0\}} d\hat{B}_{T_s} = \int_0^t 1_{\{X_s \neq 0\}} d\hat{B}_{T_s} = \int_0^t 1_{\{X_s > 0\}} d\hat{B}_{T_s}; \end{aligned} \tag{3.2.5}$$

since $X_t = |x + \hat{B}_{T_t}| > 0$. Now, we apply Tanaka formula to $x + \hat{B}_{T_t}$ for $a = 0$,

$$\begin{aligned}|x + \hat{B}_{T_t}| &= |x + \hat{B}_0| + \int_0^t \operatorname{sgn}(x + \hat{B}_{T_s}) d\hat{B}_{T_s} + \ell^0_{T_t}(x + \hat{B}) \\ &= |x + \hat{B}_0| + \int_0^t \operatorname{sgn}(x + \hat{B}_{T_s}) d\hat{B}_{T_s} + \frac{1}{2}\ell^0_{T_t}(|x + \hat{B}|) \\ &= x + \int_0^t \operatorname{sgn}(x + \hat{B}_{T_s}) 1_{\{X_s>0\}} d\hat{B}_{T_s} + \frac{1}{2}\ell^0_{T_t}(|x + \hat{B}|)\end{aligned}$$

where in the third equality we use Equation (3.2.5), $\hat{B}_0 = 0$ and $|x| = x$. Defining $\hat{B}^1_{T_t} := \int_0^t \operatorname{sgn}(x + \hat{B}_{T_s}) d\hat{B}_{T_s}$, it becomes

$$|x + \hat{B}_{T_t}| = x + \int_0^t 1_{\{X_s>0\}} d\hat{B}^1_s + \frac{1}{2}\ell^0_t(X). \tag{3.2.6}$$

Equation (3.2.6) is similar to the first part of the SDE, Equation (3.1.2), but we need a Brownian motion. Then, start with checking quadratic variation of $\hat{B}^1_{T_t}$ which is

$$\begin{aligned}\langle \hat{B}^1, \hat{B}^1 \rangle_{T_t} &= \langle \operatorname{sgn}(x + \hat{B}_T)\hat{B}_T, \operatorname{sgn}(x + \hat{B}_T)\hat{B}_T \rangle_t = \langle \hat{B}_T, \hat{B}_T \rangle_t = T_t \\ &= \int_0^{T_t} 1_{\{W_s\neq 0\}} ds = \int_0^{T_t} 1_{\{W_s\neq 0\}} d(s + \frac{1}{2a}\ell^0_s(\hat{W}) = \int_0^{T_t} 1_{\{W_s\neq 0\}} dA_s \\ &= \int_0^t 1_{\{W_{T_s}\neq 0\}} dA_{T_s} = \int_0^t 1_{\{X_s\neq 0\}} ds = \int_0^t 1_{\{X_s>0\}} ds.\end{aligned}$$

Then, take another one dimensional standard Brownian motion $\hat{B}^0$ independent from $\hat{B}^1_T$ on a probability space $(\hat{\Omega}^0, \hat{\mathcal{F}}^0, \hat{\mathbb{P}}^0)$. Then, we can define

$$B_t = \hat{B}^1_{T_t} + \int_0^t 1_{\{X_s=0\}} d\hat{B}^0_s, \quad \text{for } t \geq 0. \tag{3.2.7}$$

The quadratic variation of B is

$$\langle B, B \rangle_t = \langle \hat{B}^1, \hat{B}^1 \rangle_{T_t} + \int_0^t 1_{\{X_s=0\}} ds = \int_0^t 1_{\{X_s>0\}} ds + \int_0^t 1_{\{X_s=0\}} ds = t.$$

By Levy's characterization theorem, B is a Brownian motion. Also, we have

$$\int_0^t 1_{\{X_s>0\}} d\hat{B}^1_s = \int_0^t 1_{\{X_s>0\}} dB_s.$$

Then, B satisfies Equation (3.2.6),

$$X_t = x + \int_0^t 1_{\{X_s>0\}} dB_s + \frac{1}{2}\ell^0_t(X).$$

Therefore, (X, B) satisfies SDE (3.1.2). Also, we have

$$\begin{aligned}\int_0^t 1_{\{X_s=0\}} ds = \int_0^t 1_{\{X_s=0\}} dA_{T_s} &= \int_0^t 1_{\{W_{T_s}=0\}} dA_{T_s} = \int_0^{T_t} 1_{\{W_s=0\}} dA_s \\ &= \int_0^{T_t} 1_{\{W_s=0\}} d(s + \frac{1}{2a}\ell^0_t(W)) \\ &= \int_0^{T_t} 1_{\{W_s=0\}} ds + \int_0^{T_t} 1_{\{W_s=0\}} \frac{1}{2a} d\ell^0_t(W) \\ &= \frac{1}{2a}\ell^0_{T_t}(W) = \frac{1}{2a}\ell^0_t(X)\end{aligned}$$

which means (X, B) satisfies SDE (3.1.3). Hence, (X, B) satisfies the SDE which means the SDE has a weak solution. $\square$

3.3 Properties of the Weak Solutions

In this section, we are interested in some properties of one-sided sticky Brownian motion. Especially, non-negativeness and (joint) uniqueness-in-law properties have a role in main proofs (Theorem 3.1.1, Theorem 3.4.1). However, the following property is also essential because it links the two-sided Brownian motion to one-sided Brownian motion. We are able to use the proofs for two-sided Brownian motion in the paper [Engelbert and Peskir, 2012] for oned-sided Brownian motion with the help of this connection.

Proposition 3.3.1. *[Engelbert and Peskir, 2012] Absolute value of the weak solution of the stochastic differential system for two-sided sticky Brownian motion* (3.1.4)+(3.1.5) *is the weak solution of the stochastic differential equation/system for one-sided sticky Brownian motion (the SDE).*

Proof. Let $\tilde{X}$ be a weak solution for system of two-sided sticky Brownian motion (3.1.4)+ (3.1.5). Apply Tanaka formula to $\tilde{X}$ for $a = 0$ with $\tilde{X}_0 = \tilde{x} \in \mathbb{R}$:

$$\begin{aligned}
|\tilde{X}_t| &= |\tilde{x}| + \int_0^t \operatorname{sgn}(\tilde{X}_s) d\tilde{X}_s + \ell_t^0(\tilde{X}) \\
&= |\tilde{x}| + \int_0^t \operatorname{sgn}(\tilde{X}_s) d\tilde{X}_s + \frac{1}{2}\ell_t^0(|\tilde{X}|) \\
&= |\tilde{x}| + \int_0^t \operatorname{sgn}(\tilde{X}_s)\big(1_{\{\tilde{X}_s \neq 0\}} d\tilde{B}_s\big) + \frac{1}{2}\ell_t^0(|\tilde{X}|)
\end{aligned}$$

where in the third equality we substitute $\tilde{X}$ with Equation (3.1.4). Define $B_t := \int_0^t \operatorname{sgn}(\tilde{X}_s) d\tilde{B}_s$ which is a Brownian motion by Levy's characterization (Theorem 2.2.1), then we have

$$\begin{aligned}
|\tilde{X}_t| &= |\tilde{x}| + \int_0^t 1_{\{\tilde{X}_s \neq 0\}} dB_s + \frac{1}{2}\ell_t^0(|\tilde{X}|) \\
&= |\tilde{x}| + \int_0^t 1_{\{|\tilde{X}_s| > 0\}} dB_s + \frac{1}{2}\ell_t^0(|\tilde{X}|)
\end{aligned}$$

using $1_{\{\tilde{X}_s \neq 0\}} = 1_{\{|\tilde{X}_s| > 0\}}$. Also, we see that

$$\int_0^t 1_{\{|\tilde{X}_s| = 0\}} ds = \int_0^t 1_{\{\tilde{X}_s = 0\}} ds = \frac{1}{a} d\ell_t^0(\tilde{X}) = \frac{1}{2a} d\ell_t^0(|\tilde{X}|)$$

where in the second equality, we have used (3.1.5). Hence, we find that $(|\tilde{X}|, B)$ satisfies the SDE. $\square$

Proposition 3.3.2. *[Engelbert and Peskir, 2012] The weak solution of the stochastic differential equation/system (the SDE) for one-sided sticky Brownian motion is non-negative.*

Proof. We have already proven that the negative part of X is equal to 0, $X^- = 0$, which is Equation (3.1.7) in the proof of Theorem 3.1.1. Therefore, stochastic differential equation/system for one-sided sticky Brownian motion has a non-negative weak solution.

□

Proposition 3.3.3. *[Engelbert and Peskir, 2012] A weak solution of the stochastic differential equation/system (the SDE) for one-sided sticky Brownian motion is unique-in-law.*

Proof. Let (X, B) be solution of the SDE. By Theorem 3.2.1 and the following Theorem 3.4.1, we know that (X, B) is a weak solution but not a function of the given Brownian motion B. If we can show that every X satisfying the SDE is a function of another Brownian motion '$\hat{B}$', then the law of X will be uniquely determined.

By Dambis-Dubins-Schwarz Theorem 2.2.2, we know that every continuous local martingale with unbounded quadratic variation is a time-changed Brownian motion. To use Dambis-Dubins-Schwarz Theorem, first let us define the process

$$M_t := \int_0^t 1_{\{X_s>0\}} dB_s, \quad t \in [0, \infty).$$

By definition, M is a continuous martingale, and its quadratic variation is

$$\langle M, M \rangle_t = \int_0^t 1_{\{X_s>0\}} ds.$$

Then, for each $t \in [0, \infty)$, we can define right inverse of $t \to \langle M, M \rangle_t$ as

$$A(t) := \inf \{ s \geq 0 \ \langle M, M \rangle_s > t \}. \tag{3.3.1}$$

Secondly, we need to prove $\langle M, M \rangle_t$ goes to infinity when t goes to infinity. Assume $\langle M, M \rangle_t < \infty$ and define $\langle M, M \rangle_t^0 := \int_0^t 1_{\{X_s=0\}} d_s$. Then, we have

$$\begin{aligned} \langle M, M \rangle_t + \langle M, M \rangle_t^0 &= \int_0^t 1_{\{X_s>0\}} d_s + \int_0^t 1_{\{X_s=0\}} d_s \\ &= \int_0^t 1_{\{X_s \geq 0\}} ds = \int_0^t 1 d_s = t. \end{aligned}$$

It leads us

$$\lim_{t\to\infty} \langle M, M \rangle_t^0 = \lim_{t\to\infty} \left(t - \langle M, M \rangle_t \right) = \infty.$$

We can write X as

$$X_t = x + a\int_0^t 1_{\{X_s=0\}}ds + \int_0^t 1_{\{X_s>0\}}dB_s = x + a\big\langle M, M\big\rangle_t^0 + M_t.$$

When we take limit of X, we get

$$\lim_{t\to\infty} X_t = \lim_{t\to\infty}(x + a\big\langle M, M\big\rangle_t^0 + M_t) = \infty.$$

However, $X \in \mathbb{R}^+$. Then, there is a contradiction with $\lim\limits_{t\to\infty}\big\langle M, M\big\rangle_t^0 = \infty$. Therefore, $\big\langle M, M\big\rangle_t = \infty$.

Now, the time-changed process $\hat{B}_t := M_{A(t)}$ is a a standard one-dimensional Brownian motion by the Dambis-Dubins-Schwarz Theorem 2.2.2. If $\lim\limits_{t\to\infty}\big\langle M, M\big\rangle_t < \infty$, we can still define $\hat{B}_t := M_{A(t)}$. However, in the situation of $t > \lim\limits_{t\to\infty}\big\langle M, M\big\rangle_t$, we get $A(t) = \infty$. Then, $M_{A(t)}$ would not a Brownian motion [Revuz and Yor, 2005, p.181,182]. To understand what is X, define

$$C_t := X_{A(t)} \text{ where } t \in [0, \infty).$$

From the SDE (3.1.1), we have

$$\begin{aligned} C_t &= x + \frac{1}{2}.\ell^0_{A(t)}(X) + \int_0^{A(t)} 1_{\{X_s>0\}}dB_s = x + \frac{1}{2}\ell^0_t(X_{A(t)} + M_{A(t)} \\ &= x + \frac{1}{2}\ell^0_t(C) + \hat{B}_t. \end{aligned} \tag{3.3.2}$$

We can use proposition of Skorokhod's Lemma in [Karatzas and Shreve, 1998, p.211], then C is a reflecting Brownian motion starting at $x \in \mathbb{R}^+$ by the definition 2.2.1. By (3.3.1), $t = \big\langle M, M\big\rangle_{A(t)}$. Now, we can write

$$\begin{aligned} t = \big\langle M, M\big\rangle_{A(t)} = \int_0^{A(t)} 1_{\{X_s>0\}}ds &= A(t) - \int_0^{A(t)} 1_{\{X_s=0\}}ds \\ &= A(t) - \frac{1}{2a}d\ell^0_{A(t)}(X) = A(t) - \frac{1}{2a}d\ell^0_t(X_A) \\ &= A(t) - \frac{1}{2a}d\ell^0_t(C) \end{aligned}$$

where in the fourth equality we use Equation (3.1.3). Then, we have

$$A(t) = t + \frac{1}{2a}d\ell^0_t(C), \quad \text{for } t \geq 0.$$

We see that $A(t)$ is continuous and strictly increasing, its inverse is well-defined as

$$\big\langle M, M\big\rangle_t = A(t)^{-1} \text{ and } A\big(\big\langle M, M\big\rangle_t\big) = t, \quad \text{for } t \geq 0.$$

Therefore,

$$X_t = X_{A\left(\left\langle M,M\right\rangle_t\right)} = C_{\left\langle M,M\right\rangle_t}, \quad \text{for } \ t \geq 0.$$

Hence, X is a well determined measurable function of C which is a a well determined measurable function of $\hat{B}$ by Equation (3.3.2). Then, X is a well determined measurable function of $\hat{B}$ so X is unique-in-law. □

Stochastic differential equation/system for one-sided sticky Brownian motion has a non-negative, unique-in-law weak solution.

Remark 3.3.4. *For the SDE, the uniqueness in law implies the jointly uniqueness-in-law [Cherny, 2001, Thm.3.1]. Therefore, the weak solution of the SDE is also jointly unique-in-law.*

Remark 3.3.5. *In the SDE, we take $a \in (0,\infty)$ because in the cases of $a = 0$ or $a = \infty$, the SDE has strong solutions [Chitashvili, 1997].*

- *The strong solution of the case $a = 0$:*

$$\begin{aligned} dX_t &= 1_{\{X_t>0\}} dB_t \\ X_t &= x + \int_0^t 1_{\{X_s>0\}} dB_s \\ &= x + B_{t\wedge T} \qquad \textit{where } T = \min\{s \geq 0,\ x + B_s = 0\}. \end{aligned}$$

- *The strong solution of the case $a = \infty$:*

$$X_t = \max\big(x + B_t, \max_{0\leq s\leq t}(B_t - B_s)\big).$$

Even though we defined one-sided sticky Brownian motion as a weak solution of the SDE, one-sided sticky Brownian motion also can be defined as a Markov process which is characterized by a generator. By [Hajri et al., 2017], generator of one-sided sticky Brownian motion is

$$(\mathcal{A}f)(x) = \begin{cases} \frac{1}{2}f''(x) & x > 0, \\ af'(0+) & x = 0 \end{cases}$$

and domain of one-sided sticky Brownian motion is

$$\begin{aligned} \mathcal{D}(\mathcal{A}) = \{f \in C^2(0,\infty) : f \in C_0([0,\infty)), f'(0+), f''(0+) \text{ exist}, \\ f'(X=0) = \frac{1}{2a}f''(0), \lim_{x\to\infty} f''(x) = 0\} \end{aligned}$$

We can think of 'a' as the rate of departure from zero and the second derivative measures stickiness of X at 0. Using the boundary condition, when $a \to 0$, $\lim_{a\to 0} f''(0) = \lim_{a\to 0} 2af'(0) = 0$ means X is absorbed at zero, when $a \to \infty$, $\lim_{a\to\infty} f'(0) = \lim_{a\to\infty} \frac{1}{2a} f''(0) = 0$ means X is reflected instantaneously at zero which is simply a reflected Brownian motion. In these cases, the SDE has strong solutions.

In the case $a \in (0, \infty)$ is given, fixed, and constant, X spends positive amount of time at zero but does not stays at zero forever and X is not representable as a function of the Brownian motion 'B' in equation. In the next section, we will prove that the SDE does not admit a strong solution when $a \in (0, \infty)$.

3.4 No Strong Solutions of the SDE

In the previous section, we have found X which satisfies the SDE but X is not a measurable function of given B. In this section, we will show that if B and constant $a \in (0, \infty)$ are given, we cannot find a solution X as a measurable function of given B.

The following steps will be followed to prove no strong solutions of the SDE:

- Define an auxiliary equation which has a strong solution. [Definition 3.4.1]
- Construct a sequence of strong solutions to the auxiliary equation which approximate the weak solution of the SDE. [Lemma 3.4.2]
- Show a sequence with strong solutions of the auxiliary equation does not converge in probability. [Lemma 3.4.3]
- Change the problem of strong existence to the problem of convergence. [Lemma 3.4.4]
- Prove that strong existence fails via contradiction method. [Theorem 3.4.1]

Definition 3.4.1. *The differential equation*

$$dX_t^n = \sigma_n(X_t^n)dW_t^n + \frac{1}{2}d\ell_t^0(X^n) \tag{3.4.1}$$

with $X_0^n = x \in \mathbb{R}^+$*, where* W^n *is a standard Brownian motion,* b_n *is any sequence with* $\lim_{n\to\infty} b_n = 0$*,* $\sigma_n(y) = \begin{cases} 1 & |y| > b_n, \\ \sqrt{2ab_n} & |y| \le b_n \end{cases}$ *for* $y \in \mathbb{R}$ *and* $n \ge 1$*, is called the auxiliary equation.*

We want to converge solution of the SDE by the solutions of the auxiliary equation. Therefore, we first construct solutions to the auxiliary equation.

Lemma 3.4.2. *We can construct a sequence of strong solutions* $\{(X^n, W^n); n \geq 1\}$ *to the auxiliary equation, in which* $\{X^n; n \geq 1\}$ *approximates the weak solution* X *of the SDE* (3.1.1) /(3.1.2)+(3.1.3)*.*

Proof. We will start with defining a new measure

$$m_n(dx) := \frac{dx}{\sigma_n^2(x)} = \begin{cases} dx & |x| > b_n, \\ \frac{dx}{2ab_n} & |x| \leq b_n \end{cases}$$

and a new functional with the measure

$$A_t^n := \int_{\mathbb{R}} \ell_t^x(|x + \hat{B}|) dm_n(dx) \tag{3.4.2}$$

where $\hat{B}$ is a standard Brownian motion defined on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$, $x \in \mathbb{R}$ fixed and constant, and $t \geq 0$, $n \geq 0$. We will use inverse of this function to define a time-changed Brownian motion.

$$\begin{aligned} A_t^n &= \int_{\mathbb{R}} \ell_t^x(|x + \hat{B}|) dm_n(dx) = \int_{\mathbb{R}} \ell_t^x(|x + \hat{B}|) \frac{dx}{\sigma_n^2(x)} \\ &= \int_0^t \frac{ds}{\sigma_n^2(|x + \hat{B}_s|)} = \int_0^t 1_{\{|x+\hat{B}_s| > b_n\}} ds + \int_0^t \frac{1}{2ab_n} 1_{\{|x+\hat{B}_s| \leq b_n\}} ds. \end{aligned} \tag{3.4.3}$$

where in third equality we use the equality in [Karatzas and Shreve, 1998, p.204]. Taking the limit, we have

$$\begin{aligned} \lim_{n\to\infty} A_t^n &= \lim_{n\to\infty} \int_0^t 1_{\{|x+\hat{B}_s| > b_n\}} ds + \lim_{n\to\infty} \int_0^t \frac{1}{2ab_n} 1_{\{|x+\hat{B}_s| \leq b_n\}} ds \\ &= \int_0^t \lim_{n\to\infty} 1_{\{|x+\hat{B}_s| > b_n\}} ds + \frac{1}{a} \lim_{n\to\infty} \int_0^t \frac{1}{2b_n} 1_{\{-b_n \leq x+\hat{B}_s \leq b_n\}} ds \\ &= \int_0^t 1_{\{|x+\hat{B}_s| > 0\}} ds + \frac{1}{a} \ell_t^0(x + \hat{B}) \\ &= t + \frac{1}{a} \ell_t^0(x + \hat{B}) = t + \frac{1}{2a} \ell_t^0(|x + \hat{B}|) := A_t. \end{aligned}$$

A_t which we recently defined is the same as A_t for one-sided sticky Brownian motion in Equation (3.2.2). Since $t \to A_t^n$ is continuous and strictly increasing so we can define its inverse as $T_t^n = (A_t^n)^{-1}$ for all $t \geq 0$ which is also continuous and increasing. Then, we have

$$\lim_{n\to\infty} T_t^n = \lim_{n\to\infty} (A_t^n)^{-1} = (\lim_{n\to\infty} A_t^n)^{-1} = (A_t)^{-1} := T_t, \quad \text{for all } t \geq 0. \tag{3.4.4}$$

We can define time-changed process as $\hat{B}_{T_t^n}$, for $t \geq 0$. Define

$$X_t^n := |x + \hat{B}_{T_t^n}| \quad \text{and} \quad X_t = |x + \hat{B}_{T_t}| \quad \text{where } x \in \mathbb{R} \text{ given and fixed.} \tag{3.4.5}$$

X_t is the same X_t in Equation (3.2.4). Dini's theorem states that if a sequence of monotone increasing continuous functions converges pointwise on a compact space and the limit function is continuous then the convergence is uniform. This leads us to the convergences $\lim_{n\to\infty} A_t^n = A_t$ and $\lim_{n\to\infty} T_t^n = T_t$ are uniform for $t \in [0, N]$ with any $N > 0$. Then, we have

$$\hat{B}_{T_t^n} \to \hat{B}_{T_t} \text{ *uniformly* as } n \to \infty \text{ and} \tag{3.4.6}$$

$$X_t^n \to X_t \text{ *uniformly* as } n \to \infty \tag{3.4.7}$$

for $t \in [0, N]$ with any $N > 0$. Also, we have

$$\ell_t^0(X^n) = \ell_t^0(|x + \hat{B}_{T^n}|) = \ell_{T_t^n}^0(|x + \hat{B}|) \to \ell_{T_t}^0(|x + \hat{B}|) = \ell_t^0(|x + \hat{B}_T|) = \ell_t^0(X) \tag{3.4.8}$$

uniformly for $t \in [0, N]$ with any $N > 0$. $\hat{B}$ is defined on Theorem 3.2.1 and $\hat{B}_{T_t^n}$ is a time-change version of $\hat{B}$. Therefore, $\hat{B}_{T_t^n}$ is a continuous martingale. Now, we can define a process

$$\tilde{W}_t^n := \int_0^t \frac{d(x + \hat{B}_{T_s^n})}{\sigma_n(x + \hat{B}_{T_s^n})} \tag{3.4.9}$$

which is also a continuous martingale. Therefore, Levy's characterization theorem can be applied. We have

$$\begin{aligned}
\langle \tilde{W}^n, \tilde{W}^n \rangle_t &= \int_0^t \frac{d\langle x + \hat{B}_{T^n}, x + \hat{B}_{T^n} \rangle_s}{\sigma_n^2(x + \hat{B}_{T_s^n}} = \int_0^t \frac{d\langle x + \hat{B}, x + \hat{B} \rangle_{T_s^n}}{\sigma_n^2(x + \hat{B}_{T_s^n}} \\
&= \int_0^{T_t^n} \frac{d\langle \hat{B}, \hat{B} \rangle_s}{\sigma_n^2(x + \hat{B}_s)} = \int_0^{T_t^n} \frac{ds}{\sigma_n^2(x + \hat{B}_s)} = \int_0^{T_t^n} \frac{ds}{\sigma_n^2(|x + \hat{B}_s|)} \\
&= A_{T_t^n}^n = t \quad \text{for } t \geq 0, \ n \geq 1
\end{aligned}$$

where in the last equality we use Equation (3.4.3). Therefore, $\tilde{W}^n$ is a standard Brownian motion for $n \geq 1$. From the definition of $\tilde{W}$ in Equation (3.4.9), $(x + \hat{B}_{T_t^n}, \tilde{W}^n)$ satisfies

$$d(x + \hat{B}_{T_t^n}) = \sigma_n(x + \hat{B}_{T_t^n}) d\tilde{W}_t^n. \tag{3.4.10}$$

Nakao [Nakao, 1972] proved that the equation $d(x + \hat{B}_{T_t^n}) = \sigma_n(x + \hat{B}_{T_t^n}) d\tilde{W}_t^n$ has a path-wise unique solution for any given standard Brownian motion $\tilde{W}_t^n$. We know path-wise uniqueness implies uniqueness-in-law (Remark 2.3.9). In the consideration of Theorem 2.3.2, $d(x+\hat{B}_{T_t^n}) = \sigma_n(x+\hat{B}_{T_t^n})d\tilde{W}_t^n$ has a unique strong solution. In view of this, we can apply Tanaka formula and the auxiliary equation has a path-wise unique solution for any given standard Brownian motion. Apply Tanaka formula to $x+\hat{B}_{T_s^n}$ for $a=0$,

$$
\begin{aligned}
|x+\hat{B}_{T_s^n}| &= |x+\hat{B}_{T_0^n}| + \int_0^t \operatorname{sgn}(x+\hat{B}_{T_s^n})d(x+\hat{B}_{T_s^n}) + \ell_t^0(x+\hat{B}_{T_s^n}) \\
|x+\hat{B}_{T_s^n}| &= |x+\hat{B}_{T_0^n}| + \int_0^t \operatorname{sgn}(x+\hat{B}_{T_s^n})d(x+\hat{B}_{T_s^n}) + \frac{1}{2}\ell_t^0(|x+\hat{B}_{T_s^n}|) \\
X_t^n &= |x| + \int_0^t \operatorname{sgn}(x+\hat{B}_{T_s^n})d(x+\hat{B}_{T_s^n}) + \frac{1}{2}\ell_t^0(X) \\
&= |x| + \int_0^t \operatorname{sgn}(x+\hat{B}_{T_s^n})\big(\sigma_n(x+\hat{B}_{T_s^n})d\tilde{W}_s^n\big) + \frac{1}{2}\ell_t^0(X)
\end{aligned}
$$

where in the third equality we insert the definition of X, and in the last equality we put Equation (3.4.10). Defining W^n as $W_t^n = \int_0^t \operatorname{sgn}(x+\hat{B}_{T_s^n})d\tilde{W}_s^n$, which is a Brownian motion by Levy's characterization (Theorem 2.2.1), yields

$$
X_t^n = |x| + \int_0^t \sigma_n(x+\hat{B}_{T_s^n} dW_s^n + \frac{1}{2}\ell_t^0(X). \tag{3.4.11}
$$

We have $\sigma_n(x+\hat{B}_{T_s^n}) = \sigma_n(|x+\hat{B}_{T_s^n}|) = \sigma_n(X_s^n)$ which leads us to

$$
X_t^n = |x| + \int_0^t \sigma_n(X_s^n)dW_s^n + \frac{1}{2}\ell_t^0(X^n).
$$

Hence, $\{(X^n, W^n); n \geq 1\}$ solves the auxiliary equation (3.4.1). Also, by Theorem 3.2.1, we know that X solves the SDE with B defined in (3.2.7). □

Lemma 3.4.3. *Let $\{X^n; n \geq 1\}$ be unique strong solutions to the auxiliary equation (3.4.1) with $X_0^n = x \in \mathbb{R}^+$, where $W_t^n = \int_0^t sgn(x+\hat{B}_{T_s^n})dW_s$ is a standard Brownian motion, and b_n is any sequence with $\lim_{n\to\infty} b_n = 0$. Then, the sequence*

$$
\big(X^n - \frac{1}{2}\ell^0(X^n) - x\big)_{n\geq 1} \text{ does not converge in probability as } n \to \infty.
$$

Proof. Set $F^n = \big(X^n - \frac{1}{2}\ell^0(X^n)\big) - \big(X^{n+1} - \frac{1}{2}\ell^0(X^{n+1}\big)$ with $F_0^n = 0$ for $n \geq 1$ given and fixed. By the auxiliary equation (3.4.1), we get

$$
F_t^n = \int_0^t \sigma_n(X_s^n)dW_s - \int_0^t \sigma_{n+1}(X_s^{n+1})dW_s, \quad \text{for } t \geq 0,
$$

which is a martingale with quadratic variation

$$\langle F^n, F^n\rangle_t = \int_0^t \big(\sigma_n(X_s^n) - \sigma_{n+1}(X_s^{n+1})\big)^2 ds \tag{3.4.12}$$

$$= \begin{cases} \big(\sqrt{2ab_n} - \sqrt{2ab_{n+1}}\big)^2 t & |X_s^n| \le b_n, |X_{s+1}^n| \le b_{n+1} \\ \big(\sqrt{2ab_n} - 1\big)^2 t & |X_s^n| \le b_n, |X_{s+1}^n| > b_{n+1} \\ \big(1 - \sqrt{2ab_{n+1}}\big)^2 t & |X_s^n| > b_n, |X_{s+1}^n| \le b_{n+1} \\ 0 & |X_s^n| > b_n, |X_{s+1}^n| > b_{n+1}. \end{cases}$$

Since $\lim_{n\to\infty} b_n = 0$, we can assume that $b_{n+1} \le b_n/4$ for all $n \ge 1$. Then, we have

$$\big(\sqrt{2ab_n} - \sqrt{2ab_{n+1}}\big)^2 \le \big(\sqrt{2ab_n} - \sqrt{2a\frac{b_n}{4}}\big)^2 = \frac{ab_n}{2}. \tag{3.4.13}$$

Also, choosing $b_n \in (0, \frac{1}{8a})$, we have

$$\big(\sqrt{2ab_n} - 1\big)^2 \ge \frac{1}{4}. \tag{3.4.14}$$

Our aim is to show that F^n does not converge in probability as $n \to \infty$. Define stopping time

$$T_c^n = \inf\{t \ge 0, |F_t^n| = c\}, \quad \text{for } c > 0 \text{ given and fixed.}$$

Then, we can write

$$\begin{aligned} \mathbb{P}\big(\sup_{0\le s\le t} |F_s^n| \ge c\big) =& \mathbb{P}(T_c^n \le t) \\ \ge& \frac{1}{t}\mathbb{E}\big(1_{\{T_c^n \le t\}} \int_0^t 1_{\{|X_s^n| \le b_n\}} ds\big) \\ =& \frac{1}{t}\mathbb{E}\big(\int_{t\wedge T_c^n}^t 1_{\{|X_s^n| \le b_n\}} ds\big) \\ =& \frac{1}{t}\mathbb{E}\big(\int_0^t 1_{\{|X_s^n| \le b_n\}} ds\big) - \frac{1}{t}\mathbb{E}\big(\int_0^{t\wedge T_c^n} 1_{\{|X_s^n| \le b_n\}} ds\big). \end{aligned}$$

Further, we have

$$\begin{aligned} \frac{1}{t}\mathbb{E}\big(\int_0^t 1_{\{|X_s^n| \le b_n\}} ds\big) =& \frac{1}{t}\mathbb{E}\big(\int_0^t 1_{\{|x+\hat{B}_{T_t^n}| \le b_n\}} ds\big) \\ =& \frac{1}{t}\mathbb{E}\big(\int_0^t 1_{\{|x+\hat{B}_{T_t^n}| \le b_n\}} dA_{T_s^n}^n\big) \\ =& \frac{1}{t}\mathbb{E}\big(\int_0^{T_t^n} 1_{\{|x+\hat{B}_s| \le b_n\}} dA_s^n\big) \\ =& \frac{1}{t}\mathbb{E}\big(\int_0^{T_t^n} \frac{1}{2ab_n} 1_{\{|x+\hat{B}_s| \le b_n\}} ds\big) \end{aligned} \tag{3.4.15}$$

where in the first equality we use the definition of X in Equations (3.2.4), in the second equality we use the definitions of A_t^n and T_t^n in Equation (3.4.2),(3.4.4) and in the last equality we use Equation (3.4.3). Now, it is possible to bound the expectation in Equation (3.4.15), using Fatou's Lemma [Çınlar, 2010, p.59]

$$\begin{aligned}
&\liminf_{n\to\infty} \mathbb{E}\big(\int_0^{T_t^n} \frac{1}{2ab_n} 1_{\{|x+\hat{B}_s|\le b_n\}}\big)ds \ge \mathbb{E}\big(\liminf_{n\to\infty} \int_0^{T_t^n} \frac{1}{2ab_n} 1_{\{|x+\hat{B}_s|\le b_n\}}\big)ds \\
&= \frac{1}{a}\mathbb{E}\big(\liminf_{n\to\infty} \frac{1}{2b_n} \int_0^{T_t} 1_{\{|x+\hat{B}_s|\le b_n\}}\big)ds = \frac{1}{a}\mathbb{E}\ell^0_{T_t}(x+\hat{B}) = \frac{1}{2a}\mathbb{E}\ell^0_{T_t}(|x+\hat{B}|) \\
&= \frac{1}{2a}\mathbb{E}\ell^0_t(X) > 0
\end{aligned}$$

where in the second inequality we use $\lim_{n\to\infty} T_t^n = T_t$ in Equation (3.4.4). We also want to bound the second expectation.

$$\begin{aligned}
&\mathbb{E}\big(\int_0^{t\wedge T_c^n} 1_{\{|X_s^n|\le b_n\}} ds\big) \\
&= \mathbb{E}\big(\int_0^{t\wedge T_c^n} 1_{\{|X_s^n|\le b_n, |X_s^{n+1}|\le b_{n+1}\}} ds\big) + \mathbb{E}\big(\int_0^{t\wedge T_c^n} 1_{\{|X_s^n|\le b_n, |X_s^{n+1}|> b_{n+1}\}} ds\big) \\
&\le \frac{2}{ab_n}\mathbb{E}\big(\int_0^{t\wedge T_c^n} 1_{\{|X_s^n|\le b_n, |X_s^{n+1}|\le b_{n+1}\}} d\langle F^n, F^n\rangle_s\big) + 4\mathbb{E}\langle F^n, F^n\rangle_{t\wedge T_c^n} \\
&\le \frac{2}{ab_n}\mathbb{E}\big(\int_0^{t\wedge T_c^n} 1_{\{|F_s^n|\le 2b_n\}} d\langle F^n, F^n\rangle_s\big) + 4\mathbb{E}\langle F^n, F^n\rangle_{t\wedge T_c^n} \\
&= \frac{2}{ab_n}\int_{-2b_n}^{2b_n} \mathbb{E}\ell^z_{t\wedge T_c^n}(F^n)dz + 4\mathbb{E}\big(F^n_{t\wedge T_c^n}\big)^2.
\end{aligned} \tag{3.4.16}$$

using quadratic variation of F in (3.4.12) and the inequalities (3.4.13) + (3.4.14) for the first inequality and $\{|X_s^n| \le b_n, |X_s^{n+1}| \le b_{n+1}\} \subseteq \{|F_s^n| \le 2b_n\}$ for the second inequality.

To understand $\mathbb{E}\ell^z_{t\wedge T_c^n}(F^n)$, we will apply Tanaka formula to $F^n_{t\wedge T_c^n} - f$ for any constant $f \in \mathbb{R}$ and $a = 0$,

$$\begin{aligned}
|F^n_{t\wedge T_c^n} - f| &= |f| + \int_0^t \operatorname{sgn}(F^n_{s\wedge T_c^n} - f)dF^n_{s\wedge T_c^n} + \ell^z_{s\wedge T_c^n}(F^n) \\
&= |f| + \int_0^{t\wedge T_c^n} \operatorname{sgn}(F^n_s - f)dF^n_s + \ell^z_{s\wedge T_c^n}(F^n).
\end{aligned}$$

Then, we have

$$\begin{aligned}
\ell^z_{s\wedge T_c^n}(F^n) &= |F^n_{t\wedge T_c^n} - f| - |f| - \int_0^{t\wedge T_c^n} \operatorname{sgn}(F^n_s - f)dF^n_s \\
&\le |F^n_{t\wedge T_c^n}| - \int_0^{t\wedge T_c^n} \operatorname{sgn}(F^n_s - f)dF^n_s.
\end{aligned} \tag{3.4.17}$$

Taking expectations on both sides of inequality (3.4.17) leads us

$$\mathbb{E}\ell^z_{s\wedge T_c^n}(F^n) \le \mathbb{E}|F^n_{t\wedge T_c^n}| \le \sqrt{\mathbb{E}|F^n_{t\wedge T_c^n}|^2} \tag{3.4.18}$$

Insert Equation (3.4.18) into Equation (3.4.16)

$$\begin{aligned}\frac{2}{ab_n}\int_{-2b_n}^{2b_n}\mathbb{E}\ell^z_{t\wedge T^n_c}(F^n)dz+4\mathbb{E}\big(F^n_{t\wedge T^n_c}\big)^2 &\leq \frac{2}{ab_n}\int_{-2b_n}^{2b_n}\mathbb{E}|F^n_{t\wedge T^n_c}|dz+4\mathbb{E}\big(F^n_{t\wedge T^n_c}\big)^2\\ &=\frac{8}{a}c+4c^2.\end{aligned}$$

All in all we have,

$$\liminf_{n\to\infty}\mathbb{P}\big(\sup_{0\leq s\leq t}|F^n_s|\geq c\big)\geq\frac{1}{2ta}\mathbb{E}\ell^0_t(X)-\frac{1}{t}\big(\frac{8}{a}c+4c^2\big)\geq 0$$

choosing $c>0$ sufficiently small,given and fixed.

Hence, $\{F^n;n\geq 1\}$ does not converge in probability which means $\big(X_n-\frac{1}{2}\ell^0(X^n)-x\big)_{n\geq 1}$ does not converge in probability as $n\to\infty$.

□

We want to reduce the existence of a strong solution problem to the problem of convergence in probability. In this sense, we need the following lemma.

Lemma 3.4.4. *[Engelbert and Peskir, 2012] It is possible to approximate the weak solution of the SDE* (3.1.1)/(3.1.2)+(3.1.3) *in distribution with the strong solutions of the auxiliary equation* (3.4.1).

Proof. For the auxiliary equation (3.4.1), $\{X^n;n\geq 1\}$, $\{\ell^0(X^n);n\geq 1\}$ are tight in $\mathbb{S}$ by (3.4.7) and (3.4.8). $\{W^n;n\geq 1\}$ is also tight in $\mathbb{S}$. Therefore, $\{\big(X^n,\ell^0(X^n),W^n\big);n\geq 1\}$ is tight in $\mathbb{S}$ [Karatzas and Shreve, 1998, p.62]. According to Prokhorov's theorem [Çınlar, 2010, p.115], a family is tight if and only if it is relatively compact. Then, $\{\big(X^n,\ell^0(X^n),W^n\big);n\geq 1\}$ is relatively compact which means every sequence of elements of $\{\big(X^n,\ell^0(X^n),W^n\big);n\geq 1\}$ contains a weakly convergent subsequence. Let $\{\big(X^{n_k},\ell^0(X^{n_k}),W^{n_k}\big);n_k\geq 1\}$ be a weakly convergent subsequence of $\{\big(X^n,\ell^0(X^n),W^n\big)|n\geq 1\}$ which converges in distribution to a random element which has the same limit law with $\big(X^{n_k},\ell^0(X^{n_k}),W^{n_k}\big)$. Let us say,

$$\big(X^{n_k},\ell^0(X^{n_k}),W^{n_k}\big)\to\big(X^*,\ell^0(X^*),W^*\big)\ \textit{in distribution}\text{ as }n_k\to\infty.$$

According to Skorokhod's representation theorem [Çınlar, 2010, p.113], there exist random variables $\{\big(Y^{n_k},\ell^0(Y^{n_k}),\hat{W}^{n_k}\big);n_k\geq 1\}$ and $\big(Y^*,\ell^0(Y^*),\hat{W}^*\big)$ on some probability space such that

$$\big(X^{n_k},\ell^0(X^{n_k}),W^{n_k}\big)\sim\big(Y^{n_k},\ell^0(Y^{n_k}),\hat{W}^{n_k}\big)\text{ for each }n_k \tag{3.4.19}$$

and $\big(X^*, \ell^0(X^*), W^*\big) \sim \big(Y^*, \ell^0(Y^*), \hat{W}^*\big)$. Then, we have

$$\big(Y^{n_k}, \ell^0(Y^{n_k}), \hat{W}^{n_k}\big) \to \big(Y^*, \ell^0(Y^*), \hat{W}^*\big) \textit{ almost surely} \text{ as } n_k \to \infty. \tag{3.4.20}$$

Now, we want to show that $(Y^*, \hat{W}^*)$ satisfies the SDE. We use Skorokhod's representation theorem because *convergence in distribution* does not implies other kind of convergences, except trivial cases, but we know *almost surely convergence* implies *convergence in probability.* Then, we get

$$\big(Y^{n_k}, \ell^0(Y^{n_k}), \hat{W}^{n_k}\big) \to \big(Y^*, \ell^0(Y^*), \hat{W}^*\big) \textit{ in probability} \text{ as } n_k \to \infty.$$

Choose a continuous function, $p_c(x) = \begin{cases} 0 & 0 \leq x \leq 1/c, \\ 1 & x \geq 2/c \end{cases}$ where $c \geq 1$ is given and fixed. By [Çınlar, 2010, Prop.3.5],

$$p_c(Y^{n_k}) \to p_c(Y^*) \textit{ in probability} \text{ as } n_k \to \infty.$$

Now, it is possible to write the following convergence which is well-defined by [Kurtz and Protter, 1996]

$$p_c(Y^{n_k}) \cdot \hat{W}^{n_k} \to p_c(Y^*) \cdot \hat{W}^* \textit{ in probability} \text{ as } n_k \to \infty.$$

Then, we get

$$\big(Y^{n_k}, \ell^0(Y^{n_k}), p_c(Y^{n_k}) \cdot \hat{W}^{n_k}\big) \to \big(Y^*, \ell^0(Y^*), p_c(Y^*) \cdot \hat{W}^*\big) \textit{ in probability}$$

as $n_k \to \infty$. By Equation (3.4.19), we have

$$\big(Y^{n_k}, \ell^0(Y^{n_k}), p_c(Y^{n_k}) \cdot \hat{W}^{n_k}\big) \sim \big(X^{n_k}, \ell^0(X^{n_k}, p_c(X^{n_k}) \cdot W^{n_k}\big) \text{ for each } n_k \geq 1.$$

Then, we have

$$\big(X^{n_k}, \ell^0(X^{n_k}), p_c(X^{n_k}) \cdot W^{n_k}\big) \to \big(Y^*, \ell^0(Y^*), p_c(Y^*) \cdot \hat{W}^*\big) \textit{ in probability}$$

as $n_k \to \infty$. [Çınlar, 2010, Prop.3.5] yields

$$\frac{1}{2}\ell^0(X^{n_k}) \to \frac{1}{2}\ell^0(Y^*) \textit{ in probability} \text{ as } n_k \to \infty.$$

Also, we know by [Çınlar, 2010, Thm.3.6]

$$X^{n_k} - \frac{1}{2}\ell^0(X^{n_k}) - p_c(X^{n_k}) \cdot W^{n_k} \to Y^* - \frac{1}{2}\ell^0(Y^*) - p_c(Y^*) \cdot \hat{W}^* \textit{ in probability}$$

as $n_k \to \infty$. Convergence in probability implies convergence in distribution [Çınlar, 2010, p.110]

$$X^{n_k} - \frac{1}{2}\ell^0(X^{n_k}) - p_c(X^{n_k}) \cdot W^{n_k} \to Y^* - \frac{1}{2}\ell^0(Y^*) - p_c(Y^*) \cdot \hat{W}^* \tag{3.4.21}$$

in distribution as $n_k \to \infty$. We can examine more

$$\begin{aligned}
&X_t^{n_k} - \frac{1}{2}\ell^0(X^{n_k}) - \int_0^t p_c(X_s^{n_k})dW_s^{n_k} \\
&= |x + \hat{B}_{T_t^{n_k}}| - \frac{1}{2}\ell_t^0(|x + \hat{B}_{T^{n_k}}|) - \int_0^t p_c(|x + \hat{B}_{T_s^{n_k}}|)dW_s^{n_k} \qquad (3.4.22)\\
&= |x + \hat{B}_{T_t^{n_k}}| - \frac{1}{2}\ell_t^0(|x + \hat{B}_{T^{n_k}}|) - \int_0^t p_c(|x + \hat{B}_{T_s^{n_k}}|)\mathrm{sgn}(\hat{B}_{T_s^{n_k}})d\tilde{W}_s^{n_k} \\
&= |x + \hat{B}_{T_t^{n_k}}| - \frac{1}{2}\ell_t^0(|x + \hat{B}_{T_t^{n_k}}|) - \int_0^t p_c(|x + \hat{B}_{T_s^{n_k}}|)\mathrm{sgn}(x + \hat{B}_{T_s^{n_k}})\frac{d(x + \hat{B}_{T_s^{n_k}})}{\sigma_n(x + \hat{B}_{T_s^{n_k}})} \\
&= |x + \hat{B}_{T_t^{n_k}}| - \frac{1}{2}\ell_t^0(|x + \hat{B}_{T_t^{n_k}}|) - \int_0^t p_c(|x + \hat{B}_{T_s^{n_k}}|)\mathrm{sgn}(x + \hat{B}_{T_s^{n_k}})d\hat{B}_{T_s^{n_k}} \\
&= |x + \hat{B}_{T_t^{n_k}}| - \frac{1}{2}\ell_{T_t^{n_k}}^0(|x + \hat{B}|) - \int_0^{T_t^{n_k}} p_c(|x + \hat{B}_s|)\mathrm{sgn}(x + \hat{B}_s)d\hat{B}_s, \qquad (3.4.23)
\end{aligned}$$

where we use in the first equality the definition of X^n in Equation (3.4.5), in the second equality the definition of W^n in Equation (3.4.11) and in the third equality the definition of $\tilde{W}^n$ in Equation (3.4.9), take n_k large enough and for $t \in [0, N]$ with any $N > 0$. Uniform convergence in (3.4.6), as $n \to \infty$, Equation (3.4.23) converges *uniformly* to

$$\begin{aligned}
&|x + \hat{B}_{T_t}| - \frac{1}{2}\ell_{T_t}^0(|x + \hat{B}|) - \int_0^{T_t} p_c(|x + \hat{B}_s|)\mathrm{sgn}(x + \hat{B}_s)d\hat{B}_s \\
&= |x + \hat{B}_{T_t}| - \frac{1}{2}\ell^0(|x + \hat{B}_{T_t}|) - \int_0^t p_c(|x + \hat{B}_{T_s}|)\mathrm{sgn}(x + \hat{B}_{T_s})d\hat{B}_{T_s} \\
&= X_t - \frac{1}{2}\ell_t^0(X) - \int_0^t p_c(X_s)dB_{T_s}^1 \\
&= X_t - \frac{1}{2}\ell_t^0(X) - \int_0^t p_c(X_s)dB_s,
\end{aligned}$$

where the second equality follows from (3.4.22), and the definition of $B_{T_s}^1$ in Equation (3.2.6), and in the third we use the definition of B in Equation (3.2.7). Therefore, we have

$$X_t^{n_k} - \frac{1}{2}\ell^0(X^{n_k}) - \int_0^t p_c(X_s^{n_k})dW_s^{n_k} \to\ X_t - \frac{1}{2}\ell_t^0(X) - \int_0^t p_c(X_s)dB_s \tag{3.4.24}$$

uniformly as $n_k \to \infty$. We can use this uniform convergence as

$$X^{n_k} - \frac{1}{2}\ell^0(X^{n_k}) - p_c(X^{n_k}) \cdot W^{n_k} \to\ X - \frac{1}{2}\ell^0(X) - p_c(X) \cdot B \ \textit{uniformly}$$

as $n_k \to \infty$. Convergence in distribution in Equation (3.4.21) leads us

$$Y^* - \frac{1}{2}\ell^0(Y^*) - p_c(Y^*) \cdot \hat{W}^* \sim \; X - \frac{1}{2}\ell^0(X) - p_c(X) \cdot B. \tag{3.4.25}$$

Now, we want to relate $p_c(X) \cdot B$ with $1_{\{X>0\}} \cdot B$. We can use Doob's maximal inequality [Çınlar, 2010, p.223]

$$\begin{aligned}
&\mathbb{P}\big(\sup_{0 \le t \le N} |\int_0^t p_c(X_s) dB_s - \int_0^t 1_{\{X_s>0\}} dB_s| > \epsilon \big) \\
&\le \frac{4}{\epsilon^2} \mathbb{E}\big(\int_0^N \big(p_c(X_s) - 1_{\{X_s>0\}}\big)^2 d\langle B, B\rangle_s \big) \\
&= \frac{4}{\epsilon^2} \mathbb{E}\big(\int_0^N \big(p_c(X_s) - 1_{\{X_s>0\}}\big)^2 ds \big) \\
&= \frac{4}{\epsilon^2} \mathbb{E}\big(\int_0^N \big(1_{\{1/2c > X_s > 0\}}\big)^2 ds \big).
\end{aligned}$$

By the dominated convergence theorem [Çınlar, 2010, p.25], for any $N > 0$, we have

$$\lim_{c\to\infty} \frac{1}{\epsilon^2} \mathbb{E}\big(\int_0^N \big(1_{\{1/2c > X_s > 0\}}\big)^2 ds \big) = \frac{1}{\epsilon^2} \mathbb{E}\Big(\int_0^N \lim_{c\to\infty} \big(1_{\{1/2c > X_s > 0\}}\big)^2 ds \Big) = 0.$$

Now, we have

$$\mathbb{P}\big(\sup_{0 \le t \le N} |\int_0^t p_c(X_s) dB_s - \int_0^t 1_{\{X_s>0\}} dB_s| \big) \to 0 \text{ as } c \to \infty.$$

That is,

$$p_c(X) \cdot B \to 1_{\{X>0\}} \cdot B \; \textit{in probability} \text{ as } c \to \infty. \tag{3.4.26}$$

It follows that

$$X - \frac{1}{2}\ell^0(X) - p_c(X) \cdot B \to X - \frac{1}{2}\ell^0(X) - 1_{\{X>0\}} \cdot B \; \textit{in probability} \text{ as } c \to \infty.$$

By Theorem 3.2.1, we know that X is a weak solution for the SDE then we get

$$X - \frac{1}{2}\ell^0(X) - p_c(X) \cdot B \to 0 \; \textit{in probability} \text{ as } c \to \infty.$$

By the equality of the distributions in (3.4.25), we can say

$$Y^* - \frac{1}{2}\ell^0(Y^*) - p_c(Y^*) \cdot \hat{W}^* \to 0 \; \textit{in probability} \text{ as } c \to \infty.$$

Similarly, we have

$$Y^* - \frac{1}{2}\ell^0(Y^*) - p_c(Y^*) \cdot \hat{W}^* \to Y^* - \frac{1}{2}\ell^0(Y^*) - 1_{\{Y^*>0\}} \cdot \hat{W}^* \; \textit{in probability}$$

as $c \to \infty$. All in all, we can conclude as

$$Y^* = \frac{1}{2}\ell^0(Y^*) - \int_0^t 1_{\{Y_s^*>0\}} d\hat{W}_s^* \tag{3.4.27}$$

which means $(Y^*, \hat{W}^*)$ satisfies Equation (3.1.2). Now, we wonder whether Y^* satisfies Equation (3.1.3) or not. Y^{n_k} converges Y^* almost surely as $n_k \to \infty$ by convergece in (3.4.20). Almost sure convergence implies convergence in probability which implies convergence in distribution. Then, we have

$$Y^{n_k} \to Y^* \textit{ in distribution} \text{ as } n_k \to \infty.$$

By (3.4.7), we know also X^n converge uniformly to X as $n \to \infty$. Uniform convergence implies almost sure convergence which implies convergence in distribution (which is true for subsequences)

$$X^{n_k} \to X \textit{ in distirbution} \text{ as } n_k \to \infty.$$

By (3.4.19), we know that $Y^{n_k} \sim X^{n_k}$. The uniqueness of limits leads us $Y^* \sim X$. Therefore, we can write Equation (3.1.3) with Y^*

$$1_{\{Y_t^*=0\}} dt = \frac{1}{2a} d\ell_t^0(Y^*). \tag{3.4.28}$$

Hence, by Equations (3.4.27)+(3.4.28), $(Y^*, \hat{W}^*)$ satisfies system of one-sided sticky Brownian motion (3.1.2)+(3.1.3).

□

Theorem 3.4.1. *The stochastic differential equation/system for one-sided sticky Brownian motion (the SDE) has no strong solution.*

Proof. We will use the method of contradiction. Assume that the SDE has a strong solution (X, B). By the definition of a strong solution, X is a function of B. Then, there exists a measurable function $f : \mathbb{S} \to \mathbb{S}$ such that

$$f(B) = X.$$

By (3.1.2), we have $dX_t - \frac{1}{2}d\ell_t^0(X) = 1_{\{X_t>0\}} dB_t$. Then, there exists a measurable function $g : \mathbb{S} \to \mathbb{S}$ such that

$$g(B) = X - \frac{1}{2}\ell^0(X).$$

We like to state the auxiliary equation (3.4.1) with strong solutions (Z^n, B^n) as

$$dZ_t^n = \sigma_n(Z_t^n)dB_t^n + \frac{1}{2}d\ell_t^0(Z^n) \tag{3.4.29}$$

where $Z_0^n = z \in \mathbb{R}$ and $B_t^n = \int_0^t \operatorname{sgn}(x + \hat{B}_{T_s^n})dB_s$. Since the auxiliary equation has a strong solution (Lemma 3.4.2), Z^n is a measurable function of B^n, which is a measurable function of B. There exists a measurable function $f_n : \mathbb{S} \to \mathbb{S}$ such that

$$f_n(B) = Z^n.$$

Also, there exists a measurable function $g_n : \mathbb{S} \to \mathbb{S}$ such that

$$g_n(B) = Z^n - \frac{1}{2}\ell_t^0(Z^n).$$

By Lemma 3.4.2, we can construct strong solutions $\{(Z^n, B); n \geq 1\}$ and by Lemma 3.4.3, $\left(Z^n - \frac{1}{2}\ell_t^0(Z^n) - z\right)_{n\geq 1}$ does not converge in probability.

Equation (3.4.29) is jointly unique-in-law by [Cherny, 2001, Thm.3.1] and similarly the auxiliary Equation (3.4.1) is. Then, $\left(Z^n, \ell^0(Z^n), B^n\right) \sim \left(X^n, \ell^0(X^n), W^n\right)$ for each $n \geq 1$. By (3.4.19) + (3.4.20), $\left(X^{n_k}, \ell^0(X^{n_k}), W^{n_k}\right) \sim \left(Y^{n_k}, \ell^0(Y^{n_k}), \hat{W}^{n_k}\right)$ which converges to $\left(Y^*, \ell^0(Y^*), \hat{W}^*\right)$ almost surely as $n_k \to \infty$. Almost sure convergence implies convergence in probability which implies converges in distribution. Therefore, we can say $\left(Y^{n_k}, \ell^0(Y^{n_k}), \hat{W}^{n_k}\right)$ converges to $\left(Y^*, \ell^0(Y^*), \hat{W}^*\right)$ *in distribution* as $n_k \to \infty$. We can say

$$\left(Z^{n_k}, \ell^0(Z^{n_k}), B^{n_k}\right) \sim \left(X^{n_k}, \ell^0(X^{n_k}), W^{n_k}\right) \sim \left(Y^{n_k}, \ell^0(Y^{n_k}), \hat{W}^{n_k}\right) \text{ and,}$$

$$\left(Z^{n_k}, \ell^0(Z^{n_k}), B^{n_k}\right) \to \left(Y^*, \ell^0(Y^*), \hat{W}^*\right) \textit{ in distribution} \text{ as } n_k \to \infty.$$

By Remark 3.3.4, the jointly uniqueness-in-law holds for the System (3.1.2) +(3.1.3) , $(X, \ell^0(X), B) \sim (Y^*, \ell^0(Y^*), \hat{W}^*)$. All in all, we get

$$\left(Z^{n_k}, \ell^0(Z^{n_k}), B^{n_k}\right) \to \left(X, \ell^0(X), B\right) \textit{ in distribution} \text{ as } n_k \to \infty \text{ and}$$

$$g_{n_k}(B) = Z^{n_k} - \frac{1}{2}\ell_t^0(Z^{n_k}) \to g(B) = X - \frac{1}{2}\ell^0(X) \textit{ in distribution} \text{ as } n_k \to \infty.$$

We can write

$$(g_{n_k}(B), B) \to (g(B), B) \textit{ in distribution} \text{ as } n_k \to \infty.$$

By [Engelbert and Peskir, 2012, Lemma 2], we have

$$g_{n_k}(B) \to g(B) \text{ } \textit{in probability} \text{ as } n_k \to \infty.$$

It is a contradiction by Lemma 3.4.3 because it means

$$Z^{n_k} - \frac{1}{2}\ell_t^0(Z^{n_k}) \to X - \frac{1}{2}\ell^0(X) \text{ } \textit{in probability} \text{ as } n_k \to \infty.$$

Hence, assumption fails and the SDE has no strong solution. □

Remark 3.4.5. *Weak solution of the SDE is not path-wise unique because of Theorem 2.3.2 (similar to Tanaka's equation 2.3.10). In Theorem 3.2.1, we prove that the SDE has a weak solution. In Proposition 3.3.3, we proved weak solution is unique in law. In Theorem 3.4.1, we proved the SDE has no strong solution. Applying Theorem 2.3.2, we can say for the SDE path-wise uniqueness does not hold.*

Remark 3.4.6. *If one can find two path-wise different weak solutions to the SDE, the non-existence of strong solution Theorem 3.4.1 will be already proven from Theorem 2.3.2 because we showed that the SDE has weak solution which is unique-in-law. However, constructing two path-wise different solutions is not easy.*

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