Study of Schrödinger Particles under the Influence of a Kratzer Molecular Potential in a Global Monopole Spacetime Background Plus a Wu-Yang Magnetic Monopole

Ahmed A. M. Yasin

Submitted to the
Institute of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Master of Science
in
Physics

Eastern Mediterranean University
July 2024
Gazimağusa, North Cyprus

Approval of the Institute of Graduate Studies and Research

Prof. Dr. Ali Hakan Ulusoy
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science in Physics.

Prof. Dr. Izzet Sakallı
Chair, Department of Physics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Master of Science in Physics.

Prof. Dr. Omar Mustafa
Supervisor

Examining Committee

1. Prof. Dr. S. Habiboallah Mazharimousavi
2. Prof. Dr. Omar Mustafa
3. Assoc. Prof. Dr. Abdullah Güvendi

ABSTRACT

Global monopoles are topological defects predicted by the Grand Unified Theory throughout the transition phase of the primordial universe. They are formed as a byproduct of spontaneous global symmetry breaking. Their behavior resembles that of the elementary particles with their energies concentrated close to the core of the monopole. They do not have gravitational interactions but they are known to modify spacetime geometry. It is an interesting topic to study Schrödinger particles in a Kratzer Molecular potential in the Global Monopole Spacetime background and under the influence of a Wu-Yang Magnetic Monopole. Some modifications in the spectroscopic structure of such molecular interactions have been observed and reported. This study presents the use of the Schrödinger equation to precisely describe the motion of a particle in the presence of Kratzer potential and a Wu-Yang magnetic monopole. The radial equation was derived and solved to yield the eigenfunctions and eigenvalues for the problem.

Keywords: PDM Shrödinger Equation, Point-like Global Monopole, Topological Defect, Kartzer Potential, Wu-Yang Magnetic Monopole.

ÖZ

Global monopoller, Büyük Birlesik Teori tarafindan erken evrenin geçis fazinda öngörülen topolojik kusurlardir. Bu topolojik kusurlar, spontan global simetri kilmasinin bir sonucu olarak ortaya ikarlar. Davranslari elementer parçaciklarinkine benzer; enerjileri monopolün gekirdegine yakin yogunlasir. Gravitasyonel etkilesim göstermese de. uzay-zaman geometrisini degistirdikleri bilinir. Global Monopole uzay-zaman arka planinda ve Wu-Yang Manyetik Monopolden etkilenen Kratzer moleküler potansiyel içinde Schrödinger parçaciklarnin incelenmesi ilging bir konudur Bu moleküler etkilesimlerin spektroskopik yapisinda bazi ayarlamalar gözlemlenmis ve belgelenmistir. Bu çalisma, Global Monopole uzay-zamaninda, Kratzer potansiyeli ve Wu-Yang manyetik monopolden etkilenen kosullarda, göreli olmayan bir kuantum parçaciginin hareketini kesin olarak tanimlamak için Schrödinger denklemini kullanir. Karsilk ge-len radyal denklem, problemin özfonksiyonlarn ve özdegerlerini belirlemek için türetilmis ve çözülmüstür.

Anahtar Kelimeler: PDM Shrödinger Denklemi, Nokta Benzeri Küresel Tek Kutup, Topolojik Kusur, Kratzer Moleküles Potansiyel, Wu-Yang Manyetik Tek Kutup.

DEDICATION

...To my father, without whom I would not be the same person I am now. For his boundless love, endless patience and unwavering support have been my anchor during tough times. To my family for their support, to my friends for their love. To everyone who has taught me, believed in me and given me opportunities. I dedicate this work with gratitude to each one of you.

ACKNOWLEDGMENTS

First of all, I would like to thank God for giving me the blessings, strength, and determination to complete this thesis. Next, I cannot adequately express my deepest gratitude and appreciation for my supervisor, Prof. Dr. Omar Mustafa. He has consistently shown me care, guidance, support, and respect, so thank you, Prof. Omar Mustafa. I also thank all the faculty members in the Department of Physics at Eastern Mediterranean University.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

INTRODUCTION

A classical particle of constant mass m moving on the x-axis can be described by its location $\mathbf{x}(t)$ and velocity $\mathbf{v}(t) = \frac{d\mathbf{x}}{dt}$. The remaining dynamical and/or kinematical features of this system only depend on the variables x and v, such as linear momentum $\mathbf{p} = m\mathbf{v}$, kinetic energy $T = \frac{1}{2}mv^2$, and total energy $E = T + V$, Where V represents the potential energy. Hence, the net force exerted on such a particle is given by $\mathbf{F} = \frac{d\mathbf{p}}{dt} = m\mathbf{a}$. In non-relativistic quantum mechanics, on the other hand, the quantum particle is described by a wave function $\psi(x,t)$ that includes all information about the particle and is solved through the Schrödinger equation [1, 2]:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi = i\hbar\frac{\partial\psi}{\partial t}. \tag{1.1}$$

Here $i = \sqrt{-1}$ and $\hbar$ is Planck's constant.

However, when the mass becomes effectively position-dependent, the dynamics of both classical and quantum particles dramatically change. In classical mechanics, for example, Mustafa [3] has stated that the net force under PDM conditions is no longer represented by the rate of change of linear momentum $\mathbf{p}$ with respect to time (i.e., $\mathbf{F} \neq \frac{d\mathbf{p}}{dt}$). The net force is instead given as:

$$\mathbf{F} = \sqrt{m(x)}\,\frac{d}{dt}\left(\sqrt{m(x)}\dot{\mathbf{x}}\right) = m(x)\ddot{\mathbf{x}} + \frac{1}{2}m'(x)\,\dot{x}^2 . \tag{1.2}$$

Hence, Newton's second law in the standard one-dimensional format takes the form

$$m(x)\ddot{x} = F - \frac{1}{2}m'(x)\,\dot{x}^2; \quad m'(x) = \frac{dm(x)}{dx} . \tag{1.3}$$

The quadratic velocity term works as an internal force introduced by the PDM-particle itself. It has a pronounced effect on the dynamics of the PDM system. Moreover, since $\dot{x}^2 \geq 0$, the system accelerates or decelerates for $m'(x)$ negative or positive, respectively. A particle's mass being a coordinate-dependent affects the net force $\mathbf{F}_{net}$, which is the sum of the external force $\mathbf{F}$ and an internal force term $(-m'(x)\dot{x}^2/2)$.

In quantum mechanics, however, the one-dimensional kinetic energy operator is replaced by the so-called von-Roos PDM kinetic energy operator [5] :

$$\hat{T} = -\frac{\hbar^2}{4}[m(x)^f \partial_x m(x)^g \partial_x m(x)^h + m(x)^h \partial_x m(x)^g \partial_x m(x)^f] \tag{1.4}$$

where $(\partial_x = \frac{\partial}{\partial x})$ and x is the spatial coordinate. Position-dependent mass, in quantum mechanics, has significant consequences across a broad spectrum of physical systems. For instance, in semiconductor physics, variations in the effective mass of charge carriers can occur spatially, often due to impurities or heterostructures, resulting in intricate quantum transport phenomena. Similarly, in the realm of atomic and molecular physics, fluctuations in mass within distinct regions of molecules or nanoscale structures can markedly influence their electronic and vibrational characteristics. Moreover, position-dependent mass is essential for investigation unusual phenomena such as localization effects in disordered media, quantum tunneling in spatially variable potentials, and the formation of topological phases in manufactured materials [4–7]. Physicists can learn more about the behavior of quantum systems and open up new possibilities for scientific and technological advancement by including position-dependent mass in theoretical models and numerical simulations.

The Schrödinger equation with position-dependent mass, introduced by von Roos [5], offers a theoretical basis for characterizing quantum systems where the particle's mass changes depending on its position.

The PDM von-Roos kinetic energy operator [5]; the PDM can rewrite $m(x) = m_0 f(x)$, where m_0 is the usual constant mass of the Schr$\ddot{o}$dinger particle. The ordering ambiguity parameters $f, g,$ and h align with the von-Roos restriction that $f + g + h = -1$ [5], yet the continuity conditions at heterojunctions imply that $f = h$. In this regard, Mustafa-Mazharimousavi's ordering [7–9] (i.e., $f = -1/4 = h$ and $g = -1/2$) satisfies such conditions.

In the current thesis proposal, we shall present a transformation in the radial coordinate of the global monopole spacetime and prove that the associated Schrödinger equation becomes a one-dimensional PDM von Roos Schrödinger equation. In so doing, we are motivated by Tymchyshyn's and Khlevniuk [10] observation that a point mass in motion within the curved space transforms to a PDM in Euclidean Space. By employing Mustafa Mazharimousavi's ordering, the PDM kinetic energy operator collapses into [7]:

$$\hat{T}_{MM} = -\frac{\hbar^2}{2}[m(x)^{-\frac{1}{4}}\partial_x m(x)^{-\frac{1}{2}}\partial_x m(x)^{-\frac{1}{4}}] \tag{1.5}$$

The organization of this thesis as follows: In Chapter 2, we introduce global monopole spacetime and demonstrate that a deformation or transformation in the radial coordinate of the global monopole spacetime would produce an efficacious PDM Schrödinger equation. Specifically, we show that a point canonical transformation of the radial coordinate of the PGM yields what is called in the literature Mustafa-Mazharimousavi's kinetic energy operator $\hat{T}_{MM}$ (1.5). We begin with the Schrödinger equation in the background of transformed global monopole spacetime. We then combine our results with the PDM von Roos Schrödinger equation to obtain Mustafa-Mazharimousavi's kinetic energy operator (1.5). Next, in Chapter 3. we consider the PDM Schrödinger equation with a Kratzer potential in global monopole spacetime. in Chapter 4, We discuss the consequences of such a PDM Schrödinger equation (with $\hat{T}_{MM}$) using the krartzer potential in global monopole spacetime and a Wu-Yang magnetic monopole [12]. Our conclusion is given in Chapter 5.

Chapter 2

SCHRÖDINGER PARTICLES IN A PGM-SPACETIME BACKGROUND

2.1 Global Monopole Spacetime

One of the intriguing phenomena that plays a significant role in current physics is the global monopole, which arises from the breaking of global O(3) symmetry [20]. Other examples of such phenomena include cosmic strings and domain walls [4, 13–15]. Generally, a global monopole is a topological defect that can form due to the phase changes in the early universe [20]. This topic has been the subject of several notable investigations. Harari and Loust'o discovered that the gravitational potential of topological defects is negative [21].

Einstein's equations can be exactly solved to obtain the PGM spacetime. The metric for PGM spacetime emerges as a fascinating synthesis of gravitational curvature and topological properties [22], capturing the subtle interplay between geometry and topology on cosmic scales. The metric for global monopole spacetime is characterized by features that reflect the gravitational and topological effects caused by these cosmic relics. The presence of a global monopole, in particular, induces anisotropic curved and non-trivial topological structures, which manifest as deviations from the typical flat Minkowski spacetime metric [23].

The metric characterizing global monopole spacetime can be represented as:

$$ds^2 = g_{ab}dx^a dx^b = -dt^2 + B(r)dr^2 + r^2 d\Omega^2 \tag{2.1}$$

where g_{ab} is the metric tensor, r is the radial separation from the PGM's center, and $d\Omega^2$ represents the solid angle. The function $B(r)$ represents the gravitational and topological effects caused by the global monopole, showing the divergence from spherical symmetry and the presence of a deficit angle, which is a characteristic of topological defects. Moreover, the function $B(r)$ exhibits different behaviors in the vicinity of and away from the global monopole center, representing the transition from asymptotically flat spacetime to localized gravitational effects. The metric shows a notable deviation from the Minkowski metric [20, 24] in the central region, indicating robust topological and gravitational effects due to the global monopole. As one moves away from the center, the metric approximates the asymptotically flat form typical of locations far from large objects [25, 26].

Vilenkin and Barriola [20] have found that:

$$B(r) = \frac{1}{1-8\pi G\mu^2} \quad , \tag{2.2}$$

where G is the Newton's gravitational constant and μ is a constant represents the energy scale [24, 25]. We can rewrite the metric as:

$$ds^2 = -dt^2 + \frac{1}{\alpha^2}dr^2 + r^2(d\theta^2 + sin^2\theta \; d\phi^2) \tag{2.3}$$

Here $1 \geq \alpha^2 = 1 - 8\pi G\mu^2 > 0$, α is called the global monopole parameter [24]. Then, the metric tensor g_{ab} can be written as

$$g_{ab} = \begin{pmatrix} \frac{1}{\alpha^2} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 sin^2\theta \end{pmatrix} ; \quad a, b = r,\ \theta, \phi,$$

and it's inverse is

$$g^{ab} = \begin{pmatrix} \alpha^2 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 sin^2\theta}, \end{pmatrix}$$

Essentially, the metric characterizing global monopole spacetime captures the complex interaction between topological defects and gravitational curvature, providing insights into the significant impacts of global monopoles on the evolution of the universe [21, 23]. We can better comprehend the gravitational and topological features that constitute the cosmic environment by clarifying the mathematical formalism and physical consequences associated with this metric [27].

2.2 Schrödinger Particle in a Deformed or Transformed PGM Spacetime

Let's consider Schrödinger particles interact with a PGM spacetime metric as in (2.3) and subjected to a point canonical transformation (PCT) in the form of:

$$r = \int \sqrt{f(\rho)} d\rho = \sqrt{q(\rho)}\rho \tag{2.4}$$

where $f(\rho)$ and $q(\rho)$ are two non-zero positive valued functions that are correlated through

$$\sqrt{f(\rho)} = \sqrt{q(\rho)} \left[1 + \frac{q^{'}(\rho)}{2q(\rho)}\rho \right].$$

To show this correlation, we just take the derivative for both sides of the integral that is given in (2.4). According to this transformation, the PGM metric becomes:

$$ds^2 = -dt^2 + \frac{f(\rho)}{\alpha^2} d\rho^2 + q(\rho)\rho^2 (d\theta^2 + sin^2\theta \; d\phi^2) \tag{2.5}$$

Then, the transformed metric tensor of the global monopole spacetime takes the form:

$$g_{ab} = \begin{pmatrix} \frac{f(\rho)}{\alpha^2} & 0 & 0 \\ 0 & \rho^2 q(\rho) & 0 \\ 0 & 0 & q(\rho)\rho^2 sin^2\theta \end{pmatrix}; \quad a,b = \rho,\theta,\phi \tag{2.6}$$

The determinant of which is:

$$g = |g_{ab}| = \frac{f(\rho)}{\alpha^2} q(\rho)^2 \rho^4 sin^2\theta$$

The inverse metric tensor is :

$$g^{ab} = \begin{pmatrix} \frac{\alpha^2}{f(\rho)} & 0 & 0 \\ 0 & \frac{1}{\rho^2 q(\rho)} & 0 \\ 0 & 0 & \frac{1}{q(\rho)\rho^2 sin^2\theta} \end{pmatrix} \tag{2.7}$$

The general form of the Schrödinger equation in any spacetime can be written as:

$$\left[\left(-\frac{\hbar^2}{2m_0}\frac{1}{\sqrt{g}}\partial_a\sqrt{g}g^{ab}\partial_b\right) + V(\rho)\right]\Psi(\rho,\theta,\phi,t) = i\hbar\,\partial_t\,\Psi(\rho,\theta,\phi,t) \tag{2.8}$$

Here, $V(\rho)$ is the potential energy, and $\Psi(\rho,\theta,\phi,t)$ is the wave function of the particle, which depends on the spacetime coordinates. Equation (2.8) can be extended to give:

$$\left[-\frac{\hbar^2}{2m_0}\frac{1}{\sqrt{g}}\left(\partial_\rho\sqrt{g}g^{\rho\rho}\partial_\rho + \partial_\theta\sqrt{g}g^{\theta\theta}\partial_\theta + \partial_\phi\sqrt{g}g^{\phi\phi}\partial_\phi\right) + V(\rho)\right]\Psi = i\hbar\,\partial_t\Psi.$$

By substituting the elements of the metric tensor of the transformed PGM spacetime in the Schrödinger equation above we get

$$\left[-\frac{\hbar^2}{2m_0}\left(\frac{\alpha^2}{q(\rho)\rho^2\sqrt{f(\rho)}}\partial_\rho\frac{\rho^2 q(\rho)}{\sqrt{f(\rho)}}\partial_\rho + \frac{1}{sin\theta\rho^2 q(\rho)}\partial_\theta sin\theta\partial_\theta\right.\right.$$

$$\left.\left. + \frac{1}{q(\rho)\rho^2 sin^2\theta}\partial_\phi^2\right) + V(\rho)\right]\Psi = i\,\hbar\,\partial_t\Psi$$

Using separation of variable, we use $\Psi(\rho,\theta,\phi,t) = \psi(\rho,\theta,\phi)f(t)$ to get:

$$\left[-\frac{\hbar^2}{2m_0}\left(\frac{\alpha^2}{q(\rho)\rho^2\sqrt{f(\rho)}}\partial_\rho\frac{\rho^2 q(\rho)}{\sqrt{f(\rho)}}\partial_\rho\psi + \frac{1}{sin\theta\rho^2 q(\rho)}\partial_\theta sin\theta\partial_\theta\psi\right.\right.$$

$$\left.\left.+\frac{1}{q(\rho)\rho^2 sin^2\theta}\partial^2_\phi\psi\right) + V(\rho)\psi\right]\frac{1}{\psi} = i\,\hbar\;\frac{1}{f(t)}f'(t)$$

Note that the R.H.S. of this equation is a function of t only, and the L.H.S. is a function of ρ, θ and ϕ only. Thus, the two sides must be equal to the same constant, which we will call E. Then we have:

$$f^{'}(t) = \frac{-iE}{\hbar}f(t) \Rightarrow f(t) = e^{-\frac{iEt}{\hbar}} \tag{2.9}$$

and

$$\left[-\frac{\hbar^2}{2m_0}\left(\frac{\alpha^2}{q(\rho)\rho^2\sqrt{f(\rho)}}\partial_\rho\frac{\rho^2 q(\rho)}{\sqrt{f(\rho)}}\partial_\rho\psi\right.\right.$$

$$\left.\left.+\frac{1}{sin\theta\rho^2 q(\rho)}\partial_\theta sin\theta\partial_\theta\psi + \frac{1}{q(\rho)\rho^2 sin^2\theta}\partial^2_\phi\psi\right) + V(\rho)\psi\right] = E\psi. \tag{2.10}$$

Next, for equation (2.10) we assume that $\psi(\rho,\theta,\phi) = \psi(\rho)Y(\theta,\phi)$, to obtain:

$$\left[-\frac{\hbar^2}{2m_0}\left(\frac{\alpha^2}{\rho^2 q(\rho)\sqrt{f(\rho)}\psi(\rho)}\partial_\rho\frac{\rho^2 q(\rho)}{\sqrt{f(\rho)}}\partial_\rho\psi(\rho)\right.\right.$$

$$\left.\left.+\frac{1}{sin\theta\rho^2 q(\rho)Y}\partial_\theta sin\theta\partial_\theta Y + \frac{1}{q(\rho)\rho^2 sin^2\theta Y}\partial^2_\phi Y\right) + V(\rho)\right] = E \tag{2.11}$$

Multiplying by $\frac{-2m_0 q(\rho)\rho^2}{\hbar^2}$, from the left, we get:

$$\frac{1}{\psi(\rho)}\left[\frac{\alpha^2}{\sqrt{f(\rho)}}\partial_\rho\frac{\rho^2 q(\rho)}{\sqrt{f(\rho)}}\partial_\rho\psi(\rho) - \frac{2m_0 q(\rho)\rho^2}{\hbar^2}(V(\rho)-E)\right]$$

$$+\frac{1}{Y}\left[\frac{1}{sin\theta}\partial_\theta sin\theta\partial_\theta Y + \frac{1}{sin^2\theta}\partial^2_\phi Y\right] = 0$$

The term in the first square brackets depends on ρ only. but the remainder depends on θ and ϕ only. Thus, each must be constant. we will write the separation constant in the form $l(l+1)$, so we have:

$$\frac{1}{Y}\left[\frac{1}{sin\theta}\partial_\theta sin\theta\partial_\theta Y+\frac{1}{sin^2\theta}\partial_\phi^2 Y\right]=-l(l+1) \tag{2.12}$$

and

$$\frac{1}{\psi(\rho)}\left[\frac{\alpha^2}{\sqrt{f(\rho)}}\partial_\rho\frac{\rho^2 q(\rho)}{\sqrt{f(\rho)}}\partial_\rho\psi(\rho)-\frac{2m_0 q(\rho)\rho^2}{\hbar^2}(V(\rho)-E)\right]=l(l+1) \tag{2.13}$$

Equation (2.12) represents the dependence of $\Psi(r,\theta,\phi)$ on θ and ϕ , and multiplying it from the left by $-\hbar^2 Y$, it becomes:

$$-\hbar^2\left[\frac{1}{sin\theta}\partial_\theta sin\theta\partial_\theta Y+\frac{1}{sin^2\theta}\partial_\phi^2 Y\right]=\hbar^2 l(l+1)Y \tag{2.14}$$

The given equation is the angular momentum equation, which may be expressed as $L^2 Y_{lm}(\theta,\phi)=\hbar^2 l(l+1)Y_{lm}(\theta,\phi)$. Where $\hat{L}$ is the angular momentum operator in spherical coordinates. The spherical harmonics, denoted as $Y_{lm}(\theta,\phi)$, represent a set of functions used in the study of spherical symmetry. The angular momentum quantum number is represented by l.

According to our PCT (2.4) we have $\partial_\rho=\left(\frac{dr}{d\rho}\right)\partial_r=\left(\sqrt{f(\rho)}\right)\partial_r$, and $r^2=q(\rho)\rho^2$, so equation (2.13) in terms of r is now given by:

$$\left[\alpha^2\partial_r r^2\partial_r\psi(r(\rho))-\frac{2m_0 r^2}{\hbar^2}(V(r(\rho))-E)\ \psi(r(\rho))\right]=l(l+1)\ \psi(r(\rho))$$

If we multiply by $\frac{-\hbar^2}{2\ m_0\ r^2\ \alpha^2}$, we obtain

$$\left[-\frac{\hbar^2}{2m_0}\frac{1}{r^2}\ \partial_r\ r^2\ \partial_r\ \psi(r)+\frac{1}{\alpha^2}(V(r)-E)\,\psi(r)\right]=-\frac{\hbar^2}{2m_0}\frac{l(l+1)}{\alpha^2\ r^2}\ \psi(r)$$

This can be written in the form:

$$\frac{\hbar^2}{2m_0}\left(-\frac{1}{r^2}\ \partial_r\ r^2\ \partial_r\ \psi+\frac{l(l+1)}{\alpha^2\ r^2}\ \psi\right)+\frac{1}{\alpha^2}V(r)\ \psi=\frac{1}{\alpha^2}E\ \psi$$

Now, by substituting $\frac{E}{\alpha^2}=\mathscr{E}$ and $\frac{l\ (l+1)}{\alpha^2}=\beta\ (\beta+1)$, last equation becomes:

$$\frac{\hbar^2}{2m_0}\left(-\frac{1}{r^2}\ \partial_r\ r^2\ \partial_r\ \psi+\frac{\beta\ (\beta+1)}{r^2}\ \psi\right)+\frac{1}{\alpha^2}V(r)\ \psi=\mathscr{E}\ \psi \tag{2.15}$$

This can be simplified using $\psi(r) = \frac{U(r)}{r}$ to obtain :

$$\frac{\hbar^2}{2m_0}\left(-\,\partial_r^2\, U(r) + \frac{\beta\ (\beta+1)}{r^2}\ U(r)\right) + \frac{1}{\alpha^2}V(r)\ U(r) = \mathscr{E}\ U(r) \quad (2.16)$$

To express β in terms of l and α, we use the definition $\frac{l\ (l+1)}{\alpha^2} = \beta\ (\beta+1)$

$$\beta = -\,\frac{1}{2} + \frac{\sqrt{\alpha^2 + 4l\ (l+1)}}{2\alpha}$$

This equation calculates the value of the regular angular momentum quantum number l for flat Minkowski spacetime when $\alpha = 1$. Furthermore, the two quantum mechanical systems described in equations (2.13) and (2.15) have the same spectrum and remain unchanged. In other words, if you know the solution to one of them, you can quickly determine the solution to the other. However, they both possess identical energy.

It should be noted that the angular component of the wave function $Y(\theta,\phi)$, remains constant for all spherically symmetric potentials. The specific form of the potential, $V(r)$, only influences the radial component of the wave function, $\psi(\rho)$, which is determined by Equation 2.15, Thus $\psi(r,\theta,\phi,t) = \ \psi(r)\ Y_{lm}(\theta,\phi)\ e^{-i\ Et/\hbar}$.

2.3 Position-Dependent Mass in the Radially Transformed PGM Spacetime Metric

When dealing with situations in which the particle's mass changes depending on its position, it is necessary to adjust the kinetic energy operator in (1.4) to accommodate this variability. In Chapter 1, it has been shown that by choosing MM-ordering, $f = h = -\frac{1}{4}$ and $g = -\frac{1}{2}$, the PDM von-Roos kinetic energy operator takes the form of Mustafa-Mazharimousavi's PDM one, $\hat{T}_{MM}$ given in (1.5) [8,9].

We begin with recalling the radial equation in (2.16):

$$\frac{\hbar^2}{2m_0}\left(-\,\partial_r^2\, U(r) + \frac{\beta\ (\beta+1)}{r^2}\ U(r)\right) + \frac{1}{\alpha^2}V(r)\ U(r) = \mathscr{E}\ U(r)$$

and substituting:

$$U(r) = U(r(\rho)) = f(\rho)^{-\frac{1}{4}}\ \phi(\rho) \tag{2.17}$$

Moreover, our PCT $r = \int \sqrt{f(\rho)}d\rho$ allows us to write $\partial_r U(r)$ as

$$\partial_r U(r) = \left(\frac{d\rho}{dr}\right)\ \partial_\rho U = f(\rho)^{-\frac{1}{2}}\ \partial_\rho \left(f(\rho)^{-\frac{1}{4}}\ \phi(\rho)\right)$$

and

$$\begin{aligned}
\partial_r^2 U(r) &= \partial_r \left(\partial_r U\right) \\
&= \partial_r \left[f(\rho)^{-\frac{1}{2}}\ \partial_\rho \left(f(\rho)^{-\frac{1}{4}}\ \phi(\rho)\right)\right] \\
&= \frac{d\rho}{dr}\ \partial_\rho \left[f(\rho)^{-\frac{1}{2}}\ \partial_\rho \left(f(\rho)^{-\frac{1}{4}}\ \phi(\rho)\right)\right] \\
&= f(\rho)^{-\frac{1}{2}}\ \left(\partial_\rho \left[f(\rho)^{-\frac{1}{2}}\ \partial_\rho \left(f(\rho)^{-\frac{1}{4}}\ \phi(\rho)\right)\right]\right),
\end{aligned}$$

to obtain

$$\begin{aligned}
\left(-\frac{\hbar^2}{2m_0}\ f(\rho)^{-\frac{1}{2}}\ \partial_\rho\ f(\rho)^{-\frac{1}{2}}\ \partial_\rho\ +\frac{\hbar^2}{2m_0}\ \frac{\beta\ (\beta+1)}{q(\rho)\ \rho^2}\ +\frac{1}{\alpha^2}V(r)\right)\ f(\rho)^{-\frac{1}{4}}\ \phi(\rho) \\
= \mathscr{E}\ f(\rho)^{-\frac{1}{4}}\ \phi(\rho)
\end{aligned} \tag{2.18}$$

Next, we multiply this equation, from the left, by $f(\rho)^{\frac{1}{4}}$ and get:

$$\left(-\frac{\hbar^2}{2m_0}\ f(\rho)^{-\frac{1}{4}}\ \partial_\rho\ f(\rho)^{-\frac{1}{2}}\ \partial_\rho\ f(\rho)^{-\frac{1}{4}} + V_{eff}(\rho)\right)\ \phi(\rho) = \mathscr{E}\ \phi(\rho) \tag{2.19}$$

Where the effective potential V_{eff} is now given by:

$$V_{eff} = \frac{\hbar^2}{2m_0}\ \frac{\beta\ (\beta+1)}{q(\rho)\ \rho^2}\ + \frac{1}{\alpha^2}V(\rho)$$

that includes the central repulsive term. The particle is often pushed far from the origin, similar to the repulsive force seen in classical mechanics. Thus, the resulting operator for the effective PDM kinetic energy is:

$$\hat{T}_{MM} = -\frac{\hbar^2}{2m_0}\ f(\rho)^{-\frac{1}{4}}\ \partial_\rho\ f(\rho)^{-\frac{1}{2}}\ \partial_\rho\ f(\rho)^{-\frac{1}{4}} \tag{2.20}$$

Which is known as Mustafa-Mazharimousavi's (MM) PDM-kinetic- energy operator [8] Then, the two systems, identified in (2.16) and (2.19), exhibit isospectrality, meaning they possess identical energy levels and remain unchanged under certain transformations.

Chapter 3

PDM SCHRÖDINGER PARTICLE IN A KRATZER POTENTIAL IN A PGM SPACETIME

3.1 Kratzer Potential

Molecular potentials are used to describe the interactions between atoms or molecules. They provide a foundation for studying the behaviour and properties of matter on a molecular scale. They are essential in the field of physics, allowing us to explore the intricacies of molecular behaviour and acquire crucial knowledge about the fundamental forces that control matter. Some examples of various molecular potentials and their usage also include Yukawa potential [35–38, 40], Morse potential [44–49], Hulthén potential [43, 50–54], and Pöschl-Teller potential [55–60]. The Kratzer potential [41], which we are going to study, demonstrates a distinctive property of tending towards infinity as the distance between nuclei collapses to zero [61, 62]. This behaviour is a result of the repulsive interactions that exist between the individual molecules within the potential.

The Kratzer potential is defined as [41, 63]:

$$V_K(r) = -2D\left(\frac{A}{r} - \frac{A^2}{2r^2}\right) \tag{3.1}$$

where D is the energy required to separate two atoms in a diatomic molecule, when they are at an equilibrium inter-nuclear distance A. Both A and D are positive values. The potential under consideration is a combination of the attractive Coulomb-like term potential at long distances and a repulsive centrifugal like barrier (the second term) at small distances. Their combination of which introduces an effective potential well, which functions as a model for studying the molecular structure, molecular energy spectra, chemical interactions, and inter-nuclear vibration of diatomic molecules.

3.2 PDM Schrödinger Particle with Kratzer Potential and in a PGM Spacetime Background

We have shown, in the previous chapter, that the radial the Schrödinger equation takes the form of (2.16). If we substitute the Kratzer potential in (2.16) we obtain:

$$\frac{\hbar^2}{2m_0}\left(-\,\partial_r^2\,U(r)+\frac{\beta\,(\beta+1)}{r^2}\,U(r)\right)-\frac{2D}{\alpha^2}\left(\frac{A}{r}-\frac{A^2}{2r^2}\right)U(r)=\mathscr{E}\,U(r) \tag{3.2}$$

We aim to find the solution for $U(r)$, and determine the allowed energies. To simplify this equation we will use $\hbar=2m_0=1$ it becomes:

$$\left[-\,\partial_r^2\,+\frac{\beta\,(\beta+1)}{r^2}\,-\frac{2D}{\alpha^2}\left(\frac{A}{r}-\frac{A^2}{2r^2}\right)\right]U(r)=\mathscr{E}\,U(r) \tag{3.3}$$

We can rewrite it as:

$$\left(-\,\partial_r^2\,+\frac{L\,(L+1)}{r^2}-\frac{\omega}{r}\right)U(r)=\mathscr{E}\,U(r) \tag{3.4}$$

where $\omega=2DA/\alpha^2$ and

$$L(L+1)=\beta\,(\beta+1)+\frac{DA^2}{\alpha^2}=\frac{l(l+1)+DA^2}{\alpha^2}.$$

Thus:

$$L=-\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{l(l+1)+DA^2}{\alpha^2}}$$

Next, we define a new variable $z=\sqrt{-\mathscr{E}}r$ so that:

$$\left(\,\partial_z^2\,-\frac{L\,(L+1)}{z^2}-\frac{\tilde{\omega}}{z}-1\right)U=0 \tag{3.5}$$

where $\tilde{\omega}=\frac{\omega}{\sqrt{-\mathscr{E}}}$.

Next, we will study the asymptotic behavior. As z goes to infinity the constant term dominates, so:

$$\partial_z^2 \, U = U \tag{3.6}$$

The general solution is: $U = A_1 \, e^{-z} + B_1 \, e^{z}$ but e^{z} is unbounded as z goes to infinity, so B must be zero. Thus, for large z we have:

$$U = A_1 \, e^{-z}$$

On the other hand, as z goes to zero the second term in equation 3.5 dominates, then:

$$\partial_z^2 \, U - \frac{L\,(L+1)}{z^2} \, U = 0 \tag{3.7}$$

The general solution is

$$U = C \, z^{L+1} + D \, z^{-L}$$

But U blows up as z goes to zero , so $D = 0$, Thus, for small z:

$$U = C \, z^{L+1}$$

Finally, we can write $U(z)$ as:

$$U(z) = \; z^{L+1} \; e^{-z} \, F(z)$$

In the hope that the new function $F(z)$ will simplify the equation.

$$\partial_z \, U = z^L \, e^{-z} \left[(L+1-z) \, F + z \, \partial_z \, F \right]$$

and

$$\partial_z^2 \, U = \; z^L \, e^{-z} \left[\left(-2L - 2 + z + \; \frac{L(L+1)}{z} \right) \; F + 2(L+1-z) \, \partial_z \, F + z \, \partial_z^2 \, F \right]$$

Then the radial equation in terms of $F(z)$ becomes:

$$z \, \partial_z^2 F + 2(L+1-z) \, \partial_z F + [\tilde{\omega} - 2(L+1)] \; F = 0 \tag{3.8}$$

Equation (3.8) is a confluent hypergeometric differential equation with the confluent hypergeometric function/series solution ${}_1F_1(a,b,x)$:

$$F(z) = \; C_{nl} \; {}_1F_1 \left(L - \frac{\tilde{\omega}}{2} + 1, \; 2 + 2L, 2z \right) \tag{3.9}$$

If this solution is to be a physically acceptable one, it should be finite and square integrable. This would once again necessitate that the confluent hypergeometric series be terminated to a polynomial of degree $n = 0, 1, 2, 3...$ so that:

$$L - \frac{\tilde{\omega}}{2} + 1 = -n \tag{3.10}$$

We can easily solve it to get the value of $\mathscr{E}$

$$\mathscr{E} = -\left(\frac{\omega}{\tilde{\omega}}\right)^2 = -\frac{\omega^2}{4(1+L+n)^2} \tag{3.11}$$

which implies the energy levels:

$$E_{n,l} = -\frac{\omega^2\alpha^2}{4(1+L+n)^2} \tag{3.12}$$

Then the solution is written as:

$$\psi_{nl}(z) = C_{nl}\ z^{L+1}\ e^{-z}\ {}_1F_1\left(L - \frac{\tilde{\omega}}{2} + 1,\ 2+2L, 2z\right) \tag{3.13}$$

Obviously, equation (3.12) suggests that the energies tend to converge to $E_{n,l} \sim -D\alpha^2$ as the coupling constant $A \to \infty$. This would indicate that all energy levels cluster at the value $E_{n,l} \sim -D\alpha^2$ as $A \to \infty$. Such a tendency is observed in Figures 3.1(a) and 3.1(b), where the energies $E_{n,l} \sim -1/4$ for the parametric values used therein.

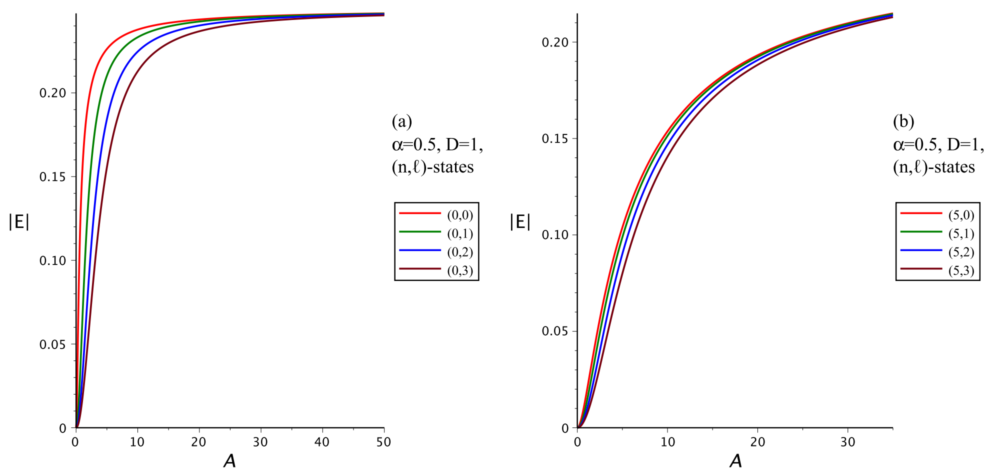

Figure 3.1: The energy levels for (n,l)-states against the coupling constant A for $l = 0, 1, 2, 3$ so that Fig. 3.1(a) for $n = 0$, and Fig. 3.1(b) for n=5.

In Figure 3.2, we show the energies $|E_{n,l}|$ against different values of the monopole parameter and within the range $0 < \alpha \leq 1$. The comparison between Figures 3.2(a), (b), and (c) suggests that the energies $|E_{n,l}|$ decrease as the radial quantum number increases. Indicating a general tendency of the energy levels towards the value $E_{n,l} \to -1/n^2$ for $n \gg 1$. This trend can be clearly observed in Fig. 3.2 (d). Moreover, the spacing between the energy levels narrows as the radial quantum number increases.

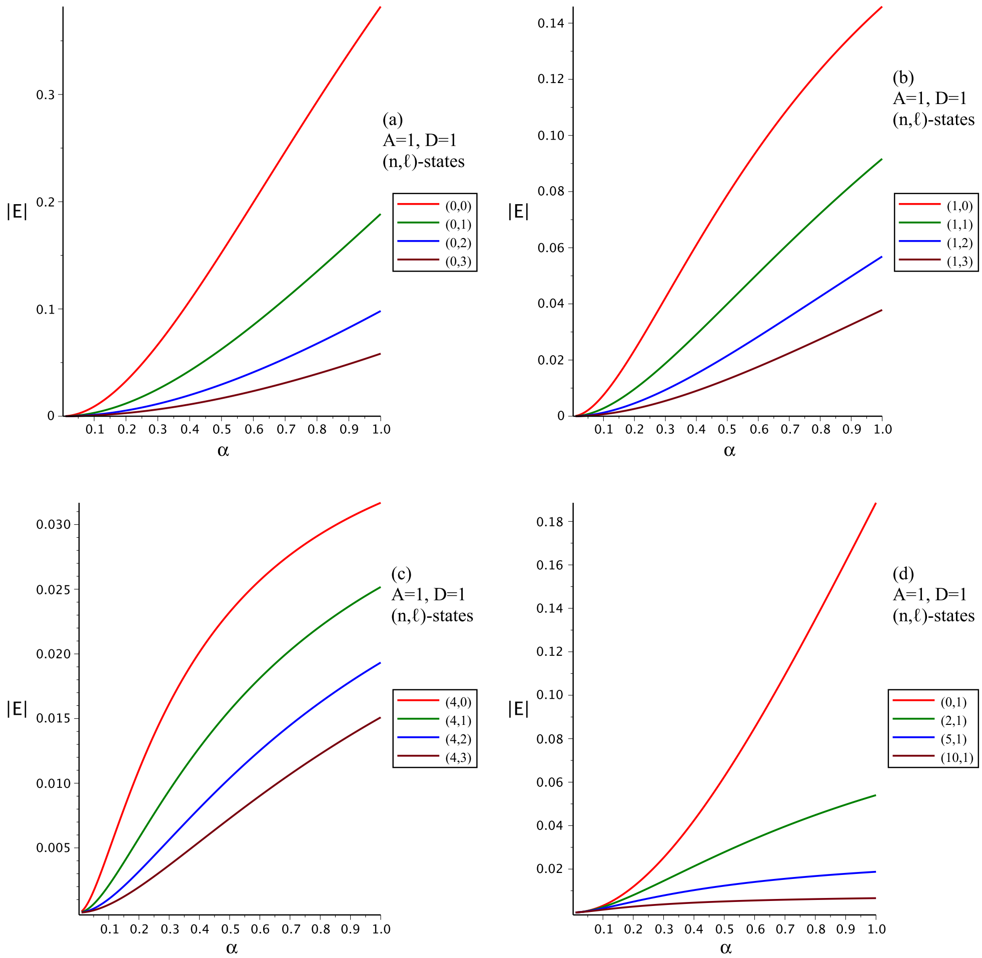

Figure 3.2: The energy levels for (n,l)-states against the monopole parameter α for $l = 0,1,2,3$ so that Fig. 3.2(a) for $n = 0$, Fig. 3.2(b) for n=1, Fig. 3.2(c) for n=4, and Fig. 3.2(d) for $l = 1$ and $n = 0,2,5,10$.

Chapter 4

PDM SCHRÖDINGER PARTICLES WITH KRATZER POTENTIAL IN A PGM SPACETIME AND A WU-YANG MAGNETIC MONOPOLE

4.1 Wu- Yang Magnetic Monopole

The choice of the vector potential $\mathbf{A} = (0, 0, A_\phi)$ around a global monopole necessarily involves singularities. This situation is analogous to selecting a coordinate system on the surface of a sphere, such as longitude and latitude. Even though the sphere itself is free of intrinsic singularities, any coordinate system will have some singularities. To avoid introducing singularities, one can divide the sphere into overlapping regions and define a coordinate system without singularities in each region [64–66].

Following this strategy, Wu and Yang divided the space outside a magnetic monopole into two regions, R_A and R_B.They defined a vector potential A_A in R_A and a vector potential A_B in R_B, covering the entire space exterior to the magnetic monopole. These regions overlap in R_{AB} with the monopole at the origin as they have chosen [64]

$$R_A : 0 \leq \theta < \frac{\pi + 2\,\eta}{2} \qquad , r > 0, \quad 0 \leq \phi < 2\pi$$

$$R_B : \frac{\pi \_ 2\eta}{2} < \theta \leq \pi \qquad , r > 0, \quad 0 \leq \phi < 2\pi$$

$$R_{AB} : \frac{\pi - 2\eta}{2} < \theta \leq \frac{\pi + 2\eta}{2} \quad , r > 0, \quad 0 \leq \phi < 2\pi$$

Where $0 < \eta \leq \frac{\pi}{2}$. The vector potential components are chosen to be:

$$A_{r,\,A} = A_{\theta,\,A} = 0\;, \qquad A_{\phi,A} = \; g\;(1 - cos\theta) \tag{4.1}$$

$$A_{r,\,B} = A_{\theta,\,B} = 0\;, \qquad A_{\phi,B} = \; -g\;(1 + cos\theta) \tag{4.2}$$

where g is the strength of the Wu-Yang magnetic monopole. To simplify the notation and make it quite handy, we use the form $A_\phi \;=\; sg - g\; cos\theta$, such that $s = 1$ for $A_{\phi,A}$ and $s = -1$ for $A_{\phi,B}$.

4.2 PDM Schrödinger Equation with Kratzer Potential and Wu-Yang Magnetic Monopole

The Schrödinger equation with potential $V(r)$, in a vector potential $\mathbf{A} = (0,0,A_\phi)$ is given by:

$$\left[\left(-\frac{1}{\sqrt{g}}\,(\partial_a - ieA_a)\,\sqrt{g}g^{ab}\,(\partial_b - ieA_b)\right) + V(r)\right]\Psi(r,\theta,\phi,t) = i\partial_t\Psi(r,\theta,\phi,t) \tag{4.3}$$

Equation (4.3) can be simplified :

$$\Bigg[-\frac{1}{\sqrt{g}}\Big((\partial_r \;- i\,e\,A_r)\,\sqrt{g}g^{rr}\,(\partial_r \;-\; i\,eA_r)\; + (\partial_\theta \;- i\,e\,A_\theta)\,\sqrt{g}g^{\theta\theta}\,(\partial_\theta \;-\; i\,eA_\theta)$$

$$+ \left(\partial_\phi \;- i\,e\,A_\phi\right)\sqrt{g}g^{\phi\phi}\left(\partial_\phi \;-\; i\,eA_\phi\right)\Big) + V(r)\Bigg]\Psi(r,\theta,\phi,t) = i\;\;\partial_t\;\Psi(r,\theta,\phi,t)$$

By substituting the elements of the metric tensor of global monopole spacetime and the Wu-Yang vector potential the Schrödinger equation becomes:

$$[-\frac{\alpha^2}{r^2}\,\partial_r(r^2\partial_r) \;-\; \frac{1}{r^2}\left(\frac{1}{sin\theta}\;\partial_\theta sin\theta\partial_\theta \;+\; \frac{1}{sin^2\theta}\;\left[\partial_\phi \;- i\,e\,A_\phi\right]^2\right)$$

$$+V(r)\,]\,\Psi(r,\theta,\phi,t) = i\;\;\partial_t\;\Psi(r,\theta,\phi,t) \tag{4.4}$$

Where r is given by (2.4). We will use the separation of variables and assume that $\Psi(r,\theta,\phi,t) = \psi(r)\;Y_{\tilde{q}lm}(\theta,\phi)\;e^{-\,i\,Et}$, where $q = eg \geq 0$ and $\tilde{q} = sq$. Then, the time-independent Schrödinger equation becomes:

$$\frac{1}{\psi(r)}\;\left[-\;\alpha^2\;\partial_r(r^2\partial_r) + \;r^2(V(r) - E)\right]\;\psi(r)$$
$$-\;\frac{1}{Y_{\tilde{q}lm}(\theta,\phi)}\left(\frac{1}{sin\theta}\;\partial_\theta sin\theta\partial_\theta \;+\; \frac{1}{sin^2\theta}\;\left[\partial_\phi \;- i\,e\,A_\phi\right]^2\right)Y_{\tilde{q}lm}(\theta,\phi) = 0$$

The terms in the first square brackets depends only on r, whereas the remainder depends on θ *and* ϕ. Hence, we may take:

$$\left(\frac{1}{sin\theta}\,\partial_\theta sin\theta\partial_\theta\;+\;\frac{1}{sin^2\theta}\,\left[\partial_\phi\;-i\,e\,A_\phi\right]^2\right)Y_{\tilde{q}lm}(\theta,\phi)\;=\;-\,\lambda\;Y_{\tilde{q}lm}(\theta,\phi) \tag{4.5}$$

and

$$\left[-\,\frac{\alpha^2}{r^2}\,\partial_r(r^2\partial_r)+\;\frac{\lambda}{r^2}\;+\;V(r)\right]\;\psi(r)=\;E\psi(r) \tag{4.6}$$

Now, we have to find the values of λ by using the substitution $Y_{\tilde{q}lm}(\theta,\phi)=e^{i\;(m+\tilde{q})\;\phi}\;\mathscr{G}_{qlm}(\theta)$ into Equation (4.6) to obtain:

$$\left(\frac{1}{sin\theta}\,\partial_\theta sin\theta\partial_\theta\;-\;\frac{1}{sin^2\theta}\,\left[m\;+\;q\;cos\theta\right]^2\right)\,\mathscr{G}_{qlm}(\theta)\;=\;-\lambda\;\mathscr{G}_{qlm}(\theta) \tag{4.7}$$

The above equation is independent of the values of s, thus the solution for $\mathscr{G}_{qlm}(\theta)$ are the same in the two regions A and B. This means

$$\left[\mathscr{G}_{qlm}(\theta)\right]_A\;=\;\mathscr{G}_{qlm}(\theta)=\;\left[\mathscr{G}_{qlm}(\theta)\right]_B.$$

To solve Equation 4.7 we are going to use the substitution $x=\cos\theta$, it becomes:

$$\left(\left[1-x^2\right]\,\partial_x^2\;-\;2x\,\partial_x\;-\;\frac{\left[m\;+\;q\;x\right]^2}{1-x^2}\right)\,\mathscr{G}_{qlm}(x)\;=\;-\lambda\;\mathscr{G}_{qlm}(x) \tag{4.8}$$

This differential equation has singularities at $x=1$ *and* $x=\;-1$. We define now

$$\mathscr{G}_{qlm}(x)\;=\;(1+x)^{a/2}\,(1-x)^{b/2}\;\;P_{qlm}(x) \tag{4.9}$$

where $a=\;(|m|-q)$ *and* $b=\;(|m|+q)$.

By using the substitution (4.9) Equation (4.8) becomes:

$$(x^2-1)\,P''_{qlm}(x)\;+\;2[q+(m+1)x]\,P'_{qlm}(x)\;+\;(m+m^2-\lambda-q^2)\,P_{qlm}=0 \tag{4.10}$$

The solution of the above equation is the hypergeometric functions/series:

$$P_{qlm}=\;C_1\;F_1\left(|m|+\frac{1}{2}\pm\frac{1}{2}\sqrt{4\lambda+4q^2+1},\;|m|-q+1,\;\frac{1}{2}(1+x)\right) \tag{4.11}$$

For the sake of having a finite and square integrable wave functions, the power series of the hypergeometric function must be terminated, thus:

$$-\tilde{n} = |m| + \frac{1}{2} \pm \frac{1}{2}\sqrt{4q^2 + 4\lambda + 1}; \qquad \tilde{n} = 0, 1, 2, 3...$$

We solve this equation to get λ:

$$\lambda = (|m| + \tilde{n} + 1)(|m| + \tilde{n}) - q^2 = l(l+1) - q^2 \tag{4.12}$$

Where $l = |m| + \tilde{n} = 0, 1, 2, 3....$ is the angular momentum quantum number. That is, when $q = 0$ (this means no Wu-Yang magnetic monopole) then $\lambda = l(l+1)$ represents the regular eigenvalues for the textbook spherical harmonics. Now, we can write the solution Y_{qlm} as:

$$Y_{qlm}(\theta, \phi) = \begin{cases} (1+\cos\theta)^{\frac{a}{2}}(1-\cos\theta)^{\frac{b}{2}} P_{qlm}(\cos\theta) e^{i(m+q)\phi} & \text{,in region A} \\ (1+\cos\theta)^{\frac{a}{2}}(1-\cos\theta)^{\frac{b}{2}} P_{qlm}(\cos\theta) e^{i(m-q)\phi} & \text{,in region B} \end{cases} \tag{4.13}$$

Wu and Yang have called Y_{qlm} as the monopole harmonics.

Now, we are going to write the radial equation 4.6, with the Kratzer potential $V_K(r) = -2D\left(\frac{A}{r} - \frac{A^2}{2r^2}\right)$ and $\lambda = l(l+1) - q^2$. Then the radial equation becomes:

$$\left[-\frac{\alpha^2}{r^2}\,\partial_r(r^2\partial_r) + \frac{l(l+1)-q^2}{r^2} - 2D\left(\frac{A}{r} - \frac{A^2}{2r^2}\right)\right]\psi(r) = E\psi(r) \tag{4.14}$$

We may rewrite it as:

$$\left[-\frac{1}{r^2}\,\partial_r(r^2\partial_r) + \frac{\zeta(\zeta+1)}{r^2} - \frac{\omega}{r}\right]\psi(r) = \mathscr{E}\psi(r) \tag{4.15}$$

where $\mathscr{E} = \frac{E}{\alpha^2}$, $\omega = \frac{2DA}{\alpha^2}$ and $\zeta(\zeta+1) = \frac{l(l+1)-q^2+DA^2}{\alpha^2}$. If we solve it for ζ we can easily get

$$\zeta = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{A^2D + l(l+1) - q^2}{\alpha^2}} \tag{4.16}$$

Furthermore, the solution to Equation (4.15) would read:

$$\psi(\rho(r)) = C_{nl}\, r^{\zeta} e^{-\sqrt{-\mathscr{E}}r}\; {}_1F_1\!\left(1+\zeta - \frac{\omega}{2\sqrt{-\mathscr{E}}},\, 2+2\zeta, 2\sqrt{-\mathscr{E}}r\right) \tag{4.17}$$

Hereby, the condition that the wavefunction be finite and square integrable would necessitate that the confluent hypergeometric series be terminated to a polynomial of degree $n \geq 0$ to obtain:

$$1+\zeta-\frac{\omega}{2\sqrt{-\mathscr{E}}} = -n \tag{4.18}$$

We can easily solve it to get the value of $\mathscr{E}$

$$\mathscr{E} = -\frac{\omega^2}{4(1+\zeta+n)^2} \tag{4.19}$$

Which implies the energy levels:

$$E_{n,l,q} = -\frac{\omega^2\alpha^2}{4(1+\zeta+n)^2} \tag{4.20}$$

At this point, one should notice that such energy levels collapse into those without the Wu-Yang magnetic monopole (i.e., $q=0$) given in Eq.(3.12). In Figures 4.1(a) and (b), we show the energies $|E_{n,l,q}|$ at $\alpha = 0.5$ for different values of the strengths q of the Wu-Yang magnetic monopole. The values of q are chosen so that the square root in Eq. (4.16) is a real value. We observe that the Wu-Yang magnetic monopole yields non-equally spaced energy levels. Additionally, the spacing between the energy levels increases as the strength of the Wu-Yang magnetic monopole increases. Moreover, the spacing between the energy levels decreases as the radial quantum number increases. Furthermore, in Figures 4.1 (c) and (d) we plot the energy levels against the coupling parameter A for $\alpha = 0.5$ and $\alpha = 0.1$, respectively at $q = 1$. It is clear that as the value of the coupling parameter A increases, the energy values increase until reaching their maximum. This maximum energy value depends on the value of α. This tendency is evident from equation (4.20), which suggests that the energies tend to converge to $E_{n,l} \to -D\alpha^2$ as the coupling constant $A \to \infty$.

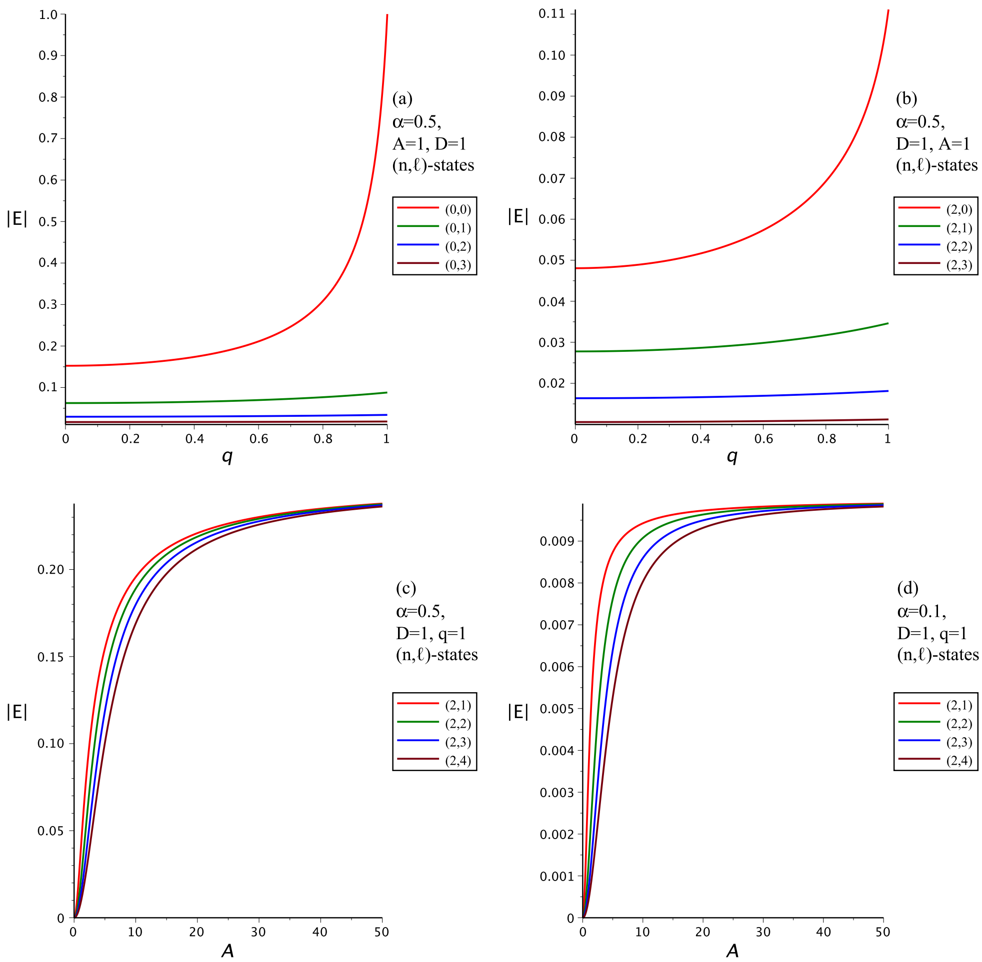

Figure 4.1: The energy levels for (n,l)-states against the WY-magnetic monopole parameter q for $D = 1 = A$ so that Fig. 4.1(a) for $n = 0$ and $\alpha = 0.5$, Fig. 4.1(b) for $n = 2$ and $\alpha = 0.5$, and the energy levels against the coupling parameter A for $q = 1 = D$ so that Fig. 4.1(c) for $n = 2$, $\alpha = 0.5$, and Fig. 4.1(d) for $n = 2$, $\alpha = 0.1$

Chapter 5

CONCLUSION

This work demonstrates that a particular transformation (2.4) of a global monopole spacetime yields a von Roos position-dependent mass Schrödinger equation (2.19). Additionally, we have solved the Schrödinger equation with Kratzer potential, a significant molecular potential describing the vibration-rotation spectra of diatomic molecules in global monopole spacetime, to obtain the energy levels spectrum. Here, the radial wave equation is derived in terms of the confluent hypergeometric function ${}_1F_1(a,b,x)$, and the angular part is expressed in terms of spherical harmonics.

Subsequently, we examined the Shrödinger equation with the Kratzer potential and Wu-Yang magnetic monopole in global monopole spacetime. Here, the radial wave function is still given in terms of the confluent hypergeometric function, but the angular part is expressed in terms of the Wu-Yang monopole harmonics.

Regarding the energy levels, when we plot the energy levels resulting from solving the radial part of the Shrödinger equation with the Kratzer potential equation (3.3) against the global monopole parameter α for all allowed values of α (i.e $1 \geq \alpha > 0$), we observe that the energies $|E_{n,l}|$ decrease as the radial quantum number increases (documented in Figures 3.2(a), and (b)). Moreover, in Figure 3.2(d), we notice that the spacing between the energy levels narrows as the radial quantum number n increases.

Next, we plot the energy levels resulting by solving the radial part of the Shrödinger equation with Kratzer potential and Wu-Yang magnetic (equation 4.15) against the strength q of the Wu-Yang magnetic at $\alpha = 0.5$. We observe that the energy levels are not equally spaced. Additionally, the gap between the energy levels widens as the strength of the Wu-Yang magnetic monopole increases. Furthermore, the gap between the energy levels narrows as the radial quantum number grows as shown in Figure 4.1(a) and (b).

When we plot the energies against the parameter A as shown in Figure 4.1 (c) and (d) for $\alpha = 0.5$ and $\alpha = 0.1$, respectively, at a constant q (we set q=1). We observe that as the coupling parameter A increases, the energies also increase until they reach their maximum value. Additionally, the highest possible value of the energy depends on the value of α. This trend is evident from equation (4.20), indicating that the energies approach convergence towards $E_{n,l} \to -D\alpha^2$ as the coupling constant $A \to \infty$.

Overall, this study improves our comprehension of the substantial impacts on the movement of charged particles subjected to topological defects and interacting via the Kratzer potential, with or without Wu-Yang magnetic monopole, specifically within the framework of a point-like global monopole.

REFERENCES

[1] Messiah, Albert. Quantum mechanics. Courier Corporation, 2014.

[2] Corben, Herbert Charles, and Philip Stehle. Classical mechanics. Courier Corporation, 2013.

[3] Mustafa, Omar. "Comment on 'Nonlinear dynamics of a position-dependent mass-driven Duffing-type oscillator'." Journal of Physics A: Mathematical and Theoretical 46.36 (2013): 368001.

[4] Mustafa, Omar. "Schrödinger oscillators in a deformed point-like global monopole spacetime and a Wu–Yang magnetic monopole: Position-dependent mass correspondence and isospectrality." Annals of Physics 459 (2023): 169550

[5] BenDaniel, D. J., and C. B. Duke. "Space-charge effects on electron tunneling." Physical review 152.2 (1966): 683.

[6] Koç, Ramazan, Mehmet Koca, and G. Şahinoğlu. "Scattering in abrupt heterostructures using a position dependent mass Hamiltonian." The European Physical Journal B-Condensed Matter and Complex Systems 48 (2005): 583-586.

[7] Mustafa, Omar, and S. Habib Mazharimousavi. "Ordering ambiguity revisited via position dependent mass pseudo-momentum operators." International Journal of Theoretical Physics 46 (2007): 1786-1796.

[8] Mustafa, Omar, and Zeinab Algadhi. "Position-dependent mass momentum operator and minimal coupling: point canonical transformation and isospectrality." The European Physical Journal Plus 134.5 (2019): 228.

[9] Mustafa, Omar. "PDM creation and annihilation operators of the harmonic oscillators and the emergence of an alternative PDM-Hamiltonian." Physics Letters A 384.13 (2020): 126265.

[10] Khlevniuk, A., and V. Tymchyshyn. "Classical treatment of particle with position-dependent mass $m(r) = 1/(1 + r^4)$ in 1D and 2D subjected to harmonic potential." Journal of Mathematical Physics 59.8 (2018).

[11] Hamil, B., and M. Merad. "Dirac and Klein-Gordon oscillators on anti-de Sitter space." The European Physical Journal Plus 133.5 (2018): 174.

[12] Wu, Tai Tsun, and Chen Ning Yang. "Dirac monopole without strings: monopole harmonics." Nuclear Physics B 107.3 (1976): 365-380.

[13] Kibble, Thomas WB. "Geometrization of quantum mechanics." Communications in Mathematical Physics 65 (1979): 189-201.

[14] Vilenkin, Alexander. "Cosmic strings and domain walls." Physics reports 121.5 (1985): 263-315.

[15] Vilenkin, Alexander. "Gravitational field of vacuum domain walls and strings." Physical Review D 23.4 (1981): 852.

[16] Vilenkin, Alexander. "Cosmological density fluctuations produced by vacuum strings." Physical Review Letters 46.17 (1981): 1169.

[17] Vilenkin, Alexander. "Gravitational field of vacuum domain walls." Physics Letters B 133.3-4 (1983): 177-179.

[18] Hiscock, William A. "Exact gravitational field of a string." Physical Review D 31.12 (1985): 3288.

[19] Linet, B. "The static metrics with cylindrical symmetry describing a model of cosmic strings." General Relativity and Gravitation 17 (1985): 1109-1115.

[20] Barriola, Manuel, and Alexander Vilenkin. "Gravitational field of a global monopole." Physical Review Letters 63.4 (1989): 341.

[21] Harari, Diego, and Carlos Lousto. "Repulsive gravitational effects of global monopoles." Physical Review D 42.8 (1990): 2626.

[22] Peskin, Michael E. An introduction to quantum field theory. CRC press, 2018.

[23] Bezerra de Mello, E. R. "Physics in the global monopole spacetime." Brazilian Journal of Physics 31 (2001): 211-222.

[24] de Oliveira, AL Cavalcanti, and ER Bezerra de Mello. "Exact solutions of the Klein–Gordon equation in the presence of a dyon, magnetic flux and scalar

potential in the spacetime of gravitational defects." Classical and Quantum Gravity 23.17 (2006): 5249.

[25] Vitória, R. L. L., and H. Belich. "Harmonic oscillator in an environment with a pointlike defect." Physica Scripta 94.12 (2019): 125301.

[26] Furtado, Claudio, and Fernando Moraes. "Harmonic oscillator interacting with conical singularities." Journal of Physics A: Mathematical and General 33.31 (2000): 5513.

[27] Barriola, Manuel, and Alexander Vilenkin. "Gravitational field of a global monopole." Physical Review Letters 63.4 (1989): 341.

[28] Spinelly, J., U. De Freitas, and ER Bezerra de Mello. "Gravitating magnetic monopole in the global monopole spacetime." Physical Review D 66.2 (2002): 024018.

[29] Bragança, E. A. F., et al. "Relativistic quantum oscillators in the global monopole spacetime." The European Physical Journal C 80 (2020): 1-11.

[30] Moshinsky, Marcos, and Adam Szczepaniak. "The dirac oscillator." Journal of Physics A: Mathematical and General 22.17 (1989): L817.

[31] de A Marques, Geusa, and Valdir B. Bezerra. "Non-relativistic quantum systems on topological defects spacetimes." Classical and Quantum Gravity 19.5 (2002): 985.

[32] Alves, Saulo S., et al. "Approximate analytical solutions of the Schrödinger equation with Hulthén potential in the global monopole spacetime." Universe 9.3 (2023): 132.

[33] De Mello, ER Bezerra, and C. Furtado. "Nonrelativistic scattering problem by a global monopole." Physical Review D 56.2 (1997): 1345.

[34] Carvalho, J., C. Furtado, and F. Moraes. "Dirac oscillator interacting with a topological defect." Physical Review A—Atomic, Molecular, and Optical Physics 84.3 (2011): 032109.

[35] Yukawa, Hideki, and Shoichi Sakata. "On the Theory of the β-Disintegration and the Allied Phenomenon." Proceedings of the Physico-Mathematical Society of Japan. 3rd Series 17 (1935): 467-479.

[36] Lam, C. S., and Y. P. Varshni. "Energies of s eigenstates in a static screened Coulomb potential." Physical Review A 4.5 (1971): 1875.

[37] Adebimpe, O., et al. "Eigensolutions, scattering phase shift and thermodynamic properties of Hulthen-Yukawa potential." Results in Physics 14 (2019): 102409.

[38] Ita, Benedict I., et al. "Evaluation of the bound state energies of some diatomic molecules from the approximate solutions of the Schrodinger equation with Eckart plus inversely quadratic Yukawa potential." Journal of Molecular Modeling 26 (2020): 1-7.

[39] Ahmed, Faizuddin. "Quantum effects with Kratzer plus generalised Yukawa potential in a point-like global monopole using different approximation schemes." Molecular Physics 121.6 (2023): e2198617.

[40] Edet, C. O., et al. "Magneto-transport and thermal properties of the Yukawa potential in cosmic string space-time." Results in Physics 39 (2022): 105749.

[41] Ahmed, Faizuddin. "Topological effects produced by point-like global monopole with Hulthen plus screened Kratzer potential on Eigenvalue solutions and NU-method." Physica Scripta 98.1 (2022): 015403.

[42] Hassanabadi, H., et al. "The semi-relativistic scattering states of the Hulthén and Hyperbolic-type potentials." Acta Physica Polonica A 124.1 (2013): 20-22.

[43] Hosseinpour, Mansoureh, et al. "Scattering and bound states for the Hulthén potential in a cosmic string background." The European Physical Journal C 77.5 (2017): 1-6.

[44] Morse, Philip M. "Diatomic molecules according to the wave mechanics. II. Vibrational levels." Physical review 34.1 (1929): 57.

[45] Bakke, K., and H. Belich. "Nonrelativistic quantum effects of the Lorentz symmetry violation on the Morse potential." Communications in Theoretical Physics 75.5 (2023): 055202.

[46] Ikot, A. N., et al. "Thermal properties of 2D Schrödinger equation with new

Morse interacting potential." The European Physical Journal D 76.11 (2022): 208.

[47] Qadeer, Saad, et al. "Vibrational levels of a generalized Morse potential." The Journal of Chemical Physics 157.14 (2022).

[48] Onate, C. A., et al. "Non-relativistic molecular modified shifted Morse potential system." Scientific Reports 12.1 (2022): 15188.

[49] Isonguyo, Cecilia N., et al. "Eigensolutions and thermodynamic properties of Kratzer plus generalized Morse potential." Frontiers in Physics 10 (2022): 962717.

[50] Hassanabadi, H., et al. "The semi-relativistic scattering states of the Hulthén and Hyperbolic-type potentials." Acta Physica Polonica A 124.1 (2013): 20-22.

[51] Oyewumi, Kayode John, and Oluwatimilehin Joshua Oluwadare. "The scattering phase shifts of the Hulthén-type potential plus Yukawa potential." The European Physical Journal Plus 131.9 (2016): 295.

[52] Jia, Chun-Sheng, Jian-Yi Liu, and Ping-Quan Wang. "A new approximation scheme for the centrifugal term and the Hulthén potential." Physics Letters A 372.27-28 (2008): 4779-4782.

[53] Ahmadov, A. I., et al. "Approximate bound state solutions of the Klein-Gordon equation with the linear combination of Hulthén and Yukawa potentials." Physics

Letters A 383.24 (2019): 3010-3017.

[54] Ferkous, N., and A. Bounames. "2D Pauli equation with Hulthén potential in the presence of Aharonov—Bohm effect." Communications in Theoretical Physics 59.6 (2013): 679.

[55] Omugbe, E., et al. "Fisher information entropies and the strength of an oscillator under a mixed hyperbolic Pöschl–Teller potential function." Indian Journal of Physics 97.12 (2023): 3411-3417.

[56] Hien, Nguyen Dinh. "Optical properties of a single quantum well with Pöschl–Teller confinement potential." Physica E: Low-dimensional Systems and Nanostructures 145 (2023): 115504.

[57] Al Sakkaf, L., and U. Al Khawaja. "Bound-states spectrum of the nonlinear Schrödinger equation with Pöschl-Teller and square-potential wells." Physical Review E 106.2 (2022): 024206.

[58] Churilova, M. S., R. A. Konoplya, and A. Zhidenko. "Analytic formula for quasinormal modes in the near-extreme Kerr-Newman–de Sitter spacetime governed by a non-Pöschl-Teller potential." Physical Review D 105.8 (2022): 084003.

[59] Oliveira, M. D., and Alexandre GM Schmidt. "Exact solutions of Schrödinger and Pauli equations on a spherical surface and interacting with Pöschl-Teller double-

ring-shaped potential." Physica E: Low-dimensional Systems and Nanostructures 120 (2020): 114029.

[60] Kratzer, Adolf. "Die ultraroten rotationsspektren der halogenwasserstoffe." Zeitschrift für Physik 3 (1920): 289-307.

[61] Edet, C. O., et al. "Bound state solutions of the Schrödinger equation for the modified Kratzer potential plus screened Coulomb potential." Indian journal of Physics 94 (2020): 425-433.

[62] Edet, C. O., and A. N. Ikot. "Effects of topological defect on the energy spectra and thermo-magnetic properties of CO diatomic molecule." Journal of Low Temperature Physics 203.1 (2021): 84-111.

[63] Hamzavi, Majid, Sameer M. Ikhdair, and B. I. Ita. "Approximate spin and pseudospin solutions to the Dirac equation for the inversely quadratic Yukawa potential and tensor interaction." Physica Scripta 85.4 (2012): 045009.

[64] Wu, Tai Tsun, and Chen Ning Yang. "Dirac monopole without strings: monopole harmonics." Nuclear Physics B 107.3 (1976): 365-380.

[65] de Oliveira, AL Cavalcanti, and ER Bezerra de Mello. "Exact solutions of the Klein–Gordon equation in the presence of a dyon, magnetic flux and scalar potential in the spacetime of gravitational defects." Classical and Quantum Gravity 23.17 (2006): 5249.

[66] de Mello, ER Bezerra. "Vacuum polarization effects in the global monopole spacetime in the presence of the Wu–Yang magnetic monopole." Classical and Quantum Gravity 19.20 (2002): 5141.