DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

SOME KINEMATIC ASPECTS OF UNDERWATER SWIMMING FROGS

by
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May, 2020
İZMİR

SOME KINEMATIC ASPECTS OF UNDERWATER SWIMMING FROGS

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We have read the thesis entitled "**SOME KINEMATIC ASPECTS OF UNDERWATER SWIMMING FROGS**" completed by **NESLİHAN YILDIRIM** under supervision of **ASSOC. PROF. DR. İLHAN KARAKILIÇ** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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SOME KINEMATIC ASPECTS OF UNDERWATER SWIMMING FROGS

ABSTRACT

The relationship between the kinematics and performance of frogs make them worthy for underwater swimming. Their underwater motion is thrust-drag based. By using the hydrodynamic equations of experimental results of frogs' underwater swimming, we obtain the speed and the distance formula for such a motion. We obtain an approximation for the curve traced by the frog by using The Frenet Approximation.

Keywords: Kinematics, underwater swimming, frogs, Frenet Approximation

SU ALTI KURBAĞA YÜZÜŞÜNÜN BAZI KİNEMATİK YÖNLERİ

ÖZ

Kurbağaların performansları ile kinematiği arasındaki ilişki onları su altı yüzüşü için önemli kılmaktadır. Kurbağaların su altı hareketi itme-direnç temellidir. Bu tezde kurbağaların su altı yüzüşündeki deney sonuçlarına ait hidrodinamik denklemler kullanılarak böyle bir harekete ait hız ve mesafe denklemleri elde edilmiştir. Ayrıca bu çalışmada Frenet Yaklaşımı kullanılarak kurbağanın izlediği eğri için bir yaklaşım elde edilmiştir.

Anahtar kelimeler: Kinematik, su altı yüzüşü, kurbağalar, Frenet Yaklaşımı

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CHAPTER ONE
INTRODUCTION

The articles published by Gal & Blake are keys to the studies for frog swimming. In the studies Gal & Blake (1988a), Gal & Blake (1988b), experiments done with frogs (Hymenochicus boettgeri), establish the relation between thrust and drag depending on the water density, wetted surface area, drag coefficient and speed. The hydrodynamic mechanism of frog swimming and the hind limb kinematics (in the experimental observations of Xenopus leavis) are given in Richards (2008). There is a comparation of the swimming kinematics and hydrodinamics between the purely aquatic (X. leavis and H. boettgeri) and semi-aquatic/terrestrial(R. pipiens and B. americanus) in Richards (2010). Gal & Blake, in 1980's, observed the swimming behavior of the frog Hymenochirus boettgeri. To calculate the resistive forces they subdivided the limb segments into eight elements which is shown in the Figure1.1
Gal & Blake measured the positional angles of the frog. They did some experiments with frogs, observed underwater swimming movements and formulated the drag force as $D = \frac{1}{2} g S_w C_d v^2$ where g is the fluid density, S_w is the wetted surface area of the frog, C_d is the drag coefficient and v is the speed of the frog.

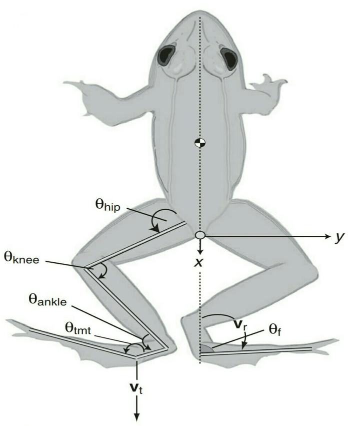

Figure 1.1 Vectorial and angular components of a frog's left foot (Richards, 2010)

Using the datas from the Gal & Blake's studies, Richards worked on the following equalities to derive foot velocity components from joint angles calculated in a local coordinate system defined as the snout-vent axis(x) and the medio-lateral axis(y).

$$d_{t,hip} = L_{fem} cos(\pi - \theta_{hip}) \tag{1.1}$$

$$d_{t,knee} = L_{tib} cos(\theta_{hip} - \theta_{knee}) \tag{1.2}$$

$$d_{t,ankle} = L_{tars} cos(\phi) \tag{1.3}$$

where $\theta_{hip}, \theta_{knee}$ and θ_{ankle} are joint angles and $d_{t,hip}, d_{t,knee}, d_{t,ankle}$ are hip, knee and ankle contributions to foot translational displacement d_t; $L_{fem}, L_{tib}, L_{tars}$ are lengths of the femur, tibia-fibula and proximal tarsal hind limb segments, and here the angle ϕ is given as $\phi = \pi - \theta_{hip} + \theta_{knee} + \theta_{ankle}$ and $dt = d_{t,hip} + d_{t,knee} + d_{t,ankle}$. This is a way to compute the speed of underwater frog swimming but in this thesis we compute the speed from the hydrodinamics of such a motion.

In Chapter 2, we get a nonlinear differential equation using the hydrodinamic equations obtained from the previous studies. Then we solve this differential equation to get the speed formula for underwater frog swimming. After then we integrate the result and get the distance formula for underwater frog swimming.

In Chapter 3, we aim to find the curve which is traced by the frog. We calculate the Frenet-Serret apparatus and then approach the curve by using Frenet Approximation.

CHAPTER TWO
UNDERWATER FROG SWIMMING

When a frog jumps into the water with thrust F, a drag D occurs on the frog's body. A nonzero acceleration causes a net force, F_{net}.

$$F_{net} = F - D \tag{2.1}$$

where F is the thrust, the total forward force, and D is the drag, the resistive force. The formula of the drag is given as

$$D = \frac{1}{2} g S_w C_d v^2 \tag{2.2}$$

where g is the fluid density, S_w is the wetted surface area of the frog, C_d is the drag coefficient and v is the speed of the frog. Here the drag coefficient can be taken as

$$C_d = 3.64 Re^{-0.378} \tag{2.3}$$

This is the drag coefficient of H. boettgeri computed in the drop-tank experiments and Re is the Reynolds number based on the snout-vent lenght,

$$Re = 10^6 \times speed(\frac{m}{s}) \times length(m) \tag{2.4}$$

calculated by Alexander, 1971.

S_w in m^2 is the surface area of a frog measured by geometric surface area determination (Gal & Blake, 1987)

$$S_w = 0.188 \lambda^{1.52} \tag{2.5}$$

where λ is the snout-vent length in meters.

Newton's laws of motion reveals that

$$F_{net} = m.a = m\frac{dv}{dt} \tag{2.6}$$

where m is the mass of a frog and a is the acceleration of the motion.

Hence we have the equality

$$m\frac{dv}{dt} = F - \frac{1}{2}gS_wC_dv^2 \tag{2.7}$$

Because of the infinitesimal effect of v on C_d we can take the phrase below as a constant, say α. That is,

$$\frac{1}{2}gS_wC_d = \alpha \tag{2.8}$$

where α is constant.

Now the equation can be written as

$$m\frac{dv}{dt} = F - \alpha v^2 \tag{2.9}$$

This is a first order nonlinear differential equation. The method of seperation of variables is used to solve this differential equation. Rearranging the terms we get

$$\frac{dv}{F - \alpha v^2} = \frac{dt}{m} \tag{2.10}$$

Integrating both sides yields

$$\int \frac{1}{F - \alpha v^2}dv = \int \frac{1}{m}dt \tag{2.11}$$

To solve this integral equation, let use the substitution $v = \sqrt{\frac{F}{\alpha}}sin\theta$ and from here we have

$$dv = \sqrt{\frac{F}{\alpha}}cos\theta d\theta \tag{2.12}$$

After these substitutions we get the new integral equality

$$\int \frac{\sqrt{\frac{F}{\alpha}}cos\theta}{F - Fsin^2\theta}d\theta = \int \frac{1}{m}dt \tag{2.13}$$

Some simplifications in (2.13) implies that

$$\frac{1}{\sqrt{\alpha F}} \int sec\theta d\theta = \int \frac{1}{m}dt \tag{2.14}$$

Then integration in (2.14) gives the equality such that

$$\frac{1}{\sqrt{\alpha F}} \ln |sec\theta + \tan\theta| + c = \frac{t}{m} \tag{2.15}$$

where c is the integral constant.

Since $v = \sqrt{\frac{F}{\alpha}} \sin\theta$, the equality $\sin\theta = \frac{v}{\sqrt{\frac{F}{\alpha}}}$ holds. Replacing θ by arcsin $\left(\frac{v}{\sqrt{\frac{F}{\alpha}}}\right)$ we obtain

$$\frac{1}{\sqrt{\alpha F}} \ln \left(\frac{\sqrt{\frac{F}{\alpha}} + v}{\sqrt{\frac{F-\alpha v^2}{\alpha}}}\right) + c = \frac{t}{m} \tag{2.16}$$

Rearranging the equation (2.16) gives

$$\frac{\sqrt{\frac{F}{\alpha}} + v}{\sqrt{\frac{F-\alpha v^2}{\alpha}}} = e^{\sqrt{\alpha F}\left(\frac{t}{m} - c\right)} \tag{2.17}$$

If we take the square of both sides in (2.17) we obtain the second order equation depending on v.

$$\alpha\left(1 + e^{2\sqrt{\alpha F}\left(\frac{t}{m} - c\right)}\right)v^2 + \left(2\sqrt{\alpha F}\right)v + F\left(1 - e^{2\sqrt{\alpha F}\left(\frac{t}{m} - c\right)}\right) = 0 \tag{2.18}$$

The discriminant Δ of the equation (2.18) is

$$\begin{aligned}
\Delta &= 4\alpha F - 4\alpha\left(1+e^{2\sqrt{\alpha F}\left(\frac{t}{m}-c\right)}\right)\left(1-e^{2\sqrt{\alpha F}\left(\frac{t}{m}-c\right)}\right)F \\
&= 4\alpha F - 4\alpha F(1-e^{4\sqrt{\alpha F}\left(\frac{t}{m}-c\right)}) \\
&= 4\alpha F e^{4\sqrt{\alpha F}\left(\frac{t}{m}-c\right)}
\end{aligned} \tag{2.19}$$

The roots v_1 and v_2 are found using the discriminant of the second order equation.

$$\begin{aligned}
v_{1,2} &= \frac{-2\sqrt{\alpha F} \pm \sqrt{4\alpha F e^{4\sqrt{\alpha F}\left(\frac{t}{m}-c\right)}}}{2\alpha\left(1+e^{2\sqrt{\alpha F}\left(\frac{t}{m}-c\right)}\right)} \\
&= \frac{-2\sqrt{\alpha F} \pm 2\sqrt{\alpha F} e^{2\sqrt{\alpha F}\left(\frac{t}{m}-c\right)}}{2\alpha\left(1+e^{2\sqrt{\alpha F}\left(\frac{t}{m}-c\right)}\right)} \\
&= \sqrt{\frac{F}{\alpha}}\;\frac{\pm e^{2\sqrt{\alpha F}\left(\dfrac{t}{m}-c\right)}-1}{e^{2\sqrt{\alpha F}\left(\dfrac{t}{m}-c\right)}+1}
\end{aligned} \tag{2.20}$$

Then the speed v in the direction of motion is

$$v = \sqrt{\frac{F}{\alpha}}\;\frac{e^{2\sqrt{\alpha F}\left(\dfrac{t}{m}-c\right)}-1}{e^{2\sqrt{\alpha F}\left(\dfrac{t}{m}-c\right)}+1} \tag{2.21}$$

Now let's use the substitution $s = e^{2\sqrt{\alpha F}\left(\frac{t}{m}-c\right)}$ to integrate the equation of the speed (2.21) with respect to t to obtain the distance traveled at underwater frog swimming.

Then we have $ds = \frac{2\sqrt{\alpha F}}{m} s dt$ and $dt = \frac{m}{2s\sqrt{\alpha F}} ds$

Thus we get the integral equality

$$\int v(t)dt = \frac{m}{2\alpha}\int \frac{s-1}{s(s+1)}ds \tag{2.22}$$

Integrating by simple fractions method and substituting

$$s = e^{2\sqrt{\alpha F}\left(\frac{t}{m}-c\right)} \tag{2.23}$$

we obtain the distance d

$$d = \frac{m}{2\alpha}\ln\left(\frac{(e^{2\sqrt{\alpha F}\left(\frac{t}{m}-c\right)}+1)^2}{e^{2\sqrt{\alpha F}\left(\frac{t}{m}-c\right)}}\right) + c_1 \tag{2.24}$$

where c_1 is the integral constant.

CHAPTER THREE

THE FRENET APPROXIMATION TO UNDERWATER FROG SWIMMING

Our aim is to find a curve for an underwater motion of a frog. To find this curve we use Frenet Approximation (See O'Neill (2014) for details). Actually, Frenet Approximation uses the Taylor expansion of the curve at the choosen point in terms of the Frenet frame apparatus. For each coordinate γ_i of a given curve γ

$$\gamma_i(t) \approx \gamma_i(t_0) + (t-t_0)\frac{d\gamma_i}{dt}(t_0) + \frac{(t-t_0)^2}{2!}\frac{d^2\gamma_i}{dt^2}(t_0) + \frac{(t-t_0)^3}{3!}\frac{d^3\gamma_i}{dt^3}(t_0) \qquad (3.1)$$

is an approximation near t_0.

Because each coordinate approximates closely we have the following approximation which is called Frenet Approximation.

$$\gamma(t) \approx \gamma(t_0) + (t-t_0)\dot{\gamma}(t_0) + \frac{(t-t_0)^2}{2!}\ddot{\gamma}(t_0) + \frac{(t-t_0)^3}{3!}\gamma^{(3)}(t_0) \qquad (3.2)$$

Since we have $\dot{\gamma} = T\|\dot{\gamma}\| = vT, \quad \ddot{\gamma} = \kappa v^2 N \quad and \quad \gamma^{(3)} = \kappa\tau v^3 B - \kappa^2 v^3 T,$ by taking the dominant terms we can write the approximation such that

$$\gamma(t) \approx \gamma(t_0) + (t-t_0)vT + \frac{(t-t_0)^2}{2!}\kappa v^2 N + \frac{(t-t_0)^3}{3!}\kappa\tau v^3 B \qquad (3.3)$$

where T, N, B, κ, τ are Frenet apparatus belongs to γ.
When the frog's speed is v and the frog jumps into water with the angle θ then we have the statements

$$\begin{aligned}
\dot{\gamma} &= (v\cos\theta, v_r, v\sin\theta) \\
\ddot{\gamma} &= (\dot{v}\cos\theta - v\dot{\theta}\sin\theta, \dot{v}_r, \dot{v}\sin\theta + v\dot{\theta}\cos\theta) \\
\gamma^{(3)} &= \Big((\ddot{v} - v\dot{\theta}^2)\cos\theta - (2\dot{v}\dot{\theta} + v\ddot{\theta})\sin\theta, \ddot{v}_r, (\ddot{v} - v\dot{\theta}^2)\sin\theta + (2\dot{v}\dot{\theta} + v\ddot{\theta})\cos\theta\Big)
\end{aligned} \qquad (3.4)$$

Here v_r is the speed created by rotational movement of the ankle of the frog.

Now we need to find Frenet frame apparatus to approach the curve. For any arbitrary curve the Frenet frame apparatus are calculated with the following equalities

$$\kappa = \frac{\|\dot\gamma \times \ddot\gamma\|}{\|\dot\gamma\|^3}, \qquad \tau = \frac{(\dot\gamma, \ddot\gamma, \gamma^{(3)})}{\|\dot\gamma \times \ddot\gamma\|^2} \tag{3.5}$$

$$T = \frac{\dot\gamma}{\|\dot\gamma\|}, \quad B = \frac{\dot\gamma \times \ddot\gamma}{\|\dot\gamma \times \ddot\gamma\|}, \quad N = B \times T \tag{3.6}$$

Putting the equalities (3.4) in the formulas (3.5) and (3.6) we get these apparatus for the curve γ that is tracing by the frog. The curvature (κ) of the curve γ is

$$\kappa = \frac{\sqrt{(v\dot v_r - v_r\dot v)^2 + (v v_r\dot\theta)^2 + v^4(\dot\theta)^2}}{\left(\sqrt{v^2 + v_r^2}\right)^3} \tag{3.7}$$

The torsion (τ) of the curve γ is

$$\tau = \frac{(\ddot v - v(\dot\theta)^2) v v_r\dot\theta + (2\dot v\dot\theta + v\ddot\theta)(v\dot v_r - \dot v v_r) - v^2\ddot v_r\dot\theta}{(v\dot v_r - v_r\dot v)^2 + (v v_r\dot\theta)^2 + v^4(\dot\theta)^2} \tag{3.8}$$

The tangent unit vector (T) of the curve γ is

$$T = \frac{1}{\sqrt{v^2 + v_r^2}}(v\cos\theta, v_r, v\sin\theta) \tag{3.9}$$

The binormal unit vector (B) of the curve γ is

$$B \;=\; k\left(A\sin\theta + v_r v\dot\theta\cos\theta, -v^2\dot\theta, -A\cos\theta + v_r v\dot\theta\sin\theta\right) \tag{3.10}$$

The normal unit vector (N) of the curve γ is

$$N = \frac{1}{k(v^2+v_r^2)}(A\sin\theta + v_r v\dot\theta\cos\theta, -v^2\dot\theta, -A\cos\theta + v_r v\dot\theta\sin\theta) \tag{3.11}$$

where $k = \frac{1}{(v\dot v_r - v_r\dot v)^2 + (v v_r\dot\theta)^2 + v^4(\dot\theta)^2}$ and $A = \dot v v_r - \dot{v_r} v$.

The equalities (3.7), (3.8), (3.9), (3.10), (3.11) are replaced in the approximation (3.3) and following result is found.

Finally we approach the curve which is traced by the frog for underwater swimming.

$$\gamma(t) \approx \gamma(t_0) + vT(t-t_0) + \kappa v^2 N\frac{(t-t_0)^2}{2!} + \kappa\tau v^3 B\frac{(t-t_0)^3}{3!} \tag{3.12}$$

$$\begin{aligned}\gamma(t) &\approx \gamma(t_0) + \frac{v}{\sqrt{v^2+v_r^2}}(t-t_0)\vec{a} + (\frac{v^2}{v^2+v_r^2})\frac{(t-t_0)^2}{2!}\vec{b} \\ &\quad + \frac{v^3[(\ddot{v}-v(\dot{\theta})^2)vv_r\dot{\theta} - A(2\dot{v}\dot{\theta}+v\ddot{\theta}) - v^2\ddot{v}_r\dot{\theta}]}{(v^2+v_r^2)[A^2+(vv_r\dot{\theta})^2+v^4(\dot{\theta})^2]}\frac{(t-t_0)^3}{3!}\vec{c}\end{aligned} \tag{3.13}$$

where

$\vec{a} = (v\cos\theta, v_r, v\sin\theta)$
$\vec{b} = (-(v^3\dot{\theta}\sin\theta + (v_r)^2 v\dot{\theta})\sin\theta + v_r A\cos\theta, vA, (v^3\dot{\theta} + (v_r)^2 v\dot{\theta})\cos\theta + v_r A\sin\theta)$
$\vec{c} = (A\sin\theta + v_r v\dot{\theta}\cos\theta, -v^2\dot{\theta}, -A\cos\theta + v_r v\dot{\theta}\sin\theta)$

and $A = \dot{v}v_r - \dot{v_r}v$.

CHAPTER FOUR
CONCLUSION

We used the hydrodinamic equations belongs to underwater swimming frogs and obtained a nonlinear differential equation. Then we solved it to find speed and distance formulas for underwater swimming. Our main goal was finding an approximation to curve traced by the frog in the water. Calculating the Frenet-Serret formulas for this motion and using the Frenet Approximaton we got an approximation to curve. When a starting point is given and we have the given angle θ at any time t, we can find an approximate location for underwater swimming frog.

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