DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

CONTINUED FRACTIONS AND FAREY TREE

by
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İZMİR

CONTINUED FRACTIONS AND FAREY TREE

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by
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We have read the thesis entitled "**CONTINUED FRACTIONS AND FAREY TREE**" completed by **BESTE AKDOĞAN** under supervision of **ASST. PROF. DR. CELAL CEM SARIOĞLU** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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CONTINUED FRACTIONS AND FAREY TREE

ABSTRACT

In this thesis, the relation between the continued fractions and Farey tree has been studied with aid of the Möbius transformations acting on the upper half plane. Beside this, the ergodicity and also the arithmetic properties of continued fractions has been studied. Finally, the relation between the modular group and algorithms on the Farey tree has been exhibited.

Keywords: Continued fractions, Farey tree, modular group, Möbius transformation, ergodicity

SÜREKLİ KESİRLER VE FAREY AĞACI

ÖZ

Bu tezde, üst yarı düzleme etki eden Möbius dönüşümleri yardımıyla sürekli kesirler ve Farey ağacı arasındaki ilişki çalışıldı. Bunun yanısıra, sürekli kesirlerin ergodikliği ve aritmetik özellikleri çalışıldı. Son olarak, Modüler grup ve Farey ağacı üzerindeki algoritmalar arasındaki ilişki sergilendi.

Anahtar kelimeler: Sürekli kesirler, Farey ağacı, modüler grup, Möbius dönüşümleri, ergodiklik

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CHAPTER ONE
INTRODUCTION

Continued fractions are the fractions of the form

$$\beta = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{\ddots}}}},$$

with b_i's are non-negative integers, is called simple continued fractions. We study continued fractions of this form. The expansion is denoted by $[b_0; b_1, b_2, b_3, \ldots]$ for simplicity. The continued fraction expansion may be finite or infinite. One of the oldest arguments where the question ”How is continued fraction gotten?” is related to is the Euclidean algorithm. For example, the problem that is obtaining the biggest squares from any rational numbers continuously, which is known jigsaw-puzzle problem. Consider a rectangle of 26 by 7 units, there are three 7×7 squares, a 5×5 square , two 2×2 squares and two 1×1 squares.

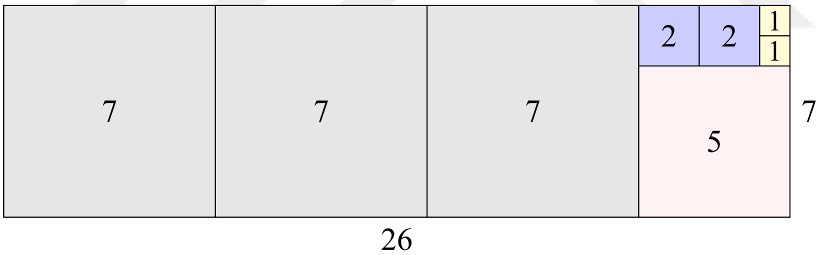

Figure 1.1 Squaring the rectangle of dimensions 26×7

Indeed,

$$\frac{26}{7} = \frac{3 \times 7 + 5}{7} = 3 + \frac{5}{7},$$
$$\frac{7}{5} = \frac{1 \times 5 + 2}{5} = 1 + \frac{2}{5},$$
$$\frac{5}{2} = \frac{2 \times 2 + 1}{2} = 2 + \frac{1}{2}.$$

So we get a continued fraction expansion of $\frac{26}{7}$ as

$$\frac{26}{7} = 3 + \frac{5}{7} = 3 + \frac{1}{\frac{7}{5}} = 3 + \frac{1}{1 + \frac{2}{5}} = 3 + \frac{1}{1 + \frac{1}{\frac{5}{2}}} = 3 + \frac{1}{1 + \frac{1}{2 + \frac{1}{2}}}.$$

The work of John Wallis(1616-1703) and Christiaan Huygens(1629-1695), the first person to indicated a practical application of continued fractions, started the work on continued fractions. Then the mathematicians Leonard Euler (1707-1783), Johan Heinrich Lambert (1728-1777), and Joseph Louis Lagrange (1736-1813) attended the group which worked on continued fraction.

In the above example for a rational number, its continued fraction expansion is finite. Swiss mathematician Euler exposed much the of modern every rational number has a finite continued fraction expansion in his book De Fractionlous Continious (1737). In addition, the elements of continued fraction expansion may be regular or irregular. Italian mathematicians Rafael Bombelli and Pietro Cataldi contributed to this field, Bombelli expressed the square root of 13 as a repeating continued fraction, similarly Cataldi did same for square root of 18. After Euler's work, Lagrange proved that the quadratic irrationals are exactly the real numbers which can be represented by infinite periodic continued fractions. For example, $\sqrt{19} = [4; \overline{2, 1, 3, 1, 2, 8}]$. Euler also expressed the continued fraction expansion of the number e and showed it is irrational number. Continued fraction expansions have been used for describing irrational numbers by series of rational numbers. For example , $22/7$ is used approximately instead of the number of π. Actually, the rational number $22/7$ is one of convergents of continued fraction expansion of π.

The nineteenth century can probably be described as the golden age of continued fractions. In addition, Jacobi, Perron, Hermite, Gauss, Cauchy and Stieljes are best known mathematicians worked on this subject.

While the mathematicians was working arithmetic properties of continued fractions, in nineteenth century , with exploring of hyperbolic geometry extensively, working on geometry of continued fractions started.

In the Chapter 2 we will introduce the Projective space, projective linear transformations and also Möbius transformations. In the Chapter 3 we will study the hyperbolic geometry, and introduce the upper half plane model and introduce its geodesics and also isometries.In the Chapter 4 we will study the continued fractions, their arithmetic properties, ergodicity and the relation with Farey tree. Finally, we approach the geometric properties of continued fractions.

CHAPTER TWO
PROJECTIVE SPACE AND PROJECTIVE TRANSFORMATIONS

In this chapter we will summarize some facts of the projective geometry and the projective linear transformations.

2.1 General Linear Group

Let R be a ring with unity and $n \in \mathbb{Z}^+$. Let $M(n, R)$ be the set of all $n \times n$ matrices whose entries are from the ring R. $M(n, R)$ is a group under matrix multiplication. The *general linear group of degree* n *over* R, which is denoted by $GL(n, R)$, is the set of all units of $M(n, R)$. In other words, $GL(n, R)$ is a subgroup of $M(n, R)$, and it consists of all $n \times n$ invertible matrices, whose entries are from the ring R. It is clear that $GL(1, R)$ is the set of all units of R and $GL(n, R)$ is not abelian for $n \geq 2$. If V is an n-dimensional module over the the ring R (or a vector space over the field R), each element $A \in M(n, R)$ defines a linear transformation $A : V \to V$ and vice versa. Beside this, if $A \in GL(n, R)$ the corresponding linear transformation $A : V \to V$ is bijective.

From now on we will take $\mathbb{C}^n$ or $\mathbb{R}^n$ instead of V, and take either $\mathbb{C}$, or $\mathbb{R}$ or $\mathbb{Z}$ instead of R.

For example, a matrix $A \in GL(n, \mathbb{C})$ is an $n \times n$ invertible matrix with complex entries and the corresponding linear transformation is $A : \mathbb{C}^n \to \mathbb{C}^n$ which is defined as

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}. \tag{2.1}$$

In addition, for any matrix $A \in GL(n, \mathbb{Z})$ we have determinant of A, $|A| = \mp 1$. Because since both A and A^{-1} belongs to the group $GL(n, \mathbb{Z})$, then both of $|A|$ and $|A^{-1}|$ must take integer values, but $|A^{-1}| = \dfrac{1}{|A|}$.

Furthermore, since $\mathbb{Z} < \mathbb{Q} < \mathbb{R} < \mathbb{C}$, then it is clear that $GL(n, \mathbb{Z}) < GL(n, \mathbb{Q}) < GL(n, \mathbb{R}) < GL(n, \mathbb{C})$.

2.2 Special Linear Group

The special linear group of degree n over a given ring R with identity is denoted by $SL(n, R)$ and it is a subgroup of the general linear group $GL(n, R)$, consisting of all $n \times n$ matrices whose determinant is 1. In other words,

$$SL(n, R) = \{A \in GL(n, R) : \ |A| = 1\}.$$

In particular, $SL(1, R) = \{1\}$ and

$$SL(2, R) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \ ad - bc = 1 \text{ and } a, b, c, d \in R \right\}.$$

Later we will show that $SL(2, \mathbb{R})$ acts on the complex upper half plane by fractional linear transformations.

2.3 Complex and Real Projective Spaces

Let n be a positive integer and $\mathbb{C}^{n+1}$ be the $(n + 1)$-fold cartesian product of the complex plane $\mathbb{C}$. Define an equivalence relation $\sim$ in $\mathbb{C}^{n+1} \setminus \{0\}$ as follows :

$$x \sim y \Leftrightarrow \exists \lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\} \text{ such that } y = \lambda x, \tag{2.2}$$

where $x = (x_0, x_1, ..., x_n), y = (y_0, y_1, ..., y_n) \in \mathbb{C}^{n+1} \setminus \{0\}$. Clearly, the relation $\sim$ is reflexive, symmetric and transitive. So it is an equivalence relation. In addition, the relation $\sim$ corresponds the $\mathbb{C}^*$ action on $\mathbb{C}^{n+1} \setminus \{0\}$.

The equivalence class of x under the equivalence relation "$\sim$" is denoted by $[x]$ and it is defined as

$$[x] = \{y \in \mathbb{C}^{n+1} \setminus \{0\} : \ x \sim y\}.$$

Notice that, the equivalence class $[x]$ contains all points on the (complex) line through the origin and the point x, except the origin.

The quotient space

$$(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^* = (\mathbb{C}^{n+1} \setminus \{0\})/_{\sim} = \{[\mathbf{x}] : \mathbf{x} \in \mathbb{C}^{n+1} \setminus \{0\}\}$$

is denoted by $\mathbb{CP}^n$ and called as the *n-dimensional complex projective space*.

On the other hand, if $y \in [x]$, then there exist a nonzero complex number λ such that $y = \lambda x$. So, if $y = (y_0, y_1, \ldots, y_n)$ and $x = (x_0, x_1, \ldots, x_n)$ then for any $i, j \in \{1, 2, \ldots, n\}$, $\dfrac{y_i}{y_j} = \dfrac{\lambda x_i}{\lambda x_j} = \dfrac{x_i}{x_j}$. This means that for any point y in $[x]$, ratios $\dfrac{y_i}{y_j}$ of the coordinates of y depends only on the ratios of the coordinates of x. For this reason, we denote the equivalence class $[x]$ of x as $[x_0 : x_1 : \ldots : x_n]$. This notation is also known as the *homogeneous coordinates* of x. Therefore

$$\mathbb{CP}^n = \{[x_0 : x_1 : \ldots : x_n] : \ (x_0, x_1, \ldots, x_n) \in \mathbb{C}^{n+1} \setminus \{0\}\}.$$

Furthermore, for any $[x_0 : x_1 : \ldots : x_n] \in \mathbb{CP}^n$ and for any $\lambda \in \mathbb{C}^*$ we have

$$[\lambda x_0 : \lambda x_1 : \cdots : \lambda x_n] = [x_0 : x_1 : \cdots : x_n]$$

and if $x_i \neq 0$, then we obtain

$$[x_0 : x_1 : \cdots : x_{i-1} : x_i : x_{i+1} : \cdots : x_n] = \left[\frac{x_0}{x_i} : \frac{x_1}{x_i} : \cdots \frac{x_{i-1}}{x_i} : 1 : \frac{x_{i+1}}{x_i} : \cdots : \frac{x_n}{x_i}\right].$$

This relation gives us a bijection σ_i from $V_i = \{[x_0 : x_1 : \cdots : x_n] \in \mathbb{CP}^n \ : \ x_i \neq 0\}$ to $\mathbb{C}^n$ via

$$[x_0 : x_1 : \cdots : x_{i-1} : x_i : x_{i+1} : \cdots : x_n] \to \left(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \cdots \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_n}{x_i}\right).$$

So, for each $i \in \{0, 1, \ldots, n\}$ the pairs (σ_i, V_i) form a patch for $\mathbb{CP}^n$.

As in the complex case, n *dimensional real projective space* $\mathbb{RP}^n$ is defined as the quotient space

$$\begin{aligned}\mathbb{RP}^n &= (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^* = (\mathbb{R}^{n+1} \setminus \{0\})/_{\sim} \\ &= \{[x_0 : x_1 : \cdots : x_n] : \ (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}\},\end{aligned}$$

where "$\sim$" is the equivalence relation

$$x \sim y \Longleftrightarrow \exists \lambda \in \mathbb{R}^* = \mathbb{R} \setminus \{0\} \text{ such that } y = \lambda x$$

corresponding to the $\mathbb{R}^*$ action on $\mathbb{R}^{n+1} \setminus \{0\}$.

From now on, we will interest only the case $n = 1$. Take a point $[x_0 : x_1]$ on $\mathbb{CP}^1$. By the definition of the projective space, both of x_0 and x_1 can not be zero at the same time. So, we have either $[x_0 : x_1] = \left[1 : \frac{x_1}{x_0}\right]$ while $x_0 \neq 0$, or $[x_0 : x_1] = \left[\frac{x_0}{x_1} : 1\right]$ while $x_1 \neq 0$. Therefore, we have bijections

$$\sigma_i : V_i = \{[x_0 : x_1] \in \mathbb{CP}^1 : \ x_i \neq 0\} \to \mathbb{C}, \quad i = 0, 1,$$

given by $\sigma_0([x_0 : x_1]) = \frac{x_1}{x_0}$ and $\sigma_1([x_0 : x_1]) = \frac{x_0}{x_1}$, respectively. Notice that $V_0 \cup V_1 = \mathbb{CP}^1$, $V_0 \setminus V_1 = \{[1 : 0]\}$, and $V_1 \setminus V_0 = \{[0 : 1]\}$. For this reason we may identify $\mathbb{CP}^1$ with $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\text{a point}\}$. On the other hand the projective point $[x_0 : x_1]$ corresponds a line through the origin $(0, 0)$ and the point (x_0, x_1). The intersection points of this line with respect to the lines $x_0 = 1$ and $x_1 = 1$ are the points $\left(1, \frac{x_1}{x_0}\right)$ and $\left(\frac{x_0}{x_1}, 1\right)$, respectively (See Figure 2.1). Notice that this line does not intersect the line $x_i = 1$ if $x_i = 0$, $i = 1, 2$, respectively. Indeed, they are parallel and, we may assume that they meet at the infinity. For this reason, we may identify V_1 with the line $x_1 = 1$ and so with $\mathbb{C}$, and identify the point $[1 : 0] \in V_0 \setminus V_1$ by ∞. Because of this fact, the point $[1 : 0]$ is known as *the point at infinity*. Thus we get a bijection

$$\sigma : \mathbb{CP}^1 \to \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \tag{2.3}$$

given by $\sigma([z : 1]) = z$ and $\sigma([1 : 0]) = \infty$.

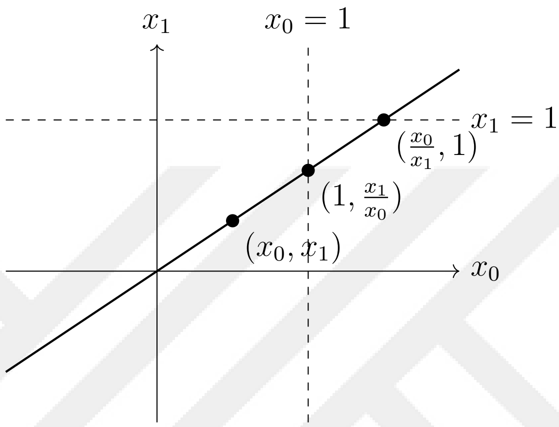

Figure 2.1 Projective point $[x_0 : x_1]$ and the corresponding affine line with slope $\frac{x_1}{x_0}$

By using a similar argument, one can identify $\mathbb{RP}^1$ with $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ via the map $\sigma : \mathbb{RP}^1 \to \widehat{\mathbb{R}}$ defined by $\sigma([z : 1]) = z$ and $\sigma([1 : 0]) = \infty$.

2.4 Projective General Linear Group

In the sections 2.1 and 2.3 we have introduced respectively that what is a general linear group and the projective space are. Recall that, for each $A \in GL(n, \mathbb{C})$ there is a linear transformation $A : \mathbb{C}^n \to \mathbb{C}^n$ defined as in the equation (2.1). On the other hand, since $\mathbb{CP}^{n-1} = (\mathbb{C}^n \setminus \{0\})/\mathbb{C}^*$ we have a $\mathbb{C}^*$ action on $\mathbb{C}^n \setminus \{0\}$, that is $x \sim \lambda x$ for each $\lambda \in \mathbb{C}^*$ and $x \in \mathbb{C}^n \setminus \{0\}$. So, the $\mathbb{C}^*$ action on $\mathbb{C}^n \setminus \{0\}$ must be consistent with the $GL(n, \mathbb{C})$ action on $\mathbb{C}^n$. In other words, for each $A \in GL(n, \mathbb{C})$, each $x \in \mathbb{C}^n \setminus \{0\}$ and each $\lambda \in \mathbb{C}^*$, Ax must be equivalent to $A(\lambda x)$ due to the relation "$\sim$". Note that $A(\lambda x) = (\lambda A)x$, which is clear from the matrix multiplication. Therefore, we can write $Ax \sim (\lambda A)x$ for each $\lambda \in \mathbb{C}^*$. That is, both the matrices A and λA have same effect on $\mathbb{CP}^{n-1}$. For this reason, we will define a relation "$\sim$" on $GL(n,\mathbb{C})$ as

$$A \sim B \Leftrightarrow \exists\lambda \in \mathbb{C}^* \text{ such that } B = \lambda A. \tag{2.4}$$

This relation corresponds to the $\mathbb{C}^*$ action on $GL(n,\mathbb{C})$, and it is an equivalence relation. The quotient group $GL(n,\mathbb{C})/\sim = GL(n+1,\mathbb{C})/\mathbb{C}^*$ is called the *n-dimensional complex projective linear group* and it is denoted by $PGL(n,\mathbb{C})$. That is

$$PGL(n,\mathbb{C}) = GL(n,\mathbb{C})/\mathbb{C}^* = GL(n,\mathbb{C})/\{\lambda I : \lambda \in \mathbb{C}^*\}. \tag{2.5}$$

Similarly, the *n-dimensional real projective linear group* is defined as

$$PGL(n,\mathbb{R}) = GL(n,\mathbb{R})/\mathbb{R}^* = GL(n,\mathbb{R})/\{\lambda I : \lambda \in \mathbb{R}^*\}. \tag{2.6}$$

Note that a matrix $A \in GL(n,\mathbb{C})$ is an $n \times n$-matrix with $|A| \neq 0$ and it has n^2 entries. The action of $\mathbb{C}^*$ on $GL(n,\mathbb{C})$ allows us to choose one of these n^2 entries of $A \in PGL(n,\mathbb{C})$ as 1.

More precisely,

$$PGL(2,\mathbb{C}) = GL(2,\mathbb{C})/\mathbb{C}^* = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc \neq 0 \right\} \Big/ \mathbb{C}^*. \tag{2.7}$$

and it consists of the 2×2 matrices such that at least one of its complex entries a, b, c, d is 1 and for the remaining three none of them is equal to the multiple of the other two, up to the relation "$\sim$".

2.5 Projective Special Linear Group

For a group G, the center of this group is the group $Z(G) = \{g \in G : hg = gh \text{ for all } h \in G\}$ is the subgroup of G that commutes with all elements of G. The projectivization PG of this group is defined as the quotient group $G/Z(G)$.

Lemma 2.5.1. *$Z(GL(n,\Bbbk)) = \{\lambda I : \lambda \in \Bbbk^*\} \cong \Bbbk^*$, where $\Bbbk$ is the field $\mathbb{C}$ or $\mathbb{R}$.*

Proof. Let the field $\Bbbk$ represent either the field $\mathbb{C}$ or $\mathbb{R}$, respectively, $\Bbbk^*$ represents either $\mathbb{C}^*$ or $\mathbb{R}^*$. By the definition of center, the group $Z(GL(n,\Bbbk))$ consists of all matrices commuting with the matrices in $GL(n,\Bbbk)$. By checking the commutativity with the elementary matrices E_{ij} whose i-th row, j-th column entry is 1 and all other entries are 0, we will see that it consists of diagonal matrices. In addition all diagonal entries must be same, otherwise it does not stabilize the sum of the vectors that each of is stabilized. Thus, $Z(GL(n,\Bbbk)) = \{\lambda I : \lambda \in \Bbbk^*\} \cong \Bbbk^*$. □

Theorem 2.5.2. *$Z(SL(n,\mathbb{C})) = \{\lambda I : \lambda^n = 1,\ \lambda \in \mathbb{C}^*\} \cong \{\lambda \in \mathbb{C} : \lambda^n = 1\}$ is a cyclic group of order n; and $Z(SL(n,\mathbb{R})) = \{\lambda I : \lambda^n = 1,\ \lambda \in \mathbb{R}^*\} \cong \mathbb{R}^*$. In particular $Z(SL(n,\mathbb{R})) = \{\mp I\} \cong \mathbb{Z}_2$ if n is even and $Z(SL(n,\mathbb{R})) = \{I\}$ is trivial if n is odd.*

Proof. Let $\Bbbk$ be the fields either $\mathbb{C}$ or $\mathbb{R}$. Recall that $SL(n,\Bbbk) \leq GL(n,\Bbbk)$, then

$$Z(SL(n,\Bbbk)) = Z(GL(n,\Bbbk)) \cap SL(n,\Bbbk) = \{\lambda I : \lambda^n = 1,\ \lambda \in \Bbbk^*\}.$$

If $\Bbbk = \mathbb{C}$, then it is clear that $Z(SL(n,\mathbb{C})) = \{\lambda I : \lambda^n = 1,\ \lambda \in \mathbb{C}^*\} \cong \{\lambda \in \mathbb{C} : \lambda^n = 1\}$ is a cyclic group of order n generated by the roots of unity. In case of $\Bbbk = \mathbb{R}$, depending on the oddness/evenness of n we have two cases. Because, the equation $\lambda^n = 1$ has a unique solution $\lambda = 1$ in $\mathbb{R}$ if n is odd. Otherwise, there are two solutions $\lambda = \mp 1$ in $\mathbb{R}$ if n is even. So, $Z(SL(n,\mathbb{R})) = \{I\}$ is trivial if n is odd, and $Z(SL(n,\mathbb{R})) = \{\mp I\} \cong \mathbb{Z}_2$ if n is even. □

As a consequence of the Lemma 2.5.1 and Theorem 2.5.2 we can state the following corollary without proof.

Corollary 2.5.3. *$PSL(n,\mathbb{C}) = SL(n,\mathbb{C})/\{\lambda I : \lambda \in \mathbb{C},\ \lambda^n = 1\}$, $PSL(n,\mathbb{R}) = SL(n,\mathbb{R})$ while n is odd and $PSL(n,\mathbb{R}) = SL(n,\mathbb{R})/\{\mp I\}$ while n is even. In particular, $PSL(2,\mathbb{C}) = PGL(2,\mathbb{C})$*

The group $PGL(2, \mathbb{C})$ is also known the group of Möbius transformations. (See the section 2.6.) The groups PGL and PSL can also be defined over a ring, e.g., over the ring $\mathbb{Z}$ of integers. Since $GL(2, \mathbb{Z})$ consists of the 2×2 matrices with integer entries having determinant ∓ 1, then $PGL(2, \mathbb{Z}) = GL(2, \mathbb{Z})/\{\mp I\}$. On the other hand, since the subgroup $SL(2, \mathbb{Z})$ of $GL(2, \mathbb{Z})$ consist of only the 2×2 matrices with integer entries having determinant 1, then $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\{\mp I\}$. This group is known as the *modular group*. We will study this group in the chapter 3

2.6 Möbius Transformations

A *linear fractional transformation* is an invertible transformation of the form

$$z \mapsto \frac{az+b}{cz+d}.$$

If $a, b, c, d, z \in \mathbb{C}$, it is called as *Möbius transformation*. Since $ad - bc \neq 0$, dividing the coefficients a, b, c, d with $\sqrt{ad - bc}$ do not change this map. For this reason, we may assume that $ad - bc = 1$. By using the book (Jones & Singerman, 1987) we will summarize the details about Möbius transformations.

Definition 2.6.1. A Möbius transformation (or a Möbius map) is a linear fractional transformation $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined in the form of $f(z) = \dfrac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$ (in particular we may assume $ad - bc = 1$).

Note that for any $\lambda \in \mathbb{C}^*$, the linear fractional transformations $\dfrac{\lambda az + \lambda b}{\lambda cz + \lambda d}$ and $\dfrac{az+b}{cz+d}$ over $\mathbb{C}$ coincides. So, by using the cancellation law, we may assume that the linear terms $az + b$ and $cz + d$ in the polynomial ring $\mathbb{C}[z]$ are relatively prime, i.e, none of each is a complex multiple of other.

On the other hand, for any Möbius transformation $f(z) = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$, the inverse transformation $f^{-1}(z) = \dfrac{-dz+b}{cz-a}$ is also a Möbius transformation. Notice that, $f(0) = \frac{b}{d}$, $f(\infty) = \frac{a}{c}$, $f^{-1}(0) = -\frac{b}{a}$ and $f^{-1}(\infty) = -\frac{d}{c}$. In addition, one can easily show that the composition of two Möbius transformations is also a Möbius transformation. So, the set of Möbius transformations form a group under the composition of maps, and its identity is the identity transformation. This group called as the *Möbius group*.

Furthermore, recall the bijective map $\sigma : \mathbb{CP}^1 \to \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ in the equation (2.3). Its inverse map $\sigma^{-1} : \widehat{\mathbb{C}} \to \mathbb{CP}^1$ maps any complex number z to a point $[z : 1]$ and, ∞ to the point $[1 : 0]$ at infinity. Therefore, for each Möbius transformation $f : \widehat{C} \to \widehat{C}$, there is a projective complex linear transformation A such that the following diagram commutes

$$\begin{tikzcd} \widehat{\mathbb{C}} \arrow[r, "f"] & \widehat{\mathbb{C}} \arrow[d, "\sigma^{-1}"] \\ \mathbb{CP}^1 \arrow[u, "\sigma"] \arrow[r, "A"] & \mathbb{CP}^1 \end{tikzcd}$$

In other words, $A = \sigma^{-1} \circ f \circ \sigma$. Then,

$$\begin{aligned} A([z:1]) &= \sigma^{-1} \circ f(z) = \sigma^{-1}\left(\frac{az+b}{cz+d}\right) = \left[\frac{az+b}{cz+d} : 1\right] = [az+b : cz+d], \\ A([1:0]) &= \sigma^{-1} \circ f(\infty) = \sigma^{-1}\left(\frac{a}{c}\right) = \left[\frac{a}{c} : 1\right] = [a : c]. \end{aligned}$$

Therefore, without lost of generality, we can say that $A([x_0 : x_1]) = [ax_0 + bx_1 : cx_0 + dx_1]$ for any point $[x_0 : x_1] \in \mathbb{CP}^1$, and its linear transformation matrix is the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2, \mathbb{C}) = PGL(2, \mathbb{C})$. So we can state the following proposition without proof.

Proposition 2.6.2. *There is a one to one correspondence between the Möbius group and the projective general linear group $PGL(2, \mathbb{C})$, that is*

$$f(z) = \frac{az+b}{cz+d} \longleftrightarrow A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PGL(2, \mathbb{C}).$$

By using this one to one correspondence we can say that Möbius transformations map any non-collinear distinct 3 points to any other non-collinear distinct 3 points in a unique way. To understand this fact, the following well-known proposition is useful and meaningful.

Proposition 2.6.3. *Let z_1, z_2, z_3 be any distinct non-collinear three points in $\widehat{\mathbb{C}}$, then there is a unique Möbius transformation such that $f(z_1) = \infty$, $f(z_2) = 0$ and $f(z_3) =$ 1. Such Möbius transformation is*

$$f(z) = \frac{z_3 - z_1}{z_3 - z_2} \cdot \frac{z - z_2}{z - z_1}.$$

Remark 2.6.4. Let's take a closer look at the following transformations from $\mathbb{C}$ to itself. Then the Möbius transformations will be more meaningful.(We denote the conjugate of any complex number z by $\overline{z}$. Note that, any complex number a can be written in the polar form such that $a = re^(i\theta)$ where r is the length of the complex number and θ is the angle made with the real axis.

- $P_x(z) = \overline{z}$ is a reflection with respect to the x-axis.
- $P_y(z) = -\overline{z}$ is a reflection with respect to the y-axis.
- $T_b(z) = z + b$ is a translation by a complex number b.
- $J(z) = \dfrac{1}{\overline{z}} = \dfrac{z}{|z|^2}$ is a reflection with respect to the unit circle centered at the origin. Notice that $J(\overline{z}) = \overline{J(z)}$.
- $f(z) = \dfrac{1}{z} = J(\overline{z}) = J \circ P_x(z) = P_x \circ J(z)$ is a composition of the reflections with respect to the unit circle and the x-axis.
- $D_r(z) = rz$, $r \in \mathbb{R}^*$, is a dilatation or homothety in the ratio r.
- $R_\theta(z) = e^{i\theta}z$, $\theta \in [0, 2\pi)$, is a rotation about the origin by an angle θ in the positive sense.
- If $f(z) = az$ for some $a \in \mathbb{C}^*$, then f is a composition of the dilatation D_r with the rotation R_θ, i.e, $f = D_r \circ R_\theta$.

- If $f(z) = az + b$ for some $a, b \in \mathbb{C}$, $a \neq 0$, then f is a composition of the translation T_b with the dilatation D_r and the rotation R_θ. That is $f = T_b \circ D_r \circ R_\theta = T_b \circ R_\theta \circ D_r$.

- If $f(z) = \dfrac{b}{z} = b \cdot \dfrac{1}{z} = re^{i\theta}\dfrac{1}{z}$,where $b = re^{i\theta}$ is the polar form of b for some $r > 0$ and $\theta \in [0, 2\pi)$, then $f = D_r \circ R_\theta \circ J \circ P_x$.

- If $f(z) = \dfrac{az+b}{z} = a + \dfrac{b}{z} = a + b \cdot \dfrac{1}{z} = a + re^{i\theta}\dfrac{1}{z}$, where $b = re^{i\theta}$ is the polar form of b for some $r > 0$ and $\theta \in [0, 2\pi)$, then $f = T_a \circ D_r \circ R_\theta \circ J \circ P_x$.

- $f(z) = \dfrac{1}{z+d} = \dfrac{1}{T_d(z)} = J \circ P_x \circ T_d(z)$.

- $f(z) = \dfrac{b}{z+d} = re^{i\theta} \cdot \dfrac{1}{T_d(z)} = D_r \circ R_\theta \circ I \circ P_x \circ T_d$, where $b = re^{i\theta}$ is the polar form of b for some $r > 0$ and $\theta \in [0, 2\pi)$.

- $f(z) = \dfrac{1}{cz+d} = (I \circ P_x) \circ T_d(cz) = (J \circ P_x) \circ T_d \circ (D_r \circ R_\theta)(z)$, where $c = re^{i\theta}$ is the polar form of c.

- $f(z) = \dfrac{az+b}{cz+d} = \dfrac{a}{c} + \dfrac{bc-ad}{c} \cdot \dfrac{1}{cz+d} = T_{a/c} \circ (D_\lambda \circ R_\alpha) \circ (J \circ P_x \circ T_d \circ D_r \circ R_\theta)(z)$, where $\dfrac{bc-ad}{c} = \lambda e^{i\alpha}$ and $c = re^{i\theta}$.

As it is seen from the above examples, every Möbius transformation can be written as a composition of some of the following transformations: translation, dilatation, rotation about a fixed point, inversion and the conjugation map.

2.7 Cross Ratio

The *cross ratio* is an invariant of lines in the projective plane. Let ℓ_1, ℓ_2, ℓ_3 and ℓ_4 be four distinct lines passing through the origin O, and P and Q be two distinct lines not passing through the origin. Assume the intersection points of these lines are the points $p_i = P \cap \ell_i$ and $q_i = Q \cap \ell_i$, $i = 1, 2, 3, 4$. It is clear that the points p_i and q_i have same homogeneous coordinates. By using the sine theorem for areas of triangles one

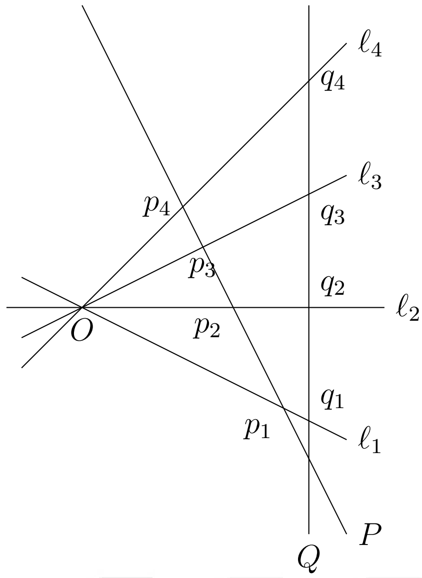

Figure 2.2 Cross ratio as projective invariant of four lines

can easily see that

$$\frac{|p_1p_3|}{|p_2p_3|}\cdot\frac{|p_2p_4|}{|p_1p_4|}=\frac{|q_1q_3|}{|q_2q_3|}\cdot\frac{|q_2q_4|}{|q_1q_4|},$$

or equivalently

$$\frac{|p_1p_3|}{|p_2p_3|}:\frac{|p_1p_4|}{|p_2p_4|}=\frac{|q_1q_3|}{|q_2q_3|}:\frac{|q_1q_4|}{|q_2q_4|}.$$

So it is a projective invariant. This will allow us to define the cross ratio of four points in $\widehat{\mathbb{C}}$ as follows.

Definition 2.7.1. The *cross ratio* of distinct four points z_1, z_2, z_3, z_4 in $\widehat{\mathbb{C}}$ is defined as

$$(z_1, z_2; z_3, z_4)=\frac{z_3-z_1}{z_3-z_2}\cdot\frac{z_4-z_2}{z_4-z_1}=\frac{z_3-z_1}{z_3-z_2}:\frac{z_4-z_1}{z_4-z_2}. \tag{2.8}$$

In particular, $(z_1, z_2; z_3, \infty)=\dfrac{z_3-z_1}{z_3-z_2}$.

The cross ratio of four distinct points can be written in a $4! = 24$ different ways and satisfy the relations in the following theorem.

Theorem 2.7.2. *Cross ratio of four distinct points $z_1, z_2, z_3, z_4 \in \widehat{\mathbb{C}}$ satisfy the following relations. In addition, if we denote the ratio in the first item by ω, then we write the other items in ω.*

i. $(z_1, z_2; z_3, z_4) = (z_2, z_1; z_4, z_3) = (z_3, z_4; z_1, z_2) = (z_4, z_3; z_2, z_1) = \omega$,

ii. $(z_1, z_2; z_4, z_3) = (z_2, z_1; z_3, z_4) = (z_4, z_3; z_1, z_2) = (z_3, z_4; z_2, z_1) = \dfrac{1}{\omega}$,

iii. $(z_1, z_3; z_2, z_4) = (z_3, z_1; z_4, z_2) = (z_2, z_4; z_1, z_3) = (z_4, z_2; z_2, z_1) = 1 - \omega$,

iv. $(z_1, z_3; z_4, z_2) = (z_3, z_1; z_2, z_4) = (z_4, z_2; z_1, z_3) = (z_2, z_4; z_3, z_1) = \dfrac{1}{1-\omega}$,

v. $(z_1, z_4; z_2, z_3) = (z_4, z_1; z_3, z_2) = (z_2, z_3; z_1, z_4) = (z_3, z_2; z_4, z_1) = \dfrac{\omega - 1}{\omega}$,

vi. $(z_1, z_4; z_3, z_2) = (z_4, z_1; z_2, z_3) = (z_3, z_2; z_1, z_4) = (z_2, z_3; z_4, z_1) = \dfrac{\omega}{\omega - 1}$.

Combining the Proposition 2.6.3 with the cross ratio, we can state the following corollary of the Proposition 2.6.3.

Corollary 2.7.3. *For any given non-collinear, distinct three points $z_1, z_2, z_3 \in \widehat{\mathbb{C}}$, the cross ratio $(z_1, z_2; z_3, z) = \dfrac{z_3 - z_1}{z_3 - z_2} \cdot \dfrac{z - z_2}{z - z_1}$ is the Möbius transformation sending z_1, z_2 and z_3 to ∞, 0 and 1, respectively.*

.

Theorem 2.7.4. *Möbius transformations preserve the cross ratio.*

Proof. Let z_1, z_2, z_3, z_4 be four distinct points in $\widehat{\mathbb{C}}$ and $f(z) = \dfrac{az+b}{cz+d}$ be a Möbius transformation. Then for any $i \neq j \in \{1, 2, 3, 4\}$ we have

$$\begin{aligned} f(z_i) - f(z_j) &= \frac{az_i + b}{cz_i + d} - \frac{az_j + b}{cz_j + d} = \frac{(az_i + b) \cdot (cz_j + d) - (az_j + b) \cdot (cz_i + d)}{(cz_i + d) \cdot (cz_j + d)} \\ &= \frac{(ad + bc) \cdot (z_i - z_j)}{(cz_i + d) \cdot (cz_j + d)} \end{aligned}$$

Therefore, by using the definition of the cross ratio we obtain

$$
\begin{aligned}
[f(z_1), f(z_2); f(z_3), f(z_4)] &= \frac{f(z_3)-f(z_1)}{f(z_3)-f(z_2)} \cdot \frac{f(z_4)-f(z_2)}{f(z_4)-f(z_1)} \\
&= \frac{\dfrac{(ad+bc)\cdot(z_3-z_1)}{(cz_1+d)\cdot(cz_3+d)}}{\dfrac{(ad+bc)\cdot(z_3-z_2)}{(cz_2+d)\cdot(cz_3+d)}} \cdot \frac{\dfrac{(ad+bc)\cdot(z_4-z_2)}{(cz_2+d)\cdot(cz_4+d)}}{\dfrac{(ad+bc)\cdot(z_4-z_1)}{(cz_1+d)\cdot(cz_4+d)}} \\
&= \frac{z_3-z_1}{z_3-z_2} \cdot \frac{z_4-z_2}{z_4-z_1} \\
&= [z_1, z_2; z_3, z_4].
\end{aligned}
$$

$\square$

CHAPTER THREE
HYPERBOLIC GEOMETRY

In this chapter we will study the facts of hyperbolic geometry such as distances, isometries, tessellations, etc.

3.1 Historical Background

The Egyptian Rhind Papyrus (2000–1800 BC), Moscow Papyrus (1890 BC) and the Babylonian clay tablets (1900 BC) are the known earliest geometry texts and these show that early geometry was a collection of empirically discovered principles concerning lengths, angles, areas, and volumes, which were developed to meet some practical need. In the 3rd century BC, geometry was put into an axiomatic form by Euclid. In his books, Elements, Euclid started with some assumptions (definitions, common notions and five postulates) and proved the other results (propositions) by using these assumptions. The most important postulate of Euclid is his fifth postulate, which is also known as *parallel postulate* and states:

“In a plane, through a point not on a given straight line, at most one line can be drawn that never meets the given line.´´

After Euclid, his followers continued to prove new theorems in his axiomatic style. For this reason, after Euclid, the classical geometry is renamed as *Euclidean geometry*. Some mathematicians, such as Khayyam (12th century), al-Tusi (13th century), Saccheri (18th century) were suspicious from Euclid’s fifth postulate whether it is really a postulate or a proposition, but no one is sure yet. However, in the beginnings of the 19th century, some mathematicians such as Bolyai, Lobachevsky , Beltrami, Gauss, and later, Klein, Cayley, Coxeter, Hilbert and Riemann tried to change the fifth postulate by its negation. This approach led to the birth of *non-Euclidean geometries*.

Around 1830, Lobachevsky replaced the fifth postulate with “For any given line and point not on it, there are at least two distinct lines through this point that do not intersect the given line.´´ and studied this geometry. On the same time period, Bolyai published his works on this subject independently from Lobachevsky. So this geometry is known as *Bolyai-Lobachevkian geometry*. Felix Klein, renamed this geometry as *hyperbolic geometry*.

After the works of Lobachevsky, Gauss, and Bolyai, the question ”Does such a model exist for hyperbolic geometry?” was raised. The model for hyperbolic geometry was answered by Beltrami, in 1868, and he used pesudosphere as a model. After some years later, Beltrami introduced four models for the hyperbolic plane.

The replacement of the fifth postulate with “Through any point in the plane, there exist no lines parallel to a given line.´´ caused to the birth of Elliptic geometry or Projective geometry. The pioneers of this geometry are Cayley and Riemann.

For more details about the historical development of the non-Euclidean geometries, read the references Anderson (2005), Dillon (2018), and the web sites https://en.wikipedia.org/wiki/Non-Euclidean_geometry and https://en.wikipedia.org/wiki/Hyperbolic_geometry.

3.2 Hyperbolic plane models

As it is explained in the Section 3.1, Hyperbolic geometry is a part of the non-Euclidean geometries. The mathematicians Bolyai, Lobachevsky, Klein and Beltrami are the pioneers of this area. There are many models for the hyperbolic plane but four of them are most common. These are *Beltrami-Klein model*, *Poincaré Disk model*, *Poincaré half-plane model* and *Lorentz or Hyperboloid model*, and they can be extendible to higher dimensions. All of these models were introduced by Beltrami. In this section we will explain only the *Poincaré half-plane model* and its isometries.

3.2.1 Poincaré half-plane model and its isometries

In the Euclidean geometry, the shortest distance between two points is the length of the straight line segment connecting these points. So the straight lines are also known as geodesics of the Euclidean spaces. On the other hand, a straight line is an undefined notion. As in the Euclid's Elements, all of us accept a straight line in the Euclidean geometry as a non-thickness line connecting two points which has infinite length. To define a model, Beltrami changed his point of view, and accepted these notions in a different way. In his point of view, Beltrami considered the hyperbolic plane as the upper part of the Euclidean plane, and the straight lines as either the straight lines perpendicular to the x-axis or upper half circles whose centres are on the x-axis in the Euclidean plane.

Therefore, *The Poincaré half plane model* is the upper half plane

$$\mathcal{H} = \{z = x + iy \in \mathbb{C} : \ x, y \in \mathbb{R} \text{ and } \mathrm{Im}(z) = y > 0\}, \tag{3.1}$$

equipped with the metric

$$ds = \frac{|dz|}{\mathrm{Im}(z)} = \frac{\sqrt{dx^2 + dy^2}}{y}. \tag{3.2}$$

In this model, the geodesic lines are the semi-lines or semi-circles of the Euclidean plane that are orthogonal to the x-axis. See Figure 3.1. Recall the notion that in the plane if two lines do not meet, they are parallel. So, in the Figure 3.1, there are 3 parallel lines to ℓ passing through the point A. In addition, all geodesics are orthogonal to the boundary (x-axis).

Definition 3.2.1. Let $\xi : [0,1] \to \mathcal{H}$ be any piecewise differentiable curve having a parametrization $\xi(t) = z(t) = x(t) + iy(t)$, $t \in [0,1]$. Then the length of ξ is defined as

$$\ell(\xi) = \int_0^1 \frac{|\xi'(t)|}{y(t)} = \int_0^1 \frac{\sqrt{(x'(t))^2 + (y'(t))^2}}{y(t)}\, dt. \tag{3.3}$$

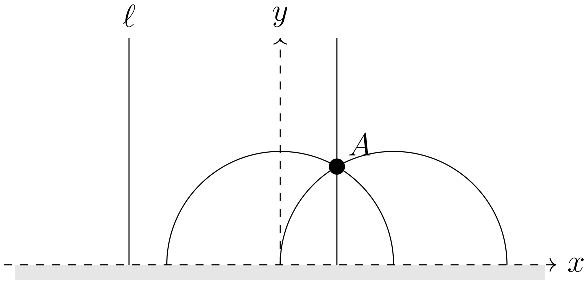

Figure 3.1 The Poincaré half plane $\mathcal{H}$ and its geodesics

In addition, the hyperbolic distance between two points $z_1, z_2 \in \mathcal{H}$, is defined as

$$d_{\mathcal{H}}(z_1, z_2) := \inf\{\ell(\xi) : \ \xi \text{ is a smooth curve with } \xi(0) = z_1,\ \xi(1) = z_2\}. \tag{3.4}$$

From the Proposition 2.6.2, we know that there is a one to one correspondence between the Möbius transformations and $PSL(2, \mathbb{C})$. By using affine version of this correspondence, one may define the special linear group $SL(2, \mathbb{C})$ action on the complex plane $\mathbb{C}$. But we are interested with the upper half plane $\mathcal{H}$, not the whole plane $\mathbb{C}$. So, we must be careful, because some Möbius transformations map the upper half plane to the lower half plane. See Remark 2.6.4. For this reason, we will use the subgroup $SL(2, \mathbb{R})$, not the whole group $SL(2, \mathbb{C})$.

The group action of $SL(2, \mathbb{R})$ on the upper half plane $\mathcal{H}$ by Möbius transformations is defined as follows: For each $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{R})$ there is a Möbius transformation $T_A(z) = \dfrac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. The following theorem makes this action more meaningful.

Theorem 3.2.2. *Any Möbius transformation* $T_A(z) = \dfrac{az+b}{cz+d}$ *with* $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$ *maps* $\mathcal{H}$ *into itself.*

Proof. Let $z \in \mathcal{H}$ and $\omega = T_A(z) = \dfrac{az+b}{cz+d}$. Then

$$\omega = \frac{az+b}{cz+d} = \frac{(az+b)\cdot\overline{(cz+d)}}{|cz+d|^2} = \frac{ac|z|^2 + adz + bc\overline{z} + bd}{|cz+d|^2}$$

Therefore,

$$\operatorname{Im}(\omega) = \frac{1}{2i} \cdot (\omega - \overline{\omega}) = \frac{1}{2i} \cdot \frac{(ad-bc)(z-\overline{z})}{|cz+d|^2} = \frac{1}{2i} \cdot \frac{2i\operatorname{Im}(z)}{|cz+d|^2} = \frac{\operatorname{Im}(z)}{|cz+d|^2}.$$

Hence, $\omega \in \mathcal{H}$ if $z \in \mathcal{H}$. $\square$

On the other hand, note that $\mp A \in SL(2,\mathbb{R})$ have the same action on $\mathcal{H}$. So we will use the quotient group $SL(2,\mathbb{R})\big/\{\mp I_2\}$, the projective special linear group $PSL(2,\mathbb{R})$. Notice also that $PSL(2,\mathbb{R})$ contains all matrices A with determinant $ad - bc = \Delta = 1$, since A and $\frac{1}{\sqrt{\Delta}}A$ same action on $\mathcal{H}$ and det $\left(\frac{1}{\sqrt{\Delta}}A\right) = 1$. This fact, also implies that $PSL(2,\mathbb{R})$ contains the elements corresponding to the transformation $z \to az + b$, where $a, b \in \mathbb{R}$, $a > 0$.

Definition 3.2.3. The translation, dilation and the inversion maps of $\mathcal{H}$ are defined as follows and they are bijections:

- The map $T_k : \mathcal{H} \to \mathcal{H}$ given by $T_k(z) = z + k$ is called a translation map by a real number k. The corresponding transformation matrix is $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$.
- The map $D_\lambda : \mathcal{H} \to \mathcal{H}$ given by $D_\lambda(z) = \lambda z$, $\lambda > 0$, is called a dilation map by the factor λ. The corresponding transformation matrix is $\begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}$.
- The map $I : \mathcal{H} \to \mathcal{H}$ given by $I(z) = \frac{-1}{z} = -\frac{\overline{z}}{|z|^2}$ is called the inversion map. The corresponding transformation matrix is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Recall from the Remark 2.6.4 that $J(z) = \frac{1}{\overline{z}}$ is the reflection with respect to the unit circle. These two maps $J(z)$ and $I(z)$ both reverse inside to outside, and outside to inside of the unit circle. Because of this reason both of them are known as inversion. These maps in the Definition 3.2.3 are Möbius transformations.

Theorem 3.2.4. *Möbius transformation in $\mathfrak{H}$ with real coefficients can be written as a composition of tranlation, dilation and inversion maps.*

Proof. Let $M(z) = \dfrac{az+b}{cz+d}$ be a Möbius transformation $M : \mathcal{H} \to \mathcal{H}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. (Because of the nature of Möbius transformations one may assume $\Delta = 1$.) Then we have the following cases:

Case 1: If $a = 0$ the $b, c \neq 0$. So we can write

$$M(z) = \frac{b}{cz+d} = (D_b \circ I)(cz+d) = D_b \circ I \circ T_d \circ D_c(z).$$

Notice also that, such a map can also be written as $M = D_{b/c} \circ I \circ T_{d/c}$.

Case 2: If $b = 0$ then $a, d \neq 0$. So we can write

$$M(z) = \frac{az}{cz+d}.$$

In case of $c = 0$, it is clear that M is the dilation map $D_{a/d}$. So now assume $c \neq 0$. Then we can write

$$M(z) = \frac{az}{cz+d}\frac{a}{c} + \frac{-\frac{ad}{c}}{cz+d} = T_{a/c} \circ D_{ad/c} \circ I \circ T_d \circ D_c(z).$$

Or equivalently

$$M(z) = \frac{az}{cz+d}\frac{a}{c} + \frac{-\frac{ad}{c^2}}{z+\frac{d}{c}} = T_{a/c} \circ D_{ad/c^2} \circ I \circ T_{d/c}.$$

Case 3: If $c = 0$ then $a, d \neq 0$. So we can write

$$M(z) = \frac{az+b}{d} = \frac{a}{d}z + \frac{b}{d} = T_{b/d} \circ D_{a/d}.$$

Case 4: If $d = 0$ then $b, c \neq 0$. So we can write

$$M(z) = \frac{az+b}{cz} = \frac{a}{c} + \frac{\frac{b}{c}}{z} = T_{a/c} \circ D_{-b/c} \circ I(z).$$

Case 5: If $a, b, c, d \neq 0$ then

$$M(z) = \frac{az+b}{cz+d} = \frac{a}{c} - \frac{\frac{ad-bc}{c}}{cz+d} = \frac{a}{c} - \frac{ad-bc}{c^2 z + cd} = T_{a/c} \circ D_{ad-bc} \circ I \circ T_{cd} \circ D_{c^2}(z).$$

$\square$

Definition 3.2.5. A map $f : \mathcal{H} \to \mathcal{H}$ is called an isometry of $\mathcal{H}$ if it preserves the distances, i.e., $d_{\mathcal{H}}(f(z_1), f(z_2)) = d_{\mathcal{H}}(z_1, z_2)$ for any $z_1, z_2 \in \mathcal{H}$.

Theorem 3.2.6. *The Möbius transformations* $M(z) = \dfrac{az+b}{cz+d}$ *with* $a, b, c, d \in \mathbb{R}$ *and* $ad - bc = 1$ *are isometries of* $\mathcal{H}$.

Proof. Recall from the Theorem 3.2.4 that every Möbius transformation can be written as a composition of a translation T_k, a dilation D_λ and the inversion I. On the other hand, it is clear from the definition of the isometry that composition of two isometries is an isometry. So it is enough to show that these three maps T_k, D_λ and I are isometries. Let z_1, z_2 be any two points and $\xi(t) = (x(t), y(t))$ (or equivalently $z(t) = x(t)+iy(t)$) be any smooth path from z_1 to z_2.

- $T_k \circ \xi(t) = (x(t) + k, y(t))$ and $(T_k \circ \xi)'(t) = (x'(t), y'(t)) = \xi'(t)$, and so

  $$\ell(T_k \circ \xi) = \int_0^1 \frac{|(T_k \circ \xi)'(t)|}{y(t)}\, dt = \int_0^1 \frac{\xi'(t)|}{y(t)}\, dt = \ell(\xi).$$

  Therefore,

  $$d_{\mathcal{H}}(T_k(z_1), T_k(z_2)) = \inf\{\ell(T_k \circ \xi)\} = \inf\{\ell(\xi)\} = d_{\mathcal{H}}(z_1, z_2).$$

  Hence, the translation map T_k is an isometry.

- $D_\lambda \circ \xi(t) = (\lambda x(t), \lambda y(t))$ and $(D_\lambda \circ \xi)'(t) = (\lambda x'(t), \lambda y'(t)) = \lambda \xi'(t)$, and so

  $$\ell(D_\lambda \circ \xi) = \int_0^1 \frac{|(D_\lambda \circ \xi)'(t)|}{\lambda y(t)}\, dt = \int_0^1 \frac{|\xi'(t)|}{y(t)}\, dt = \ell(\xi),$$

where $\lambda > 0$. Therefore,

$$d_{\mathcal{H}}(D_\lambda(z_1), D_\lambda(z_2)) = \inf\{\ell(D_\lambda \circ \xi)\} = \inf\{\ell(\xi)\} = d_{\mathcal{H}}(z_1, z_2).$$

Hence, the dilation map D_λ is an isometry.

- $I \circ \xi(t) = -\dfrac{1}{z(t)} = \dfrac{-x(t) + iy(t)}{x^2(t) + y^2(t)} = \left(\dfrac{-x(t)}{x^2(t) + y^2(t)}, \dfrac{y(t)}{x^2(t) + y^2(t)}\right)$. Then,

$$\mathrm{Im}(I \circ \xi(t)) = \mathrm{Im}\left(-\frac{1}{z(t)}\right) = \frac{y(t)}{x^2(t) + y^2(t)},$$

and

$$(\mathrm{Im}(I \circ \xi(t)))' = \left(-\frac{1}{z(t)}\right)' = \frac{z'(t)}{z(t)^2}.$$

So we get

$$|(I \circ \xi(t))'| = \left|\frac{z'(t)}{z^2(t)}\right| = \frac{|z'(t)|}{|z(t)|^2} = \frac{|\xi'(t)|}{x^2(t) + y^2(t)}.$$

Therefore,

$$\ell(I \circ \xi) = \int_0^1 \frac{|(I \circ \xi)'(t)|}{\mathrm{Im}(I \circ \xi(t))}\, dt = \int_0^1 \frac{\dfrac{|\xi'(t)|}{x^2(t) + y^2(t)}}{\dfrac{y(t)}{x^2(t) + y^2(t)}}\, dt = \int_0^1 \frac{|\xi'(t)|}{y(t)}\, dt = \ell(\xi).$$

Thus,

$$d_{\mathcal{H}}(I_\lambda(z_1), I_\lambda(z_2)) = \inf\{\ell(I_\lambda \circ \xi)\} = \inf\{\ell(\xi)\} = d_{\mathcal{H}}(z_1, z_2),$$

and the inversion map I is an isometry of $\mathcal{H}$.

$\square$

CHAPTER FOUR
CONTINUED FRACTIONS AND FAREY TREE

Every rational number can be written as a quotient of two integers, but irrational numbers can not. However, both of the set of rational numbers and the set of irrational numbers are dense in $\mathbb{R}$. This means that there exists a sequence of rational numbers converging to any (positive) real number. Our goal is to find a sequence of rational numbers converging to a given real number.

For example, the sequence

$$\left\{3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \frac{104348}{33215}, \dots\right\}$$

converges to π. Similarly, the sequence

$$\left\{1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \frac{577}{408}, \dots\right\}$$

converges to $\sqrt{2}$.

To obtain such a kind of sequence of rational numbers, the continued fractions are the most convenient and a natural tool. Now we will firstly introduce the concept of continued fractions and a notation for them.

4.1 Simple continued fractions

A simple continued fraction of a real number β is a finite or an infinite expression

$$\beta = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{b_4 + \cfrac{1}{b_5 + \cfrac{1}{b_6 + \cfrac{1}{\dots}}}}}}},$$

and it is shortly written as $\beta := [b_0; b_1, b_2, b_3, ...]$. The terms in this sequence naturally arose in the division algorithm.

Each rational number in this sequence can be expressed by a finite sequence of (positive) integers , for example $3 = [3]$, $\frac{22}{7} = [3, 7]$, $\frac{333}{106} = [3, 7, 15]$, etc.

Example 4.1.1. Let us see how we obtain the fraction sequence of the rational numbers 3, $\dfrac{22}{7}$, $\dfrac{333}{106}$ and $\dfrac{103993}{33102}$.

- The integer part of 3 is 3, so $3 = [3]$
- $\dfrac{22}{7} = 3 + \dfrac{1}{7}$. So we will denote this number by the sequence $[3, 7]$ and notice that the leading term 3 in the sequence $[3, 7]$ is the integer part of $\dfrac{22}{7}$.
- $$\frac{333}{106} = 3 + \frac{15}{106} = 3 + \cfrac{1}{\cfrac{106}{15}} = 3 + \cfrac{1}{7 + \cfrac{1}{15}}.$$

  So we will denote this number by the sequence $[3, 7, 15]$, and notice that the first two terms 3 and 7 are the integer parts of the rational numbers $\dfrac{333}{106}$ and $\dfrac{106}{15}$, respectively.

- $$\frac{103993}{33102} = 3 + \frac{4687}{33102} = 3 + \cfrac{1}{\cfrac{33102}{4687}} = 3 + \cfrac{1}{7 + \cfrac{293}{4687}} = 3 + \cfrac{1}{7 + \cfrac{1}{\cfrac{4687}{293}}}$$
$$= 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{292}{293}}} = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{\cfrac{293}{292}}}}$$
$$= 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292}}}}.$$

  So we will denote this number by the sequence $[3, 7, 15, 1, 292]$, and notice that the first four terms 3,7,15 and 1 are the integer parts of the rational numbers $\frac{103993}{33102}, \frac{33102}{4687}, \frac{4687}{293}$ and $\frac{293}{292}$, respectively.

This division technique can be extended to real numbers. The following examples show how to obtain the continued fraction expansions of $\sqrt{2}$ and $\sqrt{3}$.

Example 4.1.2. Since $(\sqrt{2} - 1)(\sqrt{2} + 1) = 1$, we can write $\sqrt{2} = 1 + \frac{1}{1 + \sqrt{2}}$.

Therefore, repeating this relation we obtain

$$
\begin{aligned}
\sqrt{2} &= 1 + \cfrac{1}{1+\sqrt{2}} \\
&= 1 + \cfrac{1}{1+1+\cfrac{1}{1+\sqrt{2}}} = 1 + \cfrac{1}{2+\cfrac{1}{1+\sqrt{2}}} \\
&= 1 + \cfrac{1}{2+\cfrac{1}{1+1+\cfrac{1}{1+\sqrt{2}}}} = 1 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\sqrt{2}}}} \\
&= 1 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+1+\cfrac{1}{1+\sqrt{2}}}}} = 1 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\sqrt{2}}}}} \\
&= 1 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\sqrt{2}}}}}} = 1 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{\ddots}}}}}} \\
&= [1; 2, 2, 2, 2, 2, \ldots].
\end{aligned}
$$

Example 4.1.3. To write the continued fraction expansion of $\sqrt{3}$, first notice that $1 < \sqrt{3} < 2$ and so the fractional part of $\sqrt{3}$ is $\sqrt{3}-1$. On the other hand, we know the identity $(\sqrt{3}-1)\cdot(\sqrt{3}+1) = 2$. Using these, we can write

$$\sqrt{3} = 1 + (\sqrt{3}-1) = 1 + \frac{2}{1+\sqrt{3}} = 1 + \frac{1}{\frac{\sqrt{3}+1}{2}}.$$

In addition, since $1 < \dfrac{\sqrt{3}+1}{2} < 2$, then we can write

$$\frac{\sqrt{3}+1}{2} = 1 + \frac{\sqrt{3}-1}{2} = 1 + \frac{1}{1+\sqrt{3}}.$$

Therefore, recursively writing the value of $\sqrt{3}$ we obtain

$$\sqrt{3} = 1 + (\sqrt{3}-1) = 1 + \frac{2}{1+\sqrt{3}} = 1 + \frac{1}{\dfrac{\sqrt{3}+1}{2}} = 1 + \cfrac{1}{1 + \cfrac{1}{1+\sqrt{3}}}$$

$$= 1 + \cfrac{1}{1 + \cfrac{1}{1 + 1 + \cfrac{1}{1 + \cfrac{1}{1+\sqrt{3}}}}} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1+\sqrt{3}}}}}$$

$$= 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + 1 + \cfrac{1}{1 + \cfrac{1}{1+\sqrt{3}}}}}}} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1+\sqrt{3}}}}}}}$$

Thus, the continued fraction expansion of $\sqrt{3}$ as

$$\sqrt{3} = [1; 1, 2, 1, 2, 1, 2, 1, 2, 1, \ldots].$$

In general, the algorithm to write the simple continued fraction of β is as follows:

- First, find the integer part $\lfloor \beta \rfloor$ and the fractional part $\beta - \lfloor \beta \rfloor$ of β. Say $b_0 = \lfloor \beta \rfloor$

and write

$$\beta = b_0 + (\beta - b_0) = b_0 + \cfrac{1}{\cfrac{1}{\beta - b_0}}.$$

- Second, for simplicity say $\beta_0 = \beta$, and $\beta_1 = \dfrac{1}{\beta_0 - b_0}$. Then find the integer and fractional parts of β_1. Say $b_1 = \lfloor \beta_1 \rfloor$ and write $\beta_1 = b_1 + \cfrac{1}{\cfrac{1}{\beta_1 - b_1}}$.

- For each i, set $\beta_i = b_i + \cfrac{1}{\cfrac{1}{\beta_i - b_i}}$, $b_{i+1} = \left\lfloor \cfrac{1}{\cfrac{1}{\beta_i - b_i}} \right\rfloor$. This recursive relation will give us the continued fraction expansion

$$\beta = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{b_4 + \cfrac{1}{\ddots}}}}}.$$

of β.

Notice that the integer part of $\beta = [b_0; b_1, b_2, b_3 \ldots]$ is b_0 and the fractional part is $[0; b_1, b_2, b_3 \ldots]$ or shortly $[b_1, b_2, b_3 \ldots]$. More generally, by using the above notation, the integer and fractional parts of β_i are b_i and $[0; b_{i+1}, b_{i+2}, \ldots]$ or shortly $[b_{i+1}, b_{i+2}, \ldots]$, respectively.

The following lemma is useful and also important for the characterization of simple continued fractions.

Lemma 4.1.4. *A real number β is rational if and only if β has a finite continued fraction expansion.*

Proof. $(\Leftarrow)$: This part is clear. Because, by using just a few calculations one can write β as a ratio of two integers.

$(\Rightarrow)$: For this part, we will use the Euclidean algorithm. The Euclid algorithm must stop after steps since the remainder term decrease and reaches to 0. Assume the euclidean algorithms stops at the nth step.

$$\begin{aligned}
p &= b_0q + r_1, \qquad 0 \leq r_1 < q \\
q &= b_1r_1 + r_2, \qquad 0 \leq r_2 < r_1 \\
r_1 &= b_2r_2 + r_3, \qquad 0 \leq r_2 < r_1 \\
&\vdots \\
r_{n-2} &= b_{n-1}r_{n-1} + r_n, \qquad 0 \leq r_n < r_{n-1} \\
r_{n-1} &= b_nr_n.
\end{aligned}$$

Therefore, we can write

$$\begin{aligned}
\frac{p}{q} &= b_0 + \frac{r_1}{q} = b_0 + \frac{1}{\frac{q}{r_1}} \\
\frac{q}{r_1} &= b_1 + \frac{r_2}{r_1} = b_1 + \frac{1}{\frac{r_1}{r_2}} \\
\frac{r_1}{r_2} &= b_2 + \frac{r_3}{r_2} = b_2 + \frac{1}{\frac{r_2}{r_3}} \\
&\vdots \\
\frac{r_{n-2}}{r_{n-1}} &= b_{n-1} + \frac{r_n}{r_{n-1}} = b_{n-1} + \frac{1}{\frac{r_{n-1}}{r_n}} \\
\frac{r_{n-1}}{r_n} &= b_n.
\end{aligned}$$

Then one get

$$\beta = \frac{p}{q} = b_0 + \cfrac{1}{\cfrac{q}{r_1}} = b_0 + \cfrac{1}{b_1 + \cfrac{1}{\cfrac{r_1}{r_2}}} = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{\cfrac{r_2}{r_3}}}}$$

$$= b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{\ddots\, b_{n-1} + \cfrac{1}{\cfrac{r_{n-1}}{r_n}}}}}} = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{\ddots\, \cfrac{1}{b_{n-1} + \cfrac{1}{b_n}}}}}}$$

$$= [b_0; b_1, b_2, \ldots, b_n].$$

□

Definition 4.1.5. For a given (infinite) simple continued fraction expansion $\beta = [b_0; b_1, b_2, b_3, \ldots]$, the rational number $\frac{p_k}{q_k}$ corresponding to the fraction $[b_0; b_1, b_2, b_3, \ldots, b_k]$ is called the k-th convergent of β, with $p_{-1} = 0$, $p_0 = 1$, $q_{-1} = 1$ and $q_0 = 1$. The real number $r_k = [b_k; b_{k+1}, b_{k+2}, \ldots]$ is called the k-th remainder.

In the proof of Lemma 4.1.4, $n-$th convergent of a rational numbers have finite continued fraction expansion $[b_0; b_1, \ldots, b_n]$ which is equal to itself. In addition, we observe that irrational numbers has infinite continued fraction expansion, since we see that the remainders of Euclidean algorithm must be irrational number. So, Euclidean algorithm does not stop after finite steps if β is irrational.

Theorem 4.1.6. *For $k \geq 2$ and a continued fraction $[b_0; b_1, b_2, b_3, \ldots]$, the convergents $\frac{p_k}{q_k}$ and the remainder r_k satisfies the equations*

$$p_k = b_k p_{k-1} + p_{k-2} \tag{4.1}$$

$$q_k = b_k q_{k-1} + q_{k-2} \tag{4.2}$$

Proof. We prove this the continued fraction expansion $[b_0; b_1, b_2, \ldots, b_n]$. Let the $r-$th convergent of $[b_0; b_1, b_2, \ldots, b_n]$ be $\dfrac{p_r}{q_r}$, for $r \geq 0$. First note that

$$\begin{aligned}
\frac{p_0}{q_0} &= b_0, \\
\frac{p_1}{q_1} &= b_0 + \frac{1}{b_1} = \frac{b_0 b_1 + 1}{b_1} \\
\frac{p_2}{q_2} &= b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_3}} = b_0 + \frac{b_2}{b_1 b_2 + 1} = \frac{b_0 b_1 b_2 + b_0 + b_2}{b_1 b_2 + 1}.
\end{aligned}$$

We use an induction on k. The above equations show that equations (4.2) and (4.1) are satisfied for $k = 2$. Now assume that,The equations (4.2) and (4.1) are true for $2 \leq k < m$. Let us consider the continued fraction expansion $[b_1; b_2, \ldots, b_m]$. For simplicity, we say α instead of $[b_1; b_2, \ldots, b_m]$. Let $\dfrac{p'_r}{q'_r}$ be $r-$th convergent of α. By induction hypothesis, the $(m-1)-$th convergent $\dfrac{p'_{m-1}}{q'_{m-1}}$ of α satisfies the following relations for $2 \leq m$.

$$\begin{aligned}
p'_{m-1} &= b_m p'_{m-2} + p'_{m-3} && (4.3) \\
q'_{m-1} &= b_m q'_{m-2} + q'_{m-3} && (4.4)
\end{aligned}$$

We also get the following relation between the continued fraction expansions $[b_0; b_1, b_2, \ldots, b_m]$ and $[b_1; b_2, \ldots, b_m]$

$$\begin{aligned}
p_m &= b_0 p'_{m-1} + q'_{m-1} && (4.5) \\
q_m &= p'_{m-1}. && (4.6)
\end{aligned}$$

When we combine (4.5) and (4.3) , we have the following results.

$$\begin{aligned}
p_m &= b_0 p'_{m-1} + q'_{m-1} = b_0(b_m p'_{m-2} + p'_{m-3}) + b_m q'_{m-2} + q_{m-3} \\
&= b_m(b_0 p'_{m-2} + q'_{m-2}) + (b_0 p'_{m-3} + q_{m-3}) = b_m p_{m-1} + p_{m-2} \\
q_m &= p'_{m-1} = b_m p'_{m-2} + p'_{m-3} = b_m q_{m-1} + q_{m-2}.
\end{aligned}$$

□

Thus, we observe from Theorem (4.1.6) the following equation for any rational number $\beta = [b_0; b_1, b_2, \ldots, b_n]$.

$$\beta = \frac{b_n p_{n-1} + p_{n-2}}{b_n q_{n-1} + q_{n-2}}. \tag{4.7}$$

Similarly, any irrational number whose continued fraction expansion $[b_0; b_1, b_2, \ldots, b_{n-1}, r_n]$ can be written in terms of its convergent and its the remainder term , as follows:

$$\beta = \frac{r_n p_{n-1} + p_{n-2}}{r_n q_{n-1} + q_{n-2}}. \tag{4.8}$$

In addition ,we said that there exists a sequence of rational numbers for a irrational number β. The elements of this sequence may be obtained from convergents of continued fraction expansion of β. So, k-convergent of β converge to β as $k \to \infty$.

An immediate consequence of the equations (4.1) and (4.2), we can obtain the relation

$$p_k q_{k-1} - q_k p_{k-1} = (b_k p_{k-1} + p_{k-2}) q_{k-1} - (b_k q_{k-1} + q_{k-2}) p_{k-1} = -(q_{k-2} p_{k-1} - p_{k-2} q_{k-1}).$$

By setting $p_{-1} = 1$ and $q_{-1} = 0$ we will see that $p_2 q_1 - p_1 q_2 = -1$. Then inductively we get the following equation for consecutive convergents

$$q_k p_{k-1} - p_k q_{k-1} = (-1)^k. \tag{4.9}$$

From the equations (4.9), (4.2) and (4.1) one obtains the following properties.

- The numerator and denominator of every convergent must be relatively prime.
- $\dfrac{p_{k-1}}{q_{k-1}} - \dfrac{p_k}{q_k} = \dfrac{(-1)^k}{q_{k-1} q_k}$. In other words, all of odd order convergents greater than all of even order convergents.

- $\dfrac{p_k}{q_k}-\dfrac{p_{k-2}}{q_{k-2}}=\dfrac{q_kp_{k-2}-p_kq_{k-2}}{q_{k-1}q_k}=\dfrac{(-1)^kb_k}{q_{k-2}q_k}$. In other words sequence of odd order convergents is decreasing and sequence of even order convergents is increasing.

- We get the following inequation with combination last two of above properties.

$$\frac{p_0}{q_0}<\frac{p_2}{q_2}<\frac{p_4}{q_4}<\cdots<\frac{p_{2k}}{q_{2k}}<\cdots<\frac{p_{2k-1}}{q_{2k-1}}<\cdots<\frac{p_5}{q_5}<\frac{p_3}{q_3}<\frac{p_1}{q_1}$$

- $q_k \geq q_{k-1}+q_{k-2} > q_{k-1}$ for $k \geq 1$. In other words, sequence of denominators of convergents is increasing.

Now we study difference between a irrational number β with and its convergents. By using the equations (4.7), (4.8) and (4.9), we get the following inequality

$$\beta-\frac{p_n}{q_n}=\frac{(-1)^{n-2}(r_n-b_n)}{(r_nq_{n-1}+q_{n-2})(b_nq_{n-1}+q_{n-2})}$$

If n is odd, then the above difference is negative. If n is even the above difference is positive. Therefore, the above equation gives that any irrational number β must be in the consecutive convergents. In other words,

$$\frac{p_0}{q_0}<\frac{p_2}{q_2}<\frac{p_4}{q_4}<\cdots<\frac{p_{2k}}{q_{2k}}<\cdots<\beta<\cdots<\frac{p_{2k-1}}{q_{2k-1}}<\cdots<\frac{p_5}{q_5}<\frac{p_3}{q_3}<\frac{p_1}{q_1}.$$

Now, our goal is getting a upper bound for difference between a real number β and its convergents. We have the following inequality for the value of the convergent infinite continued fraction expansion bu using the equation 4.8.

$$\left|\beta-\frac{p_n}{q_n}\right|<\frac{1}{q_nq_{n+1}} \qquad \text{for arbitrary} \quad n>0. \tag{4.10}$$

These above results are due to (Khinchin, 1997).

Mediant of two fraction $\frac{k}{l}$ and $\frac{y}{w}$ is defined as

$$\frac{k}{l} \oplus \frac{y}{w} = \frac{k+y}{l+w}$$

with positive denominator and this mediant is between $\frac{k}{l}$ and $\frac{y}{w}$. Consider the mediant

$$\frac{p_{k-2}+p_{k-1}}{q_{k-2}+q_{k-1}}$$

of two consecutive convergents $\frac{p_{k-2}}{q_{k-2}}$ and $\frac{p_{k-1}}{q_{k-1}}$ of β. The mediant of $\frac{p_{k-2}+p_{k-1}}{q_{k-2}+q_{k-1}}$ with the convergent $\frac{p_{k-1}}{q_{k-1}}$ is

$$\frac{p_{k-2}+2p_{k-1}}{q_{k-2}+2q_{k-1}}.$$

Continuing this process, we have a sequence of mediants:

$$\frac{p_{k-2}}{q_{k-2}} < \frac{p_{k-2}+p_{k-1}}{q_{k-2}+q_{k-1}} < \frac{p_{k-2}+2p_{k-1}}{q_{k-2}+2q_{k-1}} < \frac{p_{k-2}+3p_{k-1}}{q_{k-2}+3q_{k-1}}, \ldots, < \frac{p_{k-1}}{q_{k-1}}.$$

The mediants $\frac{p_{k-2}+tp_{k-1}}{q_{k-2}+tq_{k-1}}$ are called the intermediate fractions of β for arbitrary integer t. By using the equation (4.9), we get the difference between two consecutive mediants

$$\frac{p_k+(t+1)p_{k-1}}{q_k+(t+1)q_{k-1}} - \frac{p_k+tp_{k-1}}{q_k+tq_{k-1}} = \frac{(-1)^k}{[q_{k-1}(t+1)+q_{k-2}][q_{k-1}t+q_k]}.$$

Clearly the difference does not depend on t, just depend on k. Without loss of generality, we consider t is positive integer. Let us consider the sequence of intermediate fractions of finite continued fraction expansion

$$\frac{p_{k-2}}{q_{k-2}}, \frac{p_{k-2}+p_{k-1}}{q_{k-2}+q_{k-1}}, \frac{p_{k-2}+2p_{k-1}}{q_{k-2}+2q_{k-1}}, \frac{p_{k-2}+3p_{k-1}}{q_{k-2}+3q_{k-1}}, \ldots, \frac{p_{k-2}+b_kp_{k-1}}{q_{k-2}+b_kq_{k-1}} = \frac{p_k}{q_k}.$$

Since mediant of two consecutive convergents must be between these convergents and an irrational number must be between consecutive convergents, we can say that the irrational number between a convergent and the mediant of the convergent and its preceding convergent. In other words, we can find a rational number is more closer to the irrational number.

Consider the convergents $\frac{p_k}{q_k}$ and $\frac{p_{k+1}}{q_{k+1}}$ and their mediant, we obtain the lower bound for difference of β and any convergent $\frac{p_k}{q_k}$ (Khinchin, 1997).

$$\left|\beta - \frac{p_k}{q_k}\right| \geq \left|\frac{p_k + p_{k+1}}{q_k + q_{k+1}} - \frac{p_k}{q_k}\right| = \frac{1}{q_k(q_k + q_{k+1})}. \tag{4.11}$$

When we see to the inequalities (4.11) and (4.10) , we can say that while the denominator of convergent is raising , we can find a rational number such that it is more closer to the irrational number. Now, we will try to find a rational number which is closer to any irrational number than other rational number.

Definition 4.1.7. Consider the two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ such that $0 < d \leq b$ and $\frac{a}{b} \neq \frac{c}{d}$, the rational number $\frac{a}{b}$ which satisfies the following inequality

$$\left|\beta - \frac{a}{b}\right| \leq \left|\beta - \frac{c}{d}\right|.$$

is called a best approximation of first kind.

Definition 4.1.8. Consider the two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ such that $0 < d \leq b$ and $\frac{a}{b} \neq \frac{c}{d}$, the rational number $\frac{a}{b}$ which satisfies the following inequality

$$|d\beta - c| > |b\beta - c|$$

is called a best approximation of second kind.

Every best approximation of first kind is a convergent of β or an intermediate fraction of β (Khinchin, 1997, Theorem 15). Every best approximation coincides a convergent(Khinchin, 1997, Theorem 16). Clearly, every approximation of the second kind is also approximation of first kind except trivial case $\beta = b_0 + \frac{1}{2}$ and $\dfrac{p_0}{q_0} = \dfrac{b_0}{1}$ (Khinchin, 1997, Theorem 17).

We work to find the smallest upper bound value of the difference $\left|\beta - \dfrac{p_k}{q_k}\right|$. We interest the question "How many solution in p and q?" for our finding upper bound. We consider the following inequality instead of $\left|\beta - \dfrac{p_k}{q_k}\right| < \dfrac{1}{q_k^2}$ (since,every best approximation coincides a convergent).

$$\left|\beta - \frac{p}{q}\right| < \frac{c}{q^2} \tag{4.12}$$

Theorem 4.1.9. *(Khinchin, 1997, Theorem 18) For any real number β, its convergents satisfy at least one of the following inequalities.*

$$\left|\beta - \frac{p_k}{q_k}\right| < \frac{1}{2q_k^2} \quad \textit{and} \quad \left|\beta - \frac{p_{k-1}}{q_{k-1}}\right| < \frac{1}{2q_{k-1}^2}$$

Proof. Since the number β is between $\dfrac{p_k}{q_k}$ and $\dfrac{p_{k-1}}{q_{k-1}}$, the arithmetic mean of $\dfrac{1}{q_k^2}$ and $\dfrac{1}{q_{k-1}^2}$ greater than geometric mean ,We have

$$\left|\beta - \frac{p_k}{q_k}\right| + \left|\beta - \frac{p_{k-1}}{q_{k-1}}\right| = \left|\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}}\right| = \frac{1}{q_k q_{k-1}} < \frac{1}{2q_k^2} + \frac{1}{2q_{k-1}^2}.$$

□

Converse of the above theorem is true. If any fraction $\dfrac{a}{b}$ satisfies the inequalitis in the Theorem (4.1.9) ,it must be a convergent of β. It is enough to see that the fraction $\dfrac{a}{b}$ which satisfies the inequality is the best approximation of second kind.

Now,our goal is to find a positive real constant $c < \dfrac{1}{2}$ which satisfies the inequality (4.12) for infinitely many p and q.

Theorem 4.1.10. *(Khinchin, 1997, Theorem 20) For any real number β, its convergents satisfy at least one of the following inequalities.*

$$\left|\beta - \frac{p_k}{q_k}\right| < \frac{1}{\sqrt{5}q_k^2} \quad , \left|\beta - \frac{p_{k-1}}{q_{k-1}}\right| < \frac{1}{\sqrt{5}q_{k-1}^2} \quad \textit{and} \quad \left|\beta - \frac{p_{k-2}}{q_{k-2}}\right| < \frac{1}{\sqrt{5}q_{k-2}^2}$$

Proof. Let us define

$$\frac{q_{k-2}}{q_{k-1}} = \varphi_k \qquad \text{and} \qquad \varphi_k + r_k = \Psi_k$$

If $k \geq 2,\ \Psi_k \leq \sqrt{5}$ and $\Psi_{k-1} \leq \sqrt{5}$ then $\varphi_k > \frac{\sqrt{5}-1}{2}$

Assume the contrary, that is $\left|\beta - \frac{p_n}{q_n}\right| \geq \dfrac{1}{\sqrt{5}p_n^2}$ for $n = k$,$k-1$ and $k-2$. Then

$$\left|\beta - \frac{p_n}{q_n}\right| = \left|\frac{r_{n+1}p_n + p_{n-1}}{r_{n+1}q_n + q_{n-1}} - \frac{p_n}{q_n}\right| = \frac{1}{q_n(r_{n+1}q_n + q_{n-1})} = \frac{1}{q_n^2(r_{n+1} + \varphi_{n+1})} = \frac{1}{q_n^2\Psi_{n+1}}$$

So that the $\varphi_k > \dfrac{\sqrt{5}-1}{2}$ and $\varphi_{k+1} > \dfrac{\sqrt{5}-1}{2}$ Since we can write $a_k = \frac{1}{\varphi_{k+1}} + \varphi_k < 1$. We obtain a contradiction. Thus above inequality is satisfied by at least one of convergents. □

Can we find a constant $c < \dfrac{1}{\sqrt{5}}$ for the inequality (4.12) which have infinitely many solutions in integers p and q? The answer is "No!". For example, consider the golden ratio $\dfrac{1+\sqrt{5}}{2}$. Its continued fraction expansion is $\alpha = [1, 1, 1...]$. The inequality $\left|\beta - \dfrac{p_k}{q_k}\right| < \dfrac{c}{q_k^2}$ has no solution in convergents $\dfrac{p_k}{q_k}$ of β for $c < \dfrac{1}{\sqrt{5}}$ while it has infinitely many solutions for $c \geq \dfrac{1}{\sqrt{5}}$.

Therefore, the following inequality has infinitely many solutions in p and q if $c \leq \dfrac{1}{\sqrt{5}}$, but it has finitely many solutions for suitably chosen β if $c < \dfrac{1}{\sqrt{5}}$. Now, we consider the elements of the continued fraction expansion of β for finding bounds for the difference of β and any convergent of β. Since by combining the equations (4.10) and (4.11) and applying some arithmetic operations we have

$$\frac{1}{q_k(q_k+q_{k+1})} < \left|\beta - \frac{p_k}{q_k}\right| \leq \frac{1}{q_k q_{k+1}}$$

or

$$\frac{1}{{q_k}^2(b_{k+1}+1+\frac{q_{k-1}}{q_k})} < \left|\beta - \frac{p_k}{q_k}\right| \leq \frac{1}{{q_k}^2 b_{k+1}+\frac{q_{k-1}}{q_k}}$$

or

$$\frac{1}{q_k(b_{k+1}+2)} < \left|\beta - \frac{p_k}{q_k}\right| \leq \frac{1}{q_k^2 b_{k+1}}.$$

Clearly we get a good approximation of real number β whose elements of its continued fraction expansion contain large number. Conversely, any real number which have bounded elements gives worst approximation.

We have characterized the order of approximation by $\dfrac{1}{\sqrt{5}q_k^2}$ for all possible real number β. But it is insufficient. There may be an irrational number for which the approximation of higher order. When we investigate the irrational number β , we obtain that if the elements of continued fraction expansion of β are chosen such that $b_{k+1} > \dfrac{1}{q_k^2\varphi(q_k)}$ with any positive function $\varphi(q_k)$, the following inequality has infinitely many solutions in postive integer p and q

$$\left|\beta - \frac{p}{q}\right| < \varphi(q).$$

(Khinchin, 1997, Theorem 22).

We mentioned the boundedness of an element of an irrational number. In addition ,if a real number with bounded elements and suitably chosen small c (e.g., $b_{k+1} < M$ then choose $c < \dfrac{1}{(M+1)^2(M+2)}$) the following inequality has no solution

$$\left|\beta - \frac{p_k}{q_k}\right| < \frac{c}{q_k^2}$$

or for irrational numbers with unbounded elements and arbitrary $c > 0$ the inequality has infinitely many solution in positive integers p and q (Khinchin, 1997, Theorem 23).

We can consider an irrational number in two cases ; algebraic or transcendental. French mathematician Liouville found a result for approximation of an algebraic irrational number. For any algebraic irrational number degree n there exists a positive number C for arbitrary positive integers p and q

$$\left|\beta - \frac{p}{q}\right| > \frac{C}{q^n} \tag{4.13}$$

The above result shows the existence of transcendental numbers. If for any arbitrary real contant C there exist positive integers p and q such that

$$\left|\beta - \frac{p}{q}\right| \leq \frac{C}{q^n}$$

then the number β is transcendental.

Let us take $n = 2$. For a quadratic irrational number,there exist constant c depending on β ,so that (4.13) has no solution in positive integers p and q. Thus we obtain that elements of a quadratic number are bounded.

Theorem 4.1.11. *(Khinchin, 1997, Theorem 28) The continued fraction of β is periodic if and only if β is a quadratic irrational.*

Proof. We denote the periodic continued fraction by $\beta = [b_0, b_1, ...b_n, \overline{b_{n+1}, ..., b_{n+h}}]$. Let $\alpha = [\overline{b_{n+1}, ...b_{n+h}}]$. It is enough to show that α is a quadratic irrational. Clearly $r_{n+h+1} = \alpha$. Thus

$$\alpha = \frac{\alpha p_{n+h} + p_{n+h-1}}{\alpha q_{n+h} + q_{n+h-1}}. \tag{4.14}$$

Therefore, we obtain the quadratic equation $q_{n+h}\alpha^2 + (q_{n+h-1} - p_{n+h})\alpha - p_{n+h-1} = 0$ with root α. Consequently, β is quadratic irrational. Proof of the other part was given Lagrange in 1770. Consider the quadratic number β which satisfy $a\beta^2 + b\beta + c = 0$, $a \neq 0$. Putting (4.14) into β, we obtain the equation which is satisfied by the n-th order remainder r_n

$$K_n r_n^2 + L_n r_n + M_n = 0,$$

where

$$\begin{aligned} K_n &= ap_{n-1}^2 + bp_{n-1}q_{n-1} + cq_{n-1}^2, \\ L_n &= 2ap_{n-1}p_{n-2} + b(p_{n-1}q_{n-2} + p_{n-2}q_{n-1}), +2cq_{n-1}q_{n-2}, \\ M_n &= K_{n-1}. \end{aligned}$$

Discriminant of the equations $q_{n+h}\alpha^2 + (q_{n+h-1} - p_{n+h})\alpha - p_{n+h-1} = 0$ and $K_n r_n^2 + L_n r_n + M = 0$ are equal since

$$\left|\beta - \frac{p_{n-1}}{q_{n-1}}\right| < \frac{1}{q_n q_{n-1}}.$$

Then $p_{n-1} = \beta q_{n-1} + \dfrac{\delta}{q_{n-1}}$ with $\mid \delta \mid < 1$. We get by some operations $|K_n| < 2|a\beta| + |a| + |b|$ and $|M_n| = |K_{n-1}|$ and $|L_n| = \sqrt{b^2 - 4(ac - K_n L_n)}$. All constants of the equation $K_n r_n^2 + L_n r_n + M = 0$ are bounded. Then there are finitely many triple (K_n, L_n, M_n). So that finitely many r_n. Consequently, β has a periodic continued fraction expansion. □

4.2 Geometric representation of continued fractions

Now, we will give the relations between convergents of continued fractions and horocycles in upper half plane $\mathcal{H}$.

Definition 4.2.1. A horocycle in the upper half plane $\mathcal{H}$ are either Euclidean circles that are tangent to x-axis or the horizontal lines.

Note that, circular horocycles are perpendicular to circular geodesics, and horizontal horocycles are prependicular to the vertical geodesic. See FigureFigure 4.1.

The following Proposition states the some well-known properties of horocycles.

Proposition 4.2.2. *i. The horocycles at x_k are the euclidean circles of radius r_k*

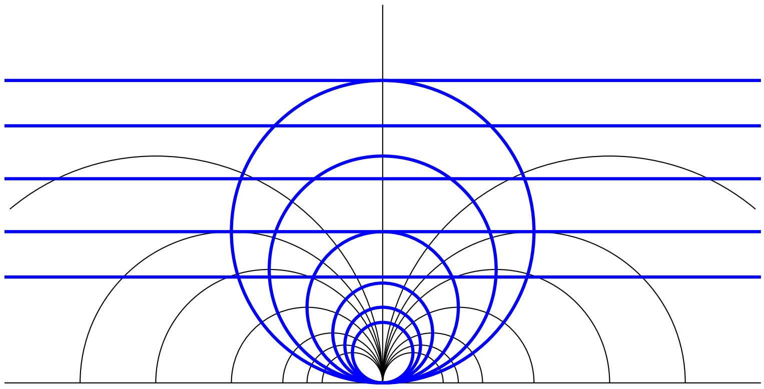

Figure 4.1 Horocycles and geodesic lines

centered at (x_k, r_k)*, where* $r_k \in \mathbb{R}^+$*,* $k = 1, 2$*. These horocycles orthogonal to all geodesics passing through* x_k*. (See Figure 4.1).*

ii. *The horocycles* $\{z : \operatorname{Im}(z) = y_k\} \cup \infty$ *are orthogonal to geodesics which are vertical lines. (See Figure 4.1).*

iii. *Two horocycles at* x_1 *and* x_2 *are tangent if their radii satisfy the equality*

$$|x_1 - x_2|^2 = 4r_1r_2$$

(See Figure 4.2).

iv. *The geodesic from* x_1 *to* x_2 *in* $\mathcal{H}$ *passes through the tangency points of horocycles (See Figure 4.2).*

v. *Two given tangent horocycles at* x_1 *and* x_2*, there is a unique horocycle and its radius satisfies the relation*

$$\frac{1}{\sqrt{r_3}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}$$

(See Figure 4.3).

Definition 4.2.3. A Ford circle $\mathcal{C}_{a/b}$ is a circle with center at $\left(\frac{a}{b}, \frac{1}{2b^2}\right)$ and radius $\frac{1}{2b^2}$, where $\frac{a}{b}$ is a rational number.

Theorem 4.2.4. *For given two rational numbers* $\frac{a}{b}$ *and* $\frac{c}{d}$ *in* $[0, 1]$*, the Ford circles* $\mathcal{C}_{a/b}$ *and* $\mathcal{C}_{c/d}$ *are tangent to each-other if and only if* $|ad - bc| = 1$*.*

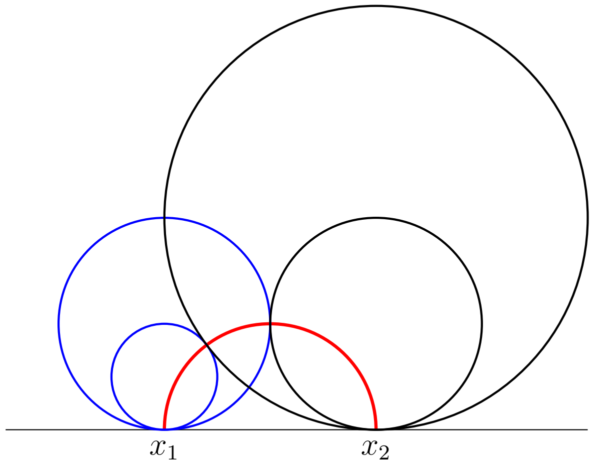

Figure 4.2 Tangency of horocycles and the geodesic line

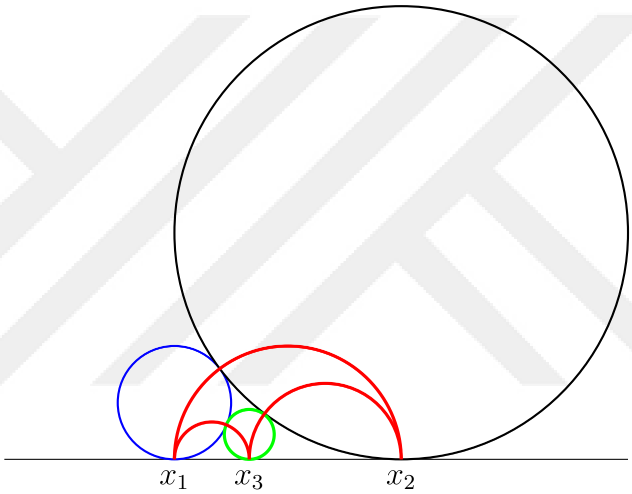

Figure 4.3 Tangent horocycles and geodesics

Proof. Ford circles are special horocycles. So, from the Proposition 4.2.2.*iii* we obtain that

$$|x_1 - x_2|^2 = 4r_1r_2 \Leftrightarrow \left|\frac{a}{b} - \frac{c}{d}\right|^2 = 4\frac{1}{4b^2d^2} \Leftrightarrow |ad - bc| = 1.$$

$\square$

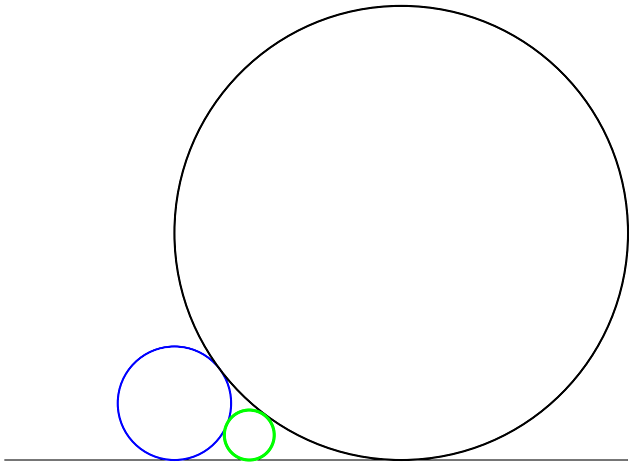

Figure 4.4 Horocycles corresponding the two consecutive convergents

Let β be a real number and $\dfrac{p_n}{q_n}$ be n^{th} convergent of continued fraction expansion of β. Let Π_0 be the horocycle at $\{z : \mathrm{Im}(z) = 1\} \cup \{\infty\}$, i.e, the horizontal line $y = 1$. If $q_n \neq 0$, define Π_n as a circular horocycle with the base point $\dfrac{p_n}{q_n}$ and Euclidean radius $\dfrac{1}{2{q_n}^2}$,i.e., Π_n is a Ford circle $\mathcal{C}_{p_n/q_n}$. Besides , if $q_n = 0$, define $\Pi_n = \{z : \mathrm{Im}[z] = {p_n}^2\} \cap \{\infty\}$.

Consider two consecutive convergents $\dfrac{p_k}{q_k}$ and $\dfrac{p_{k-1}}{q_{k-1}}$, $n \geq 1$, of β. Then, there are two horocycles Π_{k-1} and Π_k based at the points $\dfrac{p_k}{q_k}$ and $\dfrac{p_{k-1}}{q_{k-1}}$. We have known that the consecutive convergents $\dfrac{p_k}{q_k}$ and $\dfrac{p_{k-1}}{q_{k-1}}$ satisfies the equation **??**. Therefore from the Theorem 4.2.4, it is clear that the Ford circles $\mathcal{C}_{p_k/q_k}$ and $\mathcal{C}_{p_{k-1}/q_{k-1}}$ are tangent to each other. Notice that these Ford circles $\mathcal{C}_{p_k/q_k}$ and $\mathcal{C}_{p_{k-1}/q_{k-1}}$ are the horocyles Π_k and Π_{k-1}, respectively. Therefore, by using the convergents p_k/q_k of β, we can draw the chain of horocycles $\Pi_0, \Pi_1, \Pi_2, ...,$ so that each is tangent to previous and next horocyle. Since the denominator of convergents are increasing, then the radius of horocycles Π_n will be decreasing.

On the other hand, for the consecutive convergents $\dfrac{p_{n-1}}{q_{n-1}}, \dfrac{p_n}{q_n}$, we have the horocycles π_{n-1} and Π_n. By the Descartes' circle theorem, there is a unique circle which is tangent to x-axis and the circles Π_{n-1} and Π_n. Its base point is corresponding to the mediant of $\dfrac{p_{n-1}}{q_{n-1}}$ and $\dfrac{p_n}{q_n}$, which is also an intermediate fraction of β (See Figure 4.4).

4.3 Farey Tree and the Modular Group

A Farey tree is a tree with rational number vertices in the sets F_n on the interval $\mathbb{Q} \cap [0, 1]$.

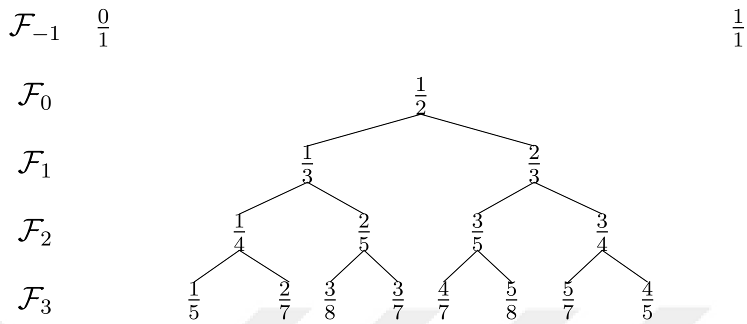

Figure 4.5 Farey Tree

The root of Farey tree is denoted by $\mathcal{F}_{-1}$ and contains $\frac{0}{1}$ and $\frac{1}{1}$. The $k-$th element of the Farey tree is denoted by r_k. The elements are arranged increasing order on the Farey tree

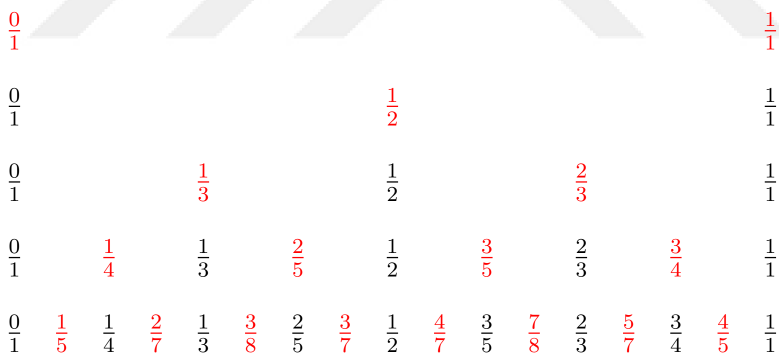

Figure 4.6 Farey sequences

In other words, Farey tree is $\cup_{k=-1}^{n-1} \mathcal{F}_k$. Notice that every set $\mathcal{F}_n$ has 2^n elements except the root of the tree.

- $\mathcal{F}_{-1} := \{\frac{0}{1}, \frac{1}{1}\}$,
- $\mathcal{F}_0 := \{\frac{1}{2}\}$,
- $\mathcal{F}_1 := \{\frac{1}{3}, \frac{2}{3}\}$,

- $\mathcal{F}_2 := \{\frac{1}{4}, \frac{2}{5}, \frac{3}{5}, \frac{3}{4}\}$,
- $\mathcal{F}_3 := \{\frac{1}{5}, \frac{2}{7}, \frac{3}{8}, \frac{3}{7}, \frac{4}{7}, \frac{5}{8}, \frac{5}{7}, \frac{4}{5}\}$,
- $\mathcal{F}_4 := \{\frac{1}{6}, \frac{2}{9}, \frac{3}{11}, \frac{3}{10}, \frac{4}{11}, \frac{5}{13}, \frac{5}{12}, \frac{4}{9}, \frac{5}{9}, \frac{7}{12}, \frac{8}{13}, \frac{7}{11}, \frac{7}{10}, \frac{8}{11}, \frac{7}{9}, \frac{5}{6}\}$,
- $\ldots$.

For each n, $\mathcal{F}_n$ is constructed by using Farey sum (mediant) $\oplus$ is the binary operation on $\mathbb{Q}$ defined as

$$\frac{k}{l} \oplus \frac{y}{w} = \frac{k+y}{l+w}.$$

We use the Farey sum for obtaining the Farey tree (See the Figure 4.7).

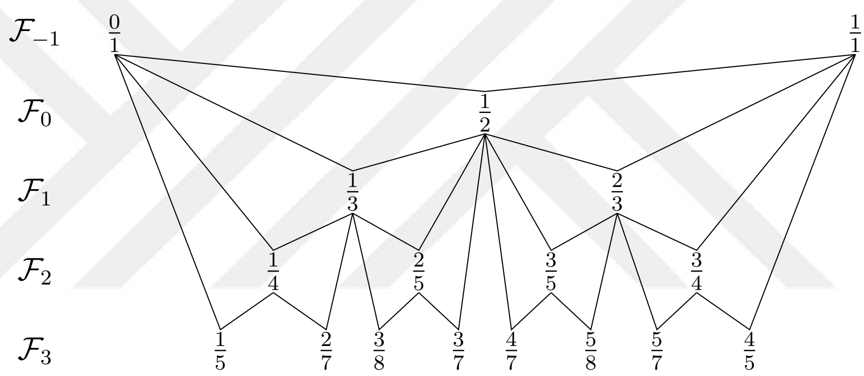

Figure 4.7 Construction of the Farey Tree with Farey sum (mediant)

Let us split the Farey tree into two parts and denote the left side by $\mathcal{F}^0$ (0-subtree) and the right side by $\mathcal{F}^1$ ($1-$subtree) (See the Figure 4.8).

Two rational numbers $\dfrac{k}{l}$ and $\dfrac{y}{w}$ are called adjacent if $|kw - ly| = 1$. It is easy to see that the Farey sum $\dfrac{k+y}{l+w}$ is also adjacent to both of the rationals $\dfrac{k}{l}$ and $\dfrac{y}{w}$ (Kocić & Stefanovska, 2005). Since we study on $[0,1]$, we write the continued fraction expansion of a real number in $[0,1]$ as $[0; x_0, x_1, x_2, x_3, \ldots]$.

Let us examine the following three transformations from $\mathbb{Q} \cap [0,1]$ to $\mathbb{Q} \cap [0,1]$ and

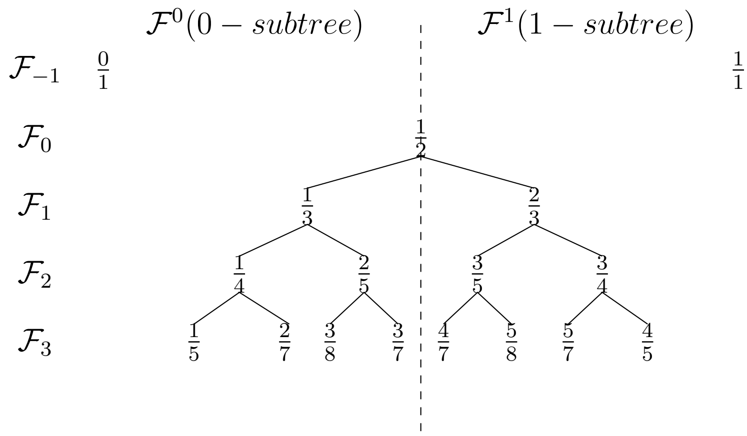

Figure 4.8 $\mathcal{F}^0$ and $\mathcal{F}^1$ subtrees of the Farey tree

their actions on the Farey tree.

$$\varphi \ : \ [x_0, x_1, x_2, x_3, \ldots, x_k] \mapsto [x_0 + 1, x_1, x_2, x_3, x_4, \ldots, x_k] \tag{4.15}$$

$$\Phi \ : \ [x_0, x_1, x_2, x_3, \ldots, x_k] \mapsto [1, x_0, x_1, x_2, \ldots, x_k] \tag{4.16}$$

$$\psi \ : \ [x_0, x_1, x_2, x_3 \ldots, x_k] \mapsto [1, x_0 - 1, x_1, x_2, x_3, \ldots, x_k]. \tag{4.17}$$

Theorem 4.3.1. *The mapping ψ maps $\mathcal{F}^0_n$ onto $\mathcal{F}^0_n$, that is $\psi(\mathcal{F}^0) = \mathcal{F}^1$ or inversely $\psi^{-1}(\mathcal{F}^1) = \mathcal{F}^0$. In the other words, $\psi(r_{2^{n+1}-k}) = r_{2^n-k-1}$ where $k = 1, 2, \ldots, 2^{n-1}$.*

Proof. See Figure 4.9. □

Theorem 4.3.2. *The mapping φ maps $\mathcal{F}_{n-1}$ onto $\mathcal{F}^0_n$ that is, $\varphi(r_{2^{n-1}+k}) = r_{3.2^{n-1}+k}$ for $k = 0, 1, \ldots, 2^{n-1}, n \in \mathbb{N}$*

Proof. See Figure 4.9 and Figure 4.10. □

In the Figure 4.9 red arrows show the action of φ and blue arrows show the action of ψ on Farey tree.

Theorem 4.3.3. *The mapping Φ maps $\mathcal{F}_{n-1}$ onto $\mathcal{F}^1_n$ s.t. $\Phi(r_{2^{n-1}+k}) = r_{3.2^{n-1}-k-1}$ for $k = 0, 1, \ldots, 2^{n-1} - 1, n \in \mathbb{N}$.*

Proof. See Figure 4.10. □

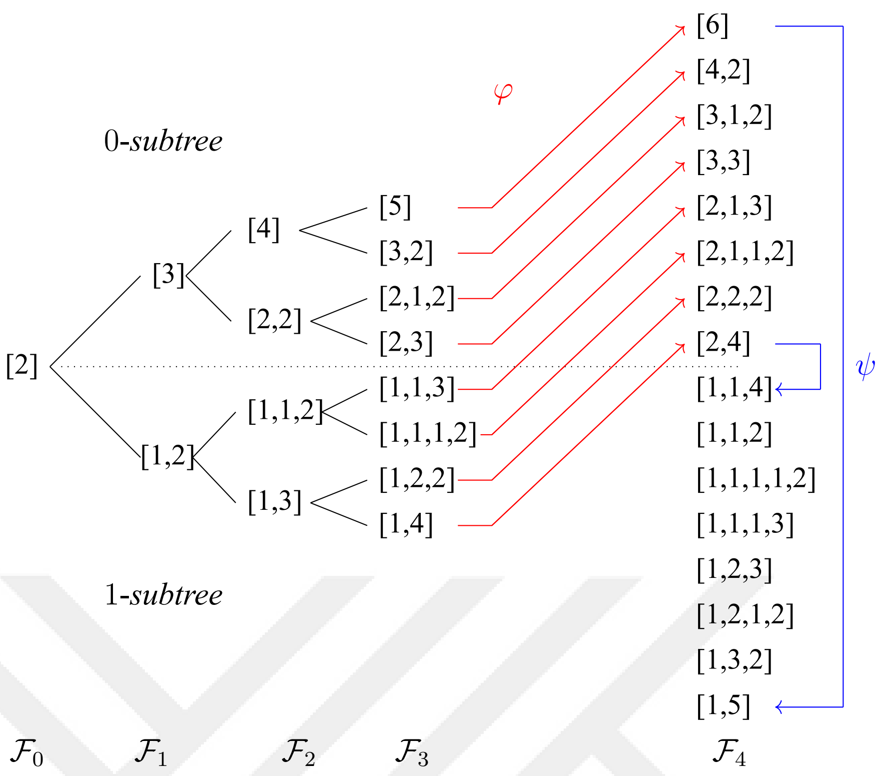

Figure 4.9 Action of the maps φ and ψ on the Farey tree

In the Figure 4.10, the green arrows show the action of Φ and blue arrows show the action of ψ^{-1} on the Farey tree.

Note that, if we apply the transformation Φ starting by $\frac{1}{1} = [1]$, then we have the following sequence

$$[1], [1,1], [1,1,1], [1,1,1,1], [1,1,1,1,1], \dots \tag{4.18}$$

which is the sequence of ratios of Fibonacci numbers, $\dfrac{F_i}{F_{i+1}}$ for $i \in \mathbb{Z}^+$:

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \frac{21}{34}, \dots .$$

The limit of the sequence (4.18) give us the Golden Ratio $\dfrac{\sqrt{5}-1}{2}$. We know the sequence (4.18) lies in "1-subtree" ($\mathcal{F}^1$).

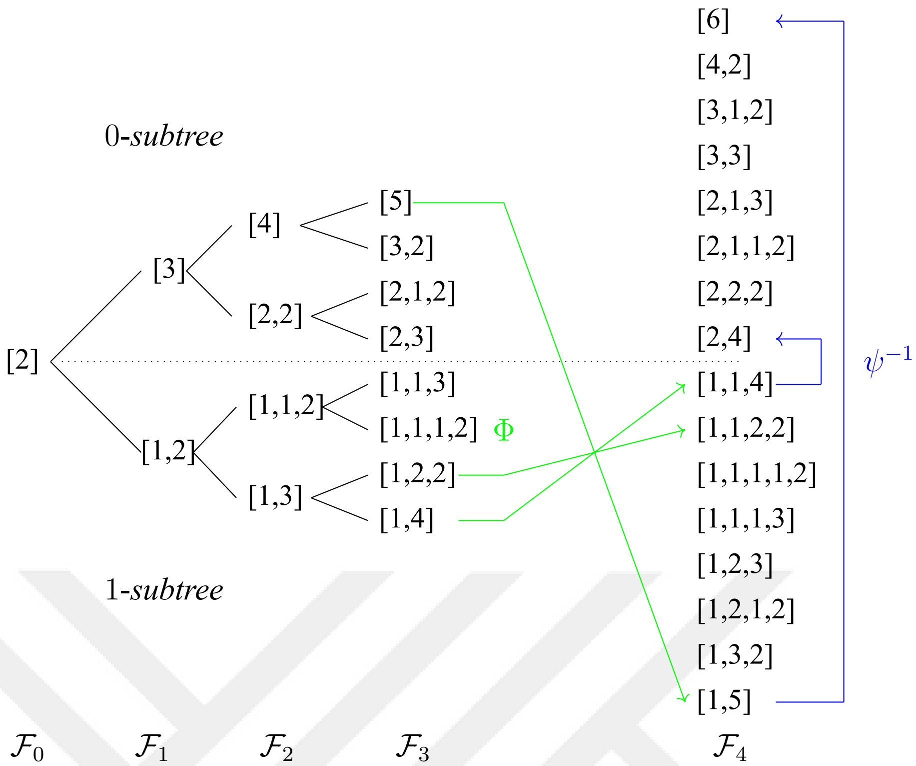

Figure 4.10 Action of the maps Φ and inverse of ψ on the Farey tree

In addition the complement of the sequence (4.18) is the sequence

$$[0], [2], [2, 1], [2, 1, 1], [2, 1, 1, 1], [2, 1, 1, 1, 1], \dots \tag{4.19}$$

or

$$\frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{3}{8}, \frac{5}{13}, \frac{8}{21}, \frac{13}{34}, \dots$$

Clearly, the above sequence gives the ratios $\dfrac{F_{i-1}}{F_{i+1}}, i \in \mathbb{Z}^+$ and its limit is equal to $1 - \left(\dfrac{\sqrt{5}-1}{2}\right) = \dfrac{3-\sqrt{5}}{2}$.

Notice that these maps satisfy the property $\Phi = \psi\varphi$. So, we can say that the Farey tree can be generated by the algorithms φ and ψ or Φ and ψ^{-1}. Linear fractional transformations corresponding to these maps are

$$\varphi : z \mapsto z + 1 \qquad \Phi : z \mapsto \frac{z+1}{z} \quad \psi : z \mapsto \frac{z}{z-1}$$

whose corresponding matrices, respectively.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \in PSL(2, \mathbb{Z}).$$

Since the matrix representations of the above algorithms on the Farey tree are in $PSL(2, \mathbb{Z})$, we can write these by type of generators of modular group . Therefore, modular group remains invariant the Farey tree.

We know the following recurrence relation about the convergents of a continued fraction

$$\frac{p_{n+1}}{q_{n+1}} = \frac{r_n p_n + p_{n-1}}{r_n q_n + q_{n-1}}.$$

If the remainder term goes to ∞, we have $\dfrac{p_n}{q_n}$ and if remainder term goes to 0, we have $\dfrac{p_{n-1}}{q_{n-1}}$. Therefore, we use the matrix notation $\begin{bmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{bmatrix}$. It has factorization as follows

$$\begin{bmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & b_1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & b_2 \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ 1 & b_{n-1} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & b_n \end{bmatrix}. \tag{4.20}$$

Notice that determinant of the matrix $\begin{bmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{bmatrix}$ is $(-1)^n$, so it belongs to the two dimensional orthogonal matrix group $O(2)$.

Now, we will relate all odd ordered convergents which are greater than even ordered convergents (Khinchin, 1997, Corollary of the Theorem 2) by using the above matrix notation of convergents of a continued fraction expansion. Also we will show that $\beta = [b_0; b_1, b_2, \ldots]$.

Consider the generators $M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $N = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ of the modular group $PSL(2,\mathbb{Z})$. We know N corresponds to the the translation map $z \mapsto z+1$, and M corresponds to the inversion $z \mapsto -\dfrac{1}{z}$. Also, we denote the map $z \mapsto z+t$ for some $t \in \mathbb{Z}$ by N^t. For example,

$$N^{-b_0}(\beta) = [0; b_1, b_2, \ldots].$$

Similarly ,

$$\begin{aligned} MN^{-b_0} &= -[b_1; b_2, b_3, \ldots]. \\ MN^{-b_1}MN^{-b_0} &= [b_2; b_3, \ldots]. \\ MN^{-b_2}MN^{-b_1}MN^{-b_0} &= -[b_3; b_4, \ldots]. \end{aligned}$$

Inductively,

$$MN^{-}b_{2n-1}MN^{-}b_{2n-2}MN^{-}b_{2n-3}\ldots MN^{-}b_2MN^{-}b_1MN^{-}b_0 = [b_{2n}; b_{2n+1}, b_{2n+3}, \ldots].$$

For simplicity, we say

$$K_{2n-1,\beta} = MN^{-b_{2n-1}}MN^{-b_{2n-2}}MN^{-b_{2n-3}}\ldots MN^{-b_2}MN^{-b_1}MN^{-b_0}.$$

Thus we have $K_{2n-1,\beta}(\beta) = r_{2n}$, since we can write $\beta = [b_0; b_1, b_2, \ldots, b_{2n-1}, r_{2n}]$, where r_{2n} is the remainder of order $2n$.

Since the remainder term may be 0 (this case is satisfied for a finite continued fraction expansion) or ∞ (this case is satisfied for an infinite continued fraction expansion) after for some step, we have.

$$K_{2n-1,\beta}(0) = [b_0; b_1, b_2, \ldots, b_{2n-1}]$$

and

$$K_{2n-1,\beta}(\infty) = [b_0; b_1, b_2, \dots, b_{2n}].$$

On the other hand , $K_{2n-1,\beta}(0) = \dfrac{p_{2n-1}}{q_{2n-1}}$ and $K_{2n-1,\beta}(\infty) = \dfrac{p_{2n}}{q_{2n}}$. We get matrix form of inverse of $K_{2n-1,\beta}$, say matrix K, from (4.1) and (4.2) such that $K = \begin{bmatrix} \lambda p_{2n-1} & \lambda p_{2n} \\ \lambda q_{2n-1} & \lambda q_{2n} \end{bmatrix}$ for some $\lambda \in \mathbb{Z}$ since the denominator and the numerator of convergents must be integers. Since $\det(K) = 1$, we have $\lambda = \mp 1$. So, $K = \begin{bmatrix} p_{2n-1} & p_{2n} \\ q_{2n-1} & q_{2n} \end{bmatrix} \in SL(2, \mathbb{Z})$. This gives $\dfrac{p_{2n-1}}{q_{2n-1}} > \dfrac{p_{2n}}{q_{2n}}$. We also say that odd order convergents are greater than even order convergents in the another way.

4.4 Ergodicy of Continued Fractions

Definition 4.4.1. A measure space is a triple $(X, \mathcal{B}, \mu)$ where

- X is a set.
- $\mathcal{B}$ is a σ–algebra: A collection of subsets of X which contains the empty set, and which is closed under complements, countable unions and countable intersections. The elements of $\mathcal{B}$ are called measurable sets.
- $\mu : \mathcal{B} \mapsto [0, \infty]$, called the measure, is a σ–additive function: If $A_1, A_2, \ldots \in \mathcal{B}$ are pairwise disjoint, then $\cup_i \mu(A_i) = \sum_i \mu(A_i)$. If $\mu(X) = 1$ then we say that μ is a probability measure and $(X, \mathcal{B}, \mu)$ is a probability space.

Definition 4.4.2. Let $T : X \mapsto X$ be a measure preserving map on a probability space $(X, \mathcal{B}, \mu)$. Then T is called ergodic if $T^{-1}(A) = A$, $\mu(A) = 1$ or $\mu(A) = 0$ for every $A \in \mathcal{B}$.

For any real number $x \in [0, 1)$, ts continued fraction expansion is

$$x = \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3 + \cfrac{1}{\ddots}}}} = [0; x_1, x_2, x_3, \ldots],$$

where $x_i \in \mathbb{Z}^+$, $x_0 = 0 \ \forall i \geq 0$.

The *Gauss map* is defined as

$$\begin{aligned} G : [0,1) &\mapsto [0,1) \\ x &\mapsto G(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \text{ and } G(0) = 0 \end{aligned}$$

Note that $\left\lfloor \frac{1}{x} \right\rfloor = n \in \mathbb{Z}^4$ if and only if $\frac{1}{n+1} < x \leq \frac{1}{n}$.

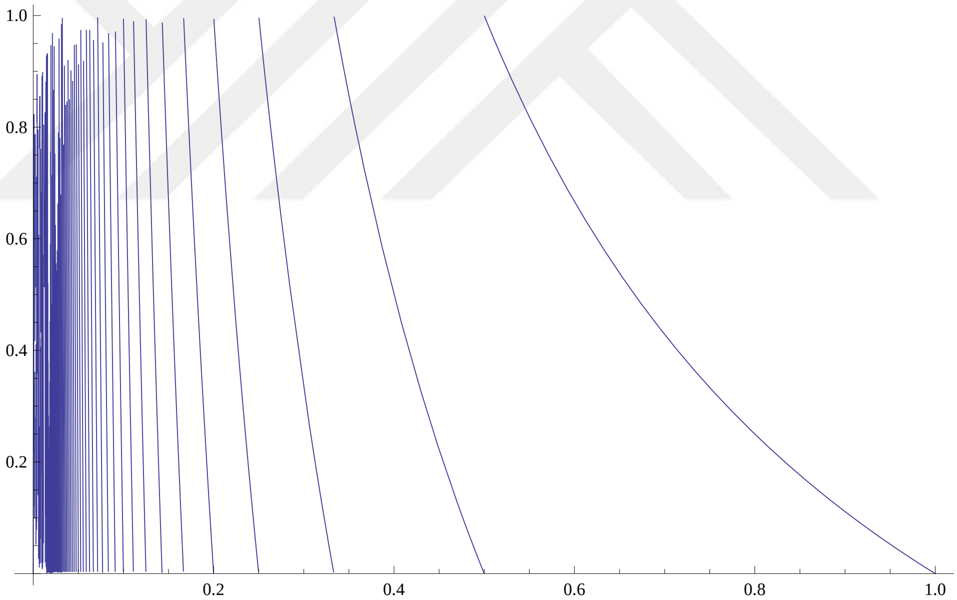

Figure 4.11 Graph of Gauss map

Some properties of Gauss map are as follows

- $G\left(\frac{1}{x}\right) = \frac{1}{\frac{1}{x}} - \left\lfloor \frac{1}{\frac{1}{x}} \right\rfloor = x - x = 0$ for all $x \in \mathbb{Z}^+$
- $\lim\limits_{x \to \frac{1}{n}^+} G(x) = \lim\limits_{x \to \frac{1}{n}^+} \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor = 1$ for all $x \in \left(\frac{1}{n}, \frac{1}{n-1}\right]$. Thus, G has discontinuity at each rational point $\frac{1}{n}$

- G is monotone decreasing on each interval $\left(\frac{1}{n}, \frac{1}{n-1}\right], n \geq 2$

Let us examine the action of Gauss map on the interval $[0, 1)$.

- If $\left\lfloor \frac{1}{x} \right\rfloor = x_1$, then $G(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor = \frac{1}{x} - x_1$. This implies that

$$x = \frac{1}{x_1 + G(x)} \Rightarrow x = [0; x_1 + G(x)].$$

- If $\left\lfloor \frac{1}{G(x)} \right\rfloor = x_2$, then $G^2(x) = \frac{1}{G(x)} - \left\lfloor \frac{1}{G(x)} \right\rfloor = \frac{1}{G(x)} - x_2$. This implies that $G(x) = \frac{1}{x_1 + G^2(x)}$. Therefore we get,

$$x = \cfrac{1}{x_1 + \cfrac{1}{x_2 + G^2(x)}} = [0; x_1, x_2 + G^2(x)].$$

- If $\left\lfloor \frac{1}{G^2(x)} \right\rfloor = x_3$, then $G^3(x) = \frac{1}{G^2(x)} - \left\lfloor \frac{1}{G^2(x)} \right\rfloor = \frac{1}{G^2(x)} - x_3$, which implies that $G^2(x) = \frac{1}{x_3 + G^3(x)}$. Therefore,

$$x = \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3 + G^3(x)}}} = [0; x_1, x_2, x_3 + G^3(x)].$$

  Continuing to this process and letting $\left\lfloor \frac{1}{G^{k-1}(x)} \right\rfloor = x_k$, we obtain the continued fraction expansion of x as $x = [0; x_1, x_2, \ldots, x_{k-1}, x_k + G^k(x)]$.

- Gauss map is one to one on the interval $\left[\frac{1}{n}, \frac{1}{n-1}\right)$. So every real number in $\left[\frac{1}{n}, \frac{1}{n-1}\right)$ has a unique continued fraction expansion.

- Let $x = [0; x_1, x_2, \ldots]$ be a real number in $[0, 1)$ then,

$$\frac{1}{x} = \cfrac{1}{\cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3 + \cfrac{1}{\ddots}}}}} = x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3 + \cfrac{1}{\ddots}}} = [x_1; x_2, x_3, \ldots]$$

and its integer part is $\left\lfloor \frac{1}{x} \right\rfloor = x_1$. Therefore,

$$G(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor = \cfrac{1}{x_2 + \cfrac{1}{x_3 + \cfrac{1}{x_4 + \cfrac{1}{\ddots}}}} = [0; x_2, x_3, x_4, \ldots].$$

Thus

$$G^{-1}(x) = \{[0; k, x_2, x_3, \ldots] : k \in \mathbb{Z}^+\} = \left\{ \frac{1}{k + x} : k \in \mathbb{Z}^+ \right\}$$

Definition 4.4.3.

$$\mu_G(S) = \frac{1}{\log 2} \int_S \frac{1}{1+x} dx \tag{4.21}$$

is called Gauss measure where S is measurable set.

Definition 4.4.4. The symbol $'' \asymp''$ is used to saying two expression A and B are asymptotic. It is notated by $A \asymp B$. In other words, if $A \asymp B$, there exists two positive constants c_1 and c_2 such that $c_1 B < A < c_2 B$.

Theorem 4.4.5. *The following inequality holds for all measurable set $S \subseteq [0, 1]$*

$$\frac{1}{2\log 2}\mu(S) < \mu_G(S) < \frac{1}{log2}\mu(S)$$

In other words, $\mu_G(S) \asymp \mu(S)$ where μ_G is Gauss measure and μ is Lebesgue measure.

Proposition 4.4.6. *(Isola, 2002) Gauss map G is a measure preserving map with respect to Gauss measure.*

Proof. We must show that $\mu_G([a, b]) = \mu_G(G^{-1}([a, b]))$ for any interval $[a, b]$.

$$
\begin{aligned}
\mu_G(G^{-1}([a,b])) &= \sum_{k=1}^{\infty} \frac{1}{\log 2} \int_{\frac{1}{b+k}}^{\frac{1}{a+k}} \frac{1}{x+1} dx \\
&= \frac{1}{\log 2} \sum_{k=1}^{\infty} \left[\log\left(1+\frac{1}{a+k}\right) - \log\left(1+\frac{1}{b+k}\right) \right] \\
&= \frac{1}{\log 2} \sum_{k=1}^{\infty} [\log(a+k+1) - \log(a+k) - \log(b+k+1) + \log(b+k)] \\
&= \frac{1}{\log 2} \lim_{K\mapsto\infty} \sum_{k=1}^{K} [\log(a+k+1) - \log(a+k) - \log(b+k+1) + \log(b+k)] \\
&= \frac{1}{\log 2} \lim_{K\mapsto\infty} [\log(a+K+1) - \log(a+1) - \log(b+K+1) + \log(b+1)] \\
&= \frac{1}{\log 2} \left[\log(b+1) - \log(a+1) + \lim_{K\mapsto\infty} \log\left(\frac{a+K+1}{b+K+1}\right) \right] \\
&= \frac{1}{\log 2} [\log(b+1) - \log(a+1)] \\
&= \frac{1}{\log 2} \int_b^a \frac{1}{x+1} dx \\
&= \mu_G([a,b])
\end{aligned}
$$

□

We observe that Gauss map is ergodic and $(X, \mathcal{B}, \mu_G, G)$ is a dynamical system where $X = [0,1) - \mathbb{Q}$ and μ_G is Gauss measure.

Theorem 4.4.7. *(Billingsley, 1978) Let $(X, \mathcal{B}, \mu, T)$ be a dynamic system with measure preserving ergodic transformation T.Then almost every $x \in X$*

$$\frac{1}{n} \sum_{i=1}^{n} f \circ T^i(x) \to \int f \, d\mu$$

where f is real valued measurable function.

Example 4.4.8. Now, we examine the application of ergodic property of continued fractions. Let $\beta = [b_0; b_1, b_2, \ldots, b_n, \ldots]$. Clearly $b_0 = 0$ if $\beta \in [0,1]$. Then

$$[b_1 + b_2 + \ldots + b_n]^{\frac{1}{n}} \to \prod_{k=1}^{\infty} \left(1 + \frac{1}{k(k+2)}\right)^{\frac{\log k}{\log 2}}$$

as $n \to \infty$ for almost real numbers β.

Proof. If the element b_1 of continued fraction expansion of β is n, then the real numbers lie in the interval $\left[\frac{1}{n+1}, \frac{1}{n}\right]$.

$$\begin{aligned}
\mu_G\{\beta[b_0; b_1, b_2, \ldots, b_n, \ldots] : b_0 = 0\} =& \mu_G\left\{\left[\frac{1}{n+1}, \frac{1}{n}\right]\right\} \\
=& \frac{1}{\log 2}\int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{dx}{1+x} \\
=& \frac{1}{\log 2}\log\left(\frac{(n+1)^2}{n(n+2)}\right) \\
=& \mu\{\beta[b_0; b_1, b_2, \ldots, b_n, \ldots] : b_0 = 0\} \\
=& \mu\left\{\left[\frac{1}{n+1}, \frac{1}{n}\right]\right\} \\
=& \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n^2+n} \asymp \frac{1}{n^2}.
\end{aligned}$$

However,

$$\int b_1(\beta) d\mu_G = \sum_{n=1}^{\infty} n\mu_G\{\beta[b_0; b_1, b_2, \ldots, b_n, \ldots] : b_0 = 0\} \asymp \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

$b_1(\beta)$ is not measurable with respect to both of Gauss and Lebesgue measures. Now we consider $\log b_1(\beta)$ instead of $b_1(\beta)$.

$$\int \log(b_1(\beta)) d\mu_G = \sum_{n=1}^{\infty} \log n\mu_G\{\beta[b_0; b_1, b_2, \ldots, b_n, \ldots] : b_0 = 0\} \asymp \sum_{n=1}^{\infty} \frac{\log n}{n} < \infty$$

Thus $\log(b_1(\beta))$ is real valued measurable function. So,when we use the Theorem 4.4.7 , we get

$$
\begin{aligned}
\frac{1}{m}\sum_{i=1}^{m-1} log(b_1(\beta))\circ G^i(\beta) &= \frac{1}{m}\sum_{i=1}^{m-1} log(b_1(G^i(\beta)) \\
&\to \int \log(b_1(\beta))d\mu_G = \sum_{n=1}^{\infty} b_1(\beta)\mu_G\{b_1(\beta)=1\} \\
&= \sum_{n=1}^{\infty}\frac{\log n}{\log 2}\log\left(\frac{n(n+2)}{(n+1)^2}\right) \\
a.e\ as\ m\to\infty &
\end{aligned}
$$

Thus , when we rewrite the above result, we get

$$
\frac{1}{n}\log[b_1(\beta)+b_2(\beta)+\ldots+b_n(\beta)] \to \sum_{k=1}^{\infty}\frac{\log k}{\log 2}\log\left(1+\frac{1}{k(k+2)}\right)
$$

or

$$
[b_1(\beta)+b_2(\beta)+\ldots+b_n(\beta)]^{\frac{1}{n}} \to \prod_{k=1}^{\infty}\left(1+\frac{1}{k(k+2)}\right)^{\frac{\log k}{\log 2}}
$$

In addition,

$$
\prod_{k=1}^{\infty}\left(1+\frac{1}{k(k+2)}\right)^{\frac{\log k}{\log 2}} \approx 2.685452001\ldots
$$

This constant is called Khinchin's constant (Khintchine, 1936). $\square$

CHAPTER FIVE
CONCLUSION

We have discussed the arithmetic properties of continued fractions and their geometric meaning. Convergents of a continued fraction expansion of a real or an irrational number give a series which converge to the real or the irrational number. We have characterized the odd indexed and even indexed convergents. In addition, we introduced their properties and bounds on the order of denominators of convergents. This bound is considered as a rate of error while we are approximating to the real or the irrational number. Convergents of a continued fraction expansion correspond to the end points of geodesics on the upper half plane $\mathcal{H}$.

Farey tree is a tree consisting of rational numbers which are obtained by Farey sum or hyperbolic mediant. The elements of the Farey tree are arranged in decreasing order. When we reach an element of the Farey tree to the another element, we use the element of Modular group which is also known two dimensional special projective linear group ($PSL(2,\mathbb{Z})$). The action of these elements of modular group corresponds to move from a rational number to another one on boundary of $\mathcal{H}$, along geodesics on $\mathcal{H}$. Besides, modular group remains invariant the Farey tree. By Farey sum (mediant) of convergents of a continued fraction expansion, we get a rational number, a more closer to the real number, and a new geodesic can be drawn with end points the rational number on $\mathcal{H}$.

On the other hand, Poincaré-Kobe theorem states that Riemann surfaces of genus $g > 0$ become a branch cover of the upper half plane. So triangulations of the upper half plane will correspond to triangulations of the Riemann surfaces. Since, ideal geodesic triangulations of the upper half plane is determined by the end points of the geodesics on the boundary, then it is completely determined by three real numbers. The continued fraction expansions of them determine the geodesic area. Therefore, a mappings of the continued fractions will correspond to an automorphism of the upper half plane and so the automorphism of the Riemann surface. These are related with the Teichmüller spaces,where there are many open questions.

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