REPLENISHMENT DECISIONS UNDER QUANTITY DISCOUNTS AND EXPIRY DATES

A Master's Thesis

by

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Ankara

August 2006

To my parents, Adile and Erol

REPLENISHMENT DECISIONS UNDER QUANTITY DISCOUNTS AND EXPIRY DATES

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of

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by

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August 2006

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science in Management.

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ABSTRACT

Replenishment Decisions under Quantity Discounts and Expiry Dates

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An M.S. Thesis

Supervisor: Assist. Prof. Dr. Emre Berk

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In this study, we model a single buyer-single seller system under a quantity discount schedule with expiry dates. The buyer faces constant demand and shortages are not allowed. The seller offers a quantity discount which is effective on the buyer's next purchase to be exercised by a certain expiry date given that the current purchase exceeds a minimum quantity. We derive the optimal replenishment policy structure for the buyer which minimizes his/her total cost rate. We obtain the expressions for the operating characteristics of the buyer and investigate the structural properties of the objective function. Under a specific inventory holding cost structure, we also investigate the impact of the buyer's decisions on the entire buyer-seller inventory system. We conduct an extensive numerical study. Our findings show that quantity discount schedules with expiry dates may be beneficial for both the buyer and the seller in a variety of operating cost settings.

Keywords: Inventory Theory, Quantity Discounts, Discount Expiry Date

ÖZET

Süreli Miktar İskontoları ile İkmal Kararları

Erim Ergene

Yüksek Lisans Tezi

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Bu çalışmada, süreli miktar iskontosu uygulanan tek müşterili ve tek tedarikçili bir sistem incelenmektedir. Müşteri sabit talep altında olup, sistemde yokluğa izin verilmemektedir. Tedarikçi bir sonraki siparişte kullanılmak üzere miktara bağlı bir iskonto teklif etmektedir. Bu teklife göre, müşteri belli bir miktar üzerinde sipariş verdiği takdirde, takip eden siparişinde iskontolu alış yapabilecekdır. Ancak iskontolu siparişin belli bir süre içerisinde gerçekleşmesi gerekmektedir. Bu sistemde, müşterinin toplam maliyet oranını en aza indirgeyen optimal tedarik politikası elde edilmiştir. Müşterinin maliyetini ilgilendiren denklemler bulunmuş ve amaç fonksiyonunun yapısal özellikleri incelenmiştir. Belirli bir stok tutma yapısında, müşterinin kararlarının tüm tedarik zincirinin envanter sistemine olan etkileri de araştırılmıştır. Kapsamlı bir sayısal çalışma sonucunda, süreli miktar iskontolarının; çeşitli maliyet parametrelerinde, hem müşteri hem de tedarikçi için faydalı olabileceği gösterilmiştir.

Anahtar Kelimeler: Envanter Kuramı, Miktar İskontosu, Süreli Miktar İskontosu

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CHAPTER 1

INTRODUCTION AND LITERATURE

1.1 Introduction

Quantity discounts have been studied for many years under different conditions. Majority of these studies have focused on quantities that are required for a discount, and the discount rates that are required to lure buyers to switch their order schedules to the ones that are offered by the suppliers. However not many have looked at expiring discounts.

Our motivation for this study comes from internet companies offering discounts to potential buyers. Many online stores offer individualized or general public discounts that are based on the number of past and/or current purchases made by customers. For example drugstore.com offered a general discount to all of its customers a "buy-one get-one free discount" on selected products on a sale that expired on June 9th, 2006. The author of this thesis frequently receives emails from Amazon.com for discounts that are customized for his past purchases. There are also offers for "preferred customers" where a discount or a coupon is offered by a company based on the purchases made by the customer. An example is cardsorder.com, an online memorabilia store, offered a 5% discount to be used for an order made in 2006, only available to customers that had purchases in 2005.

Such discounts gave the initial motivation that led to this research. We look at a single buyer, single seller system where the seller is offering a quantity discount to a buyer that is currently a customer of the seller. This discount offer is modeled as such: The seller will offer a discount on the conditions that if the buyer makes an order larger than a specified minimum, than the seller will give a discount on the next purchase if and only if this next purchase comes within a specified amount of time, basically an expiration date on the offer. If the offer is not used than it is lost. Cardsorder.com is a perfect example of the model that we use here. Since the offer was only sent to customers with purchases made, this could be looked at as the minimum order quantity requirement, and the discount is only good for a year, the expiration date.

We have a deterministic setting, constant demand rate for the buyer, an order cost, a variable per unit cost for the buyer and the buyers holding cost. We do not allow any shortages or lead times. Our model could be extended easily to handle multiple identical buyers as well as constant lead times.

Our model incorporates quantity discounts with expiry dates and compares them to the economic order quantity model (EOQ). Under the assumptions that we have stated, we know that the EOQ model will provide the optimal solution to the buyer's decisions where no discounts are presented. This model is used to calculate the optimal ordering quantity, which then enables us to calculate the cost rates associated with that order quantity. Quantity discounts are offers from the supplier, where purchasing a minimum quantity gives eligibility to the buyer to exercise cost reductions on his order. Under such conditions the EOQ model is augmented and decision criterion are calculated for the buyer to optimize his/her decisions for the offered minimum quantity and the discount for the costs. In our model, the discount is not offered on the purchase where the minimum quantity requirement is met, but rather at the next order that the buyer places if the time between these two orders is less than the expiry date that the supplier set.

We first provide information on the current literature, chapter two provides the details and assumptions on the model and its theoretical construction, chapter three extends our research to investigate the supplier's motivation and decisions, chapter four looks at our numerical analysis, and finally chapter five concludes with limitations and future work.

1.2 Literature Review

Quantity discounts have been studied extensively under different contexts for years. Researchers have focused on different aspects of quantity discounts and their characteristics, such as the number of customers that have the discount offer, the homogeneity - or the lack of it - among customers, the price and quantity breaks for the discount offer, all-unit or incremental discounts, number of periods where the discount is offered, buyer and/or seller perspectives, and in the recent years coordination between the buyers and suppliers decisions. There are also other streams of research where quantity discounts are investigated for different purposes other than the inventory theory, which we are also focusing on, but the marketing side of these discounts and their effectiveness in changing customer buying patterns and behaviors.

We are going to limit our literature review as it is impossible to discuss such a wide literature in this thesis. Munson and Rosenblatt (1998) provides an extensive literature review, where studies are categorized according to three perspectives; buyer's, seller's and joint buyer-seller perspectives. Under these three sections they further categorize each study according to the discount characteristics and the assumptions that have been used in the studies. This paper provides a valuable reference point. Benton and Park (1996) and Dolan (1987) also provide other reviews of the literature for quantity discounts.

There are some papers that have similar characteristics with our research. Wang (2002) looks at a single supplier-multiple buyer system under an EOQ setting where the supplier offers multiple discount points for the buyers utilizing a Stackelberg game. Supplier has full knowledge of the buyers' demand and cost parameters. In his model, Wang shows that none of the buyers will be worse off using any of the discount points that are provided; larger buyers will gain more at a larger discount point than the smaller buyers will; and that supplier and the buyers will be better off if the discount offer is utilized. Wang solves a nonlinear programming problem and calculates the break points which the buyers would choose, and investigates the empirical study that was first solved in Lal and Staelin (1984). This empirical study, composed of 1128 companies grouped into seven categories of buyers, hence seven non-identical buyers, shows that when such grouping is done, both the buyers and the supplier benefits. If each buyer is offered an individual discount rate, supplier benefits from the increased sales but the buyers have no additional gain. The main difference between the works by Wang (2002) and Lal and Staelin (1984) is that, the latter models the problem using a continuous approximation of a discrete quantity discount schedule where the price of the product is a monotonically decreasing function of the quantity ordered.

Monahan (1984) investigates a single buyer-single seller system where the buyer is offered to increase his/her order size by a factor of K to be eligible for an all unit single break point discount, and investigates the savings in transportation costs. Lee and Rosenblatt (1986) criticizes Monahan; under Monahan's model undiscounted prices could be lower than the discounted prices and the supplier employs an order-for-order manufacturing policy which might not be optimal. An integer multiple manufacturing policy is formed; a constraint on the discount is proposed and the problem is solved. Joglekar (1988) extends the works of Monahan (1984) and Lee and Rosenblatt(1986) for the supplier's inventory cost to be affected by the order size. Monahan (1988) acknowledged above criticisms and noted that his introductory work was not attempting to increase total orders for the supplier but rather to decrease order frequencies.

Dada and Srikanth (1987) relaxed two assumptions proposed by Lal and Staelin (1984): Holding cost of the supplier is less than approximately 70% of the buyer's holding cost and the buyer's holding cost is less than one-third of the undiscounted purchase price. Only restriction that is used by Dada and Srinkanth is that holding cost of the buyer is larger than the holding cost of the supplier. Kohli and Park (1989) extended the work of Dada and Srinkanth by analyzing the case as a bargaining problem where the parties negotiated for both all unit and incremental quantity discounts.

Weng (1995) investigates both the buyer's and the supplier's profit functions under an EOQ setting. Their joint profit function is found and the profit is split to the parties using a quantity discount model where the buyer makes a fixed payment to the supplier, i.e. a franchise fee. Parlar and Wang (1994) investigate a single supplier-multiple homogeneous buyer system under an EOQ setting, where the model is analyzed for two cases: a stackelberg game without discounts, and with a single discount break point. Joint profit function is also solved, and it is shown that the joint maximization provides the most profits for both parties.

There is limited research on time dependent discounts in the literature. Matta (1994) have looked at lead-time dependent discounts, where the time of the order is important and "early" orders are offered a discount. Sirias and Mehra (2005) have compared quantity discounts and lead-time dependent discounts as an incentive in the coordination of the supply chain. Such lead-time dependent discounts are investigated for their effectiveness of the transfer of information and channel coordination, see Whang(1993), Chen (1999), and Sahin and Robinson (2002).

Marketing literature have also looked at such discounts but their focus have been coupons and their effects on the buyer's purchase behavior. Coupons are not necessarily quantity discounts, but one could argue that they would have similar effects on the buyer, just because they both induce the buyer to purchase before the discount could be achieved. One paper that should be noted is Krishna and Zhang (1999), as they have looked at the effects of a coupon's expiration date on the profitability of such promotions. They show that the duration of the coupon's life will affect the customer's purchase behavior, and different firms should utilize different durations for maximum profitability.

Our research extends current stream of literature by it's use of the expiry date. We have followed the trend of using EOQ settings where there is no backordering or lost sales and zero lead times, with deterministic cost parameters but we introduce the time factor into the setting to better illustrate the problem.

CHAPTER 2

THE MODEL

In this chapter. we introduce the basic assumptions of the model, derive its operating characteristics, obtain some structural results toward the optimization of the problem and present the optimal replenishment policy for this model.

2.1 Basic Assumptions and the Model

Consider a single buyer-single supplier system. Shortages are not allowed and there is no lead time. The buyer faces a constant demand rate, D, that is not affected by his supply or inventory level. The buyer incurs three costs: an inventory holding cost, a fixed order cost and a (variable) purchase cost. Holding cost is charged at h per unit of stock held per unit time. The buyer faces quantity discount schedule stated as follows: If a minimum of Q_T units are ordered, the buyer becomes eligible for a future discount on the fixed ordering and unit purchasing costs. The discount eligibility expires in T time units following the order placement. If discounts are not exercised, the buyer incurs the fixed ordering cost of K_l and a unit purchasing cost of c_l. In case that the discount is exercised, the buyer incurs the fixed ordering cost of K_d and a unit purchasing cost of c_d.

In the proposed discount schedule, the seller specifies (i) the discount eligibility quantity, Q_T, (ii) the expiry date, T, and (iii) the discount depth, r. We set $r = 1 - c_d/c_l = 1 - K_d/K_l$ through the analysis. We shall refer to each combination of (Q_T, T, r) as an offering. Note that $K_l > K_d$ and $c_l > c_d$ and clearly $K_l - K_d$ maybe viewed as a payment from the supplier to the buyer as part of the discount offered, such as a rebate or a reduction in shipment costs.

We shall use the renewal theoretic approach to model the system. We define a cycle as two consecutive instances at which a purchase sequence ends. A purchase sequence consists of two subcycles. The first subcycle, named the "n subcycle", is the period where no discounts are offered to the buyer. Buyer is purchasing, for a total of n purchases, a constant quantity from the supplier under the EOQ setting. The supplier then offers a discount under specific terms which states that if the buyer purchases more than a specified minimum quantity, the supplier will offer a discount for both the order and the variable costs on his next purchase, given that the next purchase is done within a specified expiry date. Each purchase greater than the minimum quantity qualifies the buyer for another discount on his next purchase done within the expiry date. For example if the buyer makes $(m-1)$ purchases of minimum quantity or more, and the time between each order is less than or equal to the expiration time, than only the first order will be at the list prices while others are discounted and the buyer will have an option to make another purchase, m^{th} purchase, at the discounted price if that order comes within the specified time frame. If the next purchase is not made by the end of this time period the discount offer expires.

Buyer now faces a decision to make (see Figure 2-1). The offer has three parameters, the minimum quantity Q_T, the expiration date T, and the discount rate r and the buyer will make his decision based on these three values. These parameters affect the inventory level of the buyer and he needs to investigate what will happen if he chooses to utilize this offer. If Q_T is less than his demand occurring during the time frame that the discount is available, i.e. DT, then he will utilize this discount without any further analysis as he needs to purchase more than Q_T to meet the demand from his customers. However if Q_T is greater than DT, then the buyer is going to be increasing his inventory level if he chooses to use this offer, as extra supply will be purchased before the buyer needs to purchase again to meet customer demand.

The supplier has full knowledge of this situation as she is aware of the buyers past purchasing behavior. She could give an offer which would have a minium quantity less than the buyer's DT but in this case the problem is trivial, the buyer will modify his purchase, either he would buy more than Q_T at the end of each expiration date, or he would make purchases of size Q_T more frequently, while enjoying the discounted prices. We will be investigating the interesting problem where Q_T is greater than DT. At this case, the buyer, if chooses to utilize the offer, will be increasing his inventory indefinitely and is expected to stop inflating his inventory at some point. This subcycle is called the "m subcycle" continuing

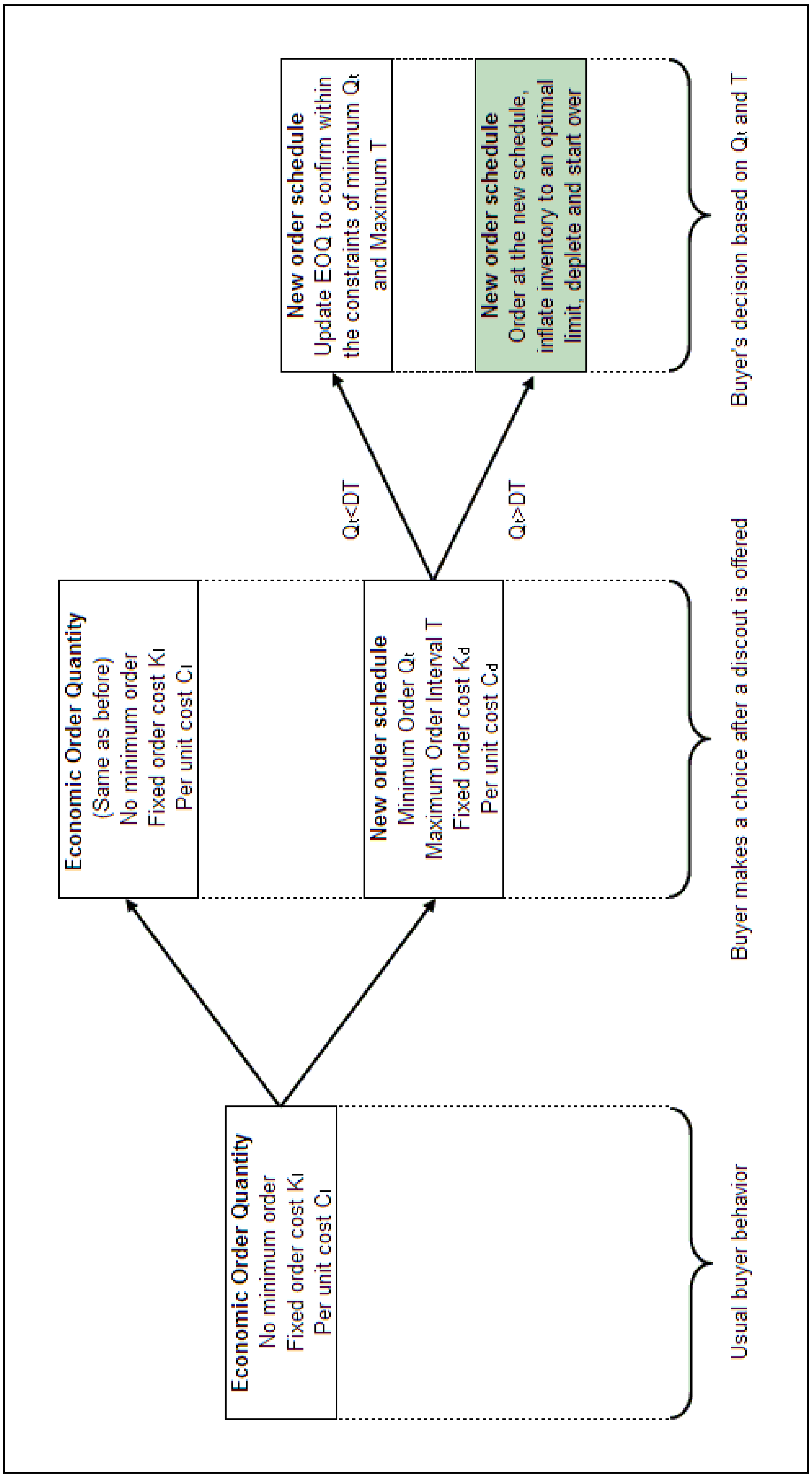

Figure 2-1: Buyer's decisions

for m purchases. The buyer first makes a purchase of $Q_0 > Q_T$ at the list prices and then makes a series of purchases greater than Q_T at the discounted prices. After the m^{th} purchase, the buyer stops this cycle, makes his last purchase, where the order size is independent of Q_T, at the discount prices and starts meeting his demand from his inventory until it is completely depleted; Figure 2-2 depicts the inventory level of the buyer through one full cycle.

Our problem definition states that there is an expiration date on the discount offer, T. During the construction of our cost functions we use this expiration date as the length of each period during the m subcycle. One might wonder if the buyer would purchase before this deal expires, i.e. the time between two orders could be less than T. However this would not be applicable to the case we are investigating; we know that $Q_T > DT$ meaning that inventory purchased will not be depleted by the end of the expiration date. This indicates that ordering before the deal expires would not be optimal to the buyer as he does not have any additional benefits from this behavior: He will be increasing his inventory, and his inventory carrying costs, earlier than he would need to if he chose to continue with the m subcycle. We will assume that the buyer would make his next purchase using the discount at the end of the expiration time, thus the time between two orders will be exactly T.

We are interested in finding whether this cycle repeats itself, first the n subcycle then the m subcycle. Our aim is to see whether the n subcycle affects the m subcycle, calculate the maximum inventory level that the buyer will be reaching and investigate under which conditions such a discount will be optimal to the

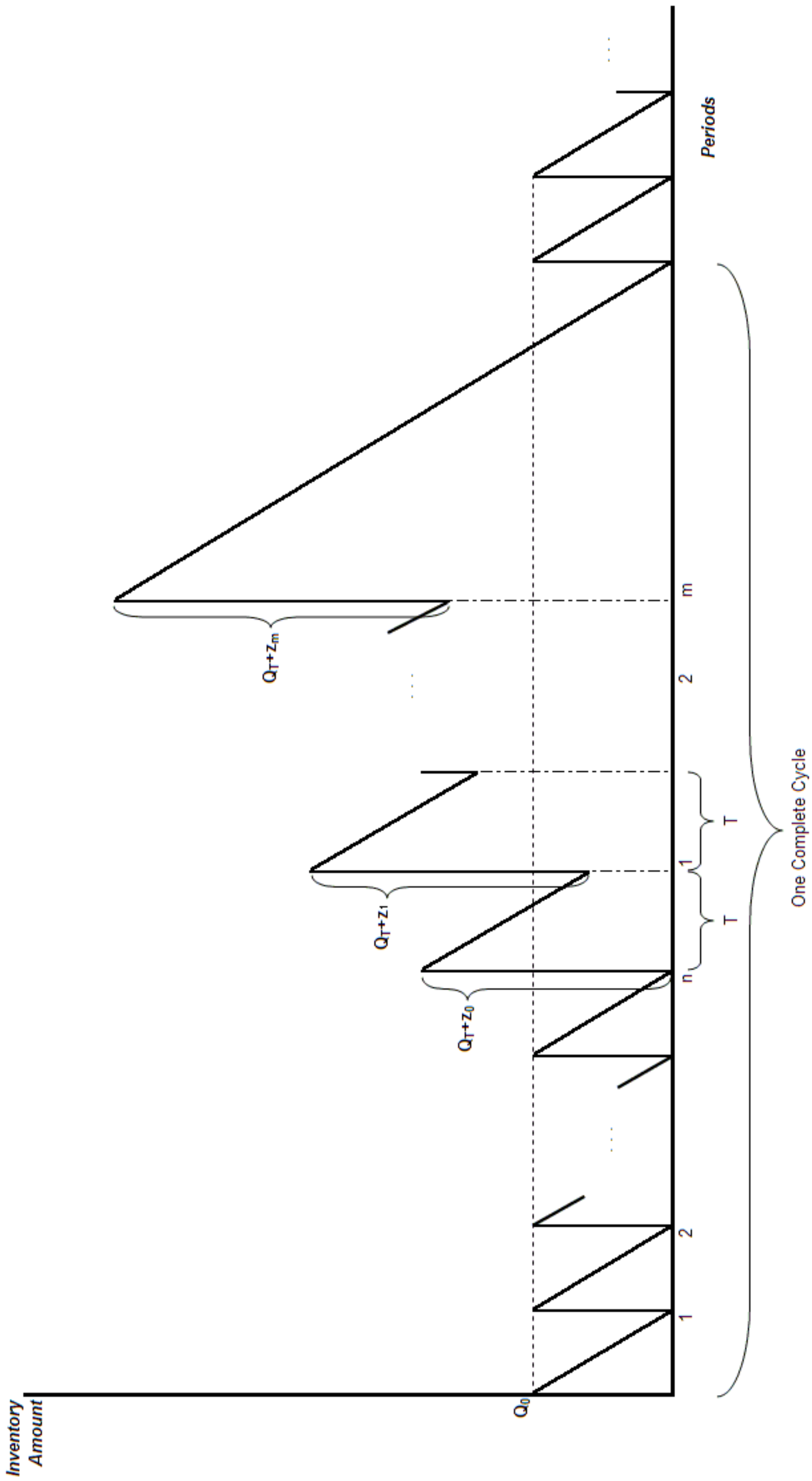

Figure 2-2: Inventory level of the buyer throughout one full cycle of acquisition

buyer. We will be looking at the total cost rate, calculated by total costs incurred through the complete cycle divided by the cycle length.

We introduce some notation below:

Q'_0: Usual ordering quantity

Q_T: Discount eligible quantity

Q_i: Order quantity in the cycle except last, $i = 0$ to $m - 1$

Q_m: Last order quantity

n: Number of times Q'_0 is ordered in the cycle

m: Number of orders of minimum size Q_T, $m \geq 1$

T: Expiry date for the offer, length of each period during m subcycle

z_i: Excess in Q_i over Q_T such that $Q_i = Q_T + z_i$

D: Demand rate

c_l: Unit list price

c_d: Unit discount price

K_l: List fixed ordering cost

K_d: Discount fixed ordering cost

h: Inventory holding cost per unit per unit of time

The objective from the buyer's perspective is to obtain the optimal replenishment policy parameters (n^*, Q'^*_0, m^*, Q^*_i and Q^*_m) which minimize the total cost rate. To this end, we will first derive the expressions for the operating characteristics of the system, and then derive the optimization results on the decision variables.

2.2 Operating Characteristics

In this section, we compute the operating characteristics of our model.

The cycle length, CL, consists of two segments: (i) the one corresponding to n many orders of the size Q_0' during which no discount eligibility occurs and (ii) the one corresponding to m orders during which discounts are taken into account, thus:

$$\begin{aligned} CL &= \tfrac{n}{D}Q_0' + mT + \tfrac{1}{D}[mQ_T + \sum_{i=0}^{m-1} z_i - mDT + Q_m] \\ &= \tfrac{n}{D}Q_0' + \tfrac{1}{D}mQ_T + \tfrac{1}{D}\sum_{i=0}^{m-1} z_i + \tfrac{1}{D}Q_m \end{aligned} \tag{2.1}$$

The ordering cost per cycle, OC, consists of the fixed ordering costs associated with list and discount orders, hence:

$$\begin{aligned} OC &= nK_l + K_l + mK_d \\ &= (n+1)K_l + mK_d \end{aligned} \tag{2.2}$$

The holding cost per cycle, HC, is composed of the inventory related costs carried over the two subcycles. It is found by computing the area under the buyers inventory curve, see Figure 2-2, and is given by:

$$HC = \frac{nh}{2D}Q_0'^2 + \{\frac{(Q_T+z_0+Q_T+z_0-DT)}{2}hT + (\frac{2(Q_T+z_0-DT+Q_T+z_1)-DT}{2})hT+ (\frac{2(2Q_T+z_0+z_1-2DT+Q_T+z_2)-DT}{2})hT + ... + \frac{hT}{2}(2[mQ_T + \sum_{i=0}^{k-1} z_i - (k-1)DT] - DT)\} + \frac{h}{2D}[mQ_T + \sum_{i=0}^{m-1} z_i - mDT + Q_m]^2$$

$$HC = \frac{nh}{2D}Q_0'^2 + \{\sum_{k=1}^{m} \frac{2[kQ_T+\sum_{i=0}^{k-1} z_i-(k-1)DT]-DT}{2}\}hT + \frac{[mQ_T+\sum_{i=0}^{m-1} z_i-mDT+Q_m]^2h}{2D} \tag{2.3}$$

The purchase cost per cycle, PC, consists of expenses for purchasing units at the list and the discounted prices.

$$PC = nQ_0'c_l + c_l(Q_T + z_0) + c_d \sum_{i=1}^{m-1}(Q_T + z_i) + c_dQ_m \tag{2.4}$$

The total cost per cycle, CC, is given by CC=OC+HC+PC. Then the total cost rate, TC, can be written as follows:

$$\begin{aligned} TC &= \frac{CC}{CL} \\ &= \frac{OC+HC+PC}{CL} \end{aligned} \tag{2.5}$$

2.3 Structural Results

In this section we obtain the analytical results on the structure of an optimal replenishment cycle for the buyer. We begin our results by showing that the buyer will not order more than Q_T during the m subcycle except for the last order.

Theorem 1 $z_i = 0$ *for all* $i = 0$ *to* $m - 1$.

Proof. Inventory position at the beginning of the m^{th} period in the m subcycle depends on the cost values, since the inventory accumulated (either through acquisition in the previous periods or purchased at the beginning of the m^{th} period) will only have effect on the future periods and is not dependent on the past periods. As we know that during the n-subcyle the inventory will be depleted by the nature of our setup, let us look at the inventory accumulated during the m-subcycle at the beginning of the m^{th} period after the last purchase is made, IL_m:

$$IL_m = \sum_{i=0}^{m} Q_i - mDT = mQ_T + \sum_{i=0}^{m-1} z_i - mDT + Q_m$$

We know that orders are made every period until the m^{th} period where the buyer decides to stop inflating his inventory, and purchases for the last time at the discounted prices and ends the cycle. Excess inventory purchased - purchases with order sizes larger than Q_T - during the m subcycle will be stored in the inventory, only to be used at the m^{th} period. We know that this is the case since we already assume that $Q_T > DT$, hence a portion of that order will not be used, adding to the extra inventory to be used after the last purchase of size Q_m. However if that excess is to be used at the m^{th} period, the buyer has the opportunity to purchase those items at the m^{th} period, and he does not need to carry the inventory cost for more than necessary. Thus it is obviously seen that if $z_i = 0$ for all $i = 0$ to $m - 1$ then total inventory accumulated at the end of the $(m-1)^{th}$ period will be decreased, which will reduce the total cost of the cycle. Hence we can conclude that $z_i = 0$ for all $i = 0$ to $m - 1$. ■

Since we know that $z_i = 0$ for all $i = 0$ to $m - 1$ from Theorem 1, we will rewrite the costs associated with the total cost rate, where for all $i = 0$ to $m - 1$, $Q_i = Q_T$. We leave Q_m as is, it could have any value, as it is the last purchase at the discounted prices, the buyer could order any quantity independent of Q_T. For notational convenience, we will denote Q_0', the order size during the n subcycle, as Q_0.

$$CL = \frac{n}{D}Q_0 + \frac{mQ_T}{D} + \frac{1}{D}Q_m \tag{2.6}$$

$$OC = (n+1)K_l + mK_d \tag{2.7}$$

$$HC = \frac{nh}{2D}Q_0^2 + \{\sum_{k=1}^{m} \frac{2[kQ_T - (k-1)DT] - DT}{2}\}hT + [m(Q_T - DT) + Q_m]^2\frac{h}{2D} \tag{2.8}$$

$$PC = nQ_0c_l + c_lQ_T + (m-1)c_dQ_T + c_dQ_m \tag{2.9}$$

$$\begin{aligned} CC = \quad & (n+1)K_l + mK_d + \tfrac{nh}{2D}Q_0^2 + \{\sum_{k=1}^{m} \tfrac{2[kQ_T-(k-1)DT]-DT}{2}\}hT + \\ & [m(Q_T - DT) + Q_m]^2 \tfrac{h}{2D} + nQ_0c_l + c_lQ_T + (m-1)c_dQ_T + c_dQ_m \end{aligned} \tag{2.10}$$

We have computed our optimality condition and realized that the total cycle does not depend on n. Hence we know that the decision of the buyer does depend on his past purchases made in the n subcycle, but rather lies solely with the m subcycle.

Theorem 2 *A mixed policy is not optimal in one cycle. Either the buyer will continue to purchase his EOQ or he will use the discounted schedule offered by the supplier: Cycle will only use one of the two purchase offers.*

Proof. Employing the first order condition on n reveals that whichever cost rate is smallest in the subcycles will dominate. The detailed derivations are given in Appendix A. ■

Our initial setup was such that both subcycles will occur one after the other. However our finding for n shows us that the n subcycle is not significant in finding the total cost rate of the complete cycle as the decision of using a discount offer does not lie with previous purchases but with the decision variables that are associated with the m subcycle. The proof of the theorem 2 is trivial in showing that past purchases do not affect the future decisions. However, we have kept this part in this study, as it was the beginning motivation for our research. We wanted to be able to see a cyclic motion, a cycle that is composed of two subcycles, basically a mixed order policy. We now see that this mixed policy does not occur, and either the EOQ policy or the discounted policy will be optimal to the buyer. The only instance where we could have an optimal mixed order policy will be if the supplier had an additional criterion on the buyer's eligibility of the discount schedule; i.e. a definition of a "preferred customer". If this was the case, where the supplier made the discount offer only to customers she deemed "preferred" based on a criterion that included past purchases, than such a mixed policy might occur.

We have shown that our cycle composed of n and m subcycles do not repeat, and the decision of using the discounted ordering pattern depends on the m subcycle only. We will now proceed to formulate m subcycle alone, and find m and Q_m through the total cost rate, i.e. total cost divided by cycle length. Our functions related to the total cost are the same as before, we only assume that $n = 0$.Let us look at the buyer's cost functions once again.

Ordering cost per cycle is given by:

$$OC = K_l + mK_d \tag{2.11}$$

Purchasing cost per cycle is given by:

$$PC = c_l Q_T + (m-1)c_d Q_T + c_d Q_m \tag{2.12}$$

Note that we have simplified the expression for the holding cost per cycle, see Appendix A, proof of Theorem 3; then the holding cost is given by:

$$HC = m^2\left(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2\right) + m\left(\frac{hT}{2}Q_T + \frac{h}{D}Q_mQ_T - hQ_mT\right) + \frac{h}{2D}Q_m^2 \quad (2.13)$$

Hence the cycle cost is:

$$\begin{aligned} CC = \quad & K_l + mK_d + c_lQ_T + (m-1)c_dQ_T + c_dQ_m + m^2(-\tfrac{hT}{2}Q_T + \tfrac{h}{2D}Q_T^2) + \\ & m(\tfrac{hT}{2}Q_T + \tfrac{h}{D}Q_mQ_T - hQ_mT) + \tfrac{h}{2D}Q_m^2 \end{aligned} \quad (2.14)$$

Finally the cycle length is given by:

$$CL = \frac{1}{D}mQ_T + \frac{1}{D}Q_m \quad (2.15)$$

So the total cost rate is the ratio of the cycle cost to cycle length:

$$\begin{aligned} TC = \quad & \frac{1}{\frac{1}{D}mQ_T + \frac{1}{D}Q_m} \quad [K_l + mK_d + c_lQ_T + (m-1)c_dQ_T + c_dQ_m + \\ & m^2(-\tfrac{hT}{2}Q_T + \tfrac{h}{2D}Q_T^2) + m(\tfrac{hT}{2}Q_T + \tfrac{h}{D}Q_mQ_T - hQ_mT) \\ & + \tfrac{h}{2D}Q_m^2] \end{aligned} \quad (2.16)$$

Before investigating the optimal values of m and Q_m, let us look at the convexity of the buyer's total cost rate.

Theorem 3 *For all* $m \geq 1$, *TC is quasi-convex in* Q_m. *Furthermore, for all* $Q_m \geq 0$, *TC is quasi-convex in* m.

Proof. This result follows from computing the second derivatives of the cycle cost and the cycle length with respect to Q_m and m. We show that cycle cost is convex in Q_m and cycle length is concave in Q_m for $m \geq 1$, for the quasi-convexity of the cost rate in Q_m. Similarly we show that cycle cost is convex in m and cycle length is concave in m for $Q_m \geq 0$, see appendix A for the proof. ■

The quasi-convexity of the buyer's total cost rate in m and Q_m indicates that the total cost rate function is unimodal, we could find the extrema for m and Q_m for the total cost rate. To find the these values we will look at the first order condition of the total cost rate. We simplify the first order condition as follows: Assume that A and B are functions in x, then the first order condition will be $\frac{\partial}{\partial x}(\frac{A}{B}) = 0$.

$$\frac{\partial}{\partial x}(\frac{A}{B}) = \frac{1}{B^2}(\frac{\partial A}{\partial x}B - A\frac{\partial B}{\partial x}) = 0$$

$$\frac{\partial A}{\partial x}B\frac{1}{B^2} - A\frac{\partial B}{\partial x}\frac{1}{B^2} = 0$$

$$\frac{\partial A}{\partial x}\frac{1}{B} = A\frac{\partial B}{\partial x}\frac{1}{B^2}$$

$$\frac{\partial A}{\partial x} = A\frac{\partial B}{\partial x}\frac{1}{B}$$

$$\frac{\frac{\partial A}{\partial x}}{\frac{\partial B}{\partial x}} = \frac{A}{B} \tag{2.17}$$

We will be using the above result in finding the extrema of the total cost rate. We first investigate the extrema of Q_m, which we denote by $\hat{Q}_m$.

Lemma 1 *For any* $m > 1$, $\hat{Q}_m$ *is equal to the single positive root to the equation:*

$$\frac{h}{2D}Q_m^2 + m\frac{h}{D}Q_TQ_m - K_l - mK_d - c_lQ_T + c_dQ_T - m^2\frac{hT}{2}Q_T + m^2\frac{h}{2D}Q_T^2 - m\frac{hT}{2}Q_T = 0$$

if it exists; otherwise, Q_m *is zero.*

Proof. Follows from the first order condition, see Appendix A. ■

We have found the $\hat{Q}_m$ for any value of m. We then set the first order equations with respect to m and Q_m to each other and find another result.

Lemma 2 *When* $m = \hat{m}$, $\hat{Q}_m$ *is also given by the equation* $\frac{1}{hT}K_d + \frac{1}{2}Q_T$.

Proof. Follows from equalizing the first order condition of the buyer's cost rate with respect to m and Q_m, see Appendix A. ■

As we have found Q_m for any positive m, we will now compute m. But note that m is integer valued. Therefore we could find the optimal m through the first difference condition; m^* is the minimum m such that:

$$TC(m+1) - TC(m) \geq 0 \tag{2.18}$$

Due to its tedium, we will hereafter treat m as a real number.

Lemma 3 *When* $Q_m = \frac{1}{hT}K_d + \frac{1}{2}Q_T$, $\hat{m}$ *is given by unique positive root that solves the following:*

$$m^2(-\tfrac{hT}{2}Q_T + \tfrac{h}{2D}Q_T^2) + m(\tfrac{1}{DT}K_dQ_T + \tfrac{h}{2D}Q_T^2 - K_d - \tfrac{hT}{2}Q_T) + \tfrac{1}{2DThT}K_d^2 + \tfrac{1}{2TD}Q_TK_d + c_dQ_T - c_lQ_T - K_l + \tfrac{h}{8D}Q_T^2 = 0$$

Proof. Proof rests on computing the first order condition of the buyer's total cost rate with respect to m when $Q_m = \frac{1}{hT}K_d + \frac{1}{2}Q_T$. See Appendix A for details. ■

Theorem 3 does not prove that $\hat{m}$ will always have a positive root, it only shows that $\frac{-b-\sqrt{b^2-4ac}}{2a}$ will always be negative when $Q_T > DT$. Positivity of $\frac{-b+\sqrt{b^2-4ac}}{2a}$ depends on the coefficients of the quadratic equation, i.e. the costs, quantities and the discounts associated with any given problem, so we could possibly find a negative value for m. This would be meaningless, in such a case, a non-optimal but positive m could be calculated where the buyer still has incentive to utilize the given discount offer. We will not look further into this detail, during our numerical study any negative m found will be discarded and those cases will not be used.

Following Theorem 2, note that Q_m is independent of the number of orders placed before within a cycle. This does not imply that Q_m in general, for any m, does not depend on m. But rather the optimal Q_m in an optimal cycle with m does not.

Remark:

There are two cases that need to be answered to show that when the m subcycle ends, the discount offer will no longer be available to the buyer.

1. $Q_m \geq Q_T$: If Q_m is greater than or equal to Q_T then, as Q_T is strictly greater than DT, we know that when the buyer's inventory depletes the discount offer that is presented due to the order $Q_m \geq Q_T$ will be withdrawn. This is due to the fact that the expiry date would have passed, so the buyer could only restart the cycle at this point.

2. $Q_m < Q_T$: If Q_m is less than Q_T, the buyer's inventory could deplete before or after the expiry date, this depends on previous inventory accumulated by the buyer. However in this case, as the last purchase made does not meet the minimum order criteria, the buyer will not be offered a discount, and independent of the time that the new order comes, he will need to restart the cycle.

The above results are based on the identification of the extrema of TC. Next, we examine the convexity properties of the total cost rate. We will be looking at the positivity of the second derivative of the buyer's total cost rate.

Theorem 4 *For any Q_m set at $[\frac{1}{hT}K_d + \frac{1}{2}Q_T]$, the buyer's total cost rate function is convex in m if:*

$Q_T^2 H_2 \frac{h}{D} m + Q_T H_2^2 h + Q_T H_1 \frac{2}{D} + Q_T^2 \frac{H_1}{H_2} \frac{2}{D^2} m > Q_T H_2 hTm + H_2^2 hTD + 2H_2H_3 + Q_T H_3 \frac{2}{D} m$

where

$H_1 = \frac{1}{2DhT^2}K_d^2 + \frac{h}{8D}Q_T^2 + \frac{1}{2TD}K_dQ_T + c_lQ_T + c_d\frac{1}{hT}K_d - \frac{1}{2}c_dQ_T + K_l$

$H_2 = \frac{1}{DhT}K_d + \frac{1}{2D}Q_T$

$H_3 = \frac{1}{DT}K_d Q_T + \frac{h}{2D}Q_T^2 + c_d Q_T$

Proof. The proof follows from calculation of the Hessian of the total cost rate as given in Appendix A. ∎

Theorem 4 provides the necessary condition for the convexity of the buyer's total cost rate. Below result provides the sufficient condition for this convexity.

Corollary 1 *For any Q_m set at $[\frac{1}{hT}K_d+\frac{1}{2}Q_T]$, it suffices to have $K_l > \frac{h}{2D}[\frac{1}{hT}K_d+\frac{1}{2}Q_T]^2$ for TC to be convex in m.*

Proof. Follow immediately by considering individual terms in Theorem 4 separately, as given in Appendix A. ∎

We can show that the first and the second terms on the left hand side of the inequality in Theorem 4 are greater than their corresponding terms on the right hand side, as long as $Q_T > DT$. However, we can not say the same for the third and the fourth terms. As seen through Corollary 1, one could identify cases where the convexity does not hold. However we are certain that the convexity holds for out intents and purposes, as we have performed extensive numerical checks. During the numerical analysis that we have performed, none of the experiment instances considered violated the necessary condition for convexity.

We have also looked at cases where the above equation might not hold, and our analysis indicate that this equation holds as long as we don't have a small discount rate with a high Q_T with respect to T and the demand rate. If the buyer is offered a small discount, that has an associated Q_T and T couplet which increases his inventory drastically, i.e. Q_T is very large compared to DT, then convexity might not hold. We have not done an in depth analysis where we could identify cutoff points or conditions where this would occur, but we can argue that under such circumstances the buyer would not be interested in using the discount offer. Hence, we can conclude that for our purposes the payoff function is convex.

We now know the sufficient condition for the convexity of the buyer's total cost rate and moreover we know the extrema of the total cost rate in m and Q_m.Then we could write the following Theorem 5 about the optimality of the total cost rate function

Theorem 5 *For any $Q_m = \frac{1}{hT}K_d + \frac{1}{2}Q_T$ and m given by the quadratic equation given in Lemma 3; if the sufficiency condition given by Corollary 1 holds, then:*

1. $Q_m^* = \frac{1}{hT}K_d + \frac{1}{2}Q_T$
2. m^* *is either*
   (a) 1, *or*
   (b) $\lceil m \rceil$, *or*
   (c) $\lfloor m \rfloor$.

when $\lceil x \rceil$ and $\lfloor x \rfloor$ denote the ceiling and floor functions, respectively.

Proof. We have shown that the buyer's total cost rate is quasi-convex in Q_m and m indicating that this function is unimodal. We have then found the values of Q_m and m that minimize the total cost rate function of the buyer. We have also given the necessary and sufficient conditions for the convexity of the buyer's total cost rate. As we know the unimodality, we know that the optimal values of Q_m and m will either be given by the extrema that we have computed or the boundary conditions that are found. If the value of m exists when $Q_m = \frac{1}{hT}K_d + \frac{1}{2}Q_T$ then we know that there exists an optimal discount offer for the buyer given by the couplet (Q_m^*, m^*) where $Q_m^* = \frac{1}{hT}K_d + \frac{1}{2}Q_T$ and m^* is given by one of the following, whichever provides a smaller TC, 1, $\lceil m \rceil$ or $\lfloor m \rfloor$. ■

We have identified the decision variables of the buyer's total cost rate. By computing Q_m^* and m^* the buyer enables to find the optimal values of his decision variables, and make a comparison of the discounted schedule with his EOQ order. He can then decide whether he will utilize the discount offer or not.

CHAPTER 3

SUPPLIER'S CASE

We have so far concentrated on the buyer and his decisions and have assumed a discount schedule offered by the supplier without investigating her decisions and cost functions. In this section we will look at the supplier and her possible motives to provide such a discount.

We know that the offered schedule does not alter the total quantity that the buyer purchases over time. The supplier is not selling more units and because of the discount, her revenues are decreasing. Even under this condition, there could be motives for the supplier to offer this discounted schedule. We know that such discounts alter the buyer's purchasing frequency. Over the length of a cycle we know that if he orders his EOQ, the buyer's order frequency will be uniformly distributed but if he follows the discounted schedule, he will be purchasing more frequently at the beginning of the cycle and there would be a longer period at the end of the cycle where he does not make any purchases. This would affect the supplier's inflow of cash, as in an EOQ schedule her cashflows would be uniformly distributed whereas in the discounted schedule she will be receiving the total payments in a shorter amount of time. Due to this case we can safely assume that if the supplier is in need of cash earlier, she might have incentives to sell more earlier to feed this need. We will not look into this case in this thesis, as it is out of its scope, but this case remains an important point as an incentive for the supplier.

The second incentive would be the holding costs that the supplier incurs. Supplier, through the use of this discount, is minimizing her inventory related costs. This approach has been previously used by Wang (2002), Dara and Srinkanth (1987) and Lal and Staelin (1984). We do not consider the supplier's inventory replenishment policies, but basically assume that she has inventory stored, and she is meeting the demand of the buyer from this stock. As the buyer is offered to inflate his orders, the inventory held by the supplier is transferred to the buyer before the buyer needs that stock. This policy enables the supplier to save on her inventory related costs, and through these additional gains she is able to offer a discount to the buyer.

We propose that the decision criteria for the supplier's decision is based on her holding cost ratio to the buyer's holding cost. As the supplier is going to receive a smaller total payment, she needs to break even through the transferred costs. We know that the buyer is also paying extra in holding costs. Then the ratio of these extra holding costs to the supplier's loss is the same as the ratio of the buyer's holding cost to the supplier's holding cost.

We will first look at the supplier's profit under the two ordering schemes. Let SR_E denote the supplier's profit when buyer orders his EOQ per unit time and let SR_D denote the supplier's profit when buyer orders discounted schedule per unit time.

Let us look at the supplier's profit under the EOQ policy, which is given by the sum of the revenue from sales and gains from the transfer of inventory related costs:

$$SR_E = Dc_l + \frac{Q_0}{2}h \tag{3.1}$$

Let us look at the supplier's profit under the discount policy, which is given by the sum of the revenue from sales under discount and gains from the transfer of inventory related costs, minus the lost revenue from fixed cost:

$$\begin{aligned} SR_D = \quad & Dc_d + \tfrac{1}{CL}[m^2(-\tfrac{hT}{2}Q_T + \tfrac{h}{2D}Q_T^2) + m(\tfrac{hT}{2}Q_T + \tfrac{h}{D}Q_mQ_T - hQ_mT) + \tfrac{h}{2D}Q_m^2] \\ & -\tfrac{1}{CL}m(K_l - K_d) \end{aligned} \tag{3.2}$$

where the cycle length, CL, is equal to $\frac{1}{D}mQ_T + \frac{1}{D}Q_m$.

We know that the supplier will only offer a discount if she is not worse off. Let us look at the supplier's break-even point, the supplier will offer a discount if:

$$SR_D - SR_E \geq 0 \tag{3.3}$$

If the above condition holds, then the supplier, for a given discount schedule, will be profiting. If, at the same time, this discount schedule is profitable to the buyer as well, then both parties will have a motivation to use the discount policy.

Now let us look at the inventory costs of the buyer; we denote the buyer's inventory cost following EOQ schedule as HCB_{EOQ}, the buyer's inventory cost following discounted schedule as HCB_D, and the supplier's unit-time holding cost as h_0.

Buyer's inventory cost rate following EOQ:

$$HCB_{EOQ} = \frac{Q_0}{2}h \tag{3.4}$$

Buyer's inventory cost rate following discounted schedule:

$$HCB_D = \frac{[m^2(-\frac{hT}{2}Q_T+\frac{h}{2D}Q_T^2)+m(\frac{hT}{2}Q_T+\frac{h}{D}Q_mQ_T-hQ_mT)+\frac{h}{2D}Q_m^2]}{\frac{1}{D}mQ_T+\frac{1}{D}Q_m} \tag{3.5}$$

Buyer's additional cost on inventory - as $Q_T > DT$ - is equal to $HCB_D - HCB_{EOQ}$. This cost is also equal to the inventory cost that is not paid by the supplier, as this is the quantity that the supplier transferred to the buyer through the discount offer. If the supplier is profiting, i.e. $SR_D - SR_E \geq 0$, then we could say that:

$(SR_D - SR_E) * h = h_0 * (HCB_D - HCB_{EOQ})$

hence:

$$\frac{h_0}{h} = \frac{SR_D - SR_E}{HCB_D - HCB_{EOQ}} \tag{3.6}$$

This ratio gives us a decision criterion as it acts as a break-even point for any given discount offer. If, for a given discount offering which benefits the buyer, the actual holding cost ratio of the supplier to the buyer is higher than the calculated percentage then the supplier will also benefit from the discount. The breakeven point gives the supplier the criterion to offer a discount that benefits both parties.

CHAPTER 4

NUMERICAL ANALYSIS

Our theoretical findings have given us the basic rules in determining our system. We will now undertake a numerical study to further investigate the effects of our discount schedule.

Our study will be comparing an EOQ case versus a schedule with expiring discounts. Our initial problem setting defined a cycle as a combination of two distinct schedules. First buyer purchases a number of EOQ orders and then purchases the increased orders. As we have previously shown, this does not happen; the EOQ orders do not have any impact on the later decisions faced by the buyer when a discounted schedule is offered. Hence the buyer will either choose to continue his EOQ schedule or will switch to an increased order that will enlarge his inventory until it is no longer profitable to do so where he will quit purchasing until his inventory depletes. Our numerical study will investigate this case.

We have first developed 50 reference points as a basis to compare our further investigation. Each of these reference points is termed a "case" for convenience in later discussions. Each case had a demand of 300 units per month and an ordering cost of $10 per order. Five different holding costs used, varied as 2.5, 5, 7.5, 10, and 15 percent of the ordering cost. Similarly ten different unit variable costs are used from 5% to 50%, increasing with 5% increments, again as a percent of the ordering cost.

The reason behind using a constant ordering cost and holding cost represented as a percentage of the ordering cost is as follows: The order cost, as it is incurred every order regardless of order size, has a big impact on the frequency of orders. If the order cost to holding cost ratio is high, the calculated EOQ will be larger than it would be in the opposite case where such a ratio has a lower value. So varying holding cost with respect to a constant order cost captures this phenomena. If we look at the variable cost, similarly as the as aforementioned argument, it could be argued that order cost to variable cost ratio might change the ensuing EOQ size to change.

Summarizing, we had a total of 50 cases, as we had 5 different holding cost and 10 different variable costs a total of 50 combinations, to be used as benchmarks when we concluded our numerical analysis. For each case we have calculated the economic order quantity and the time between two orders along with a total cost per month which is a linear term; total cost per order projected over a month.

Each of our 50 cases were subjected to 125 different "offerings". For convenience we define an offering as a specific instant of a given discount percentage on the order and variable costs, an order interval T, and a minimum purchase amount Q_t. Each of these variables were changed 5 times hence a total of 125 combinations for each case.

The discount percentage, r, $r = 1 - c_d/c_l = 1 - K_d/K_l$, is used to find K_d and c_d from K_l and c_l, varying from a 5% discount to 25% discount in equal increments of 5%. Similarly Q_t is found using the order sizes calculated for each case, i.e. the economic order quantities, increased by 5% to 25% in equal increments of 5%. For the order interval T we have shortened the order intervals calculated for the cases by 5% to 25% in equal increments of 5%. See Table 4.1 for a summary of the parameters used in the numerical analysis.

We have investigated a total of 6250 offerings. For each offering Q_m and m are first found. Then according to our problem setting, m orders of Q_t is made followed by the last order Q_m. First Q_t is purchased by the buyer at the list prices, and following orders are purchased at the discounted costs. Total cost per cycle is calculated where a cycle defined as the total time between the time that the first order is made until all of the inventory is depleted after the last order Q_m is made. Then this cost is linearly converted to a per month cost to be compared against the original case.

Note that m could be found to be negative, as this is meaningless we will not take these offerings into consideration. Moreover it should be noted that we have used real numbered values for each case and offering, hence ordering a fraction of a unit is allowed and similarly m takes on real values i.e. the buyer could order a real valued number of times, this shortcoming will be addressed later.

Table 4.1: Parameters for the Numerical Analysis

Parameters	Modifications in the Tested Parameters
h/K_l	2.5, 5, 7.5, 10, 15%
c_l/K_l	5, 10, 15,...45, 50%
Q_T	%105, %110, %115, %120, %125 of the Buyers EOQ
T	%95, %90, %85, %80, %75 of the Buyer's EOQ Order Intervals
r	%5, %10, %15, %80, %75 of the List Prices
$D = 300$ units, $K_l = \$10/order$	

4.1 Results

We will investigate our results in three sections: (i) Sensitivity,(ii) Impact of real value approximation, and (iii) Advantages to the seller. Note that our results hold for the sufficient condition for the convexity that we have discussed earlier.

4.1.1 Sensitivity

We will examine the sensitivity of the optimal policy parameter values with the First let us look at the relationship between T, Q_T, m, and r. Note that in our previous discussions, we used a different notation for the list and discounted prices, however, they could have easily been replaced by multiplying list prices by a discount percentage where $r = 1 - c_d/c_l = 1 - K_d/K_l$.

We find that T and r have a direct relationship with m - as they are increased m also increases - whereas Q_T has an inverse relationship with m. We also observe that m and the total cost rate have a direct relationship and increasing m decreases the total benefits (since total cost rate increases). The reader is referred to Figure 4-1 for the above conclusions.

Note that Figure 4-1 is using data produced for case 1, where the demand is

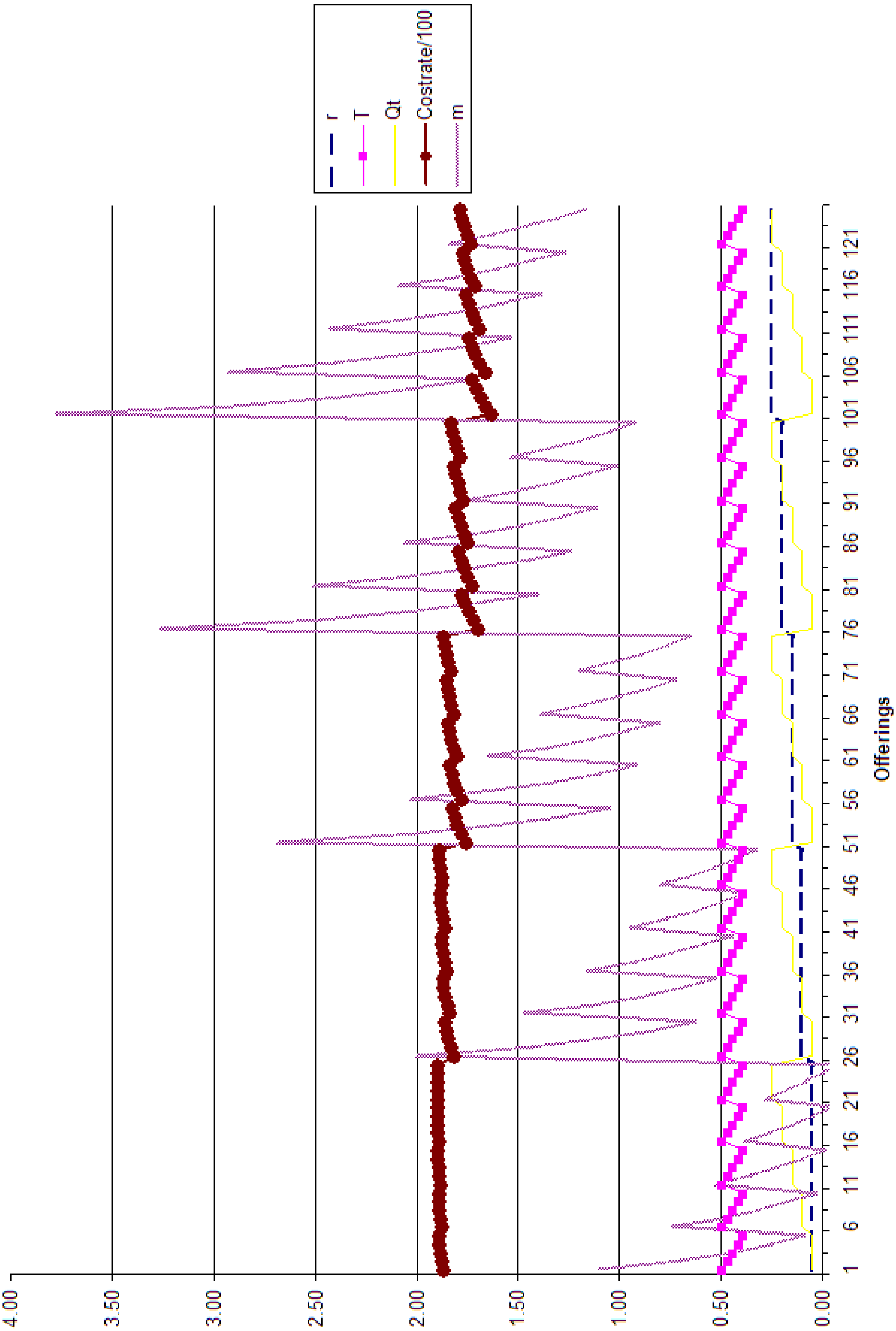

Figure 4-1: Changes in T, Qt, m and Discount Percentage and their relationship with Total Cost Rate for Case 1

300, order cost \$10, variable cost \$0.5, holding cost \$0.25, and economic order quantity 155 units, and the time between two EOQ orders at 0.516 months. This graph is representative among all cases, we nevertheless have included sample graphs for cases 7, 15, and 32 for comparisons as well.

These findings are very logical, as m increases, the cycle length increases and the buyer is forced to keep additional inventory for a longer period of time hence the total cost increase, thus higher values of m is not beneficial. As T increases, m also increases, this could be attributed to the fact that as T increases, the time between two orders becomes closer in value to the order interval during the EOQ pattern. Increases in Q_T decreases m, which is due to the fact that increasing Q_T forces the buyer to keep a larger inventory, hence his inventory level increases to his optimal maximum quicker, thus m decreases. Finally discount percentage and m has a direct relationship, indicating that higher discounts make it possible for the buyer to stay in the cycle for a longer period of time. Table 4.2 summarizes the above findings.

Observing the differences between cases, we see that increasing holding cost increases the total cost rate, decreasing the savings, which is expected. Another observation is that as the variable cost percentage (c_l/K_l) increases the number cases that show savings increase. Variable cost percentage is the cost per item that is calculated as a percentage of the fixed order cost. As we increase this percentage, the effect of the fixed cost decreases, and the cycle cost becomes less sensitive to the fixed cost since the overall number of orders has limited affect on the cycle cost. Some obvious results also were observed, we have a smaller number

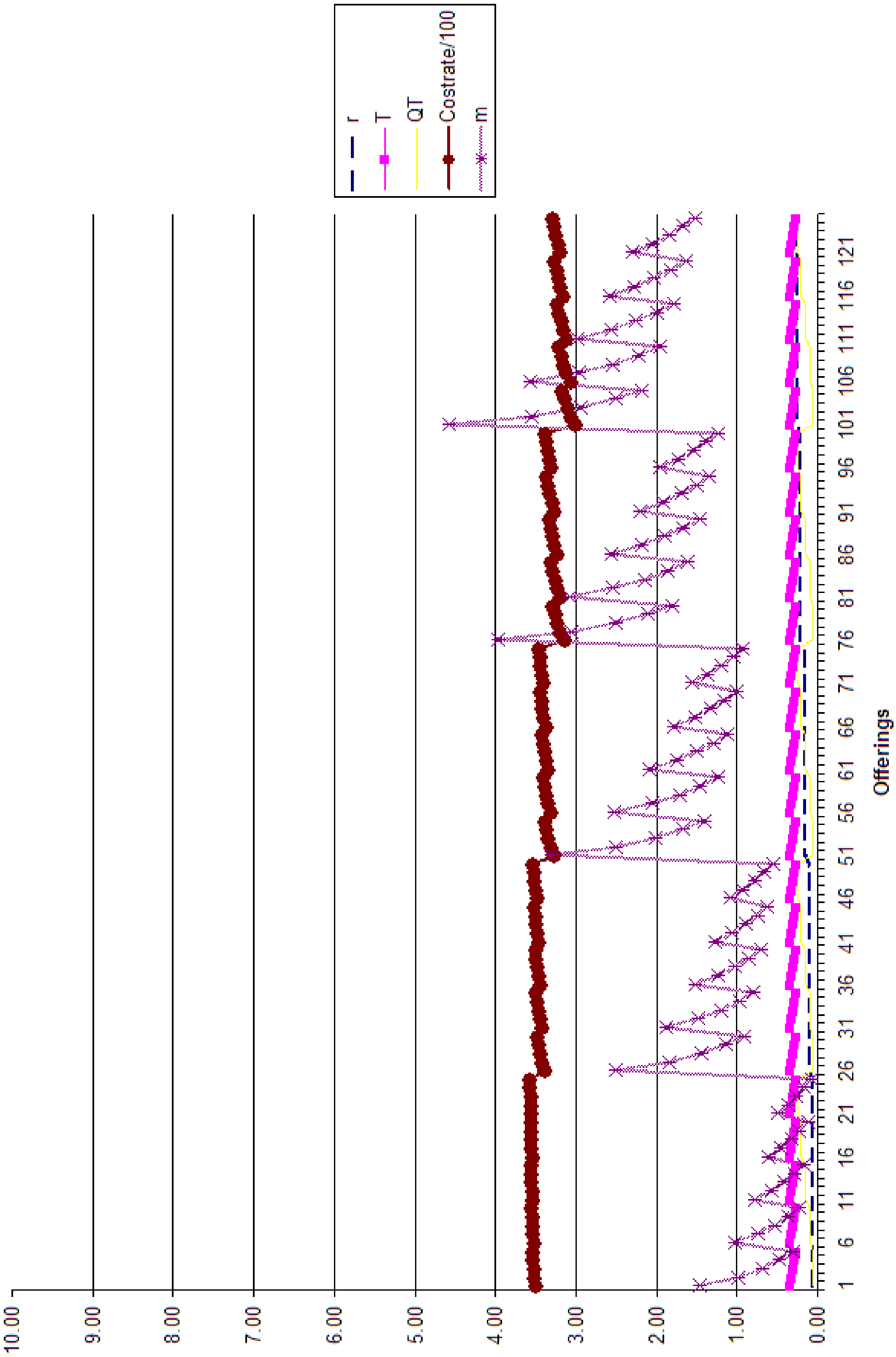

Figure 4-2: Changes in T, Qt, m and Discount Percentage and their relationship with Total Cost Rate for Case 7

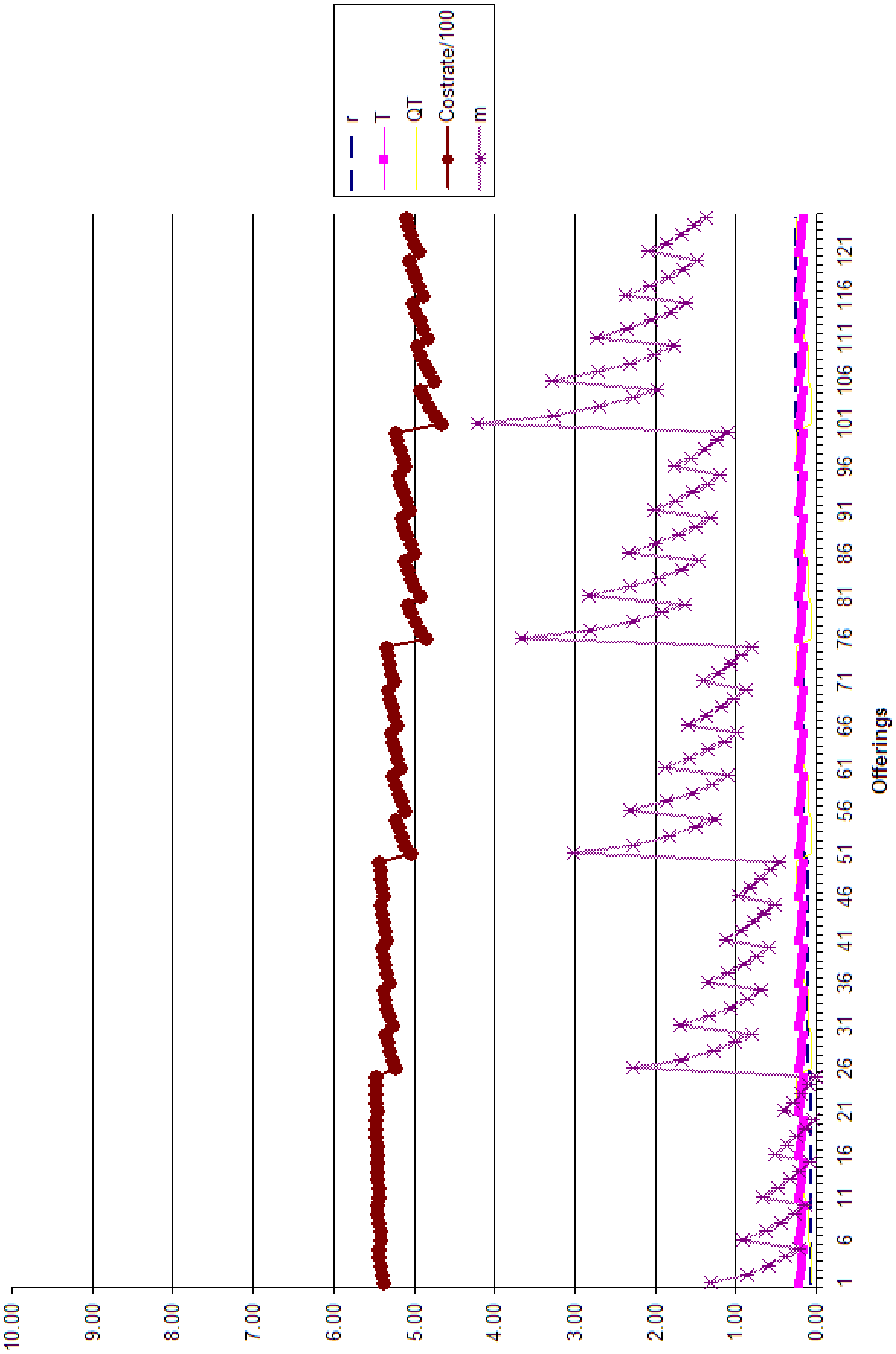

Figure 4-3: Changes in T, Qt, m and Discount Percentage and their relationship with Total Cost Rate for Case 15

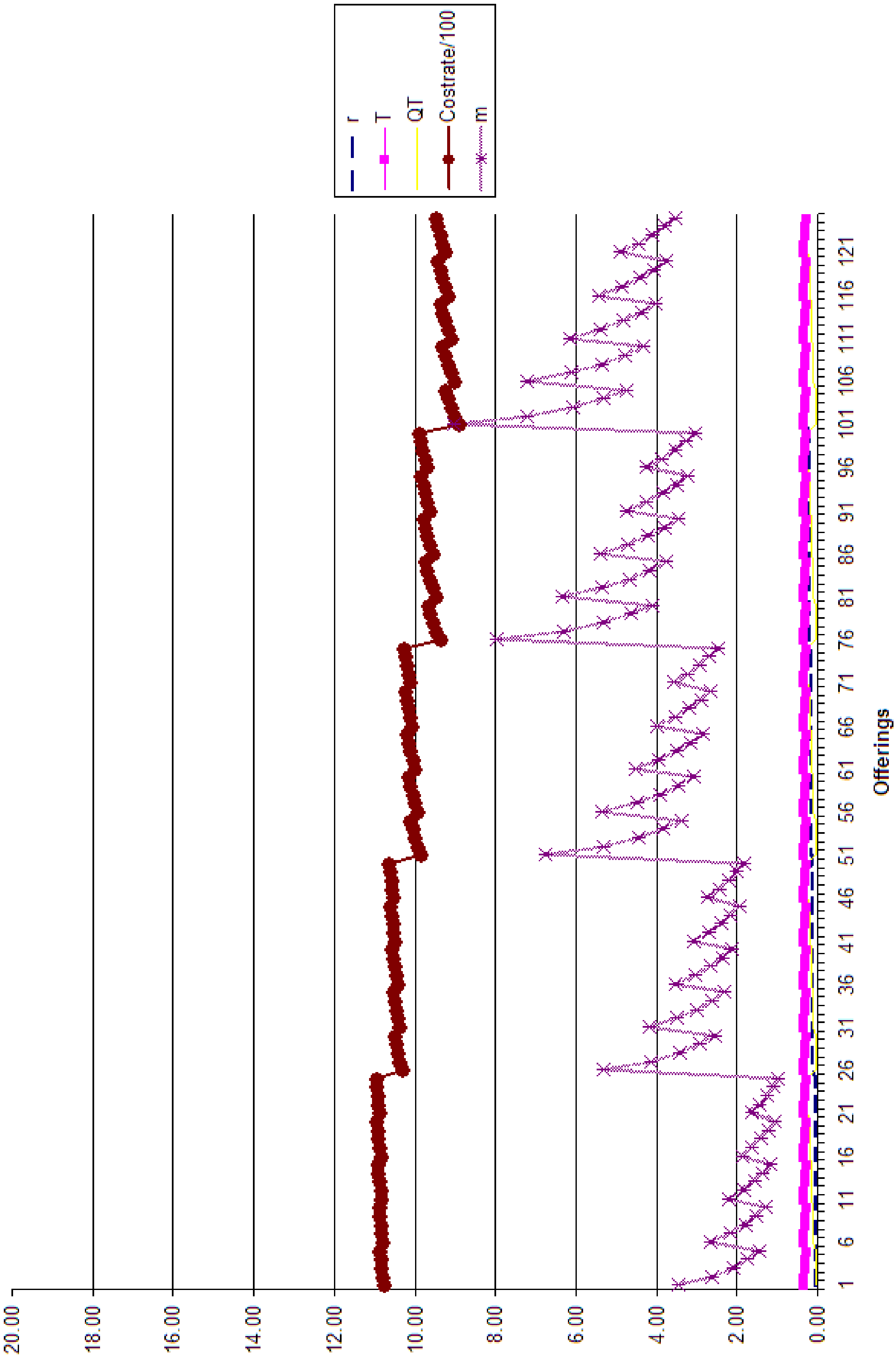

Figure 4-4: Changes in T, Qt, m and Discount Percentage and their relationship with Total Cost Rate for Case 32

Table 4.2: Summary of Sensitivity Results

Parameters	Effect on m	Effect on Total Cost Rate
$Q_T \uparrow$	$m \downarrow$	$TC \uparrow$
$T \uparrow$	$m \uparrow$	$TC \downarrow$
$m \uparrow$	$-$	$TC \uparrow$
$r \uparrow$	$m \uparrow$	$TC \downarrow$

of offerings that show savings with increased Q_T, and we have a higher number of offerings showing savings with higher discount percentages.

Overall, of the 6250 offerings, 4313 of those showed savings. Table 4.3 contains descriptive statistics cases when m was a real number. Table 4.4 contains the number of discount offerings that were profitable for each case and statistical information on the savings percentages. Average savings were 7.29%, and were as high as 20.79%. We see that the highest concentration of offerings showed less than 5% savings, however almost 6% of all offerings had more than 15% savings. We also see that increasing Q_T had a negative effect on the number of offerings with savings while the discount rate r had a positive effect, almost all of the offerings with 25% discount (1127/1250) showed positive savings for the buyer. Of all cases, only cases 4 and 5 did not have any offerings with a positive discount, these two cases had high holding costs ($h/K_l = 0.1$ and 0.15 respectively) and low unit variable cost ratios with the fixed cost ($c_l/K_l = 0.05$ for both).

4.1.2 Impact of real value approximation

Note that we have used real valued m's for this analysis. However, we have previously shown that total cost rate function is convex when around the optimal

Table 4.3: Descriptive Statistics of the results when m is real valued

Total Number of Offerings with Savings			4313
Average Savings per Offering			7.29%
Offerings with Savings		Number of Offerings	Average Savings
Less than	5.0%	1680	2.38%
Less than	10.0%	1336	7.43%
Less than	15.0%	948	12.32%
More than	15.0%	349	16.76%
Qt %	Number of Offerings	Disc.%	Number of Offerings
5%	933	5%	338
10%	896	10%	801
15%	859	15%	974
20%	829	20%	1073
25%	796	25%	1127

m hence the optimal m provides for a minimum for the total cost rate. This allows us to compute the optimal m and then round it to the 2 nearest integers, one of which will be the unique integer that minimizes the total cost rate, and compute the cycle cost using these rounded-up and rounded-down m values. Then the decision is to basically use the m which provides the lower total cost rate.

This was investigated during our numerical analysis. We see that the total cost rate that is calculated using the optimal m value creates a lower bound for the total cost rate calculated with rounded up and down values of m. We also noted that the overall trend of the total cost rate is similar in all three calculations, except for certain offerings where m was not rounded down (if $m < 1$ then $m = 1$) to be able to provide meaningful results. See Figure 4-5.

We found that when m is rounded down the number of discount offerings with savings dropped to 4251 and when m is rounded up the number of offerings with

Table 4.4: Sensitivity Analysis when m is real valued

Case #	Number of Offerings with Savings	% Savings			
		Minimum	Mean	Median	Maximum
1	46	0.04	3.01	2.71	9.28
2	7	0.20	1.37	0.56	3.97
3	1	0.34	0.34	0.34	0.34
4	0	0.00	0.00	0.00	0.00
5	0	0.00	0.00	0.00	0.00
6	90	0.02	5.68	5.47	14.38
7	59	0.21	4.00	3.57	10.97
8	38	0.09	2.82	2.33	8.57
9	22	0.07	2.08	1.61	6.67
10	6	0.04	1.27	0.98	3.69
11	103	0.06	7.18	7.21	16.60
12	84	0.07	5.65	5.41	14.01
13	70	0.06	4.59	4.29	12.18
14	57	0.06	3.87	3.37	10.72
15	37	0.02	2.74	2.28	8.41
16	113	0.02	7.83	7.28	17.88
17	99	0.07	6.54	6.43	15.76
18	85	0.12	5.85	5.68	14.26
19	77	0.04	5.09	4.96	13.06
20	60	0.24	4.15	3.78	11.15
21	124	0.02	7.99	7.64	18.74
22	103	0.00	7.51	7.50	16.92
23	97	0.17	6.53	6.32	15.64
24	88	0.02	6.02	5.96	14.60
25	76	0.03	5.05	4.85	12.96
26	125	0.32	8.57	8.29	19.37
27	107	0.01	8.11	7.66	17.76
28	101	0.14	7.33	7.23	16.63
29	98	0.00	6.55	6.34	15.72
30	85	0.02	5.86	5.72	14.27
31	125	0.67	9.07	8.81	19.84
32	116	0.03	8.14	7.71	18.40
33	103	0.21	8.00	7.92	17.38
34	101	0.04	7.26	7.14	16.56
35	94	0.07	6.32	6.22	15.26
36	125	0.94	9.48	9.23	20.22
37	123	0.02	8.20	7.81	18.90
38	109	0.02	8.19	7.81	17.97
39	103	0.05	7.83	7.76	17.23
40	99	0.20	6.83	6.67	16.04
41	125	1.17	9.81	9.58	20.53
42	125	0.16	8.50	8.22	19.31
43	116	0.00	8.19	7.75	18.46
44	106	0.09	8.19	7.93	17.77
45	101	0.04	7.38	7.23	16.68
46	125	1.37	10.09	9.87	20.79
47	125	0.42	8.86	8.60	19.66
48	122	0.04	8.21	7.77	18.86
49	111	0.00	8.30	8.00	18.22
50	101	0.40	7.96	7.75	17.20

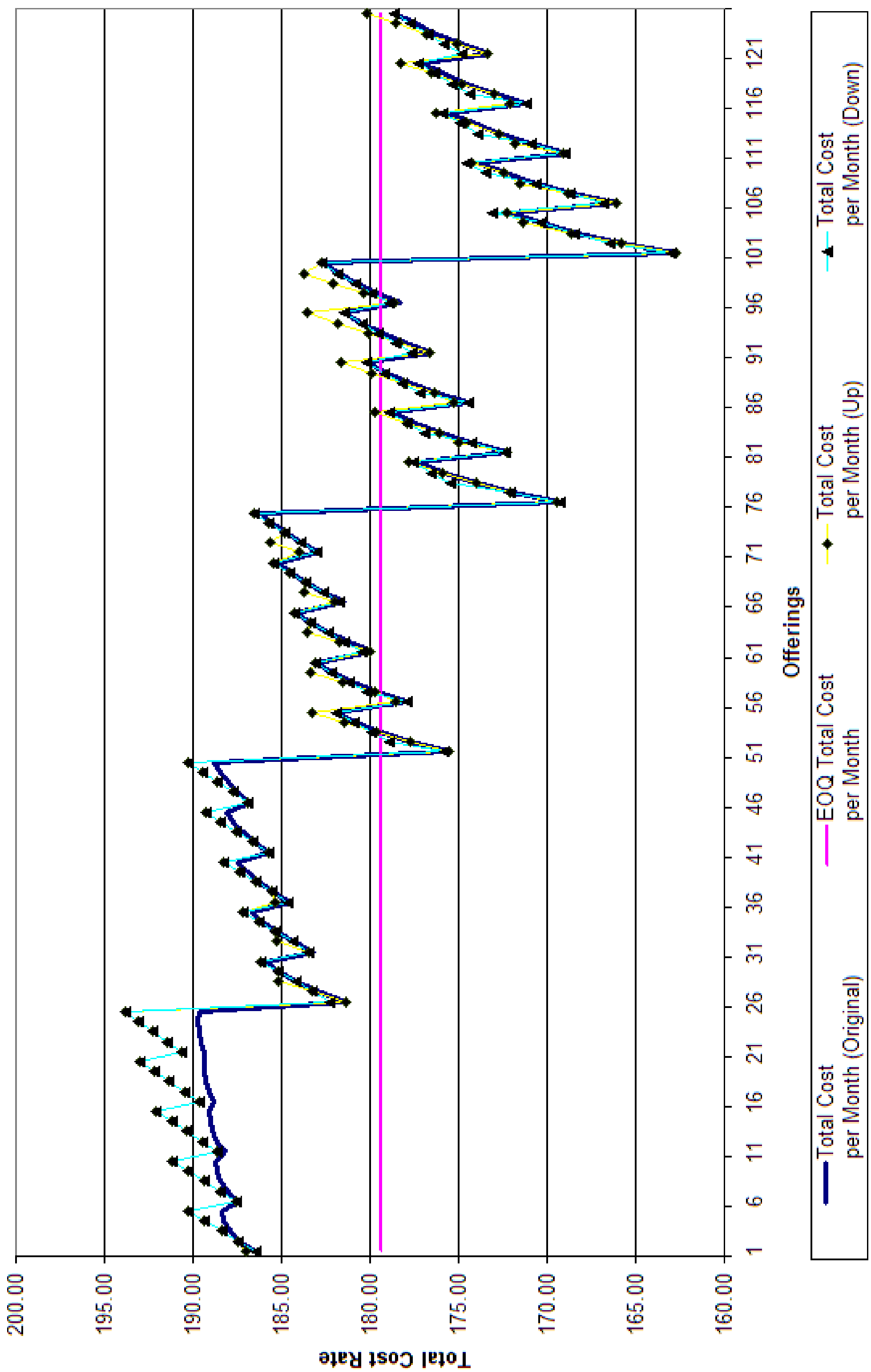

Figure 4-5: Total cost rate calculated for real valued m, rounded-up m and rounded-down m and the EOQ

savings was 4274. As we know, if m is optimal and real valued these numbers would have been 4313. Sensitivity statistics are provided in the Tables 4.5 and 4.6.

We have also run our analysis for the optimal value of the integer m. We have calculated both instances of the total cost rate when m was rounded, and selected the one with a better result for the total cost rate. In this case we see that the total number of discount offerings with savings for the buyer was 4296 with an average savings per discount offering of 7.31%. Note that mean savings are higher in the cases when m is rounded; this is due to the fact that discount offerings with small savings, where the calculated total cost rate is very close to the EOQ cost, are eliminated during rounding, hence the mean increases. Table 4.7 has the sensitivity information for the analysis when we looked at the optimal integer value of m.

4.1.3 Advantages of the Seller

We have calculated the holding cost ratios in our numerical analysis, Figure 4-6 presents this ratio for offerings made in Case 1 - note that the offerings where buyer does not make a profit has no calculation of the ratio. Here we see that as discount rates increase, the ratio increases, which is logical because the increase in discount rate decreases supplier revenues. On the contrary as T decreases, we see that the ratio decreases; making it more profitable for the supplier, which coincides with our previous result where decreases in T made buyer worse off. Q_T has a

Table 4.5: Descriptive Statistics of the results when m is rounded down

Total Number of Offerings with Savings			4251
Average Savings per Offering			7.31%
Offerings with Savings		**Number of Offering**	**Average Savings**
Less than	5.0%	1640	2.37%
Less than	10.0%	1325	7.42%
Less than	15.0%	944	12.33%
More than	15.0%	342	16.77%
Qt %	**Number of Offering**	**Disc.%**	**Number of Offering**
5%	920	5%	319
10%	882	10%	784
15%	845	15%	963
20%	816	20%	1064
25%	788	25%	1121
Savings Trials for each Case			
Case #	**Offerings**	**Case #**	**Offerings**
1	45	26	125
2	4	27	105
3	1	28	101
4	0	29	97
5	0	30	81
6	88	31	125
7	57	32	115
8	35	33	103
9	18	34	100
10	5	35	92
11	101	36	125
12	82	37	123
13	68	38	106
14	55	39	103
15	34	40	99
16	112	41	125
17	99	42	125
18	82	43	114
19	76	44	105
20	57	45	100
21	124	46	125
22	102	47	125
23	97	48	121
24	85	49	108
25	75	50	101

Table 4.6: Descriptive Statistics of the results when m is rounded up

Total Number of Offerings with Savings			4274
Average Savings per Offering			7.30%
Offerings with Savings		Number of Offerings	Average Savings
Less than	5.0%	1663	2.39%
Less than	10.0%	1320	7.43%
Less than	15.0%	945	12.31%
More than	15.0%	346	16.76%
Qt %	Number of Offerings	Disc.%	Number of Offerings
5%	928	5%	321
10%	886	10%	795
15%	853	15%	970
20%	820	20%	1065
25%	787	25%	1123
Savings Offerings for each Case			
Case #	Offerings	Case #	Offerings
1	42	26	125
2	7	27	106
3	1	28	101
4	0	29	97
5	0	30	85
6	88	31	125
7	59	32	115
8	37	33	103
9	20	34	101
10	4	35	94
11	103	36	125
12	84	37	122
13	69	38	109
14	56	39	102
15	36	40	99
16	110	41	125
17	97	42	125
18	85	43	115
19	75	44	104
20	59	45	101
21	122	46	125
22	101	47	125
23	97	48	120
24	87	49	110
25	75	50	101

Table 4.7: Sensitivity Statistics for the optimal integer value of m

Case Number	Buyer's Optimal Decision			Total Discount Savings (m rounded)	Total Discount Savings (m real)
	EOQ	M Down	M Up		
1	80	21	24	45	46
2	118	3	4	7	7
3	124	0	1	1	1
4	125	0	0	0	0
5	125	0	0	0	0
6	37	40	48	88	90
7	66	19	40	59	59
8	88	13	24	37	38
9	104	9	12	21	22
10	119	3	3	6	6
11	22	46	57	103	103
12	41	38	46	84	84
13	56	28	41	69	70
14	68	27	30	57	57
15	89	16	20	36	37
16	13	53	59	112	113
17	26	45	54	99	99
18	40	41	44	85	85
19	49	35	41	76	77
20	65	28	32	60	60
21	1	68	56	124	124
22	23	52	50	102	103
23	28	41	56	97	97
24	38	36	51	87	88
25	50	33	42	75	76
26	0	57	68	125	125
27	19	48	58	106	107
28	24	50	51	101	101
29	28	43	54	97	98
30	40	37	48	85	85
31	0	58	67	125	125
32	9	61	55	116	116
33	22	50	53	103	103
34	24	51	50	101	101
35	31	38	56	94	94
36	0	58	67	125	125
37	2	51	72	123	123
38	16	53	56	109	109
39	22	50	53	103	103
40	26	50	49	99	99
41	0	51	74	125	125
42	0	71	54	125	125
43	10	48	67	115	116
44	19	53	53	106	106
45	24	45	56	101	101
46	0	61	64	125	125
47	0	54	71	125	125
48	4	48	73	121	122
49	15	58	52	110	111
50	24	48	53	101	101

Table 4.8: Results on the holding cost ratios of the seller to the buyer

	Ratio
Average	159.28%
Minimum	44.24%
Maximum	653.28%
Median	136.36%
Less than 50%	19
Between 50-100%	1555
More than 100%	2739

similar effect to the seller, contrary to the buyer, increases in Q_T is profitable to the supplier. We have calculated the holding cost ratios for all discount offerings where the buyer was saving, Table 4.8 contains the descriptive statistics. Note that 1574 out of the 4313 discount offerings which showed savings has a calculated ratio less than 100%. In the literature, it is assumed that the supplier will have a smaller holding cost rate than the buyer. If this assumption holds, 25.2% of all discount offerings or 36.5% of the discount offerings that had savings for the buyer, will be profitable to the supplier as well.

We have conducted other studies for additional results. Note that we have previously remarked that the buyer will not be able to continue his cycle when his inventory depleted following his Q_m purchase. We have confirmed this result during our numerical study. Even though there were cases when Q_m was less than Q_T, the inventory accumulated after the Q_m purchase have always depleted after the expiry date T. The time it took to deplete ranged from 0.03 to 1.58 time units (in our case months) *after* T time units have passed.

We have also computed our results for cases where $K_d = K_l$ to be able to remove the affects of the order cost from the discount schedule. As expected

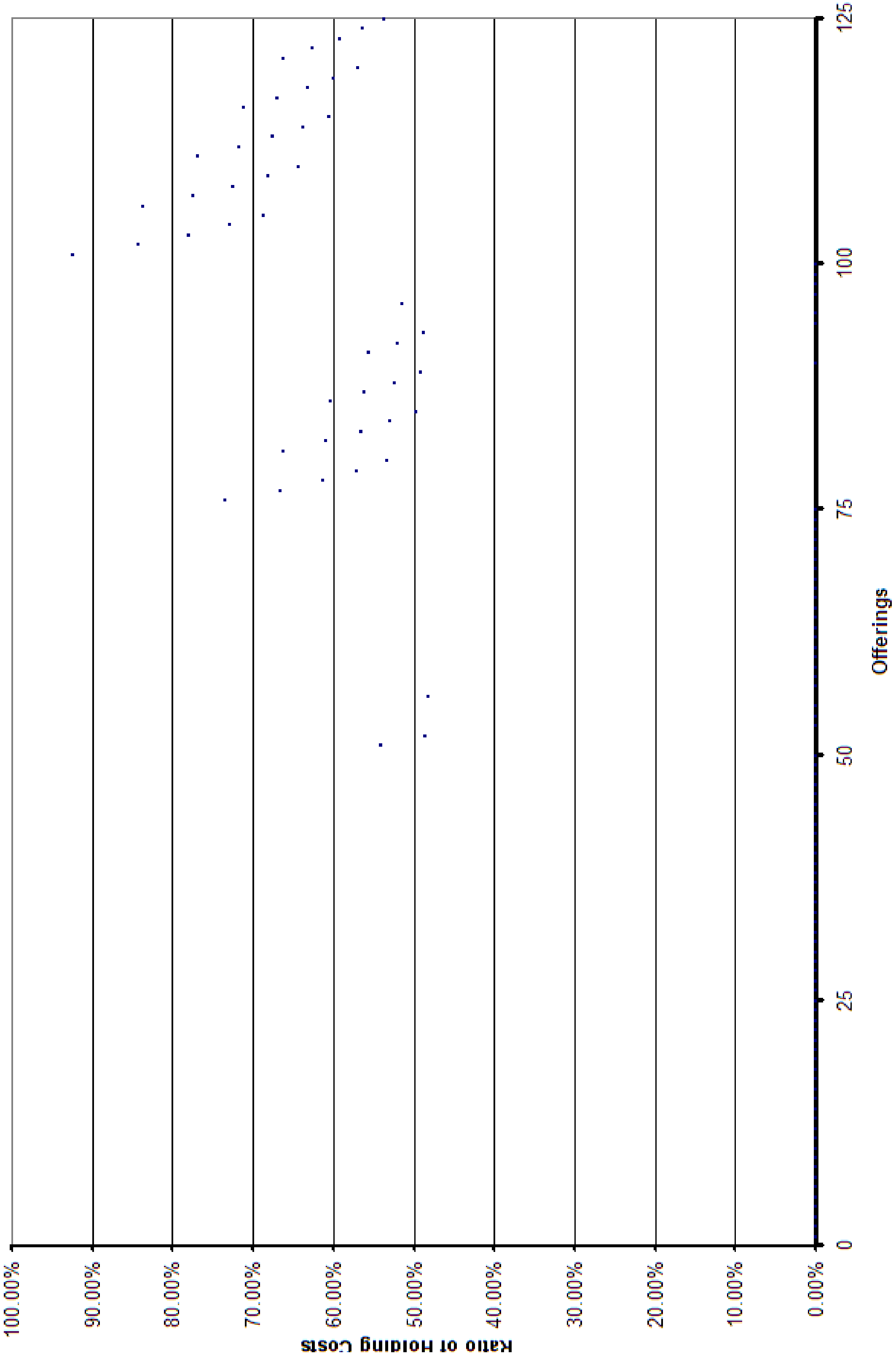

Figure 4-6: Ratio of the holding cost of the supplier to the holding cost of the buyer, computed for offerings in Case 1

the number of offerings where savings occurred for the buyer was smaller, 4187 offerings vs. 4313 offerings with order cost discounted.

CHAPTER 5

CONCLUSIONS AND FUTURE WORK

We have looked at a single buyer-single supplier system under an EOQ setting with quantity discounts and expiry dates. We obtained the optimal replenishment policy parameters for the buyer. We have also extended our research discussing the supplier's motivation and decisions in some detail and with a numerical example to show that this model captures our intuitive results.

The contribution of this study to the current stream of literature lies with the expiry dates. Expiring quantity discounts have not been studied within the literature and we provide a basic set of decisions that could be further investigated by academicians in detail.

Our research has shortcomings as well, basically due to our starting assumptions. We assumed the demand that the buyer faces does not change with his supply, which could be restrictive however one should consider that if the demand rate increases, the buyer will be better off than what we have provided here - as it would be unrealistic to assume that his demand will decrease due to increase in supply.

Another shortcoming for our research is that we use the EOQ policy to compare our results, instead of a stochastic system. Although it has been shown in the previous literature that the EOQ model is a good approximation of the real world scenarios, possible future work on this area might be including stochastic elements to this model.

We also believe that the supplier's case should be investigated more in detail as it was an extension for us, we have not looked at this case in the required detail. We have not solved for the supplier's decisions Q_T, T, and r, but only provided some insight for her decision based on the holding cost ratio of the supplier to her customer's. This insight does not take into consideration of the supplier's inventory replenishment policy and assumes that the supplier's ordering (or producing) frequency would not be changed when offering such a discount schedule. This simplification, as it is suboptimal, should be tackled in future research.

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Appendix A

Proofs

A.1 Proof of Theorem 2

Proof. 2

$CC = (n+1)K_l + mK_d + \frac{nh}{2D}Q_0^2 + \{\sum_{k=1}^{m} \frac{2[kQ_T-(k-1)DT]-DT}{2}\}hT + [m(Q_T - DT) + Q_m]^2\frac{h}{2D} + nQ_0c_l + c_lQ_T + (m-1)c_dQ_T + c_dQ_m$

$CL = \frac{n}{D}Q_0 + \frac{1}{D}mQ_T + \frac{1}{D}Q_m$

Our optimality condition is: $\frac{\frac{\partial CC}{\partial n}}{\frac{\partial CL}{\partial n}} = \frac{CC}{CL}$

$\frac{\partial CC}{\partial n} = K_l + \frac{h}{2D}Q_0^2 + Q_0c_l$

$\frac{\partial CL}{\partial n} = \frac{1}{D}Q_0$

$\frac{\frac{\partial CC}{\partial n}}{\frac{\partial CL}{\partial n}} = \frac{DK_l}{Q_0} + \frac{h}{2}Q_0 + c_lD$

Then,

$\frac{DK_l}{Q_0} + \frac{h}{2}Q_0 + c_l D =$

$\frac{1}{\frac{n}{D}Q_0+\frac{1}{D}mQ_T+\frac{1}{D}Q_m}[(n+1)K_l+mK_d+\frac{nh}{2D}Q_0^2+\{\sum_{k=1}^{m}\frac{2[kQ_T-(k-1)DT]-DT}{2}\}hT+[m(Q_T- DT) + Q_m]^2\frac{h}{2D} + nQ_0c_l + c_lQ_T + (m-1)c_dQ_T + c_dQ_m]$

Let us look at the left-hand side:

$LHS = (\frac{DK_l}{Q_0} + \frac{h}{2}Q_0 + c_lD)(\frac{n}{D}Q_0 + \frac{1}{D}mQ_T + \frac{1}{D}Q_m)$

$\Longrightarrow (\frac{DK_l}{Q_0} + \frac{h}{2}Q_0 + c_lD)(\frac{1}{D}mQ_T + \frac{1}{D}Q_m) + \frac{DK_l}{Q_0}\frac{n}{D}Q_0 + \frac{h}{2}Q_0\frac{n}{D}Q_0 + c_lD\frac{n}{D}Q_0$

$\Longrightarrow (\frac{DK_l}{Q_0} + \frac{h}{2}Q_0 + c_lD)(\frac{1}{D}mQ_T + \frac{1}{D}Q_m) + K_ln + \frac{nh}{2D}Q_0^2 + c_lnQ_0$

Now let us look at the right-hand side:

$RHS = (n+1)K_l + mK_d + \frac{nh}{2D}Q_0^2 + \{\sum_{k=1}^{m}\frac{2[kQ_T-(k-1)DT]-DT}{2}\}hT + [m(Q_T - DT) + Q_m]^2\frac{h}{2D} + nQ_0c_l + c_lQ_T + (m-1)c_dQ_T + c_dQ_m$

$\Longrightarrow nK_l + \frac{nh}{2D}Q_0^2 + nQ_0c_l + K_l + mK_d + \{\sum_{k=1}^{m}\frac{2[kQ_T-(k-1)DT]-DT}{2}\}hT + [m(Q_T - DT) + Q_m]^2\frac{h}{2D} + c_lQ_T + (m-1)c_dQ_T + c_dQ_m$

Then LHS=RHS

$(\frac{DK_l}{Q_0} + \frac{h}{2}Q_0 + c_lD)(\frac{1}{D}mQ_T + \frac{1}{D}Q_m) + K_ln + \frac{nh}{2D}Q_0^2 + c_lnQ_0 = nK_l + \frac{nh}{2D}Q_0^2 + nQ_0c_l + K_l + mK_d + \{\sum_{k=1}^{m}\frac{2[kQ_T-(k-1)DT]-DT}{2}\}hT + [m(Q_T-DT)+Q_m]^2\frac{h}{2D} + c_lQ_T + (m-1)c_dQ_T + c_dQ_m$

$\Longrightarrow (\frac{DK_l}{Q_0}+\frac{h}{2}Q_0+c_lD)(\frac{1}{D}mQ_T+\frac{1}{D}Q_m) = K_l+mK_d+\{\sum_{k=1}^{m}\frac{2[kQ_T-(k-1)DT]-DT}{2}\}hT+ [m(Q_T - DT) + Q_m]^2\frac{h}{2D} + c_lQ_T + (m-1)c_dQ_T + c_dQ_m$

We see that the terms that include n has cancelled out of the equation. This means that the decision of the buyer to accept a discounted offer does not depend on his past purchases but rather at the discounted schedule. This also means that, if the buyer finds the discounted schedule to be profitable, he will switch to this schedule as long as it is available, indicating that he will either repeat his undiscounted cycle or the discounted cycle. Our definition of one complete cycle will not happen. ■

A.2 Proof of Theorem 3

Proof. 3

We will now show the quasi-convexity properties of the total cost rate in m and Q_m. However, let us further simplify the cycle cost before we show our results on the convexity.

$CC = K_l + mK_d + c_l Q_T + (m-1)c_d Q_T + c_d Q_m + \{\sum_{k=1}^{m} \frac{2[kQ_T-(k-1)DT]-DT}{2}\}hT + [m(Q_T - DT) + Q_m]^2 \frac{h}{2D}$

Let us look at the holding cost component of the cycle cost, CC.

$HC = \{\sum_{k=1}^{m} \frac{2[kQ_T-(k-1)DT]-DT}{2}\}hT + [m(Q_T - DT) + Q_m]^2 \frac{h}{2D}$

$\Longrightarrow hT \sum_{k=1}^{m} \frac{2kQ_T}{2} - hT \sum_{k=1}^{m} \frac{2(k-1)DT}{2} - hT \sum_{k=1}^{m} \frac{DT}{2} + [m(Q_T - DT) + Q_m]^2 \frac{h}{2D}$

$\Longrightarrow hTQ_T \frac{(m+1)m}{2} - hT \sum_{k=1}^{m} kDT + hT \sum_{k=1}^{m} DT - \frac{hT}{2} \sum_{k=1}^{m} DT + [m(Q_T - DT) + Q_m]^2 \frac{h}{2D}$

$\Longrightarrow hTQ_T \frac{(m+1)m}{2} - hTDT \frac{(m+1)m}{2} + \frac{hT}{2} \sum_{k=1}^{m} DT + [m(Q_T - DT) + Q_m]^2 \frac{h}{2D}$

$\Longrightarrow hT \frac{(m+1)m}{2} (Q_T - DT) + hTm \frac{DT}{2} + [m(Q_T - DT) + Q_m]^2 \frac{h}{2D}$

$\Longrightarrow \frac{hT}{2}(m^2+m)(Q_T - DT) + hTm\frac{DT}{2} + [m^2(Q_T^2 - 2Q_T DT + D^2T^2) + Q_m^2 + 2m(Q_T - DT)Q_m]\frac{h}{2D}$

$\Longrightarrow \frac{hT}{2}(m^2+m)Q_T - \frac{hT}{2}(m^2+m)DT + hTm\frac{DT}{2} + m^2\frac{h}{2D}Q_T^2 - m^2\frac{h}{2D}2Q_T DT + m^2\frac{h}{2D}D^2T^2 + \frac{h}{2D}Q_m^2 + 2m\frac{h}{2D}Q_mQ_T - 2m\frac{h}{2D}Q_m DT$

$\Longrightarrow \frac{hT}{2}Q_Tm^2 + \frac{hT}{2}Q_Tm - \frac{hT}{2}DTm^2 - \frac{hT}{2}DTm + hTm\frac{DT}{2} + m^2\frac{h}{2D}Q_T^2 - m^2hQ_TT + m^2\frac{h}{2}DT^2 + \frac{h}{2D}Q_m^2 + m\frac{h}{D}Q_mQ_T - mhQ_mT$

$\Longrightarrow \frac{hT}{2}Q_Tm + m^2\frac{h}{2D}Q_T^2 - \frac{1}{2}m^2hQ_TT + \frac{h}{2D}Q_m^2 + m\frac{h}{D}Q_mQ_T - mhQ_mT$

$\Longrightarrow m^2(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + m(\frac{hT}{2}Q_T + \frac{h}{D}Q_mQ_T - hQ_mT) + \frac{h}{2D}Q_m^2$

Then the cycle cost is equal to:

$CC = K_l + mK_d + c_lQ_T + (m-1)c_dQ_T + c_dQ_m + m^2(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + m(\frac{hT}{2}Q_T + \frac{h}{D}Q_mQ_T - hQ_mT) + \frac{h}{2D}Q_m^2$

Now let us look at the second derivatives of the cycle cost and the cycle length with respect to Q_m.

$CC = K_l + mK_d + c_lQ_T + (m-1)c_dQ_T + c_dQ_m + m^2(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + m(\frac{hT}{2}Q_T + \frac{h}{D}Q_mQ_T - hQ_mT) + \frac{h}{2D}Q_m^2$

$\frac{\partial CC}{\partial Q_m} = c_d + m\frac{h}{D}Q_T - mhT + 2\frac{h}{2D}Q_m$

$\frac{\partial^2 CC}{\partial Q_m^2} = \frac{h}{D}$

$CL = \frac{1}{D}mQ_T + \frac{1}{D}Q_m$

$\frac{\partial CL}{\partial Q_m} = \frac{1}{D}$

$\frac{\partial^2 CL}{\partial Q_m^2} = 0$

As the second derivative of the cycle cost with respect to Q_m is always positive, CC is convex in Q_m for all values of m. We also see that the second derivative of the cycle length with respect to Q_m is zero, then CL is concave in Q_m for all values of m. Thus TC, the total cost rate, is quasi-convex in Q_m for all m.

Now let us look at the second derivatives of the cycle length and the cycle cost with respect to m.

$\frac{\partial CC}{\partial m} = K_d + c_d Q_T + 2m(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + \frac{hT}{2}Q_T + \frac{h}{D}Q_m Q_T - hQ_m T$

$\frac{\partial^2 CC}{\partial m^2} = -hTQ_T + \frac{h}{D}Q_T^2 = hQ_T(\frac{1}{D}Q_T - T)$

$\frac{\partial CL}{\partial m} = \frac{1}{D}Q_T$

$\frac{\partial^2 CL}{\partial m^2} = 0$

We know that, as $Q_T > DT$, the second derivative of the cycle cost with respect to m is positive, thus CC is convex in m for all values of Q_m. Similarly as $\frac{\partial^2 CL}{\partial m^2} = 0$, cycle length is concave in m for all values of Q_m. Thus TC, the total cost rate, is quasi-convex in m for all values of Q_m. ■

A.3 Proof of Lemma 1

Proof. 1

We will now find the optimal value of Q_m. Let us look at the cycle cost and the cycle length.

$CC = K_l + mK_d + c_l Q_T + (m-1)c_d Q_T + c_d Q_m + m^2(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) +$

$m(\frac{hT}{2}Q_T + \frac{h}{D}Q_mQ_T - hQ_mT) + \frac{h}{2D}Q_m^2$

$\frac{\partial CC}{\partial Qm} = c_d + m\frac{h}{D}Q_T - mhT + \frac{h}{D}Q_m$

$\frac{\partial CL}{\partial Q_m} = \frac{1}{D}$

Then the optimality condition is:

$\frac{\frac{\partial CC}{\partial Qm}}{\frac{\partial CL}{\partial Qm}} = \frac{CC}{CL}$

$\implies \frac{1}{\frac{1}{D}}(c_d + \frac{h}{D}Q_m + m\frac{h}{D}Q_T - mhT) = \frac{1}{\frac{mQ_T}{D}+\frac{1}{D}Q_m}(K_l + mK_d + c_lQ_T + (m-1)c_dQ_T + c_dQ_m + m^2(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + m(\frac{hT}{2}Q_T + \frac{h}{D}Q_mQ_T - hQ_mT) + \frac{h}{2D}Q_m^2)$

$\implies (c_d + \frac{h}{D}Q_m + m\frac{h}{D}Q_T - mhT)(mQ_T + Q_m) = K_l + mK_d + c_lQ_T + (m-1)c_dQ_T + c_dQ_m + m^2(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + m(\frac{hT}{2}Q_T + \frac{h}{D}Q_mQ_T - hQ_mT) + \frac{h}{2D}Q_m^2$

$\implies \frac{h}{D}Q_m(mQ_T+Q_m)+Q_m(c_d+m\frac{h}{D}Q_T-mhT)+mQ_T(c_d+m\frac{h}{D}Q_T-mhT) = K_l + mK_d + c_lQ_T + (m-1)c_dQ_T + m^2(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + c_dQ_m + m\frac{hT}{2}Q_T + m\frac{h}{D}Q_mQ_T - mhQ_mT + \frac{h}{2D}Q_m^2$

$\implies \frac{h}{D}Q_m(mQ_T + Q_m) + Q_m(c_d + m\frac{h}{D}Q_T - mhT) - m\frac{h}{D}Q_mQ_T + mhQ_mT - \frac{h}{2D}Q_m^2 - c_dQ_m = K_l+mK_d+c_lQ_T+(m-1)c_dQ_T+m^2(-\frac{hT}{2}Q_T+\frac{h}{2D}Q_T^2)+m\frac{hT}{2}Q_T - mQ_T(c_d + m\frac{h}{D}Q_T - mhT)$

$\implies \frac{h}{D}Q_mmQ_T + \frac{h}{D}Q_m^2 + Q_m(c_d + m\frac{h}{D}Q_T - mhT) - m\frac{h}{D}Q_mQ_T + mhQ_mT - \frac{h}{2D}Q_m^2 - c_dQ_m = K_l+mK_d+c_lQ_T+(m-1)c_dQ_T+m^2(-\frac{hT}{2}Q_T+\frac{h}{2D}Q_T^2)+m\frac{hT}{2}Q_T - mQ_T(c_d + m\frac{h}{D}Q_T - mhT)$

$\implies \frac{h}{2D}Q_m^2+Q_m(c_d+m\frac{h}{D}Q_T-mhT+\frac{h}{D}mQ_T-m\frac{h}{D}Q_T+mhT-c_d) = K_l+mK_d+c_lQ_T + (m-1)c_dQ_T + m^2(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + m\frac{hT}{2}Q_T - mQ_T(c_d + m\frac{h}{D}Q_T - mhT)$

$\implies \frac{h}{2D}Q_m^2 + Q_mm\frac{h}{D}Q_T = K_l + mK_d + c_lQ_T + mc_dQ_T - c_dQ_T - m^2\frac{hT}{2}Q_T +$

$m^2 \frac{h}{2D} Q_T^2 + m\frac{hT}{2} Q_T - mQ_T c_d - mQ_T m\frac{h}{D} Q_T + mQ_T mhT$

$\implies \frac{h}{2D} Q_m^2 + Q_m m\frac{h}{D} Q_T = K_l + mK_d + c_l Q_T - c_d Q_T + m^2 \frac{hT}{2} Q_T - m^2 \frac{h}{2D} Q_T^2 + m\frac{hT}{2} Q_T$

$\implies \frac{h}{2D} Q_m^2 + m\frac{h}{D} Q_T Q_m - K_l - mK_d - c_l Q_T + c_d Q_T - m^2 \frac{hT}{2} Q_T + m^2 \frac{h}{2D} Q_T^2 - m\frac{hT}{2} Q_T = 0$

So Q_m has a solution through above quadratic equation where:

$a = \frac{h}{2D}$

$b = m\frac{h}{D} Q_T$

$c = -K_l - mK_d - c_l Q_T + c_d Q_T - m^2 \frac{hT}{2} Q_T + m^2 \frac{h}{2D} Q_T^2 - m\frac{hT}{2} Q_T$

Note that as both a and b are positive, only the positive root, i.e. $\frac{-b+\sqrt{b^2-4ac}}{2a}$, will be used to find Q_m - since $\frac{-b}{2a}$ is negative. Hence there is a single positive root to the equation. ∎

A.4 Proof of Lemma 2

Proof. 2

We know that at optimality, where we have $m = m^*$ and $Q_m = Q_m^*$ we have:

$\frac{\frac{\partial CC}{\partial m}}{\frac{\partial CL}{\partial m}} = \frac{\frac{\partial CC}{\partial Q_m}}{\frac{\partial CL}{\partial Q_m}}$

$CC = K_l + mK_d + c_l Q_T + (m-1)c_d Q_T + c_d Q_m + m^2(-\frac{hT}{2} Q_T + \frac{h}{2D} Q_T^2) + m(\frac{hT}{2} Q_T + \frac{h}{D} Q_m Q_T - hQ_m T) + \frac{h}{2D} Q_m^2$

$CL = \frac{1}{D} mQ_T + \frac{1}{D} Q_m$

$\frac{\partial CC}{\partial Q_m} = c_d + \frac{h}{D}Q_m + m\frac{h}{D}Q_T - mhT$

$\frac{\partial CC}{\partial m} = K_d + c_d Q_T + 2m(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + (\frac{hT}{2}Q_T + \frac{h}{D}Q_m Q_T - hQ_m T)$

$\frac{\partial CL}{\partial Q_m} = \frac{1}{D}$

$\frac{\partial CL}{\partial m} = \frac{Q_T}{D}$

Then we have:

$\implies \frac{1}{\frac{Q_T}{D}}[K_d + c_d Q_T + 2m(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + (\frac{hT}{2}Q_T + \frac{h}{D}Q_m Q_T - hQ_m T)] = \frac{1}{D}(c_d + \frac{h}{D}Q_m + m\frac{h}{D}Q_T - mhT)$

$\implies K_d + c_d Q_T + 2m(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + (\frac{hT}{2}Q_T + \frac{h}{D}Q_m Q_T - hQ_m T) = Q_T(c_d + \frac{h}{D}Q_m + m\frac{h}{D}Q_T - mhT)$

$\implies K_d + c_d Q_T - 2m\frac{hT}{2}Q_T + 2m\frac{h}{2D}Q_T^2 + \frac{hT}{2}Q_T + \frac{h}{D}Q_m Q_T - hQ_m T = Q_T c_d + Q_T\frac{h}{D}Q_m + Q_T m\frac{h}{D}Q_T - Q_T mhT$

$\implies -Q_T m\frac{h}{D}Q_T + Q_T mhT - 2m\frac{hT}{2}Q_T + 2m\frac{h}{2D}Q_T^2 + \frac{h}{D}Q_m Q_T - hQ_m T - Q_T\frac{h}{D}Q_m = Q_T c_d - K_d - c_d Q_T - \frac{hT}{2}Q_T$

$\implies (-Q_T\frac{h}{D}Q_T + Q_T hT - 2\frac{hT}{2}Q_T + 2\frac{h}{2D}Q_T^2)m + Q_m(\frac{h}{D}Q_T - hT - Q_T\frac{h}{D}) = -K_d - \frac{hT}{2}Q_T$

$\implies (-\frac{h}{D}Q_T^2 + hTQ_T - hTQ_T + \frac{h}{D}Q_T^2)m + Q_m(-hT) = -K_d - \frac{hT}{2}Q_T$

$\implies Q_m(-hT) = -K_d - \frac{hT}{2}Q_T$

So when $m = m^*$, we have:

$\implies Q_m^* = \frac{1}{hT}K_d + \frac{1}{2}Q_T$ ■

A.5 Proof of Lemma 3

Proof. 3

Now we will look at the optimal value of m. Note that in theorem **1,** we have simplified the cycle cost expression.

$CC = K_l + mK_d + c_lQ_T + (m-1)c_dQ_T + c_dQ_m + m^2(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + m(\frac{hT}{2}Q_T + \frac{h}{D}Q_mQ_T - hQ_mT) + \frac{h}{2D}Q_m^2$

$CL = \frac{1}{D}mQ_T + \frac{1}{D}Q_m$

Our optimality condition is $\frac{\frac{\partial CC}{\partial m}}{\frac{\partial CL}{\partial m}} = \frac{CC}{CL}$

$\frac{\partial CL}{\partial m} = \frac{Q_T}{D}$

$\frac{\partial CC}{\partial m} = K_d + c_dQ_T + 2m(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + \frac{hT}{2}Q_T + \frac{h}{D}Q_mQ_T - hQ_mT$

$\Longrightarrow K_d + c_dQ_T - hTQ_Tm + \frac{h}{D}Q_T^2m + \frac{hT}{2}Q_T + \frac{h}{D}Q_mQ_T - hQ_mT$

$\Longrightarrow K_d + c_dQ_T - hTQ_Tm + \frac{h}{D}Q_T^2m + \frac{hT}{2}Q_T + \frac{h}{D}Q_mQ_T - hQ_mT$

When $Q_m^* = \frac{1}{hT}K_d + \frac{1}{2}Q_T$

$\Longrightarrow K_d + c_dQ_T - hTQ_Tm + \frac{h}{D}Q_T^2m + \frac{hT}{2}Q_T + \frac{h}{D}(\frac{1}{hT}K_d + \frac{1}{2}Q_T)Q_T - hT(\frac{1}{hT}K_d + \frac{1}{2}Q_T)$

$\Longrightarrow K_d + c_dQ_T - hTQ_Tm + \frac{h}{D}Q_T^2m + \frac{hT}{2}Q_T + \frac{1}{DT}K_dQ_T + \frac{h}{D}\frac{1}{2}Q_T^2 - K_d - \frac{hT}{2}Q_T$

$\Longrightarrow c_dQ_T - hTQ_Tm + \frac{h}{D}Q_T^2m + \frac{1}{DT}K_dQ_T + \frac{h}{D}\frac{1}{2}Q_T^2$

$\Longrightarrow m(\frac{h}{D}Q_T^2 - hTQ_T) + \frac{1}{DT}K_dQ_T + \frac{h}{2D}Q_T^2 + c_dQ_T$

Also substituting $Q_m = \frac{1}{hT}K_d + \frac{1}{2}Q_T$ to CC and CL

$CC = K_l + mK_d + c_lQ_T + (m-1)c_dQ_T + c_d(\frac{1}{hT}K_d + \frac{1}{2}Q_T) + m^2(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + m(\frac{hT}{2}Q_T + \frac{h}{D}(\frac{1}{hT}K_d + \frac{1}{2}Q_T)Q_T - h(\frac{1}{hT}K_d + \frac{1}{2}Q_T)T) + \frac{h}{2D}(\frac{1}{hT}K_d + \frac{1}{2}Q_T)^2$

$\implies K_l + mK_d + c_lQ_T + mc_dQ_T - c_dQ_T + c_d(\frac{1}{hT}K_d + \frac{1}{2}Q_T) + m^2(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + m(\frac{hT}{2}Q_T + \frac{1}{hT}K_d\frac{h}{D}Q_T + \frac{1}{2}Q_T\frac{h}{D}Q_T - \frac{1}{hT}K_dhT - \frac{1}{2}hTQ_T) + \frac{h}{2D}(\frac{1}{hT}K_d + \frac{1}{2}Q_T)^2$

$\implies K_l + mK_d + c_lQ_T + mc_dQ_T - c_dQ_T + c_d(\frac{1}{hT}K_d + \frac{1}{2}Q_T) + m^2(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + m(\frac{1}{DT}K_dQ_T + \frac{h}{2D}Q_T^2 - K_d) + \frac{h}{2D}(\frac{1}{hT}K_d + \frac{1}{2}Q_T)^2$

$\implies m^2(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + m(\frac{1}{DT}K_dQ_T + \frac{h}{2D}Q_T^2 + c_dQ_T) + K_l + c_lQ_T - c_dQ_T + c_d\frac{1}{hT}K_d + \frac{1}{2}Q_Tc_d + \frac{1}{h^2T^2}K_d^2\frac{h}{2D} + \frac{1}{4}Q_T^2\frac{h}{2D} + 2\frac{h}{2D}\frac{1}{2hT}Q_TK_d$

$\implies m^2(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + m(\frac{1}{DT}K_dQ_T + \frac{h}{2D}Q_T^2 + c_dQ_T) + K_l + c_lQ_T - c_dQ_T + c_d\frac{1}{hT}K_d + \frac{1}{2}Q_Tc_d + \frac{1}{2DhT^2}K_d^2 + Q_T^2\frac{h}{8D} + \frac{1}{2DT}Q_TK_d$

$\implies m^2(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + m(\frac{1}{DT}K_dQ_T + \frac{h}{2D}Q_T^2 + c_dQ_T) + K_l + c_lQ_T + c_d\frac{1}{hT}K_d - \frac{1}{2}Q_Tc_d + \frac{1}{2DhT^2}K_d^2 + Q_T^2\frac{h}{8D} + \frac{1}{2DT}Q_TK_d$

$CL = \frac{1}{D}mQ_T + \frac{1}{D}(\frac{1}{hT}K_d + \frac{1}{2}Q_T)$

$\implies \frac{1}{D}mQ_T + \frac{1}{hDT}K_d + \frac{1}{2D}Q_T$

We know that $\frac{\frac{\partial CC}{\partial m}}{\frac{\partial CL}{\partial m}} = \frac{CC}{CL}$, hence:

$\frac{1}{\frac{Q_T}{D}}[m(\frac{h}{D}Q_T^2 - hTQ_T) + \frac{1}{DT}K_dQ_T + \frac{h}{2D}Q_T^2 + c_dQ_T] = \frac{1}{\frac{mQ_T}{D} + \frac{1}{DhT}K_d + \frac{1}{2D}Q_T}[m^2(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + m(\frac{1}{DT}K_dQ_T + \frac{h}{2D}Q_T^2 + c_dQ_T) + K_l + c_lQ_T + c_d\frac{1}{hT}K_d - \frac{1}{2}Q_Tc_d + \frac{1}{2DhT^2}K_d^2 + Q_T^2\frac{h}{8D} + \frac{1}{2DT}Q_TK_d]$

$\implies \frac{D}{Q_T}(\frac{mQ_T}{D} + \frac{1}{DhT}K_d + \frac{1}{2D}Q_T)[m(\frac{h}{D}Q_T^2 - hTQ_T) + \frac{1}{DT}K_dQ_T + \frac{h}{2D}Q_T^2 + c_dQ_T] = m^2(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + m(\frac{1}{DT}K_dQ_T + \frac{h}{2D}Q_T^2 + c_dQ_T) + K_l + c_lQ_T + c_d\frac{1}{hT}K_d - \frac{1}{2}Q_Tc_d + \frac{1}{2DhT^2}K_d^2 + Q_T^2\frac{h}{8D} + \frac{1}{2DT}Q_TK_d$

$\implies (m + \frac{1}{Q_T hT}K_d + \frac{1}{2})[m(\frac{h}{D}Q_T^2 - hTQ_T) + \frac{1}{DT}K_dQ_T + \frac{h}{2D}Q_T^2 + c_dQ_T] = m^2(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + m(\frac{1}{DT}K_dQ_T + \frac{h}{2D}Q_T^2 + c_dQ_T) + K_l + c_lQ_T + c_d\frac{1}{hT}K_d - \frac{1}{2}Q_Tc_d + \frac{1}{2DhT^2}K_d^2 + Q_T^2\frac{h}{8D} + \frac{1}{2DT}Q_TK_d$

Let us first look at the left hand side:

$LHS = (m + \frac{1}{Q_ThT}K_d + \frac{1}{2})(m(\frac{h}{D}Q_T^2 - hTQ_T) + \frac{1}{DT}K_dQ_T + \frac{h}{2D}Q_T^2 + c_dQ_T)$

$\implies m^2(\frac{h}{D}Q_T^2 - hTQ_T) + m\frac{1}{DT}K_dQ_T + m\frac{h}{2D}Q_T^2 + mc_dQ_T + m\frac{1}{Q_ThT}K_d(\frac{h}{D}Q_T^2 - hTQ_T) + \frac{1}{DT}K_dQ_T\frac{1}{Q_ThT}K_d + \frac{h}{2D}Q_T^2\frac{1}{Q_ThT}K_d + \frac{1}{Q_ThT}K_dc_dQ_T + \frac{1}{2}m(\frac{h}{D}Q_T^2 - hTQ_T) + \frac{1}{2}\frac{1}{DT}K_dQ_T + \frac{1}{2}\frac{h}{2D}Q_T^2 + \frac{1}{2}c_dQ_T$

$\implies m^2(\frac{h}{D}Q_T^2 - hTQ_T) + m\frac{1}{DT}K_dQ_T + m\frac{h}{2D}Q_T^2 + mc_dQ_T + m(\frac{1}{Q_ThT}K_d\frac{h}{D}Q_T^2 - \frac{1}{Q_ThT}K_dhTQ_T) + \frac{1}{DThT}K_d^2 + Q_T\frac{1}{2DT}K_d + \frac{1}{hT}K_dc_d + m(\frac{h}{2D}Q_T^2 - \frac{1}{2}hTQ_T) + \frac{1}{2DT}K_dQ_T + \frac{h}{4D}Q_T^2 + \frac{1}{2}c_dQ_T$

$\implies m^2(\frac{h}{D}Q_T^2 - hTQ_T) + m(\frac{1}{DT}K_dQ_T + \frac{h}{2D}Q_T^2 + c_dQ_T + \frac{1}{DT}K_dQ_T - K_d + \frac{h}{2D}Q_T^2 - \frac{1}{2}hTQ_T) + \frac{1}{DThT}K_d^2 + \frac{1}{hT}K_dc_d + \frac{1}{DT}K_dQ_T + \frac{h}{4D}Q_T^2 + \frac{1}{2}c_dQ_T$

$\implies m^2(\frac{h}{D}Q_T^2 - hTQ_T) + m(\frac{2}{DT}K_dQ_T + \frac{h}{D}Q_T^2 + c_dQ_T - K_d - \frac{1}{2}hTQ_T) + \frac{1}{DThT}K_d^2 + \frac{1}{hT}K_dc_d + \frac{1}{DT}K_dQ_T + \frac{h}{4D}Q_T^2 + \frac{1}{2}c_dQ_T$

Then:

$m^2(\frac{h}{D}Q_T^2 - hTQ_T) + m(\frac{2}{DT}K_dQ_T + \frac{h}{D}Q_T^2 + c_dQ_T - K_d - \frac{1}{2}hTQ_T) + \frac{1}{DThT}K_d^2 + \frac{1}{hT}K_dc_d + \frac{1}{DT}K_dQ_T + \frac{h}{4D}Q_T^2 + \frac{1}{2}c_dQ_T = m^2(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + m(\frac{1}{DT}K_dQ_T + \frac{h}{2D}Q_T^2 + c_dQ_T) + K_l + c_lQ_T + c_d\frac{1}{hT}K_d - \frac{1}{2}Q_Tc_d + \frac{1}{2DhT^2}K_d^2 + Q_T^2\frac{h}{8D} + \frac{1}{2DT}Q_TK_d$

$\implies m^2(\frac{h}{D}Q_T^2 - hTQ_T + \frac{hT}{2}Q_T - \frac{h}{2D}Q_T^2) + m(\frac{2}{DT}K_dQ_T + \frac{h}{D}Q_T^2 + c_dQ_T - K_d - \frac{1}{2}hTQ_T - \frac{1}{DT}K_dQ_T - \frac{h}{2D}Q_T^2 - c_dQ_T) + \frac{1}{DThT}K_d^2 + \frac{1}{hT}K_dc_d + \frac{1}{DT}K_dQ_T + \frac{h}{4D}Q_T^2 +$

$\frac{1}{2}c_dQ_T - K_l - c_lQ_T - c_d\frac{1}{hT}K_d + \frac{1}{2}Q_Tc_d - \frac{1}{2DhT^2}K_d^2 - Q_T^2\frac{h}{8D} - \frac{1}{2DT}Q_TK_d = 0$

$\implies m^2(\frac{h}{2D}Q_T^2 - \frac{hT}{2}Q_T) + m(\frac{1}{DT}K_dQ_T + \frac{h}{2D}Q_T^2 - K_d - \frac{1}{2}hTQ_T) + \frac{1}{2DThT}K_d^2 + \frac{1}{2DT}K_dQ_T + \frac{h}{8D}Q_T^2 + c_dQ_T - K_l - c_lQ_T$

Hence we get a quadratic equation for m where:

$a = \frac{h}{2D}Q_T^2 - \frac{hT}{2}Q_T$

$b = \frac{1}{DT}K_dQ_T + \frac{h}{2D}Q_T^2 - K_d - \frac{hT}{2}Q_T$

$c = \frac{1}{2DThT}K_d^2 + \frac{1}{2TD}Q_TK_d + \frac{h}{8D}Q_T^2 + c_dQ_T - K_l - c_lQ_T$

We know that the roots for this quadratic equation are: $m^*_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

Let us investigate a and b in depth:

$Q_T > DT$

$\implies \frac{h}{D}Q_T > hT$

$\implies \frac{h}{2D}Q_T^2 > \frac{hT}{2}Q_T$

$\frac{h}{2D}Q_T^2 - \frac{hT}{2}Q_T > 0$

Then, $a > 0$

Since we are only interested in the case when $Q_T > DT$ we know that a is always positive. Now let us look at b:

$Q_T > DT$

$\implies \frac{1}{DT}Q_T > 1 \implies \frac{1}{DT}K_dQ_T > K_d$

and we also know that $\frac{h}{2D}Q_T^2 > \frac{hT}{2}Q_T$ then:

$\Longrightarrow \frac{1}{DT}K_dQ_T + \frac{h}{2D}Q_T^2 > K_d + \frac{hT}{2}Q_T$

$\Longrightarrow \frac{1}{DT}K_dQ_T + \frac{h}{2D}Q_T^2 - K_d - \frac{hT}{2}Q_T > 0$

$b > 0$

So we also know that b is always positive when $Q_T > DT$ thus one of the roots for m, i.e.$\frac{-b-\sqrt{b^2-4ac}}{2a}$, will be negative when $Q_T > DT$ because $\frac{-b}{2a}$ is negative and we will need $+\sqrt{b^2-4ac}$ to find a meaningful m. So m has a single positive root at $\frac{-b+\sqrt{b^2-4ac}}{2a}$. ■

A.6 Proof of Theorem 4

Proof. 4

To prove the convexity of the total cost rate we need to compute its Hessian. As we set $Q_m = \frac{1}{hT}K_d + \frac{1}{2}Q_T$ the total cost rate, TC, is only dependent on m hence we need to look at the second derivative of the total cost rate, and if we are able to show that it is greater than 0, then we know that TC is convex in m.

$CC = K_l + mK_d + m^2(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + m[\frac{hT}{2}Q_T + \frac{h}{D}Q_T(\frac{1}{hT}K_d + \frac{1}{2}Q_T) - hT(\frac{1}{hT}K_d + \frac{1}{2}Q_T)] + \frac{h}{2D}(\frac{1}{hT}K_d + \frac{1}{2}Q_T)^2 + c_lQ_T + (m-1)c_dQ_T + c_d\frac{1}{hT}K_d + \frac{1}{2}c_dQ_T$

$\Longrightarrow m^2(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + m(\frac{hT}{2}Q_T + \frac{h}{D}Q_T\frac{1}{hT}K_d + \frac{h}{D}Q_T\frac{1}{2}Q_T - hT\frac{1}{hT}K_d - hT\frac{1}{2}Q_T + c_dQ_T + K_d) + \frac{h}{2D}\frac{1}{h^2T^2}K_d^2 + \frac{h}{2D}\frac{1}{4}Q_T^2 + \frac{h}{2D}\frac{1}{hT}K_dQ_T + c_lQ_T - c_dQ_T + c_d\frac{1}{hT}K_d + \frac{1}{2}c_dQ_T + K_l$

$\Longrightarrow m^2(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + m(\frac{1}{DT}K_dQ_T + \frac{h}{2D}Q_T^2 + c_dQ_T) + \frac{1}{2DhT^2}K_d^2 + \frac{h}{8D}Q_T^2 + \frac{1}{2DT}K_dQ_T + c_lQ_T + c_d\frac{1}{hT}K_d - \frac{1}{2}c_dQ_T + K_l$

Let $H_1 = \frac{1}{2DhT^2}K_d^2 + \frac{h}{8D}Q_T^2 + \frac{1}{2TD}K_dQ_T + c_lQ_T + c_d\frac{1}{hT}K_d - \frac{1}{2}c_dQ_T + K_l$

Then $CC = m^2(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + m(\frac{1}{DT}K_dQ_T + \frac{h}{2D}Q_T^2 + c_dQ_T) + H_1$

$\frac{\partial CC}{\partial m} = 2m(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + \frac{1}{DT}K_dQ_T + \frac{h}{2D}Q_T^2 + c_dQ_T = m(-hTQ_T + \frac{h}{D}Q_T^2) + \frac{1}{DT}K_dQ_T + \frac{h}{2D}Q_T^2 + c_dQ_T$

$CL = \frac{mQ_T}{D} + \frac{1}{DhT}K_d + \frac{1}{2D}Q_T$

Let $H_2 = \frac{1}{DhT}K_d + \frac{1}{2D}Q_T$

Then $CL = \frac{mQ_T}{D} + H_2$

$\frac{\partial CL}{\partial m} = \frac{Q_T}{D}$

Now let us look at the first derivative of the total cost rate with respect to m.

$\frac{\partial}{\partial m}(\frac{CC}{CL}) = \frac{\partial CC}{\partial m}\frac{CL}{CL^2} - \frac{\partial CL}{\partial m}\frac{CC}{CL^2}$

$\frac{\partial}{\partial m}(\frac{CC}{CL}) = \frac{1}{CL^2}[(m(-hTQ_T + \frac{h}{D}Q_T^2) + \frac{1}{DT}K_dQ_T + \frac{h}{2D}Q_T^2 + c_dQ_T)(\frac{mQ_T}{D} + H_2) - \frac{Q_T}{D}(m^2(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + m(\frac{1}{DT}K_dQ_T + \frac{h}{2D}Q_T^2 + c_dQ_T) + H_1)]$

Let $H_3 = \frac{1}{DT}K_dQ_T + \frac{h}{2D}Q_T^2 + c_dQ_T$

$\frac{\partial}{\partial m}(\frac{CC}{CL}) = \frac{1}{CL^2}[(m(-hTQ_T + \frac{h}{D}Q_T^2) + H_3)(\frac{mQ_T}{D} + H_2) - \frac{Q_T}{D}(m^2(-\frac{hT}{2}Q_T + \frac{h}{2D}Q_T^2) + m(H_3) + H_1)]$

$\implies \frac{1}{CL^2}(-mhTQ_T\frac{mQ_T}{D} + \frac{h}{D}mQ_T^2\frac{mQ_T}{D} + H_3\frac{mQ_T}{D} - H_2mhTQ_T + H_2m\frac{h}{D}Q_T^2 + H_2H_3 + \frac{Q_T}{D}m^2\frac{hT}{2}Q_T - \frac{Q_T}{D}m^2\frac{h}{2D}Q_T^2 - \frac{Q_T}{D}mH_3 - \frac{Q_T}{D}H_1)$

$\implies \frac{1}{CL^2}(-H_2mhTQ_T + H_2m\frac{h}{D}Q_T^2 + H_2H_3 - m^2\frac{hT}{2D}Q_T^2 + m^2\frac{h}{2D^2}Q_T^3 - \frac{Q_T}{D}H_1)$

$\frac{\partial}{\partial m}(\frac{CC}{CL}) = \frac{1}{CL^2}(-H_2mhTQ_T + H_2m\frac{h}{D}Q_T^2 + H_2H_3 - m^2\frac{hT}{2D}Q_T^2 + m^2\frac{h}{2D^2}Q_T^3 - \frac{Q_T}{D}H_1)$

where $CL^2 = \frac{m^2Q_T^2}{D^2} + H_2^2 + 2\frac{mQ_T}{D}H_2$

Now let us look at the second derivative:

$\frac{\partial}{\partial m}(-H_2mhTQ_T + H_2m\frac{h}{D}Q_T^2 + H_2H_3 - m^2\frac{hT}{2D}Q_T^2 + m^2\frac{h}{2D^2}Q_T^3 - \frac{Q_T}{D}H_1) =$
$-H_2hTQ_T + H_2\frac{h}{D}Q_T^2 - 2m\frac{hT}{2D}Q_T^2 + 2m\frac{h}{2D^2}Q_T^3$

$\frac{\partial}{\partial m}(CL^2) = 2m\frac{Q_T^2}{D^2} + 2\frac{Q_T}{D}H_2$

$\frac{\partial^2}{\partial m^2}(\frac{CC}{CL}) = \frac{1}{CL^4}[(-H_2hTQ_T + H_2\frac{h}{D}Q_T^2 - 2m\frac{hT}{2D}Q_T^2 + 2m\frac{h}{2D^2}Q_T^3)(\frac{m^2Q_T^2}{D^2} + H_2^2 + 2\frac{mQ_T}{D}H_2) - (-H_2mhTQ_T + H_2m\frac{h}{D}Q_T^2 + H_2H_3 - m^2\frac{hT}{2D}Q_T^2 + m^2\frac{h}{2D^2}Q_T^3 - \frac{Q_T}{D}H_1)(2m\frac{Q_T^2}{D^2} + 2\frac{Q_T}{D}H_2)]$

$\Longrightarrow \frac{1}{CL^4}(A - B)$

$A = (-H_2hTQ_T + H_2\frac{h}{D}Q_T^2 - 2m\frac{hT}{2D}Q_T^2 + 2m\frac{h}{2D^2}Q_T^3)(\frac{m^2Q_T^2}{D^2} + H_2^2 + 2\frac{mQ_T}{D}H_2)$

$\Longrightarrow -H_2hTQ_T\frac{m^2Q_T^2}{D^2} + H_2\frac{h}{D}Q_T^2\frac{m^2Q_T^2}{D^2} - 2m\frac{hT}{2D}Q_T^2\frac{m^2Q_T^2}{D^2} + 2m\frac{h}{2D^2}Q_T^3\frac{m^2Q_T^2}{D^2} - H_2hTQ_T2\frac{mQ_T}{D}H_2 + H_2\frac{h}{D}Q_T^22\frac{mQ_T}{D}H_2 - 2m\frac{hT}{2D}Q_T^22\frac{mQ_T}{D}H_2 + 2m\frac{h}{2D^2}Q_T^32\frac{mQ_T}{D}H_2 - H_2hTQ_TH_2^2 + H_2\frac{h}{D}Q_T^2H_2^2 - 2m\frac{hT}{2D}Q_T^2H_2^2 + 2m\frac{h}{2D^2}Q_T^3H_2^2$

$\Longrightarrow -H_2\frac{hT}{D^2}m^2Q_T^3 + H_2\frac{h}{D^3}m^2Q_T^4 - \frac{hT}{D^3}m^3Q_T^4 + \frac{h}{D^4}m^3Q_T^5 - H_2^2\frac{2hT}{D}mQ_T^2 + H_2^2\frac{2h}{D^2}mQ_T^3 - H_2\frac{2hT}{D^2}m^2Q_T^3 + H_2\frac{2h}{D^3}m^2Q_T^4 - H_2^3hTQ_T + H_2^3\frac{h}{D}Q_T^2 - H_2^2\frac{hT}{D}mQ_T^2 + H_2^2\frac{h}{D^2}mQ_T^3$

$\Longrightarrow \frac{h}{D^4}m^3Q_T^5 + Q_T^4(-\frac{hT}{D^3}m^3 + H_2\frac{3h}{D^3}m^2) + Q_T^3(H_2^2\frac{3h}{D^2}m - H_2\frac{3hT}{D^2}m^2) + Q_T^2(H_2^3\frac{h}{D} - H_2^2\frac{3hT}{D}m) - H_2^3hTQ_T$

$B = (-H_2mhTQ_T + H_2m\frac{h}{D}Q_T^2 + H_2H_3 - m^2\frac{hT}{2D}Q_T^2 + m^2\frac{h}{2D^2}Q_T^3 - \frac{Q_T}{D}H_1)(2m\frac{Q_T^2}{D^2} + 2\frac{Q_T}{D}H_2)$

$\Longrightarrow -2m\frac{Q_T^2}{D^2}H_2mhTQ_T + 2m\frac{Q_T^2}{D^2}H_2m\frac{h}{D}Q_T^2 + 2m\frac{Q_T^2}{D^2}H_2H_3 - 2m\frac{Q_T^2}{D^2}m^2\frac{hT}{2D}Q_T^2 + 2m\frac{Q_T^2}{D^2}m^2\frac{h}{2D^2}Q_T^3 - 2m\frac{Q_T^2}{D^2}\frac{Q_T}{D}H_1 - 2\frac{Q_T}{D}H_2H_2mhTQ_T + 2\frac{Q_T}{D}H_2H_2m\frac{h}{D}Q_T^2 + 2\frac{Q_T}{D}H_2H_2H_3 - 2\frac{Q_T}{D}H_2m^2\frac{hT}{2D}Q_T^2 + 2\frac{Q_T}{D}H_2m^2\frac{h}{2D^2}Q_T^3 - 2\frac{Q_T}{D}H_2\frac{Q_T}{D}H_1$

$\implies -H_2\frac{2hT}{D^2}m^2Q_T^3 + H_2\frac{2h}{D^3}m^2Q_T^4 + H_2H_3\frac{2}{D^2}mQ_T^2 - \frac{hT}{D^3}m^3Q_T^4 + \frac{h}{D^4}m^3Q_T^5 - H_1\frac{2}{D^3}mQ_T^3 - H_2^2\frac{2hT}{D}mQ_T^2 + H_2^2\frac{2h}{D^2}mQ_T^3 + H_2^2H_3\frac{2}{D}Q_T - H_2\frac{hT}{D^2}m^2Q_T^3 + H_2\frac{h}{D^3}m^2Q_T^4 - H_2H_1\frac{2}{D^2}Q_T^2$

$\implies \frac{h}{D^4}m^3Q_T^5 + Q_T^4(H_2\frac{3h}{D^3}m^2 - \frac{hT}{D^3}m^3) + Q_T^3(-H_2\frac{3hT}{D^2}m^2 - H_1\frac{2}{D^3}m + H_2^2\frac{2h}{D^2}m) + Q_T^2(H_2H_3\frac{2}{D^2}m - H_2^2\frac{2hT}{D}m - H_2H_1\frac{2}{D^2}) + H_2^2H_3\frac{2}{D}Q_T$

Then:

$A - B = \frac{h}{D^4}m^3Q_T^5 + Q_T^4(-\frac{hT}{D^3}m^3 + H_2\frac{3h}{D^3}m^2) + Q_T^3(H_2^2\frac{3h}{D^2}m - H_2\frac{3hT}{D^2}m^2) + Q_T^2(H_2^3\frac{h}{D} - H_2^2\frac{3hT}{D}m) - H_2^3hTQ_T - \frac{h}{D^4}m^3Q_T^5 + Q_T^4(-H_2\frac{3h}{D^3}m^2 + \frac{hT}{D^3}m^3) + Q_T^3(H_2\frac{3hT}{D^2}m^2 + H_1\frac{2}{D^3}m - H_2^2\frac{2h}{D^2}m) + Q_T^2(-H_2H_3\frac{2}{D^2}m + H_2^2\frac{2hT}{D}m + H_2H_1\frac{2}{D^2}) - H_2^2H_3\frac{2}{D}Q_T$

$\implies Q_T^5(\frac{h}{D^4}m^3 - \frac{h}{D^4}m^3) + Q_T^4(-\frac{hT}{D^3}m^3 + H_2\frac{3h}{D^3}m^2 - H_2\frac{3h}{D^3}m^2 + \frac{hT}{D^3}m^3) + Q_T^3(H_2^2\frac{3h}{D^2}m - H_2\frac{3hT}{D^2}m^2 + H_2\frac{3hT}{D^2}m^2 + H_1\frac{2}{D^3}m - H_2^2\frac{2h}{D^2}m) + Q_T^2(H_2^3\frac{h}{D} - H_2^2\frac{3hT}{D}m - H_2H_3\frac{2}{D^2}m + H_2^2\frac{2hT}{D}m + H_2H_1\frac{2}{D^2}) + Q_T(-H_2^3hT - H_2^2H_3\frac{2}{D})$

$\implies Q_T^3(H_2^2\frac{h}{D^2}m + H_1\frac{2}{D^3}m) + Q_T^2(H_2^3\frac{h}{D} - H_2^2\frac{hT}{D}m - H_2H_3\frac{2}{D^2}m + H_2H_1\frac{2}{D^2}) + Q_T(-H_2^3hT - H_2^2H_3\frac{2}{D})$

If we can show that the second derivative of the total cost rate with respect to m is positive, than we know that the m found using it's quadratic equation is the minimum, and the total cost rate is convex.

$\frac{\partial^2}{\partial m^2}(\frac{CC}{CL}) = \frac{1}{(\frac{mQ_T}{D}+H_2)^4}[Q_T^3(H_2^2\frac{h}{D^2}m + H_1\frac{2}{D^3}m) + Q_T^2(H_2^3\frac{h}{D} - H_2^2\frac{hT}{D}m - H_2H_3\frac{2}{D^2}m + H_2H_1\frac{2}{D^2}) + Q_T(-H_2^3hT - H_2^2H_3\frac{2}{D})]$

We know that $\frac{1}{CL^4}$ is positive then we need to show that $A - B$ is positive.

$Q_T^3(H_2^2\frac{h}{D^2}m + H_1\frac{2}{D^3}m) + Q_T^2(H_2^3\frac{h}{D} - H_2^2\frac{hT}{D}m - H_2H_3\frac{2}{D^2}m + H_2H_1\frac{2}{D^2}) + Q_T(-H_2^3hT -$

$H_2^2 H_3 \frac{2}{D}) >^? 0$

$\implies Q_T^2 H_2 \frac{h}{D} m + Q_T H_2^2 h + Q_T H_1 \frac{2}{D} + Q_T^2 \frac{H_1}{H_2} \frac{2}{D^2} m >^? Q_T H_2 h T m + H_2^2 h T D +$

$2 H_2 H_3 + Q_T H_3 \frac{2}{D} m$ ■

A.7 Proof of Corollary 1

Proof. 1 We have investigated each term on either sides of the inequality found in Theorem 4 separately.

Let us look at the first terms of the inequality on either side:

$Q_T^2 H_2 \frac{h}{D} m >^? Q_T H_2 h T m \implies Q_T \frac{1}{D} >^? T \implies Q_T > DT$

Let us look at the second terms of the inequality on either side:

$Q_T H_2^2 h >^? H_2^2 h T D \implies Q_T > DT$

As we are only interested in the case when $Q_T > DT$ we know the first two terms satisfy the positivity requirement.

Let us look at the third terms of the inequality on either side

$Q_T H_1 \frac{2}{D} >^? 2 H_2 H_3 \implies Q_T H_1 >^? D H_2 H_3$

Finally, let us look at the fourth terms of the inequality on either side

$Q_T^2 \frac{H_1}{H_2} \frac{2}{D^2} m >^? Q_T H_3 \frac{2}{D} m \implies Q_T \frac{H_1}{H_2} \frac{1}{D} > H_3$

$\implies Q_T H_1 >^? D H_2 H_3$ (Note that this is the same as the third-term inequality)

$\implies Q_T(\frac{1}{2DhT^2} K_d^2 + \frac{h}{8D} Q_T^2 + \frac{1}{2TD} K_d Q_T + c_l Q_T + c_d \frac{1}{hT} K_d - \frac{1}{2} c_d Q_T + K_l) >^?$

$D(\frac{1}{DhT}K_d + \frac{1}{2D}Q_T)(\frac{1}{DT}K_dQ_T + \frac{h}{2D}Q_T^2 + c_dQ_T)$

$\implies \frac{1}{2DhT^2}K_d^2 + \frac{h}{8D}Q_T^2 + \frac{1}{2TD}K_dQ_T + c_lQ_T + c_d\frac{1}{hT}K_d - \frac{1}{2}c_dQ_T + K_l >^? (\frac{1}{hT}K_d + \frac{1}{2}Q_T)(\frac{1}{DT}K_d + \frac{h}{2D}Q_T + c_d)$

$\implies \frac{1}{2DhT^2}K_d^2 + \frac{h}{8D}Q_T^2 + \frac{1}{2TD}K_dQ_T + c_lQ_T + c_d\frac{1}{hT}K_d - \frac{1}{2}c_dQ_T + K_l >^? \frac{1}{hT}K_d\frac{1}{DT}K_d + \frac{1}{2}Q_T\frac{1}{DT}K_d + \frac{1}{hT}K_d\frac{h}{2D}Q_T + \frac{1}{2}Q_T\frac{h}{2D}Q_T + \frac{1}{hT}K_dc_d + \frac{1}{2}Q_Tc_d$

$\implies \frac{1}{2DhT^2}K_d^2 + \frac{h}{8D}Q_T^2 + \frac{1}{2TD}K_dQ_T + c_lQ_T + c_d\frac{1}{hT}K_d - \frac{1}{2}c_dQ_T + K_l >^? \frac{1}{hDT^2}K_d^2 + \frac{1}{2DT}K_dQ_T + \frac{1}{2TD}K_dQ_T + \frac{h}{4D}Q_T^2 + \frac{1}{hT}K_dc_d + \frac{1}{2}Q_Tc_d$

$\implies c_lQ_T + K_l >^? \frac{1}{2hDT^2}K_d^2 + \frac{1}{2DT}K_dQ_T + \frac{h}{8D}Q_T^2 + Q_Tc_d$

We know that $c_lQ_T > Q_Tc_d$:

$\implies K_l >^? \frac{1}{2hDT^2}K_d^2 + \frac{1}{2DT}K_dQ_T + \frac{h}{8D}Q_T^2$

$\implies K_l >^? \frac{h}{2D}(\frac{1}{h^2T^2}K_d^2 + \frac{1}{hT}K_dQ_T + \frac{1}{4}Q_T^2)$

$\implies K_l >^? \frac{h}{2D}(\frac{1}{hT}K_d + \frac{1}{2}Q_T)^2$

$\implies K_l >^? \frac{h}{2D}Q_m^{*2}$

So $K_l > \frac{h}{2D}Q_m^{*2}$ is the sufficient condition for the convexity of the buyer's cost function in m. ■

Appendix B

Visual Basic Code for the Numerical Analysis

The numerical analysis was performed using Microsoft Excel, and its Visual Basic Editor. The code analyzes the input values, and then calculates m and Q_m and then finds the Total Cost Rate. This is the master file, as there were other smaller files that were created to compare cases, and simple modifications were done to this code to find the total cost rate when m was rounded.

```
Sub eoqheuristic()
Worksheets("Initial").Activate
'write out counters start
nosavings05 = 0
savings05 = 0
nosavings10 = 0
savings10 = 0
nosavings15 = 0
```

```
savings15 = 0
nosavingsmore = 0
savingsmore = 0
i = 2
si = 3
qp5 = 0
qp10 = 0
qp15 = 0
qp20 = 0
qp25 = 0
dp5 = 0
dp10 = 0
dp15 = 0
dp20 = 0
dp25 = 0
'write out counters end
For dpc = 1 To 5      'discount percentage counter
For qpc = 1 To 5      'percentage counter
For tc = 1 To 5      'time counter for the discounted period
For trc = 3 To 52      'original eoq cases
Worksheets("Initial").Activate
i = i + 1
d = Cells(trc, 2).Value
h = Cells(trc, 4).Value
Kl = Cells(trc, 5).Value
cl = Cells(trc, 7).Value
eoq = Cells(trc, 8).Value
oi = Cells(trc, 10).Value      'Order interval
Kd = Kl * (1 - dpc * 0.05)      'Fixed cost discounted
cd = cl * (1 - dpc * 0.05)      'Variable cost discounted
```

```
qt = eoq * (1 + qpc * 0.05)        'qt increased and rounded up
TInv = 0
t = oi * (1 - tc * 0.05)           'Order interval for qt found
TotalCost = 0
Worksheets("Results").Activate
ma = -h * t / 2 * qt + h / 2 / d * qt ^2
mb = 1 / d / t * Kd * qt + h / 2 / d * qt ^2 - Kd - h * t / 2 * qt
mc = 1 / 2 / d / t ^2 / h * Kd ^2 + 1 / 2 / t / d * qt * Kd + cd * qt - cl * qt - Kl + h / 8 / d * qt ^2
m = (-mb + (mb ^2 - 4 * ma * mc) ^0.5) / (2 * ma)       'm calculated
qm = (1 / (h * t)) * Kd + 0.5 * qt                      'Qm calculated
'Purchase Cost Calculation
PurcCost = Kl + m * Kd + qt * cl + (m - 1) * qt * cd + qm * cd
        'Holding Cost Calculation
HoldCost = m ^2 * (h / 2 / d * qt ^2 - h * t * qt / 2) + m * (h * t / 2 * qt + h / d * qm * qt - h * qm * t) + h / 2 / d * qm ^2
TotalTime = m * qt / d + qm / d
TotalCost = PurcCost + HoldCost
costpertime = TotalCost / TotalTime      'Total Cost Rate found
eoqcost = Worksheets("Initial").Cells(trc, 15).Value
savings = eoqcost - costpertime
'Writing out results with savings to another worksheet
If savings > 0
Savingsperc = savings / eoqcost 'if there are savings achieved using qt
Worksheets("SavingsResults").Activate
Cells(si, 1).Value = trc - 2
Cells(si, 2).Value = Kl
Cells(si, 3).Value = cl
Cells(si, 4).Value = Kd
Cells(si, 5).Value = cd
Cells(si, 6).Value = qt
```

```
Cells(si, 7).Value = dpc * 0.05
Cells(si, 8).Value = h
Cells(si, 9).Value = eoq
Cells(si, 10).Value = oi
Cells(si, 11).Value = t
Cells(si, 12).Value = m
Cells(si, 13).Value = TotalCost
Cells(si, 14).Value = TotalTime
Cells(si, 15).Value = costpertime
Cells(si, 16).Value = eoqcost
Cells(si, 17).Value = savings
Cells(si, 18).Value = Savingsperc
Cells(si, 19).Value = qpc * 0.05
Cells(4, 24).Value = si - 2
'Some statistics on cases that had positive statistics
If Savingsperc < 0.15 Then
If Savingsperc < 0.1 Then
If Savingsperc < 0.05 Then
nosavings05 = nosavings05 + 1
savings05 = savings05 + Savingsperc
Else
nosavings10 = nosavings10 + 1
savings10 = savings10 + Savingsperc
End If
Else
nosavings15 = nosavings15 + 1
savings15 = savings15 + Savingsperc
End If
Else
nosavingsmore = nosavingsmore + 1
```

```
savingsmore = savingsmore + Savingsperc
End If
If qpc = 1 Then
qp5 = qp5 + 1
ElseIf qpc = 2 Then
qp10 = qp10 + 1
ElseIf qpc = 3 Then
qp15 = qp15 + 1
ElseIf qpc = 4 Then
qp20 = qp20 + 1
Else
qp25 = qp25 + 1
End If
If dpc = 1 Then
dp5 = dp5 + 1
ElseIf dpc = 2 Then
dp10 = dp10 + 1
ElseIf dpc = 3 Then
dp15 = dp15 + 1
ElseIf dpc = 4 Then
dp20 = dp20 + 1
Else
dp25 = dp25 + 1
End If
si = si + 1
Worksheets("Results").Activate
Else
Savingsperc = 0
End If
        'Writing out results regardless savings
```

```
Cells(i, 1).Value = trc - 2
Cells(i, 2).Value = Kl
Cells(i, 3).Value = cl
Cells(i, 4).Value = Kd
Cells(i, 5).Value = cd
Cells(i, 6).Value = qt
Cells(i, 7).Value = dpc * 0.05
Cells(i, 8).Value = h
Cells(i, 9).Value = eoq
Cells(i, 10).Value = oi
Cells(i, 11).Value = t
Cells(i, 12).Value = m
Cells(i, 13).Value = TotalCost
Cells(i, 14).Value = TotalTime
Cells(i, 15).Value = costpertime
Cells(i, 16).Value = eoqcost
Cells(i, 17).Value = savings
Cells(i, 18).Value = Savingsperc
Cells(i, 19).Value = qpc * 0.05
Cells(i, 20).Value = qm
        'Qm calculated using the quadratic equation
qa = h / 2 / d
qb = m * h / d * qt
qc = -Kl - m * Kd - cl * qt + cd * qt - m ^2 * h * t / 2 * qt + m ^2 * h / 2 / d * qt ^2 - m * h * t / 2 * qt
qmcalc = (-qb + (qb ^2 - 4 * qa * qc) ^0.5) / (2 * qa)
Cells(i, 21).Value = qmcalc
Next
Next
Next
Next
```

```
Worksheets("SavingsResults").Activate
Cells(7, 24).Value = nosavings05
If nosavings05 = 0 Then
Cells(7, 25).Value = 0
Else
Cells(7, 25).Value = savings05 / nosavings05
End If
Cells(8, 24).Value = nosavings10
If nosavings10 = 0 Then
Cells(8, 25).Value = 0
Else
Cells(8, 25).Value = savings10 / nosavings10
End If
Cells(9, 24).Value = nosavings15
If nosavings15 = 0 Then
Cells(9, 25).Value = 0
Else
Cells(9, 25).Value = savings15 / nosavings15
End If
Cells(10, 24).Value = nosavingsmore
If nosavingsmore = 0 Then
Cells(10, 25).Value = 0
Else
Cells(10, 25).Value = savingsmore / nosavingsmore
End If
Cells(6, 28).Value = qp5
Cells(7, 28).Value = qp10
Cells(8, 28).Value = qp15
Cells(9, 28).Value = qp20
Cells(10, 28).Value = qp25
```

```
Cells(6, 30).Value = dp5
Cells(7, 30).Value = dp10
Cells(8, 30).Value = dp15
Cells(9, 30).Value = dp20
Cells(10, 30).Value = dp25
Worksheets("Initial").Activate
End Sub
```