

FORAGING MOTION OF SWARMS AS NASH EQUILIBRIA OF DIFFERENTIAL GAMES

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DIFFERENTIAL GAMES

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We certify that we have read this dissertation and that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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ABSTRACT

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The question of whether foraging swarms can form as a result of a non-cooperative game played by individuals is shown here to have an affirmative answer. A dynamic (or, differential) game played by N agents in one-dimensional motion is introduced and models, for instance, a foraging ant colony. Each agent controls its velocity to minimize its total work done in a finite time interval. The agents in the game start from a set of initial positions and migrate towards a target foraging location. Such swarm games are shown to have unique Nash equilibria under two different foraging location specifications and both equilibria display many features of a foraging swarm behavior observed in biological swarms. Explicit expressions are derived for pairwise distances between individuals of the swarm, swarm size, and swarm center location during foraging.

Foraging swarms in one-dimensional motion with four different information structures are studied. These are complete and partial information structures, hierarchical leadership and one leader structures. In the complete information structure, every agent observes its distance to every other agent and makes use of this information in its effort optimization. In partial information structure, the agents know the position of only its neighboring agents. In the hierarchical leadership structure, the agents look only forward and measures its distance to the agents ahead. In single leader structure, the agents know the position of only leader. In all cases, a Nash equilibrium exists under some realistic assumptions on the sizes of the weighing parameters in the cost functions.

The consequences of having a “passive” leader in a swarm are also investigated. We model foraging swarms with leader and followers again as non-cooperative, multi-agent differential games. We consider two types of leadership structures, namely, hierarchical leadership and a single leader structure. In both games, the type of leadership is assumed to be passive since a leader is singled out only due to its rank in the initial queue. We identify the realistic assumptions under which a unique Nash equilibrium exists in each game and derive the properties of the

Nash equilibriums in detail. It is shown that having a passive leader economizes in the total information exchange at the expense of aggregation stability in a swarm.



Keywords: Differential games, Dynamic games, Nash equilibrium, Multi-agent systems, Swarm modeling, Swarming behavior, Social foraging, Artificial potentials, Rendezvous problem, Optimal control theory.

ÖZET

SÜRÜLERİN BESİN KAYNAĞI ARAMA HAREKETİNİN DİFERANSİYEL OYUNLARIN NASH DENGESİ OLARAK MODELLENMESİ

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Sürü hareketi çok oyunculu işbirliksiz oyunun sonucunda ortaya çıkabilir mi sorusunun cevabının olumlu olduğu gösterilmiştir. N oyuncu tarafından oynanan bir boyutlu dinamik (ya da diferansiyel) oyun, karınca sürülerinin hareketini modelleyebilmektedir. Her oyuncu kendi hızını kontrol ederek sınırlı bir zaman aralığında harcadığı çabayı azaltmaktadır. Sürünün üyeleri belirli ilk pozisyonlardan harekete başlamakta ve hedefe doğru yönelmektedir. Bu tarz sürü davranışının iki besin kaynağı modeli için tek Nash dengesinin var olduğu gösterilmiştir. Her iki Nash dengesi de biyolojik sürülerin özelliklerini taşımaktadır. Sürü üyelerinin çiftlerarası mesafesi, sürü ebatı ve sürünün merkezinin hareket eğrisi için açık ifadeler bulunmuştur.

Bir boyuttaki sürülerin besin kaynağı arayışı için dört veri yapısı ele alınmıştır. Bunlar tam ve kısmi veri yapıları; hiyerarşik ve tek liderli yapılardır. Tam veri yapısı altında, her oyuncunun diğer tüm oyuncuların pozisyonunu bildiği varsayılmaktadır. Kısmi veri yapısında, her oyuncunun yalnızca komşularının pozisyonunu bildiği varsayılmaktadır. Hiyerarşik lider yapısı altında, sürü üyelerinin sadece ileri baktıkları ve yalnızca önündeki sürü üyeleriyle mesafesini ölçtükleri varsayılmaktadır. Tek liderli yapı altında ise, lider sabit hızla hedefe yönelmektedir ve diğer oyuncular sadece liderin pozisyonunu bilmektedir. Her dört durum için, itme-çekme parametreleriyle ilgili gerçekçi varsayımlar altında Nash dengesinin var olduğu gösterilmiştir.

Sürüde pasif liderin varlığının sonuçları da detaylı bir şekilde ele alınmıştır. Lider ve takipçilerden oluşan sürülerin besin kaynağı arayışı da işbirliksiz çok oyunculu diferansiyel oyun olarak modellenmiştir. Bu kapsamda, hiyerarşik lider yapısı ve tek liderli yapı olmak üzere iki farklı lider yapısı incelenmiştir. Her iki oyunda da pasif liderlik ele alınmıştır, çünkü lider diğer oyuncularla mesafesini ölçmemektedir. Birtakım gerçekçi varsayımlar tanımlanarak, bu durumda Nash

dengesinin tek olduđu gösterilmiş ve bu Nash dengesinin özellikleri incelenmiştir. Pasif liderin varlığının toplam pozisyon bilgisi paylaşımından kazanç sağlarken sürü kararlılığından kayba sebep olduđu sonucuna varılmıştır.



Anahtar sözcükler: Diferansiyel oyunlar, Dinamik oyunlar, Nash dengesi, Çok parçacıklı sistemler, Sürü modellemesi, Sürü davranışı, Toplu besin arayışı, Yapay potansiyel, Randevu problemi, Optimal kontrol kuramı.

Dedicated to my father Aşır Yıldız



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Chapter 1

Introduction

Motivating a study of swarm behavior through a dynamic game model and via the concept of Nash equilibrium is our objective in this chapter. We first draw the boundaries of the swarm behavior that is examined, list possible applications of the model studied here and elsewhere, verbally describe the problems to be solved and summarize the results obtained, and finally describe the organization of the thesis.

1.1 Swarming Behavior

There are a number of motives for collective movements such as schooling of fish, flocking of birds, and herding of sheep. Some of these are having protection from predators, saving energy, and locating food sources with ease [1]. The following features of a swarm are most remarkable [2]: i) No member in a swarm views the whole picture, but their decentralized actions result in a collective behavior. ii) Simple actions of the members result in a complex behavior of the swarm. iii) There are no leaders commanding the others so that many swarms are self-propelled. iv) There is limited communication based on local information among members. Such features of swarms are expressed by the notions of coordinated

group behavior, self organization, stability, collision avoidance and distributed control [3]. Engineers have based their designs of multi-robot or multi-vehicle systems mainly on these concepts [4], [5], [6], [7].

The term “swarming behavior” is defined as the cooperative coordination of animals of the same species to achieve aggregation by forming clusters [8]. This behavior has many advantages such as reducing individual efforts, increasing the immigration distances, providing safety of the animals, and also enhancing the foraging performance [9]. For instance, the reason behind the flocking of birds in a v-formation is effort reduction to be able to cover longer immigration distances [10]. Initially studied for the purpose of biological modeling, the swarming behavior has been the basis for modeling of multi robot systems, multi vehicle systems, and also optimization algorithms.

Social foraging is defined as the searching act of a group of animals for food or better environment. In [11], the problem of the animal decision making in social foraging is modeled in a game theoretical framework. In that work, the effect of the ratio of the producers and scroungers on foraging performance is investigated. In [12], foraging is modeled as the minimization of a scalar field which represents the toxicity and food characteristics of the environment. A biological swarm is defined as a cluster that emigrates as a group via communications among individuals [13]. Foraging is such emigration to more favorable (less toxic or richer in nutrients) territory of the environment [12]. This collective behavior consists of three main stages: Formation of a cluster, emigration to a new territory, and constructing a nest [13]. There are two types of clusters, namely self organized swarms and leader-follower swarms. In self organized swarms, there are no leaders commanding the others, but there are some simple rules resulting in coordination [2].

In the leader-follower type of swarms, there is an agent that may be called an active leader, who guides the coordination of the whole cluster [14]. A passive leader in a swarm may be a member that makes no attempt at coordination but which all other members follow voluntarily in possibly varying degrees of submissiveness. In a foraging swarm, there may be different leaders in different

intervals of the journey or the same agent may play the role during the whole journey. A passive leader does not interact with the rest of the swarm members. Further, some members of the swarm may emerge with varying degrees of leadership, thereby resulting in a hierarchical structure in the swarm. There are certain advantages of having a leader in a swarm. The leader may initiate the route and the remaining group members follow that path [14]. The leader designates the search direction and, by its guidance, a wider area can be covered and the collisions can be avoided [15]. Moreover, leader-follower swarms reach consensus more rapidly [14]. There are also cases, where consensus may not even be guaranteed by only simple rules and choices of specific leaders become necessary to ensure consensus [16]. Leadership also provides orientation improvement and coordination via communication in the group [10], [17]. Leader-follower swarms have a multitude of practical applications such as robot teams, ship flocks, UAVs and vehicle platoons. The leader may play various roles in such systems. In robot teams, a leader is generally an active one, who itself is motion-controlled by an external control input [18]. In ship flocks, leader may enable coordination of possibly under-actuated followers [19]. In unmanned aerial vehicles, leader may provide reference position and velocity for followers [20]. In vehicle platoons, leader ensures string stability where tight formations are maintained [21]. In optimization techniques such as PSO, leader usually follows the shortest path, i.e., the line towards the minimum and the followers perform the search around that line [22]. In all these types, leaders constitutes a small subset of the group that guides the coordination of the whole network [14].

1.2 Applications of Swarming Behavior

Swarm modeling is a research topic that has attracted the attention of many diverse disciplines like physics, biology, and engineering. The application areas of swarm modeling range from biological modeling ([23]) to optimization ([24]) and locomotion design for autonomous systems ([25]). One of the most important applications of swarming is the motion planning of teams of robots. In a multiple robot system, the robots keep a formation while navigating to a target location.

In this setting, the agents achieve a cooperative task by exchanging information with the others while controlling their individual dynamics, [26], [27] and [28]. Here, using a team of simple robots instead of one sophisticated robot increases the robustness and resilience against communication errors [29]. An example of optimal motion planning for multiple robots is [30].

Another biologically inspired field related to swarms is the coordination of multiple vehicle systems. The swarm theory has been applied to both platooning of vehicles and air traffic control. Conflicts in the intersection crossings have been resolved by swarm theory in [31] and [32] for vehicle platooning on automated highways. In current air traffic control mechanism, the planes fly in predefined paths, which may deviate from the shortest path significantly. In future free flight paradigm that is discussed in [33] and [34], the air vehicles will arbitrarily select the elevation, speed and path, and the conflicts will be resolved by intelligent collision avoidance algorithms. Such future multi vehicle systems, namely the Unmanned Aerial Vehicles are studied in [35]. Another important application of swarming behavior is optimization. The recent versions of such an algorithm, Particle Swarm Optimization, are [36], [37].

1.3 A Survey of Swarm Modeling and Simulation

Simulation of swarming motion dates back to 1986 where an artificial environment called Boids has been created by Craig Reynolds [38]. This environment simulates swarming motion based on three simple rules: The particles should move in the same direction as, should not move too far away from, and should not approach too close to the adjacent particles. These simple rules resulted in satisfactory simulation of swarming motion. These are also the guiding principles of model-based approaches to foraging swarms like the attraction and repulsion potentials approaches in [39], [40], [41], [23], [42], and [43]. Compared to model

based approaches, simulation based approaches suffer from convergence, accuracy, and computational complexity issues. On the other hand, while Lyapunov based methods (e.g. [44], [45], [46]) remain confined to the stability (boundedness) analysis, a model based approach allows a more comprehensive theoretical analysis that may reveal important structural properties.

Artificial potentials are commonly used to model the interaction between individuals in multi agent systems. In this technique, the interaction is modeled as attractions and repulsions between the individuals so that a cluster form is maintained, [42], [47]. The individuals repel the neighbors in near field, and attract them in the far field. One of the first works that exploited artificial potentials is [48]. In that work, a set of individuals is selected as virtual leaders, so that the system is semi decentralized to achieve scalability. Another work that employs artificial potentials and that includes stability analysis is [40].

Open form algorithms are also widely used to analyze multi agent systems. However, convergence, accuracy, and computational complexity can be problematic in algorithm based techniques as opposed to closed form solutions. An example of a collision avoidance algorithm based on near field repulsion is [49]. Lyapunov based techniques are also applied [23], [50] and focus on the stability of the system but do not yield explicit solutions of the dynamics of the system. A method that yields an explicit solution of the system is, of course, preferable since it would lead to a simulation with low complexity and display the stability of the system with ease. A resource on obtaining explicit solutions of linear quadratic games is [51].

Game theory, in particular evolutionary game theory, has been extensively applied to analysis of swarm behavior and animal decision making, [11] and [52]. The use of game theory in social foraging, such as in [52], is limited to two person games since the objective is to predict and explain the foraging behaviors of animals while in groups. A combination of game theory and optimal control theory has also been applied to the modeling of dynamic behaviors of multi agent systems such as in [53]. The cooperative control of a multi agent system has been formulated as Hamilton-Jacobi form in a differential game framework in [54] and

[55]. In [56], game theory is employed for the optimal network consensus problem.

The notion of Nash equilibrium is actually ideally suited for studying collective behaviors that are caused by individual motives and actions. It thus seems that quests into the nature and the origin of collective behavior in swarms is a natural application area for game theoretical models; but, such studies are surprisingly rare. While it is true that game theory has been extensively applied to studies on animal decision making and social foraging ([11], [52]), the application has been limited to two-person games since the objective was mainly to understand the “motive formation” of animals. In studying multi-robot, multi-vehicle systems *cooperative* game theory has been the main tool applied since the emphasis there [57] is on the “design” of a swarm system, rather than an analysis which strives to “explain” collective behavior. Vehicle platooning or air traffic control in automated environments require conflict resolution so that game theory is used in [58], [59], [33] and [60] for the purpose of coordination.

One of the first studies of leader-follower swarms is [48], where Lyapunov stability test is applied to an artificial potentials framework to show that stable aggregation occurs under the presence of passive leaders. Numerous works have followed similar Lyapunov methodology under certain relaxations such as [61], [62], and [63]. The disadvantage of such techniques is that they are restricted to merely stability analysis, i.e., whether a cluster is actually formed and maintained. Some recent works such as [64], [65], and [66] have modeled leader-follower teams as a quadratic optimal control problem with a passive leader and obtained numerical solutions via Riccati equations.

Some limited success in obtaining analytic solutions to a unique Nash equilibrium has been demonstrated (see e.g., [51], [9], [55]) exploiting the advantage that the cost functionals are quadratic. In case of more general cost functionals, serious effort has been spent in studying the existence and uniqueness of a Nash equilibrium. Such problems are challenging because the Necessary Conditions of Optimality approach usually results in a set of nonlinear equations that do not obey any Lipschitz condition, so that the existence of a solution is by no means obvious. It should also be mentioned that whether one allows continuous

or discontinuous strategies to be available to agents is of crucial importance, as discussed in [67].

Current research on the existence and uniqueness of a Nash equilibrium in dynamic games focuses on two main techniques, which are the viscosity solution and regularity synthesis methods. In viscosity solution technique, optimality is checked by using properties of conservation laws for hyperbolic solutions [68]. If hyperbolicity conditions are satisfied, existence is guaranteed under some Lipschitz like regularity conditions. The existence is based on Glimm's Theorem [69], [70]. A successful example of a viscosity solution approach is [71], where the structure of the problem posed is rather simple.

The demonstration that a swarming behavior may result from a Nash equilibrium is to our knowledge a novel contribution of this thesis to swarm literature. As we will elaborate below, each agent in a swarm is assumed to optimize its total effort (total work done) in a finite time interval by minimizing a personal objective functional that encompasses the control effort, attraction, and repulsion profile, and a foraging location profile of the agent. Such games are shown to have a unique Nash equilibrium in the sense of [53] and the equilibrium trajectories of agents display many features of members in foraging swarm behavior. The attraction, repulsion, and foraging profiles are modeled based on the artificial potential approach of [12], where, in effect, swarm formation and its stability have been studied through the optimization of a collective (global) objective functional.

We remark that the nonlinear system that results from the Necessary Conditions of Optimality in this thesis does not obey any Lipschitz condition, so that existence of a solution is not automatic. However, the fact that we are dealing with a specific system dynamics helps and, in all four games considered here, we are able to establish the existence and uniqueness of a Nash equilibrium (i) for a class of attraction/repulsion profiles and (ii) under the assumption that strategies (control inputs) available to agents are continuous with respect to initial conditions.

The results in this thesis have been reported before in a series of publications

consisting of [72],[73],[74], and [75].

1.4 Main Contributions of the Thesis

In this thesis, a game theoretical model is introduced to examine how swarms form as in, for instance, the foraging behavior of ant colonies or in platooning of vehicles on automated highways. This is an individual focused study of swarms that questions whether a swarm can form in a time interval by non-cooperative actions of a finite number of individuals or agents. Here, we assume that each agent in a group, while in search of, say, food, minimizes its total effort by using the force it applies as a control input. This leads to an N -person infinite-dimensional dynamic game, [53], and to the question of whether this game has a Nash equilibrium that carries the features of a swarm. An affirmative answer means that non-cooperative optimization by N individuals results in a collective behavior, namely swarming behavior. The answer indeed turns out to be affirmative for particular individual cost functionals into which artificial potential energy [12] terms that represents the trade-off between repulsion and attraction is incorporated.

The modeling effort here is constrained to one-dimensional swarms, i.e., the motion of the agents are constrained to a line as is similar in the foraging behavior of ant colonies or in the platooning of vehicles on automated highways. This exercise, although purely theoretical, actually helps us to gain insight into swarms that form in higher dimensions. The main contribution of this thesis is to model foraging swarm behavior as four different non-cooperative dynamic games played by N individuals and show that these games have Nash equilibria that are unique under some reasonable assumptions and with respect to a class of strategies. These indicate that swarming behavior can result from non-cooperative actions of individuals. The Nash Equilibrium solutions for these games are described explicitly, i.e., expressions for optimal trajectories, swarm size and center trajectory are obtained. The games are also analyzed under different terminal conditions that correspond to whether the agents have partial or full desire to reach their target foraging location.

The information exchange structure among the agents is the main distinguishing feature of the four games described in detail in Chapter 3 below. In Game-1, each agent has a complete information of its pairwise distances to all other agents, which was also assumed in [12]. The swarm in Game 1 is thus assumed to have the structure of a complete topology network. The assumption that a member interacts with (exchanges information with or has sensory perception of) all of the remaining members of a swarm may be a realistic assumption when the swarm size is not too large or when designing a swarm system from scratch. It may not, however, be realistic in large biological swarms or if the cost of communication is substantial, in which cases it is more natural that the interaction takes place with adjacent agents only. Thus, in Game 2, this assumption is relaxed by considering that each agent has a partial information access and knows its pairwise distances to neighboring agents only. The swarm in Game 2 is thus assumed to have the structure of a line topology communication network. In both Game 1 and 2, there is no hierarchy among agents except the one imposed by their initial ordering or queuing. Games 1 and 2 will sometimes be referred to as *complete* and *partial information games*, respectively.

In Games L1 and L2, we study two further information structures that assumes a hierarchy among agents and that can be interpreted as games with *passive leaders*. These are members that *are* singled out by the other members, not because they command, coordinate, or organize, but because of their present geographical position in the group. Below, Games L1 and L2 will occasionally be referred to as *hierarchical* and *single passive leader games* due to their distinct information structures. The swarm members in both games are allowed to be “nonidentical” and each member measures its distance only to those members that are ahead. Both games may be compared with the v-formation of birds (although we limit our study to one-dimensional swarms) because an agent’s (level of) leadership depends on how close it is to the top of the hierarchy, [15], [76]. These games have a loose information structure as very little amount of attention span is needed from an agent during its journey. One consequence of this sparsity in intra-swarm communication is economy in energy expenditure.

Power and energy expenditure reduction is indeed an essential feature of a v-formation [77], [10], [78], and [79].

The main conclusion of our study, given in Chapter 4 below, is that all these four games without or with leader(s) have unique Nash equilibria under intuitive realistic constraints. This may be interpreted to mean that independent motives by the members of a group give rise to a swarm behavior that is characterized by aggregation stability and achievement of the foraging task. The appointment of a subset of the agents as leader(s) seems to remedy the sparsity of information or communication among members. In so far as the foraging task is concerned, the most distinguishing feature of a successful leader is that its desire to reach the target location must be at least as good as its followers'. The followers desire to reach the target location is of minor importance. Finally, in the game with a single leader, a change of order among the followers is also a valid Nash equilibrium. This feature has not been present in the other three games (complete, partial information and hierarchical leadership games) considered so far. The price payed by the loose information structure is that the formed swarm is less stable (looser aggregation) than those in which information exchange is denser. This is shown in Chapter 6 below by a thorough comparison of the swarm behavior that follow from the solutions to our four games.

1.5 Organization of the Thesis

The rest of this thesis is organized as follows. Chapter 2 contains some preliminaries on non-cooperative differential games and their Nash equilibrium. Chapter 3 includes the definition of the one dimensional game in its generality and the definitions of Games 1, 2, L1, and L2, for which we have been able to establish the existence of Nash equilibriums. Chapter 4 is devoted to the main results on these four games. The proofs of the theorems in Chapter 4 are presented in Chapter 5. Comparison of the obtained results on these four proposed games is covered in Chapter 6. The simulation results that illustrate the main characteristics of

the Nash equilibriums as well as some negative examples in which no Nash equilibrium exists are presented in Chapter 7. Finally, Chapter 8 is dedicated to the conclusions and possible future works.



Chapter 2

Preliminaries

This chapter describes a common framework for the dynamic (or, differential) games that will serve as a basis of models for foraging swarms. We give a definition of Nash equilibrium for a class of dynamic games and state a set of necessary conditions for a Nash equilibrium to exist.

2.1 Nash Equilibrium and Necessary Conditions for Existence

Consider dynamic state equations for $i \in \{1, 2, \dots, N\}$

$$\dot{x}^i(t) = f^i(t, x(t), u^1(t), \dots, u^N(t)); \quad x^i(0) = x_0^i \in R, \quad (2.1)$$

where $x^i(t) \in R$ is the state vector and $u^i(t) \in U^i$, is the input in the interval $[0, T]$ that lives in a function space U^i and is available to agent $i \in \{1, 2, \dots, N\}$. Let $\mathbf{x}_0 := [x_0^1, \dots, x_0^N]'$, where prime denotes transpose.

Also consider a set of cost functionals

$$L^i(u^1, \dots, u^N) = q^i(x(T)) + \int_0^T g^i(t, \mathbf{x}(t), u^1(t), \dots, u^N(t))dt, \quad (2.2)$$

where the functions $q^i(\cdot)$ and $g^i(t, \cdot, u^1(t), \dots, u^N(t))$ satisfy certain continuity requirements, that each agent $i \in \{1, 2, \dots, N\}$ minimizes. This defines a an N-agent dynamic game of fixed duration T . The game is said to admit a *Nash equilibrium* $\{u^{1*}, \dots, u^{N-1*}, u^{N*}\}$ if the following inequalities hold:

$$\begin{aligned} L^{1*} &= L^1(u^{1*}, u^{2*}, \dots, u^{N*}) \leq L^1(u^1, u^{2*}, \dots, u^{N*}) \\ L^{2*} &= L^2(u^{1*}, u^{2*}, u^{3*}, \dots, u^{N*}) \leq L^2(u^{1*}, u^2, u^{3*}, \dots, u^{N*}) \\ &\vdots \\ L^{N*} &= L^N(u^{1*}, \dots, u^{N-1*}, u^{N*}) \leq L^N(u^{1*}, \dots, u^{N-1*}, u^N), \end{aligned} \quad (2.3)$$

where u^{i*} is the *best response* (or, optimal) input of i -th agent, $u^i \in U^i$, and L^{i*} is the optimal cost of i -th agent incurred by the best responses. The resulting trajectories $\{x^{i*}, \dots, x^{N*}\}$ will also be referred to as *Nash trajectories*.

The following is based on Chapter 5 of [80], which we state for the case of agent- i 's optimal control problem.

Theorem 2.1 Suppose $u^{i*}(t), \dots, u^{(i-1)*}(t), u^{(i+1)*}(t), \dots, u^{N*}(t)$ are given. If $u^{i*}(t)$ is such that $L(u^{1*}, \dots, u^{N*})$ is the optimal cost of agent- i incurred by the best responses, and if $(x^{i*}(t), t \in [0, T])$ is the corresponding trajectory, then there exists a costate function $p^{i*}(t) : [0, T] \rightarrow R$ such that

$$\dot{x}^{i*}(t) = f^i(t, x^*(t), u^{1*}(t), \dots, u^{N*}(t)), \quad x^{i*}(0) = x_0^i, \quad (2.4)$$

$$0 = \frac{\partial}{\partial u^i} H^i(t, p^i(t), x^*(t), u^{1*}(t), \dots, u^{i-1*}(t), u^i(t), u^{i+1*}(t), \dots, u^{N*}(t)) \quad (2.5)$$

$$\dot{p}^i(t) = -\frac{\partial}{\partial x} H^i(t, p^i(t), x^*(t), u^{1*}(t), \dots, u^{N*}(t)), \quad (2.6)$$

$$p^i(T) = \frac{\partial}{\partial x} q^i(x^*(T)) \quad i \in \{1, 2, \dots, N\}, \quad (2.7)$$

where

$$H^i(t, p^i, x, u^1, \dots, u^N) = g^i(t, x, u^1, \dots, u^N) + p^i f^i(t, x, u^1, \dots, u^N) \quad (2.8)$$

with $t \in [0, T], i \in \{1, 2, \dots, N\}$.

Proof: Defining the Hamiltonian as in (2.8), the necessary conditions on pp. 180 of [80] and the boundary condition for fixed final time yield equations (2.4)-(2.7).

In Chapter 5, we determine the best response and the corresponding optimal trajectory of each agent- i for $i = 1, \dots, N$ using (2.4)-(2.7) and combine them to get a nonlinear state equation, among the solution of which, one searches Nash equilibrium of the game (2.1) (2.2).

We remark that Theorem 6.11 of [53] gives necessary conditions for the existence of a Nash equilibrium of games similar to ours. However, we can not directly use this result of [53] due to the differentiability assumption for the function $g(\cdot)$.

Chapter 3

Problem Definition

We first define a general one-dimensional swarm game and list four specializations of this general game for which we have been able to determine explicit Nash equilibria, under various assumption on the foraging efforts.

3.1 A General Swarm Game

A possible mathematical model of a one dimensional swarm, such as an ant colony in a queue, is now described. This is an infinite-dimensional, dynamic, non-cooperative, N-agent game: Minimize for $i = 1, \dots, N$,

$$L^i := \beta_i x^i(T)^2 + \int_0^T \left\{ \frac{u^i(t)^2}{2} + \sum_{j \in P^i} \left(a_{ij} \frac{[x^i(t) - x^j(t)]^2}{2} - r_{ij} |x^i(t) - x^j(t)| \right) \right\} dt, \quad (3.1)$$

subject to

$$u^i = \dot{x}^i, \quad (3.2)$$

where β_i, a_{ij} , and r_{ij} are all nonnegative numbers. Here, T is the duration of swarming journey, $x^i(t)$ is the position of the i^{th} agent, $u^i(t)$ is the control input of i^{th} agent. The summation is performed over the set $P^i \subset \{1, \dots, N\}$, which is the set of agents that agent- i measures its distance to. This determines the

information structure of the game. If $P^i = \{1, \dots, N\} \setminus \{i\}$ for instance, then we have a “completely connected graph” because every agent measures its distance to every other agent and uses this information in its optimization.

This formulation of the swarm game specifies a very simple “attractant/repellent profile,” [12]. The first term in (3.1) penalizes the distance to the foraging location at the final time, which is assumed to be the origin in $x^1 \dots x^N$ -space, i.e., $x^i = 0$ for $i = 1, \dots, N$. This component of the total work is the “environment potential” which monitors the toxicity or the amount of food source at position x . Here, it is selected as a quadratic profile as in [12]. The higher the coefficient β_i is, the stronger is the desire of agent- i to reach the target location. The second term in the integrand is the kinetic energy term which measures the dynamic effort of the i^{th} member. The minimization of this effort term implies that the swarm members use their energy efficiently which is an essential feature of actual biological swarms [10]. This term of the integrand is the contribution to the total work done by agent’s kinetic energy due to (3.2). Using velocity as a control input $u^i(t) = \dot{x}^i(t)$ arises from applying force in a viscous environment at which particle mass is neglected [12]. The third term in the total work done is the attraction potential energy and the last term is the repulsion potential energy. These terms are introduced as a result of the assumption that each Agent- i measures its distance to every other agent in P^i and optimizes these distances so as to remain as close as possible to every other agent in P^i without getting too close to any one of them. Introduction of such terms into the *total* potential energy and its (cooperative) minimization have been shown to lead to stable swarms in the stability analysis of [12]. Thus, each agent minimizes its total effort, total work done, during the foraging process or time interval $[0, T]$. The cost functional (3.1) actually models the *motive* of Agent- i , which is to remain close to its neighbors while avoiding collision without spending too much effort. Existence of a Nash equilibrium and its characteristics will then hopefully provide hints about the swarm *behavior that results from the individual motives*.

The parameter β_i will be referred to as the *foraging coefficient* of the i^{th} agent. The parameters a_{ij} and r_{ij} , on the other hand, will be called the *attraction and repulsion coefficients*, respectively, associated with the i^{th} and j^{th} agents.

Observe that this swarm game is in the context of the dynamic game and Theorem 2.1 of Chapter 2 with the identifications

$$\begin{aligned} x(t) &= [x^1 \ x^2 \ \dots \ x^N]', (n = N) \\ f(t, x(t), u^1(t), \dots, u^N(t)) &= [u^1 \ u^2 \ \dots \ u^N]', q^i(x(T)) = \beta_i x^i(T)^2, \\ g^i(t, x(t), u^1(t), \dots, u^N(t)) &= \frac{u^i(t)^2}{2} + \sum_{j \in P^i} (a_{ij} \frac{[x^i(t) - x^j(t)]^2}{2} - r_{ij} |x^i(t) - x^j(t)|). \end{aligned}$$

Our investigations indicate that in this general formulation of the swarm game, a Nash equilibrium fails to exist and suitable assumptions on the information structure as well as on the relative sizes of the weights in the individual cost functions are necessary for existence. A collection of examples in which a Nash equilibrium fails to exist is given in Section 4.4 below.

Note that in defining the above game, we have not specified the foraging target (food supply) location but added a simple quadratic term in the cost functional that penalizes the distances to the target location, which is the origin $x^i = 0$ for $i = 1, \dots, N$. In this swarm game, the indexing of the agents indicate the ranking in the initial queue of the agents. The agent of index 1 starts at the closest position to the foraging target and that with index N , to be at the farthest. A solution, if it exists, should have the property that the swarm gets progressively closer to the origin. If it exists, we will refer to a solution of this swarm game problem as *a Nash equilibrium with free terminal condition*. The specification of the origin as the target would mean that each agent has full desire to reach the target location. Thus, if we consider the same cost functional (3.1), but without the foraging term, and specify $x^i(T) = 0$ for $i = 1, \dots, N$ as the terminal condition, then we obtain a different game and a new problem. We will refer to a solution of this new problem as *a Nash equilibrium with specified terminal condition*.

3.2 Four Special Swarm Games

All four games that are defined now are those that admit a Nash equilibrium and are obtained by specializations of the swarm game above.

In the first game $P^i = \{1, \dots, N\} - \{i\}$ and the foraging, attraction, and repulsion coefficients are uniformly the same for all agents. Note that this amounts to the assumption that all swarm members are alike in their motives and holds true for most biological swarms. All games are subject to (3.2) so that the control input of each agent is its speed.

Game 1 (Complete Information): Determine $\min_{u^i} \{L^i\}$ subject to $\dot{x}^i = u^i$, $\forall i = 1, \dots, N$, where

$$L^i := [x^i(T)]^2 \beta + \int_0^T \left[\frac{u^i(t)^2}{2} + \sum_{j=1, j \neq i}^N \left(\frac{a[x^i(t) - x^j(t)]^2}{2} - r|x^i(t) - x^j(t)| \right) \right] dt, \quad (3.3)$$

In the second game $P^1 = \{2\}$, $P^N = \{N-1\}$ and, for $i=2, \dots, N-1$, $P^i = \{i-1, i+1\}$. It is again assumed that the foraging, attraction, and repulsion coefficients are uniformly the same for all agents. The main difference between Games 1 and 2 is that, in Game 2, the swarm members measure their own distance only with respect to the adjacent members instead of exchanging position information with all of the swarm members. This feature is preferable in large swarms since sensing all of the member locations may not be possible in large swarms. This more realistic incomplete information assumption actually makes the problem technically much more challenging than that in Game 1.

Game 2 (Partial information): Determine $\min_{u^i} \{L^i\}$ subject to $\dot{x}^i = u^i$, $\forall i = 1, \dots, N$, where

$$L^i(u^i, x^{i-1}, x^i, x^{i+1}) := [x^i(T)]^2 \beta + \int_0^T \left\{ \frac{u^i(t)^2}{2} + \sum_{j=i-1, j \neq i}^{i+1} \left(\frac{a[x^i(t) - x^j(t)]^2}{2} - r|x^i(t) - x^j(t)| \right) \right\} dt, \quad (3.4)$$

with the convention that $x_0(t) = x_{N+1}(t) = 0$.

The next two games permit swarm members that are not alike since the coefficients of the cost functionals minimized by different agents are allowed to be different. In Game L1, $P^1 = \emptyset$, $P^i = \{j : 1 \leq j \leq i-1\}$ and in Game L2

$$P^i = \begin{cases} \{\emptyset\} & \text{for } i = 1, \\ \{1\} & \text{for all } 2 \leq i \leq N \end{cases}$$

Game L1 (Hierarchical Leadership): Determine $\min_{u^i} \{L^i\}$ subject to $\dot{x}^i = u^i$, $\forall i = 1, \dots, N$, where

$$\begin{aligned} L^1 &:= \gamma \frac{x^1(T)^2}{2} + \int_0^T \frac{u^1(t)^2}{2} dt, \\ L^i &:= \beta \frac{x^i(T)^2}{2} + \int_0^T \left\{ \frac{u^i(t)^2}{2} + \sum_{j=1}^{i-1} \left(a_j \frac{[x^i(t) - x^j(t)]^2}{2} - r_j |x^i(t) - x^j(t)| \right) \right\} dt, \\ 2 &\leq i \leq N. \end{aligned} \tag{3.5}$$

Game L2 (A Single Leader): Determine $\min_{u^i} \{L^i\}$ subject to $\dot{x}^i = u^i$, $\forall i = 1, \dots, N$, where

$$\begin{aligned} L^1 &:= \gamma \frac{x^1(T)^2}{2} + \int_0^T \frac{u^1(t)^2}{2} dt, \\ L^i &:= \beta \frac{x^i(T)^2}{2} + \int_0^T \left\{ \frac{u^i(t)^2}{2} + a_i \frac{[x^i(t) - x^1(t)]^2}{2} - r_i |x^i(t) - x^1(t)| \right\} dt, \\ 2 &\leq i \leq N. \end{aligned} \tag{3.6}$$

The information structures of these four swarm games are illustrated in Figure 3.1, where an arrow emanating from agent- i to agent- j indicates that i keeps track of its distance to j during the foraging journey.

The cost functions considered in Games L1 and L2 are similar to Games 1 and 2 but with important differences. In all games, the indexing of the agents indicate the ranking in the initial queue of the agents. The agent of index 1 starts at the closest position to the foraging target and that with index N , to be at the farthest. In Games L1 and L2, agent-1 and others have different cost function structures, as opposed to the uniform structure in Game 2. Second, we extend the identical agent form of Game 2 to nonidentical agents by allowing coefficients a and r to vary among different agents. The desire to reach the target location can be at differing levels among the agents. Above all, we alter the undistinguished structure in Games 1 and 2 to a leader-follower structure. The agent of index 1 is distinguished by its ignorance of the position of any other member in the group in the duration of the whole journey. Each agent in Game L1 is assumed to observe (measure) and know the positions of the agents ahead of it, whereas in Game L2, it is assumed to observe the position of agent-1 only. The latter is the

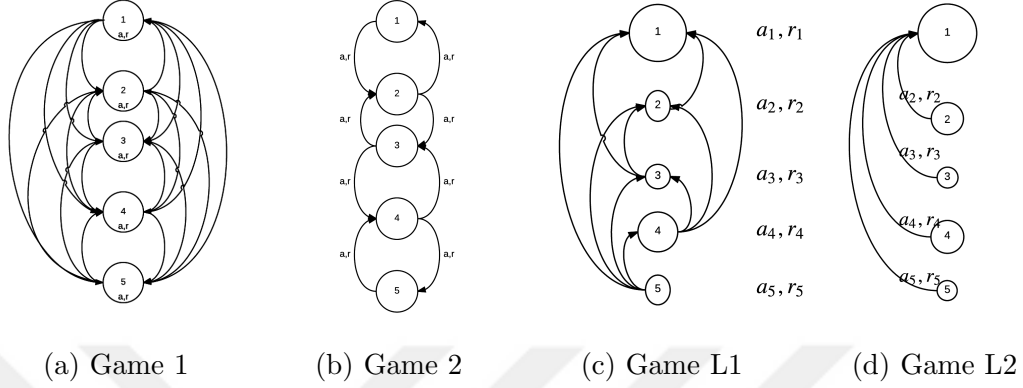


Figure 3.1: Information structures of four swarm games

loosest information structure among those in Games 1, 2, L1, and L2. One way to view Game L1 is that each agent exhibits a different level of leadership based on its rank in the swarm. In other words, all the agents except the rearmost agent perform leadership by being under surveillance by the agents at its back. The agent in front is a full leader relied upon by all remaining agents in Game L2. Therefore, the passive leadership is somewhat hierarchical in the first case, whereas one distinguished agent is the passive leader and all others are followers in the latter case.

The problem that faces each agent is an optimal control problem and necessary conditions are obtained by Pontryagin's minimum principle (see [80] or [51]) and Theorem 2.1 of [53] given in Chapter 2. A Nash equilibrium solution exists provided the optimal solutions of N agents result, when simultaneously considered, in well-defined position trajectories for every given $x^i(0) \in \mathbf{R}$, $i = 1, \dots, N$, [53], Section 6.3. Here, we limit the permissible strategies $u^i(t) = \dot{x}^i(t)$ available to agents to be continuous with respect to the initial conditions $x^i(0)$, (see [67] and p. 227 of [53]).

As in the general swarm game, if we drop the foraging term from the cost functionals and specify instead $x^i(T) = 0$ for $i = 1, \dots, N$ as the terminal condition, then we obtain a corresponding game *with specified terminal condition*. We will consider the solutions of both free and specified terminal condition games in the next chapter.

Chapter 4

Main Results

Solving Games 1, 2, L1, and L2 via minimizing the inter-dependent non convex cost functions by which they are defined is challenging due to several reasons. While it is relatively easy to transform the problem posed by the general swarm game into a problem of finding solutions to systems of differential equations, these turn out to be nonlinear and unfortunately do not obey any Lipschitz conditions. A further difficulty is that these systems have mixed boundary conditions.

In case of the four games defined in the previous chapter, we are able to surmount these difficulties. We first postulate that the initial ordering of the agents in the queue during the whole journey is preserved. This eliminates the nonlinearity of the system of differential equations obtained for our games. We then verify that our postulate holds true in the Nash equilibrium obtained. This ensures that the Nash equilibrium satisfies the nonlinear state equation given in (5.2).

Existence and uniqueness of a Nash equilibrium in dynamic games is indeed a difficult problem. The reader is referred to [71] as an example of a result on existence and uniqueness in a much simpler problem than the one considered here.

The results below show that there is a unique Nash equilibrium for our games

with free or specified terminal condition for every initial positions of the agents. These equilibria display many known characteristics of a swarm behavior. In each game, explicit expressions for instantaneous pairwise distances between agents, the swarm size, and the distance of the swarm center to the foraging location are obtained.

4.1 Nash equilibrium for Game 1 of Complete Information

Free Terminal Condition

Theorem 4.1 *A Nash equilibrium for Game 1 with free terminal condition exists, is unique, and is such that the initial ordering among the N agents in the queue is preserved during $[0, T]$. The Nash equilibrium has the following properties:*

P1. *The distance between any two agents i, j at time t is given by*

$$x^i(t) - x^j(t) = b(t)[x^i(0) - x^j(0)] + r c(t)[s^i(0) - s^j(0)], \quad (4.1)$$

where, with $\alpha := \sqrt{Na}$,

$$\begin{aligned} b(t) &= \frac{\alpha \cosh[\alpha(T-t)] + \beta \sinh[\alpha(T-t)]}{\alpha \cosh(\alpha T) + \beta \sinh(\alpha T)}, \\ c(t) &= \frac{1}{2\alpha^2} \frac{[\beta \{\sinh(\alpha T) - \sinh(\alpha t) - \sinh[\alpha(T-t)]\} + \alpha \{\cosh(\alpha T) - \cosh[\alpha(T-t)]\}]}{\alpha \cosh(\alpha T) + \beta \sinh(\alpha T)}, \\ s^i(0) &:= \sum_{k=1, k \neq i}^N \operatorname{sgn}[x^i(0) - x^k(0)], \quad i = 1, \dots, N. \end{aligned} \quad (4.2)$$

P2. *For every T and as $T \rightarrow \infty$, the swarm size $d(t) := \max_{i,j} |x^i(t) - x^j(t)|$ remains bounded in $[0, T]$:*

$$d(t) = b(t)d(0) + c(t)rm(0) \leq b(t^*)d(0) + c(t^*)(r)m(0),$$

where

$$t^* = \frac{1}{2\alpha} \ln \left(\frac{e^{\alpha T} \{[\beta(e^{\alpha T} - 1) + \alpha e^{\alpha T}](r)m(0) - 2\alpha^2(\beta + \alpha)e^{\alpha T}d(0)\}}{[\beta(e^{\alpha T} - 1) + \alpha](r)m(0) + 2\alpha^2(\beta - \alpha)d(0)} \right), \quad (4.3)$$

$d(0) := \max_{i,j} |x^i(0) - x^j(0)|$ is the distance between the first and the last agent in the queue at the initial time, and $m(0) := \max_{i,j} |s^i(0) - s^j(0)| = 2N - 2$. The bound is attained if and only if $0 \leq t^* \leq T$. Maximum swarm size is attained at 0 if $t^* < 0$.

The expression for the swarm size at the final time is

$$d(T) = \frac{\cosh(\alpha T) - 1}{2\alpha[\alpha \cosh(\alpha T) + \beta \sinh(\alpha T)]} (r) m(0) + \frac{\alpha}{\alpha \cosh(\alpha T) + \beta \sinh(\alpha T)} d(0).$$

P3. The swarm center $x_c(t) := \frac{x^1(t) + \dots + x^N(t)}{N}$ is given by

$$x_c(t) = x_c(0) \left(1 - \frac{\beta t}{T\beta + 1} \right), \quad (4.4)$$

which monotonically approaches the origin as $t \rightarrow T$ and ends up at the origin as $T \rightarrow \infty$.

P4. As $T \rightarrow \infty$, the distances between the consecutive agents in the queue are the same and is equal to $\frac{(r)}{\alpha(\alpha + \beta)}$.

Remark 1. The main result above is that a swarming behavior, an act of aggregation, does follow from non-cooperative actions of the N agents for Game 1. The fact that the game has a *unique* Nash equilibrium is also significant. The initial ordering of the agents in the queue is preserved at all times in this Nash equilibrium. This is of course a consequence of the attraction and repulsion terms in each agent's cost functional, the effect of which turns out to be similar to connecting the agents in the queue to each other by translational springs [81].

Remark 2. The swarm size throughout the foraging activity is given in (P2). The foraging activity of the swarm is accomplished increasingly better given sufficient time by (P3). In (P1), an explicit expression is given for pairwise distances. It is also possible to describe the individual paths $x^i(t)$ explicitly. However, the formula is rather lengthy and is not included here. By (P4), given sufficient time, the foraging swarm will be more regular as it gets closer to the foraging location since distances between adjacent agents will be progressively more uniform. A closer examination of $d(T)$ reveals an additional property of the swarm. If the

agents start far apart from each other at the initial time, then the attraction term becomes effective and they end up closer together at the final time. Conversely, if they start close enough together, then the repulsion term is more effective and they later get apart from each other.

Specified Terminal Condition

The Nash equilibrium for Game 1 with specified terminal condition $x^i(T) = 0$ for $i = 1, \dots, N$ is described next. We remark that the expressions for distances between agents, swarm size, swarm center, etc., are quite different than those in Theorem 4.1. This is because, due to the difference in the terminal condition, a new (but related) game is obtained.

Theorem 4.2 *A Nash equilibrium for Game 1 with specified terminal condition exists, is unique, and is such that the initial ordering among the N agents in the queue is preserved during $[0, T]$. The Nash equilibrium has the following properties:*

P1. *The distance between any two agents i, j at time t is given by*

$$x^i(t) - x^j(t) = b(t)[x^i(0) - x^j(0)] + rc(t)[s^i(0) - s^j(0)], \quad (4.5)$$

where

$$b(t) := \frac{\sinh[\alpha(T-t)]}{\sinh(\alpha T)},$$

$$c(t) := \frac{1}{2\alpha^2} \frac{\sinh(\alpha T) - \sinh(\alpha t) - \sinh[\alpha(T-t)]}{\sinh(\alpha T)},$$

P2. *For every T and as $T \rightarrow \infty$, the swarm size $d(t) := \max_{i,j} |x^i(t) - x^j(t)|$ remains bounded in $[0, T]$:*

$$d(t) = b(t)d(0) + c(t)rm(0)$$

$$\leq b(t^*)d(0) + c(t^*)rm(0),$$

where

$$t^* = \frac{1}{2\alpha} \ln \left(\frac{e^{\alpha T}[(e^{\alpha T} - 1)rm(0) - 2\alpha^2 e^{\alpha T}d(0)]}{(e^{\alpha T} - 1)rm(0) + 2\alpha^2 d(0)} \right) \quad (4.6)$$

The bound is attained if and only if $0 \leq t^* \leq T$. Maximum swarm size is attained at 0 if $t^* < 0$. The swarm size at the final time is $d(T) = 0$.

P3. The swarm center is given by

$$x_c(t) = x_c(0) \left(1 - \frac{t}{T}\right), \quad (4.7)$$

so that $x_c(T) = 0$.

Remark 3. It will be noticed that the above expressions are all obtained by letting $\beta \rightarrow \infty$ in the corresponding expressions of Theorem 4.1. This is somewhat expected. Specifying the cost of each agent being away from the target location as infinity is as good as requiring that each agent is exactly at that location at the terminal time. The expressions of Theorem 4.2 are, nevertheless, derived independent of Theorem 4.1 in the Chapter 5 by solving the game with the specified terminal condition.

Remark 4. The properties (P1)-(P3) of Theorem 4.2 shows that the swarm that is formed with specified terminal condition has entirely similar features to the swarm formed with free terminal condition; the major difference is that the foraging target is reached exactly at the final time, as specified in the set-up of the game.

Dense vs. Sparse Swarms

The degree of cohesion in our swarm model can be tuned by the levels of attraction and repulsion between the agents. The model is flexible in the sense that it can result in both dense and sparse swarms by selecting different values for the attraction constant a and the repulsion constant r . It is expected that if the ratio $\frac{a}{r}$ increases, then the swarm will get denser, and the swarm will get sparser as it decreases, which is confirmed by the following result.

Corollary 1: (i) The maximum swarm size is always attained in the interval $[0, T)$. (ii) The swarm size monotonically decreases in the interval $[0, T]$ if and

only if

$$\frac{a}{r}d(0) \geq \frac{N-1}{N} \frac{(e^{\alpha T} - 1)^2 \beta + \alpha(e^{2\alpha T} - 1)}{(e^{2\alpha T} + 1)\beta + \alpha(e^{2\alpha T} - 1)}, \quad (4.8)$$

$$\frac{a}{r}d(0) \geq \frac{N-1}{N} \frac{(e^{\alpha T} - 1)^2}{e^{2\alpha T} + 1} \quad (4.9)$$

in the free and specified terminal condition cases, respectively.

Thus, by (i) and (ii), the value obtained when equality is achieved in (4.8) or in (4.9) is a critical value of the ratio $\frac{a}{r}$. The maximum swarm size is attained at $t = 0$ for values larger than this critical value and it is attained in the open interval $(0, T)$ for values smaller. Note that this conclusion follows by the fact that the right hand sides in both (4.8) and (4.9) are less than 1 for each value of $t \in [0, T]$ and for all values of $\alpha = \sqrt{Na}$. An asymptotic analysis of (i) and (ii) indicates that swarm size, the pairwise distances (4.1), and (4.5) all grow hyperbolically and parabolically with time for $\frac{a}{r}$ sufficiently large and small, respectively.

4.2 Nash equilibrium for Game 2 of Partial Information

Consider the vector of positions of the N agents

$$\mathbf{x}(t) := \begin{bmatrix} x^1(t) & \dots & x^N(t) \end{bmatrix}',$$

and the vector of pairwise distances and sum

$$\mathbf{y}(t) := \begin{bmatrix} x^1(t) - x^2(t) & | & \dots & | & x^{N-1}(t) - x^N(t) & | & \sum_{j=1}^N x^j(t) \end{bmatrix}'$$

where “prime” denotes transpose. Let $M \in \mathbf{R}^{(N-1) \times N}$ be such that $M_{i,i} = 1$, $M_{i,i+1} = -1$, $M_{i,j} = 0$ for $i = 1, \dots, N-1$, $j = 1, \dots, N$, $i \neq j \neq i+1$. Thus, the i -th row of M has all zeros except a 1 and a -1 at its i -th and $(i+1)$ -st positions, respectively. Consider the singular value decomposition

$$M = U\Sigma V', \quad (4.10)$$

for unitary matrices $V \in \mathbf{R}^{N \times N}$, $U \in \mathbf{R}^{(N-1) \times (N-1)}$. The matrix M has one zero singular value and $N-1$ distinct singular values all in the open interval $(0, 2)$ (see Lemma A.1 in Appendix). The N singular values $\sigma_1 > \sigma_2 > \dots > \sigma_{N-1} > \sigma_N$ are non-degenerate so that the columns of U and of V are unique up to sign. Let

$$\sigma_k := 2 \cos\left(\frac{k\pi}{2N}\right), \quad \alpha_k := \sigma_k \sqrt{a}, \quad k = 1, \dots, N-1, \quad (4.11)$$

and $\sigma_N = \alpha_N := 0$. The time constants α_k^{-1} will determine how $\mathbf{x}(t)$ and $\mathbf{y}(t)$ evolve in time.

We first describe the solution in the specified-terminal condition case, in which the swarm members has full desire to reach the target location. Define

$$\begin{aligned} \tilde{b}_k(t) &:= \frac{\sinh[\alpha_k(T-t)]}{\sinh(\alpha_k T)}, \\ \tilde{c}_k(t) &:= \frac{1}{\alpha_k^2} \left[1 - \tilde{b}_k(t) - \frac{\sinh(\alpha_k t)}{\sinh(\alpha_k T)} \right], \end{aligned} \quad (4.12)$$

and consider

$$\begin{aligned} \tilde{B}(t) &:= \text{diag} \left[\tilde{b}_1(t), \dots, \tilde{b}_{N-1}(t), \frac{T-t}{T} \right], \\ \tilde{C}(t) &:= r \text{diag} \left[\tilde{c}_1(t), \dots, \tilde{c}_{N-1}(t), \frac{(T-t)t}{2} \right], \\ \tilde{Q} &:= \begin{bmatrix} U & \mathbf{0} \\ \mathbf{0}' & 1 \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} 1 & 0 & \dots & 0 & 1 & 0 \end{bmatrix}' \in \mathbf{R}^N. \end{aligned} \quad (4.13)$$

In the free terminal condition case, the following results are obtained.

Theorem 4.3 *Given any $r \in (0, \infty)$, there exists $a_0 \in (0, \infty)$ such that for each value $a \in (0, a_0)$ of the attraction coefficient, a Nash equilibrium with specified terminal condition of the Game 2 in (3.4) exists. This Nash equilibrium has the following properties:*

P1. *The initial ordering among the N agents in the queue is preserved during $[0, T]$.*

P2. *The vector of pairwise distances and sum at time t is given by*

$$\mathbf{y}(t) = \tilde{Q} \tilde{B}(t) \tilde{Q}' \mathbf{y}(0) + \tilde{Q} \tilde{C}(t) \tilde{Q}' \mathbf{r}. \quad (4.14)$$

P3. *For every T and as $T \rightarrow \infty$, the swarm size $\tilde{d}(t) := \max_{i,j} |x^i(t) - x^j(t)|$ remains bounded in $[0, T]$.*

It follows that self-organized (no leader) agents, each individually optimizing its effort, end up in a coordinated movement towards the foraging location. The swarm that results is such that the initial ordering among agents is preserved. It is stable (its size is bounded) by P3. The distance between the consecutive agents can be computed by P2 at any given time. Also by P2, the last entry of $\mathbf{y}(t)$ gives the swarm-center $x_c(t) := \frac{x^1(t) + \dots + x^N(t)}{N}$ as

$$\mathbf{x}_c(t) = \frac{T-t}{T} \mathbf{x}_c(0), \quad (4.15)$$

which monotonically approaches the target location as $t \rightarrow T$ and ends up at the origin at T .

The assumption that the agents have full desire to reach the foraging location is now relaxed. In the free terminal condition case, the agents get progressively closer to the target location by penalizing their distance to the location through the environment potential term. The solution generalizes the result of Theorem 4.3 since Theorem 4.4 yields Theorem 4.3 in the limiting case of $\tilde{\beta} \rightarrow \infty$, which sets the penalty of not being at the origin at the terminal time to be infinity for each agent.

For $k = 1, \dots, N-1$, define

$$\begin{aligned} \tilde{b}_k(t) &:= \frac{2\beta \sinh[\alpha_k(T-t)] + \alpha_k \cosh[\alpha_k(T-t)]}{2\beta \sinh(\alpha_k T) + \alpha_k \cosh(\alpha_k T)}, \\ \tilde{c}_k(t) &:= \frac{1}{\alpha_k^2} \left[1 - \tilde{b}_i(t) - \frac{2\beta \sinh(\alpha_k t)}{2\beta \sinh(\alpha_k T) + \alpha_k \cosh(\alpha_k T)} \right], \end{aligned} \quad (4.16)$$

and consider

$$\begin{aligned} \tilde{B}(t) &:= \text{diag} \left[\tilde{b}_1(t), \dots, \tilde{b}_{N-1}(t), \frac{1 + 2\beta(T-t)}{1 + 2\beta T} \right], \\ \tilde{C}(t) &:= \tilde{r} \text{diag} \left[\tilde{c}_1(t), \dots, \tilde{c}_{N-1}(t), \frac{T[1 + 2\beta(T-t)] + T-t}{2(1 + 2\beta T)} t \right], \end{aligned}$$

Theorem 4.4 *Given any $\tilde{r} \in (0, \infty)$, there exists $a_0 \in (0, \infty)$ such that for each value $\tilde{a} \in (0, a_0)$ of the attraction coefficient a Nash equilibrium with free terminal condition of Game 2 in (3.4) exists. This Nash equilibrium has the following properties:*

P1. *The initial ordering among the N agents in the queue is preserved during $[0, T]$.*

P2. *The vector $\mathbf{y}(t)$ of pairwise distances and sum is given by*

$$\mathbf{y}(t) = \tilde{Q}\tilde{B}(t)\tilde{Q}'\mathbf{y}(0) + \tilde{Q}\tilde{C}(t)\tilde{Q}'\tilde{\mathbf{r}}. \quad (4.17)$$

P3. *For every T and as $T \rightarrow \infty$, the swarm size $\tilde{d}(t) := \max_{i,j} |x^i(t) - x^j(t)|$ remains bounded in $[0, T]$.*

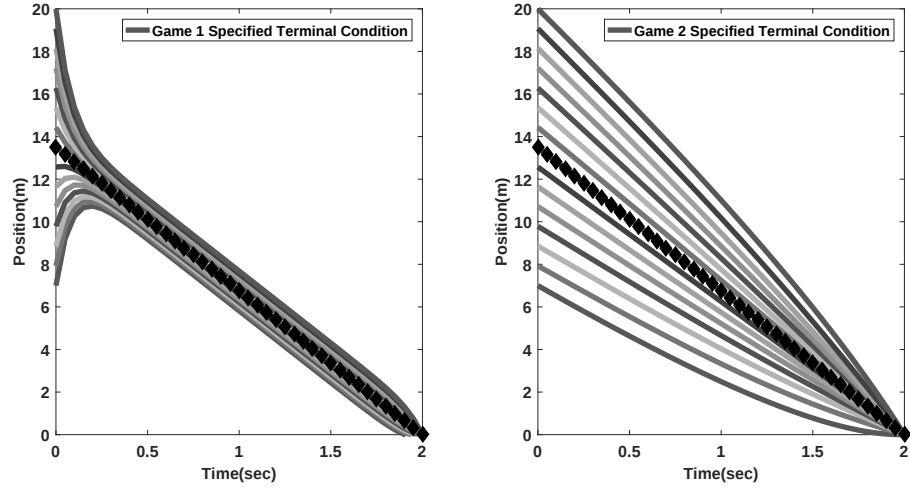
By P2, the last entry of $\mathbf{y}(t)$ gives the swarm-center $\tilde{x}_c(t) := \frac{x^1(t) + \dots + x^N(t)}{N}$ as

$$\tilde{x}_c(t) = \frac{1 + 2\beta(T - t)}{1 + 2\beta T} \tilde{x}_c(0), \quad (4.18)$$

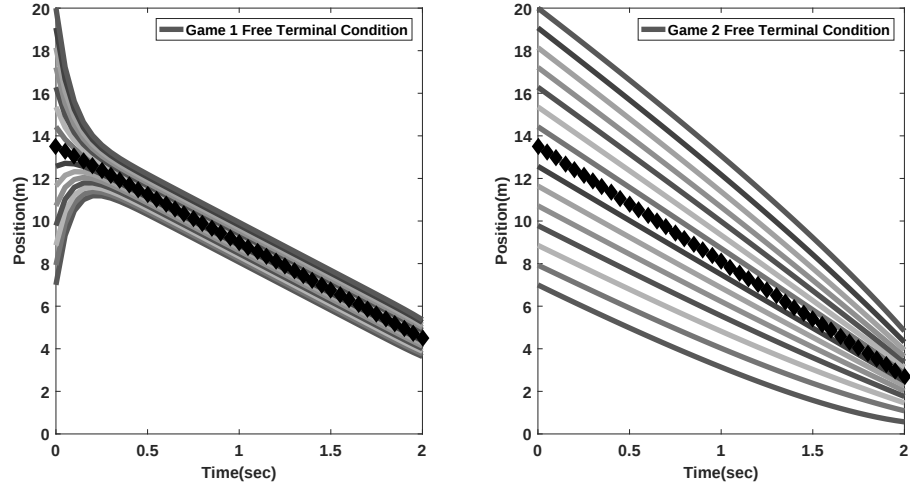
which monotonically approaches the target location as $t \rightarrow T$ and ends up at the origin as $T \rightarrow \infty$.

Remark 5. In Theorem 4.3 and 4.4 both, the convergence of trajectories $x^i(t)$ to the foraging location (and divergence from the initial positions if the agents start their foraging journey too close together) are determined by a combination of the $N - 1$ distinct time constants $(\sigma_k \sqrt{\tilde{a}})^{-1}$ of (4.11). The same comment also applies to pairwise distances $x^i(t) - x^j(t)$ for $i \neq j$. $\triangle$

Remark 6. A closer look (see Remark 9 in Chapter 5) into the proof of existence of the Nash equilibria of Theorems 4.3 and 4.4 reveals that *for sufficiently large values of the attraction coefficient $\tilde{a}$, Nash equilibria (for both specified and free terminal conditions) also exist* $\triangle$



(a) Game 1 and Game 2 are compared for specified terminal condition. The swarm population, the duration of the game, and the sampling period are selected as $N = 15$, $T = 2$, and $T_s = 0.05$ for both graphs. Both attraction and repulsion coefficients are equal to 10. Initial positions are equispaced between 7 and 20.



(b) Game 1 and Game 2 are compared for free terminal condition. All the coefficients are the same as in Figure 4.1a. Additionally, the foraging coefficients are chosen as $\beta = 1$

Figure 4.1: Games 1 and 2 under the same initial conditions

In Figure 4.1, we observe that neither Game 1 nor Game 2 exhibits ordering changes. In both cases, swarm center follows a straight line. If the agents have

a full desire to reach the target, they indeed end up at the target as shown in graphs with specified terminal condition, i.e. Figure 4.1a. If they have a partial desire to reach the target, they approach towards the target, but they do not end up exactly at the target as shown in Figure 4.1b.

4.3 Nash equilibrium for Game L1 of Hierarchical Leadership

Let $\alpha_1 := 0$, $\alpha_k := \sqrt{a_1 + \dots + a_{k-1}}$, $k = 2, \dots, N$ be called convergence rates for Game L1 and suppose that $x^N(0) > \dots > x^1(0)$. Define

$$\begin{aligned} \rho_j(t) &:= \begin{cases} \frac{\gamma-\beta}{\gamma T+1} \left[\frac{\sinh(\alpha_{j+1}t)}{\beta \sinh(\alpha_{j+1}T) + \alpha_{j+1} \cosh(\alpha_{j+1}T)} \right], & j = 1 \\ 0, & j = 2, \dots, N, \end{cases} \\ \hat{b}_k(t) &:= \begin{cases} 1 - \frac{\gamma t}{\gamma T+1}, & k = 1 \\ \frac{\beta \sinh[\alpha_k(T-t)] + \alpha_k \cosh[\alpha_k(T-t)]}{\beta \sinh(\alpha_k T) + \alpha_k \cosh(\alpha_k T)}, & k = 2, \dots, N, \end{cases} \\ \hat{c}_k(t) &:= \frac{1}{\alpha_k^2} \left[1 - \hat{b}_k(t) - \frac{\beta \sinh(\alpha_k t)}{\beta \sinh(\alpha_k T) + \alpha_k \cosh(\alpha_k T)} \right], \quad k = 2, \dots, N \end{aligned} \quad (4.19)$$

Theorem 4.5 *There exists a unique Nash equilibrium for the hierarchical leadership game under continuous strategies if and only if $\gamma \geq \beta \geq 0$. The Nash equilibrium has the following features:*

P1. *The initial ordering among the agents is preserved during $0 \leq t \leq T$.*

P2. *The leader trajectory and the distances to the leader are given by*

$$\begin{aligned} x^1(t) &= \hat{b}_1(t)x^1(0), \\ x^i(t) - x^1(t) &= \rho_1(t)x^1(0) + \sum_{k=2}^i \{ \hat{b}_k(t)[x^k(0) - x^{k-1}(0)] + \hat{c}_k(t)r_{k-1} \}, \quad (4.20) \\ 2 \leq i &\leq N. \end{aligned}$$

P3. *The swarm size is given by*

$$|x^N(t) - x^1(t)| = \rho_1(t)|x^1(0)| + \sum_{k=2}^N \{\hat{b}_k(t)|x^k(0) - x^{k-1}(0)| + \hat{c}_k(t)|r_{k-1}|\}.$$

P4. *The swarm center $x_c := (1/N)(x^1 + \dots + x^N)$ follows the trajectory*

$$x_c(t) = \hat{b}_1(t)x^1(0) + \rho_1(t)x^1(0) + \frac{1}{N} \sum_{i=1}^N \sum_{k=2}^i \{\hat{b}_k(t)[x^k(0) - x^{k-1}(0)] + \hat{c}_k(t)r_{k-1}\},$$

$$t \in [0, T]$$

P5. *If the foraging target is specified for all agents including the followers, then there is a unique Nash equilibrium of Game L1 for continuous strategies. The distance expressions are obtained from (4.20) in the limit as $\gamma = \beta \rightarrow \infty$ in (4.19).*

P6. *If the foraging task is dropped, then there still exists a unique Nash equilibrium for continuous strategies. The distance expressions are obtained by (4.20) by substituting $\gamma = 0$ and $\beta = 0$ in (4.19).*

Remark 7.

- i) Note that the Nash equilibrium is valid when $\beta = 0$ and $\gamma \geq 0$. If in addition $\gamma \rightarrow \infty$, then this is the case in which only the leader has full desire to reach the foraging target. If $\gamma = 0$, then there is no foraging task at all, which is the situation considered by **P6**. In case there is no foraging task, then the leader's optimal trajectory is $x^1(t) = x^1(0) \forall t \in [0, T]$, i.e., the leader preserves its initial position at all times. In the resulting Nash equilibrium, other agents progressively get closer to the leader in time.

The effect of attraction parameters and foraging parameters on swarming is demonstrated in Figure 4.2. It is observed that β has a more immense effect on foraging than a_j by dictating that all agents get closer to target. As the attraction parameters become larger, the size of the swarm becomes smaller, because all the agents approach to each other. However, large attraction parameters does not cause the agents to meet exactly at foraging target.

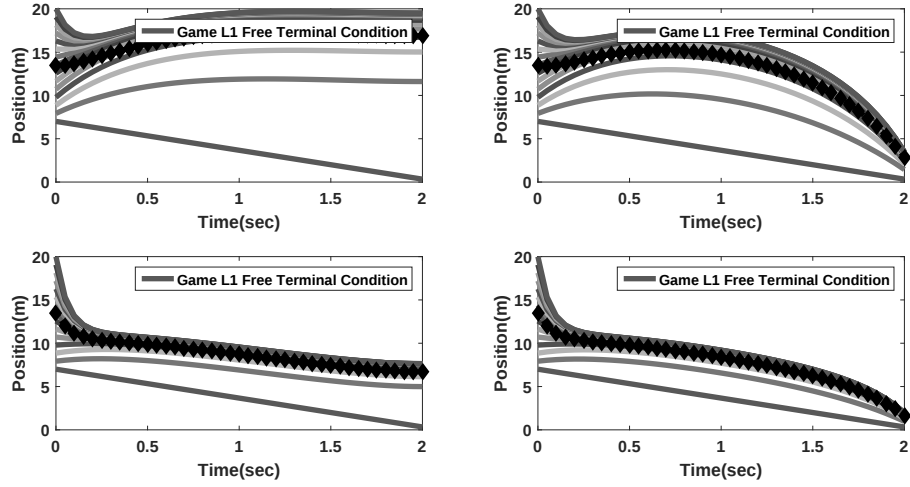


Figure 4.2: The parameters are the same as in Figure 4.7a except that attraction parameters of two plots in the second row are selected equispaced between 0.6 and 60, γ is equal to 10 in each plot, and β is equal to 0 in the first column of plots and β is equal to 10 in the second column of plots.

- ii) The necessity of $\gamma \geq \beta$, i.e., the foremost leader having a stronger desire to reach the foraging target is quite intuitive since otherwise, under certain initial conditions, the Agent-1 will fall behind. However, Agent-1 does not observe its distance to the other agents so that a consensus (a swarm) is not formed at all. In order to support this claim, let $\gamma = 1$, $\beta = 5$ so that $\gamma < \beta$. Also let $N = 10$, $T = 2$, $a_j = r_j = j$ for $j = 1, 2, \dots, N - 1$, and $x^i(0) = 0.1i + 9$ for $i = 1, 2, \dots, N$. Figure 4.3 shows that in the resulting trajectories an order change occurs. The reason behind this ordering change is rather intuitive. Since the followers have a more precise knowledge of the target than the leader, they get closer to the target than the leader. The followers should approach towards target slower than the leader to ensure no ordering change.

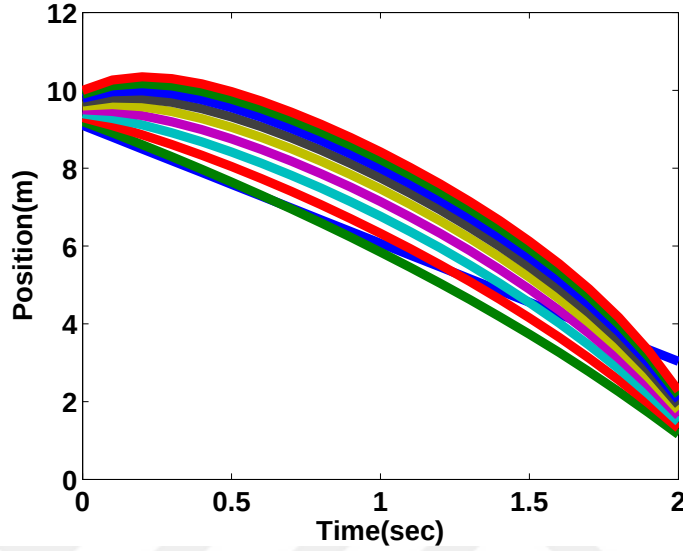


Figure 4.3: Optimal trajectories of Game L1 with $N=10$ particles for $\gamma < \beta$

- iii) If attraction increases as $a_j \rightarrow \infty$ for all $j = 1, \dots, N$, then all agents instantaneously stick to each other and move towards the target location altogether.
- iv) Under constant attractions, if $r_j \rightarrow \infty$ for all $j = 1, \dots, N$, then the agents depart from each other so much that the initial positions become immaterial. However, swarming and foraging tasks are accomplished adequately, no matter how large r_j is.

Effects of large attraction parameters and large repulsion parameters are demonstrated in Figure 4.4. The left bottom plot in Figure 4.4 shows that the agents indeed stick to each other if the attraction parameters are large. The right top plot shows that the initial positions indeed become immaterial for large repulsion parameters.

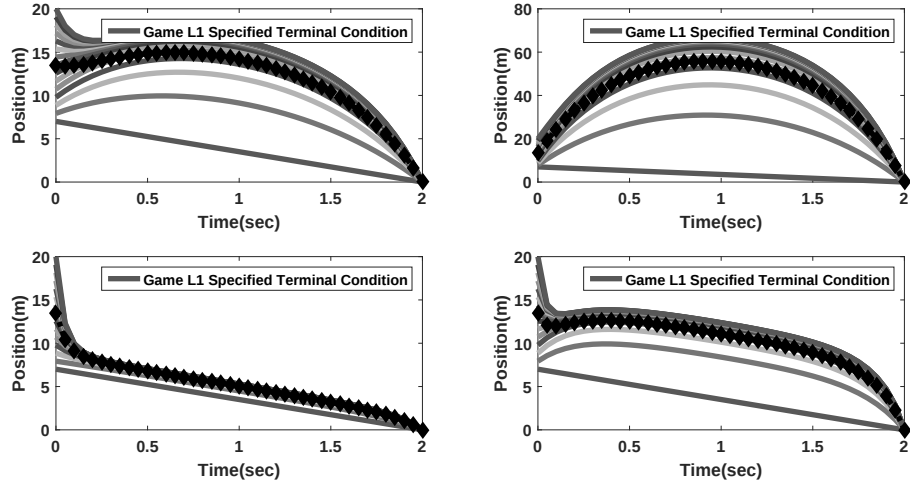


Figure 4.4: The parameters are the same as in Figure 4.7a except that attraction coefficients are selected equispaced between 20 and 200 in the second row and repulsion parameters are selected equispaced between 100 and 10 in the second column of plots.

- v) If the target is specified to all agents ($\gamma, \beta \rightarrow \infty$), then starting at any set of initial positions, all agents end up precisely at the foraging target. If $\gamma \rightarrow \infty$ and $\beta = 0$, then the followers still move towards the target location, but do not end up exactly at the target.

4.4 A Negative Example of Swarm Formation

In Section 3.1, we claimed that a Nash equilibrium with the postulated order fails to exist for general attraction and repulsion coefficients a_{ij} and r_{ij} . In other words, some general information structures are inconsistent in the sense that the postulated order is not preserved. Here, we endorse this claim by a system of 3 agents where each agent measures the distance with respect to only forward agents by coefficients a_{ij} and r_{ij} given below. First, we present the analytic expressions of optimal trajectories for such a system. The trajectories where ordering change occurs although we postulated no ordering change are then plotted. The optimal

trajectories of the agents are given by

$$x^1(t) = b_1(t)x^1(0),$$

$$x^2(t) = [b_1(t) - b_2(t)]x^1(0) + b_2(t)x^2(0) + c_2(t)r_{21},$$

$$x_3(t) = \frac{1}{q^*}[p^*x^1(0) + a_{32}[b_2(t) - b_3(t)]x^2(0) + q^*b_3(t)x^3(0) + r_{21}a_{32}[c_2(t) - c_3(t)] + [r_{31} + r_{32}]q^*c_3(t)]$$

where $b_1(t) = 1 - \frac{t}{T}$, $b_2(t) = \frac{\sinh[\sqrt{a_{21}}(T-t)]}{\sinh(\sqrt{a_{21}}T)}$, and $b_3(t) = \frac{\sinh[\sqrt{a_{31}+a_{32}}(T-t)]}{\sinh(\sqrt{a_{31}+a_{32}}T)}$.

Moreover, $c_1(t) = \frac{(T-t)t}{2}$, $c_2(t) = \frac{1}{a_{21}}[1 - b_2(t) - \frac{\sinh(\sqrt{a_{21}}t)}{\sinh(\sqrt{a_{21}}T)}]$, and $c_3(t) = \frac{1}{a_{31}+a_{32}}[1 - b_3(t) - \frac{\sinh(\sqrt{a_{31}+a_{32}}t)}{\sinh(\sqrt{a_{31}+a_{32}}T)}]$

Finally, $q^* = a_{31} + a_{32} - a_{21}$ and $p^* = q^*b_1(t) - a_{32}b_2(t) + (a_{21} - a_{31})b_3(t)$

Figure 4.5 demonstrates such a system for $N = 3$, $T = 2$, $\mathbf{x}(0) = [2 \ 6 \ 9]'$, $a_{21} = 0.1$, $a_{31} = 100$, $a_{32} = 10$, $r_{21} = 4$, $r_{31} = 0.1$, $r_{32} = 1$ where $a_{ij} = r_{ij} = 0$ for $j \geq i$. Ordering changes occur in such systems since m^{th} agent can approach towards k^{th} agent while n^{th} agent moves away from k^{th} agent. This results in an ordering change between m^{th} and n^{th} agent. Such nonuniform motion is not allowed in Game L1, where each m^{th} and n^{th} agent measure its distance from k^{th} agent under same attraction and repulsion coefficients satisfying $a_{mk} = a_{nk}$ and $r_{mk} = r_{nk}$.

4.5 Nash equilibrium for Game L2 of a Single Passive Leader

Let $\alpha_k := \sqrt{a_k}$, which will figure as convergence rates for Game L2, and suppose that $x^i(0) > x^1(0)$ for $1 < i \leq N$. Define

$$\begin{aligned} \bar{\rho}_k(t) &= \frac{\gamma-\beta}{\gamma T+1} \left[\frac{\sinh(\alpha_{k+1}t)}{\beta \sinh(\alpha_{k+1}T) + \alpha_{k+1} \cosh(\alpha_{k+1}T)} \right], & k = 1, \dots, N-1, \\ \bar{c}_k(t) &:= \frac{1}{\alpha_k^2} \left[1 - \bar{b}_k(t) - \frac{\beta \sinh(\alpha_k t)}{\beta \sinh(\alpha_k T) + \alpha_k \cosh(\alpha_k T)} \right], & k = 2, \dots, N, \\ \bar{b}_k(t) &:= \begin{cases} 1 - \frac{\gamma t}{\gamma T+1}, & k = 1 \\ \frac{\beta \sinh[\alpha_k(T-t)] + \alpha_k \cosh[\alpha_k(T-t)]}{\beta \sinh(\alpha_k T) + \alpha_k \cosh(\alpha_k T)}, & k = 2, \dots, N. \end{cases} \end{aligned} \tag{4.21}$$

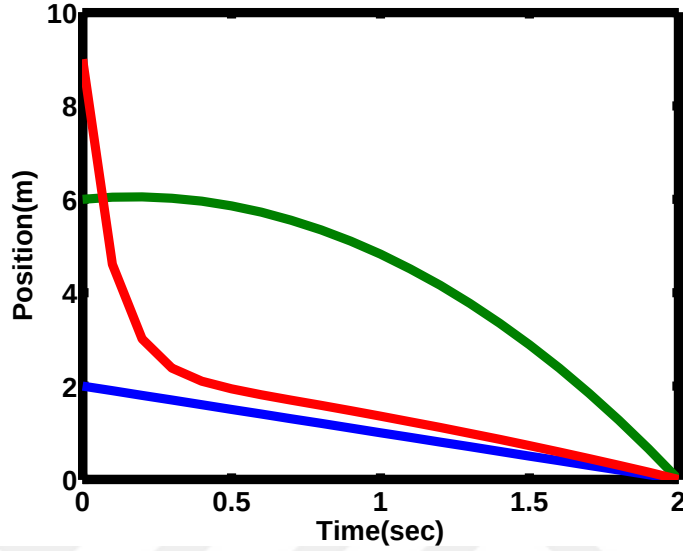


Figure 4.5: Optimal trajectories with an ordering change in Game L1

Theorem 4.6 *There is a unique Nash equilibrium for Game L2 under continuous strategies if and only if $\gamma \geq \beta \geq 0$. The Nash equilibrium has the following properties:*

P1. *The Agent-1 remains the leader throughout the journey. There are initial conditions that lead to legitimate ordering changes among the agents unless $a_i = a$ for all $i = 2, \dots, N$.*

P2. *The leader trajectory and distances of the followers to the leader are given by*

$$\begin{aligned} x^1(t) &= \bar{b}_1(t)x^1(0), \\ x^i(t) - x^1(t) &= \bar{\rho}_{i-1}(t)x^1(0) + \bar{b}_i(t)[x^i(0) - x^1(0)] + \bar{c}_i(t)r_i \quad 2 \leq i \leq N. \end{aligned} \tag{4.22}$$

P3. *An upper bound on the swarm size $d(t)$ is given by*

$$d(t) \leq \max_i \{\bar{\rho}_{i-1}(t)\} |x^1(0)| + \max_i \{\bar{b}_i(t)\} |x^i(0) - x^1(0)| + \max_i \{\bar{c}_i(t)r_i\}.$$

P4. *The swarm center $x_c = (1/N)(x^1 + \dots + x^N)$ follows the trajectory*

$$x_c(t) = \bar{b}_1(t)x^1(0) + \frac{1}{N} \sum_{i=1}^N \{\bar{\rho}_{i-1}(t)x^1(0) + \bar{b}_i(t)[x^i(0) - x^1(0)] + \bar{c}_i(t)r_i\}, \quad t \in [0, T].$$

P5. *If the foraging target is specified for all agents including the followers, then there is a unique Nash equilibrium of Game L2 for continuous strategies. The distance expressions are obtained from (4.22) in the limit as $\gamma = \beta \rightarrow \infty$ in (4.21).*

P6. *If the foraging task is dropped, then there still exists a unique Nash equilibrium for continuous strategies. The distance expressions are obtained by (4.22) by substituting $\gamma = 0$ and $\beta = 0$ in (4.21).*

Remark 8.

- i) In this loose information structure, a unique Nash equilibrium is still reached if and only if the leader has a greater desire to reach the target location ($\gamma \geq \beta$).
- ii) The ordering in the resulting swarm is such that the leader maintains its first rank at all times. On the other hand, changes of ordering among followers are permissible in this Nash equilibrium.
- iii) If $a_i = a$ and $r_i = r$ for all $i = 2, \dots, N$, then no ordering change occurs among the agents since Game L2 becomes a special case of Game L1.

It is shown in the second plot of Figure 4.6 that indeed no ordering change occurs in Game L2 for equal attraction parameters and equal repulsion parameters. In the first plot of Figure 4.6, it is observed that no ordering change occurs when a_j is decreasing and r_j is increasing as j increases. The case that is most likely to have ordering changes is when a_j increases and r_j decreases as j increases. This situation is depicted in third plot of Figure 4.6.

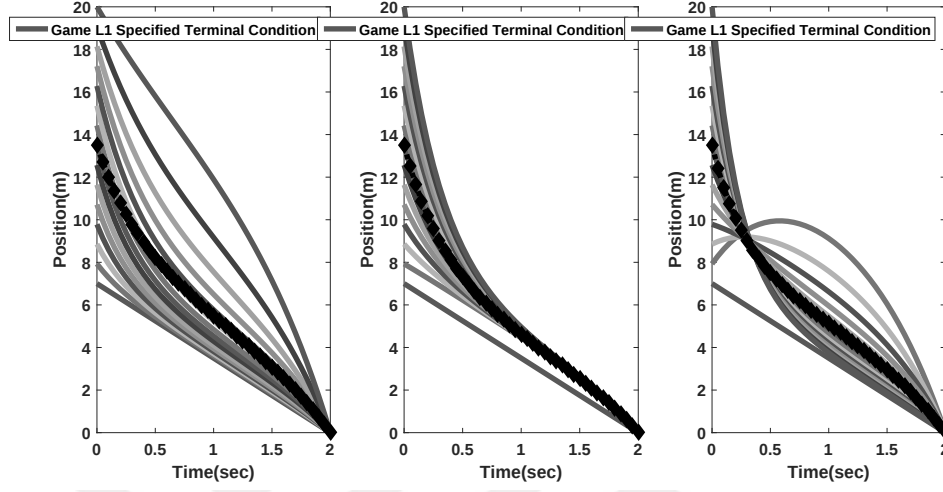
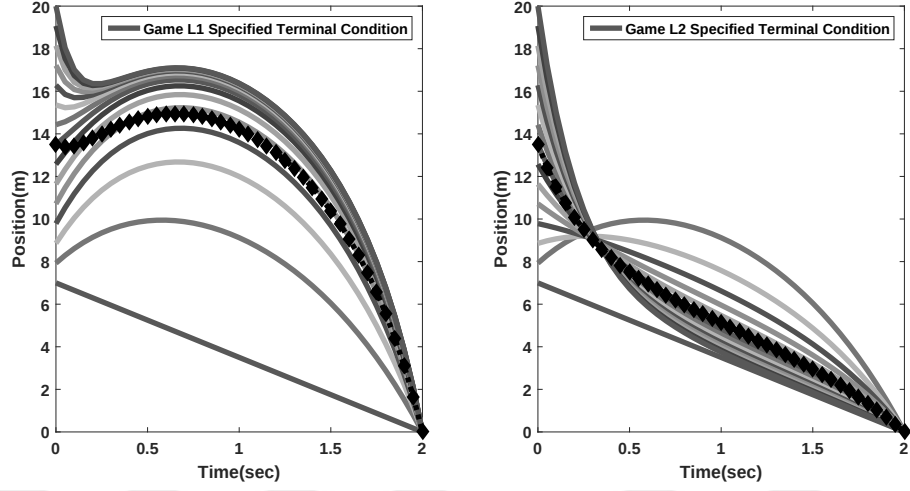


Figure 4.6: All parameters are the same as in Figure 4.7a except that attraction and repulsion parameters are all equal to 10 in the second plot and attraction parameters are equispaced between 20 and 2 and repulsion parameters are equispaced between 2 and 20 in the first plot.

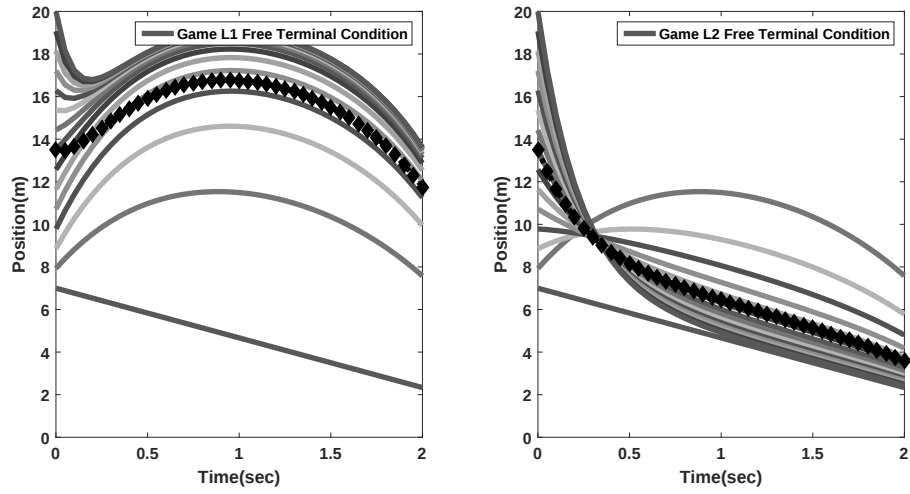
- iv) The trajectory dynamics in Game L2 (under all three types of target specification) are always dominated by hyperbolic functions. This is a consequence of the hypothesized types of the artificial energy components in the cost functions (3.5) and (3.6) as well as the dynamic constraint. In fact, the same kind of dynamics dominate the other trajectories resulting in Game L1 as well as in Games 1 and 2.
- v) If attraction is large such that $a_i \rightarrow \infty$ for some $i \in \{1, \dots, N\}$, then agent i instantaneously sticks to the leader and move towards the target location with the leader.
- vi) If $r_i \rightarrow \infty$ for some $i \in \{1, \dots, N\}$, then the agent i suddenly departs from the swarm, stays there for $0 < t < T$ and suddenly moves towards the target location as $t \rightarrow T$ when the foraging term becomes more effective in the cost functions.

Figure 4.7 juxtaposes two swarm trajectories obtained in Games L1 and L2 for both specified and free terminal condition. It is observed that legitimate ordering

changes can occur in Game L2 as opposed to Game L1. If the agents have a partial desire for the target, then they progressively approach towards the target without exactly reaching it.



(a) Games L1 and L2 are compared for specified terminal condition. The swarm population, the duration of the game, and the sampling period are selected as $N = 15$, $T = 2$, and $T_s = 0.05$ for both graphs. Attraction coefficients are equispaced between 0.1 and 20, repulsion coefficients are equispaced between 20 and 0.1, and initial positions are also equispaced between 7 and 20



(b) Games L1 and L2 are compared for free terminal condition. All the coefficients are same as in Figure 4.7a except that the foraging coefficients are $\gamma = \beta = 1$

Figure 4.7: Games L1 and L2 under the same initial conditions

Chapter 5

Proofs of Existence and Uniqueness of Nash Equilibrium

5.1 Explicit Nash equilibrium of Four Games for Free Terminal Condition

The most general cost function concerning the proposed games is given in (3.1). The relation between cost functions in (3.3), (3.4), (3.5), and (3.6) to (3.1) is presented below.

Game 1

$$\begin{aligned}\beta_i &= \beta \quad \forall 1 \leq i \leq N, \\ P^i &= \{1 \leq j \leq N, j \neq i\}, \\ a_{ij} &= a \quad \forall 1 \leq i \leq N, j \in P^i, \\ r_{ij} &= r \quad \forall 1 \leq i \leq N, j \in P^i.\end{aligned}$$

Game 2

$$\begin{aligned}
\beta_i &= \beta \quad \forall 1 \leq i \leq N, \\
P^1 &= \{2\} \\
P^i &= \{i-1 \leq j \leq i+1, j \neq i\}, \\
P^N &= \{N-1\} \\
a_{ij} &= a \quad \forall 1 \leq i \leq N, j \in P^i, \\
r_{ij} &= r \quad \forall 1 \leq i \leq N, j \in P^i.
\end{aligned}$$

Game L1

$$\begin{aligned}
\beta_1 &= \gamma, \\
\beta_i &= \beta \quad \forall 2 \leq i \leq N, \\
P^1 &= \emptyset \\
P^i &= \{1 \leq j \leq i-1\}, \\
a_{ij} &= a_j \quad \forall 2 \leq i \leq N, j \in P^i, \\
r_{ij} &= r_j \quad \forall 2 \leq i \leq N, j \in P^i.
\end{aligned}$$

Game L2

$$\begin{aligned}
\beta_1 &= \gamma, \\
\beta_i &= \beta \quad \forall 2 \leq i \leq N, \\
P^i &= \begin{cases} \{\emptyset\} & \text{for } i = 1, \\ \{1\} & \text{for all } 2 \leq i \leq N \end{cases}, \\
a_{i1} &= a_i \quad \forall 2 \leq i \leq N, j \in P^i \\
r_{i1} &= r_i \quad \forall 2 \leq i \leq N, j \in P^i..
\end{aligned}$$

The Hamiltonian associated with the cost function (3.1) is

$$H^i = p^i u^i + \frac{u^i(t)^2}{2} + \sum_{j \in P^i} (a_{ij} \frac{[x^i(t) - x^j(t)]^2}{2} - r_{ij} |x^i(t) - x^j(t)|)$$

for $i = 1, \dots, N-1$, where p^i is a Lagrange multiplier associated with the i^{th} agent. By Theorem 2.1, necessary conditions for the game to have a Nash equilibrium are $\frac{\partial H^i}{\partial u^i} = 0$, $\dot{p}_i = -\frac{\partial H^i}{\partial x^i}$, which yield

$$\begin{aligned}
u^i &= -p^i, \\
\dot{p}^i &= -\frac{\partial H^i}{\partial x^i}, \\
\dot{x}^i &= u^i, \\
\beta x^i(T) &= p^i(T) \text{ for all } 1 \leq i \leq N.
\end{aligned} \tag{5.1}$$

These equations in (5.1) can be written in compact form as

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} 0 & -I \\ -A & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{p}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{s}(t) \end{bmatrix}, \quad (5.2)$$

Here, $A \in \mathbf{R}^{N \times N}$ matrix depends on the information structure and the A matrices for each game is given below. It is not difficult to compute the eigenvalues and the eigenvectors in each case. One eigenvalue turns out to be zero in every matrix. The remaining $N - 1$ eigenvalues α_k , $k = 1, \dots, N - 1$ are nonzero and are given next to the description of each A matrix below.

Game 1

$$A = a \begin{bmatrix} N-1 & -1 & -1 & \dots & -1 \\ -1 & N-1 & -1 & \dots & -1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -1 & \dots & -1 & N-1 & -1 \\ -1 & \dots & -1 & -1 & N-1 \end{bmatrix}, \quad \alpha_k = \sqrt{Na} \quad \forall 1 \leq k \leq N-1. \quad (5.3)$$

Game 2

$$A = a \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ & \ddots & \ddots & \ddots \\ & & -1 & 2 & -1 \\ & & 0 & -1 & 1 \end{bmatrix}, \quad \alpha_k = 2\cos\left(\frac{k\pi}{2N}\right)\sqrt{a} \quad \forall 1 \leq k \leq N-1. \quad (5.4)$$

Game L1

$$A = \begin{bmatrix} 0 \\ -a_1 & a_1 \\ -a_1 & -a_2 & a_1 + a_2 \\ \vdots & \vdots & & \ddots \\ -a_1 & -a_2 & -(a_1 + a_2) & \dots & \sum_{j=1}^{N-1} a_j \end{bmatrix}, \quad \begin{aligned} \alpha_k &= \sqrt{a_1 + a_2 + \dots + a_k} \\ \forall 1 \leq k \leq N-1. \end{aligned} \quad (5.5)$$

Game L2

$$A = \begin{bmatrix} 0 & & & & \\ -a_2 & a_2 & & & \\ -a_3 & 0 & a_3 & & \\ \vdots & \vdots & & \ddots & \\ -a_N & 0 & 0 & \dots & a_N \end{bmatrix}, \quad \alpha_k = \sqrt{a_{k+1}} \quad \forall 1 \leq k \leq N-1. \quad (5.6)$$

Suppose that $x^1(0) < x^2(0) < \dots < x^N(0)$ and let $\mathbf{s}(t) \in \mathbf{R}^{N \times 1}$. Then $\mathbf{s}(t) = r[1 - N \quad 3 - N \dots N - 3 \quad N - 1]'$ for Game 1. Likewise, $\mathbf{s}(t) = r[-1 \quad 0 \quad 0 \dots 0 \quad 0 \quad 1]'$ for Game 2, $\mathbf{s}(t) = [0 \quad r_1 \quad r_1 + r_2 \dots \sum_{j=1}^{N-1} r_j]'$ for Game L1, and $\mathbf{s}(t) = [0 \quad r_2 \quad r_3 \dots r_N]'$ for Game L2. In these expressions for $\mathbf{s}$, we assume that $(x_i - x_j)$ never becomes zero and does not change sign.

Since A matrix in (5.3) and (5.4) are symmetric, they are diagonalizable under similarity. The eigenvalues of A matrix in (5.5) and (5.6) are distinct, therefore they are also diagonalizable. Thus, we can write

$$A = V \begin{bmatrix} 0 & \mathbf{0}' \\ \mathbf{0} & D^2 \end{bmatrix} V^{-1}, \quad (5.7)$$

$$D := \text{diag}[\alpha_1, \dots, \alpha_{N-1}],$$

where $V \in \mathbf{R}^{N \times N}$ is the matrix of eigenvectors of A . The first column of V is the eigenvector associated with the eigenvalue 0 and the remaining $N - 1$ columns are those associated with $\alpha_1, \dots, \alpha_{N-1}$, respectively. It turns out that, in Game 1, V attains the following form.

$$V = \begin{bmatrix} 1 & \mathbf{w}' \\ \mathbf{w} & -I \end{bmatrix}, \quad (5.8)$$

where $\mathbf{w} \in \mathbf{R}^{(N-1) \times 1}$ is a vector with all entries equal to 1. In Game L1, V is a lower triangular matrix such that $v_{ij} = 1$ for $i \geq j$ and in Game L2, it is a matrix with entries in its first column all equal to 1, diagonal entries all equal to 1, and all other entries equal to zero. (An explicit description of the V matrix in Game 2 will not be needed.)

Let us now postulate that $\mathbf{s}(t) = \mathbf{s}(0)$ $t \in [0, T]$, i.e., the initial ordering among the agents is preserved for all $t \in [0, T]$. The system (5.2) then becomes linear under this assumption and its solution is

$$\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{p}(t) \end{bmatrix} = \phi(t) \begin{bmatrix} \mathbf{x}(0) \\ \mathbf{p}(0) \end{bmatrix} + \psi(t, 0)\mathbf{s}(0), \quad (5.9)$$

where $\phi(t)$ is the state transition matrix of (5.2). In block partitioned form

$$\begin{aligned} \phi(t) &= \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{bmatrix} := \mathcal{L}^{-1} \left\{ \begin{bmatrix} sI & I \\ A & sI \end{bmatrix}^{-1} \right\}, \\ \psi(t, t_0) &:= \int_{t_0}^t \begin{bmatrix} \phi_{12}(t - \tau) \\ \phi_{22}(t - \tau) \end{bmatrix} d\tau. \end{aligned}$$

The state transition matrix can be obtained explicitly via

$$\begin{bmatrix} sI & I \\ A & sI \end{bmatrix}^{-1} = \begin{bmatrix} s(s^2I - A)^{-1} & -(s^2I - A)^{-1} \\ -A(s^2I - A)^{-1} & s(s^2I - A)^{-1} \end{bmatrix},$$

which gives

$$\begin{aligned} \phi_{11}(t) &= \phi_{22}(t) = V \operatorname{diag} [1, \cosh(\alpha_1 t), \dots, \cosh(\alpha_{N-1} t)] V^{-1}, \\ \phi_{12}(t) &= V \operatorname{diag} \left[-t, -\frac{\sinh(\alpha_1 t)}{\alpha_1}, \dots, -\frac{\sinh(\alpha_{N-1} t)}{\alpha_{N-1}} \right] V^{-1}, \\ \phi_{21}(t) &= V \operatorname{diag} [0, -\alpha_1 \sinh(\alpha_1 t), \dots, -\alpha_{N-1} \sinh(\alpha_{N-1} t)] V^{-1}. \end{aligned} \quad (5.10)$$

$$\begin{aligned} \psi_1(t, 0) &= V \operatorname{diag} \left[-\frac{t^2}{2}, \frac{1 - \cosh(\alpha_1 t)}{\alpha_1^2}, \dots, \frac{1 - \cosh(\alpha_{N-1} t)}{\alpha_{N-1}^2} \right] V^{-1}, \\ \psi_2(t, 0) &= V \operatorname{diag} \left[t, \frac{\sinh(\alpha_1 t)}{\alpha_1}, \dots, \frac{\sinh(\alpha_{N-1} t)}{\alpha_{N-1}} \right] V^{-1}. \end{aligned} \quad (5.11)$$

Using the boundary condition $\beta \mathbf{x}(T) = \mathbf{p}(T)$ in (5.9) for $t = T$ gives

$$[\beta \phi_{11}(T) - \phi_{21}(T)]\mathbf{x}(0) + [\beta \phi_{12}(T) - \phi_{22}(T)]\mathbf{p}(0) + [\beta \psi_1(T, 0) - \psi_2(T, 0)]\mathbf{s}(0) = 0,$$

which can be solved for $\mathbf{p}(0)$ since $[\beta \phi_{12}(T) - \phi_{22}(T)]$ is nonsingular. It follows that there is a candidate solution of (5.2) for every $\mathbf{x}(0)$. This solution is

$$\begin{aligned} \mathbf{x}(t) &= \\ &\{ \phi_{11}(t) - \phi_{12}(t)[\beta \phi_{12}(T) - \phi_{22}(T)]^{-1}[\beta \phi_{11}(T) - \phi_{21}(T)] \} \mathbf{x}(0) + \\ &\{ \psi_1(t, 0) - \phi_{12}(t)[\beta \phi_{12}(T) - \phi_{22}(T)]^{-1}[\beta \psi_1(T, 0) - \psi_2(T, 0)] \} \mathbf{s}(0). \end{aligned} \quad (5.12)$$

If we plug in the state transition matrix blocks in (5.10) and (5.11) to (5.18), we have

$$\mathbf{x}(t) = VB(t)V^{-1}\mathbf{x}(0) + VC(t)V^{-1}\mathbf{s}(0). \quad (5.13)$$

Here, $B(t) \in \mathbf{R}^{N \times N}$ is given by

$$B(t) = \text{diag}[1 - \frac{\beta t}{\beta T + 1}, b_1(t), \dots, b_{N-1}(t)], \quad (5.14)$$

where

$$b_k(t) = \frac{\beta \sinh[\alpha_k(T-t)] + \alpha_k \cosh[\alpha_k(T-t)]}{\beta \sinh(\alpha_k T) + \alpha_k \cosh(\alpha_k T)}. \quad (5.15)$$

On the other hand, $C(t) \in \mathbf{R}^{N \times N}$ is given by

$$C(t) = \text{diag}[\frac{-t^2}{2} + \frac{t(\beta T^2 + 2T)}{2\beta T + 2}, c_1(t), \dots, c_{N-1}(t)], \quad (5.16)$$

where

$$c_k(t) = \frac{1}{\alpha_k^2} [1 - b_k(t) - \frac{\beta \sinh(\alpha_k t)}{\beta \sinh(\alpha_k T) + \alpha_k \cosh(\alpha_k T)}]. \quad (5.17)$$

5.2 Nash equilibrium for Specified and Unspecified Terminal Condition

The equations of specified terminal condition and unspecified terminal condition are limiting cases of equations of free terminal condition as $\beta \rightarrow \infty$ and $\beta \rightarrow 0$ respectively. The equations of specified terminal condition are given here. The proof for unspecified condition can be done similarly.

The terminal condition in (5.1) boils down to $x^i(T) = 0$ for all $1 \leq i \leq N$ for specified terminal condition. Using this boundary condition for $t = T$ gives

$$\phi_{11}(T)\mathbf{x}(0) + \phi_{12}(T)\mathbf{p}(0) + [\psi_1(T, 0) - \psi_2(T, 0)]\mathbf{s}(0) = 0,$$

which can be solved for $\mathbf{p}(0)$ since $\phi_{12}(T)$ is nonsingular. It follows that there is a candidate solution of (5.2) for every $\mathbf{x}(0)$. This solution is

$$\begin{aligned} \mathbf{x}(t) = & \{\phi_{11}(t) - \phi_{12}(t)[\phi_{12}(T)]^{-1}\phi_{11}(T)\}\mathbf{x}(0) \\ & + \{\psi_1(t, 0) - \phi_{12}(t)[\phi_{12}(T)]^{-1}\psi_1(T, 0)\}\mathbf{s}(0). \end{aligned} \quad (5.18)$$

If we plug in the state transition matrix blocks in (5.10) and (5.11) to (5.18), we have

$$\mathbf{x}(t) = VB(t)V^{-1}\mathbf{x}(0) + VC(t)V^{-1}\mathbf{s}(0). \quad (5.19)$$

Here, $B(t) \in \mathbf{R}^{N \times N}$ is given by

$$B(t) = \text{diag}[1 - \frac{t}{T}, b_1(t), \dots, b_{N-1}(t)], \quad (5.20)$$

where

$$b_k(t) = \frac{\sinh[\alpha_k(T-t)]}{\sinh(\alpha_k T)}. \quad (5.21)$$

On the other hand, $C(t) \in \mathbf{R}^{N \times N}$ is given by

$$C(t) = \text{diag}[\frac{(T-t)t}{2}, c_1(t), \dots, c_{N-1}(t)], \quad (5.22)$$

where

$$c_k(t) = \frac{1}{\alpha_k^2} [1 - b_k(t) - \frac{\sinh(\alpha_k t)}{\sinh(\alpha_k T)}]. \quad (5.23)$$

Thus, we have obtained the optimal trajectory components in P1 of Theorem 4.2 and P2 of Theorem 4.3.

5.3 Pairwise Distances of Four Games

Let

$$P = \begin{bmatrix} \mathbf{w}' \\ M \end{bmatrix}, \quad (5.24)$$

be the matrix to transform optimal trajectories to pairwise distances. Here, $\mathbf{w} \in \mathbf{R}^{N \times 1}$ is a vector with all entries equal to 1. $M \in \mathbf{R}^{(N-1) \times N}$ be such that $M_{i,i} = 1$, $M_{i,i+1} = -1$ for $i = 1, \dots, N-1$, and all other entries of M are equal to zero. Thus, the i -th row of M has all zeros except a 1 and a -1 at its i -th and $(i+1)$ -st positions, respectively.

Transforming the solution in (5.13) by this P matrix, we obtain the trajectory of swarm center and pairwise distances for Game 1 as follows.

$$x_c(t) = x_c(0)(1 - \frac{\beta t}{\beta T + 1}), \quad (5.25)$$

where $x_c(t) \in \mathbf{R}$.

$$x^i(t) - x^j(t) = b(t)[x^i(0) - x^j(0)] + c(t)[s^i(0) - s^j(0)].$$

Transforming the solution in (5.13) by the same P matrix, we obtain same trajectory for swarm center given in (5.25). The pairwise distances of Game 2 are given by

$$\mathbf{y}_d(t) = U\tilde{B}(t)U'\mathbf{y}_d(0) + U\tilde{C}(t)U'\mathbf{r}_d(0),$$

where $\mathbf{y}_d = [x_1 - x_2 \dots x_{N-1} - x_N]$ and $\mathbf{r}_d = [-r, 0, 0, \dots -r]$ for Game 2. Here, $\tilde{B}(t) = \text{diag}[b_1(t), \dots, b_{N-1}(t)]$, and $\tilde{C}(t) = \text{diag}[c_1(t), \dots, c_{N-1}(t)]$. U matrix is given in (4.10).

For Game L1, if P is selected as

$$P = \begin{bmatrix} \mathbf{v}' \\ -M \end{bmatrix}, \quad (5.26)$$

where $\mathbf{v} \in \mathbf{R}^{N \times 1}$ is a vector with first entry equal to 1, and all other entries equal to 0. The trajectory of the leader is

$$x^1(t) = x^1(0) \left(1 - \frac{\beta t}{\beta T + 1}\right), \quad (5.27)$$

and pairwise distances are

$$x^i(t) - x^{i-1}(t) = b_i(t)[x^i(0) - x^{i-1}(0)] + c_i(t)[s^i(0) - s^{i-1}(0)].$$

If P is selected as

$$P = \begin{bmatrix} 1 & \mathbf{0}' \\ -\mathbf{w} & I \end{bmatrix}, \quad (5.28)$$

for Game L2, we obtain same trajectory for leader as (5.27). Distance from leader turns out to be

$$x^i(t) - x^1(t) = b_i(t)[x^i(t) - x^1(t)] + c_i(t)s^i(0),$$

for Game L2.

5.4 Existence of Nash Equilibrium for Game 1

Since $b(t) > 0$ and $c(t) > 0$, pairwise distance does not change sign. Hence, ordering does not change which establishes the existence of Nash equilibrium.

5.5 Existence of Nash Equilibrium for Game L1 and Game L2

Since $b_i(t) > 0$ and $c_i(t) > 0$ for Game L1 and Game L2, pairwise distances do not change sign in Game L1, and distance from the leader does not change sign in Game

L2. This establishes the existence of Nash equilibrium for Game L1 and Game L2

5.6 Existence of Nash equilibrium for Game 2

The existence of Nash equilibrium in case of Game 2 is rather lengthy since establishing the positivity of two time varying matrices turns out to be difficult.

Let $B_n(t)$ denote the n -th Bernoulli polynomial (see e.g., [82], Ch. 12).

Lemma 5.1. *It holds that*

$$b_k(t) = \sum_{n=0}^{\infty} \beta_n(t) \sigma_k^{2n}, c_k(t) = \sum_{m=0}^{\infty} \gamma_m(t) \sigma_k^{2m}, \quad (5.29)$$

where

$$\begin{aligned} \beta_n(t) &= B_{2n+1}\left(1 - \frac{t}{2T}\right) \frac{2^{2n+1}}{(2n+1)!} T^{2n} a^n, \\ \gamma_m(t) &= -[B_{2m+3}\left(1 - \frac{t}{2T}\right) + B_{2m+3}\left(\frac{1}{2} + \frac{t}{2T}\right)] \frac{2^{2m+3}}{(2m+3)!} T^{2m+2} a^{m+1}. \end{aligned} \quad (5.30)$$

Moreover,

$$\text{sign}\{\beta_n(t)\} = \text{sign}\{\gamma_n(t)\} = (-1)^n, \forall t \in [0, T]. \quad (5.31)$$

Proof. By the defining equation for Bernoulli polynomials

$$\frac{ye^{xy}}{e^y - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{y^n}{n!},$$

we can write

$$\frac{2\alpha_k T e^{(2\tau-1)\alpha_k T}}{e^{\alpha_k T} - e^{-\alpha_k T}} = \sum_{n=0}^{\infty} B_n(\tau) \frac{(2\alpha_k T)^n}{n!}, \quad \frac{2\alpha_k T e^{-(2\tau-1)\alpha_k T}}{e^{-\alpha_k T} - e^{-\alpha_k T}} = \sum_{n=0}^{\infty} B_n(\tau) (-1)^n \frac{(2\alpha_k T)^n}{n!}.$$

Subtracting and dividing by $2\alpha_k T$, and letting $\tau := 1 - \frac{t}{2T}$, we get

$$\frac{e^{(2\tau-1)\alpha_k T} - e^{-(2\tau-1)\alpha_k T}}{e^{\alpha_k T} - e^{-\alpha_k T}} = \sum_{n=0}^{\infty} B_n(\tau) [1 - (-1)^n] \frac{(2\alpha_k T)^{n-1}}{n!}$$

which leads to the expression for $b_k(t)$ in (5.29) and (5.30). A similar procedure applied to $c_k(t)$ also leads to the expansion in (5.29), (5.30) for $c_k(t)$. The odd-numbered Bernoulli polynomials have constant sign in the interval $(0.5, 1)$ with $B_1(\tau)$ having positive sign (Chapter 23 of [83]). Now, $B'_{2n+1}(\tau) = (2n+1)B_{2n}(\tau)$ for $n \geq 1$ so that

$\text{sign}\{B'_{2n+1}(1)\} = \text{sign}\{B_{2n}(1)\}$. Since $B_{2n}(1)$ are the second Bernoulli numbers, it follows that $\text{sign}\{B'_{2n+1}(1)\} = (-1)^{n-1}$ for $n \geq 1$. Thus, $B_{2n+1}(\tau)$ is decreasing to 0 as $\tau \rightarrow 1$ if and only if n is even, which gives

$$\text{sign}\{B_{2n+1}(\tau)\} = (-1)^n, \forall \tau \in (0.5, 1).$$

This implies (5.31) by the expressions in (5.30). $\square$

Lemma 5.2. *If the attraction parameter a is sufficiently small, then it holds that*

$$\hat{b}_k(t) = \sum_{n=0}^{\infty} \hat{\beta}_n(t) \sigma_k^{2n}, \hat{c}_k(t) = \sum_{m=0}^{\infty} \hat{\gamma}_m(t) \sigma_k^{2m},$$

where

$$\begin{aligned} \hat{\beta}_n(t) &= \frac{2fT(T-t)+T}{2fT(T-t)+(T-t)} B_{2n+1} \left(1 - \frac{t}{2T}\right) \frac{2^{2n+1}}{(2n+1)!} T^{2n} a^n, \\ \hat{\gamma}_m(t) &= - \left[\frac{2fT(T-t)+T}{2fT(T-t)+(T-t)} B_{2m+3} \left(1 - \frac{t}{2T}\right) + \frac{2fT}{2fT+1} B_{2m+3} \left(\frac{1}{2} + \frac{t}{2T}\right) \right] \frac{2^{2m+3}}{(2m+3)!} T^{2m+2} a^{m+1}, \end{aligned}$$

in which

$$\text{sign}\{\hat{\beta}_n(t)\} = \text{sign}\{\hat{\gamma}_n(t)\} = (-1)^n, \forall t \in [0, T].$$

Proof. The expression (4.16) can be written as

$$\begin{aligned} \hat{b}_k(t) &= \frac{2f+\alpha_k \coth(u)}{2f+\alpha_k \coth(v)} b_k(t) \\ &= \frac{2f+\alpha_k(u^{-1}+\frac{u}{3}+\dots)}{2f+\alpha_k(v^{-1}+\frac{v}{3}+\dots)} b_k(t) \end{aligned}$$

where $u = \alpha_k(T-t)$ and $v = \alpha_k T$. If a , and hence $\alpha_k = \sqrt{a}\sigma_k$, are small enough, then

$$\hat{b}_k(t) \sim \frac{2fT(T-t)+T}{2fT(T-t)+(T-t)} b_k(t).$$

Since the multiplying constant is positive for all $t \in (0, T)$, the sign properties of $\hat{b}_k(t)$ is similar to those of $b_k(t)$. A similar procedure applied to $\hat{c}_k(t)$ gives the expression in terms of Bernoulli polynomials and the sign equality. $\square$

We now show that with such $\mathbf{y}(t)$, the ordering of the agents indeed remains the same, i.e., $\text{sign } y^i(t) = \text{sign } y^i(0)$ for all $i = 1, \dots, N-1$ and $t \in [0, T]$. We establish this for some (small enough) values of the attraction parameter $a > 0$. Let us consider the sub-vector $\mathbf{y}_d := [x_1 - x_2 \dots x_{N-1} - x_N]'$ of $\mathbf{y}$, Then, with $\mathbf{r}_d := [1 \ 0 \dots 0 \ 1]'$, (4.17) gives

$$\begin{aligned} \mathbf{y}_d(t) &= K(t)\mathbf{y}_d(0) + L(t)\mathbf{r}_d, \\ K(t) &:= U \text{diag}[b_1(t), \dots, b_{N-1}(t)]U', \\ L(t) &:= U \text{diag}[c_1(t), \dots, c_{N-1}(t)]U', \end{aligned}$$

which are both positive definite matrices for every $t \in [0, T]$ by the fact that $b_i(t)$ and $c_i(t)$ are positive functions of $t \in [0, T]$ for $i = 1, \dots, N-1$. The matrix U of (4.10) that occurs here can be written explicitly by [84], p. 514, as

$$U = [U_{ij}], \quad U_{ij} = \sqrt{\frac{2}{N}} \sin\left(\frac{(N-j)i\pi}{N}\right),$$

so that, for $i, j = 1, \dots, N-1$,

$$\begin{aligned} K_{ij}(t) &= \frac{2}{N} \sum_{k=1}^{N-1} b_{N-k}(t) \sin\left(\frac{ki\pi}{N}\right) \sin\left(\frac{kj\pi}{N}\right), \\ L_{ij}(t) &= \frac{2}{N} \sum_{k=1}^{N-1} c_{N-k}(t) \sin\left(\frac{ki\pi}{N}\right) \sin\left(\frac{kj\pi}{N}\right). \end{aligned}$$

We now show that, there exist values for the attraction parameter $a > 0$ such that for all $i, j = 1, \dots, N-1$ and $t \in [0, T]$, $K_{ij}(t) > 0$ and $L_{ij}(t) > 0$. Consider

$$\begin{aligned} K_{ij}(t) &= \frac{2}{N} \sum_{k=1}^{N-1} b_k(t) \sin\left(\frac{(N-k)i\pi}{N}\right) \sin\left(\frac{(N-k)j\pi}{N}\right) \\ &= \frac{2}{N} (-1)^{i+j} \sum_{k=1}^{N-1} b_k(t) \sin\left(\frac{ki\pi}{N}\right) \sin\left(\frac{kj\pi}{N}\right) \\ &= \frac{(-1)^{i+j}}{N} (-1)^{i+j} \sum_{k=1}^{N-1} b_k(t) \left\{ \cos\left[\frac{(i-j)k\pi}{N}\right] - \cos\left[\frac{(i+j)k\pi}{N}\right] \right\}. \end{aligned}$$

By these expressions it follows that $K_{ij}(t) = K_{N-j, N-i}(t)$, $L_{ij}(t) = L_{N-j, N-i}(t)$ for all i, j , i.e., K and L are centrosymmetric (or bisymmetric) matrices, [85]. This allows us to only show the positivity of the entries with

$$j < i \leq N-j, \quad j = 1, \dots, \left\lfloor \frac{N-1}{2} \right\rfloor. \quad (5.32)$$

Substituting (5.29) into $K_{ij}(t)$ and employing the trigonometric identity

$$\cos^{2m}(\theta) = \frac{1}{2^{2m}} \binom{2m}{m} + \frac{2}{2^{2m}} \sum_{l=0}^{m-1} \binom{2m}{l} \cos[2(m-l)\theta],$$

we have

$$\begin{aligned} K_{ij}(t) &= \frac{(-1)^{i+j}}{N} \sum_{n=0}^{\infty} \beta_n \sum_{k=1}^{N-1} \sigma_k^{2n} \left\{ \cos\left[\frac{(i-j)k\pi}{N}\right] - \cos\left[\frac{(i+j)k\pi}{N}\right] \right\} \\ &= \frac{(-1)^{i+j}}{N} \sum_{n=0}^{\infty} \beta_n \sum_{k=1}^{N-1} 2^{2n} \cos^{2n}\left(\frac{k\pi}{2N}\right) \left\{ \cos\left[\frac{(i-j)k\pi}{N}\right] - \cos\left[\frac{(i+j)k\pi}{N}\right] \right\} \\ &= \frac{(-1)^{i+j}}{N} \sum_{n=0}^{\infty} \beta_n \binom{2n}{n} \sum_{k=1}^{N-1} \left\{ \cos\left[\frac{(i-j)k\pi}{N}\right] - \cos\left[\frac{(i+j)k\pi}{N}\right] \right\} \\ &\quad + \frac{2(-1)^{i+j}}{N} \sum_{n=0}^{\infty} \beta_n \sum_{l=0}^{n-1} \binom{2n}{l} \sum_{k=1}^{N-1} \cos\left[\frac{(n-l)k\pi}{N}\right] \left\{ \cos\left[\frac{(i-j)k\pi}{N}\right] - \cos\left[\frac{(i+j)k\pi}{N}\right] \right\} \end{aligned}$$

We now compute the finite sums over k and t . Let $\mathcal{E}(N)$ read as “an even multiple of N .” The first sum is

$$\begin{aligned} & \sum_{k=1}^{N-1} \left\{ \cos\left[\frac{(i-j)k\pi}{N}\right] - \cos\left[\frac{(i+j)k\pi}{N}\right] \right\} \\ &= \left\{ \begin{matrix} -1, & i-j \neq \mathcal{E}(N) \\ N-1, & i-j = \mathcal{E}(N) \end{matrix} \right\} - \left\{ \begin{matrix} -1, & i+j \neq \mathcal{E}(N) \\ N-1, & i+j = \mathcal{E}(N) \end{matrix} \right\} = 0, \end{aligned}$$

where the last equality is by (5.32). Let

$$\begin{aligned} t_1 &:= t - i + j, \quad t_2 := t + i - j, \\ t_3 &:= t + i + j, \quad t_4 := t - i - j. \end{aligned}$$

The second sum is

$$\begin{aligned} & \sum_{l=0}^{n-1} \binom{2n}{l} \sum_{k=1}^{N-1} \cos\left[\frac{(n-l)k\pi}{N}\right] \left\{ \cos\left[\frac{(i-j)k\pi}{N}\right] - \cos\left[\frac{(i+j)k\pi}{N}\right] \right\} \\ &= \sum_{t=1}^n \binom{2n}{n-t} \sum_{k=1}^{N-1} \cos\left(\frac{tk\pi}{N}\right) \left\{ \cos\left[\frac{(i-j)k\pi}{N}\right] - \cos\left[\frac{(i+j)k\pi}{N}\right] \right\} \\ &= \frac{1}{2} \sum_{t=1}^n \binom{2n}{n-t} \sum_{k=1}^{N-1} \left\{ \cos\left(\frac{t_1 k\pi}{N}\right) - \cos\left(\frac{t_2 k\pi}{N}\right) + \cos\left(\frac{t_3 k\pi}{N}\right) - \cos\left(\frac{t_4 k\pi}{N}\right) \right\} \\ &= \frac{1}{2} \sum_{t=1}^n \binom{2n}{n-t} \left(\left\{ \begin{matrix} -1, & t_1 \neq \mathcal{E}(N) \\ N-1, & t_1 = \mathcal{E}(N) \end{matrix} \right\} - \left\{ \begin{matrix} -1, & t_2 \neq \mathcal{E}(N) \\ N-1, & t_2 = \mathcal{E}(N) \end{matrix} \right\} \right. \\ & \quad \left. + \left\{ \begin{matrix} -1, & t_3 \neq \mathcal{E}(N) \\ N-1, & t_3 = \mathcal{E}(N) \end{matrix} \right\} - \left\{ \begin{matrix} -1, & t_4 \neq \mathcal{E}(N) \\ N-1, & t_4 = \mathcal{E}(N) \end{matrix} \right\} \right). \end{aligned}$$

By (5.32), it is easy to see that if $t_l = \mathcal{E}(N)$ for some $l = 1, 2, 3, 4$, then $t_k \neq \mathcal{E}(N)$ for all three $k \neq l$. Therefore,

$$\begin{aligned} K_{ij}(t) &= (-1)^{i+j} \sum_{n=0}^{\infty} \beta_n \left[\sum_{\substack{p=1 \\ p = \mathcal{E}(N) + i - j}}^n \binom{2n}{n-p} + \sum_{\substack{p=1 \\ p = \mathcal{E}(N) - i - j}}^n \binom{2n}{n-p} \right. \\ & \quad \left. - \sum_{\substack{p=1 \\ p = \mathcal{E}(N) - i + j}}^n \binom{2n}{n-p} - \sum_{\substack{p=1 \\ p = \mathcal{E}(N) + i + j}}^n \binom{2n}{n-p} \right]. \end{aligned}$$

At this stage, rather than $K_{ij}(t)$, it will be more convenient to consider the expression for $K_{i,N-j}(t)$ for

$$N - j < i \leq j, \quad j = \lfloor \frac{N+1}{2} \rfloor, \dots, N-1. \quad (5.33)$$

With this change of index, we are still considering the same subset of entries of K but their expressions will be simpler. Substituting $N - j$ for j in the above expression, we

have

$$S := K_{i,N-j}(t)(-1)^N(-1)^{i+j} = \sum_{n=0}^{\infty} \beta_n \left[\sum_{\substack{p=1 \\ p=O(N)-i-j}}^n \binom{2n}{n-p} + \sum_{\substack{p=1 \\ p=O(N)-i+j}}^n \binom{2n}{n-p} \right. \\ \left. - \sum_{\substack{p=1 \\ p=O(N)-i+j}}^n \binom{2n}{n-p} - \sum_{\substack{p=1 \\ p=O(N)+i+j}}^n \binom{2n}{n-p} \right],$$

where $O(N)$ reads "odd multiple of N ." Writing out a few terms of each summation in the expression of S , it is not difficult to see that

$$S = \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \sum_{k=0}^{2N-1} \left\{ \beta_{mN-i-j+k} \sum_{t=0}^{\frac{m-1}{2}} \binom{2mN-2(i+j)+2k}{2tN+k} + \beta_{mN-i+j+k} \sum_{t=0}^{\frac{m-1}{2}} \binom{2mN-2(i-j)+2k}{2tN+k} \right. \\ \left. - \beta_{mN+i-j+k} \sum_{t=0}^{\frac{m-1}{2}} \binom{2mN+2(i-j)+2k}{2tN+k} - \beta_{mN+i+j+k} \sum_{t=0}^{\frac{m-1}{2}} \binom{2mN+2(i+j)+2k}{2tN+k} \right\}$$

We now separate the even and odd k in the summations with respect to k , to obtain

$$S = \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \sum_{k=0}^{N-1} \left\{ \beta_{mN-i-j+2k} \sum_{t=0}^{\frac{m-1}{2}} \binom{2mN-2(i+j)+4k}{2tN+2k} + \beta_{mN-i+j+2k} \sum_{t=0}^{\frac{m-1}{2}} \binom{2mN-2(i-j)+4k}{2tN+2k} \right. \\ \left. - \beta_{mN+i-j+2k} \sum_{t=0}^{\frac{m-1}{2}} \binom{2mN+2(i-j)+4k}{2tN+2k} - \beta_{mN+i+j+2k} \sum_{t=0}^{\frac{m-1}{2}} \binom{2mN+2(i+j)+4k}{2tN+2k} \right. \\ \left. + \beta_{mN-i-j+2k+1} \sum_{t=0}^{\frac{m-1}{2}} \binom{2mN-2(i+j)+4k+2}{2tN+2k+1} + \beta_{mN-i+j+2k+1} \sum_{t=0}^{\frac{m-1}{2}} \binom{2mN-2(i-j)+4k+2}{2tN+2k+1} \right. \\ \left. - \beta_{mN+i-j+2k+1} \sum_{t=0}^{\frac{m-1}{2}} \binom{2mN+2(i-j)+4k+2}{2tN+2k+1} - \beta_{mN+i+j+2k+1} \sum_{t=0}^{\frac{m-1}{2}} \binom{2mN+2(i+j)+4k+2}{2tN+2k+1} \right\}.$$

For fixed m and k , the smallest indexed β occurs in the first term in the brackets. By the expression in (5.6), the sign of S is determined by the sign of $\beta_{mN-i-j+2k}$ for small enough attraction parameter $a > 0$ because $\beta_{mN-i-j+2k}$ is divisible by the smallest power of a among all β that occur in the above expression. It follows by (5.31) that, $\text{sign}(\beta_{mN-i-j+2k}) = (-1)^{mN-i-j}$ for all $t \in [0, T]$. Since m is odd, we have $\text{sign}(S) = (-1)^N(-1)^{i+j}$. This establishes that, there exists $a > 0$ such that for all i, j as in (5.33), $K_{i,N-j}(t) > 0$, $t \in [0, T]$. The proof of positivity of the matrix L is obtained in exactly the same manner since $\gamma_n(t)$ of Lemma 5.1 replacing $\beta_n(t)$ in the last expression above yields $L_{i,N-j}(t)$. This proves that there is a Nash equilibrium in which the initial ordering among the agents is preserved in the whole interval $[0, T]$.

Remark 9 The infinite summation expression for S , crucially used in establishing the existence of Nash equilibria, also indicates that $\text{sign}(S)$ is determined by the sign of $-\beta_{mN+i+j+2k+1}$ for large enough attraction parameter $a > 0$. This is because $\beta_{mN+i+j+2k+1}$ is divisible by the largest power of a among all β that occur in that expression. It follows by (5.31) that, $\text{sign}(-\beta_{mN+i+j+2k+1}) = (-1)^{mN+i+j}$ for all $t \in [0, T]$. Since m is odd, we again have $\text{sign}(S) = (-1)^N (-1)^{i+j}$. This establishes that, for large $a > 0$ and for all i, j as in (5.33), $K_{i,N-j}(t) > 0$, $t \in [0, T]$. Similarly, one can conclude the positivity of other relevant matrices L , $\hat{K}$, and $\hat{L}$. Therefore, Nash equilibria also exist for sufficiently large values of the attraction parameter as well. $\triangle$

5.7 Proof of Uniqueness of Nash Equilibria

We finally show that Nash equilibrium is unique with respect to strategies that are continuous against initial positions. Again, we give the proof in detail for the specified terminal condition case but a similar procedure proves the uniqueness also in the free terminal condition case.

Suppose that there are changes in the ordering of the agents at the $n - 1$ time instants $\{t_1, \dots, t_{n-1}\} \in (0, T)$, with $n \geq 2$. The integer n is finite since the terminal condition should be satisfied exactly, not asymptotically. Let $t_0 := 0$ and $t_n := T$. For $k = 1, \dots, n - 1$, the response at $t \in (t_{k-1}, t_k)$ can be expressed in terms of the response at t_{k-1} as

$$\mathbf{z}(t) = \phi^k(t - t_{k-1})\mathbf{z}(t_{k-1}) + \psi^k(t, t_{k-1})\mathbf{s}_{k-1}, \quad t \in (t_{k-1}, t_k), \quad (5.34)$$

where $\mathbf{z}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{p}(t) \end{bmatrix}$, $\phi^k(t - t_{k-1})$ is the state transition matrix for $t \in (t_{k-1}, t_k)$ and is related to the state transition matrix $\phi^{k-1}(t - t_{k-2})$, $t \in (t_{k-2}, t_{k-1})$, by

$$\phi^k(t - t_{k-1}) = \begin{bmatrix} P\phi_{11}^{k-1}(t - t_{k-2})P' & P\phi_{12}^{k-1}(t - t_{k-2})P' \\ P\phi_{21}^{k-1}(t - t_{k-2})P' & P\phi_{22}^{k-1}(t - t_{k-2})P' \end{bmatrix},$$

where P is a permutation matrix and the sizes of four partitions are all $N \times N$. The matrix $\psi^k(t, t_k)$ is

$$\psi^k(t, t_k) := \int_{t_k}^t r \begin{bmatrix} P\phi_{12}^{k-1}(t - t_{k-2})P' \\ P\phi_{22}^{k-1}(t - t_{k-2})P' \end{bmatrix} d\tau.$$

It follows that at $t_n = T$, we have

$$\mathbf{z}(T) = \left[\prod_{l=1}^n \phi^l(t_l - t_{l-1}) \right] \mathbf{z}(0) + \sum_{l=1}^n \left[\prod_{m=l}^n \phi^m(t_m - t_{m-1}) \right] \psi^l(t_l, t_{l-1}) \mathbf{s}_{l-1}.$$

Multiplying both sides on the left by $[I \ 0]$, where I has size N , we have

$$[I \ 0] \mathbf{z}(T) = [I \ 0] \left[\prod_{l=1}^n \phi^l(t_l - t_{l-1}) \right] \mathbf{z}(0) + [I \ 0] \sum_{l=1}^n \left[\prod_{m=l+1}^n \phi^m(t_m - t_{m-1}) \right] \psi^l(t_l, t_{l-1}) \mathbf{s}_{l-1}.$$

We employ the boundary condition $\mathbf{x}(T) = \mathbf{0}$ and obtain

$$0 = \Omega_{11} \mathbf{x}(0) + \Omega_{12} \mathbf{p}(0) + \sum_{l=1}^n \Gamma_1^l \mathbf{s}_{l-1},$$

where Ω_{ij} is the ij^{th} block of $\Omega := \prod_{l=1}^n \phi^l(t_l - t_{l-1})$ and Γ_i^l is the i^{th} block of

$\left[\prod_{m=l+1}^n \phi^m(t_m - t_{m-1}) \right] \psi^l(t_l, t_{l-1})$. We now show that Ω_{12} is nonsingular for small enough a so that $\mathbf{p}(0)$ is uniquely determined. In fact, as $a \rightarrow 0$, the state transition matrix in each interval $l = 1, \dots, n$ asymptotically approaches

$$\phi^l(t) \rightarrow \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} I & -tI \\ 0 & I \end{bmatrix} \begin{bmatrix} P' & 0 \\ 0 & P' \end{bmatrix} = \begin{bmatrix} I & -tI \\ 0 & I \end{bmatrix}, \quad (5.35)$$

for the permutation matrix P that represents the ordering change passing from the interval $l-1$ to l . It follows that as $a \rightarrow 0$

$$\Omega \rightarrow \begin{bmatrix} I & -(t_n - t_{n-1})I \\ 0 & I \end{bmatrix} \begin{bmatrix} I & -(t_{n-1} - t_{n-2})I \\ 0 & I \end{bmatrix} \dots \begin{bmatrix} I & -(t_1 - t_0)I \\ 0 & I \end{bmatrix}$$

so that $\Omega_{12} \rightarrow -\sum_{l=1}^n (t_l - t_{l-1})I = -TI$, which implies that Ω_{12} is nonsingular for sufficiently small $a > 0$. Therefore,

$$\mathbf{p}(0) = -\Omega_{12}^{-1} \left(\Omega_{11} \mathbf{x}(0) + \sum_{l=1}^n \Gamma_1^l \mathbf{s}_{l-1} \right). \quad (5.36)$$

Let us now consider the response in the vicinity of t_1 , the first change of ordering instant, at which (5.34) gives

$$\mathbf{x}(t) = [\phi_{11}^1(t) \mathbf{x}(0) + \phi_{12}^1(t) \mathbf{p}(0)] + \psi_1^1(t, 0) \mathbf{s}_0.$$

Suppose $x_i(t_1) = x_j(t_1)$, i.e., the i^{th} and the j^{th} agents change positions at t_1 . Substituting $p(0)$ obtained in (5.36) and multiplying both sides of this equation by the row

vector $\mathbf{w}_{ij}^T$, all entries of which are 0 except 1 in its i^{th} entry and -1 in its j^{th} entry, we obtain

$$\begin{aligned} x_i(t) - x_j(t) = & \mathbf{w}_{ij}^T (\phi_{11}^1(t) + \phi_{12}^1(t)\Omega_{12}^{-1}\Omega_{11}) \mathbf{x}(0) \\ & + \mathbf{w}_{ij}^T \phi_{12}^1(t)\Omega_{12}^{-1} \sum_{l=1}^n \Gamma_1^l \mathbf{s}_{l-1} + \mathbf{w}_{ij}^T \psi_1^1(t, 0) \mathbf{s}_0. \end{aligned} \quad (5.37)$$

For ϵ sufficiently small and $t \in (t_1, t_1 + \epsilon)$, the left hand side can be made as small as desired without any permutation in Ω and Γ_1^l since no change of ordering occurs in this time interval. By continuity of strategies with respect to $\mathbf{x}(0)$, $x_i(t) - x_j(t)$ and the first term on the right hand side vary continuously and can assume an infinity of values, whereas the last term can only take a finite number of values. It follows that (5.37) can not hold. This contradiction implies that the solution with no ordering change is unique for all $0 < a < a_0$ for some $a_0 > 0$. $\square$

5.8 Proof Sketches for Properties of Theorems in Section 4

Here, we present summary of proofs of Theorem properties in Section 4. For complete proofs, we refer the reader to [73], [86], [74], and [75].

The swarm size expressions in Theorems 4.1, 4.2, 4.4, 4.3, 4.5, and 4.6 are obtained by computing pairwise distances in (4.1), (4.5), (4.17), (4.14)(4.20), and (4.22) for $i = N$, $j = 1$, and taking the absolute value of the resulting expression. We deduce that the swarm size is bounded by noting that each term in the swarm size expression is bounded.

Swarm center trajectories in Theorems 4.1, 4.2, 4.4, and 4.3, are obtained by setting $\alpha_k \rightarrow 0$ in (4.1), (4.5), (4.17), (4.14). On the other hand, swarm size is obtained by computing $x_c(t) = \frac{1}{N} \sum_{i=1}^N x^i(t)$ where $x^i(t)$ is given in (5.13) for Theorems 4.5, and 4.6.

Chapter 6

A Comparison Among Swarm Characteristics

6.1 Nash Equilibria of Game 1 versus Game 2

We observe three main differences in how distance evolves in $[0, T]$ between agents:

i) In swarms with complete information, convergence rate of all pairwise distances are dictated by the same inverse time constant $\sqrt{Na}$, while in swarms with partial information the convergence rate is determined by a combination of $N - 1$ distinct constants $\sigma_k\sqrt{a}$ of (4.11). This implies that complete information results in a more uniform convergence and, hence, gives a more harmonious appearance to the swarm.

ii) If $N \geq 4$, then $\sigma_k < \sqrt{N}$ for all $k = 1, \dots, N - 1$, and consequently, each $|x^i(t) - x^j(t)|$ converges to zero strictly faster in the complete information case than in the partial information case. (We mention in passing that the same conclusion also applies to each trajectory $x^i(t)$, not only to pairwise distances). This further implies that the size of swarms with complete information decays or expands faster than the size of swarms with partial information.

iii) While $x^i(t) - x^j(t)$ depends only on the initial distance of that pair in complete information, it depends on the initial conditions of all other pairs for swarms in partial information case. The restricted information during the journey makes the swarm members more dependent on the initial conditions—a rather interesting consequence.

We also note, by Theorem 4.2 that, the swarm-center expressions in both cases are exactly the same, (4.7). This is of course due to the preservation of initial ranks in the swarm and the resulting “symmetry in information” in both types of swarms.

6.2 Nash Equilibria in Four Games of Swarm

Games 1, 2, L1, and L2 will now be compared focusing on the resulting Nash equilibria. We compare only the specified terminal condition versions of these four games. This is merely for convenience since under specified target location, the trajectories end up exactly at $x^1(T) = 0, \dots, x^N(T) = 0$ so that the resulting trajectory expressions are all more compact. Similar analyses and conclusions are valid also in the cases of unspecified or free terminal condition.

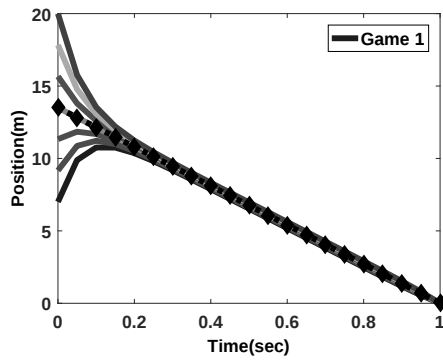
The first significant difference of the four games is that the initial ordering may change in Game L2 when at least one agent is different from the rest, i.e., it is not the case that $a_i = a$ for all i ! This is not possible in Games L1, 1, or 2. In Games 1 and 2, the attraction and the repulsion parameters were assumed to be the same across the swarm population, i.e., the individuals across each swarm were assumed to be identical. The above conclusion was observed to be valid in Games 1 and 2, even when we allowed different values for attraction coefficients in the same swarm. To be able to compare other differences and similarities among the four games, we now make the assumption that $a_i = a_j = a$ and $r_i = r_j = r$, for all i, j in Games L1 and L2. Moreover, we assume that the population N is “relatively large” in making the comparisons among dependence on initial conditions and among maximum swarm sizes. This has the effect of making the information structure of Game 2 disadvantageous because the bordering agents, agents 1 and N do not keep track of the positions of each other, except in a very indirect manner. The following table is formed using Theorems 4.1 and 4.2 above and the properties and formulae in Games 1 and 2.

	Game 1	Game 2	Game L1	Game L2
Change of Order	No	No	No	No*
Convergence Rate (α_k)	$\sqrt{N}\sqrt{a}$	$2 \cos(\frac{k\pi}{N})\sqrt{a}$	$\sqrt{k-1}\sqrt{a}$	$\sqrt{a}$
Trajectory of the Swarm Center	Line	Line	Hyperbolic	Hyperbolic
Correlation with $x^i(0)$	Very High	Very low	Low	High
Maximum Swarm Size	Very small	Very large	Small	Large
* Only in case of identical agents				

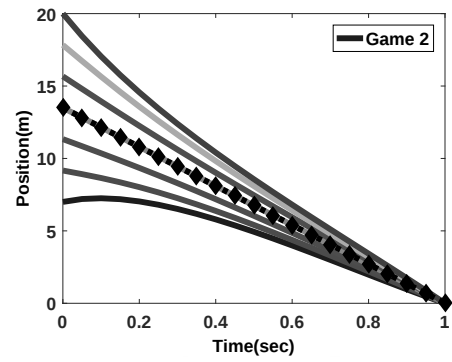
Table 6.1: Four swarm games compared

We can observe from this table that the convergence rate in all cases is directly proportional to the square root of adhesion. Its dependence on swarm population N is interesting. Only in Game L2, the convergence rate is independent of the population of the swarm, whereas in Game 1 it is proportional to the square root of N . In Games 2 and L1, the convergence rate of the agent's trajectory depends on its position in the queue and increases as we go down to the last agent in the queue. The back-and-forth symmetry in the information structure implies that the center of the swarm follows a line. In the leader games, it plots a function of the same type as agent trajectories. As the information exchange among the swarm members gets sparser, the dependence of the trajectories on the initial conditions get lower. The consequence of this is that the initial positional configuration in a swarm is less preserved in looser information structures. Maximum swarm size values follow a similar pattern to this, in that, looser the information exchange, larger is the maximum swarm size.

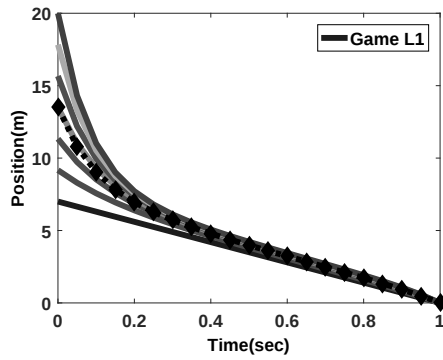
Figure 6.1 verifies these conclusions on example trajectories that one observes in the resulting Nash equilibria in all four games of swarm with population $N = 7$ and $T = 1$ under the same initial positions, attraction, and repulsion parameter values $a = 40$, $r = 10$. The trajectories of the swarm centers are shown marked with diamonds.



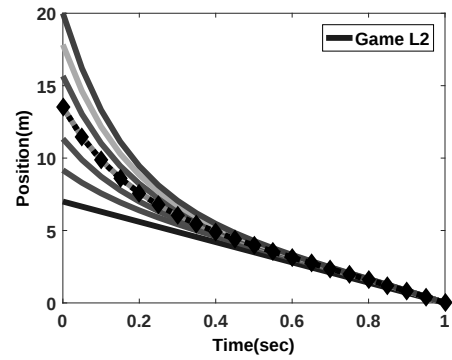
(a) Game 1.



(b) Game 2.



(c) Game L1.



(d) Game L2.

Figure 6.1: Comparison of optimal trajectories of Games 1, 2, L1, and L2

Chapter 7

Some Additional Simulations

Simulations have been performed to verify the formulas derived in Theorems 4.1 and 4.2 and to detect other features of the swarming behavior than those already mentioned above. The simulations were conducted for $N = 10$ agents for random

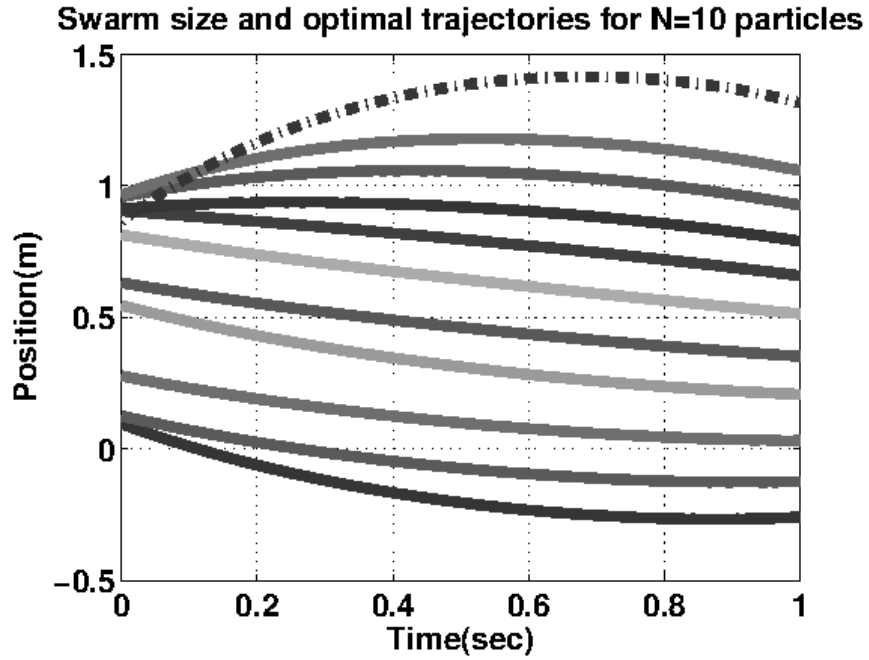


Figure 7.1: Game 1: Optimal trajectories and the swarm size for ten agents for the free terminal condition case.

initial conditions between 0 and 1. The swarm model parameters were selected as

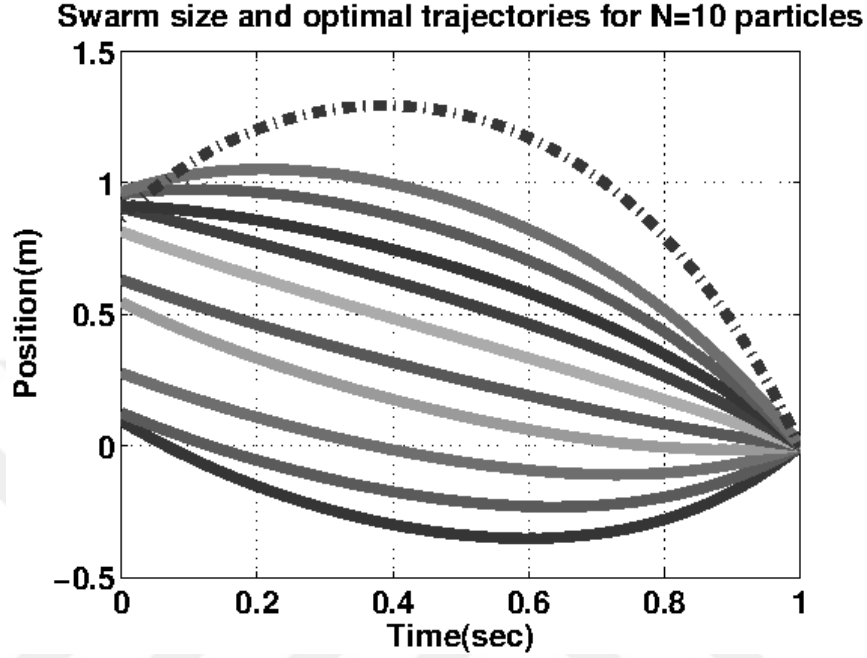


Figure 7.2: Optimal trajectories and the swarm size for ten agents for the specified terminal case.

$f = 1/2$, $a = 1/2$ and $r = 1$. The simulation duration was chosen as $T = 1$. The same set of initial conditions and parameters were used for the two cases in order to make comparisons unbiased. Examples of the optimal trajectories are shown in Figures 7.1 and 7.2 for one set of initial conditions. The swarm size plots are shown with dashed lines in these figures. The features observed were similar in all other simulations.

The maximum swarm size for the free terminal state case is 1.4149 and the corresponding value for specified terminal state case is 1.2938. All the predicted features by Theorems 4.1 and 4.2 are confirmed. One additional observation, in the free terminal case is that no matter how nonuniform the swarm initial conditions are, the swarm evolves into a regular form, i.e., the distances between consecutive agents get more uniform after some time [87]. It is an important property of formation control [88] that the regularity is also maintained, given sufficient time, in case of departures or new entries into the swarm. This is also verified in Figure 7.3 in which three individuals depart at second 1 from the agents of Figure 7.1 and two new agents join the swarm at second 2.

Although formal expressions for the inputs $u^i(t) = \dot{x}^i(t)$'s can be derived, these

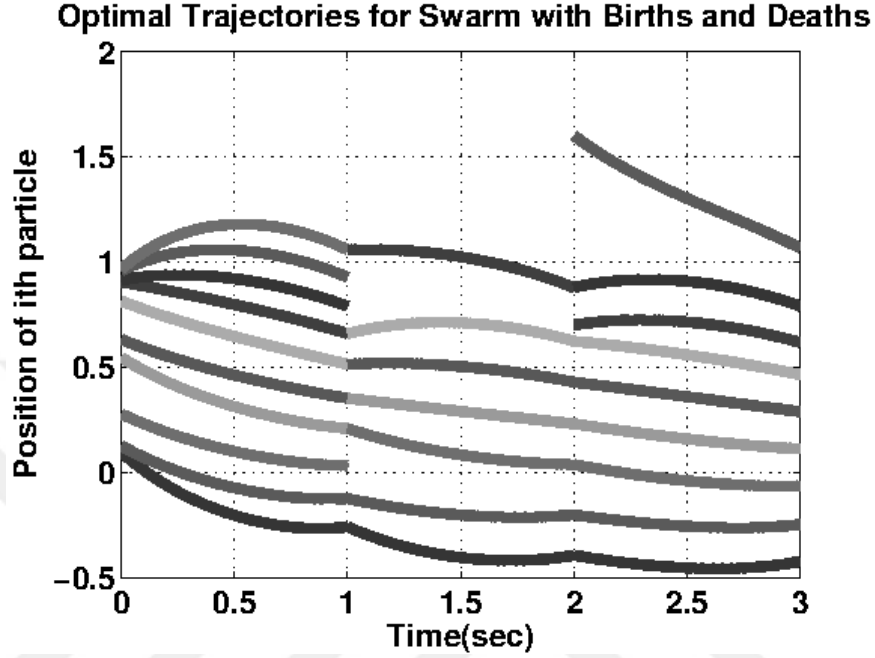


Figure 7.3: Trajectories for a swarm with departures and arrivals at different times.

are lengthy and not given here. However, in all simulations, their sizes remain within reasonable limits. For instance in the free terminal state simulation above, the plots of inputs (velocities of the agents) are as given in Figure 7.4.

Note that the sign of the velocities change for each agent and the change occurs at a different time instant for each agent. This implies that each agent changes direction during foraging activity. This can also be seen in Figures 7.1 and 7.2 as the positions first increase and then decrease after reaching a maximum at t^* in the open interval $(0, T)$. The change of direction during swarming is a well known phenomena commonly observed in many actual swarms [89]. For instance, abrupt changes occur in the direction of birds in the foraging flocks. After their maneuvers, the agents start to navigate towards the foraging location. In Figure 7.4, an example of such a behavior is observed, but in case of a one-dimensional motion.

Simulations also indicate that the Nash equilibria are still confined to those in which initial ordering is preserved even if the strategy spaces of agents are enlarged to include those that are discontinuous with respect to initial positions. A theoretical justification of this observation is left for future work.

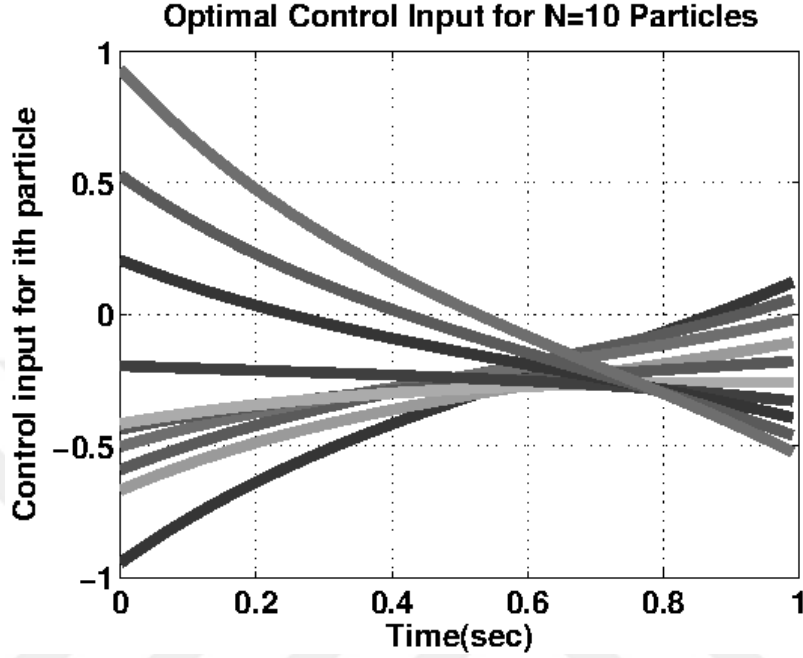


Figure 7.4: Game 1: Optimal control inputs (velocities) for the free terminal condition case.

Additional simulations were performed to compare swarms with complete and partial position information regarding issues like time constants of the trajectories, swarm size, and pairwise-distances.

As an example, we present in figures 7.5 and Fig. 7.6, a swarm with $N = 7$ members, whose initial positions were selected uniformly between 1 and 20. In this particular example, the model parameters were selected as $T = 1$, $a = 4$, $r = 2$, and $f = 0.4$. Optimal trajectories of swarms with complete and partial information were plotted in the same graph for the same terminal condition specification. The curves with solid lines represent the optimal trajectories of a swarm with complete information and dashed lines represent the optimal trajectories of a swarm with partial information.

The trajectories demonstrate that both swarms have a decent swarming behavior with cohesive aggregation and flocking at the target. The trajectory of the swarm center turns out to follow the same curve for both type of swarms due to symmetry. It is also clearly observed that the swarm with complete information converges to the target location faster. The swarm members with partial information are more likely to preserve their initial structure as observed in Figures 7.5 and 7.6. This is due to the

**Optimal Trajectories
of Swarms with Complete and Incomplete Information
for Specified Terminal Condition**

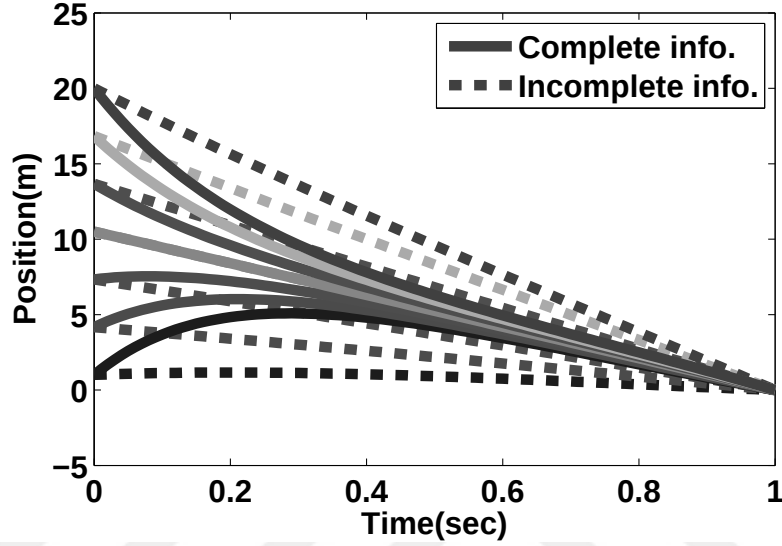


Figure 7.5: Optimal trajectories juxtaposed in Games 1 and 2 under specified terminal condition

fact that swarms with partial information are more dependent on the initial conditions as was shown in Section 6.

We have also examined the effect of change in swarm parameters during the journey as well as that of arrivals and departures to a swarm. In our example scenario, the swarm parameters, namely, a , r , and f , are assumed to be piecewise constant during four stages. Thus, these parameters have constant values in each time segment and they change at each boundary. Additionally, some members depart from the swarm while some other members arrive due to, for instance, birth or death. Simulations have been performed under these relaxations and the resulting trajectories are as shown in Figure 7.7, which demonstrates that the swarm members maintain their cohesive structure and they get progressively closer to the target. Thus, a reasonable swarming behavior occurs even under such relaxations.

**Optimal Trajectories
of Swarms with Complete and Incomplete Information
for Free Terminal Condition**

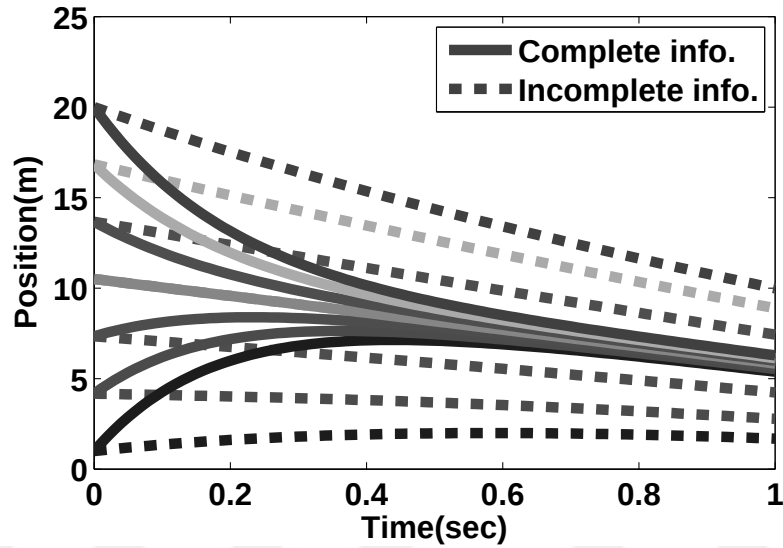


Figure 7.6: Optimal trajectories juxtaposed in Games 1 and 2 under free terminal condition.

**Optimal Trajectories for a Time Varying Swarm with Departures and Arrivals
Free Terminal Condition
Incomplete Information Case**

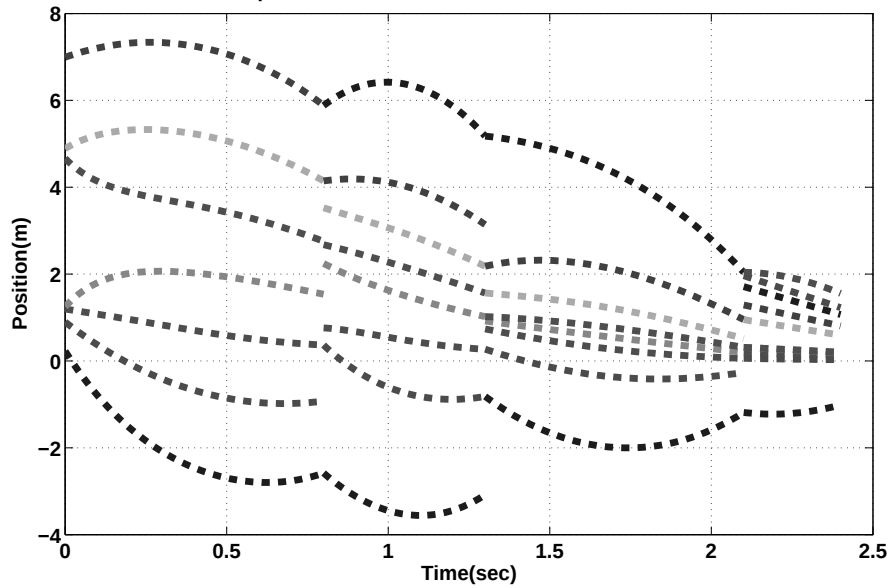


Figure 7.7: Game 2: Effect of departures and arrivals

Chapter 8

Conclusions

We have been able to establish that a swarm behavior may result as a Nash equilibrium, under four different assumptions on the topology of the information structure. Even when most members have partial information and observe their distance to one or few other members, as in Games 2, L1, and L2, Nash equilibrium still results under moderate assumptions on the size of the foraging, attraction, and repulsion coefficients. This is a significant instance of non-collaborative (no active leader, no cooperation) actions ending up in a collective behavior in a dynamic environment.

Comparing Games 1 and 2, we observe that the prices paid in going from the complete to partial information assumptions are: a slower convergence to the foraging location, more dependence on the initial conditions, and having to additionally assume an ordering relation such as the attraction parameter a is small, or equivalently, that the repulsion parameter r is large. It is of course an open question whether a Nash equilibrium in Game 2 exists without the assumption of a small or large attraction parameter a . The simulations and our intuition indicate that the answer is “yes.”

In the games of leader-follower information structures, the existence of a foraging task is irrelevant for aggregation stability and a swarm-like behavior still results. The accomplishment of the foraging task, when it is there, depends on whether the desire of the foremost member is at least as strong as the other swarm members.

The dynamic game model introduced in this thesis can be generalized in different

directions. The first interesting generalization would be to extend the results on the one-dimensional motion presented here to two and three dimensional space. The model would then be applicable to robot motion planning. Furthermore, the cost functionals used by the agents and the foraging terms in them can be made more general to cover other possible and credible motives for agents. The main result of this study, that a swarming behavior can result as a Nash equilibrium of a non-cooperative game played by individuals, is expected to be true in all these generalizations.

All features of swarm behavior that have been inferred through this one-dimensional study are expected to shed light on two- or three-dimensional swarm behavior. The conclusions reached under different information structures, their similarities and differences, we expect will also hold in higher dimensions. In higher dimensions, our efforts will focus on explaining, for instance, the v-formation observed in flocks of birds. Which motive, i.e., the cost functional used by a member in our case, leads to this energy saving formation as explained in [10]. Another question we may focus on is: What type of spontaneous geometric pattern formations are possible if the agents use similar cost functionals as we have used here in one-dimension. Because, even in the two-dimensional case of planar motion, the possibilities are enormous when compared with the simple queue formation of one dimension.

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