Resolvable Uniform Cycle Systems in Complete Graphs

by

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and have found that it is complete and satisfactory in all respects,
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ABSTRACT

Cycle decomposition of a graph G is a collection of edge-disjoint cycles G_1, G_2,...., G_r of G such that each edge of G belongs to exactly one of those cycles. If the cycles can be partitioned into classes in such a way that the cycles in a given class are vertex disjoint, and their union is a spanning subgraph of G, then this decomposition is called *resolvable cycle decomposition of* G. Each class in a resolvable cycle decomposition is called a *parallel class* of that decomposition. If all the cycles have the same length in a decomposition, then the decomposition is called a *uniform cycle decomposition*. This thesis is a survey on uniform resolvable cycle decompositions of complete graphs.

ÖZETÇE

G çizgesinin döngü parçalanması, kenarları G çizgesinin kenarlarını kenar ayrık bölen, her biri G çizgesinin döngüleri olan G_1,G_2,....,G_r ile gösterilen topluluktur. Eğer bu döngüler, köşe ayrık ve kapsayan alt çizgeler şeklinde sınıflara ayrılabiliyorsa, bu parçalanmaya, G'nin *çözünebilir döngü parçalanması* adı verilir. Bu çözünebilir döngü parçalanmasının her bir sınıfına ise *paralel sınıf* denir. Eğer parçalanıştaki tüm döngüler aynı uzunlukta ise bu parçalanışa *düzgün çözünebilir döngü parçalanışı* denir. Bu tez, tam çizgelerin düzgün çözünebilir döngü parçalanışları üzerine bir araştırma niteliğindedir.

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INTRODUCTION

Combinatorial design theory is a part of combinatorial mathematics that deals with the arrangement of elements of a finite set satisfying generalized concepts of balance and symmetry. Although it has roots in recreational mathematics and statistical theory of experimental design, it evolved in the twentieth century into a mathematical discipline with various applications in statistics and computer science with its own aims, methods and problems. Some of its applications can be stated as scheduling, lotteries, mathematical biology, algorithm design and analysis, networking, group testing, and cryptography [3]. Main problems in design theory are the existence, enumeration and classification of discrete objects, structural properties and applications of them [18, 19].

Design theory has a strong connection with graph theory. Most of the problems in design theory can be solved using graph theory concepts and tools. Since the problem considered in this text is solved using those tools and concepts, some useful and related definitions are given here. It is assumed that some basic notions such as a graph, a complete graph, a cycle, etc. are already known by the reader.

1.1 Basic Definitions

An *isomorphism* of graphs G and H is a bijection between the vertex sets of G and H. If this bijection is denoted by f, then

$f:\ V(G) \to V(H)$

with the property that any two vertices u and v are adjacent if and only if $f(u)$ and $f(v)$ are adjacent in H. If such an isomorphism exists between G and H, then these graphs are said to be *isomorphic*. Since isomorphism preserves structure of the graphs, all of the properties that those graphs have are exactly the same.

A *regular graph* is a graph where each vertex has the same number of adjacent vertices, in other words, every vertex has the same degree.

A *factor of a graph* G is a subgraph which has exactly the same vertex set as G. A k*-factor of a graph* is a spanning k-regular subgraph. If the edges of the graph can be partitioned into edge-disjoint k-factors, then this partition is called a k*-factorization of* G. In particular, a 1-factor is a set of edges, no two of which are incident to a common vertex with the additional property that every vertices of the graph are seen. A 2-factor is a set of cycles that spans every vertices of the graph. A 2-factorization is a collection of 2-factors such that the edge-disjoint union of these factors give the original graph.

A *bipartite graph* is a graph whose vertices can be divided into two disjoint sets U and V such that all the edges of the graph connects a vertex in U and a vertex in V. Equivalently, a graph that does not contain any odd cycles is called bipartite. By the latter definition, the bipartite 2-factor can be considered as a 2-factor consisting only of even length cycles.

Let $\mathbf{H}$ be a family of graphs. $\mathbf{H} = \{m_1H_1, m_2H_2, ..., m_lH_l\}$ where each H_i appears m_i times in $\mathbf{H}$. The decomposition of G is defined to be the partition of the edges of G into $\sum_{i=1}^{l} m_i$ edge-disjoint subgraphs such that m_i of them are isomorphic to H_i for each $i = 1, 2, ...l$.

A *cycle decomposition of a graph* is a decomposition such that each subgraph in the decomposition is a cycle. In other words, if all of the H_i's of the previous definition are selected to be cycles, then the decomposition of G is said to be cycle decomposition. Since every vertex in a cycle has a degree of 2, every vertex in a graph should have even degree to be cycle decomposable. When the complete graphs are taken into consideration, by the fact previously stated, the graph should have an odd number of vertices. If the total number of vertices is even, then to make all the degrees even and equal, a 1-factor should be removed from the graph.

Let $\mathbf{H} = \{H_1, H_2, ..., H_l\}$ be a cycle decomposition of a graph G. If the graphs H_i can be partitioned into classes in such a way that the H_i in a given class are vertex-disjoint, and their union is a spanning subgraph of G, then $\mathbf{H}$ is said to be *resolvable decomposition of* G and each class is called a *parallel class* of that decomposition. If all cycles of a resolvable cycle decomposition of the graph have the same length, that is $|H_i| = k$, the decomposition is then called *uniform resolvable cycle decomposition* or equivalently, *resolvable k-cycle decomposition.*

1.2 Example: Resolvable 3-cycle Decomposition of K_9

In order to concretize the definitions and explanations above, an example of a uniform resolvable 3-cycle decomposition of K_9 is given below by Table 1.1 [15]. Let $V(K_9) = \{1, 2, ..., 9\}$ and the cycle length is fixed to be 3, then all of the parallel classes are shown below denoted by π_i. Since there are totally 9 vertices and each cycle contains 3 vertices, the number of cycles in each parallel class is $\frac{9}{3} = 3$. Since each vertex has a degree of 8 and in each parallel class 2 degrees are considered, there are totally $\frac{8}{2} = 4$ parallel classes.

Table 1.1: Parallel Classes

π_1	π_2	π_3	π_4
1 2 3	1 4 7	1 5 9	1 6 8
4 5 6	2 5 8	2 6 7	2 4 9
7 8 9	3 6 9	3 4 8	3 5 7

This example is known as Kirkman Triple System of order 9 which is the resolvable decomposition of complete graphs into cycles of length 3.

Throughout the text, the uniform and resolvable cycle decomposition of the complete graph K_n or complete graph subtracted a 1-factor $K_n - F$ depending the parity of n will be considered. The number of the parallel classes is calculated by dividing the degree of a vertex by 2 which is found to be $\frac{n-1}{2}$ and $\frac{n-2}{2}$ respectively. The number of cycles of each parallel class is calculated by dividing the total number of vertices by the length of the cycles. Let the cycle length be m, then there are $\frac{n}{m}$ cycles in each parallel class. The fact that $m|n$ is known as the necessary condition for the resolvable uniform cycle decomposition of a graph with n vertices.

1.3 Oberwolfach Problem

One of the most popular problems in combinatorics, which is known as the Oberwolfach Problem is related to the cycle decompositions. So, this famous problem and the relation of it with the main subject is presented here.

The Oberwolfach Problem was first posed in 1967, when mathematicians studying graph theory gathered for a conference in Germany, Oberwolfach. The problem is really simple to visualize but not straightforward to solve. The general Oberwolfach Problem: Is it possible to seat an odd number n of mathematicians at m round tables in $\frac{n-1}{2}$ meals so that each mathematician sits next to everyone else exactly once. If the m round tables are of sizes $k_1, k_2, ..., k_m$ (with $k_1 + k_2 + ... + k_m = n$) then the corresponding Oberwolfach problem is denoted as OP$(k_1, k_2, ..., k_m)$ [20].

For example, let's consider 11 people with three tables one having 3 seats and the other two having 4 seats each. The solution for this problem is given by Table 1.2 below [20]. The people are labeled by the set $\{0, 1, ..., 10\}$ and the table below shows that everyone is sat with the others exactly once in different meals.

Table 1.2: Meal Arrangement

Meal 1	Meal 2	Meal 3	Meal 4	Meal 5
0,10,5	1,10,6	2,10,7	3,10,8	4,10,9
1,4,6,9	2,5,7,0	3,6,8,1	3,7,9,2	5,8,0,3
8,7,3,2	9,8,4,3	0,9,5,4	1,0,6,5	2,1,7,6

The Oberwolfach problem can be considered in a graph theoretical perspective. Each person can be viewed as a vertex, the edges are between those vertices and each table is a cycle. This leads us to think it as a complete graph with n vertices, the number of the seats of each table as the length of the cycles. So the question is turned into a combinatorial problem: Is it possible to decompose the complete graph K_n into cycles of length $k_1, k_2, ..., k_m$ (with $k_1 + k_2 + ... + k_m = n$) in a resolvable way? If we let $k_1 = 3$ and $k_2 = k_3 = 4$, a solution of the above problem can be viewed as decomposing K_{11} into one 3-cycle and two 4-cycles for five times such that in each decomposition, different edges of K_{11} are used. Table 1.2 given above represents the meal arrangements, or, in other words, a solution to $\mathrm{OP}(3, 4, 4)$.

When the Oberwolfach problem was first posed, the cycle length (number of mathematicians around tables) was not considered to be uniform. However, the difficulty of having different cycle lengths led mathematicians study this problem with fixed length cycles. This case is clearly equivalent to the resolvable decomposition of complete graph into cycles of uniform length.

As stated above, the necessary conditions for the uniform m-cycle decomposition of complete graph K_n is $m|n$ and n should be odd. However, the parity problem is accomplished by removing a 1-factor from K_n when n is even. The main question should be then whether the first condition is sufficient or not. In other words, if the cycle length divides the number of vertices of a complete graph, is it possible to decompose the complete graph or complete graph with one factor removed to those cycles in a resolvable way? This question is answered for different cases of m and n by several mathematicians and the aim of this thesis is to exhibit these studies.

1.4 Studies on Resolvable Cycle Decompositions

The first result was published by D. K. Ray-Chaudhuri and R. M. Wilson in 1971 in their paper *"Solution of Kirkman's schoolgirl problem"* [17]. They have shown that there exists a 3-cycle factorization of K_{3m} for every odd integer m. The case where m is even is studied by several mathematicians and by the contribution of R. D. Baker, R. M. Wilson, A. F. Brouwer, E. Mendelshon, C. Huang, A. Kotzig and A. Rosa, it is shown that $K_{6m} - F$ has a resolvable 3-cycle decomposition except for $m = 1$ and $m = 2$ [4, 8, 12, 13]. By these results, the Oberwolfach problem for uniform cycle length of 3 is completely solved and it is proven that a resolvable uniform 3-cycle decomposition of K_n for odd n and $K_n - F$ for even n exists except for $n = 6$ and $n = 12$.

Another study concerning to this famous problem was done by B. Alspach and R. Haggkvist and published in 1985 with the paper titled *"Some Observations on the Oberwolfach Problem"* [1]. This paper contains the proof of a 2-factorization of $K_n - F$ where the components of each factor are only even cycles of length m. This paper has its motivation from the paper published by Haggkvist, *"A Lemma on Cycle Decomposition"*, in the same year [9]. An important proposition proven in the latter paper led the authors to prove the uniform even cycle case of the Oberwolfach Problem. This proof is exhibited in Section 2.

The case where the cycle length considered to be odd is proven 4 years after the proof of the even case is completed. B. Alspach, P. J. Schellenberg, D. R. Stinson and D. Wagner published a paper with title *"The Oberwolfach Problem and Uniform Odd Length Cycles"* dealing with the odd cycle decomposition of complete graphs [2]. This paper proved that the necessary condition is also sufficient for cycles of odd length except possibly when $n = 4m$. This paper is introduced and the proof is given in Section 3.

The remaining case, that is the C_m-factorization of $K_n - F$ where $n = 4m$ and m is odd is given by D. G. Hoffman and P. J. Schellenberg by the paper *"The Existence of C_k-factorizations of $K_{2n} - F$"* [10]. This paper includes a direct construction of the factorization. By this work, all of the cases are considered and the factorization of the complete graphs into cycles of uniform length is completed. This construction and examples of different cases are presented in the last section of this thesis.

Chapter 2

C_{2M} FACTORIZATION OF $K_{2MN} - F$

We first present some definitions and notions which will be used throughout this section.

The *wreath product* $G \wr H$ is obtained from the graph G by replacing each vertex of G with a copy of H and putting edges between all vertices of distinct copies if and only if the corresponding vertices in G are adjacent [1]. The wreath product throughout this text is generally used to replace each vertex of a graph by a fixed number of isolated vertices. For example $G \wr \overline{K}_n$ is used to replace each vertex of G by $\overline{K}_n$ which can be considered as n isolated vertices.

Two graphs G and H with the same vertex set are said to be *edge-disjoint* if they have no edge in common. The *edge disjoint union* of these two graphs is the graph with the same vertex set and the union of the edge sets of G and H. The edge-disjoint union of G and H is denoted by $G \oplus H$ [1].

A Hamiltonian cycle of a graph G is a cycle that contains all the vertices of the graph G. Hamiltonian factorization is a 2-factorization of the graph such that all of the 2-factors include only one Hamiltonian cycle.

Let G be a labeled graph with edge set $E(G)$ and $L = (H_1, H_2, ..., H_n)$ be a list of unlabeled graphs where the repetition is possible. The list L is *proper* for G if each H_i is a subgraph of G and $|E(G)| = \sum_{i=1}^{n} |E(H_i)|$. The list L is said to *pack* G if G is the edge-disjoint union of the labeled graphs $G_1, G_2, ..., G_n$ (as stated above, this may be written as $G_1 \oplus G_2 .... \oplus G_n$) where G_i is a labeled copy of H_i, $i = 1, 2, ..., n$. If the proper list $L = (M, M, ..., M)$ packs G, then the graph is said to have an $M - decomposition$. This is sometimes written as $M|G$ and read as M divides G. If every entry in the list occurs an even number of times, then the list L is said to be *even* [1]. Since the decomposition of complete graphs into cycles of a fixed length is considered throughout the text, the list will contain the factors consisting of cycles. For example, if the graph has a C_m factorization and the number of the cycles for each factor is calculated to be n, then the list will be $(nC_m, nC_m, ..., nC_m)$.

2.1 Main Theorem and Earlier Studies

For the resolvable even length cycle decomposition of a complete graph with one factor removed, the main theorem is stated below.

Theorem 2.1.1. *[1] Let $n \geq 2$ be an integer. The graph $K_{2mn} - F$ has a C_{2m} decomposition for every natural number m.*

The proof of Theorem 2.1.1 uses the following results.

Proposition 2.1.2. *[16] The graph $K_{2n} - F$ has a Hamiltonion decomposition for $n \geq 2$.*

Proof. The proof is done by direct construction.

The vertices of the graph except one are enumerated from 0 to $2n-2$ and all of the vertices are arranged in such a way that they construct a polygon with $2n-1$ corners as shown in Figure 2.1. The remaining vertex, labeled by ∞ in this case, is placed in the middle and the other vertices are placed on the $2n-1$ corners of the polygon where labels increase in the clockwise direction. Since the degree of each vertex is $2n-1$ in K_{2n}, when a 1-factor is removed, the degree will decrease to $2n-2$. So, the number of the Hamiltonian cycles will be $\frac{2n-2}{2} = n-1$. The mathematical representation of these cycles is given by C_i below and the first two cycles are sketched in Figure 2.1.

$C_i = (\infty, i, i+1, i-1, i+2, i-2, ..., i+n)$.

Note that the vertex numbers are calculated in $\mathbb{Z}_{2n-1}$.

Each C_i for $0 \leq i \leq n-2$ gives a Hamiltonian cycle and each of these cycles gives a factor of Hamiltonian factorization. The 1-factor removed for this direct construction is $(0, 2n-2), (1, 2n-3), (2, 2n-4), ..., (n-2, n)$ and $(n-1, \infty)$.

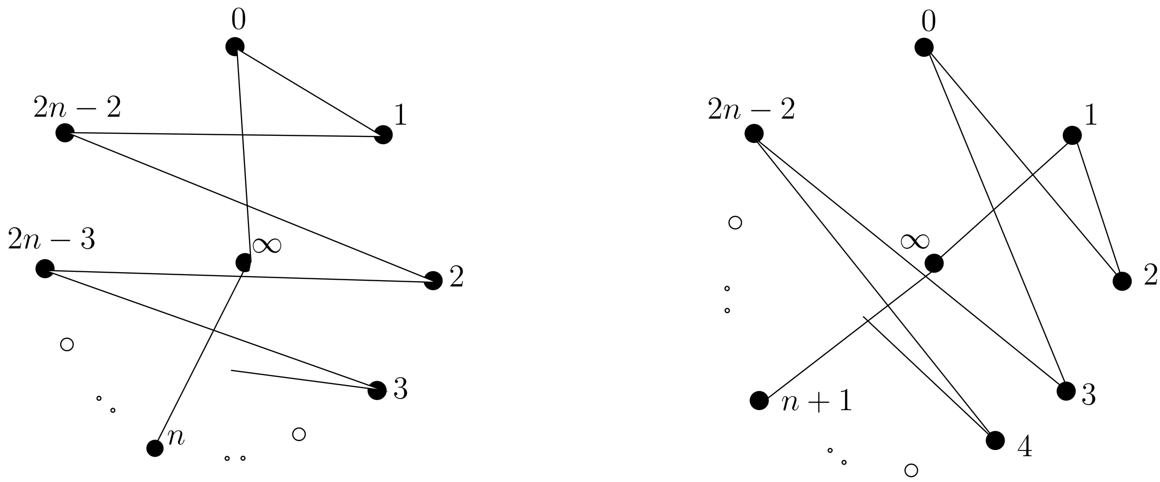

Figure 2.1: First 2 Cycles for Hamiltonian Decomposition of $K_{2n} - F$

$\square$

Proposition 2.1.3. *[14] The graph $G \wr \overline{K}_m$ has a Hamiltonian decomposition whenever G has a Hamiltonian decomposition.*

Proposition 2.1.4. *[11] The graph $K_{4m} - F$ has a C_{2m} decomposition for every $m \geq 2$.*

Proposition 2.1.5. *[9] Any proper even list of bipartite 2-factors packs $G \wr \overline{K}_2$ for every Hamilton decomposable graph G.*

The proof of this proposition is important since it is used several times in the proof of the main theorem. The proof depends on the following lemma and after the lemma is proven, the proof of Proposition 2.1.5 will be almost trivial.

Lemma 2.1.6. *[9] Let G be a path or a cycle with n edges and let H be a 2-regular graph on $2n$ vertices with all components even. Then $G \wr \overline{K}_2 = G' \oplus G''$ where $G' \simeq G'' \simeq H$.*

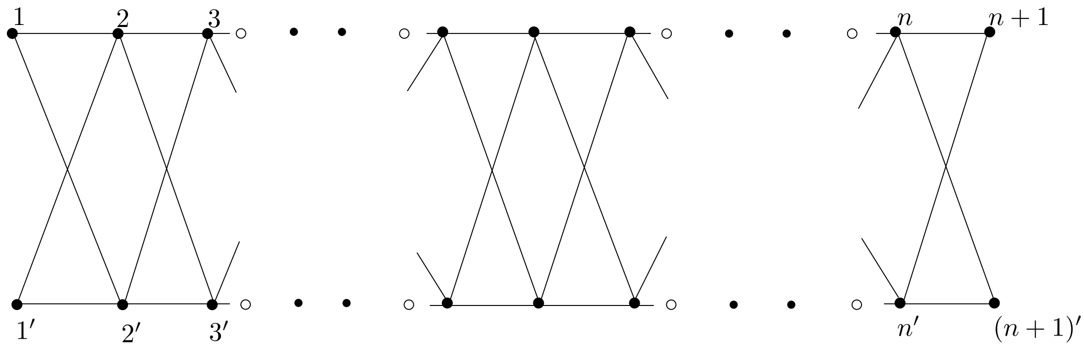

Figure 2.2: The Graph $G \wr \overline{K}_2$

Proof. The proof of the lemma is done in such a way that whether G is a path or a cycle does not make any difference. Assume without loss of generality that G is a path with n edges and $n+1$ vertices. Let the vertices of $G \wr \overline{K}_2$ be enumerated as $\{1, 2, ..., n+1\}$ and $\{1', 2', ..., (n+1)'\}$ as shown in Figure 2.2. Assume also that H consists of m disjoint cycles with lengths $2n_1, 2n_2, ..., 2n_m$ respectively with the property that $\sum_{i=1}^{m} n_i = n$. Then the cycles of H are obtained as shown in Figure 2.3. The next paragraph explains the way those cycles are obtained.

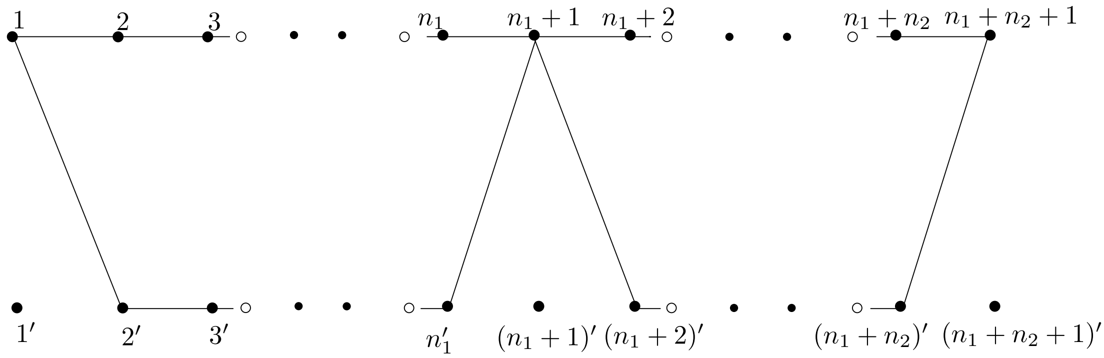

Figure 2.3: First 2 Cycles of G'

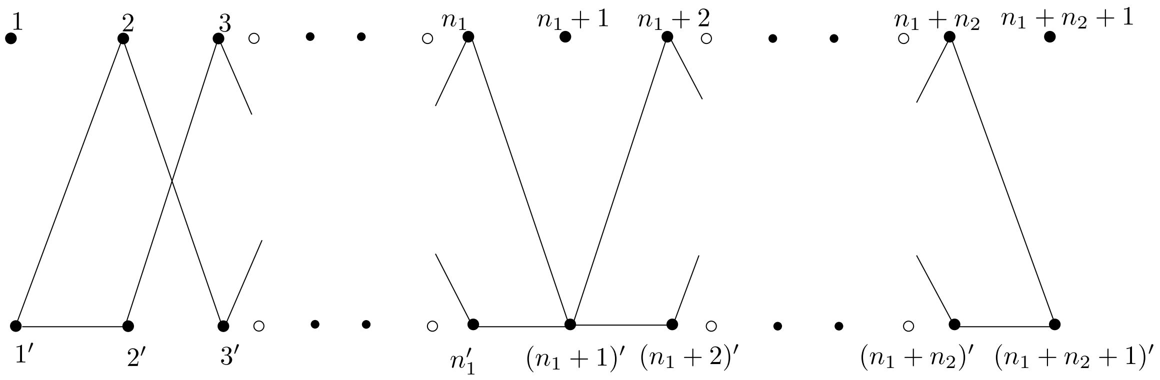

Figure 2.4: First 2 Cycles of G''

The first cycle can be represented mathematically by $(1, 2, 3, ..., n_1, n_1+1, n_1', (n_1-1)', ..., 3', 2')$. The second cycle is represented by $(n_1 + 1, n_1 + 2, ..., n_1 + n_2, n_1 + n_2 + 1, (n_1 + n_2)', (n_1 + n_2 - 1)', ..., (n_1 + 3)', (n_1 + 2)')$ All of the cycles are obtained by the same method. These two cycles are represented by Figure 2.3. In general, the cycle with length $2n_t$ takes place between the vertices $n_1 + n_2 + ... + n_{(t-1)} + 1$ and $n_1 + n_2 + ... + n_{(t-1)} + n_t + 1$ with exactly the same pattern applied in the first two cycles. The first graph G' is completed by this method.

The second graph G'' will be obtained by the edges remained. This is done by using the complement of G' in $G \wr \overline{K}_2$. The cycles are obtained without any work since the complement is already a 2-regular graph on $2n$ vertices as shown in Figure 2.4. This graph G'' again contains cycles of length $2n_1$, $2n_2$,....., $2n_m$ as its complement G' and so, $G \wr \overline{K}_2 = G' \oplus G''$ where $G' \simeq G'' \simeq H$.

One important note to be declared here is that the cycles of G' can be obtained by using any vertices and no matter which edges and vertices are used since the complement of each cycle gives an isomorphic copy of that cycle in G''. We do not have to fix a pattern and stick with it. The pattern is fixed here to simplify the explanation of the method.

Another important note which was mentioned previously is that although G is assumed to be a path in the proof, no matter whether G is a cycle or a path. If G is a cycle, then the last vertices of the graphs G' and G'' coincide. In other words, vertices 1 and $n+1$, $1'$ and $(n+1)'$ are the same but no problem can be encountered in the proof since no more edges are added. The same proof is valid for this situation.

□

Now proof of Proposition 2.1.5 does not require much work.

Proof of Proposition 2.1.5. Assume that the proper even list of bipartite 2-factors is $L = (H_1, H_1, H_2, H_2, ..., H_m, H_m)$ on $2n$ vertices. The property bipartite 2-factor guarantees that every subgraph H_i contains cycles of even length.

Since G has a Hamiltonian decomposition, let $G = G_1 \oplus G_2 \oplus .... \oplus G_m$ where each of G_i is a Hamiltonian cycle. Hence, $G \wr \overline{K}_2 = G_1 \wr \overline{K}_2 \oplus G_2 \wr \overline{K}_2 \oplus .... \oplus G_m \wr \overline{K}_2$. By Lemma 2.1.6, $G_i \wr \overline{K}_2 = H_i' \oplus H_i''$ where $H_i' \simeq H_i'' \simeq H_i$ for every $1 \leq i \leq m$. Consequently,

$G \wr \overline{K}_2 = G_1 \wr \overline{K}_2 \oplus G_2 \wr \overline{K}_2 \oplus .... \oplus G_m \wr \overline{K}_2 = H_1' \oplus H_1'' \oplus H_2' \oplus H_2'' \oplus .....H_m' \oplus H_m''$. which completes the proof.

□

Corollary 2.1.7. *Any proper even list of bipartite 2-factors packs $G \wr \overline{K}_{2m}$ whenever G has a Hamilton decomposition.*

Proof. Assume that G has a Hamilton decomposition. By proposition 2.1.3, $G \wr \overline{K}_m$ has a Hamilton decomposition. Then, the graph $G \wr \overline{K}_m$ can be put instead of G in proposition 2.1.5. Obviously $(G \wr \overline{K}_m) \wr \overline{K}_2$ is isomorphic to the graph $G \wr \overline{K}_{2m}$. Since the graph $G \wr \overline{K}_m$ has Hamilton decomposition, by proposition 2.1.5, any proper even list of bipartite 2-factors packs $(G \wr \overline{K}_m) \wr \overline{K}_2$, hence $G \wr \overline{K}_{2m}$.

□

Using these propositions and the corollary, we are ready to prove the main theorem.

Proof of Theorem 2.1.1. The proof is done considering two different cases depending on the parity of n. [1]

Case 1. Let n be odd. Then $G = K_{2nm} - F = K_{nm} \wr \overline{K}_2$ We may write G as $H_1 \oplus H_2$ where $H_1 \simeq n(K_m \wr \overline{K}_2)$ and $H_2 \simeq K_n \wr \overline{K}_{2m}$. Since $K_m \wr \overline{K}_2 \simeq K_{2m} - F$,

H_1 consists of n distinct isomorphic sets with the property that each of those sets contains complete graph consisting of $2m$ vertices with a 1-factor removed, that are isomorphic to $K_{2m} - F$. On the other hand, H_2 consists of all edges between those n sets. The edge-disjoint union of these two subgraphs gives the whole graph since all edges are considered.

Consider $H_1 \simeq n(K_m \wr \overline{K}_2)$. Since $K_m \wr \overline{K}_2 = K_{2m} - F$ and $K_{2m} - F$ has a Hamilton decomposition by proposition 2.1.2, each part of H_1 has a Hamilton decomposition. Since the cycles are Hamiltonian, each of the cycles have $2m$ vertices. Selecting an arbitrary Hamilton cycle of $K_m \wr \overline{K}_2$ for each n subset and taking the union of those Hamilton cycles give a factor of the desired 2-factorization. By this method, all of the factors are constructed.

Now consider $H_2 \simeq K_n \wr \overline{K}_{2m}$. Since n is odd, K_n has a Hamilton decomposition. Any proper even list of bipartite 2-factors packs $K_n \wr \overline{K}_{2m}$ by Corollary 2.1.7. C_{2m} factors are the candidates for the list of bipartite 2-factors. If it can be shown that a list of C_{2m} packs H_2, the proof will be completed by the definition of packing. C_{2m} is clearly a bipartite graph since it does not contain any odd cycle. The number of C_{2m} in a factor is found by simply dividing the total number of vertices of H_2 by the number of vertices of C_{2m}, that is $\frac{|V(K_n \wr \overline{K}_{2m})|}{|V(C_{2m})|}| = \frac{n \cdot 2m}{2m} = n$. So, L contains a repetitive list of nC_m, only the factors which contain n cycles of length $2m$. In other words, $L = (nC_{2m}, nC_{2m}, ..., nC_{2m})$. nC_{2m} is proper since it is a subgraph of $K_n \wr \overline{K}_{2m}$. It is remained to show that L contains an even number of nC_m's. To calculate the number of times that nC_{2m} take place in the list, the number of the edges of H_2 is divided by the number of the edges of nC_{2m}.

Since the degree of each vertex in H_2 is $(n-1) \cdot 2m$, and totally there are $n \cdot 2m$ edges in nC_m, $\frac{|E(K_n \wr \overline{K}_{2m})|}{|E(nC_{2m})|} = \frac{\frac{(n-1)\cdot 2m}{2} \cdot 2m \cdot n}{n \cdot 2m} = m \cdot (n-1)$ which is even since n is odd. So the list $(nC_{2m}, nC_{2m}, ..., nC_{2m})$ is even and proper. Hence H_2 has a C_{2m} factorization by Corollary 2.1.7.

For Case 1, both H_1 and H_2 have C_{2m} factorizations, thus, by taking the union of the factors of these factorizations, it is clear that $G = H_1 \oplus H_2$ has C_{2m} factorization and the proof for this case is completed.

Case 2. Assume that n is even. In a similar way, $G = K_{2mn} - F = H_1 \oplus H_2$ where $H_1 \simeq \frac{n}{2}(K_{2m} \wr \overline{K}_2)$ and $H_2 \simeq K_{\frac{n}{2}} \wr \overline{K}_{4m}$

$K_{2m} \wr \overline{K}_2 \simeq K_{4m} - F$, and it has C_{2m} factorization by Proposition 2.1.4. Since $\frac{|V(K_{4m}-F)|}{|V(C_{2m})|}| = 2$, each factor in the factorization of $K_{4m} - F$ consists 2 cycles of length $2m$. The union of factors from each factorization of $\frac{n}{2}$ different sets gives the factorization of H_1. As a result, the factors of factorization of H_1 consists of n cycles of length $2m$.

For H_2, two different cases on the parity of $\frac{n}{2}$ will be considered.

If $\frac{n}{2}$ is odd, $K_{\frac{n}{2}}$ has Hamilton decomposition. In order to use Corollary 2.1.7, the calculation of the number of nC_{2m} in $K_{\frac{n}{2}} \wr \overline{K}_{4m}$ is done simply by dividing the number of edges.

$\frac{|E(K_{\frac{n}{2}} \wr \overline{K}_{4m})|}{|E(nC_{2m})|} = \frac{\frac{(\frac{n}{2}-1)\cdot 4m}{2} \cdot \frac{n}{2} \cdot 4m}{n \cdot 2m} = m \cdot (n-2)$ which is even since n is even. So by corollary 2.1.7, H_2 has a C_{2m} factorization for $\frac{n}{2}$ odd.

For the remaining case, the situation is more complicated. Assume that $\frac{n}{2}$ is even. Then let $K_{\frac{n}{2}} = M_1 \oplus M_2 \oplus M_3 \oplus ...... \oplus M_{\frac{n-2}{2}}$ where $M_i = \frac{n}{4}K_2$ for $1 \leq i \leq \frac{n-2}{2}$. Since each vertex has degree $\frac{n-2}{2}$ in $K_{\frac{n}{2}}$, we are allowed to make such an edge-disjoint union and all of the edges of $K_{\frac{n}{2}}$ are considered. This edge-disjoint union can be thought as a 1-factorization of $K_{\frac{n}{2}}$ and each $M_i = \frac{n}{4}K_2$ for $1 \leq i \leq \frac{n-2}{2}$ is a 1-factor of such a factorization. Now define $M_i(4m) = \frac{n}{4}K_{4m,4m}$ for $1 \leq i \leq \frac{n-2}{2}$. Since for each $M_i(4m)$ the edges of $\frac{n}{4}$ distinct pairs of $K_{4m,4m}$ are constructed, taking the edge-disjoint union of $M_i(4m)$ for $1 \leq i \leq \frac{n-2}{2}$ will create $K_{\frac{n}{2}} \wr \overline{K}_{4m}$.

So, $K_{\frac{n}{2}} \wr \overline{K}_{4m} = M_1(4m) \oplus M_2(4m) \oplus ...... \oplus M_{\frac{n-2}{2}}(4m)$ where $M_i(4m) = \frac{n}{4}K_{4m,4m}$ for $1 \leq i \leq \frac{n-2}{2}$. Investigate that $K_{4m,4m} \simeq C_4 \wr \overline{K}_{2m}$. Since $C_4 \wr \overline{K}_{2m}$ has a Hamiltonian decomposition by Proposition 2.1.3, to be able to use Corollary 2.1.7, the number of C_{2m} factors is calculated. The number of C_{2m} for each factor is found by simply dividing $|V(K_{4m,4m})|$ by $|V(C_{2m})|$; $\frac{8m}{2m} = 4$. Then whether the proper list of bipartite 2-factors $4C_{2m}$ is even or not is found by the calculation of the number of $4C_{2m}$ factors. As in the previous case, that number is found by edge calculation: $\frac{E(K_{4m,4m})}{E(4C_{2m})} = \frac{4m \cdot 4m}{4 \cdot 2m} =$

$2m$, which is even. So according to Corollary 2.1.7, $K_{4m,4m}$ is packed by $4C_{2m}$. For each $K_{4m,4m}$, there are $2m$ factors which consist exactly of $4C_{2m}$. Thus, for each $M_i(4m) = \frac{n}{4}K_{4m,4m}$, $4C_{2m}$ factorization is obtained by taking the union of any distinct factors of $K_{4m,4m}$ from $\frac{n}{4}$ factorizations. Since the vertices of every $M_i(4m)$ $1 \leq i \leq \frac{n-2}{2}$ are common, the union of the factors of each factorizations of $M_i(4m)$ gives the desired $4C_{2m}$ factorization of H_2. Thus, for the case $\frac{n}{2}$ is even, H_2 has a C_{2m} factorization. In conclusion, H_2 has a C_{2m}-decomposition for the case n is even. Hence, the initial graph $G = K_{2mn} - F$ has a resolvable C_{2m}-decomposition for n even since both H_1 and H_2 have C_{2m} decompositions.

The proof for all cases is completed. As a result, $K_{2mn} - F$ has a 2-factorization into cycles of length $2m$. $\square$

Chapter 3

C_M FACTORIZATION OF K_{MN} OR $K_{MN} - F$ FOR ODD M

There are some definitions and notions presented below which will be used throughout this section.

If G and H are two graphs, then $G \cup H$ is the graph with properties $V(G \cup H)$=$V(G) \cup V(H)$ and $E(G \cup H)$=$E(G) \cup E(H)$ [2].

If G and H are two graphs, then the *direct product* $G \times H$ satisfies $V(G \times H)$=$V(G) \times V(H)$ and $u_{i,j}u_{r,s} \in E(G \times H)$ if and only if both $v_iv_r \in E(G)$ and $w_jw_s \in E(H)$ [2].

Let $S \subseteq \{1, 2, ..., n\}$ has the property that $i \in S$ if and only if $n - i \in S$. The *circulant graph* $Circ(n; S)$ has the vertex-set $\{u_0, u_1, ..., u_{n-1}\}$ and u_i is adjacent to u_j if and only if $(j - i) mod\ n \in S$. The *length* of an edge u_iu_j is defined as the minimum of the two elements in S $j - i$ or $i - j$ modulo n [2].

Let J be a set of positive integers. A $J - resolvable\ cycle\ design$ (denoted by $J - RCD$) is a 2-factorization of K_n,when n is odd, $K_n - F$, when n is even, such that the length of any cycle in the 2-factorization is from J [2].

If the set J consists of only one element, say m, then the cycle design consists of only m-cycles and the set parenthesis is omitted for this case. In other words, the complete graph has a resolvable factorization into cycles of length m if it has $m - RCD$.

3.1 Main Theorem and Related Constructions

The main theorem for this section is stated below.

Theorem 3.1.1. *[2] For m odd and $m \geq 5$, K_n when n is odd and $K_n - F$ when n is even has an $m - RCD$ for all $n \equiv 0\ (mod\ m)$ except possibly when $n = 4m$. In* *addition, if $m = 3$, $n \in 3\mathbb{N}\backslash\{6, 12\}$.*

Lemma 3.1.2. *[2] Let s and t be odd with $t \geq 5$ and $s \geq 3$. Then*

$$C_s \wr \overline{K}_t = (C_s \times Circ(t; \{3, 4, ..., t-3\})) \oplus C_s \times (Circ(t; \{0, 1, 2, t-2, t-1\}))$$

Proof. The graph $C_s \wr \overline{K}_t$ is obtained by replacing every vertex of C_s by t isolated vertices. So the edges are between those vertices of adjacent sets. The subgraph $(C_s \times Circ(t; \{3, 4, ..., t-3\}))$ considers the edges of length from the set $\{3, 4, ..., t-4, t-3\}$ and the other subgraph $C_s \times (Circ(t; \{0, 1, 2, t-2, t-1\}))$ considers the edges of length from the set $\{0, 1, 2, t-3, t-2\}$. By this way, all of the edges of $C_s \wr \overline{K}_t$ are considered. □

The reason that the graph $C_s \wr \overline{K}_t$ is separated into two subgraphs is to show that this graph can be decomposed into 2-factors all of whose cycles have length t when $s \leq t$ and t prime. The first subgraph and the second will be considered separately to prove this assertion. The following theorem states this claim.

Theorem 3.1.3. *[2] Let s be an odd integer and t be a prime so that $3 \leq s \leq t$. Then $C_s \wr \overline{K}_t$ has a 2-factorization so that each 2-factor is composed of s cycles of length t.*

Proof. The proof is done using the two subgraphs constructed in Lemma 3.1.2. A direct construction for the latter is done in [2]. For the first one, the parallel classes are obtained by the following construction.

The graph $(C_s \times Circ(t; \{3, 4, ..., t-3\}))$ can be considered as follows: There are exactly s number of sets where each set contains t number of vertices labeled from 1 to t with no edges between those vertices of the same set.These sets are constructed by blowing up the vertices of C_s and turning the adjacent vertices into adjacent sets. All of the edges are between these adjacent sets. The vertices with label difference from the set $\{3, 4, ..., t-3\}$ are connected by exactly one edge. In other words, there is no edge between the vertices having label difference of $\{0, 1, 2, t-2, t-1\}$, whereas the others are connected by a unique edge. These differences are calculated

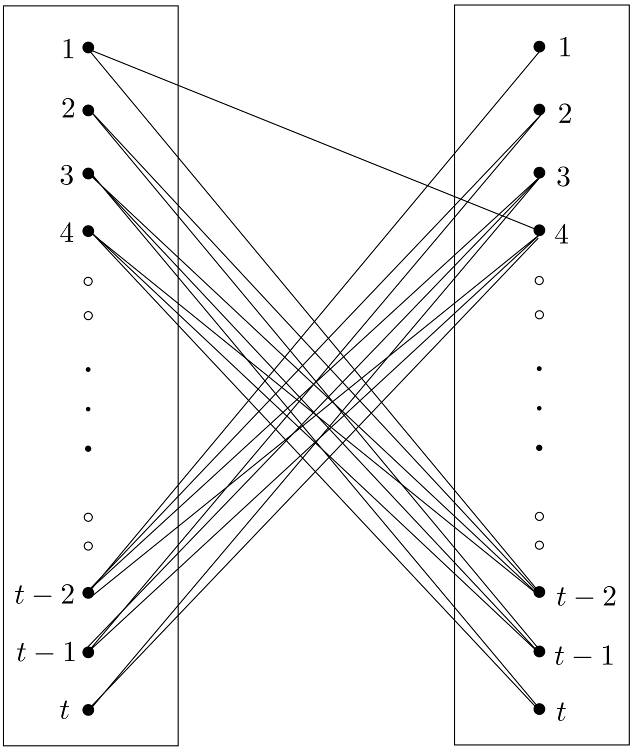

Figure 3.1: Edges Between Any Two Sets

in modulo t. Figure 3.1 shown above demonstrates the edges of any two adjacent sets. It is intended to show the edges between any two adjacent sets containing $\overline{K}_t$. However, the main picture involves s number of sets with isomorphic edge set between the adjacent sets.

The aim here is to obtain cycles of length t and there are s sets containing t isolated vertices. Since t is prime, unless $s = t$, $s \not| t$. If we select an arbitrary vertex of an arbitrary set, fix a direction to be clockwise or counterclockwise, and start edge drawings to get a t-cycle, the t^{th} vertex of that cycle will not be placed in the set we get started since $s \not| t$ for $s \neq t$. So, a t-cycle can not be obtained considering only one direction. In other words, if the direction is fixed, the t^{th} vertex will be in a different set other than the starting one.

If a number k and a difference d in $\mathbb{Z}_t$ is fixed, where $3 \leq d \leq t-3$, addition of d to k in each step will give different values until the t^{th} step is reached. This is because any d is relatively prime to the prime number t.. So, when we turn back to our situation, if an edge difference d is fixed, at t^{th} step, the vertex that we started the t-cycle drawing will be obtained. Thus, the aim for this construction should be obtaining the starting set at t^{th} step. If it is done, the last point will be exactly the starting one and a t-cycle will be obtained as desired. The construction uses this fact and the cycles are obtained by going a fixed direction, turn back at a specified set and complete the cycle using this reverse direction as shown in Figure 3.2 and Figure 3.3.

The set that the direction is reversed depends on the relationship between t and s. Let $t \equiv a \ (mod\ s)$. If a is even, then the number of edges in the first direction is $\frac{a}{2}$, whereas if a is odd, this number will be $\frac{a+s}{2}$. After these number of edges are drawn, the direction is reversed. By this way, the a^{th} edge when a is even and $(a+s)^{th}$ edge when a is odd will take place between the second set and the starting set. The two different cases for a explained are shown below in Figure 3.2 and Figure 3.3 respectively and it is easier to investigate the cycles using that figure. The number of remaining edges for each cycle ($t - a$ for a even or $t - a - s$ for a odd) will be clearly a multiple of s. Since there are s sets totally, when the t^{th} edge is drawn, the last set will coincide with the starting one and the last vertex will be the initial one as explained above. One of the important note to be declared here is that if the difference is taken to be d, the cycle will naturally contain the difference $t - d$ since the direction is reversed. Due to the fact that a vertex has a degree of 2 for each difference, it has degree of 4 totally considering differences of both d and $t - d$. Thus, two parallel classes should be obtained when one difference is taken into account.

The cycles for one parallel class are obtained by starting edge drawings from each set by a fixed vertex, say 1 for example (the vertices are enumerated from 1 to t), traveling in one direction with increasing the vertex label by d for each step, reversing the direction when $\frac{a}{2}$ (for a even) or $\frac{a+s}{2}$ (for a odd) edges are drawn, and traveling in this direction until the t-cycle is completed. A parallel class is obtained fixing a difference from the set $\{3, 4, ..., \frac{t-1}{2}\}$ and fixing the starting direction to be clockwise

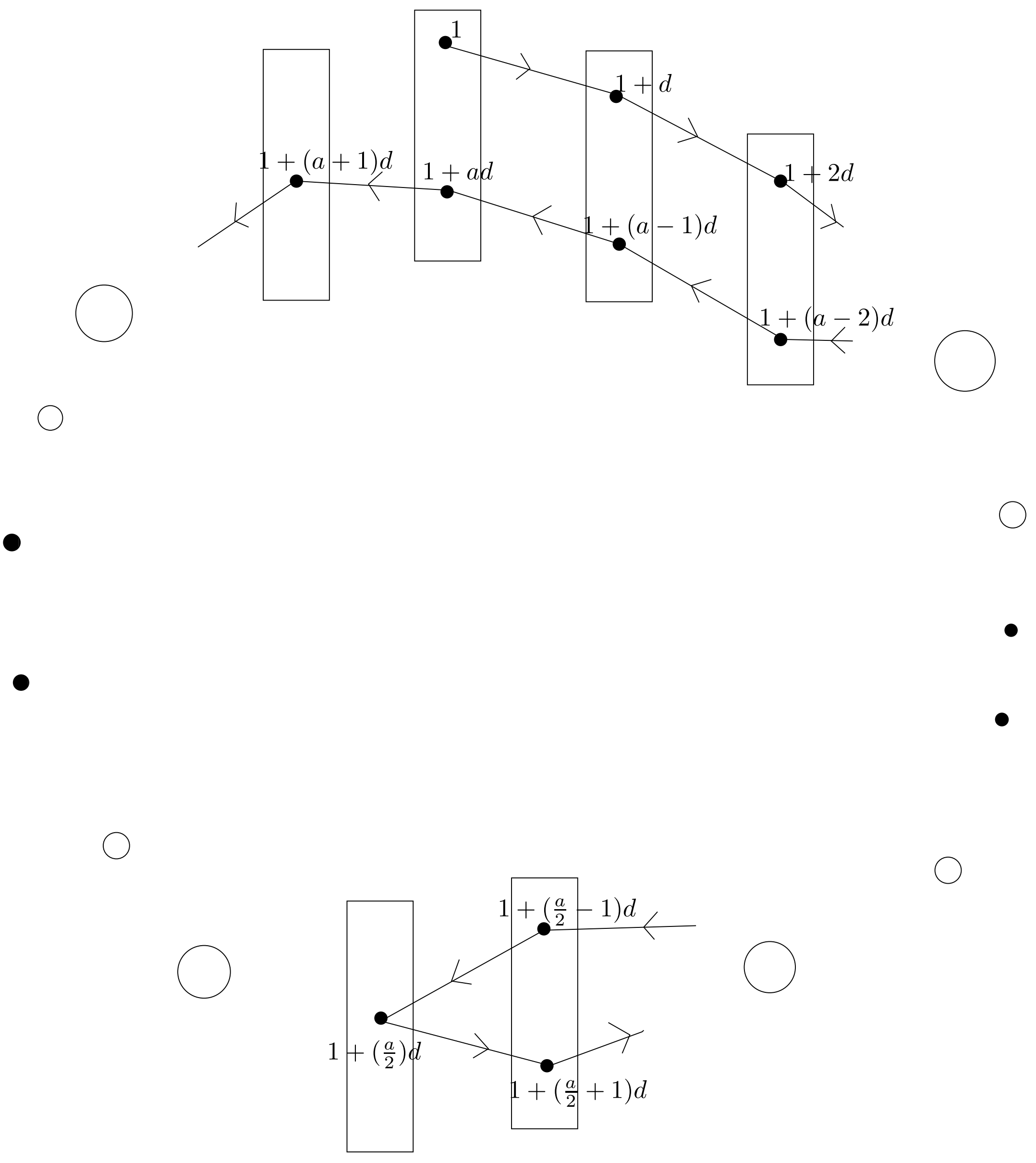

Figure 3.2: Pattern of a General Cycle for even a

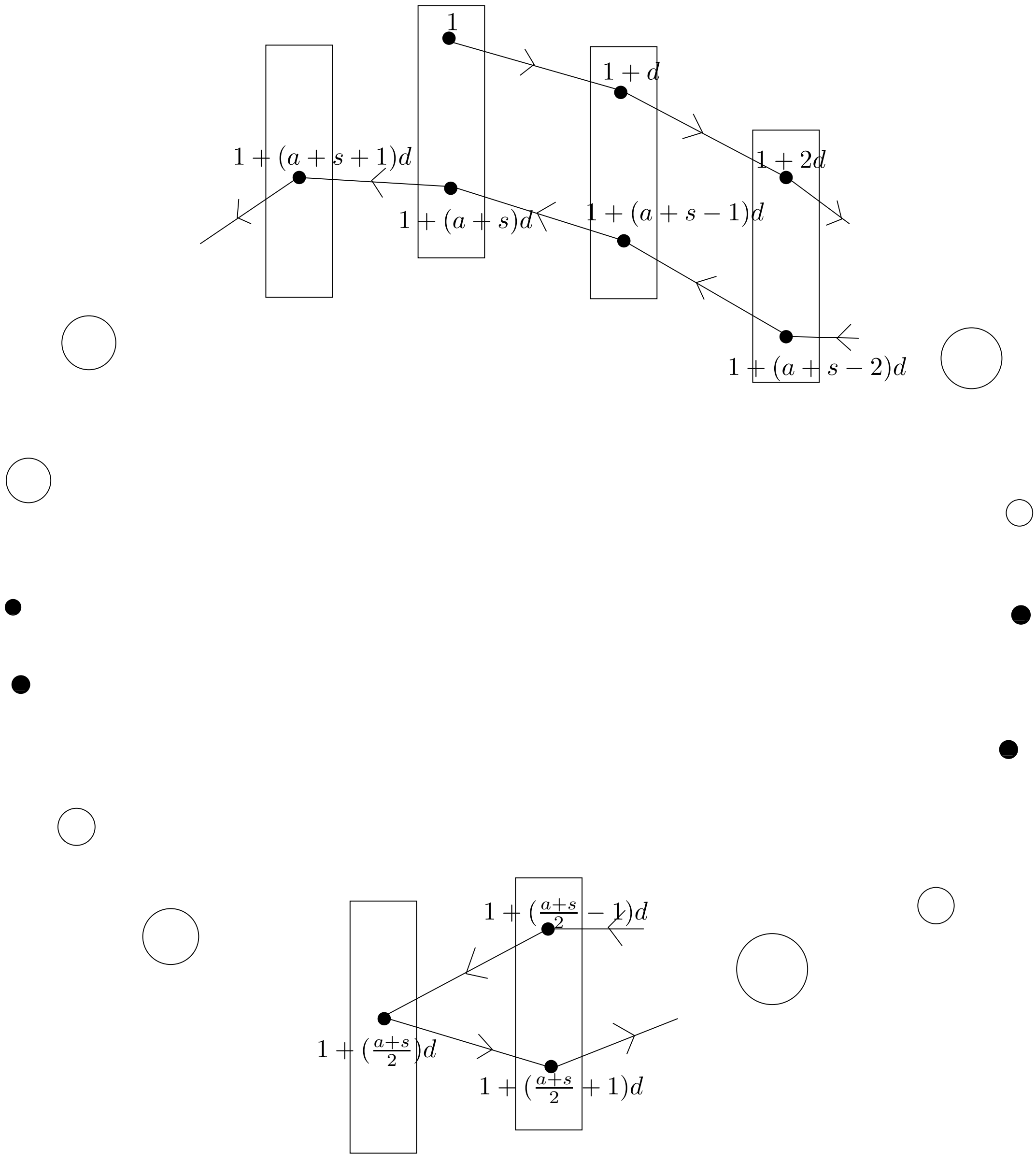

Figure 3.3: Pattern of a General Cycle for odd a

and forming the cycles using the method explained. Another parallel class will be obtained by the same method with the same difference but reversing the starting direction of the cycles to be counter-clockwise. Using the same method for every difference from the set $A = \{3, 4, ..., \frac{t-1}{2}\}$, 2 parallel classes will be obtained for each difference as explained before. So in total there will be $t-5$ parallel classes, calculated by multiplying the number of elements of A by 2, which is expected since any vertex has a degree of $2t - 10$ for this subgraph. Note that since there are totally s sets in the graph $(C_s \times Circ(t; \{3, 4, ..., t-3\}))$, there will be s cycles in each parallel class because the cycles of any parallel class are obtained by starting the cycle with the vertex enumerated 1 of each set. So totally $t \cdot s$ vertices are used in each parallel class which is expected since it is the number of total vertices of the graph.

In mathematical representation, the parallel classes are shown below. The sets each containing t vertices are enumerated from 1 to s and the vertices in those sets are enumerated from 1 to t. Let $u_{i,j}$ represents the j^{th} vertex of the i^{th} set. The edge disjoint union of the cycles C_i $1 \leq i \leq s$ constructs a parallel class for a fixed edge distance d

$C_i = (\mathbf{u}_{(i,1)}, \mathbf{u}_{(i+1,1+d)}, \mathbf{u}_{(i+2,1+2d)}, ...\mathbf{u}_{(i+\frac{a}{2},1+\frac{a}{2}d)}, \mathbf{u}_{(i+\frac{a}{2}-1,1+(\frac{a}{2}+1)d)},$
$\mathbf{u}_{(i+\frac{a}{2}-2,1+(\frac{a}{2}+2)d)}, ...\mathbf{u}_{(i+1,1+(t-1)d)}, \mathbf{u}_{(i,1+td)})$ $1 \leq i \leq s$ for a even, and
$C_i = (\mathbf{u}_{(i,1)}, \mathbf{u}_{(i+1,1+d)}, \mathbf{u}_{(i+2,1+2d)}, ...\mathbf{u}_{(i+\frac{a+s}{2},1+\frac{a+s}{2}d)}, \mathbf{u}_{(i+\frac{a+s}{2}-1,1+(\frac{a+s}{2}+1)d)},$
$\mathbf{u}_{(i+\frac{a+s}{2}-2,1+(\frac{a+s}{2}+2)d)}, ...\mathbf{u}_{(i+1,1+(t-1)d)}, \mathbf{u}_{(i,1+td)})$ $1 \leq i \leq s$ for a odd.

Note that the first subscript is taken modulo s and the second subscript is taken modulo t.

Note that the cycles represented above are obtained by fixing the starting direction to be clockwise. The other parallel class will be obtained by reversing the starting direction to be counter-clockwise following similar pattern.

In order to guarantee that all the vertices are used in each parallel class, any vertex $u_{k,m}$ is fixed. Assume that the first cycle C_1 of this parallel class contains the vertex $u_{t,m}$ and $k - t + 1 = b$ $(mod\ s)$. Then the b^{th} cycle C_b contains the vertex $u_{k,m}$ since when b steps are taken, the vertices that were encountered in set t will be encountered in set k.

The total number of edges used for each parallel class is found to be $s \cdot t$ since in a cycle, the number of vertices and the number of edges coincide. Due to the fact that there are $t - 5$ parallel classes totally, the total number of edges in the factorization is calculated to be $t \cdot s \cdot (t - 5)$ which is expected since this is consistent with the total number of edges of the graph $(C_s \times Circ(t; \{3, 4, ..., t - 3\}))$.

□

Theorem 3.1.4. *[2] If s is an odd integer and t is a prime so that $t \geq s$, then K_{st} has a $t - RCD$.*

Proof. As usual, the graph will be decomposed into two subgraphs and these subgraphs will be decomposed into t-length cycles.

$K_{st} = K_s \wr \overline{K_t} \oplus sK_t$. Since t is odd, K_t has a Hamilton decomposition. Thus, the second subgraph will have a resolvable t-cycle decomposition. The factors are constructed by pairing Hamilton cycles from each s distinct sets containing K_t. The number of 2-factors will be simply $\frac{t-1}{2}$ since any vertex in sK_t has a degree of $t - 1$.

Next, for the first subgraph, since s is odd, K_s has Hamilton decomposition. Let the Hamilton cycles be H_1, H_2, H_3,.....$H_{\frac{s-1}{2}}$. There are $\frac{s-1}{2}$ cycles of length s since every vertex has a degree of $s - 1$. So the first subgraph can be rewritten as $K_s \wr \overline{K_t} \simeq H_1 \wr \overline{K_t} \oplus H_2 \wr \overline{K_t} \oplus H_3 \wr \overline{K_t}.....H_{\frac{s-1}{2}} \wr \overline{K_t}$. The components of H_i, $1 \leq i \leq \frac{s-1}{2}$ are isomorphic to C_s. So each component of $K_s \wr \overline{K_t}$ is isomorphic to $C_s \wr \overline{K_t}$ which is known to be decomposable into cycles of length t by Theorem 3.1.3 and the result follows.

□

Another theorem which was proven by Baranyai and Szasz [5] is stated without its proof and will be used later on.

Theorem 3.1.5. *[5] If G and H can be decomposed into Hamilton cycles, then $G \wr H$ can also be decomposed into Hamilton cycles.*

Theorem 3.1.6. *[2] If st is odd and K_{st} has a $t - RCD$, then K_{stl} when l is odd, $K_{stl} - F$ when l is even have $tl - RCD$ for every positive integer l.*

Proof. Since $st \in D(t)$ given, K_{st} can be factored into cycles of length t. Let the factors be $F_1, F_2, ..., F_{\frac{ts-1}{2}}$. Due to the fact that any vertex has a degree of $ts - 1$ in K_{st}, the number of factors will be $\frac{ts-1}{2}$ since in any factor, every vertex has a degree of 2.

If l is odd, then

$K_{stl} = K_{st} \wr K_l = F_1 \wr K_l \oplus F_2 \wr \overline{K}_l..... \oplus F_{\frac{ts-1}{2}} \wr \overline{K}_l$ where the components of $F_1 \wr K_l$ are isomorphic to $C_t \wr K_l$ and the components of the other subgraphs are isomorphic to $C_t \wr \overline{K}_l$. Since C_t, K_l and $\overline{K}_l$ all have Hamilton cycles, by Theorem 3.1.5, $C_t \wr K_l$ and $C_t \wr \overline{K}_l$ have Hamilton cycles. Hence they can be decomposed into cycles of length tl by the definition of Hamilton cycle.

If l is even, then

$K_{stl} - F = K_{st} \wr (K_l - F) = F_1 \wr (K_l - F) \oplus F_2 \wr \overline{K}_l..... \oplus F_{\frac{ts-1}{2}} \wr \overline{K}_l$ where the components of $F_1 \wr (K_l - F)$ are isomorphic to $C_t \wr (K_l - F)$ and the components of the others are isomorphic to $C_t \wr \overline{K}_l$. By the same argument in the previous case, they have Hamilton decomposition according to Theorem 3.5. So for both cases, the graph can be decomposed into cycles of length tl and the proof is completed. □

In order to prove the main theorem, the following two theorems should be stated. Lemmas 3.1.9 and 3.1.10 are proven using these two theorems. In these two theorems, each factor contains cycles of length 3 or 5. Note that in a factor, both of these two cycles may appear.

Theorem 3.1.7. *[2] If n is odd, there exists a $\{3, 5\} - RCD$ of K_n if and only if $n \neq 7$ or 11.*

Theorem 3.1.8. *[2] If n is even, there exists a $\{3, 5\} - RCD$ of $K_n - F$ if and only if $n \neq 4, 6,$ or 12.*

Lemma 3.1.9. *[2] If d is an odd integer and p is an odd prime, then there is a 2-factorization of $K_d \wr \overline{K}_p$ which consists entirely of p-cycles.*

Proof. Two cases will be considered while proving the theorem depending on the value of p.

For $p = 3$, $K_d \wr \overline{K}_3$ is simply the graph obtained by removing one parallel class from Kirkman triple system of K_{3d}. For every odd integer d, there exists a Kirkman triple system of K_{3d} [11]. The remaining parallel classes give a $3 - RCD$ of $K_d \wr \overline{K}_3$.

For the case $p \geq 5$ and p prime, by Theorem 3.1.7, if $d \notin \{7, 11\}$, there exists a $\{3, 5\} - RCD$ of K_d. Let the factors of this design be $F_1, F_2, ..., F_{\frac{d-1}{2}}$, where each component of the factors is isomorphic to either C_3 or C_5. Hence

$K_d \wr \overline{K}_p = F_1 \wr \overline{K}_p \oplus F_2 \wr \overline{K}_p \oplus ...... \oplus F_{\frac{d-1}{2}} \wr \overline{K}_p.$

Each component of $F_i \wr \overline{K}_p$ is $C_3 \wr \overline{K}_p$ or $C_5 \wr \overline{K}_p$. Since $p \geq 5$, Theorem 3.1.3 implies that each of these subgraphs can be decomposed into resolvable 2-factors each consisting entirely of p-cycles. Hence, there is a $p - RCD$ of $K_d \wr \overline{K}_p$ for $d \notin \{7, 11\}$.

For the exceptional case, let $d \in \{7, 11\}$ and p be a prime satisfying $p \geq d$, then by Theorem 3.1.3, $K_d \wr \overline{K}_p$ has a $p - RCD$.

In order to complete the proof, a 2-factorization of $K_d \wr \overline{K}_p$ consisting entirely of p-cycles should be constructed for the values $(d, p) = (7, 5), (11, 5), (11, 7)$. In other words, C_5 factorization of $K_7 \wr \overline{K}_5$, C_5 factorization of $K_{11} \wr \overline{K}_5$ and C_7 factorization of $K_{11} \wr \overline{K}_7$ should be considered. This construction is done in [2]. This is a direct construction and in all three cases, a factor with its cycle components is given and the other factors are obtained by a suitable permutation of the vertices.

C_5 factorization of $K_7 \wr \overline{K}_5$: Let the vertices of K_{35} be $x_1, x_2, ..., x_5, u_0, u_1, ..., u_{14}, v_0, v_1 ......, v_{14}$. Since the graph $K_7 \wr \overline{K}_5$ is obtained from K_{35} by removing all of the edges of 7 vertex disjoint sets of K_5, these sets should be specified. Let the sets be considered as

$\{x_1, x_2, x_3, x_4, x_5\}$, $\{u_0, u_3, u_6, u_9, u_{12}\}$, $\{u_1, u_4, u_7, u_{10}, u_{13}\}$, $\{u_2, u_5, u_8, u_{11}, u_{14}\}$,

$\{v_0, v_3, v_6, v_9, v_{12}\}$, $\{v_1, v_4, v_7, v_{10}, v_{13}\}$, $\{v_2, v_5, v_8, v_{11}, v_{14}\}$.

In order to construct this factorization, all of the edges between each of the vertices of these different sets should be appeared in 5-cycles. Since there are 35 vertices totally and the number of vertices in each cycle is 5, there are $\frac{35}{5} = 7$ cycles in each factor. Let the first factor consists of the cycles below.

$(u_0, u_1, u_8, u_6, u_{10})$, $(v_0, v_1, v_8, v_6, v_{10})$ $(x_1, u_2, v_3, u_{14}, v_4)$, $(x_2, v_2, u_3, v_{14}, u_4)$,

$(x_3, u_5, v_{12}, u_9, v_{11})$, $(x_4, v_5, u_{12}, v_9, u_{11})$, $(x_5, u_7, v_{13}, u_{13}, v_7)$.

Note that all of the vertices are seen in the factor. So, this factor is a parallel class for this factorization. The other factors are obtained using a permutation. The permutation for this case is given by $(u_0u_1.....u_{14})(v_0v_1......v_{14})$. The meaning of this permutation is that the subscript of each vertex denoted by u_i and v_i for $0 \leq i \leq 14$ is increased by 1 evaluated in modulo 15 for each factor. Every vertex in a factor has a degree of 2 and every vertex in the graph has a degree of 30. So the number of the factors for this case is calculated as $\frac{30}{2} = 15$. A new factor is obtained each time this permutation is applied. Hence, the permutation will be considered for 14 times since an initial factor is given.

C_5 *factorization of* $K_{11} \wr \overline{K}_5$: The same method explained above will be applied. So, 11 different vertex-disjoint sets of K_{55} each containing 5 vertices should be considered and every edges between these vertices will be removed. Let the vertices of K_{55} be

$x_1, x_2, ..., x_5, u_0, u_1, ..., u_{24},\ v_0, v_1,\ ......, v_{24}$ similar to the previous case. Let the sets contain the vertices

$\{x_1, x_2, x_3, x_4, x_5\}$, $\{u_0, u_5, u_{10}, u_{15}, u_{20}\}$, $\{u_1, u_6, u_{11}, u_{16}, u_{21}\}$, $\{u_2, u_7, u_{12}, u_{17}, u_{22}\}$,

$\{u_3, u_8, u_{13}, u_{18}, u_{23}\}$, $\{u_4, u_9, u_{14}, u_{19}, u_{24}\}$, $\{v_0, v_5, v_{10}, v_{15}, v_{20}\}$, $\{v_1, v_6, v_{11}, v_{16}, v_{21}\}$,

$\{v_2, v_7, v_{12}, v_{17}, v_{22}\}$, $\{v_3, v_8, v_{13}, v_{18}, v_{28}\}$, $\{v_4, v_9, v_{14}, v_{19}, v_{24}\}$.

The edges in each sets are removed. The first factor is given by the cycles below.

$(u_0, u_3, u_{12}, u_{14}, u_1)$, $(v_0, v_3, v_{12}, v_{14}, v_1)$ $(u_2, u_9, u_5, u_{24}, v_{10})$, $(v_2, v_9, v_5, v_{24}, u_{10})$,

$(u_{15}, u_{23}, v_{22}, u_{20}, v_{11})$, $(v_{15}, v_{23}, u_{22}, v_{20}, u_{11})$, $(u_6, v_{13}, u_7, v_{17}, x_1)$, $(v_6, u_{13}, v_7, u_{17}, x_2)$,

$(u_8, v_{21}, u_{16}, v_{19}, x_3)$ $(v_8, u_{21}, v_{16}, u_{19}, x_4)$ $(u_4, u_{18}, v_{18}, v_4, x_5)$.

Since the number of factors in this case is calculated as $\frac{50}{2} = 25$, a suitable permutation which is given as $(u_0u_1.....u_{24})(v_0v_1......v_{24})$ should be applied 24 times to obtain the cycles of each factor similar to the previous case.

C_7 *factorization of* $K_{11} \wr \overline{K}_7$: The method is similar to the cases explained before. For this case, K_{77} is considered and 11 vertex-disjoint sets are constructed as predicted. Let the vertices of K_{77} be

$x_1, x_2, ..., x_7, u_0, u_1, ..., u_{34},\ v_0, v_1,\ ......, v_{34}$. Let the sets contain the vertices

$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, $\{u_0, u_5, u_{10}, u_{15}, u_{20}, u_{25}, u_{30}\}$,

$\{u_1, u_6, u_{11}, u_{16}, u_{21}, u_{26}, u_{31}\}$, $\{u_2, u_7, u_{12}, u_{17}, u_{22}, u_{27}, u_{32}\}$,

$\{u_3, u_8, u_{13}, u_{18}, u_{23}, u_{28}, u_{33}\}$, $\{u_4, u_9, u_{14}, u_{19}, u_{24}, u_{29}, u_{34}\}$,

$\{v_0, v_5, v_{10}, v_{15}, v_{20}, v_{25}, v_{30}\}$, $\{v_1, v_6, v_{11}, v_{16}, v_{21}, v_{26}, v_{31}\}$,

$\{v_2, v_7, v_{12}, v_{17}, v_{22}, v_{27}, v_{32}\}$, $\{v_3, v_8, v_{13}, v_{18}, v_{23}, v_{28}, v_{33}\}$,

$\{v_4, v_9, v_{14}, v_{19}, v_{24}, v_{29}, v_{34}\}$,

where the edges between the vertices of each set are removed.

The components of the first factor which are 11 vertex-disjoint cycles are shown below.

$(u_0, u_1, u_{34}, u_2, u_{33}, u_4, u_{21})$, $(v_0, v_1, v_{34}, v_2, v_{33}, v_4, v_{21})$, $(u_3, u_{22}, u_9, u_{20}, u_{32}, u_{23}, u_{30})$,

$(v_3, v_{22}, v_9, v_{20}, v_{32}, v_{23}, v_{30})$, $(u_{17}, v_{13}, u_{18}, v_{12}, u_{19}, v_{11}, x_1)$,

$(v_{17}, u_{13}, v_{18}, u_{12}, v_{19}, u_{11}, x_2)$, $(u_{26}, v_{28}, u_{15}, v_{27}, u_{16}, v_6, x_3)$,

$(v_{26}, u_{28}, v_{15}, u_{27}, v_{16}, u_6, x_4)$, $(u_{14}, v_{29}, u_{10}, v_{24}, u_{25}, v_7, x_5)$,

$(v_{14}, u_{29}, v_{10}, u_{24}, v_{25}, u_7, x_6)$, $(u_8, v_5, u_{31}, v_{31}, u_5, v_8, x_7)$.

The permutation acting on this factor is given as $(u_0u_1.....u_{34})(v_0v_1.....v_{34})$. Similar to the calculation done in the previous cases, this permutation should be applied 34 times to obtain 35 factors of this factorization.

All the exceptional cases are considered and the proof is completed.

□

Lemma 3.1.10. *[2] If $d \in 2\mathbb{N} \setminus \{4, 6\}$ and p is a prime, $p \geq 5$, then $(K_d - F) \wr \overline{K}_p$ has a $p - RCD$.*

Proof. The proof of this lemma is similar to the proof of the previous lemma. By

Theorem 3.1.8, there exists a $\{3,5\} - RCD$ of K_d for $d \in 2\mathbb{N} \setminus \{4,6,12\}$. Let the factors of this design be $F_1, F_2, ..., F_{\frac{d-2}{2}}$, where each component of the factors is isomorphic to either C_3 or C_5. Hence

$(K_d - F) \wr \overline{K}_p = F_1 \wr \overline{K}_p \oplus F_2 \wr \overline{K}_p \oplus ...... \oplus F_{\frac{d-2}{2}} \wr \overline{K}_p.$

Each component of $F_i \wr \overline{K}_p$ are either $C_3 \wr \overline{K}_p$ or $C_5 \wr \overline{K}_p$. Since $p \geq 5$, Theorem 3.1.3 implies that each of these subgraphs can be decomposed into resolvable 2-factors each consisting entirely of p-cycles. Hence, there is a $p - RCD$ of $K_d \wr \overline{K}_p$ for $d \notin \{4,6,12\}$.

For the case where $d = 12$, two cases will be considered. Huang, Kotzig and Rosa have shown that $(K_{12} - F)$ has a $\{5,7\} - RCD$ [11]. The components of the factors of $K_{12} - F$ are C_5 and C_7. Therefore, $C_5 \wr \overline{K}_p$ and $C_7 \wr \overline{K}_p$ are the components of the factors of $(K_{12} - F) \wr \overline{K}_p$ which are known to be decompose into p-cycles when $p \geq 7$ by Theorem 3.1.3. Hence, it can be concluded that $(K_{12} - F) \wr \overline{K}_p$ has a $\{p\} - RCD$ for $p \geq 7$.

The case $(K_{12} - F) \wr \overline{K}_p$ for $p \geq 7$ is explained above. The remaining case for $d = 12$ is the $5 - RCD$ of $(K_{12} - F) \wr \overline{K}_5$. A direct construction for this case is shown in [2] which is explained below.

The graph $(K_{12} - F) \wr \overline{K}_5$ is obtained by considering 6 vertex-disjoint sets each consisting of 10 vertices without any edge. In other words, the graph is obtained by 6 vertex-disjoint sets each containing $\overline{K}_{10}$ with all possible edges between those sets, that is $K_6 \wr \overline{K}_{10}$. When the vertices of $(K_{12} - F)$ are blown up by $\overline{K}_5$ and each two of the non-adjacent sets are combined, the explained graph will be obtained. Designate the vertices of $K_6 \wr \overline{K}_{10}$ by $x_1, x_2, ..., x_{10}$, $u_0, u_1, ..., u_{24}$, $v_0, v_1, ...v_{24}$.

Let these 6 sets be defined as $\{x_i : \ i = 1, 2, ...10\}$ and

$\{u_i, v_i : \ i \equiv k(mod\ 5)\}$ for $k = 0, 1, 2, 3, 4$

Let the first factor be

$(u_0, u_1, u_3, u_6, u_{14})$, $(v_0, v_1, v_3, v_6, v_{14})$,

$(x_1, u_{13}, u_7, v_{15}, v_{22})$, $(x_2, v_{13}, v_7, u_{15}, u_{22})$,

$(x_3, u_8, u_{17}, v_{16}, v_{12})$, $(x_4, v_8, v_{17}, u_{16}, u_{12})$,

$(x_5, u_{20}, v_{23}, u_2, v_4)$, $(x_6, v_{20}, u_{23}, v_2, u_4)$,

$(x_7, u_9, v_{21}, u_{10}, v_{19})$, $(x_8, v_9, u_{21}, v_{10}, u_{19})$,

$(x_9, u_5, u_{18}, v_{11}, v_{24})$, $(x_{10}, u_{24}, v_5, u_{11}, v_{18})$.

The other factors are obtained in a way similar to the exceptional cases of the previous lemma. Let the permutation $(u_o, u_1, ..., u_{24})(v_0, v_1, ..., v_{24})$ act on the cycles of the factor for 24 times and this will create 25 different factors of this factorization and this exceptional case will be completed.

$\square$

Theorem 3.1.11. *[2] (a) If d and m are both odd integers, there is an $m - RCD$ of $K_d \wr \overline{K}_m$.*

(b) If $d \in 2\mathbb{N} \setminus \{4\}$ and m is any odd number which is not a power of 3 and $(d, m) \neq (6, 5)$, then there is an $m - RCD$ of $(K_d - F) \wr \overline{K}_m$.

Proof. (a) If m is an odd prime, the proof is completed by Lemma 3.1.9. Otherwise, let $m = p \cdot a$ where p is an odd prime, $p > 3$. The case where $p = 3$ will be considered later. Then

$K_d \wr \overline{K}_m \simeq K_d \wr [\overline{K}_p \wr \overline{K}_a] \simeq [K_d \wr \overline{K}_p] \wr \overline{K}_a.$

By Lemma 3.1.9, $K_d \wr \overline{K}_p$ has a $p - RCD$. Hence, the factors of $K_d \wr \overline{K}_p$ have cycles of length p. According to this fact, $[K_d \wr \overline{K}_p] \wr \overline{K}_a$ has a factorization, where each component of the factors is $C_p \wr \overline{K}_a$. According to Theorem 3.1.5, these components can be decomposed into Hamilton cycles. Since each of $C_p \wr \overline{K}_m$ has $p \cdot a = m$ vertices, the cycles are of length m. Finally, it can be concluded that $K_d \wr \overline{K}_m$ has a 2-factorization of m-cycles only, that is there is an $m - RCD$ of $K_d \wr \overline{K}_m$.

Unless m is a power of 3, this proof works. For the remaining case, i.e. $m = 3^n$ for some $n \geq 1$, the only choice for p is 3. For this case,

$K_d \wr \overline{K}_{3^n} \simeq K_d \wr [\overline{K}_3 \wr \overline{K}_{3^{n-1}}] \simeq [K_d \wr \overline{K}_3] \wr \overline{K}_{3^{n-1}}.$

Since $K_d \wr \overline{K}_3$ can be considered as a parallel class deleted from Kirkman triple system of K_{3d}, $K_d \wr \overline{K}_3$ has a resolvable factorization into cycles of length 3. The remaining parallel classes construct that decomposition. The subgraphs $C_3 \wr \overline{K}_{3^{n-1}}$ have a Hamilton decomposition by Theorem 3.1.5, hence cycles of length 3^n. Thus,

$K_d \wr \overline{K}_{3^n}$ has a 2-factorization, where each component of the factors is cycles of length $m = 3^n$.

(b) Similar to the case (a), if m is an odd prime, the proof is completed by Lemma 3.1.10. Otherwise, let $m = p \cdot a$, where p is an odd prime, $p > 3$. If $m \neq 3^n$, always such a prime p exists. Then

$(K_d - F) \wr \overline{K}_m \simeq (K_d - F) \wr [\overline{K}_p \wr \overline{K}_a] \simeq [(K_d - F) \wr \overline{K}_p] \wr \overline{K}_a$

Again similar to the proof done in part (a), by Lemma 3.1.10, $(K_d - F) \wr \overline{K}_p$ has a $p - RCD$ for $d \notin \{4, 6\}$. Hence, the factors of $(K_d - F) \wr \overline{K}_p$ have cycles of length p. According to this fact, $[(K_d - F) \wr \overline{K}_p] \wr \overline{K}_a$ has a factorization, where each component of the factors is $C_p \wr \overline{K}_a$. By Theorem 3.1.5, these components can be decomposed into Hamilton cycles. Since each of $C_p \wr \overline{K}_m$ has $p \cdot a = m$ vertices, the cycles are of length m. The conclusion is that $(K_d - F) \wr \overline{K}_m$ has a 2-factorization consisting of m-cycles for $d \notin \{4, 6\}$. In other words, $(K_d - F) \wr \overline{K}_m$ has an $m - RCD$.

The remaining part for this theorem is to show that $(K_6 - F) \wr \overline{K}_m$ has an $m - RCD$ when $m > 5$ and odd. This is shown in [2] and the direct construction is given in the following lemma.

To sum up, $(K_d - F) \wr \overline{K}_m$, where d is even, has a 2-factorization consisting of m-cycles when m is odd for $d \neq 4$, $m \neq 3^n$ and $(d, p) \neq (6, 5)$.

□

Lemma 3.1.12. *[2] If m is an odd integer, $m > 5$, then $(K_6 - F) \wr \overline{K}_m$ has an $m - RCD$.*

Proof. The proof is done by a direct construction. Observe that $(K_6 - F) \wr \overline{K}_m \simeq K_3 \wr \overline{K}_{2m}$. Since the graph $K_3 \wr \overline{K}_{2m}$ contains 3 vertex-disjoint sets containing $2m$ vertices each, let the sets be constructed as follows.

$X = \{x_0, x_1, ..., x_{2m-1}\}$

$Y = \{y_0, y_1, ..., y_{2m-1}\}$

$Z = \{z_0, z_1, ..., z_{2m-1}\}$

The first factor is dependent on the value of m but the permutation that will act on the factors to produce the other factors is fixed. That permutation is given by

$(y_0 y_1 \ldots . y_{2m-1})(z_0 z_1 \ldots . z_{2m-1})$.

Note that since the degree of each vertex in the graph is $4m$ and the degree of each vertex in a factor is 2, there will be $2m$ factors at the end. So the permutation will act on the initial factor for $2m-1$ times totally to produce all $2m$ factors consisting of m-cycles only.

As explained before, the initial 2-factors are dependent on the value of m. There are 2 cases to be considered where the value of m is evaluated in modulo 4. In each of the 2 cases, the direct construction of the initial factor is done.

Case 1: $m = 2r+1 \equiv 1 \ (mod\ 4), \ m > 5$. Each factor contains m-cycles and since there are $6m$ vertices totally, each factor should contain six cycles. Let two of these cycles be

$(y_3, z_{2m-3}, y_4, z_{2m-4}, \ldots . y_r, z_{2m-r}, x_0, y_m, x_1, z_{r+1}, x_2)$,

$(z_3, y_{2m-3}, z_4, y_{2m-4}, \ldots . z_r, y_{2m-r}, x_3, z_0, x_4, y_{r+1}, x_5)$.

Let the remaining four m-cycles start with the four consecutive vertices

$y_1 z_{2m-1} y_2 z_{2m-2}$,

$z_1 y_{2m-1} z_2 y_{2m-2}$,

$y_0 z_m y_{m-1} z_{m+4}$,

$y_{m+3} z_{m+3} y_{m+4} z_{m-1}$.

These four m-cycles are completed by alternating series of vertices from the set X and the union of the other sets, that is $Y \cup Z$. The only restriction is that the vertices adjacent to any of x_i will be from different sets. In other words, each x_i is adjacent to a vertex from Y and a vertex from Z.

As an example, for $m = 9$, an appropriate initial 2-factor is

$(y_3, z_{15}, y_4, z_{14}, x_0, y_9, x_1, z_5, x_2)$,

$(z_3, y_{15}, z_4, y_{14}, x_3, z_0, x_4, y_5, x_5)$,

$(y_1, z_{17}, y_2, z_{16}, x_6, y_6, x_7, z_6, x_8)$,

$(z_1, y_{17}, z_2, y_{16}, x_9, z_7, x_{10}, y_7, x_{11})$,

$(y_0, z_9, y_8, z_{13}, x_{12}, y_{10}, x_{13}, z_{10}, x_{14})$,

$(y_{12}, z_{12}, y_{13}, z_8, x_{15}, y_{11}, x_{16}, z_{11}, x_{17})$.

The other factors are constructed by the action of the permutation given above.

Case 2: $m = 2r + 1 \equiv 3\ (mod\ 4), m > 5$. The method is similar to the previous case but the cycles are different. Let the two cycles of the initial factor be

$(y_1, z_{2m-1}, y_2, z_{2m-2}, \ldots .y_r, z_{2m-r}, x_0)$,

$(z_1, y_{2m-1}, z_2, y_{2m-2}, \ldots .z_r, y_{2m-r}, x_1)$.

Let the remaining four m-cycles start with the following vertices

$y_0 z_m, \quad z_{r+1} y_{r+1}, \quad y_{r+2} z_{r+3}$ and z_{r+2}, y_{r+3}

The completion of these four cycles is similar to the Case 1.

As an example, an initial 2-factor for $m = 11$ can be written as

$(y_1, z_{21}, y_2, z_{20}, y_3, z_{19}, y_4, z_{18}, y_5, z_{17}, x_0)$,

$(z_1, y_{21}, z_2, y_{20}, z_3, y_{19}, z_4, y_{18}, z_5, y_{17}, x_1)$,

$(y_0, z_{11}, x_2, y_9, x_3, z_0, x_4, y_{10}, x_5, z_9, x_6)$,

$(y_6, z_6, x_7, y_{11}, x_8, z_{10}, x_9, y_{12}, x_{10}, z_{12}, x_{11})$,

$(y_7, z_8, x_{12}, y_{13}, x_{13}, z_{13}, x_{14}, y_{14}, x_{15}, z_{14}, x_{16})$,

$(z_7, y_8, x_{17}, z_{15}, x_{18}, y_{15}, x_{19}, z_{16}, x_{20}, y_{16}, x_{21})$.

The direct construction of the factorization is done for two different cases and the proof for Lemma 3.1.12 is completed.

□

The following theorem is the result of Huang, Kotzig and Rosa which will be needed while proving the main theorem.

Theorem 3.1.13. *[11] For any integer $m > 3$, $K_{2m} - F$ has an $m - RCD$.*

The statement of Theorem 3.1.1 (main theorem for this section) is repeated here.

For m odd and $m \geq 5$, K_n when n is odd and $K_n - F$ when n is even has an $m{-}RCD$ for all $n \equiv 0\ (mod\ m)$ except $n = 4m$. In addition, if $m = 3$, $n \in 3\mathbb{N}\backslash\{6, 12\}$.

Proof of Theorem 3.1.1. It was proven by Ray-Chaudhuri and Wilson [17] that $n \in 3\mathbb{N}\backslash\{6, 12\}$ for the case $m = 3$. In other words, the complete graph K_n when n is odd, $K_n - F$ when n is even have $3 - RCD$'s for $n \in 3\mathbb{N}\backslash\{6, 12\}$.

For the general case, the proof is dependent on the parity of d. For both cases, the first essential step is to break K_{dm} or $K_{dm} - F$ into d blocks of cardinality m and to work with the resulting two subgraphs.

Case 1. If d is odd, then

$K_{dm} = dK_m \oplus K_d \wr \overline{K}_m$.

Since m is odd, each K_m has a Hamilton decomposition, so dK_m which is the collection of d distinct complete graphs with m vertices, has 2-factorization, where each components of the factors are m-cycles. The factors of C_m-factorization of dK_m are obtained by taking any Hamilton cycle of each set containing K_m. The number of the factors is $\frac{m-1}{2}$ since each vertex has a degree of $m-1$ in K_m.

The second subgraph $K_d \wr \overline{K}_m$ has a 2-factorization into m-cycles by Theorem 3.1.11 (a). The union of the factors of dK_m and $K_d \wr \overline{K}_m$ yields a 2-factorization of K_{dm}.

Case 2. For d even,

$(K_{dm} - F) = \frac{d}{2}(K_{2m} - F) \oplus (K_d - F) \wr \overline{K}_m$

$K_{2m} - F$ has a factorization into m-cycles by Theorem 3.1.13.

The second subgraph $(K_d - F) \wr \overline{K}_m$ has a 2-factorization into m-cycles by Theorem 3.1.11(b) excluding the exceptional cases. The exceptional cases are $(d, m) = (6, 5)$ and $m = 3^n$ for some positive integer n.

For the case where $m = 3^n$, let

$K_{d \cdot 3^n} - F = \frac{d}{2}(K_{2 \cdot 3^n} - F) \oplus (K_d - F) \wr \overline{K}_{3^n} = \frac{d}{2}(K_{2 \cdot 3^n} - F) \oplus [(K_d - F) \wr \overline{K}_3] \wr \overline{K}_{3^{n-1}}$.

The first subgraph $\frac{d}{2}(K_{2 \cdot 3^n} - F)$ has a decomposition into 3^n-cycles by Theorem 3.1.13 taking $m = 3^n$.

For the second subgraph $[(K_d - F) \wr \overline{K}_3] \wr \overline{K}_{3^{n-1}}$, the first part $[(K_d - F) \wr \overline{K}_3]$ is the graph obtained by deleting one parallel class of a resolvable C_3 decomposition of $K_{3d} - F$. So it can be decomposed into cycles of length 3 for $d \notin \{2, 4\}$ since it is known that neither $K_{12} - F$ nor $K_6 - F$ has a factorization into cycles of length 3,

Excluding the exceptional cases where $d \in \{2, 4\}$, $[(K_d - F) \wr \overline{K}_3] \wr \overline{K}_{3^{n-1}}$ has a decomposition into $C_3 \wr \overline{K}_{3^{n-1}}$ which has a Hamilton cycle by Theorem 3.1.5. Since the total number of vertices of $C_3 \wr \overline{K}_{3^{n-1}}$ is 3^n, the graph $[(K_d - F) \wr \overline{K}_3] \wr \overline{K}_{3^{n-1}}$ has a decomposition into cycles of length $3^n = m$.

The case which is not included in the statement of the main theorem is $d = 4$, so it will be omitted. For $d = 2$, the graph $K_{2 \cdot 3^n} - F$ can be decomposed into cycles of length 3^n for $n \geq 2$ by Theorem 3.1.13.

The construction for the case $(d, m) = (6, 5)$ that is a factorization of $K_{30} - F$ into 5-cycles is done in [2]. This direct construction is explained below.

Since there are 30 vertices totally and the cycles are of length 5, each factor consists of six 5-cycles. The initial factor and a permutation will be given as done in previous theorems and the other factors will be constructed by the action of that permutation on the first factor. Let the initial 2-factor be

$(u_0, v_2, v_{13}, v_3, v_5)$, $(v_0, u_2, u_{12}, u_3, u_4)$,

$(u_1, v_7, x_1, u_8, v_8)$, $(v_1, u_6, x_2, v_4, u_7)$,

$(u_5, v_9, u_{10}, u_{13}, u_{11})$, $(v_{10}, u_9, v_{12}, v_6, v_{11})$.

Let the permutation acting on this initial factor be $(u_0 u_1 ...... u_{13})(v_0 v_1 ..... v_{13})$. The action of this permutation on this initial factor will create a factor of the factorization in each 13 steps which we want to obtain.

All general cases and exceptional cases are considered and the proof of the main theorem for this section is completed.

□

Chapter 4

C_M FACTORIZATION OF $K_{4M} - F$ FOR ODD M

So far, the m-cycle factorization of complete graphs K_{mn} or $K_{mn} - F$ depending on the parity of mn are done in the first two sections. Clearly, C_m makes sense for $m \geq 3$ since in a simple graph, no cycles with less than 3 edges exist.

The case where m is even and $m \geq 4$ is done by Alspach and Haggkvist [1] and explained in details in Section 2. The C_3 decomposition of complete graphs K_{3n} for n odd or $K_{3n} - F$ for n even (known as Kirkman Triple System) was done by Ray Chaudhuri and Wilson [17] and it is proven that C_3 factorization of K_{3n} exists for $n \notin \{2, 4\}$. The case where m is odd and $m \geq 5$ is done by Alspach, Schellenberg, Stinson and Wagner and explained in details in Section 3 with the exceptional case that $n = 4$ [2]. In order to complete all of the cases, the C_k factorization of $K_{4k} - F$ for k odd and $k > 3$ should have been done. In 1991, Hoffman and Schellenberg have completed cycle factorization of complete graphs by giving a direct construction of C_k factorization of $K_{4k} - F$ [10]. This section consists of this construction in details for different cases of k.

The number of the factors and the cycles that a factor consists of have to be calculated before giving the direct construction. The number of cycles that each factor contains is simply found by the calculation of the vertices. Since each vertex of $K_{4k} - F$ should appear in any factor and the cycle C_k is the only component of the factor, the number of the cycles is obviously $\frac{|V(K_{4k}-F)|}{|V(K_k)|} = \frac{4k}{k} = 4$. So there are 4 cycles in each factor of this factorization. On the other hand, the number of the factors in that factorization is calculated by taking the total number of edges of the complete graph and the total number of edges in a factor into consideration, that is $\frac{|E(K_{4k}-F)|}{4 \cdot |E(C_k)|} = \frac{\frac{4k \cdot (4k-1)}{2} - 2k}{4 \cdot k} = 2k - 1$.

As a result, for each factorization in this section, every factor should contain exactly 4 cycles and totally $2k-1$ factors should be obtained.

Before explaining the general construction, two special cases to which the general construction does not apply is given below, namely a C_5 factorization of $K_{20} - F$ and a C_7 factorization of $K_{28} - F$. For the general and special cases of the construction, Bose's method of symmetrically repeated differences is used [7]. An example is shown in [6]. Before going further, some definitions and the outline of the method should be given.

In all cases, the vertices of the complete graph $K_{4k} - F$ consist two copies of $\mathbb{Z}_{2k-1}$, i.e. $\{0, 1, 2, ..., 2k-2\}$ and $\{\overline{0}, \overline{1}, \overline{2}, ..., \overline{2k-2}\}$, together with two fixed points $\{\infty_1, \infty_2\}$. The pure difference is defined to be the difference of the numbers between the endpoints of an edge if the endpoints are from the same copy of $\mathbb{Z}_{2k-1}$. For every such an edge, 2 different pure differences will be obtained one being the negative value of the other. The mixed difference is defined to be the difference between two consecutive vertices which belong to distinct copies of $\mathbb{Z}_{2k-1}$. In other words, the mixed difference for the consecutive vertices a and $\overline{b}$ is defined to be $b-a$ modulo $\mathbb{Z}_{2k-1}$, taking the positive value of vertex of the overlined one and subtracting the other vertex value. The first factor with 4 cycles will be constructed directly for each general and exceptional cases and the action of the group $\mathbb{Z}_{2k-1}$ will be applied to each of those 4 cycles of the factor. In other words, the values of the vertices will be increased by 1 in each step in modulo $\mathbb{Z}_{2k-1}$ and the corresponding cycles will be obtained. Since a 1-factor is removed from the complete graph, a fixed difference will never be appeared in the construction. The edges that have that difference constitutes that one factor removed. All of the other mixed and pure differences should be included for each factor. Since the same method is applied in the following two special cases and all the differences are shown clearly, they may be helpful to figure out the general construction which will be explained later on in this section.

4.1 Direct Construction of 2 Exceptional Cases

Lemma 4.1.1. *[10] There exists a C_5 factorization of $K_{20}-F$ and a C_7 factorization of $K_{28} - F$.*

Proof. Let the vertices of K_{20} be

$\{0, 1, 2, ..., 8\} \bigcup \{\overline{0}, \overline{1}, \overline{2}, ..., \overline{8}\} \bigcup \{\infty_1, \infty_2\}$.

Let the first factor contain the cycles

$C_1 = (0, 1, 3, 6, \overline{0})$, $C_2 = (2, 7, \overline{5}, 8, \overline{1})$, $C_3 = (\infty_1, 4, \overline{8}, \overline{2}, \overline{4})$, $C_4 = (\infty_2, 5, \overline{6}, \overline{7}, \overline{3})$.

All pure and mixed differences are calculated for this factor. Table 4.1 below shows the results.

Table 4.1: Differences in Cycles of $K_{20} - F$

	pure differences	mixed differences
1^{st} cycle	±1,±2, ±3	0,3
2^{nd} cycle	±5	7,6,2,8
3^{rd} cycle	$\pm\overline{6}$, $\pm\overline{2}$, ∞_1	4, ∞_1
4^{th} cycle	$\pm\overline{1}$, $\pm\overline{4}$, ∞_2	1, ∞_2

The table demonstrates that all of the pure differences, the mixed differences except 5 are obtained by the construction of this factor. Since the total number of the factors is $2k - 1 = 9$, there are 8 other factors to be shown. The other 8 factors are obtained by the action of the group $\mathbb{Z}_9$, that is increasing the number of each vertex by 1 (excluding ∞_1 and ∞_2) throughout 8 steps. One of the important note is that the differences between ∞_1 and a number, together with ∞_2 and a number will be obtained throughout these steps since in every step, the vertices that are adjacent to ∞_1 and ∞_2 are increased by 1 and finally all of the vertices have been adjacent to both of the fixed points.

The 1-factor removed should naturally include the mixed difference 5 and the difference between the two fixed points ∞_1 and ∞_2 since those differences are never seen in the cycles. The 1-factor consists of the edges

$\{\overline{5}, 0\}, \{\overline{6}, 1\}, \{\overline{7}, 2\}, \{\overline{8}, 3\}, \{\overline{0}, 4\}, \{\overline{1}, 5\}, \{\overline{2}, 6\}, \{\overline{3}, 7\}, \{\overline{4}, 8\}$ and $\{\infty_1, \infty_2\}$.

There are 10 edges removed which is expected since a 1-factor includes $2k$ edges for a graph with $4k$ vertices, and $k = 5$ for this example.

The construction of a C_7 factorization of $K_{28} - F$ is similar to the case that is explained above. Let the vertices be

$\{0, 1, 2, ..., 12\} \bigcup \{\overline{0}, \overline{1}, \overline{2}, ..., \overline{12}\} \bigcup \{\infty_1, \infty_2\}$.

Let the first factor contain the cycles

$C_1 = (0, 1, 3, 6, 10, 5, \overline{0})$, $C_2 = (\overline{1}, \overline{2}, \overline{4}, , \overline{10}, 9, , \overline{11}, 7)$,

$C_3 = (\infty_1, 11, 4, \overline{3}, \overline{7}, \overline{12}, , \overline{9})$, $C_4 = (\infty_2, 12, \overline{8}, 2, \overline{5}, 8, \overline{6},)$.

The edges of this factor contain every pure differences, mixed differences except 5 and differences between the fixed points and two numbers which is expected similar to the case of C_5 factorization of $K_{20} - F$. The differences for this case are shown on Table 4.2.

Table 4.2: Differences in Cycles of $K_{28} - F$

	pure differences	mixed differences
1^{st} cycle	± 1, ± 2, ± 3, ± 4, ± 5	0, 8
2^{nd} cycle	$\pm \overline{1}$, $\pm \overline{2}$, $\pm \overline{6}$	1, 2, 4, 7
3^{rd} cycle	± 6 $\pm \overline{4}$, $\pm \overline{5}$, $\pm \overline{3}$, ∞_1	12, ∞_1
4^{th} cycle	∞_2	9, 6, 3, 10, 11, ∞_2

The 1-factor for this case consists of the edges which contain the mixed difference 5 and the edge between the fixed points.

$\{\overline{5}, 0\}, \{\overline{6}, 1\}, ..., \{\overline{4}, 12\}$ and $\{\infty_1, \infty_2\}$.

Applying the same procedure; letting $\mathbb{Z}_{13}$ act on this factor generates a C_7 factorization of $K_{28} - F$.

□

The exceptional cases are considered by the previous lemma and now the direct construction of the general case will be done.

4.2 Direct Construction for General Case

Theorem 4.2.1. *[10] If $k > 3$ is an odd integer, then $K_{4k} - F$ has a C_k-factorization.*

Proof. Since C_k factorization of the complete graph $K_{4k} - F$ is desired for odd k, let $k = 2t + 1$. Let the complete graph K_{8t+4} have vertex set

$\{0, 1, ..., 4t\} \cup \{\overline{0}, \overline{1}, ..., \overline{4t}\} \cup \{\infty_1, \infty_2\}$.

Since there are four $(2t+1)$-cycles for each 2-factor of the factorization, these four cycles will be constructed directly and letting a suitable group action on this factor produces the other factors and the factorization will be completed. For this direct construction, three of the four cycles will be constructed directly independent of the value of t and the fourth cycle will be considered depending on the value of t. As done in the exceptional cases in the preceding lemma, all of the pure and the mixed differences are going to be considered and all differences will be seen in the initial factor except the mixed difference $4t$ and the difference between ∞_1 and ∞_2 which will constitute the 1-factor removed. After considering the general construction, there will be examples of this construction for different values of k at the end of this section. So, especially while considering the fourth cycle, these examples will be helpful to understand the pattern clearly.

The first cycle is

$(0, \overline{1}, -1, \overline{2}, -2, \overline{3}, -3, ..., \overline{t}, -t)$.

The edges of this cycle have the pure differences $\pm t$ and the mixed differences $1, 2, ..., 2t$. The group that will act on the factor will be $\mathbb{Z}_{4t+1}$ since $2k - 1 = 4t + 1$ so the differences are calculated in $\mathbb{Z}_{4t+1}$. All the differences of each cycle are demonstrated on the Table 4.3. This cycle is shown on Figure 4.1 at the top by thin lines.

The second cycle is

$(-t - 2, \overline{t + 1}, -t - 3, \overline{t + 2}, ..., -2t + 1, \overline{2t - 2}, -2t, \overline{2t - 1}, \overline{0}, \overline{2t}, -2t - 1)$.

The edges of this cycle have pure differences $\pm\overline{(2t - 1)}$, $\pm\overline{2t}$ and $\pm(t - 1)$ and mixed differences $2t + 3, 2t + 4, ..., 4t - 1$. The second cycle is shown on Figure 4.1

below the first cycle.

The third cycle is

$(\infty_1, \overline{4t-1}, \overline{2t+1}, \overline{4t-2}, \overline{2t+2}, ..., \overline{3t-1}, \overline{3t}, t-2)$.

The edges of this cycle have pure differences $\pm\overline{1}, \pm\overline{2}, ..., \pm\overline{(2t-2)}$ and the only mixed difference $2t+2$. It also has all differences involving the vertex ∞_1. This cycle is shown on Figure 4.1 by bold lines saturating the vertices of right bottom part.

Figure 4.1 represents and helps us to figure out the pattern considered in this construction. Note that the overlined vertices are shown on the right side of the figure whereas the others are shown on the left side. The fixed points ∞_1 and ∞_2 are placed at the bottom of the figure. A quick observation can be stated as nearly all of the mixed differences are obtained using the first and the second cycles. The third cycle and the fourth one which is not explained yet are used to construct the edges having almost all pure differences and the differences between a number and ∞_1 and ∞_2 respectively. The mixed difference $4t$ and the difference between ∞_1 and ∞_2 are not considered since these will constitute the 1-factor removes from the complete graph. That 1-factor has the edges

$\{\overline{0}, 1\}$, $\{\overline{1}, 2\}$,......,$\{\overline{4t-1}, 4t\}$, $\{\overline{4t-1}, 4t\}$, $\{\overline{4t}, 0\}$ and $\{\infty_1, \infty_2\}$.

Table 4.3 summarizes all differences of the first three cycles and the missing differences which are expected to appear on the fourth cycle can be derived by this table. The construction of the fourth cycle depends on the parity of t and there will be three subcases both for odd and even t. This cycle is shown with dashed lines in Figure 4.1.

Table 4.3: Differences in Cycles of $K_{4k} - F$

	pure differences	mixed differences
1^{st} cycle	$\pm t$	1, 2, 3,....,$2t$
2^{nd} cycle	$\pm(t-1)$, $\pm\overline{(2t-1)}$, $\pm\overline{2t}$	0, $2t+3$, $2t+4$,....,$4t-1$
3^{rd} cycle	$\pm\overline{1}$ $\pm\overline{2}$,.... $\pm\overline{(2t-2)}$ ∞_1	$2t+2$, ∞_1

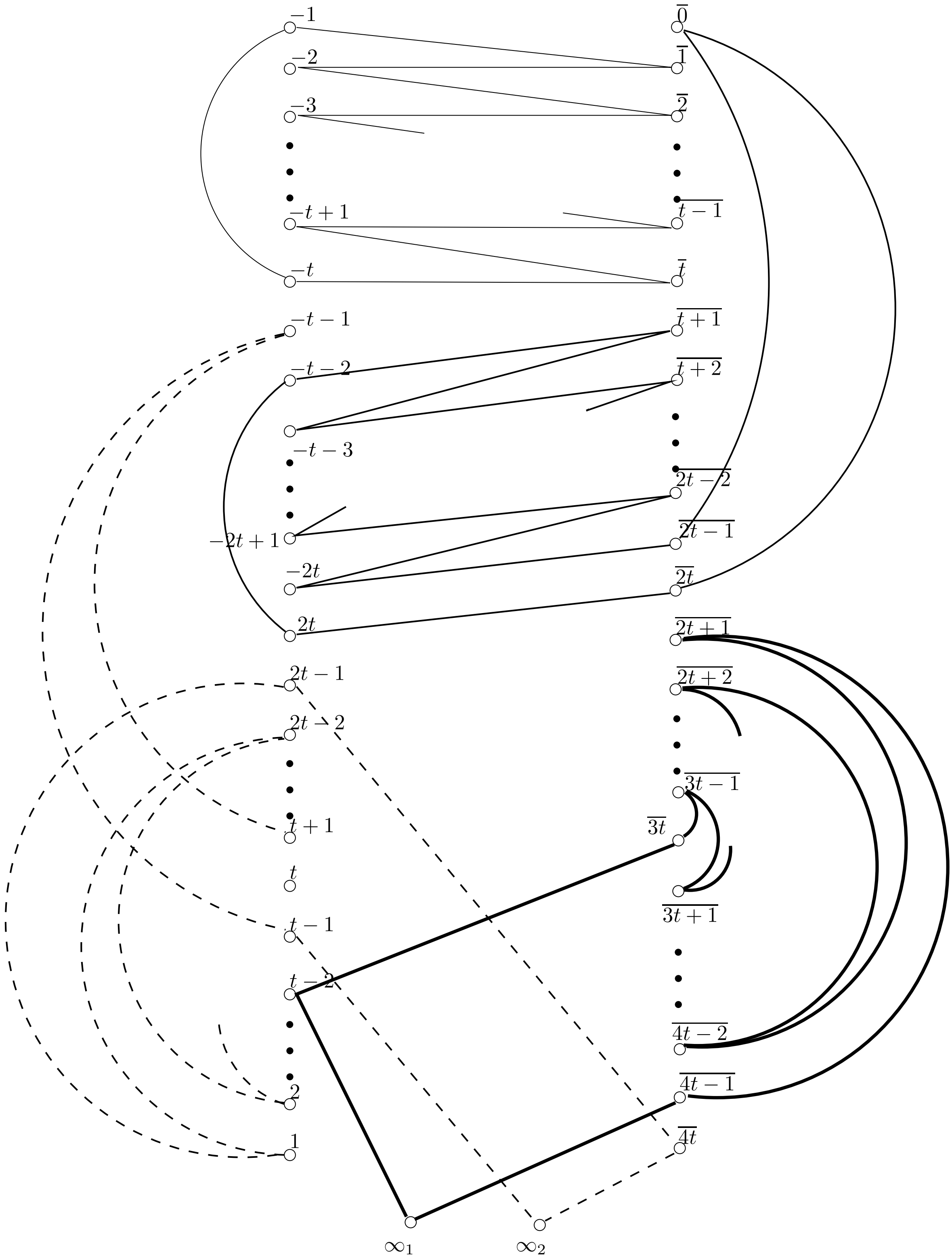

Figure 4.1: Cycles of $K_{4k} - F$

Case1: Assume that t is even.

The fourth cycle contains the path

$\{t+1, -t-1, t-1, \infty_2, \overline{4t}, 2t-1, 1, 2t-2, 2, ..., \frac{t-4}{2}, \frac{3t+2}{2}, \frac{t-2}{2}, \frac{3t}{2}, \frac{3t-2}{2}, \frac{3t-6}{2}\}$.

This path contains the pure differences $\pm 2t, \pm(2t-1), ..., \pm(t+1)$, $\pm 1, \pm 2$, difference between a number and ∞_2, and mixed difference $2t+1$.

The remaining differences for the construction to be completed are the pure differences $\pm 3, \pm 4, ..., \pm(t-3), \pm(t-2)$.

The path that will be adjoint to vertex $\frac{3t-6}{2}$ are of the form

$P_{a,b} = \{a, b, a+1, b+2, a-1, b+1, a-3\}$.

This path starts at the vertex of the form a, passes through the vertices of the form $b, a+1, b+2, a-1, b+1$ and ends at the vertex $a-3$.

Path $P_{a,b}$ represents the pure differences

$\pm(a-b+1), \pm(a-b), \pm(a-b-1), \pm(a-b-2), \pm(a-b-3), \pm(a-b-4)$.

The structure of the vertices a, b are of the form

$(a, b) = (\frac{3t-6}{2}, \frac{t}{2}), (\frac{3t-12}{2}, \frac{t+6}{2}, (\frac{3t-18}{2}, \frac{t+12}{2}), ...$

The values of a and b are limited and they take these values as long as $a > t+4$ and $b < t-7$ with an exception that if (a, b) takes the last values of $(t+6, t-9)$ then they take the values of $(t+3, t-6)$ and the path is completed. The cycle will be completed in different ways for different values of t. The 3 subcases will be considered below in order to complete the cycle, hence the construction.

When the explanation of the construction is completed, examples of each 6 different cases will be shown in order to prevent confusion.

Case 1.1: $t \equiv 0 \ (mod\ 6)$.

Consider the union of paths $P_{a,b}$ for (a, b) where

$(a, b) \in (\frac{3t-6i}{2}, \frac{t-6+6i}{2}) : i = 1, 2, ..., \frac{t-6}{6}\}$.

Note that for $t = 6$ no paths $P_{a,b}$ are used since the set is empty.

The path starts at the value of $(a, b) = (\frac{3t-6}{2}, \frac{t}{2})$ and ends at the value of $(a, b) = (t+3, t-6)$ so it actually starts at the vertex $\frac{3t-6}{2}$ and ends at the vertex t since the last vertex has the value of $a-3$. The path passes through the vertices $\frac{3t-6}{2}, \frac{t}{2}, \frac{3t-4}{2}, \frac{t+4}{2}, ...,$

$t-5, t$.

The edges of this path represents the pure differences of the set

$\{\pm 5, \pm 6, ..., \pm(t-3), \pm(t-2)\}$.

When $(a, b) = (t, t-3)$, the cycle is completed by adding the path $\{t, t-3, t+1\}$. This path represents the pure differences of ± 3 and ± 4. By the construction described, the fourth cycle is also a cycle of length $2t+1$ and all of the pure and mixed differences are obtained by these four cycles. Since these four cycles have equal length of $k = 2t+1$ and all of the cycles have different vertices, every vertices of $K_{4k} = K_{8t+4}$ are appeared by the cycles.

The number of the factors of $K_{4k} - F$ is calculated to be $2k-1 = 4t+1$ since each vertex has a degree of $4k-2$. So letting $\mathbb{Z}_{4t+1}$ act this factor produces the factors of the desired factorization and the construction is completed for case $t \equiv 0$ $(mod\ 6)$. The same method is valid even for the remaining cases and the action of $\mathbb{Z}_{4t+1}$ will be applied to all of the factors that will be considered in the following cases.

As an example for this case, the construction for $t = 18$ is shown at the end of the section.

Case 1.2: $t \equiv 2$ $(mod\ 6)$

Consider the union of paths $P_{a,b}$ for (a, b) where

$(a, b) \in (\frac{3t-6i}{2}, \frac{t-6+6i}{2}) : i = 1, 2, ..., \frac{t-14}{6}\}$.

When $(a, b) = (t+4, t-7)$, the path $P_{a,b}$ will not be considered as explained above. The cycle is completed by adding the path

$\{t+4, t-7, t+5, t-5, t+2, t-6, t+3, t-3, t-4, t+1\}$.

All differences and all of the vertices are obtained by this fourth cycle and the other three explained before. So the action of $\mathbb{Z}_{4t+1}$ will give the desired factorization except the case where $t \in \{2, 8\}$.

The case where $t = 8$ is exceptional since $\frac{3t-6}{2} = 9$ which is less than $t+4 = 12$. So no path $P_{a,b}$ will be applied in this case. The fourth cycle is given directly by

$(\infty_2, \overline{32}, 15, 1, 14, 2, 13, 3, 12, 10, 11, 5, 8, 4, 9, 24, 7)$.

The case $t = 2$ is considered in the previous lemma and the direct construction of

C_5 factorization of $K_{20} - F$ is done.

As an example, the case where $t = 20$ is shown after the cases are completed.

Case 1.3: $t \equiv 4\ (mod\ 6)$

The argument proceeds as in the previous cases. Taking the union of paths $P_{a,b}$ for

$(a, b) \in (\frac{3t-6i}{2}, \frac{t-6+6i}{2}) : i = 1, 2, ..., \frac{t-10}{6}\}$.

When $(a, b) = (t + 2, t - 5)$, the cycle is completed by adding the path $\{t + 2, t - 5, t + 3, t - 3, t, t - 4, t + 1\}$.

For the case $t = 4$, $\frac{3t-6}{2} = 3$ which is less than $t + 2 = 6$. This exceptional case is considered separately and the fourth cycle is given by $(\infty_2, \overline{16}, 7, 1, 6, 4, 3, 12, 5)$.

Look at the end of the section for an example of this case; $t = 22$.

Case2: Assume that t is odd.

The fourth cycle contains the path

$\{t+1, -t-1, t-1, \infty_2, \overline{4t}, 2t-1, 1, 2t-2, ..., \frac{3t+3}{2}, \frac{t-3}{2}, \frac{3t+1}{2}, \frac{t-1}{2}, \frac{3t-5}{2}, \frac{3t-1}{2}, \frac{3t-3}{2}, \frac{t+3}{2}\}$.

The pure differences obtained for this case are

$\pm 2t, \pm(2t - 1), \pm(2t - 2), ..., \pm(t + 1, \pm(t - 2), \pm 2, \pm 1, \pm(t - 3)$. The mixed difference $2t + 1$, as well as the difference between a number and ∞_2 are also obtained.

As in the even case of t, paths which are in the form

$Q_{a,b} = \{b, a, b - 1, a - 2, b + 1, a - 1, b + 3\}$ will be adjoint to the end of the path, that is to the vertex $\frac{t+3}{2}$.

Moreover, this path represents the pure differences of the form

$\pm(a - b + 1),\ \pm(a - b),\ \pm(a - b - 1),\ \pm(a - b - 2),\ \pm(a - b - 3),\ \pm(a - b - 4)$.

The values of a and b are of the form

$(a, b) = (\frac{3t-7}{2}, \frac{t+3}{2}), (\frac{3t-13}{2}, \frac{t+9}{2}, (\frac{3t-19}{2}, \frac{t+15}{2}), ...$

as long as the inequalities $a > t + 3$ and $b < t - 5$ hold. As in the previous case, 3 subcases will be considered depending on the value of t.

Case 2.1: $t \equiv 1\ (mod\ 6)$.

When $(a, b) = (t + 3, t - 5)$, the cycle is completed by adding the path

$\{t - 5, t + 2, t - 6, t + 3, t - 3, t, t - 4, t + 1\}$.

When $t = 7$, $\frac{(3t-7)}{2} = 7$ and this value is smaller than $t + 3 = 10$, so no paths will be added and this approach does not work. For this exceptional case, the fourth cycle is given by

$(\infty_2, \overline{28}, 13, 1, 12, 2, 11, 3, 8, 21, 6, 10, 9, 7, 4)$.

For the case where $t = 1$ the nonexistence of C_3 factorization of $K_{12} - F$, is shown by Kotzig and Rosa [13] as explained in the previous section.

The case where $t = 19$ is given as an example in the examples part below.

Case 2.2: $t \equiv 3\ (mod\ 6)$.

When $(a, b) = (t + 1, t - 3)$, the cycle is completed by adding the path

$\{t - 3, t, t - 4, t + 1\}$.

When $t = 3$, $\frac{(3t-7)}{2} = 1$ and this value is smaller than $t + 1 = 4$, so no paths will be added and this approach does not work. However, this is C_7 factorization of K_{28} and this exceptional case was shown in Lemma 3.1.

For another example, the case $t = 21$ is shown below.

Case 2.3: $t \equiv 5\ (mod\ 6)$.

When $(a, b) = (t + 2, t - 4)$, the cycle is completed by adding the path

$\{t - 4, t + 2, t - 5, t, t - 3, t + 1\}$.

When $t = 5$, $\frac{(3t-7)}{2} = 4$ and this value is smaller than $t + 2 = 7$, so no paths will be added and this approach does not work. For this case, let the fourth cycle be $(\infty_2, \overline{20}, 9, 1, 8, 2, 5, 7, 6, 15, 4)$.

The case $t = 23$ is demonstrated below in details as an example.

□

So far, the direct construction of C_k-factorization of $K_{4k} - F$ are explained. Each factor (or parallel class) contains 4 cycles. The first three cycles are given directly and the fourth cycle is considered in 6 different cases depending on the value of t where $k = 2t+1$. The other factors are obtained by applying $\mathbb{Z}_{4t+1}$ to each of these 4 cycles. So, if the first four cycles are obtained, the remaining part will be straightforward. To put in a different way, fixing the vertices ∞_1 and ∞_2, increasing each of the other vertex numbers by 1 in modulo $\mathbb{Z}_{4t+1}$ in each $4t + 1$ steps will give the desired factorization.

The direct construction considered and explained in this section is hard to visualize especially when the fourth cycle is taken into consideration. In order to prevent confusion, 6 examples for each 6 different cases is given below. These examples may help the reader to figure out the pattern examined in the fourth cycle with a less confusion.

4.3 Examples of Remaining Cycles for Each Different Cases

Case1.1: $t \equiv 0$ $(mod\ 6)$.

Let $t = 18$. Then $k = 2t + 1 = 37$. So we are looking for C_{37}-factorization of $K_{148} - F$ since $37 \cdot 4 = 148$. The first three cycles given above are clear to construct whereas the fourth cycle is considered here. The fourth cycle has the path

$\{19, -19, 17, \infty_2, \overline{72}, 35, 1, 34, 2, 33, 3, ..., 29, 7, 28, 8, 27, 26, 24\}$.

Then $(a, b) = (24, 9)$, so

$P_{a,b} = \{24, 9, 25, 11, 23, 10\}$.

Then take $(a, b) = (21, 12)$, so

$P_{a,b} = \{21, 12, 22, 14, 20, 13\}$ and finish with the path $\{18, 15, 19\}$.

To sum them up, the fourth cycle is found to be

$(19, -19, 17, \infty_2, \overline{72}, 35, 1, 34, 2, 33, 3, ..., 29, 7, 28, 8, 27, 26, 24, 9, 25, 11, 23, 10, 21, 12, 22, 14, 20, 13, 18, 15)$.

Case1.2: $t \equiv 2$ $(mod\ 6)$.

Let $t = 20$. Then $k = 2t + 1 = 41$. So we are looking for C_{41}-factorization of $K_{164} - F$. The fourth cycle has the path

$\{21, -21, 19, \infty_2, \overline{80}, 39, 1, 38, 2, 37, 3, ...32, 8, 31, 9, 30, 29, 27\}$.

Then $(a, b) = (27, 10)$, so

$P_{a,b} = \{27, 10, 28, 12, 26, 11\}$

and finish with the path $\{24, 13, 25, 15, 22, 14, 23, 17, 20, 16, 21\}$.

In conclusion, the fourth cycle is

$(21, -21, 19, \infty_2, \overline{80}, 39, 1, 38, 2, 37, 3, ..., 31, 9, 30, 29, 27, 10, 28, 12, 26, 11, 24, 13, 25,$

$15, 22, 14, 23, 17, 20, 16, 21)$.

Case1.3: $t \equiv 4$ $(mod\ 6)$.

Let $t = 22$. Then $k = 2t + 1 = 45$. So we are looking for C_{45}-factorization of $K_{180} - F$. The fourth cycle has the path

$\{23, -23, 21, \infty_2, \overline{88}, 43, 1, 42, 2, 41, 3, ...34, 10, 33, 32, 30\}$.

Then $(a, b) = (30, 11)$, so

$P_{a,b} = \{30, 11, 31, 13, 29, 12\}$.

Then $(a, b) = (27, 14)$, so

$P_{a,b} = \{27, 14, 28, 16, 26, 15\}$

and finish with the path $\{24, 17, 25, 19, 22, 18, 23\}$.

In conclusion, the fourth cycle is

$(23, -23, 21, \infty_2, \overline{88}, 43, 1, 42, 2, 41, 3, ..., 34, 10, 33, 32, 30, 11, 31, 13, 29, 12, 27, 14, 28, 16, 26, 15, 24, 17, 25, 19, 22, 18)$.

Case2.1: $t \equiv 1$ $(mod\ 6)$.

Let $t = 19$. Then $k = 2t + 1 = 39$. So we are looking for C_{39}-factorization of $K_{156} - F$. The fourth cycle has the path

$\{20, -20, 18, \infty_2, \overline{76}, 37, 1, 36, 2, 35, 3, ..., 30, 8, 29, 9, 26, 28, 27, 11\}$.

Then since $(a, b) = (25, 11)$,

$P_{a,b} = \{11, 25, 10, 23, 12, 24, 14\}$

and complete the cycle with the path $\{14, 21, 13, 22, 16, 19, 15, 20\}$.

When all these paths are adjoint, the fourth cycle is found to be

$(20, -20, 18, \infty_2, \overline{76}, 37, 1, 36, 2, 35, 3, ..., 30, 8, 29, 9, 9, 26, 28, 27, 11, 25, 10, 23, 12, 24, 14, 21, 13, 22, 16, 19, 15)$

Case2.2: $t \equiv 3$ $(mod\ 6)$.

Let $t = 21$. Then $k = 2t + 1 = 43$. So we are looking for C_{43}-factorization of $K_{172} - F$. The fourth cycle starts with the path

$\{22, -22, 20, \infty_2, \overline{84}, 41, 1, 40, 2, 39, 3, ...33, 9, 32, 10, 29, 31, 30, 12\}$.

Then $(a, b) = (28, 12)$, so

$P_{a,b} = \{12, 28, 11, 26, 13, 27\}$.

Then $(a,b) = (25,15)$, so

$P_{a,b} = \{15, 25, 14, 23, 16, 24, 18\}$.

and finish with the path $\{18, 21, 17, 22\}$.

Thus, the fourth cycle is

$(22, -22, 20, \infty_2, \overline{84}, 41, 1, 40, 2, 39, 3, ..., 33, 9, 32, 10, 29, 31, 30, 12, 28, 11, 26, 13,$
$27, 15, 25, 14, 23, 16, 24, 18, 21, 17)$.

Case2.3: $t \equiv 4\ (mod\ 6)$.

Let $t = 23$. Then $k = 2t + 1 = 47$. So we are looking for C_{47}-factorization of $K_{188} - F$. The fourth cycle has the path

$\{24, -24, 22, \infty_2, \overline{92}, 45, 1, 44, 2, 43, 3, ...36, 10, 35, 11, 32, 34, 33, 13\}$.

Then, since $(a,b) = (31,13)$,

$P_{a,b} = \{13, 31, 12, 29, 14, 30, 16\}$.

Then, since $(a,b) = (28,16)$,

$P_{a,b} = \{16, 28, 15, 26, 17, 27, 19\}$.

and finish with the path $\{19, 25, 18, 23, 20, 24\}$.

In conclusion, the fourth cycle is

$(24, -24, 22, \infty_2, \overline{92}, 45, 1, 44, 2, 43, 3, ..., 36, 10, 35, 11, 32, 34, 33, 13, 31, 12, 29, 14,$
$30, 16, 28, 15, 26, 17, 27, 19, 25, 18, 23, 20)$.

Note that the fourth cycle in each of the examples spans all the remaining vertices and when the union of all four cycles are taken, one parallel class of the desired cycle factorization is obtained. Letting $\mathbb{Z}_{2k-1}$ act on this parallel class will give us the other parallel classes and the factorization will be completed.

The cycle factorization considered in this section is a special case which was not considered in the previous section since $(K_4 - F) \wr \overline{K_p}$ can not be factored into p-cycles as explained in section 3. The direct construction of this special case is done by Hoffman and Schellenberg. We explained their construction and give examples for each case in this section, thus, all of the cases are completed.

Chapter 5

CONCLUSION

In this thesis, the studies on uniform resolvable cycle decompositions of complete graphs are demonstrated. The main theorems are proven and the direct construction of the exceptional cases are shown, examples related to these constructions are given. When the cycles are selected to be uniform, the cycle decomposition problem is completely solved. However, the general problem, which is known as Oberwolfach Problem is still open and several mathematicians still study on different cases of this famous area.

In the first chapter, basic definitions and notions are given and the historical development of the problem is explained. Examples on cycle decomposition and the Oberwolfach problem are shown and the solutions are given using tables.

In the second chapter, the first general study on this problem is given. This study is done by B. Alspach and R. Haggkvist in 1985 [1]; they considered and completely solved the problem for even length cycle case. The theorems and lemmas they used for the main theorem are demonstrated and most of them are proven. The figures related to those proofs are sketched.

In the third chapter, the proof for the odd length cycles is shown using the studies of B. Alspach, P. J. Schellenberg, D. R. Stinson and D. Wagner [2] published in 1989. They considered the resolvable odd length cycle decomposition in general; however, resolvable C_k decomposition of K_{4k} is not proven in this paper and this case is remained as an exceptional case.

In the fourth chapter, the remained exceptional case of odd length cycles is given. This case is shown by D. G. Hoffman and P. J. Schellenberg [10] in 1991 using Bose's symmetrically difference method. The general construction and the construction for two exceptional cases are explained and examples for each cases are given at the end of the chapter.

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