SOME COMBINATORIAL RESULTS IN FULL AND PARTIAL TRANSFORMATION SEMIGROUPS

by

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ABSTRACT

SOME COMBINATORIAL RESULTS IN FULL AND PARTIAL TRANSFORMATION SEMIGROUPS

Let $[n] = \{1, 2, \ldots, n\}$. A function α with domain $\mathrm{Dom}\,\alpha \subseteq [n]$ and image $\mathrm{Im}\,\alpha \subseteq [n]$ is called a partial transformation. It is a full transformation if $\mathrm{Dom}\,\alpha = [n]$. Partial transformations form a semigroup under the composition of maps. The index and period of an element α in this semigroup are the smallest values of positive integers m and r such that $\alpha^{m+r} = \alpha^m$. α is called m-potent if $r = 1$ and it is a potent element if it is m-potent for some m. The number of m-potent and potent elements are investigated. We give a discussion of the fact that every α in full transformation with index m and period r can be uniquely written as a product of a permutation of order r and an m-potent element where the permutation and potent element have disjoint shifts.

If $\mathcal{S}_n$ is the group of permutations, it acts on full transformation group by conjugation. We calculate the cardinality of equivalence classes of conjugates of 1-potent, more commonly known as idempotent, elements.

In the last two chapters we investigate the cardinality properties of nilpotent semigroups. We utilize graph theoretic methods in this investigation. Interpreting the results obtained for the cardinality of maximal symmetric inverse semigroups we obtain some relationships involving Bell numbers and Stirling numbers.

ÖZET

TAM VE KISMİ DÖNÜŞÜM YARIGRUPLARINDA BAZI KOMBİNATORYAL SONUÇLAR

$[n] = \{1, 2, \ldots, n\}$ koyalım. Tanım kümesi $\mathrm{Dom}\,\alpha \subseteq [n]$ ve görüntü kümesi $\mathrm{Im}\,\alpha \subseteq [n]$ olan bir α fonksiyonuna kısmi dönüşüm denir. Eğer $\mathrm{Dom}\,\alpha = [n]$ ise α'ya tam dönüşüm denir. Kısmi dönüşümler, transformasyonların bileşke işlemi altında bir yarı grup oluşturur. Bu yarıgruptaki bir α öğesinin indeksi ve periyodu, $\alpha^{m+r} = \alpha^m$ olacak şekilde pozitif m ve r tam sayılarının en küçük değerleridir. $r = 1$ ise α, m-potent olarak adlandırılır. Bir m için m-potent olan elemanlara potent eleman diyelim. Bu çalışmada m-potent ve potent elemanların sayısı verilmiştir.

İndeksi m ve periyodu r olan bir tam dönüşümün, kaymaları (shift) ayrık olan r dereceli bir permütasyonun ve bir m-potent elemanın, benzersiz bir şekilde, çarpımı olarak yazılabileceği gerçeğinin bir tartışması verilmiştir.

$\mathcal{S}_n$ permütasyon grubu tam dönüşüm grubu üzerinde eşlenik alma işlemi ile etki eder. 1-potent, daha yaygın olarak idempotent olarak bilinen elemanların eşleniklerinin denklik sınıflarının kardinaliteleri hesaplanmıştır.

Son iki bölümde, nilpotent yarı grupların kardinalite özellikleri araştırılmıştır. Bu incelemede çizge teorik yöntemler kullanılmıştır. Maksimal simetrik tersinir yarı grupların kardinalitesi için elde edilen sonuçlar yorumlanarak, Bell sayıları ve Stirling sayılarını içeren bazı ilişkiler elde edilmiştir.

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LIST OF SYMBOLS/ABBREVIATIONS

$[n]$	The set $\{1, 2, 3, \ldots\}$
$\langle n \rangle$	The set $\{0, 1, 2, 3, \ldots\}$
$\mathcal{P}(A)$	Power set of A
$\mathcal{P}_i(A)$	The set of all elements λ of $\mathcal{P}(A)$ with $\lvert\lambda\rvert = i$
B_n	n th Bell number
$S(n, k)$	Stirling number of the second kind
$\left\{ n \atop k \right\}$	Stirling number of the second kind
$s(n, k)$	Stirling number of the first kind
$\left[n \atop k \right]$	Stirling number of the first kind
$p(n)$	The number of partitions of a positive integer n
$x^{\underline{n}}$	The falling factorial $x(x-1)\cdots(x-n+1)$
$x^{\overline{n}}$	The rising factorial $x(x+1)\cdots(x+n-1)$
$\mathcal{T}(A)$	The semigroup of all self transformations of A
$\mathcal{PT}(A)$	The semigroup of all partial selftransformations of A
$\mathcal{IS}(A)$	Symetric inverse subsemigroup of $\mathcal{PT}(A)$
$\mathcal{T}_n$	$\mathcal{T}([n])$
$\mathcal{PT}_n$	$\mathcal{PT}([n])$
$\mathcal{IS}_n$	$\mathcal{IS}([n])$
Γ_α	The digraph associated with the transformation $\alpha \in \mathcal{PT}_n$
$k\alpha$	The image of k under α for $\alpha \in \mathcal{PT}_n$
$\mathrm{Fix}(\alpha)$	$\{x \in [n] : x\alpha = x\}$
$\mathrm{Shift}(\alpha)$	$\{x \in [n] : x\alpha \neq x\}$
$\ell_\sigma(x)$	The maximum length of a chain C for partial order σ where x is on top
$L_\sigma(i)$	$\{x \in A : \ell_\sigma(x) = i\}$ where A is the base set of σ
$\mathfrak{O}_n$	The set of antisymmetric partial order on $[n]$
$\mathfrak{O}_n^{(k)}$	The set of all elements of $\mathfrak{O}_n$ whose longest chain has length k.
0_n	The element of $\mathcal{PT}_n$ which maps each $x \in \langle n \rangle$ to 0.
$\mathrm{Nil}(S)$	Nilpotent subsemigroups of a semigroup S

$\mathrm{Nil}_k(S)$	The elements of $\mathrm{Nil}(S)$ of nilpotency degree k.
$\mathrm{nd}(X)$	The nilpotency degree of X where X is either an element or a semigroup.
$\mathfrak{M}_n^{(k)}$	The set of all ordered partitions of $[n]$ with k elements
$\mathrm{Im}\,\alpha$	For $\alpha \in \mathcal{PT}_n$ the set of images of α contained in $[n]$
$\mathrm{rank}\,\alpha$	$\lvert\mathrm{Im}\,\alpha\rvert$
$f(B)$	$\sum_{k=0}^{n} a_k B_k$ where $f(x)$ is the polynomial $\sum_{k=0}^{n} a_k x^x$
$F(m_1, \ldots, m_k)$	$(x^{\underline{m_1}} \cdots x^{\underline{m_k}})(B)$
$\mathbf{X}(m_1, \ldots, m_k)$	The number of elements in maximal nilpotent subsemigroup of $\mathcal{IS}_n$ corresponding to $(m_1, \ldots, m_k) \in \mathfrak{M}_n^{(k)}$
IDSTR0	Incoming directed spanning trees rooted at 0

1. INTRODUCTION

The mathematical literature on $\mathcal{S}_n$, the symmetric group of n objects, is vast. Most classical semigroups investigated in detail are the full symmetric semigroup $\mathcal{T}_n$, $\mathcal{PT}_n$, and $\mathcal{IS}_n$. It can be shown that every finite semigroup is isomorphic to a subsemigroup of $\mathcal{T}_n$ and every finite inverse semigroup is isomorphic to a subsemigroup of $\mathcal{IS}_n$. This shows that these semigroups are quite important subjects of mathematical research.

Once we have a mathematical object finite in nature, it is almost always an interesting question to find the cardinalities of such objects possessing various properties. In this thesis we mainly consider the cardinalities of certain subsets of $\mathcal{T}_n$, $\mathcal{PT}_n$, and $\mathcal{IS}_n$. Given a partial transformation α , let

$$\begin{aligned} b(\alpha) &= |\mathrm{Dom}\,\alpha| \\ f(\alpha) &= |\{x \in [n] : x\alpha = x\}| \\ h(\alpha) &= |\mathrm{Im}\,\alpha| \\ c(\alpha) &= \left|\left\{x \in \mathrm{Dom}\,\alpha : x\alpha = t \text{ for some } t \text{ with } \left|t\alpha^{-1}\right| \geq 2\right\}\right| \end{aligned}$$

Comprehensive coverage of research on the cardinalities of some equivalences defined by means of these parameters on $\mathcal{PT}_n$ can be found in [1].

Throughout this thesis we use the directed graph representation of transformations in $\mathcal{PT}_n$. We use this interpretation and labelled trees to prove some of the cardinality issues. We believe this interpretation make the matters more picturesque and easy to grasp.

In the second chapter we expose the work done in [2]. An amazing result in this work is the description of elements with period r and index m, called (m, r)-potent elements. It claims that every (m, r)-potent element $\alpha \in \mathcal{T}_n$ can be written uniquely as $\alpha = \sigma\beta$ where σ is an element of order r in $\mathcal{S}_n$, β is an m-potent element and $\mathrm{Shift}(\sigma) \cap \mathrm{Shift}(\beta) = \phi$. An element α in $\mathcal{T}_n$ is called m-potent if α satisfies $\alpha^{m+1} = \alpha^m$. We expose this result by using directed graph representation and make it clear why the theorem holds.

The structure of m-potent elements is described in [2] and the cardinality of such elements is given by using this description. We expand this work by considering potent elements. More specifically, we define an element α as a potent element if it is m-potent for some positive integer m. We give a formula to evaluate the cardinality of such elements in $\mathcal{T}_n$. This formula gives the sum of the cardinalities of m-potent elements for $m = 1, \ldots, n-1$.

In the third chapter we consider the action of the symmetric group $\mathcal{S}_n$ by conjugation to $\mathcal{T}_n$. More specifically if $\alpha \in \mathcal{T}_n$ and $\sigma \in \mathcal{S}_n$, the action of σ on β is given as $\sigma^{-1}\beta\sigma$ called the conjugate of β. It naturally partitions $\mathcal{T}_n$ into equivalence classes. We make some remarks on the equivalence classes of (m, r)-potent elements. It is clear that a conjugate of an element is m-potent if and only if the element itself is m-potent. A 1-potent element is called idempotent in the literature. We have a through investigation on the equivalence classes of idempotent elements as well as the cardinalities of these classes. We also give some information of variants of semigroups. Some information can be found in Lyapin's monograph [3].

One of the widely investigated topic is nilpotent elements in $\mathcal{PT}_n$. There are enormous number of results on the cardinality of such elements with varying hypothesis imposed on the nature of nilpotent elements.

According to [4] first paper written purely in the context of semigroup theory is [5].

It is argued in [6] that if A_n is the number of all finite semigroups of order n and B_n is the number of all finite nilpotent semigroups of order n, then $\lim_{n\to\infty} \frac{A_n}{B_n} = 1$. To avoid peculiar set theoretic behavior it is assumed that both A_n and B_n, as sets, are subsets of positive integers. The proof of this claim is outlined in [6] but a complete proof is not given. Determining the number A_n and/or B_n seems to be an extremely difficult question.

In fourth chapter we deal with nilpotent subsemigroups in $\mathcal{PT}_n$ and $\mathcal{IS}_n$. Here we employ labelled trees again to reprove Theorem 4.30 and Theorem 4.31.

The exposition of maximal nilpotent subsemigroups depends on the process of association of nilpotent subsemigroups and partial orders on $[n]$. Using partial orders to describe maximal nilpotent semigroups appear in [7,8,9]. Most of the material in last two chapters can be found in these papers. The ideas used in these papers gives rise an abstract study of nilpotent semigroups. [10].

The amazing result, Theorem 55 is due to M. Pavlov. It appeared in [11] for the first time, and can also be found in [12]. In [13] analogous ideas were successfully applied to determine the cardinalities of some maximal nilpotent subsemigroups of the semigroup $\mathcal{IO}_n$ of all partial order-preserving injections.

We give a direct proof of Theorem 55 for partitions of the form (a, b) and (a, b, c) and use this fact to deduce some relationships involving Bell numbers and Stirling numbers.

First chapter contains preliminary results to be used throughout out the thesis.

2. PRELIMINARIES

2.1. GRAPH THEORY BASICS

In this section, we will review graph theory to meet our needs. We are mainly interested in the basic facts about graphs, especially directed graphs. These notes follow the footsteps of [14] [15] [16].

Definition 1. *An (undirected)* ***graph*** *G is a pair (V, E) of two finite sets:*

V is a non-void set . An element of V is called a ***vertex*** *of G.*

An element of E is called an ***edge*** *of G. For each edge e there are two associated vertices, x and y, distinct or not, called the* ***end vertices*** *of e.*

Example 1. *Let $V = \{a, b, c, d, e\}$, $E = \{e_1, e_2, e_3, e_4\}$. These are the sets of vertices and edges. We must also give the associated vertices for each edge.*

Edge	*Associated vertices*
e_1	a, b
e_2	a, b
e_3	b, c
e_4	d

Graphs are conventionally illustrated on the plane by selecting distinct points on the plane for each vertex and a path for each edge e joining its associated vertices. As an example we give an illustration of the graph in Example 1 below. The paths in this illustration are by no means geometric paths of any kind. A path representing an edge e is drawn to simply point out that "two vertices are end points of the edge e."

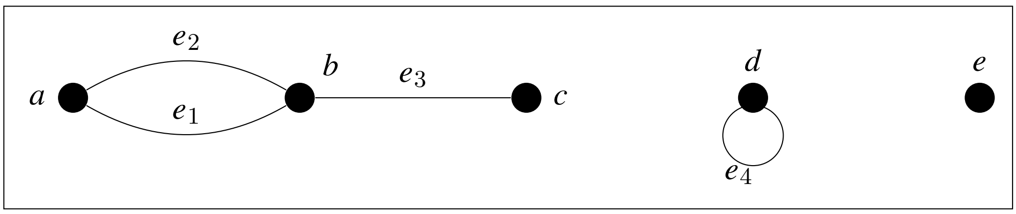


Figure 2.1. Graph of first example

Definition 2. *A **loop** is an edge whose end points are equal. **Multiple edges** are edges having the same pair of end points. A **simple graph** is a graph having no loops or multiple edges.*

In Example 1 e_1, e_2 are multiple edges with the same end points a, b. e_4 is a loop with a single end points d.

We specify a simple graph by its vertex set and edge set, treating the edge set as a set of unordered pairs of vertices and writing $e = uv$ (or $e = vu$) for an edge e with end points u and v. Let us use the notation $\mathcal{P}_i(V)$ to denote the subsets of V containing exactly i elements.

$$\varepsilon : E \to \mathcal{P}_2(V)$$

defined by $\varepsilon(e) = \{x, y\}$ where x, y are the distinct vertices associated with e, then ε is one-to-one. Thus if $G = (V, E)$ is a simple graph then we may interpret E as a subset $\mathcal{P}_2(V)$.

In general, we may think of E as a multiset whose elements are chosen from $\mathcal{P}_1(V) \cup \mathcal{P}_2(V)$ where a subset $\{x, y\} \subset V$ is repeated as many times as there are edges with associated vertices x, y.

Definition 3. *When u and v are the end points of an edge e, they are **adjacent** and are neighbours. A vertex a is **incident** to an edge e if a is one of the two end points associated with e. Two edges e and f are incident if the have common end point.*

Remark 1. *The **null graph** is the graph whose edge set is empty.*

A **walk** is a list $v_0, e_1, v_1, \ldots, e_k, v_k$ of vertices and edges such that, for $1 \le i \le k$, the edge e_i has end points v_{i-1} and v_i.

$$v_0 \xrightarrow{e_1} v_1 \to, \ldots, v_{k-1} \xrightarrow{e_k} v_k \tag{2.1}$$

We call v_0 the **initial vertex** and v_k the **final vertex** of the walk, and say the list 2.1 is a walk from v_0 to v_k. v_0 and v_k are the **end points** of the walk. The **length** of a walk is the number of edges in the walk.

A **trail** is a walk with no repeated edge. If, in addition, the vertices $v_0, v_1, \ldots, v_k$ are distinct (except, possibly, $v_0 = v_k$), then the trail is a **path**. A walk or trail is **closed** if its end points are the same. If a closed path containing at least one edge is a **cycle**. Note that any loop or pair of multiple edges is a cycle. Also note that in a simple graph a cycle must have length at least 3. A cycle of length 3 is a triangle.

A graph is **connected** if there is a walk from a to b for any pair of vertices a and b.

Theorem 2. *A graph G is connected if and only if for any pair of vertices a and b there is a path from a to b.*

Proof. Sufficiency is clear. We need to prove necessity. Let G be connected and a and b be a pair of distinct vertices. There is a walk from a to b. Let

$$a = v_0 \xrightarrow{e_1} v_1 \rightarrow, \ldots, v_{k-1} \xrightarrow{e_k} v_k = b \tag{2.2}$$

be a walk with smallest length k. Then this walk must be a path. For if $0 \le i < j \le k$ and $v_i = v_j$, then we may remove the part of walk 2.2 from v_i to v_j obtaining a path with smaller length, a contradiction. □

A **subgraph** of a graph G is a graph H such that $V(H) \subset V(G)$ and $E(H) \subset E(G)$'and the assignment of end points of edges in H is the same as in G. We then write $H \subset G$ and say that "G contains H".

Definition 4. *An* ***isomorphism*** *from a simple graph G to a simple graph H is a bijection* $f : V(G) \rightarrow V(H)$ *such that* $uv \in E(G)$ *if and only if* $f(u)f(v) \in E(H)$. *We say "G is-isomorphic to H", written* $G \cong H$, *if there is an isomorphism from G to H.*

An **isomorphism class** of graphs is an equivalence class of graphs under the isomorphism relation.

A **clique** in a graph G is a subset of the vertex set $C \subseteq V(G)$, such that for every two vertices in C, there exists an edge connecting the two. An **independent** set (or **stable set**) in a graph is a set of pairwise non adjacent vertices.

2.2. DIGRAPHS

A **directed** graph, or **digraph**, G is a pair $(V(G), E(G))$ where $V(G)$ is a non-empty finite set called the set of **vertices** or **nodes**, $E(G)$ is a finite family of ordered pairs of elements of $V(G)$ called the set of **arcs** or **edges**. An example of a digraph is shown in Figure 2.2.

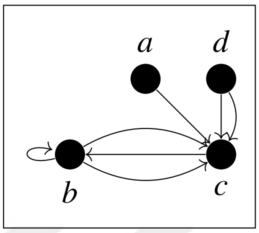


Figure 2.2. An example of a digraph

An arc (v, w) is usually abbreviated to vw. Given an edge vw, the node v is the **tail** of the edge, the node w is the **head**; together, they are the end points of vw.

If $v = w$, then vw is a **loop** at v. We say vw is an edge from its **tail** v to its **head** w.

A digraph G is a **simple digraph** if there are no multiple arcs in the family $E\ (G)$ and there are no loops. Given an edge (a, b) in a digraph, the set $\{a, b\}$ is, by definition, the edge of obtained by removing the order. The underlying graph of a digraph (V, E) is the graph (V, E') where E' is obtained from E by removing the order of each edge. In Figure 2.2 we see a digraph with its underlying graph.

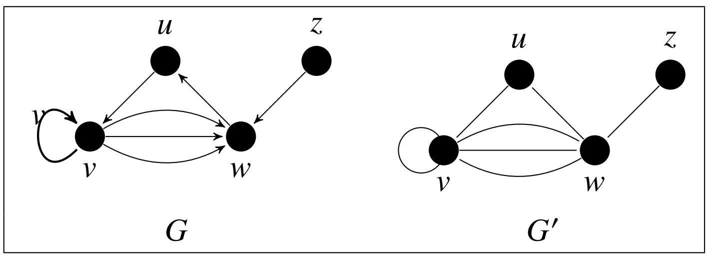


Figure 2.3. A digraph and its underlying graph

Note that the underlying graph of a simple digraph need not be a simple graph. In Figure 2.4 we see such an example.

Two digraphs are **isomorphic** if there is an isomorphism between their underlying graphs

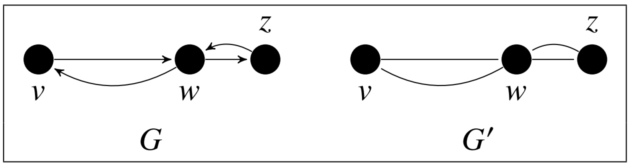


Figure 2.4. A simple digraph with non-simple underlying graph

that preserves the ordering of the vertices in each arc. Most of the terminology developed for undirected graphs carry over to digraphs almost without change.

Let G be a directed graph and $v, w \in V(G)$. v and w are **adjacent** if there is an arc in $E(G)$ of the form vw or wv. The vertices v and w are **incident** to vw and wv. If $V(G) = \{v_1, ..., v_n\}$, the **adjacency matrix** of G is the $n \times n$ matrix $A = (a_{ij})$, where a_{ij} is the number of arcs from v_i to v_j.

Let G be a digraph and $v_0, v_m \in V(G)$. A **walk** from v_0 to v_m in G is a finite sequence of arcs of the form $v_0 \to v_1 \to ... \to v_{m-1} \to v_m$. Similarly, we can define **directed trails**, **directed paths** and **directed cycles**. A trail cannot contain a given arc vw more than once, but it can contain both vw and wv.

A digraph G is **weakly connected** if its underlying path is connected. G is **strongly connected** if, for any two vertices v and w of G, there is a directed path from v to w. Every strongly connected digraph is connected, but not all weakly connected digraphs are strongly connected.

The digraph G given in Figure 2.5(a) is not strongly connected since there is no directed path from w to z. But it is weakly connected since its underlying graph is connected. The

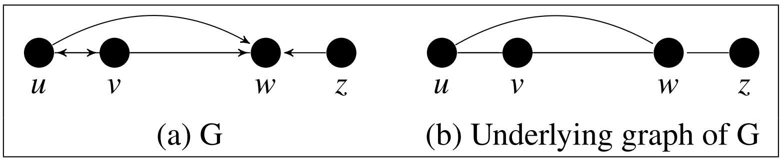


Figure 2.5. A non strongly connected digraph with connected uUnderlying graph

out-degree of a vertex v of G is the number of arcs of the form vw, and is denoted by $\operatorname{outdeg}(v)$ or $d^+(v)$. Similarly, the in-degree of v is the number of arcs of G of the form wv, and is denoted by $\operatorname{indeg}(v)$ or $d^-(v)$.

The **out-neighbourhood** or **successor set** $N^+(v)$ is $\{x \in V(G) : v \to x\}$.

The **in-neighbourhood** or **predecessor set** $N^{-}(v)$ is $\{x \in V(G) : x \rightarrow v\}$.

2.3. TUTTE'S THEOREM

Let $G = (V, E)$ be a directed graph with finite vertex set V and finite directed edge set. We assume that there are no multiple edges, that is there exists at most one directed edge between distinct vertices. We further assume that G has no loops. The **in-degree** of a vertex $v \in V$ is the number of vertices w such that (w, v) is an edge. Similarly, the **out-degree** of v is the number of vertices w such that (v, w) is an edge. A **directed cycle** of G is a collection of distinct vertices $\{v_1, v_2, \ldots, v_n\}$ such that (v_j, v_{j+1}) is an edge where $v_{n+1} = v_1$.

A directed graph $G' = (V', E')$ is a directed subgraph of G if $V' \subset V$ and $E' \subset E$. Let us pick a vertex r in V. A directed subgraph G' of G is an **outgoing** (**incoming**) **directed spanning tree rooted at** r, if $V' = V$, and if the following conditions hold:

1) Every vertex $v \neq r$ in V' has in-degree (out-degree) 1.

2) The root vertex r has in-degree (out-degree) 0.

3) G' has no directed cycles.

Given a directed graph G, **adjacency matrix** A of G is the square $m \times m$ matrix defined by

$$[A]_{vw} = \begin{cases} 1 & \text{if } v \neq w \text{ and } (v, w) \in E \\ 0 & \text{otherwise} \end{cases}$$

The indegree and outdegree diagonal matrices are square $m \times m$ matrices defined as follows:

- D_{in} is a diagonal matrix defined as $[D_{in}]_{vv}$ = in-degree of vertex v, for all $v \in V$.

- D_{out} is a diagonal matrix defined as $[D_{out}]_{vv}$ = out-degree of vertex v, for all $v \in V$.

The **Laplacians** of G are $m \times m$ matrices L_{in} and L_{out} defined as follows

$$L_{in} = D_{in} - A, \text{ and } L_{out} = D_{out} - A^T$$

where A is adjacency matrix of G and A^T is the transpose of A.

Given a vertex $r \in V$,we define the reduced Laplacians L_{in}^r and L_{out}^r by removing the rth row and r th columns from L_{in} and L_{out}, respectively.

Theorem 3 (Tutte's Directed Matrix-Tree Theorem). *[17] Let $G = (V, E)$ be a digraph. The number of outgoing and incoming directed spanning trees rooted at v_r are equal to* $\det L_1^r$ *and* $\det L_2^r$*, respectively.*

2.4. LABELLED TREES

Let V be a linearly ordered non empty finite set containing $n \geq 2$ letters or numbers. If T is a tree with vertices labelled with letters in V. There are $\binom{n}{2}$ subsets containing 2 elements. Let us denote by $\mathcal{P}_2(V)$ two element subsets of V. Any subset of E of $\mathcal{P}_2(V)$ is the edge set of the simple graph $G = (V, E)$. Thus there are $2^{\binom{n}{2}}$ of such simple graph. Let

$$\mathcal{L}(V) = \{(V, E) : E \in \mathcal{P}_2(V) \text{ and } (V, E) \text{ is a tree}\}$$

An element of $\mathcal{L}(V)$ is called a **labelled tree**.

Let us briefly recall the Prüfer code of a labelled tree in $\mathcal{L}(V)$. If T is a tree with vertices labelled with letters in V, then its **Prüfer code** $P(T) \in V^{n-2}$ is defined inductively as follows:

It will be convenient to refer to the vertices of T with their labels. If T has only two vertices, $P(T)$ is empty. Otherwise, the first term a of $P(T)$ is the label of the neighbour of the leaf of T having the smallest label b. Then $P(T)$ is the sequence $a, P(T - b)$.

As an example let us write the Prüfer code of the tree 2.6 where $V = [n]$. In the following Figure 2.7 we will use b for the smallest leaf, a for neighbour of b and S for the Prüfer sequence constructed so far.

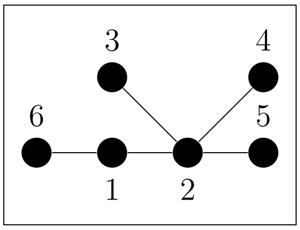


Figure 2.6. A tree

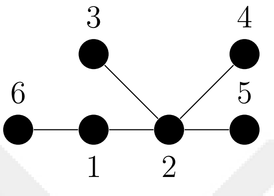
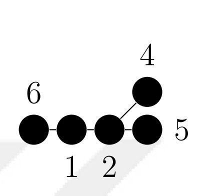
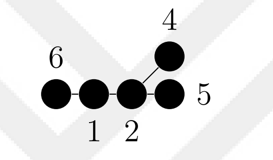
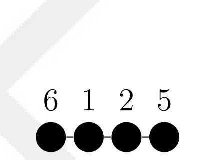
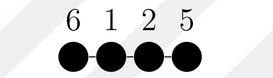
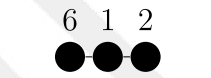
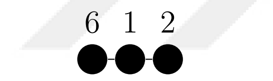
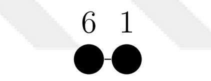
T	*b*	*a*	*S*	**T-b**
	3	2	(2)	
	4	2	(2,2)	
	5	2	(2,2,2)	
	2	1	(2,2,2,1)	

Figure 2.7. Prüfer code of the tree $\{\{2,3\},\{2,4\},\{2,5\},\{2,1\},\{1,6\}\}$

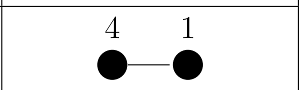
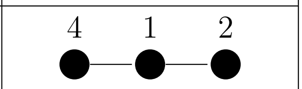
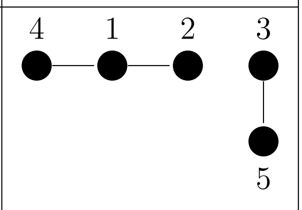
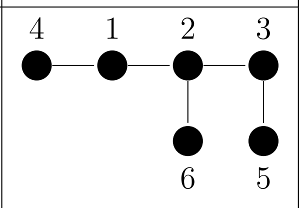
Sequence Left	**Symbols Left**	**Tree built**
1,2,3,2	1,2,3,4,5,6	none
2,3,2	1,2,3,5,6	
3,2	2,3,5,6	
2	2,3,6	
empty	2,6	

Figure 2.8. The tree with Prüfer code $s = (1,2,3,2)$

Given any $s = (a_1, a_2, \ldots, a_{n-2}) \in V^{n-2}$, we define a function $\lambda : V^{n-2} \to \mathcal{L}(V)$ as follows. First take n points labelled by V. b_1 be the smallest number that does not appear in it, and join the vertices a_1 and b_1. We then remove a_1 from the sequence, remove the number b_1 from consideration, and proceed as before. In this way we build up the tree, edge by edge. Figure 2.8 is an example of this process.

An immediate consequence of this definition is the following theorem.

Theorem 4 (Cayley, 1898)**.** *[15] There are n^{n-2} distinct labelled trees on n vertices.*

Proof. It is easy to see that $P(\lambda(s)) = s$. This means that P is one-to-one and λ is onto. Since both V^{n-2} and $\mathcal{L}(V)$ are finite we see that P is onto. $\square$

Associated with each labelled tree $T \in \mathcal{L}(V)$ and a vertex $r \in V$ designated as **root** we may associate with T a directed tree $\varphi(T, r)$ rooted at r. The vertex set of $T(r)$ is again V. For every vertex a in T there exists a unique path π_a from a to r. The length of π_a is defined as the **level** of a denoted by $\mathrm{level}(a)$. Using the level function, we may assign an orientation of vertices in T, thereby making T a directed one, as follows: Let $e = \{a, b\}$ be an edge of T. Either b is the vertex that comes immediately after a in π_a or a is the vertex that comes immediately after b in π_b. In the first case we direct the edge e as the ordered pair (a, b) and in the second case we direct the edge e as the ordered pair (b, a). Notice that in the first case $\mathrm{level}(a) = \mathrm{level}(b) + 1$ and in the second case $\mathrm{level}(b) = \mathrm{level}(a) + 1$. We also note that each vertex has outdegree 1 except r. By definition the image set $\mathrm{Im}\,\varphi(T, r)$ is the set of all vertices with positive indegree. such that in

Theorem 5 ([18])**.** *If $T \in \mathcal{L}(V)$ and $v \in V$, then v appears $\deg v - 1$ times in Prüfer code of T.*

Corollary 1. *If $T \in \mathcal{L}(V)$ and v is a vertex of T then v is in $\mathrm{Im}\,\varphi(T, r)$ if and only if v appears as a coordinate in Prüfer code.*

2.5. BELL NUMBERS

In the following three sections we will give some background on Bell numbers and Stirling numbers. The theorems can be found in the books [19] we include the proofs for completeness.

Definition 5. *Let A be a set with $|A| = n$. A non empty collection $\mathcal{P}$ of subsets of A is called a **partition** of A if satisfies the following conditions:*

(a) *if $B \in \mathcal{P}$, then $B \neq \phi$,*

(b) *the subsets in $\mathcal{P}$ are pairwise disjoint, and*

(c) $A = \cup\{P : P \in \mathcal{P}\}$.

*The subsets in $\mathcal{P}$ are called **blocks** of $\mathcal{P}$.*

We notice the simple fact that if A and B are sets with $|A| = |B| = n$ and if $\varphi{:}A \to B$ is a bijection, then $\mathcal{P}$ is a partition of A if and only if $\mathcal{P}f = \{Pf : P \in \mathcal{P}\}$ is a partition of B. So the number of distinct partitions of both sets are the same.

Definition 6. *The number of all partitions of the set $[n]$ is the n^{th} **Bell number**, denoted by B_n.*

For convenience we define $B(0) = 1$. A list of a few Bell numbers are given in the following table.

Table 2.1. First few Bell numbers

n	0	1	2	3	4	5	6	7	8	9	10
B_n	1	1	2	5	15	52	203	877	4140	21147	115975

We may easily give a recursion formula for the Bell numbers.

Theorem 6. *[19] For all $n \geq 1$*

$$B(n) = \sum_{s=0}^{n-1} \binom{n-1}{s} B(s)$$

Proof. Let $\mathcal{M}$ be the set of all partitions of $[n]$. If $\mathcal{P} \in \mathcal{M}$, then n belongs to a unique set

$C_n(\mathcal{P}) \in \mathcal{P}$. Let

$$\mathcal{M}_k = \{\mathcal{P} \in \mathcal{M} : |C_n(\mathcal{P})| = k\}$$

If $k \neq s$, then $\mathcal{M}_k \cap \mathcal{M}_s = \phi$, and

$$\mathcal{M} = \bigcup_{k=1}^{n} \mathcal{M}_k$$

So it suffices to find $|\mathcal{M}_k|$. To choose a subset of $[n]$ with k elements we have to choose a subset of $[n] - \{n\}$ with $k-1$ elements and put n into it. So there are exactly $\binom{n-1}{k-1}$ ways of choosing a subset of $[n]$ with k elements containing n. For a fixed such a subset C, remaining $n-k$ elements of $[n]$ can be partitioned in $B(n-k)$ different ways. Thus

$$|\mathcal{M}_k| = \binom{n-1}{k-1} B(n-k)$$

It follows that

$$B(n) = \binom{n-1}{0} B(n-1) + \binom{n-1}{1} B(n-2) + \cdots + \binom{n-1}{n-2} B(1) + \binom{n-1}{n-1}$$

Thus

$$B(n) = \sum_{k=1}^{n} \binom{n-1}{k-1} B(n-k) = \sum_{k=1}^{n} \binom{n-1}{n-k} B(n-k) = \sum_{s=0}^{n-1} \binom{n-1}{s} B(s)$$

□

2.6. STIRLING NUMBERS

Definition 7. *For $1 \leq k \leq n$ the* ***Stirling number of the second kind*** *$S(n,k)$ is the partitions of $[n]$ which contains exactly k blocks.*

$S(n,k)$ is also denoted by the symbol $\left\{{n \atop k}\right\}$. We set $S(0,0) = 1$. When $n > 0$, by definition of $S(n,k)$, we have $S(n,k) = 0$ for $k = 0$ and $k > n$.

Theorem 7. *[19] $S(n+1,k) = S(n,k-1) + kS(n,k)$.*

Proof. The partitions of $[n+1]$ with k blocks is a union of two disjoint sets $\mathcal{A}$ and $\mathcal{B}$. A partition $\mathcal{P}$ with k blocks belong to $\mathcal{A}$ if and only if the block in $\mathcal{P}$ which contains $n+1$ is $\{n+1\}$. A partition $\mathcal{P}$ with k blocks belong to $\mathcal{B}$ if and only if the block in $\mathcal{P}$ which contain

$n+1$ contains at least one more element. If $\mathcal{P} \in \mathcal{A}$, then $\mathcal{P} - \{\{n+1\}\}$ is a partition of $[n]$ with $k-1$ blocks. Given any partition $\mathcal{D}$ of $[n]$ with $k-1$ blocks $\mathcal{D} \cup \{\{n+1\}\} \in \mathcal{A}$. So we have $|\mathcal{A}| = S(n, k-1)$.

Given any partition $\mathcal{E}$ of $[n]$ with k blocks we have k choices of blocks to choose to add $n+1$ into the block to get an element in $\mathcal{P} \in \mathcal{B}$. Thus $|\mathcal{B}| = kS(n, k)$. It follows that $S(n+1, k) = S(n, k-1) + kS(n, k)$. □

A list of a few Stirling numbers of the second kind are given in the following table.

Table 2.2. First few Stirling numbers of second kind

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$n=0$	1					
$n=1$		1				
$n=2$		1	1			
$n=3$		1	3	1		
$n=4$		1	7	6	1	
$n=5$		1	15	25	10	1
$n=6$		1	31	90	65	15

The following formula attributed to Euler provides an explicit formula for $S(n, k)$.

Theorem 8. *[19]* $S(n, k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$.

Proof. We can make the proof by induction on $m = n + k$. By definition $S(n, k)$ we have $S(n, k) = 0$ for $k = 0$ or $k > n$. Now suppose that $m = n+1+k$ and the claim holds for smaller m. We know that $S(n+1, k) = S(n, k-1) + kS(n, k)$. If we set $L = S(n, k-1) + kS(n, k)$, then

$$\begin{aligned}
L &= \frac{1}{(k-1)!}\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}(k-1-j)^n+\frac{k}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n \\
&= \frac{1}{(k-1)!}\sum_{j=1}^{k}(-1)^{j-1}\binom{k-1}{j-1}(k-j)^n+\frac{1}{(k-1)!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n \\
&= \frac{1}{(k-1)!}\sum_{j=1}^{k}(-1)^j\left(\binom{k}{j}-\binom{k-1}{j-1}\right)(k-j)^n+\frac{1}{(k-1)!}k^n \\
&= \frac{1}{k\,(k-1)!}\sum_{j=1}^{k}(-1)^j\binom{k}{j}(k-j)^{n+1}+\frac{1}{(k-1)!}k^n \\
&= \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^{n+1}
\end{aligned}$$

This proves the claim. □

Definition 8. *[14] The polynomial* $x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1)$ *called the* ***falling factorial****.*

Theorem 9. *[14] The falling factorial satisfies*

$$x^n = \sum_{k=1}^{n} S(n,k)\,x^{\underline{k}}$$

Proof. We prove this by induction on n. Let us first check a few cases:

n	$\sum_{k=1}^{n} S(n,k)\,x^k$
1	$S(1,1)\,x^{\underline{1}} = x$
2	$S(2,1)\,x + S(2,2)\left(x^2-x\right) = x^2$

Let us assume the claim for n and prove the identity for $n+1$.

Since $S(n+1,k) = S(n,k-1) + kS(n,k)$ and $S(n,n+1) = 0$ we get

$$\begin{aligned}
\sum_{k=1}^{n+1} S(n+1,k)\, x^{\underline{k}} &= \sum_{k=1}^{n+1} S(n,k-1)\, x^{\underline{k}} + \sum_{k=1}^{n+1} kS(n,k)\, x^{\underline{k}} \\
&= \sum_{k=1}^{n} S(n,k)\, x^{\underline{k+1}} + \sum_{k=1}^{n} kS(n,k)\, x^{\underline{k}} \\
&= \sum_{k=1}^{n} S(n,k)\, x^{\underline{k}}(x-k) + \sum_{k=1}^{n} kS(n,k)\, x^{\underline{k}} \\
&= \sum_{k=1}^{n} S(n,k)\, x^{\underline{k}}(x-k+k) \\
&= x\sum_{k=1}^{n} S(n,k)\, x^{\underline{k}} = x^{n+1}
\end{aligned}$$

□

Definition 9. *[20] For $1 \leq k \leq n$ the (* ***Stirling number of the first kind*** *$s(n,k)$ is the number of permutation of $[n]$ with exactly k cycles.*

In [19] $s(n,k)$ is denoted by $c(n,k)$ and called the signless Stirling number of the first kind. We prefer to use the notation in [20]. $s(n,k)$ is also denoted by the symbol $\left[{n \atop k}\right]$. We set $s(0,0) = 1$. When $n > 0$, by definition of $s(n,k)$, we have $s(n,k) = 0$ for $k = 0$ or $k > n$.

Theorem 10. *[20][19] $s(n+1,k) = s(n,k-1) + ns(n,k)$.*

Proof. Let W be the set of permutations $[n+1]$ with exactly k cycles into two groups A and B. A consists of permutations $\sigma \in W$, for which $(n+1)$ is a cycle of length 1 in the cycle representation of σ. Clearly $|A| = s(n,k-1)$. B be the set of permutations $\sigma \in W$, for which $(n+1)$ is not a cycle of length 1 in the cycle representation of σ. Given any permutation of $[n]$ with exactly k cycles we can place $n+1$ after each element $i \in [n]$. Clearly this yields an element in B. Since there are n elements in $[n]$, then $|B| = ns(n,k)$. This proves that $s(n+1,k) = |W| = s(n,k-1) + ns(n,k)$. □

A list of a few Stirling numbers of the first kind are given in the following table.

Table 2.3. First few Stirling numbers of first kind

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$n=0$	1						
$n=1$		1					
$n=2$		1	1				
$n=3$		2	3	1			
$n=4$		6	11	6	1		
$n=5$		24	50	35	10	1	
$n=6$		120	274	225	85	15	1

Theorem 11. *[20] Stirling numbers of the first kind satisfies the following relation.*

$$x^{\underline{n}} = \sum_{k=1}^{n} (-1)^{n-k} \, s(n,k)\, x^k$$

Proof. We prove this by induction on n. Let us first check a few cases:

$$\begin{array}{cc}
n & \sum_{k=1}^{n} (-1)^{n-k} \, s(n,k)\, x^k \\
1 & s(1,1)\, x = x^{\underline{1}} \\
2 & -s(2,1)\, x + s(2,2)\, x^2 = x^2 - x = x^{\underline{2}}
\end{array}$$

Let us assume the claim for n and prove the identity for $n+1$. Since $s(n+1,k) = s(n,k-1) + ns(n,k)$ and $s(n,n+1) = 0$ we get

$$\begin{aligned}
\sum_{k=1}^{n+1} (-1)^{n+1-k} \, s(n+1,k)\, x^k &= \sum_{k=1}^{n+1} (-1)^{n+1-k} \, s(n,k-1)\, x^k + \sum_{k=1}^{n+1} (-1)^{n+1-k} \, ns(n,k)\, x^k \\
&= \sum_{k=1}^{n} (-1)^{n-k} \, s(n,k)\, x^{k+1} + \sum_{k=1}^{n} (-1)^{n+1-k} \, ns(n,k)\, x^k \\
&= \sum_{k=1}^{n} (-1)^{n-k} \, s(n,k)\, (x-n)\, x^k \\
&= x^{\underline{n}} (x-n) \\
&= x^{\underline{n+1}}
\end{aligned}$$

□

Let P_n be the vector space of all polynomials with rational coefficients of degree n and Q_n

be the subspace of P_n consisting of polynomials without constant term. Clearly the sets $\left\{x, x^2, \ldots, x^n\right\}$ and $\left\{x^{\underline{1}}, x^{\underline{2}}, \ldots, x^{\underline{n}}\right\}$ are two distinct bases of Q_n. So the theorems above give the transformation matrices between these two bases. Let us define the matrices Σ_n and Δ_n by

$$\Sigma_n = \begin{pmatrix} S(1,1) & & & \\ S(2,1) & S(2,2) & & \\ \vdots & \vdots & & \\ S(n,1) & S(n,2) & \cdots & S(n,n) \end{pmatrix}$$

$$\Delta_n = \begin{pmatrix} s(1,1) & & & \\ -s(2,1) & s(2,2) & & \\ \vdots & \vdots & & \\ (-1)^{n-1} s(n,1) & (-1)^{n-2} s(n,2) & \cdots & s(n,n) \end{pmatrix}$$

Then

$$\Sigma_n \begin{pmatrix} x^{\underline{1}} \\ x^{\underline{2}} \\ \vdots \\ x^{\underline{n}} \end{pmatrix} = \begin{pmatrix} x \\ x^2 \\ \vdots \\ x^n \end{pmatrix} \text{ and } \Delta_n \begin{pmatrix} x \\ x^2 \\ \vdots \\ x^n \end{pmatrix} = \begin{pmatrix} x^{\underline{1}} \\ x^{\underline{2}} \\ \vdots \\ x^{\underline{n}} \end{pmatrix}$$

It follows that

$$\Delta_n \Sigma_n \begin{pmatrix} x^{\underline{1}} \\ x^{\underline{2}} \\ \vdots \\ x^{\underline{n}} \end{pmatrix} = \begin{pmatrix} x^{\underline{1}} \\ x^{\underline{2}} \\ \vdots \\ x^{\underline{n}} \end{pmatrix} \text{ and } \Sigma_n \Delta_n \begin{pmatrix} x \\ x^2 \\ \vdots \\ x^n \end{pmatrix} = \begin{pmatrix} x \\ x^2 \\ \vdots \\ x^n \end{pmatrix}$$

Hence we have proved the following theorem.

Theorem 12. *$\Delta_n \Sigma_n = \Sigma_n \Delta_n = I_n$ the identity matrix of size n.*

We can now utilize this theorem to prove the inversion formula for Stirling numbers.

Theorem 13 (Inversion Formula for Stirling Numbers)**.** *[20] Let $\{a_k\}$ and $\{b_k\}$ be two sequences for $k \in \mathbb{N}$. Then $a_n = \sum_{k=1}^{n} S(n,k)\, b_k$ for every $n \in \mathbb{N}$ if and only if $b_n = \sum_{k=1}^{n} (-1)^{n-k} s(n,k)\, a_k$ for every $n \in \mathbb{N}$.*

As an application of this theorem we have the following.

Theorem 14. *For every $m \in \mathbb{N}$ we have*

$$1 = \sum_{i=1}^{m} (-1)^{m-i} s(m,i) B_i$$

Proof. Let $a_k = B_k$ and $b_k = 1$ for $k \in \mathbb{N}$. It is clear that

$$B_m = \sum_{i=1}^{m} S(m,i) = \sum_{i=1}^{m} S(m,i) \cdot 1$$

It follows from inversion formula that

$$1 = \sum_{i=1}^{m} (-1)^{m-i} s(m,i) B_i$$

□

2.7. PARTITIONS OF INTEGERS

Given an integer $n \in \mathbb{N}$ a solution $(x_1, x_2, \cdots, x_k)$ of the equation

$$n = x_1 + x_2 + \cdots + x_k \tag{2.3}$$

satisfying $x_1 \geq x_2 \geq \cdots \geq x_k \geq 1$ is called a **partition** of the integer n. x_i's are called the **parts** of the partition. The number of distinct partitions is denoted by $p(n)$. Number of partitions with k parts is denoted by $p_k(n)$. There is a considerable literature on $p(n)$ for example [21] may be used as a reference. There is now explicit formula for $p_k(n)$ or $p(n)$. But there is a recurrence relation which may be used to calculate these numbers.

Notice that

$$p(n) = \sum_{k=1}^{n} p_k(n)$$

Theorem 15. *[19] If $0 < k \leq n$, then*

$$p_k(n) = p_{k-1}(n-1) + p_k(n-k) \tag{2.4}$$

Proof. The partitions into k parts may be viewed in two A and B. A is the set of partitions in which $x_k = 1$ and B is the set of partitions in which $x_k > 1$. For any partition $(x_1, x_2, \cdots, 1) \in A$ we have that $(x_1, x_2, \cdots, x_{k-1})$ is a partition of $n-1$ into $k-1$ parts and the correspondence $(x_1, x_2, \cdots, 1) \leftrightarrow (x_1, x_2, \cdots, x_{k-1})$ is clearly bijective. If $(x_1, x_2, \cdots, x_k) \in B$ is a partition, then $(x_1 - 1, x_2 - 1, \cdots, x_k - 1)$ is a partition of $n - k$ into k parts and $(x_1, x_2, \cdots, x_k) \leftrightarrow (x_1 - 1, x_2 - 1, \cdots, x_k - 1)$ is clearly bijective. This shows that

$$p_k(n) = |A| + |B| = p_{k-1}(n-1) + p_k(n-k)$$

□

Using this formula we compute a few of the values of $p(n)$.

Table 2.4. Partition of integers $p(n)$ for $1 \leq n \leq 8$

n	1	2	3	4	5	6	7	8	$\cdots$
$p(n)$	1	2	3	5	7	11	15	22	$\cdots$

Theorem 16. *The equation*

$$nr_n + (n-1)r_{n-1} + \cdots + 2r_2 + r_1 = n \tag{2.5}$$

has $p(n)$ nonnegative integer solutions $(r_1, r_2, \ldots, r_n) \in \mathbb{N}^n$.

Proof. Given a partition $(x_1, x_2, \cdots, x_k)$ of n let k_i be the number of occurrences of i in $(x_1, x_2, \cdots, x_k)$. Then

$$n = x_1 + x_2 + \cdots + x_k = r_1 + 2r_2 + \cdots + nr_n$$

Thus $(r_1, r_2, \ldots, r_n) \in \mathbb{N}^n$ is a solution of Equation 2.5. If $(r_1, r_2, \ldots, r_n) \in \mathbb{N}^n$ is a solution of Equation 2.5, let the non zero values be $r_{j_1}, r_{j_2}, \cdots, r_{j_t}$ where $j_1 > \cdots > j_{t-1} > j_t$ we let first r_{j_t} of x_i's equal to j_t, next $r_{j_{t-1}}$of x_i's equal to j_{t-1}, etc. This give a partition $(x_1, x_2, \cdots, x_k)$ which is a solution of Equation 2.3 where $k = r_{j_1} + r_{j_2} + \cdots + r_{j_t}$. Clearly this correspondence is bijective. This proves the theorem. □

2.8. PARTIAL ORDERS

Definition 10. *A **partial order** on a set A is a relation $\sigma \subset A \times A$ satisfying:*

P1) (antisymmetry) $(x, x) \notin \sigma$ for all $x \in A$.

P2) (transitivity) If $(x, y) \in \sigma$ and $(y, z) \in \sigma$, then $(x, z) \in \sigma$.

We will assume that A is finite in the following discussion. If σ is a partial order, then we write $x <_\sigma y$ to indicate that $(x, y) \in \sigma$. A element $y \in A$ **covers** $x \in A$ if $x <_\sigma y$ and there is no $z \in A$ satisfying $x <_\sigma z <_\sigma y$. A **chain** of **length** r is a sequence $x_0, x_1, \ldots, x_r$ in A such that $x_0 <_\sigma x_1 <_\sigma \cdots <_\sigma x_r$. By a chain of length 0 we simply mean an element x_0 of A. If x is the maximum element of a chain we say x sits on the **top** of the chain. So each element $x \in A$ is on the top of the chain consisting of only $x_0 = x$. A chain is **maximal** if it is not a proper subsequence of another chain. A chain $x_0 <_\sigma x_1 <_\sigma \cdots <_\sigma x_r$ is **saturated** if x_i covers x_{i-1} for $i = 1, \ldots, r$. A partial order σ is **graded of rank** r if every maximal chain has the same length r. The **rank** or **level** $\ell_\sigma(x)$ is the maximum length of a chain C assuming x on top. So if $\ell_\sigma(x) = 0$, then x is minimal in the sense there is no element $z \in A$ with $z <_\sigma x$. Notice that if $x_0 <_\sigma x_1 <_\sigma \cdots <_\sigma x_r = x$ is a chain with $\ell_\sigma(x) = r$, then this chain is saturated.

Let σ be a partial order and $r_\sigma = \max \ell_\sigma(x)$. We call r_σ the **rank** of σ. For $0 \leq i \leq r_\sigma$ we set

$$L_\sigma(i) = \{x \in A : \ell_\sigma(x) = i\}$$

We call $L_\sigma(i)$, the **level sets** of σ.

Lemma 1. *Let σ be a partial order of rank r_σ on a finite non empty set A.*

a) $L_\sigma(i) \neq \phi$ for any $0 \leq i \leq r_\sigma$ and A is the disjoint union of these level sets.

b) If $x, y \in L_\sigma(i)$, then neither $x <_\sigma y$ nor $y <_\sigma x$, i.e. x, y are incomparable.

Proof. a) If $r_\sigma = 0$, then $L_\sigma(0) = A \neq \phi$. Suppose that $r = r_\sigma > 0$ and $x \in A$ is such that $x_0 <_\sigma x_1 <_\sigma \cdots <_\sigma x_r = x$ is a chain of maximum length. Clearly $i \leq \ell_\sigma(x_i)$ for

$0 \leq i \leq r_\sigma$. Suppose

$$y_0 <_\sigma y_1 <_\sigma \cdots <_\sigma y_s = x_i \text{ where } s = \ell_\sigma(x_i)$$

is a chain of maximum length with x_i on top. Then

$$y_0 <_\sigma y_1 <_\sigma \cdots <_\sigma y_s = x_i <_\sigma \cdots <_\sigma x_r = x$$

is a chain of length $s + (r - i)$. Hence $s + (r - i) \leq r$, which implies that $s \leq i$. Thus $s = i$. So $x_i \in L_\sigma(i)$.

b) Suppose that $x, y \in L_\sigma(i)$. If $i = 0$ then the claim is obvious since both x and y are minimal. Suppose to the contrary and say $x <_\sigma y$. Since $x \in L_\sigma(i)$, then there is a chain

$$x_0 <_\sigma x_1 <_\sigma \cdots <_\sigma x_i = x$$

So

$$x_0 <_\sigma x_1 <_\sigma \cdots <_\sigma x_i = x <_\sigma y$$

is a chain of length $i + 1$. This implies that $i + 1 \leq \ell_\sigma(y) = i$, a contradiction. This proves the claim.

□

We may represent σ pictorially. The **Hasse diagram** of σ is the directed graph drawn on the plane, say $\mathbb{R}^2$ described below. The vertex set of the graph is A. A point named a is chosen on the plane for each point $a \in A$ obeying the following rules:

i) The elements of $L_\sigma(i)$ have the same y-coordinate i.

ii) $y \to x$ is an edge if and only if y covers x.

Example 17. *Figure 2.9 shows that the Hasse diagram of a partial order σ on the set $\{a, b, c, d, e, f, x, y\}$. From the diagram we see that e covers c, c covers a, x covers f, f covers d, d covers a, y covers g, y covers b. But we do have $a <_\sigma e$, $a <_\sigma f$, $a <_\sigma x$, and $d <_\sigma x$ since σ is transitive. These edges are not shown in the graph rather these relations*

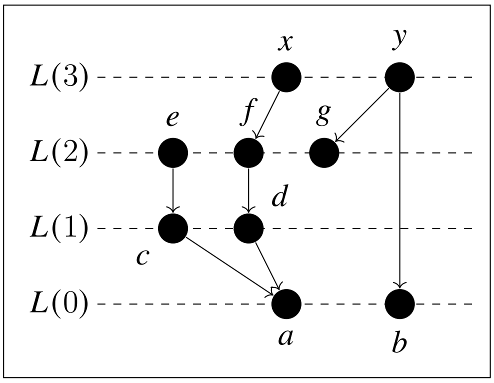


Figure 2.9. The Hasse diagram of the partial order σ

are implied by the graph taking into account that σ is transitive. Actually this can be made precise by noting that σ is the transitive closure of the relations given in the graph. Recall that if $\rho \subset A \times A$, then the transitive closure of ρ is the intersection of all transitive relations in $A \times A$ containing σ.

Definition 11. *Two partial orders λ and μ on a set A are isomorphic if there is a bijection $\varphi : A \to A$ such that*

$$a <_\lambda b \Longleftrightarrow \varphi(a) <_\mu \varphi(b)$$

3. THE SEMIGROUPS OF TRANSFORMATIONS

3.1. TRANSFORMATIONS

Let A be a finite set of cardinality n. Then the set of all self functions from A into A is denoted by $\mathcal{T}(A)$. $\mathcal{T}(A)$ is clearly a semigroup under the composition of maps. It is customary to call such a self function a **transformation** of A and the semigroup $\mathcal{T}(A)$ **full transformation semigroup**. Given $\beta \in \mathcal{T}(A)$ we will occasionally write it in tabular form:

$$\beta = \begin{pmatrix} a \\ a\beta \end{pmatrix}$$

For example if $A = \{1, 2, 3, 4, 5, 6, 7\}$ and $\beta \in \mathcal{T}(A)$ is the transformation such that

$$1\beta = 1,\ 2\beta = 1,\ 3\beta = 4,\ 4\beta = 2,\ 5\beta = 2,\ 6\beta = 4,\ 7\beta = 3$$

we represent β in tabular form as

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 1 & 4 & 2 & 2 & 4 & 3 \end{pmatrix}$$

It is easy to see that if A and B are of the same size and $\sigma : A \to B$ is a bijection, then $\mathcal{T}(A)$ is isomorphic to $\mathcal{T}(B)$ as a semigroup. Given $\alpha \in \mathcal{T}(A)$ let us define $\alpha\phi \in \mathcal{T}(B)$ by

$$\alpha\phi = \sigma^{-1}\alpha\sigma$$

$$\begin{tikzcd} B \arrow[r, "\sigma^{-1}"] \arrow[rrr, bend right, "\sigma^{-1}\alpha\sigma"'] & A \arrow[r, "\alpha"] & A \arrow[r, "\sigma"] & B \end{tikzcd}$$

$\phi : \mathcal{T}(A) \to \mathcal{T}(B)$ is an isomorphism. For let $\alpha, \beta \in \mathcal{T}(A)$.

$$\begin{aligned} (\alpha\beta)\,\phi &= \sigma^{-1}(\alpha\beta)\,\sigma \\ &= \left(\sigma^{-1}\alpha\sigma\right)\left(\sigma^{-1}\beta\sigma\right) \\ &= (\alpha\phi)\,(\beta\phi) \end{aligned}$$

Thus ϕ is a homomorphism. Similarly $\psi : \mathcal{T}(B) \to \mathcal{T}(A)$ defined by

$$\beta\psi = \sigma\beta\sigma^{-1}, \beta \in \mathcal{T}(B)$$

is a homomorphism. We also have

$$\alpha(\phi\psi) = \sigma\left(\sigma^{-1}\alpha\sigma\right)\sigma^{-1} = \alpha \text{ and } \beta(\psi\phi) = \sigma^{-1}\left(\sigma\beta\sigma^{-1}\right)\sigma = \beta$$

for all $\alpha \in \mathcal{T}(A)$ and $\beta \in \mathcal{T}(B)$. Thus ψ is the inverse of ϕ. Hence ϕ is an isomorphism.

We use the symbol $[n] = \{1, 2, \ldots, n\}$. So we will exclusively deal with $[n]$ and the semigroup $\mathcal{T}_n = \mathcal{T}([n])$.

We will denote the bijections of $[n]$ by $\mathcal{S}_n = \mathcal{S}([n])$ which is the symmetric group of permutations of n elements.

A transformation is said to be **singular** if it is not injective. Clearly product of two singular elements is also singular. The set $\mathcal{ST}_n = \mathcal{T}_n \backslash \mathcal{S}_n$ is a subsemigroup called the **singular transformation semigroup**.

3.2. PARTIAL TRANSFORMATION

Let A be a finite set of cardinality n. By adding a null symbol 0 to A, where we assume that $0 \notin A$, let us consider the set $A \cup \{0\}$. Define

$$\mathcal{PT}(A) = \{\alpha \in \mathcal{T}(A \cup \{0\}) : 0\alpha = 0\}$$

It is easy to see that $\mathcal{PT}(A)$ is a subsemigroup of $\mathcal{T}(A \cup \{0\})$ called the **the semigroup of partial transformations** on A.

For example if $A = [7]$ and $\beta \in \mathcal{PT}(A)$ is the transformation such that

$$1\beta = 1, 2\beta = 0, 3\beta = 4, 4\beta = 2, 5\beta = 0, 6\beta = 4, 7\beta = 3$$

we represent β in tabular form as

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & 4 & 2 & 0 & 4 & 3 \end{pmatrix}$$

Suppose we are given two partial transformations α and β

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 4 & 2 & 0 & 4 & 3 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & 4 & 1 & 0 & 4 & 3 \end{pmatrix}$$

then

$$\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 & 4 \end{pmatrix}$$

Note that in this notation trying to find $1\alpha\beta$ we need to find 0β. However the top row of the tabular list that represents β do not have the 0. Yet we recall that $\beta \in \mathcal{PT}(A)$ and we have agreed that $0\beta = 0$.

We set $\mathcal{PT}_n = \mathcal{PT}([n])$ which is the partial transformation semigroup of n elements.

3.3. SYMMETRIC INVERSE SEMIGROUP

For a finite set A the semigroup $\mathcal{T}(A)$ contains $\mathcal{S}(A)$, the symmetric group of all permutations of A as a subsemigroup. Similarly, $\mathcal{PT}(A)$ contains $\mathcal{IS}(A)$ consisting of injective partial transformations of A. To make things more clear let us restate what we mean by injective. Given $\alpha \in \mathcal{PT}(A)$ we define the domain of $\operatorname{dom}\alpha$ and the image set $\operatorname{Im}\alpha$ as

$$\operatorname{dom}\alpha = \{x \in A : x\alpha \in A\} = \{x \in A : x\alpha \neq 0\}$$

$$\operatorname{Im}\alpha = (\operatorname{dom}\alpha)\,\alpha$$

By definition, $\alpha \in \mathcal{IS}(A)$ if and only if $\alpha : \operatorname{dom}\alpha \to \operatorname{Im}\alpha$ is a bijection. For example if $A = [7]$, then

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 4 & 2 & 0 & 1 & 3 \end{pmatrix} \in \mathcal{IS}(A)$$

where

$$\operatorname{dom}\alpha = \{3,4,6,7\} \text{ and } \operatorname{Im}\alpha = \{4,2,1,3\}$$

So, $\alpha \in \mathcal{IS}(A)$ if and only if the inverse image $x\alpha^{-1}$ is a singleton or empty for any $x \in A$. However $0\alpha^{-1}$ may not be a singleton.

Let us now show that $\mathcal{IS}(A)$ is a subsemigroup of $\mathcal{PT}(A)$. If $\alpha, \beta \in \mathcal{IS}(A)$ we want to show that $\alpha\beta \in \mathcal{IS}(A)$. To see this let for some $z \in \operatorname{Im}(\alpha\beta) \in A$ and $x, y \in A$ be such that $x(\alpha\beta) = y(\alpha\beta) = z$. Then $x\alpha \neq 0$ and $y\alpha \neq 0$ for otherwise $z = 0(\alpha\beta) = 0\beta = 0$ contradicting to the fact that $z \in A$. This shows that $(x\alpha)\beta = (y\alpha)\beta = z$, and consequently $x\alpha$, $y\alpha \in \operatorname{dom}\beta$. Thus $x\alpha = y\alpha \in A$ since $\beta \in \mathcal{IS}(A)$. It follows that $x, y \in \operatorname{dom}\alpha$ and $x\alpha = y\alpha$. Thus $x = y$. This proves that $\alpha\beta \in \mathcal{IS}(A)$,i.e. $\mathcal{IS}(A)$ is a subsemigroup of $\mathcal{PT}(A)$.

A semigroup S is an **inverse semigroup** if for every $\alpha \in S$ there exists a unique element $\beta \in S$ such that $\alpha\beta\alpha = \alpha$ and $\beta\alpha\beta = \beta$ called the inverse of α [22]. Now, let us show that $\mathcal{IS}(A)$ is an inverse semigroup. Let $\alpha \in \mathcal{PT}(A)$ we must find $\beta \in \mathcal{PT}(A)$ such that $\alpha\beta\alpha = \alpha$ and $\beta\alpha\beta = \beta$. If $x\alpha = 0$ for all $x \in A$, then we may choose $\beta = \alpha$. So assume that $\operatorname{dom}\alpha \neq \phi$. We know that $\operatorname{Im}\alpha = (\operatorname{dom}\alpha)\alpha$ and $\alpha : \operatorname{dom}\alpha \to \operatorname{Im}\alpha$ is a bijection with inverse $\gamma : \operatorname{Im}\alpha \to \operatorname{dom}\alpha$. We define $\beta : A \cup \{0\} \to A \cup \{0\}$ by

$$x\beta = \begin{cases} 0 & x \notin \operatorname{Im}\alpha \\ x\gamma & x \in \operatorname{Im}\alpha \end{cases}$$

Clearly $0(\alpha\beta\alpha) = 0 = 0\alpha$ and $0(\beta\alpha\beta) = 0 = 0\alpha$. If $x \in A$, then either $x \in \operatorname{dom}\alpha$ or $x \in A \backslash \operatorname{dom}\alpha$. If $x \in \operatorname{dom}\alpha$, then $x\alpha \in \operatorname{Im}\alpha$. Hence

$$((x\alpha)\beta)\alpha = ((x\alpha)\gamma)\alpha = x\alpha$$

If $x \in A \backslash \operatorname{dom}\alpha$, then $x\alpha = 0$ and hence $x(\alpha\beta\alpha) = 0(\beta\alpha) = 0 = 0\alpha$. In either case $x(\alpha\beta\alpha) = x\alpha$. So $\alpha\beta\alpha = \alpha$.

Now let us show that $\beta\alpha\beta = \beta$. If $x \in A$, then either $x \in \operatorname{Im}\alpha$ or $x \in A \backslash \operatorname{Im}\alpha$. If $x \in \operatorname{Im}\alpha$, then

$$x\left(\beta\alpha\beta\right) = \left(x\gamma\right)\left(\alpha\beta\right) = \left(x\left(\gamma\alpha\right)\right)\beta = x\beta$$

If $x \in A\backslash \operatorname{Im}\alpha$, then $x\beta = 0$ and hence $x\left(\beta\alpha\beta\right) = 0\left(\alpha\beta\right) = 0 = x\beta$. In either case $x\left(\beta\alpha\beta\right) = x\beta$. So $\beta\alpha\beta = \beta$.

We must also show that β is unique. Let $\sigma \in \mathcal{IS}\left(A\right)$, $\alpha\sigma\alpha = \alpha$ and $\sigma\alpha\sigma = \sigma$. Let $x \in \operatorname{Im}\alpha$. Then $x = y\alpha$ for some $y \in \operatorname{dom}\alpha$. Thus

$$y\left(\alpha\sigma\alpha\right) = y\alpha \Longrightarrow \left(x\sigma\right)\alpha = y\alpha = x \Longrightarrow x\sigma \neq 0 \text{ and } \left(x\sigma\right)\alpha = y\alpha$$

Hence $x\sigma = y = x\gamma$ since α is injective on $\operatorname{dom}\alpha$. If $x \notin \operatorname{Im}\alpha$, then $x\sigma = 0$. For if $z = x\sigma \neq 0$. then

$$z = x\sigma = \left(\left(x\sigma\right)\alpha\right)\sigma \Longrightarrow x = \left(x\sigma\right)\alpha \Longrightarrow x \in \operatorname{Im}\alpha$$

a contradiction. So $\beta = \sigma$.

We set $\mathcal{IS}_n = \mathcal{IS}\left([n]\right)$ which is the symmetric inverse semigroup of n elements.

3.4. ORBITS

Let $\beta \in \mathcal{T}_n$. By definition we set $\beta^0 = 1_{[n]}$ the identity of $[n]$. For $x, y \in [n]$ we define $x \sim_\beta y$ if there are integers $i \geq 0$ and $j \geq 0$ such that $x\beta^i = y\beta^j$. We claim $\sim_\beta$ is an equivalence relation.

Let $x, y, z \in [n]$.

$x \sim_\beta x$: $x\beta^0 = x = x\beta^0$. Thus $x \sim_\beta x$.

$x \sim_\beta y$: There are integers $i \geq 0$ and $j \geq 0$ such that $x\beta^i = y\beta^j$. Hence $y\beta^j = x\beta^i$, i.e. $y \sim_\beta x$.

$x \sim_\beta y$, $y \sim_\beta z$: There are integers $i, k \geq 0$ and $j, s \geq 0$ such that $x\beta^i = y\beta^j$ and $y\beta^k = z\beta^s$. Then $x\beta^{i+k} = y\beta^{j+k} = z\beta^{s+j}$. so $x \sim_\beta z$.

The equivalence classes of $\sim_\beta$ are called **orbits** of β. Let $y \in [n]$ we denote the orbit of β which contains y by $\mathrm{orb}_\beta(y)$. Clearly $y \in \mathrm{orb}_\beta(y)$ and $y, y\beta, \ldots, y\beta^i, \ldots$are all in $\mathrm{orb}_\beta(y)$. But $[n]$ is finite. Hence this sequence must be finite. That is there is an integer $k \geq 1$ such that

$$y\beta^k \in \left\{y, y\beta, \ldots, y\beta^{k-1}\right\}$$

Let k be smallest such $k \geq 1$. With this choice of k we have $y, y\beta, \ldots, y\beta^{k-1}$ are pairwise distinct. To see this let us assume that $0 \leq i < j \leq k-1$ and $y\beta^i = y\beta^j$. Then

$$y\beta^{k-1} = y\beta^{k-1-j}\beta^j = \left(y\beta^j\right)\beta^{k-1-j} = \left(y\beta^i\right)\beta^{k-1-j} = y\beta^{k-1-(j-i)}$$

and $k-1-(j-i) < k-1$ contradicting the minimality of k.

Notice that if $y\beta^k = y\beta^r$ where $0 \leq r < k$, then

$$\left(y\beta^r\right)\beta^{k-r} = y\beta^k = y\beta^r$$

So if we let $x = y\beta^r$ and $i = k-r \geq 1$, then $x = x\beta^i$. Moreover if $1 \leq j < i$, then $x \neq x\beta^j$. For if $x = x\beta^j$, then $y\beta^{k-j} = y\beta^k = y\beta^r$ contradiction as $1 \leq k-j < k$ and $0 \leq r < k$. Thus $C = \left\{x, x\beta, \ldots, x\beta^{i-1}\right\}$ is a subset of the orbit $\mathrm{orb}_\beta(y)$, the elements $x, x\beta, \ldots, x\beta^{i-1}$ are pairwise distinct and β restricted to this set is the cycle $\left(x, x\beta, \ldots, x\beta^{i-1}\right)$. Suppose that an element $z \in \mathrm{orb}_\beta(y)$ satisfies the property that $z = z\beta^s$ for some $s \geq 1$ and $z, z\beta, \ldots, z\beta^{s-1}$ are distinct, the β restricted to $D = \left\{z, z\beta, \ldots, z\beta^{s-1}\right\} \subset \mathrm{orb}_\beta(y)$ is the cycle $\left(z, z\beta, \ldots, z\beta^{s-1}\right)$. We claim that $C = D$. To see this let us first note that $x, z \in \mathrm{orb}_\beta(y)$. Hence there are non-negative integers p, q such that $x\beta^p = z\beta^q$. There are non-negative integers m, t such that $p = mi + t$ and $0 \leq t < i$. Then

$$x\beta^{mi+t} = z\beta^q \Longrightarrow x\beta^t = z\beta^q \Longrightarrow x = x\beta^t\beta^{i-t} = z\beta^q\beta^{i-t} = z\beta^{q+i-t} \in D$$

Thus $C \subset D$. Similarly $D \subset C$. It follows that $C = D$.

This shows that each orbit contains a unique subset of the form $C = \left\{x, x\beta, \ldots, x\beta^{i-1}\right\}$ where the elements $x, x\beta, \ldots, x\beta^{i-1}$ are pairwise distinct and β restricted to this set is the cycle $\left(x, x\beta, \ldots, x\beta^{i-1}\right)$. So the transformation β may be represented by a simple graph Γ_β such the set of vertices of the graph is $[n]$, the edges of the graph is the set $\{\{x, x\beta\} : x \in [n]\}$ such that each component of the graph contains a unique cycle.

3.5. DIGRAPH REPRESENTATION OF TRANSFORMATIONS

With each $\alpha \in \mathcal{T}_n$ there is an associated directed (or a digraph) graph Γ_α defined as follows:

The vertex set of Γ_α is $[n] = \{1, 2, \ldots, n\}$. A pair $(r, s) \in [n] \times [n]$ is an edge in Γ_α if and only if $r\alpha = s$.

For example consider the transformation α in $\mathcal{T}_{17}$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\ 2 & 3 & 4 & 1 & 1 & 5 & 5 & 16 & 9 & 10 & 10 & 10 & 12 & 12 & 12 & 17 & 8 \end{pmatrix}$$

This transformation is pictured in Figure 3.1.

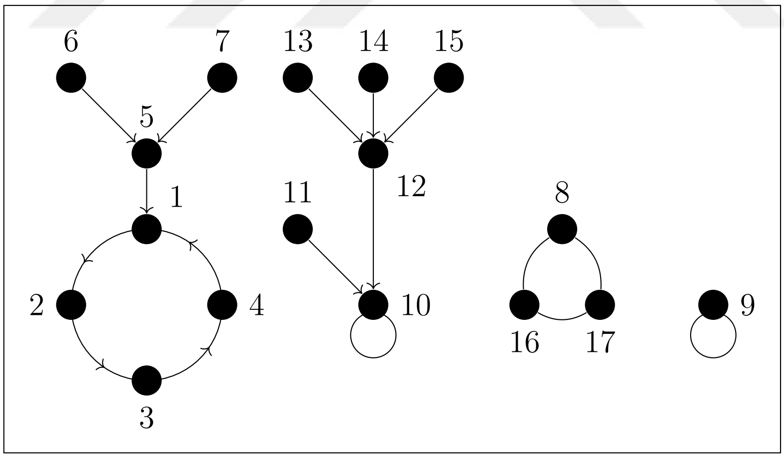


Figure 3.1. The digraph Γ_α corresponding to the transfomation α

Clearly a digraph $\Gamma = ([n], E)$ will be the graph of some $\alpha \in \mathcal{T}_n$ if and only if each vertex is the tail of exactly one arrow.

The graph Γ_α decomposes into a disjoint union of connected components. There exactly as many components as there are orbits.

Given $\beta \in \mathcal{T}_n$, since $\mathcal{T}_n$ is finite, in fact it has n^n elements, the sequence $\beta, \beta^2, \beta^3, \ldots$ is finite. So there is a smallest $k \geq 2$ such that $\beta, \beta^2, \beta^3, \ldots, \beta^{k-1}$ are distinct but $\beta^k = \beta^m$ for some $1 \leq m \leq k-1$. This integer m is called the **index** of β. If $r = k - m$, then we have $\beta^m, \ldots, \beta^{m+r-1}$ are distinct and $\beta^{m+r} = \beta^m$. We call the integer $1 \leq r$ the **period** of β. For example consider the transformation β in $\mathcal{T}_{24}$ with Γ_β is as in Figure 3.2. Given an element $x \in [n]$ and $\beta \in \mathcal{T}_n$ we make the convention that $x\beta^0 = x$. We may use the same idea to conclude that there are integers $m_x \geq 0$ and $r_x \geq 1$ such that

$$x, x\beta, \ldots, x\beta^{m_x}, \ldots, x\beta^{m_x+r_x-1}$$

are distinct and $x\beta^{m_x+r_x-1} = x\beta^{m_x}$. We see that $\left(x\beta^{m_x}, \ldots, x\beta^{m_x+r_x-1}\right)$ is a cycle.

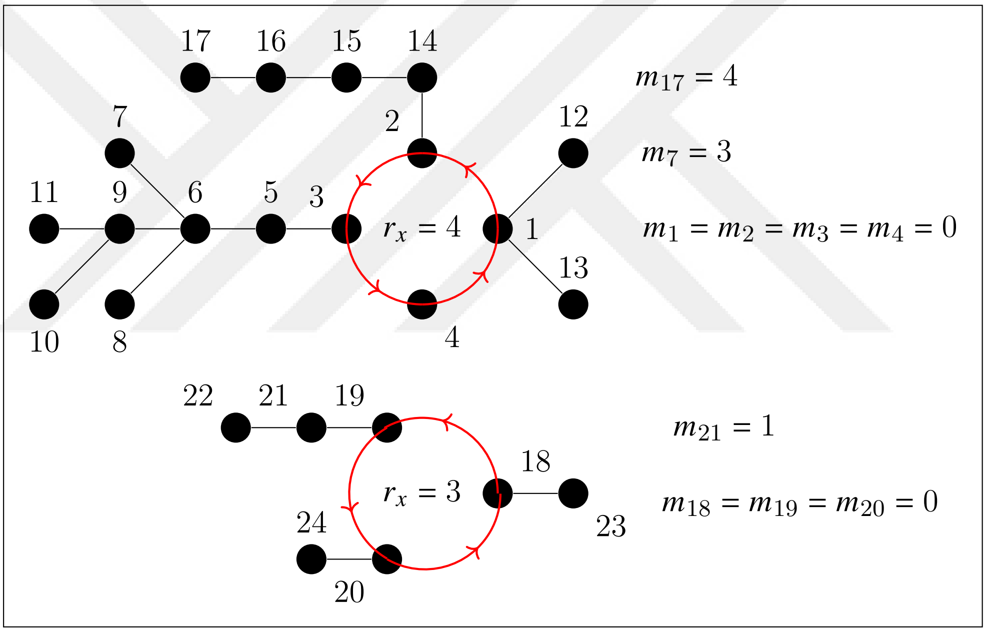


Figure 3.2. Γ_β

It is easy to see that $\sim_\beta$ is an equivalence relation and the equivalence class of $x, \operatorname{orb}_\beta(x)$ contains the sequence

$$\left\{x, x\beta, \ldots, x\beta^{m_x}, \ldots, x\beta^{m_x+r_x-1}\right\}$$

which enters in the cycle

$$\left(x\beta^{m_x}, \ldots, x\beta^{m_x+r_x-1}\right)$$

and stays there for any further application of β.

Each orbit contains a unique cycle. Thus r_x is fixed for any x in this cycle.

If β is a permutation, then $m_x = 0$ for each x and $\mathrm{orb}_\beta(x)$ consists of the cycle $\left(x, x\beta, \ldots, x\beta^{r_x-1}\right)$ alone.

Note that actually we must write $m_x(\beta)$ and $r_x(\beta)$ since these integers depend on β. But we will simple use m_x and r_x if β is clear from the context.

3.6. SHIFT, FIX AND CYCLE SET

Let us consider the transformation α with Γ_α as in Figure 3.3.

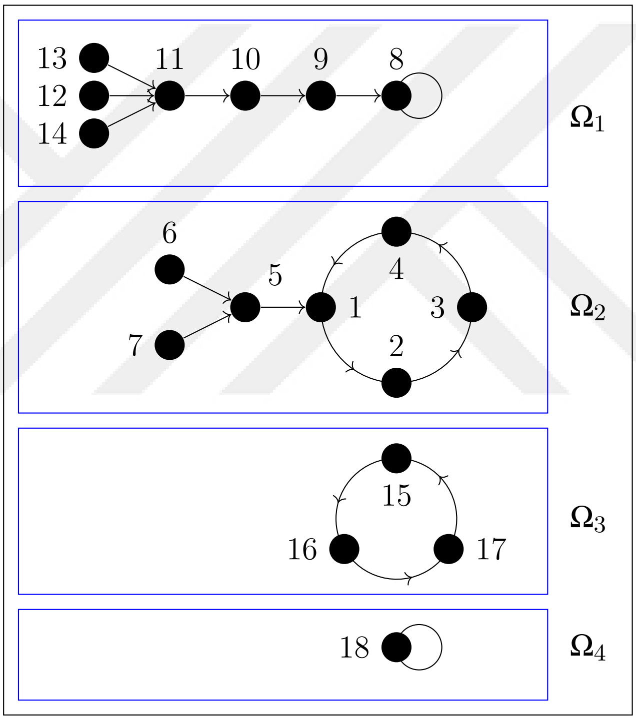


Figure 3.3. Γ_α

The orbits of α are $\Omega_1, \ldots, \Omega_k$ then there are k components of Γ_α corresponding to each orbit. In this example $k = 4$.

If (V_i, E_i) is the component corresponding to the orbit Ω_i, then V_i is the vertex set of Ω_i and $(r, s) \in E_i$ if and only if $r, s \in \Omega_i$ and $r\alpha = s$.

In the general case it is clear that, for each x in $[n]$,the sequence eventually arrives in a cycle (or a fixed point, which of course we may regard as a special case of a cycle) and remains there for all subsequent iterations.

We will use notations $Z(\alpha)$ as the set of all elements contained in cycles.

$\mathrm{Fix}(\alpha) = \{x \in [n] : x\alpha = x\}$

$\mathrm{Shift}(\alpha) = \{x \in [n] : x\alpha \neq x\} = [n] \setminus \mathrm{Fix}(\alpha)$.

In this example

$$Z(\alpha) = \{8, 1, 2, 3, 4, 15, 16, 17, 18\}, \mathrm{Fix}(\alpha) = \{8, 18\}, \mathrm{Shift}(\alpha) = [18] \setminus \mathrm{Fix}(\alpha)$$

The index m of α is the length of the longest path into $Z(\alpha)$, and the period r is the least common multiple of the lengths of the cycles.

Thus, we have $m = 4$ and $r = 12$.

3.7. *M*-POTENT ELEMENTS

It is clear also that the element α is m-potent if and only if $Z(\alpha)$ consists solely of fixed points and there exists y in $[n]$ such that

$$y\alpha^m \in Z(\alpha), y\alpha^{m-1} \notin Z(\alpha)$$

The largest m that can occur is $n - 1$.

Let $\mathcal{ST}_n = \mathcal{T}_n \backslash S_n$, the full singular transformation semigroup of degree n. If $\beta \in \mathcal{ST}_n$, then $m \geq 1$. In fact $\beta \in \mathcal{S}_n$ if and only if $m = 0$. Let $\alpha \in \mathcal{ST}_n$ be m-potent: Thus $\alpha^{m+1} = \alpha^m$, and $\alpha^{q+1} = \alpha^q$ for all $q < m$.

For each x in $[n]$, let m_x be the least integer such that $x\alpha^{m_x} \in Z(\alpha)$, and, for $j = 0, , 1, \ldots, m$

let

$$A_j = \{x \in [n] : m_x = j\}$$

It is evident that $A_0 = Z(\alpha)$, and that the sets $A_0, A_1, ..., A_m$ are mutually disjoint.

It is clear too that $\cup_{j=0}^{m} A_j = [n]$, since, for all x in $[n]$, the sequence $x, x\alpha, x\alpha^2 \ldots$ must reach $Z(\alpha)$ in at most m steps.

It is natural to call $(A_0, A_1, ..., A_m)$ of the set $[n]$ as the **level sets** of α. The arguments above proves that following theorem.

Theorem 18 (m-potent elements)**.** *The level sets $(A_0, A_1, ..., A_m)$ of an m-potent element α is a partition of the set $[n]$.*

As an example consider the following 3-potent transformation θ with Γ_θ as in Figure 3.4.

$$\theta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 4 & 4 & 4 & 5 & 5 & 8 & 8 & 10 & 10 & 10 & 11 & 1 \end{pmatrix}$$

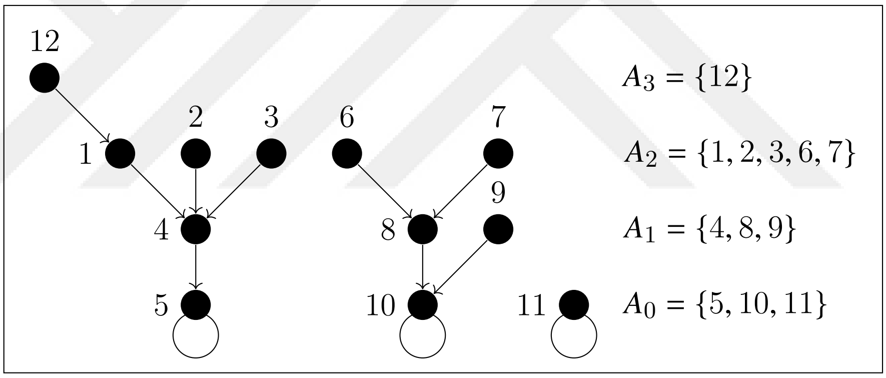


Figure 3.4. Γ_θ

Let us consider the following 4-potent transformation θ with Γ_θ as in Figure 3.5.

$$\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 4 & 4 & 4 & 4 & 3 & 3 & 1 & 7 & 8 & 8 & 1 & 1 \end{pmatrix}$$

So there is a loop at the node 4. If we ignore this loop, then remaining digraph is a tree. So this digraph deserves to be called a semi-tree with just one loop. With this interpretation we may think of an m-potent element as a forest of semitrees such that each semitree in the forest has height at most m and at least one of them with exact height m. If we forget about the restriction m we may give the following definition.

Definition 12. *A forest of semitrees in $\mathcal{T}_n$ is called a* ***potent*** *element.*

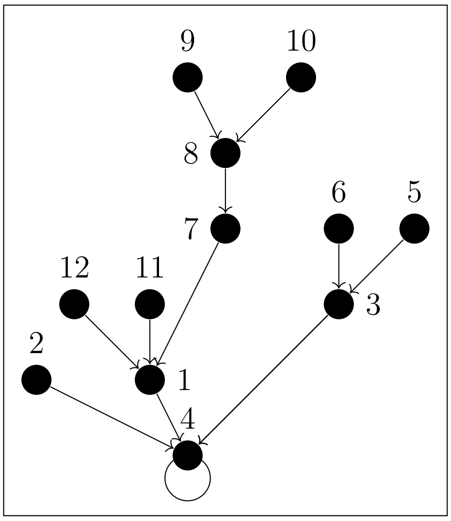


Figure 3.5. Γ_δ

By using m-potent elements Theorem, we can obtain a formula for the number of m-potents in $\mathcal{T}_n$. We let

$$P_{m+1}(n) = \{(k_0, k_1, \ldots, k_m) : 1 \le k_i \le n \text{ for } i = 0, 1, \ldots, m \text{ and } k_0 + k_1 + \ldots + k_m = n\}$$

Given a sequence $(k_0, k_1, \ldots, k_m) \in P_{m+1}(n)$ of integers in $[n]$, we can assign a partition $(A_0, A_1, \ldots, A_m)$ of $[n]$ with $\left|A_j\right| = k_j$ for each j in

$$\binom{n}{k_0, k_1, \ldots, k_m}$$

ways. Each element of A_1 must map to elements of A_0 and there are $|A_0|^{|A_1|}$ ways in which this can happen. Similarly, there are $|A_1|^{|A_2|}$ ways in which the elements of A_2 can map into A_1.

Continuing in this way, we obtain the following theorem:

Theorem 19. *The number of m-potent elements* ($m = 1, 2, ..., n-1$) *in* $\mathcal{T}_n$ *is*

$$\sum_{(k_0, k_1, \ldots, k_m) \in P_{m+1}(n)} \binom{n}{k_0, k_1, \ldots, k_m} k_0^{k_1} k_1^{k_2} \cdots k_{m-1}^{k_m}$$

3.8. POTENT ELEMENTS

Let the integer n be given. Let $\mathcal{P}_n$ denote the set of all potent elements for a given $n \geq 1$. We would like to find the cardinality of $\mathcal{P}_n$

The number of partitions $p(n)$ is the number of solutions of

$$k_1 + 2k_2 + \cdots + nk_n = n$$

where $k_i \geq 0$ for $i \in [n]$. Given $j_1 > j_2 > \cdots > j_t$ and $k_{j_1}, k_{j_2}, \ldots, k_{j_t}$ where $k_{j_i} > 0$ for $i \in [t]$ and

$$j_1 k_{j_1} + j_2 k_{j_2} + \cdots + j_t k_{j_t} = n$$

We consider a partition $C_1, \ldots, C_t$ of $[n]$ such that $|C_i| = j_i$ for $i \in [t]$. Consider all labelled trees with vertex set C_i. We may choose a vertex r of C_i to designate it as a root. Then from each vertex v in C_i there is a unique path to r. By adding a loop at the root r and directing each edge in accordance with the paths from each vertex to the root, all such directed subtrees of C_i represent an element of $\mathcal{P}_n$. Moreover each element of $\mathcal{P}_n$ is obtained in this way for varying partitions.

The permutation of $C_1, \ldots, C_t$ does not affect the cardinality. So for each such partition there are

$$j_1^{j_1-1} \cdot \cdots \cdot j_t^{j_t-1}$$

distinct potent elements. Thus we obtain the following theorem.

Theorem 20. *The number of potent elements in $\mathcal{T}_n$ is*

$$\sum_{j_1 k_{j_1} + j_2 k_{j_2} + \cdots + j_t k_{j_t} = n} \binom{n}{\underbrace{j_1, \ldots, j_1}_{k_{j_1}-\text{times}}, \underbrace{j_2, \ldots, j_2}_{k_{j_2}-\text{times}}, \ldots, \underbrace{j_t, \ldots, j_t}_{k_{j_t}-\text{times}}} \frac{1}{k_{j_1}!} \frac{1}{k_{j_2}!} \cdots \frac{1}{k_{j_t}!} j_1^{j_1-1} \cdot \cdots \cdot j_t^{j_t-1}$$

We will give some examples and compare the number of potent elements with the number of all m-potents elements for $m = 1, 2, ..., n-1$.

3.9. EXAMPLES

Given an equation of the form

$$k_1 + 2k_2 + \cdots + nk_n = n$$

k_i is the weight of i appearing in the partition of n. For a given n we use the notation $(k_1, k_2, \cdots, k_n)$ to denote weights used for the partition of n. We give specific examples for $n = 3, 4, 5$.

$$\begin{array}{cccccc}
n=3 & 1+1+1 & (3,0,0) & \frac{3!}{1!1!1!}\frac{1}{3!} & 1 & 1^0 = 1 \\
 & 2+1 & (1,1,0) & \frac{3!}{1!2!} & 3 & 3 \cdot 2^1 = 6 \\
 & 3 & (0,0,1) & \frac{3!}{3!} & 1 & 1 \cdot 3^2 = 9
\end{array}$$

So, the total number of potent elements is $= 1 + 6 + 9 = 16$.

$$\begin{array}{cccccc}
n=4 & 1+1+1+1 & (4,0,0,0) & \frac{4!}{1!1!1!1!}\frac{1}{4!} & 1 & 1^0 = 1 \\
 & 2+1+1 & (2,1,0,0) & \frac{4!}{2!1!1}\frac{1}{2!} & 6 & 6 \cdot 2^1 = 12 \\
 & 2+2 & (0,2,0,0) & \frac{4!}{2!2!}\frac{1}{2!} & 3 & 3 \cdot 2^1 \cdot 2^1 = 12 \\
 & 3+1 & (1,0,1,0) & \frac{4!}{3!1!} & 4 & 4 \cdot 3^2 = 36 \\
 & 4 & (0,0,0,1) & \frac{4!}{4!} & 1 & 1 \cdot 4^3 = 64
\end{array}$$

So, the total number of potent elements is $= 1 + 36 + 12 + 12 + 64 = 125$

$$\begin{array}{ccccccc}
n=5 & 1+1+1+1+1 & (5,0,0,0,0) & \frac{5!}{1!1!1!1!1!}\frac{1}{5!} & 1 & 1^0 = 1 & \\
 & 2+1+1+1 & (3,1,0,0,0) & \frac{5!}{2!1!1!1!}\frac{1}{3!} & 10 & 10 \cdot 2^1 = 20 & \\
 & 2+2+1 & (1,2,0,0,0) & \frac{5!}{2!2!1!}\frac{1}{2!} & 15 & 15 \cdot 2^1 \cdot 2^1 = 60 & \\
 & 3+1+1 & (2,0,1,0,0) & \frac{5!}{3!1!1!}\frac{1}{2!} & 10 & 10 \cdot 3^2 = 90 & :: \\
 & 3+2 & (0,1,1,0,0) & \frac{5!}{3!2!} & 10 & 10 \cdot 3^2 \cdot 2^1 = 180 & \\
 & 4+1 & (1,0,0,1,0) & \frac{5!}{4!1!} & 5 & 5 \cdot 4^3 = 320 & \\
 & 5 & (0,0,0,0,1) & \frac{5!}{5!} & 1 & 1 \cdot 5^4 = 625 &
\end{array}$$

So, the total number of potent elements is $1 + 320 + 180 \ + 90 \ + 60 \ + 20 \ + 625 = 1296$.

Let us recount potent elements by noting that all each potent element is an m-potent element for some $m = 1, 2, ..., n-1$. So this number is given by

$$1 + \sum_{m=1}^{n-1} \sum_{(k_0,k_1,\dots,k_m)\in P_{m+1}(n)} \binom{n}{k_0, k_1, \dots, k_m} k_0^{k_1} k_1^{k_2} \cdots k_{m-1}^{k_m}$$

Note that we need to add one as we have counted the identity element as potent element. Let us evaluate this sum for $n = 3, 4, 5$.

$n = 3$:

$$\begin{aligned} & 1 + \sum_{k_0+k_1=3} \binom{3}{k_0, k_1} k_0^{k_1} + \sum_{k_0+k_1+k_2=3} \binom{3}{k_0, k_1, k_2} k_0^{k_1} k_1^{k_2} \\ = \; & 1 + \binom{3}{1, 2} 1^2 + \binom{3}{2, 1} 2^1 + \binom{3}{1, 1, 1} 1^1 1^1 \\ = \; & 1 + 3 + 3 \cdot 2 + 6 = 16 \end{aligned}$$

$n = 4$: We first note that

$$\begin{aligned} L &= \sum_{k_0+k_1+k_2+k_3=4} \binom{4}{k_0, k_1, k_2, k_3} k_0^{k_1} k_1^{k_2} k_2^{k_3} \\ &= \binom{4}{1, 1, 1, 1} 1^1 1^1 1^1 1^1 = 24 \end{aligned}$$

Thus the formula for potent elements when $n = 4$ is

$$\begin{aligned} & 1 + \sum_{k_0+k_1=4} \binom{4}{k_0, k_1} k_0^{k_1} + \sum_{k_0+k_1+k_2=4} \binom{4}{k_0, k_1, k_2} k_0^{k_1} k_1^{k_2} + 24 \\ = \; & 1 + \frac{4!}{1!3!} 1^3 + \frac{4!}{3!1!} 3^1 + \frac{4!}{2!2!} 2^2 + \frac{4!}{2!1!1!} 2^1 1^1 + \frac{4!}{1!2!1!} 1^2 2^1 + \frac{4!}{1!1!2!} 1^1 1^2 + 24 \\ = \; & 125 \end{aligned}$$

$n = 5$: We first note that for $m = 4$ we have

$$\sum_{k_0+k_1+k_2+k_3+k_4=5} \binom{5}{k_0, k_1, k_2, k_3, k_4} k_0^{k_1} k_1^{k_2} k_2^{k_3} k_3^{k_4} = 5! = 120$$

Let use evaluate the cases $m = 1, 2, 3$ separately.

$$\sum_{k_0+k_1=5} \binom{5}{k_0, k_1} k_0^{k_1} = \sum_{k=1}^{4} \binom{5}{k} k^{5-k} = 195$$

The solutions of $k_0 + k_1 + k_2 = 5$ is $\{(1,1,3), (1,2,2), (1,3,1), (2,1,2), (2,2,1), (3,1,1)\}$

$$\begin{aligned}
&\sum_{k_0+k_1+k_2=5} \binom{5}{k_0, k_1, k_2} k_0^{k_1} k_1^{k_2} \\
= &\sum_{k=1}^{3} \sum_{r=1}^{4-k} \binom{5}{k, r, 5-k-r} k^r r^{5-k-r} \\
= &\sum_{k=1}^{3} \sum_{r=1}^{4-k} \frac{5!}{k! r! (5-k-r)!} k^r r^{5-k-r} = 560
\end{aligned}$$

The solutions of $k_0 + k_1 + k_2 + k_3 = 5$ is $\{(1,1,1,2), (1,1,2,1), (1,2,1,1), (2,1,1,1)\}$. Note that for each of them

$$\binom{5}{k_0, k_1, k_2, k_3} = \frac{5!}{2!} = 60$$

Thus

$$\sum_{k_0+k_1+k_2+k_3=5} \binom{5}{k_0, k_1, k_2, k_3} k_0^{k_1} k_1^{k_2} k_2^{k_3} = 60\,(1+2+2+2) = 420$$

So the total is $1 + 120 + 195 + 560 + 420 = 1296$.

3.10. THE FACTORIZATION OF (M, R)-POTENTS

We know that each element in $\mathcal{T}_n$ has a period r and index m. We will refer to such an element as (m, r)**-potent.**

$\alpha \in \mathcal{T}_n$; index m and period r. Orbits of $\alpha : \Omega_1, \ldots, \Omega_k$
For $j \in [k]$; $\alpha_j = \alpha|\Omega_j$; with index m_j and period r_j;
for $a \in Z(\alpha_j)$; $T_j(a) = \{x \in [n] : x\alpha^{m_x} = a\}$
$m = \max\{m_1, \ldots, m_k\}$, $r = \operatorname{lcm}\{r_1, \ldots, r_k\}$, $\cup_{a\in Z(\alpha_j)} T_j(a) = \Omega_j$

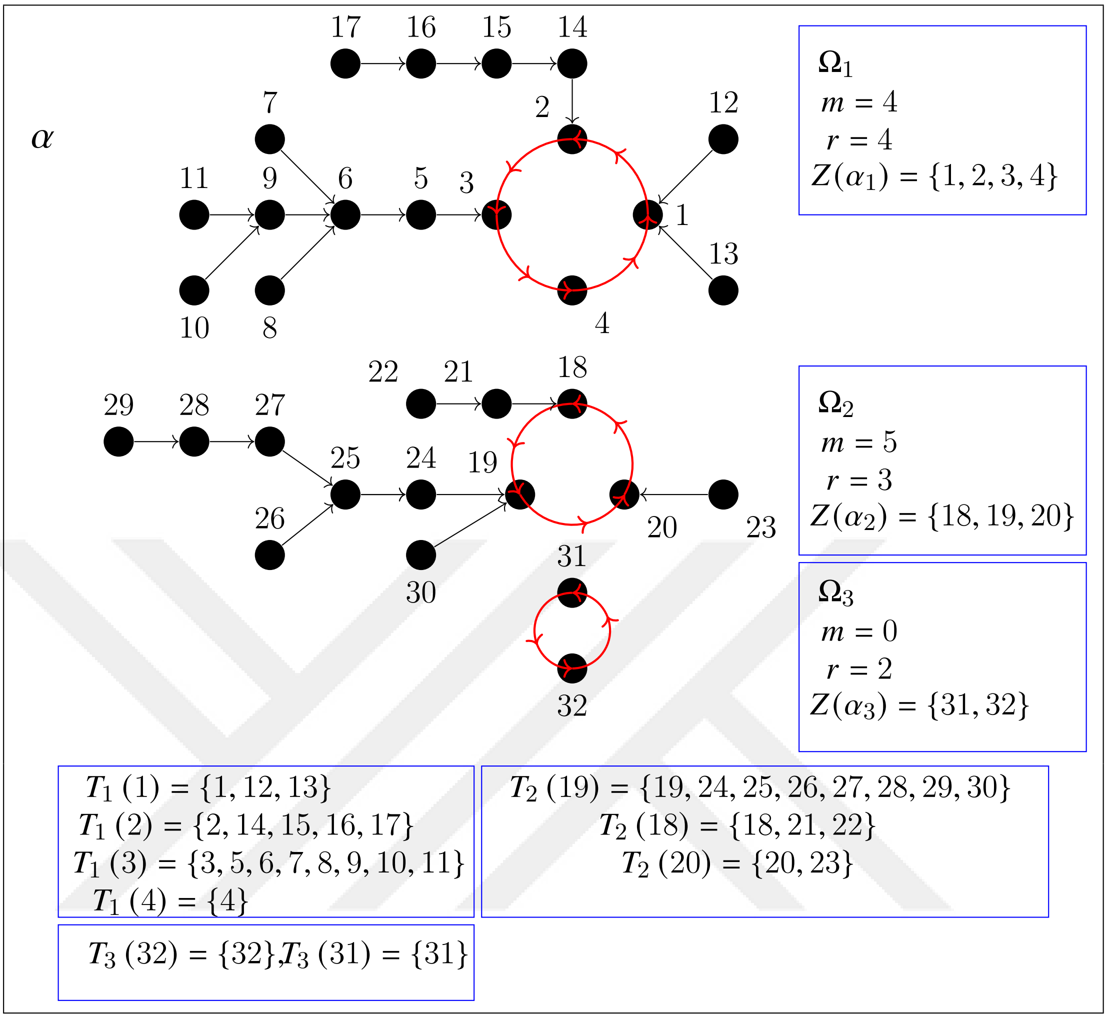


Figure 3.6. An example of an $(5, 12)$-potent element

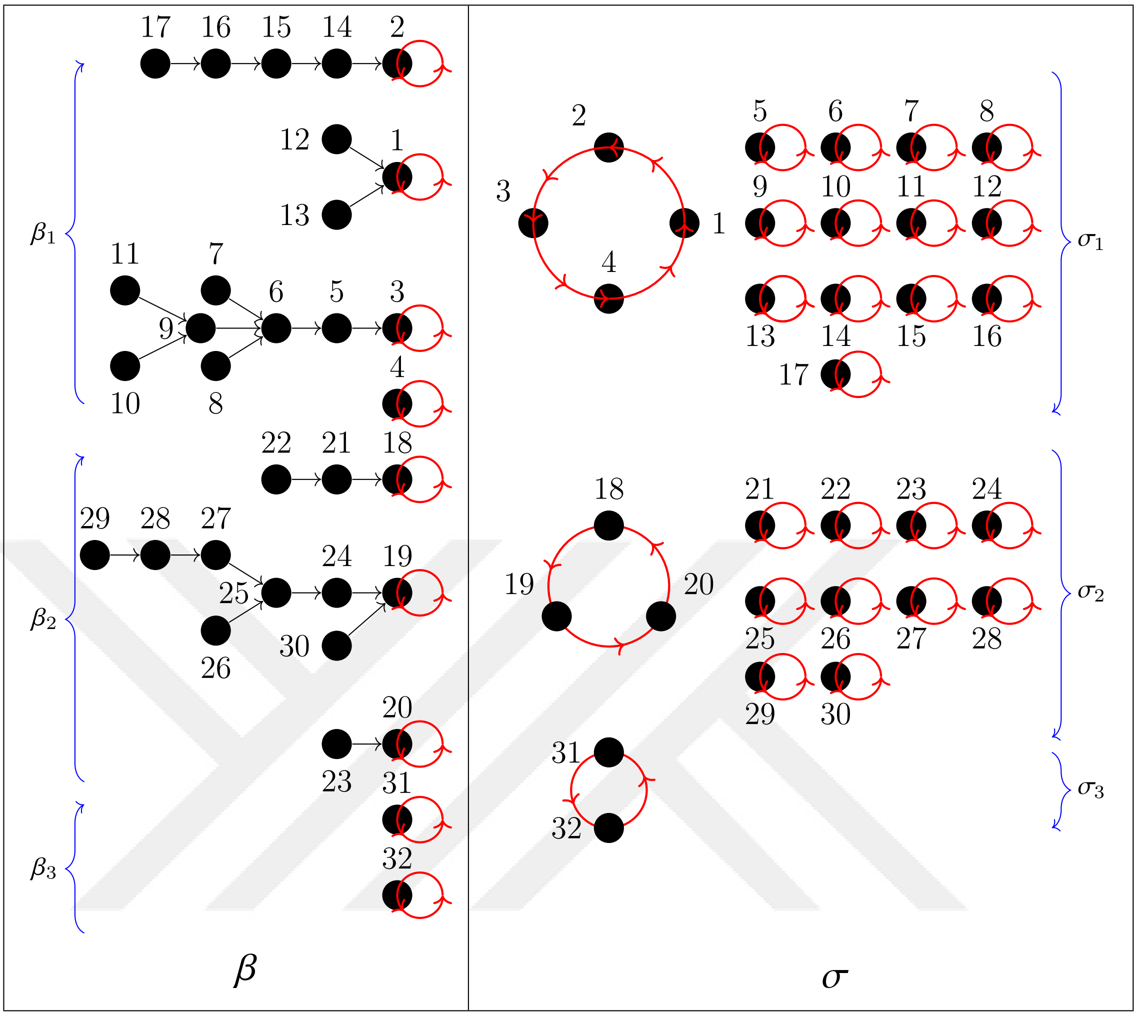


Figure 3.7. Factorization of $(5, 12)$-potent element in Figure 3.6

We then define β_j and permutation σ_j on the set Ω_j by

$$x\beta_j = \begin{cases} x\alpha_j & x \in \Omega_j \backslash Z(\alpha_j) \\ x & x \in Z(\alpha_j) \end{cases} \quad \text{and} \quad x\sigma_j = \begin{cases} x\alpha_j & x \in Z(\alpha_j) \\ x & x \in \Omega_j \backslash Z(\alpha_j) \end{cases}$$

We set

$$\beta = \beta_1 \cdots \beta_k \quad \text{and} \quad \sigma = \sigma_1 \cdots \sigma_k$$

Another example is given below by considering the transformation in Figure 3.8. The factoring is pictured in Figure 3.9.

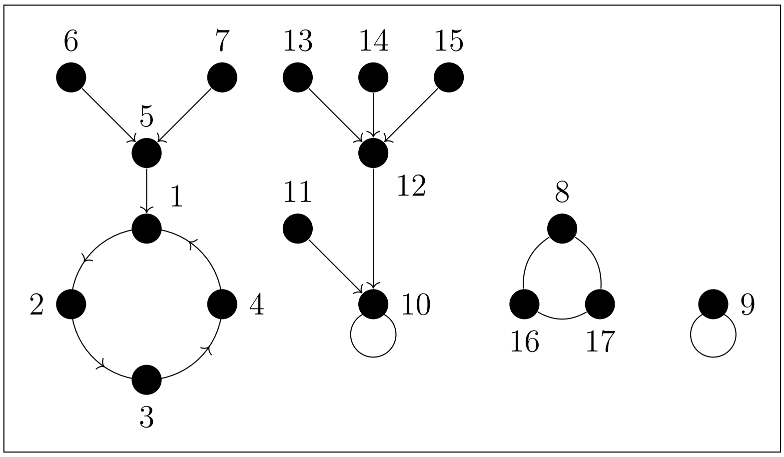


Figure 3.8. A factorization example

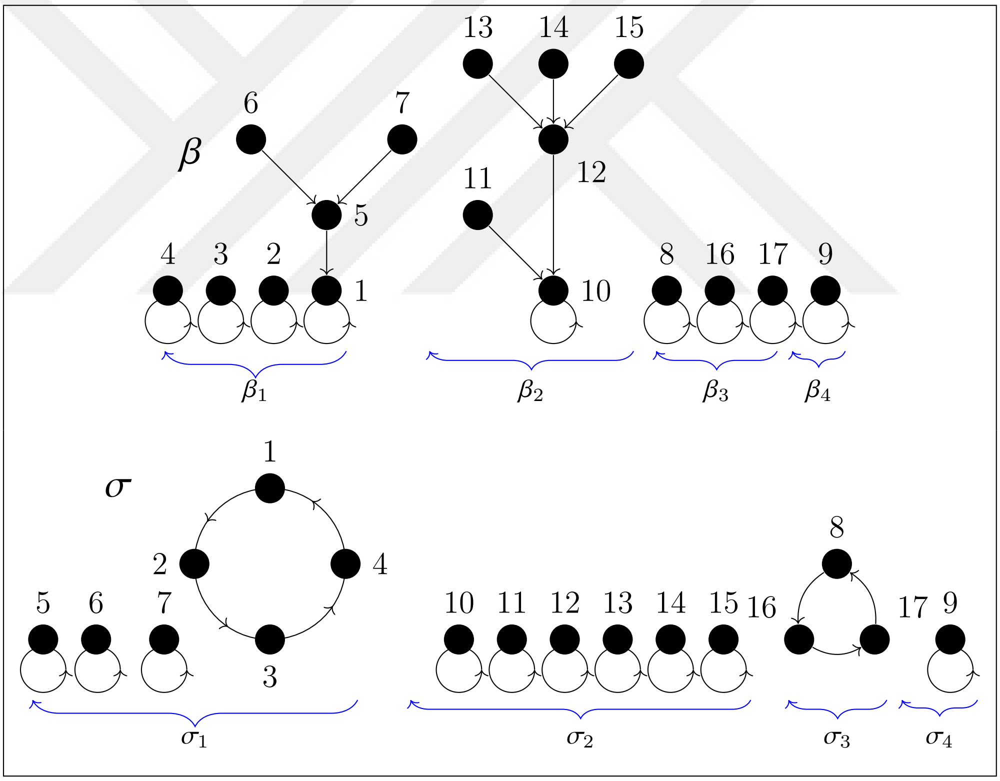


Figure 3.9. Transformation in Figure 3.8 factored

This proves the following result:

Theorem 21 (Factoring). *Let α be an element of $\mathcal{T}_n$ of index m and period r. Then there exist a permutation σ of order r and an m-potent element β such that $\alpha = \sigma\beta$ and* $\mathrm{Shift}\,(\sigma) \cap \mathrm{Shift}\,(\beta) = \phi$.

We have the following converse result:

Theorem 22 (Product)**.** *Let σ be a permutation of order r and let β be an m-potent such that* $\text{Shift}(\sigma) \cap \text{Shift}(\beta) = \phi$. *Let $\alpha = \sigma\beta$. Then*

(i) α has index m and period r;

(ii) $\text{Im}(\alpha^m)$ *is the disjoint union of* $\text{Shift}(\sigma) \cup \text{Fix}(\alpha)$.

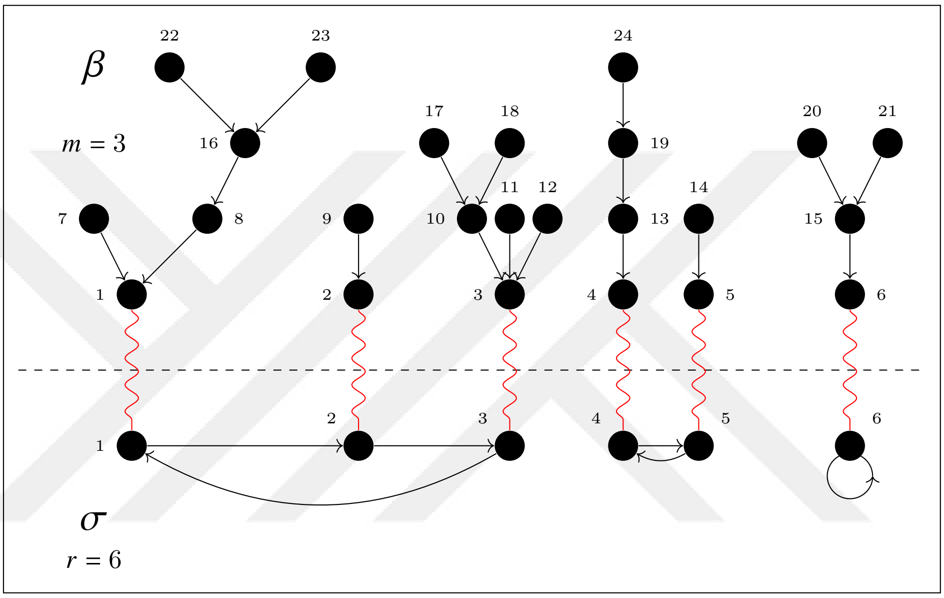


Figure 3.10. Sketch of the proof of the product

Proof. A sketch of the proof is given in Figure 3.10. Above for $x \in [n]$, let $q_x = m_x(\beta)$ and $s_x = r_x(\sigma)$. First observe that if $x \in \text{Shift}(\sigma)$ and $y = x\sigma \neq x$. Thus $y\sigma \neq x\sigma = y$. So $y \in \text{Shift}(\sigma)$. $\text{Shift}(\sigma) \cap \text{Shift}(\beta) = \phi$, then $\text{Shift}(\beta) \subset \text{Fix}(\sigma)$ and $\text{Shift}(\sigma) \subset \text{Fix}(\beta)$. So if $x \in \text{Shift}(\beta)$, then for $0 \leq j \leq q_x$

$$x\alpha^j = x\beta^j \tag{3.1}$$

Now $y = x\beta^{q_x} \in \text{Fix}(\beta)$. So either $y \in \text{Shift}(\sigma)$ or $y \in \text{Fix}(\beta) \cap \text{Fix}(\sigma)$. If $y \in \text{Fix}(\sigma)$, then $s_y = 0$. If $y \in \text{Shift}(\sigma)$

$$y\alpha = \underbrace{(y\sigma)}_{\in \text{Shift}(\sigma)} \beta = y\sigma$$

Thus

$$y\alpha^i = y\sigma^i$$

for all $0 \leq i$ Thus we have

$$x\alpha^{q_x+s_y} = y\sigma^{s_y} = y = x\alpha^{q_x}$$

We carry this out for each orbit Ω_j in turn, and obtain the orbital structure of $\sigma\beta$. The longest path into $Z(\sigma\beta)$ is determined by the corresponding path for Ω_j, and the (non-trivial) cycles are the same as those for σ. We deduce that $\sigma\beta$ has index m and period r.

(ii) If β is the identity map (in which case $m = 0$), then the result is obvious. We suppose that β is not the identity map. $\mathrm{Im}\,(\alpha^m) \subset \mathrm{Shift}\,(\sigma) \cup \mathrm{Fix}\,(\alpha)$: Let $z \in \mathrm{Im}\,(\alpha^m)$, then there exists x such that $z = x\alpha^m$. Since β is m-potent, then there exists $j \in [m]$ such that $x, x\beta, \ldots, x\beta^{j-1} \in \mathrm{Shift}\,(\beta)$ are pairwise distinct and $x\beta^j \in \mathrm{Fix}\,(\beta)$, so that and $x\alpha^j = x\beta^j \in \mathrm{Shift}\,(\sigma) \cup \mathrm{Fix}\,(\alpha)$. Certainly $z = x\alpha^m = x\alpha^j\alpha^{m-j} \in \mathrm{Shift}\,(\sigma) \cup \mathrm{Fix}\,(\alpha)$.

Conversely, suppose that $z \in \mathrm{Shift}\,(\sigma) \cup \mathrm{Fix}\,(\alpha)$. If $z \in \mathrm{Fix}\,(\alpha)$, then

$$z = z\alpha = z\alpha^2 = \cdots = z\alpha^m \in \mathrm{Im}\,(\alpha^m)$$

If $z \in \mathrm{Shift}\,(\sigma)$, then $z\sigma^{-m} \in \mathrm{Shift}\,(\sigma)$, and so, by (3.1),

$$z = (z\sigma^{-m})\,\sigma^m = (z\sigma^{-m})\,\alpha^m \in \mathrm{Im}\,(\alpha^m)$$

It is clear that the union $\mathrm{Shift}\,(\sigma) \cup \mathrm{Fix}\,(\alpha)$ is disjoint. □

We are ready now to prove our main result:

Theorem 23 (Factorization)**.** *Let α be an element of $\mathcal{T}_n$. Then α is (m, r)-potent if and only if there exist a unique permutation σ of order r and a unique m-potent element β such that $\alpha = \sigma\beta$ and* $\mathrm{Shift}\,(\sigma) \cap \mathrm{Shift}\,(\beta) = \phi$.

Proof. From Factoring Theorem we know that there is such a factorization. Also, from Product Theorem we know that if σ is a permutation of order r and β is an m-potent element such that $\alpha = \sigma\beta$ and $\mathrm{Shift}\,(\sigma) \cap \mathrm{Shift}\,(\beta) = \phi$, then α is (m, r)-potent. Thus it is enough to show that the factorization of α given by Factoring Theorem is unique.

Let $\alpha = \sigma_1\beta_1 = \sigma_2\beta_2$ where σ_j is a permutation of order r, and β_j, is an m-potent such that $\mathrm{Shift}\,(\sigma_j) \cap \mathrm{Shift}\,(\beta_j) = \phi$ for $j = 1, 2$. It follows from Product Theorem that α has index m_j and period r_j for $j = 1, 2$.. Hence $m_1 = m_2$ and $r_1 = r_2$. Moreover, it follows from Product Theorem(ii) that $\mathrm{Shift}\,(\sigma_1) = \mathrm{Im}\,(\alpha^m) \setminus \mathrm{Fix}\,(\alpha) = \mathrm{Shift}\,(\sigma_2)$. Since, $z\alpha = z\,(\sigma_j\beta_j) = z\sigma_j$ for all z in $\mathrm{Shift}\,(\sigma_j)$, it follows that $\sigma_1 = \sigma_2$, and it then follows immediately that $\beta_1 = \beta_2$. $\square$

It is important to note that the uniqueness of the factorization depends upon the condition $\mathrm{Shift}\,(\sigma) \cap \mathrm{Shift}\,(\beta) = \phi$. Figure 3.11 is an example which shows that $\sigma_1 \neq \sigma_2$ and $\beta_1 \neq \beta_2$ but $\alpha = \sigma_1\beta_1 = \sigma_2\beta_2$

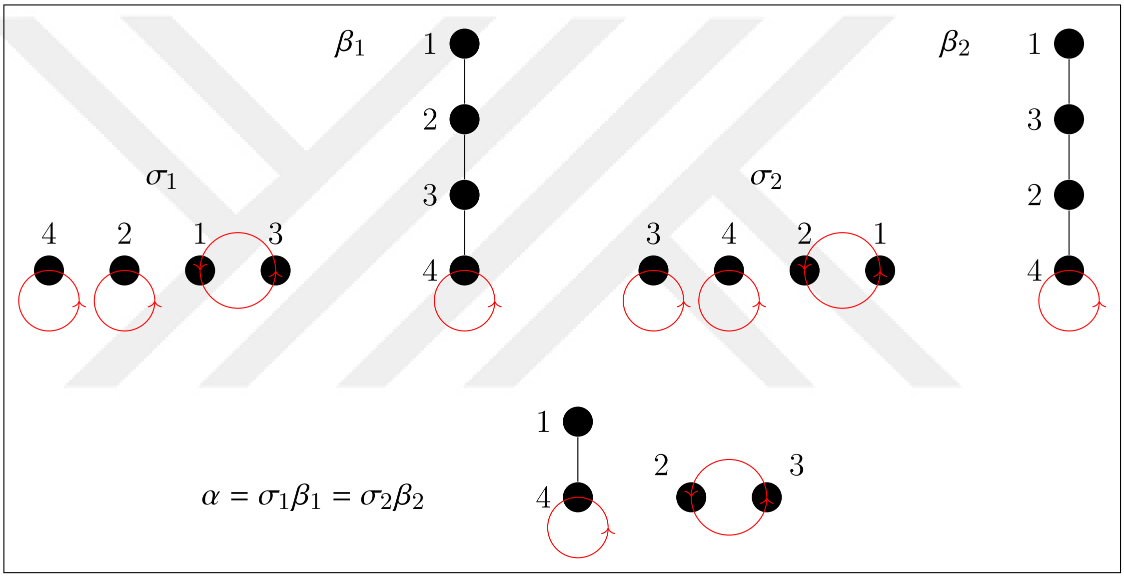


Figure 3.11. A counter example to uniqueness of factorization

So Uniqueness depends on the fact that shifts are disjoint. Product of degree r permutation and an m-potent is not necessarily an (m, r)-potent.

We may apply this factorization to determine the number of elements (m, r)-potent elements.

Let $\tau\,(k, r)$ denote the number of permutations of k elements of order r. Since $(0, r)$- potent elements are permutations, we only take into account (m, r)-potent elements with $m \geq 1$ in the following theorem.

Proof. Let $m \geq 1$ and $r \geq 1$. Then the number of (m, r)-potent elements an element of $\mathcal{ST}_n$

is

$$\sum_{(k_0,k_1,\ldots,k_m)\in P_{m+1}(n)} \binom{n}{k_0,k_1,\ldots,k_m} k_0^{k_1} k_1^{k_2} \cdots k_{m-1}^{k_m} \tau(k_0,r)$$

□

Proof. We know that each (m,r)-potent element can be uniquely written as $\sigma\beta$ where σ is a permutation of order r and β is an m-potent element and $\mathrm{Fix}(\sigma) \cup \mathrm{Fix}(\beta) = [n]$.

Let $(A_0, A_1, ..., A_m)$ be a partition of $[n]$ with $|A_j| = k_j$; $(j = 0, 1, ..., m)$, and let $\kappa = \kappa(A_0, A_1, ..., A_m)$ denote the set of all m-potent elements in $\mathcal{T}_n$ with corresponding partition $(A_0, A_1, ..., A_m)$, as in m-potent elements Theorem . For each $\beta \in \kappa$ we have $\mathrm{Fix}(\beta) = A_0$. Since, by Factoring Theorem , $\mathrm{Shift}(\sigma) \subset A_0$ there are $\tau(k_0, r)$ permutations of order r in $\mathcal{T}_n$ having $\mathrm{Shift}(\sigma) \subset A_0$. Similarly, the result follows from Factorization Theorem , as required. □

4. CONJUGATION AND VARIANTS

4.1. CONJUGATION

In this chapter we will investigate the action of the group S_n on $\mathcal{T}_n$ by conjugation. We will especially be interested in the effect on the cardinalities of this action on the idempotents in $\mathcal{T}_n$. Let us first prove a theorem concerning the digraph representation of transformations.

Theorem 24. *In the semigroup $\mathcal{T}_n$, two elements $\alpha, \beta \in \mathcal{T}_n$ are S_n-conjugate if and only if the graphs Γ_α and Γ_β are isomorphic.*

Proof. Suppose that $\alpha, \beta \in \mathcal{T}_n$ and $\sigma \in S_n$ is such that $\beta = \alpha^\sigma = \sigma^{-1}\alpha\sigma$.

If $x, y \in [n]$, then

$$(x, y) \in E(\Gamma_\alpha) \Longleftrightarrow (x, y) = (x, x\alpha) \tag{4.1}$$

$$\Longleftrightarrow (x\sigma, x\alpha\sigma) = ((x\sigma), (x\sigma)\beta) \tag{4.2}$$

$$\begin{tikzcd} x \arrow[r, "\sigma"] \arrow[d, "\alpha"'] & x\sigma \arrow[d, "\beta"] \\ x\alpha \arrow[r, "\sigma"] & x\sigma\beta = x\alpha\sigma \end{tikzcd}$$

Hence σ induces an isomorphism between Γ_α and Γ_β.

Conversely suppose that $\sigma \in S_n$ induces an isomorphism between Γ_α and Γ_β.

Then for each $x \in [n]$ the pair $(x, x\alpha)$ is an edge of the graph Γ_α. Hence is $(x\sigma, x\alpha\sigma)$ an edge of the graph Γ_β. Thus $((x\sigma), (x\sigma)\beta)) = (x\sigma, x\alpha\sigma)$. In other words $x\sigma\beta = x\alpha\sigma$ for each $x \in [n]$. Thus $\sigma\beta = \alpha\sigma$, i.e. $\beta = \alpha^\sigma$. □

Definition 13. *For an α is an (m, r)-potent element in $\mathcal{T}_n$ let us call the unique pair (τ, β) the* ***factorization*** *of α, where σ is a permutation of order r and β is m-potent with $\alpha = \sigma\beta$ and* $\mathrm{Shift}(\tau) \cap \mathrm{Shift}(\beta) = \phi$.

Recall that we for $\alpha \in \mathcal{T}_n$ and $\sigma \in S_n$ we use the notation $\alpha^\sigma = \sigma^{-1}\alpha\sigma$.

Lemma 2. *If $\sigma \in S_n$, then α is an (m, r)-potent element if and only if $\beta = \sigma^{-1}\alpha\sigma = \alpha^\sigma$ is* *an (m,r)-potent.*

Proof. First note that

$$\beta^k = \underbrace{\left(\sigma^{-1}\alpha\sigma\right)\left(\sigma^{-1}\alpha\sigma\right)\cdots\left(\sigma^{-1}\alpha\sigma\right)}_{k \text{ times}} = \sigma^{-1}\alpha^k\sigma$$

Since $\sigma : [n] \longrightarrow [n]$ is a bijection, then given any $\gamma, \delta \in \mathcal{T}_n$ we have $\sigma^{-1}\gamma\sigma = \sigma^{-1}\delta\sigma$ if and only if $\gamma = \delta$. So $\alpha, \alpha^2, \ldots, \alpha^{m+r-1}$ are distinct if and only if $\beta, \beta^2, \ldots, \beta^{m+r-1}$ are distinct. Also $\beta^{m+r} = \sigma^{-1}\alpha^{m+r}\sigma$. Thus α is an (m,r)-potent element if and only if $\beta = \sigma^{-1}\alpha\sigma$ is an (m,r)-potent. □

Lemma 3. *Let α be an (m,r)-potent element of $\mathcal{T}_n$ and $\sigma \in S_n$. If (τ, β) is the factorization of α, then $(\tau^\sigma, \beta^\sigma)$ is the factorization of α^σ.*

Proof. We know by the previous lemma that τ^σ is a permutation of order r and β^σ is m-potent. Suppose that $x \in \operatorname{Shift}(\tau^\sigma) \cap \operatorname{Shift}(\beta^\sigma)$. If we let $y = x\sigma^{-1}$, then

$$x\tau^\sigma \neq x \Longrightarrow x\sigma^{-1}\tau\sigma \neq x \Longrightarrow x\sigma^{-1}\tau \neq x\sigma^{-1} \Longrightarrow y\tau \neq y$$

Also

$$x\beta^\sigma \neq x \Longrightarrow x\sigma^{-1}\beta\sigma \neq x \Longrightarrow x\sigma^{-1}\beta \neq x\sigma^{-1} \Longrightarrow y\beta \neq y$$

Hence $y \in \operatorname{Shift}(\tau) \cap \operatorname{Shift}(\beta) = \phi$, a contradiction. Thus $\operatorname{Shift}(\tau^\sigma) \cap \operatorname{Shift}(\beta^\sigma) = \phi$. We also have

$$\alpha = \tau\beta \Longrightarrow \alpha^\sigma = \tau^\sigma\beta^\sigma$$

Thus $(\tau^\sigma, \beta^\sigma)$ is the factorization of α^σ. □

Theorem 25. *If α is an (m,r)-potent element of $\mathcal{T}_n$ with factorization $\alpha = \tau\beta$, then $C(\alpha) = C(\tau) \cap C(\beta)$.*

Proof. If $\sigma \in C(\alpha)$ then $\sigma^{-1}\alpha\sigma = \alpha$ which implies that $\tau^\sigma\beta^\sigma = \tau\beta$. So by the uniqueness of factorization that $\tau^\sigma = \tau$ and $\beta^\sigma = \beta$. Thus $\sigma \in C(\tau) \cap C(\beta)$. Conversely, if $\sigma \in C(\tau) \cap C(\beta)$, then $\tau^\sigma = \tau$ and $\beta^\sigma = \beta$. Hence $\sigma^{-1}\alpha\sigma = \alpha$ which implies that $\sigma \in C(\alpha)$. □

4.2. CONJUGATES OF IDEMPOTENTS

An $\beta \in \mathcal{T}_n$ is an idempotent if $\beta^2 = \beta$. Let I_n be the set of all idempotents in $\mathcal{T}_n$. Given $\beta \in I_n$ we define the fixed set of β by

$$\operatorname{Fix}\beta = \{x \in [n] : x\beta = x\}$$

If $y \in [n]$ and $x = y\beta$, then $x\beta = y\beta^2 = y\beta = x$. Thus $\operatorname{Im}\beta \subset \operatorname{Fix}\beta$. If $x \in \operatorname{Fix}\beta$, then $x = x\beta \in \operatorname{Im}\beta$. Thus $\operatorname{Fix}\beta \subset \operatorname{Im}\beta$. Consequently $\operatorname{Im}\beta = \operatorname{Fix}\beta$.

We define the **rank** of an idempotent β as $\operatorname{rank}\beta = |\operatorname{Im}\beta|$.

Lemma 4. *For $1 \le r \le n$ there are exactly $\binom{n}{r} r^{n-r}$ idempotents in I_n of rank r.*

Proof. Choose a subset $F \subset [n]$ with r elements. If $\delta : [n] - F \longrightarrow F$ is any function, then $\beta_\delta : [n] \to [n]$ defined by

$$k\beta_\delta = \begin{cases} k & k \in F \\ k\delta & k \in [n] - F \end{cases}$$

is an idempotent for clearly $k\beta_\delta \in F$, implies that $k\beta_\delta^2 = k\beta_\delta$. Moreover if $\delta \neq \delta'$, then $x\delta \neq x\delta'$ for some $x \in [n] - F$. Thus $x\beta_\delta = x\delta \neq x\delta' = x\beta_{\delta'}$. Clearly $\operatorname{Im}\beta_\delta = F$.

Also if β is an idempotent with $\operatorname{Im}\beta = F$ and $\delta = \beta|\,([n] - F)$, then $k\beta_\delta = \beta$. Thus there are exactly r^{n-r} idempotents in I_n with $\operatorname{Im}\beta = F$ of rank r. Since there $\binom{n}{r}$ choices of subsets of $[n]$, then there are exactly $\binom{n}{r} r^{n-r}$ idempotents in I_n of rank r. □

Theorem 26. *[4] $|I_n| = \sum_{r=1}^{n} \binom{n}{r} r^{n-r}$.*

Proof. This easily follows from Lemma 4. □

Given $\beta \in I_n$ and $\sigma \in S_n$, $\sigma^{-1}\beta\sigma$ is also an idempotent since

$$\left(\sigma^{-1}\beta\sigma\right)^2 = \sigma^{-1}\beta\sigma\sigma^{-1}\beta\sigma = \sigma^{-1}\beta^2\sigma = \sigma^{-1}\beta\sigma$$

Thus S_n acts on I_n on the right by

$$\beta \cdot \sigma = \sigma^{-1}\beta\sigma$$

This is an action, for if $\sigma, \tau \in S_n$, then

$$\beta \cdot (\sigma\tau) = (\sigma\tau)^{-1}\beta\sigma\tau = \tau^{-1}\sigma^{-1}\beta\sigma\tau = (\beta \cdot \sigma) \cdot \tau$$

and

$$\beta \cdot id = \beta$$

where $id \in S_n$ is the identity permutation. Given $\sigma \in S_n$we call $\sigma^{-1}\beta\sigma = \beta^{\sigma}$ a **conjugate** of β. It is easy to show that being conjugate is an equivalence relation. This means that if we define $\alpha \sim \beta$ for $\alpha, \beta \in I_n$ if there exists $\sigma \in S_n$ such that $\sigma^{-1}\beta\sigma = \alpha$, then $\sim$ is an equivalence relation on I_n. Equivalence classes are called **conjugacy classes**.

For $1 \leq r \leq n$ let $I_{n,r}$ be the set of idempotents β with $\operatorname{rank}\beta = r$.

Lemma 5. *Let $\sigma \in S_n$ and $\gamma = \sigma^{-1}\beta\sigma$. Then*

(a) *γ is an idempotent.*

(b) $(\operatorname{Im}\beta)\,\sigma = \operatorname{Im}\gamma$. *In particular* $\operatorname{rank}\beta = \operatorname{rank}\alpha$.

(c) *If $a \in \operatorname{Im}\beta$ and $a\sigma = b$ then $\left(a\beta^{-1}\right)\sigma = b\gamma^{-1}$.*

Proof. (a) We have already pointed out that γ is an idempotent.

(b) Let $x \in \operatorname{Im}\gamma$. Then

$$x = x\gamma = x\sigma^{-1}\beta\sigma \Longrightarrow x\sigma^{-1} = \left(x\sigma^{-1}\right)\beta \Longrightarrow x\sigma^{-1} \in \operatorname{Im}\beta \Longrightarrow x \in (\operatorname{Im}\beta)\,\sigma$$

So $(\operatorname{Im}\beta)\,\sigma \subset \operatorname{Fix}\gamma$. We have $\beta = \sigma\gamma\sigma^{-1}$. Thus $(\operatorname{Im}\gamma)\,\sigma^{-1} \subset \operatorname{Im}\beta \Longrightarrow \operatorname{Im}\gamma \subset (\operatorname{Im}\beta)\,\sigma$. Hence $(\operatorname{Im}\beta)\,\sigma = \operatorname{Im}\gamma$.

(c) Let $a \in \operatorname{Im}\beta$ and $b = a\sigma$. Let $x \in a\beta^{-1}$ and $y = x\sigma$. Then

$$y\gamma = y\left(\sigma^{-1}\beta\sigma\right) = x\sigma\left(\sigma^{-1}\beta\sigma\right) = x\beta\sigma = a\sigma = b$$

Thus $y \in b\gamma^{-1}$. This shows $\left(a\beta^{-1}\right)\sigma \subset b\gamma^{-1}$. We also have $\beta = \sigma\gamma\sigma^{-1}$. Thus $\left(b\gamma^{-1}\right)\sigma^{-1} \subset a\beta^{-1}$ which yields $b\gamma^{-1} \subset b\left(a\beta^{-1}\right)\sigma$. This proves that $b\gamma^{-1} = \left(a\beta^{-1}\right)\sigma$. □

We now fix an $r \in [n]$ and deal with $I_{n,r}$. Given an idempotent $\beta \in I_{n,r}$ we put

$$F_{\beta,s} = \left\{x \in \operatorname{Im}\beta : \left|x\beta^{-1}\right| = s\right\}$$

for $1 \leq s \leq n$. We define $r_{\beta,s} = \left|F_{\beta,s}\right|$. Naturally $F_{\beta,s}$ may be empty for some s. We define

$$J_{\beta,s} = \cup_{x\in F_{\beta,s}} x\beta^{-1}$$

Note that this is a disjoint union since $x\beta^{-1} \cap y\beta^{-1} = \phi$ when $x \neq y$. We also note that $J_{\beta,s} \cap J_{\beta,t} = \phi$ when $s \neq t$. For if $y \in J_{\beta,s} \cap J_{\beta,t}$, then $y \in u\beta^{-1}$ for some $u \in F_{\beta,s}$ and $y \in v\beta^{-1}$ for some $v \in F_{\beta,t}$. Then $u = y\beta = v$. This means that $s = \left|u\beta^{-1}\right| = \left|v\beta^{-1}\right| = t$, a contradiction. Thus $[n]$ is the disjoint union of $J_{\beta,s}$'s.

$$[n] = \cup_{s=1}^{n} J_{\beta,s}$$

Lemma 6. *Let $\beta \in I_n$ and* $\operatorname{rank}\beta = r$.

(a) *The conjugacy class of β is contained in $I_{n,r}$.*

(b) $\operatorname{rank}\beta = \sum_{s=1}^{n} r_{\beta,s}$.

Proof. (a) This follows from Lemma 5.

(b) This is clear since for each $x \in \operatorname{Im}\beta$, $1 \leq \left|x\beta^{-1}\right| \leq n$ and $F_{\beta,s} \cap F_{\beta,t} = \phi$ for $1 \leq s \neq t \leq n$. □

The following theorem says exactly when two idempotents are conjugate.

Theorem 27. *Let $\alpha, \beta \in I_{n,r}$. Then α and β are conjugates if and only if $r_{\alpha,s} = r_{\beta,s}$ for each $s \in [n]$.*

Proof. Suppose that $r_{\alpha,s} = r_{\beta,s}$ for each $s \in [n]$. By Lemma 6 $\operatorname{rank}\beta = \sum_{s=1}^{n} r_{\beta,s} = \operatorname{rank}\alpha = \sum_{s=1}^{n} r_{\alpha,s}$. Hence there is a bijection $\lambda_s : F_{\beta,s} \to F_{\alpha,s}$. Now suppose that $r_{\alpha,s} = r_{\beta,s} \geq 1$. If $x \in F_{\beta,s}$ and $y = x\lambda_s \in F_{\alpha,s}$, then $F_{\beta,s}$ and $F_{\alpha,s}$ are of the form

$$x\beta^{-1} = \{x, u_1, \ldots, u_{s-1}\} \text{ if } s \geq 2 \text{ and } x\beta^{-1} = \{x\} \text{ if } s = 1$$

$$y\alpha^{-1} = \{y, v_1, \ldots, v_{s-1}\} \text{ if } s \geq 2 \text{ and } y\alpha^{-1} = \{y\} \text{ if } s = 1$$

Extend the definition of λ_s from $J_{\beta,s} = \cup_{x \in F_{\beta,s}} x\beta^{-1}$ to $\cup_{x \in F_{\alpha,s}} x\alpha^{-1}$ by

$$u_i\lambda_s = v_i \text{ for } 1 \leq i \leq s-1$$

□

Proof. Since $[n] = \cup_{s=1}^{n} J_{\beta,s}$ we can define $\sigma \in S_n$ by

$$x\sigma = x\lambda_s \text{ for } x \in J_{\beta,s}$$

We now claim that $\alpha = \sigma^{-1}\beta\sigma$. Let $d \in [n]$. There is a unique $1 \leq s \leq n$ such that $c \in J_{\beta,s}$. Let $d = c\sigma = c\lambda_s \in J_{\alpha,s}$. There exists a unique $x \in F_{\beta,s}$ such that

$$\begin{aligned} c &\in x\beta^{-1} = \{x, u_1, \ldots, u_{s-1}\} \Longrightarrow \\ d &= c\sigma \in \{x\lambda_s, u_1\lambda_s, \ldots, u_{s-1}\lambda_s\} = \{y, v_1, \ldots, v_{s-1}\} = y\alpha^{-1} \end{aligned}$$

where $y = x\lambda_s$. Thus

$$d\sigma^{-1}\beta\sigma = x\sigma = y = d\alpha$$

□

Give an idempotent β we call the n-tuple $t_\beta = \left(r_{\beta,n}, r_{\beta,n-1}, \ldots, r_{\beta,1}\right)$ the **type** of β. By Theorem 27 two idempotents are conjugate if and only if they have the same type. So by Theorem 16 we have the following result.

Theorem 28. *The number of distinct conjugate classes of idempotents in $\mathcal{T}_n$ is $p(n)$.*

We now tackle the problem of the number of elements in the conjugate class of an idempotent β.

Theorem 29. *Given $j_1 r_{j_1} + j_2 r_{j_2} + \cdots + j_t r_{j_t} = n$ where $r_{j_i} > 0$ and $j_1 > \cdots > j_{t-1} > j_t$ for $i \in [t]$, there are exactly*

$$\binom{n}{\underbrace{j_1, \ldots, j_1}_{r_{j_1}-times}, \underbrace{j_2, \ldots, j_2}_{r_{j_2}-times}, \ldots, \underbrace{j_t, \ldots, j_t}_{r_{j_t}-times}} \frac{1}{r_{j_1}!} \frac{1}{r_{j_2}!} \cdots \frac{1}{r_{j_t}!} j_1^{r_{j_1}} j_2^{r_{j_2}} \cdots j_t^{r_{j_t}}$$

elements in the conjugacy class corresponding to the partition with type $t_\beta = \left(r_{\beta,n}, r_{\beta,n-1}, \ldots, r_{\beta,1}\right)$ where

$$r_{\beta,s} = \begin{cases} r_{j_i} & \text{if } s = j_i \\ 0 & \text{otherwise} \end{cases}$$

Stabilizer group of an idempotent We have seen that

Given $j_1 k_{j_1} + j_2 k_{j_2} + \cdots + j_t k_{j_t} = n$ where $k_{j_i} > 0$ for $i \in [t]$, there are exactly

$$\binom{n}{\underbrace{j_1, \ldots, j_1}_{k_{j_1}-\text{times}}, \underbrace{j_2, \ldots, j_2}_{k_{j_2}-\text{times}}, \ldots, \underbrace{j_t, \ldots, j_t}_{k_{j_t}-\text{times}}} \frac{1}{k_{j_1}!} \frac{1}{k_{j_2}!} \cdots \frac{1}{k_{j_t}!} j_1^{k_{j_1}} j_2^{k_{j_2}} \cdots j_t^{k_{j_t}}$$

elements in the conjugacy class corresponding to the partition determined by $j_1 k_{j_1} + j_2 k_{j_2} + \cdots + j_t k_{j_t} = n$.

Given an idempotent β of type $\underbrace{j_1, \ldots, j_1}_{k_{j_1}-\text{times}}, \underbrace{j_2, \ldots, j_2}_{k_{j_2}-\text{times}}, \ldots, \underbrace{j_t, \ldots, j_t}_{k_{j_t}-\text{times}}$ the stabilizer group $C(\beta)$ has

$$\underbrace{j_1! \cdots j_1!}_{k_{j_1}-\text{times}} \underbrace{j_2! \cdots j_2!}_{k_{j_2}-\text{times}} \cdots \underbrace{j_t! \cdots j_t!}_{k_{j_t}-\text{times}} \frac{k_{j_1}! k_{j_2}! k_{j_t}!}{j_1^{k_{j_1}} j_2^{k_{j_2}} \cdots j_t^{k_{j_t}}}$$

elements.

It is clear that we only need to understand this formula for the part

$$\underbrace{j!\cdots j!}_{k-\text{times}}\frac{k!}{j^k}$$

So suppose

$$m = jk$$

and an idempotent β have type $\underbrace{j,\ldots,j}_{k-\text{times}}$. Let $A_1,\ldots,A_k$ be the fibers of elements $a_1 \in A_1,\ldots,a_k \in A_k$. So if we are given a permutation of τ of $\{a_1,\ldots,a_k\}$ and a bijection $\rho_i : A_i - \{a_i\} \to A_{i\tau} - \{a_{i\tau}\}$ for each $i = 1,\ldots,k$ we define $\rho_i' : A_i \to A_{i\tau}$

$$j\rho_i' = \begin{cases} j\rho_i & j \neq a_i \\ a_{i\tau} & j = a_i \end{cases}$$

If we take σ whose restriction to A_i is ρ_i', then we $\sigma \in C(\beta)$.

Conversely each $\sigma \in C(\beta)$ is obtained in this way. There are $k!$ many τ's. For each choice of τ and i there are $(j-1)!$ many ρ_i. So the number of elements in $C(\beta)$ is

$$\underbrace{(j-1)!\cdots(j-1)!}_{k-\text{times}}\,k! = \underbrace{j!\cdots j!}_{k-\text{times}}\frac{k!}{j^k}$$

4.3. VARIANTS OF $\mathcal{T}_N$

Give a semigroup S and $a \in S$ a **variant** of S associated with a is the semigroup $(S^a, \circ_a)$ where $S^a = S$ and for $x, y \in S$, $x \circ_a y = xay$. For $x, y, z \in S$

$$x \circ_a (y \circ_a z) = xayaz = (x \circ_a y) \circ_a z$$

Thus $\circ_a$is an associative binary operation on S and consequently $(S^a, \circ_a)$ is a semigroup.

Below we give information about variants of $\mathcal{T}_n$.

Lemma 7. *$P = \{S_n \alpha S_n : \alpha \in \mathcal{T}_n\}$ is a partition of $\mathcal{T}_n$.*

Proof. Let $\alpha, \beta \in \mathcal{T}_n$ be such that $S_n\alpha S_n \cap S_n\beta S_n \neq \phi$. Then there are $\lambda, \mu, \sigma, \tau \in S_n$ such that $\lambda\alpha\mu = \sigma\beta\tau$. This means that $\alpha = \rho\beta\theta$ where $\rho = \lambda^{-1}\sigma$ and $\theta = \tau\mu^{-1}$. Now, if $\delta \in S_n\alpha S_n$, then $\delta \in S_n\rho\beta\theta S_n \subset S_n\beta S_n$. It follows that $S_n\alpha S_n \subset S_n\beta S_n$. Similarly $S_n\beta S_n \subset S_n\alpha S_n$. Hence $S_n\beta S_n = S_n\alpha S_n$. This proves that given two elements in P either they are equal or disjoint. Moreover $\alpha \in S_n\alpha S_n$. Thus P is a partition of $\mathcal{T}_n$. □

For $\alpha \in \mathcal{T}_n$ and $x \in [n]$ the $x\alpha^{-1} = \{y : y\alpha = x\}$ is the preimage of x under α. If $x \neq y$, then $x\alpha^{-1} \cap y\alpha^{-1} = \phi$ and $[n]$ is the union of preimages. For $k \geq 0$ we define $\mathbf{t}_k(\alpha)$ to be the number of preimages with cardinality k.

For example if

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 3 & 4 & 4 & 4 & 6 & 6 & 7 & 9 \end{pmatrix}$$

Preimages of α :

x	1	2	3	4	5	6	7	8	9
$x\alpha^{-1}$	ϕ	ϕ	$\{1,2\}$	$\{3,4,5\}$	ϕ	$\{6,7\}$	$\{8\}$	ϕ	$\{9\}$

Thus

$$\mathbf{t}_0(\alpha) = 4, \mathbf{t}_1(\alpha) = 2, \mathbf{t}_2(\alpha) = 2, \mathbf{t}_3(\alpha) = 1, \mathbf{t}_k(\alpha) = 0 \text{ for } k \geq 4$$

Another example is

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 2 & 2 & 2 \end{pmatrix}$$

Preimages of α :

x	1	2	3	4	5	6
$x\alpha^{-1}$	$\{1,2,3\}$	$\{4,5,6\}$	ϕ	ϕ	ϕ	ϕ

Thus

$$\mathbf{t}_0(\alpha) = 4, \mathbf{t}_1(\alpha) = 0, \mathbf{t}_2(\alpha) = 0, \mathbf{t}_3(\alpha) = 2, \mathbf{t}_k(\alpha) = 0 \text{ for } k \geq 4$$

For $\alpha \in \mathcal{T}_n$ we define the type $\mathbf{t}(\alpha)$ of α as $\mathbf{t}(\alpha) = (\mathbf{t}_0(\alpha), \mathbf{t}_1(\alpha), \ldots, \mathbf{t}_n(\alpha))$.

It is clear that $\mathbf{t}_0(\alpha) = |[n] - \operatorname{Im}\alpha|$

Lemma 8. *If $\alpha, \beta \in \mathcal{T}_n$, then $S_n\beta S_n = S_n\alpha S_n$ if and only* $\mathbf{t}(\alpha) = \mathbf{t}(\beta)$.

Proof. Let us first assume that $S_n\beta S_n = S_n\alpha S_n$. Then as in the proof of the previous lemma we can find $\rho, \theta \in S_n$ such that $\alpha = \rho\beta\theta$. Let $0 \leq k \leq n$. Given an $x \in [n]$, $y = x\alpha$ if and only if $y\theta^{-1} = (x\rho)\beta$. Thus $\left(x\alpha^{-1}\right)\theta^{-1} = (x\rho)\beta^{-1}$. This proves $\mathbf{t}(\alpha) = \mathbf{t}(\beta)$.

Conversely suppose that $\mathbf{t}(\alpha) = \mathbf{t}(\beta)$. Since $\mathbf{t}_0(\alpha) = \mathbf{t}_0(\beta)$, then $k_0 = |[n] - \operatorname{Im}\alpha| = |[n] - \operatorname{Im}\beta|$ and consequently $m = |\operatorname{Im}\alpha| = |\operatorname{Im}\beta|$. Let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_m \\ a_1 & a_2 & \cdots & a_m \end{pmatrix} \text{ and } \beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_m \\ b_1 & b_2 & \cdots & b_m \end{pmatrix}$$

We assume that the elements of $\operatorname{Im}\alpha$ is ordered so that $|A_i| \leq |A_{i+1}|$ for $1 \leq i < m$. Similarly the elements of $\operatorname{Im}\beta$ is ordered so that $|B_i| \leq |B_{i+1}|$ for $1 \leq i < m$. For example if

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 3 & 4 & 4 & 4 & 6 & 6 & 7 & 9 \end{pmatrix}$$

$$\alpha = \begin{pmatrix} \{8\} & \{9\} & \{1,2\} & \{6,7\} & \{3,4,5\} \\ 7 & 9 & 3 & 6 & 4 \end{pmatrix}$$

Since $\mathbf{t}(\alpha) = \mathbf{t}(\beta)$, then we have $|A_i| = |B_i|$ for $1 \leq i \leq m$. A_i's are disjoint and their union is $[n]$. Similarly B_i's are disjoint and their union is $[n]$. There is a bijection θ from $[n]$ onto $[n]$ such that $A_j = \left\{i\theta : s_{j-1} + 1 \leq i \leq s_{j-1} + k_j\right\}$ where

$$k_j = \left|A_j\right|, 1 \leq j \leq m$$

and

$$s_0 = 0, s_1 = k_1, s_2 = k_1 + k_2, \ldots, s_m = k_1 + k_2 + \cdots + k_m$$

Similarly, there is a bijection ρ from $[n]$ onto $[n]$ such that $B_j = \left\{i\theta : s_{j-1} + 1 \leq i \leq s_{j-1} + k_j\right\}$ since $\left|A_j\right| = \left|B_j\right|$ for $1 \leq j \leq m$. Let σ be a permutation of $[n]$ such that $a_j\sigma = b_j$ and the restriction of σ to $[n] - \operatorname{Im}\alpha$ is a bijection from $[n] - \operatorname{Im}\alpha$

onto $[n] - \operatorname{Im}\beta$. Then for $s_{j-1} + 1 \leq i \leq s_{j-1} + k_j$

$$i\theta\alpha\sigma = a_j\sigma = b_j \text{ and } i\rho\beta = b_j$$

Thus $\theta\alpha\sigma = \rho\beta$. Thus $\alpha = \theta^{-1}\rho\beta\sigma^{-1}$. □

Lemma 9. *If $\alpha \in \mathcal{T}_n$, then there exists an idempotent $\beta \in \mathcal{T}_n$ such that* $\mathbf{t}(\alpha) = \mathbf{t}(\beta)$.

Proof. Let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_m \\ a_1 & a_2 & \cdots & a_m \end{pmatrix}$$

We assume that the elements of $\operatorname{Im}\alpha$ is ordered so that $|A_i| \leq |A_{i+1}|$ for $1 \leq i < m$. For each $1 \leq j \leq m$ let x_j be the first element if A_j. We define β as

$$\beta = \begin{pmatrix} A_1 & A_2 & \cdots & A_m \\ x_1 & x_2 & \cdots & x_m \end{pmatrix}$$

Then clearly $\mathbf{t}(\alpha) = \mathbf{t}(\beta)$ and β is idempotent. □

For $\alpha \in \mathcal{T}_n$ we define the equivalence relation $\sim_\alpha$ on $[n]$ by $i \sim_\alpha j$ if and only if $i\alpha = j\alpha$. For $i \in [n]$,

We will use the notation $[i]_\alpha$ to denote the class of the $\sim_\alpha$that contains i.

Lemma 10. *Let α, $\beta \in \mathcal{T}_n$ and $\rho \in S_n$ be such that $i \sim_\alpha j$ if and only if $i\rho \sim_\beta j\rho$. Then* $\mathbf{t}(\alpha) = \mathbf{t}(\beta)$.

Proof. We claim that $[i]_\alpha \rho = [i\rho]_\beta$. For if $j \in [i]_\alpha$, then

$$i \sim_\alpha j \Longrightarrow i\rho \sim_\beta j\rho \Longrightarrow j\rho \in [i\rho]_\beta$$

Thus $[i]_\alpha \rho \subset [i\rho]_\beta$. But if $s \in [i\rho]_\beta$, then

$$s \sim_\beta i\rho \Longrightarrow \left(s\rho^{-1}\right)\rho \sim_\beta i\rho \Longrightarrow s\rho^{-1} \sim_\alpha i \Longrightarrow s\rho^{-1} \in [i]_\alpha \Longrightarrow s \in [i]_\alpha \rho$$

. Thus $[i]_\alpha \rho = [i\rho]_\beta$. Now let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_m \\ a_1 & a_2 & \cdots & a_m \end{pmatrix}$$

arranged in such a way that $|A_1| \leq |A_2| \leq \cdots \leq |A_m|$. Then $A_1, \ldots, A_m$ are the equivalence classes of $\sim_\alpha$. Pick any $i \in A_s$. If $j \in A_s$, then $i\alpha = j\alpha = a_s$. Thus $i \sim_\alpha j$,i.e. $A_s \subset [i]_\alpha$. If $j \in [i]_\alpha$, then $i \sim_\alpha j$,i.e.$j\alpha = i\alpha = a_s$. Thus $[i]_\alpha \subset A_s$. So $A_s = [i]_\alpha$. Pick an element $i_k \in A_s$. Then

$$A_k\rho = [i_k]_\alpha \rho = [i_k\rho]_\rho = B_k$$

$$\alpha = \begin{pmatrix} B_1 & B_2 & \cdots & B_m \\ i_1\rho & i_2\rho & \cdots & i_m\rho \end{pmatrix}$$

This shows that $\mathbf{t}(\alpha) = \mathbf{t}(\beta)$. □

Theorem 30. *For $\alpha, \beta \in \mathcal{T}_n$ the following are equivalent.*

(a) $\mathcal{T}_n^\alpha \simeq \mathcal{T}_n^\beta$.

(b) *There exists $\lambda, \mu \in S_n$ such that $\alpha = \lambda\beta\mu$.*

(c) $\mathbf{t}(\alpha) = \mathbf{t}(\beta)$.

Proof. We know that $(b) \Longleftrightarrow (c)$.

$(b) \Longrightarrow (a)$ Let $\lambda, \mu \in \mathcal{S}_n$ be such that $\alpha = \lambda\beta\mu$. Define $\phi : S^\alpha \to S^\beta$ by $\gamma\phi = \mu^{-1}\gamma\lambda^{-1}$. Then

$$(\rho \cdot_\alpha \theta)\,\phi = \mu^{-1}(\rho\alpha\theta)\,\lambda^{-1}$$

$$\begin{aligned} (\rho\phi) \cdot_\beta (\theta\phi) &= \left(\mu^{-1}\rho\lambda^{-1}\right)\beta\left(\mu^{-1}\theta\lambda^{-1}\right) \\ &= \mu^{-1}\rho\left(\lambda^{-1}\beta\mu^{-1}\right)\theta\lambda^{-1} = \mu^{-1}(\rho\alpha\theta)\,\lambda^{-1} \end{aligned}$$

Thus ϕ is a homomorphism. Similarly $\varphi : S^\beta \to S^\alpha$ defined by $\gamma\varphi = \mu\gamma\lambda$ is a homomorphism since $\beta = \lambda^{-1}\alpha\mu^{-1}$. Also

$$\gamma\left(\varphi\phi\right) = \left(\mu\gamma\lambda\right)\phi = \mu^{-1}\left(\mu\gamma\lambda\right)\lambda^{-1} = \gamma \text{ and } \gamma\left(\phi\varphi\right) = \left(\mu^{-1}\gamma\lambda^{-1}\right)\varphi = \mu\mu^{-1}\gamma\lambda^{-1}\lambda = \gamma$$

Thus ϕ is an isomorphism.

$(a) \implies (b)$ Without loss of generality we may assume that both α and β are idempotents. We will need some more detailed information about $\mathcal{T}_n^{\alpha}$. Let $J = \{\varepsilon_i : i \in [n]\}$ where ε_i is the constant map $j\varepsilon_i = i$ for every $j \in [n]$. J is an ideal of $\mathcal{T}_n^{\alpha}$ for $\varepsilon_i \circ_a \gamma = \varepsilon_i\alpha\gamma = \varepsilon_{i\alpha\gamma} \in J$ and $\gamma \circ_a \varepsilon_i = \gamma\alpha\varepsilon_i = \varepsilon_i \in J$. Also any ideal I in $\mathcal{T}_n^{\alpha}$ contains J. To see this pick any $\gamma \in I$. Then $\varepsilon_i = \gamma \circ_a \varepsilon_i \in I$. Thus if $\varphi : \mathcal{T}_n^{\alpha} \to \mathcal{T}_n^{\beta}$ is an isomorphism, then $J\varphi = J$. Since φ is a bijection, then this implies that there is $\rho \in S_n$ such that

$$\varepsilon_i\varphi = \varepsilon_{i\lambda}$$

Let us define the equivalence relation $\sim_\alpha$ on $[n]$ by $i \sim_\alpha j$ if and only if $i\alpha = j\alpha$. Similarly we define an equivalence relation $i \sim_\beta j$ on J if and only if $i\beta = j\beta$. We claim that $i \sim_\alpha j$ if and only if $i\rho \sim_\beta j\rho$ .i.e. $i\rho\beta = j\rho\beta$.

Let $i \sim_\alpha j$. Then $i\alpha = j\alpha$. Thus $i\alpha\gamma = j\alpha\gamma$ for every $\gamma \in \mathcal{T}_n^{\alpha}$. Thus $\varepsilon_i \circ_a \gamma = \varepsilon_j \circ_a \gamma$ for every $\gamma \in \mathcal{T}_n^{\alpha}$. Hence $\varepsilon_i\varphi \circ_\beta \gamma\varphi = \varepsilon_j\varphi \circ_\beta \gamma\varphi$ for every $\gamma \in \mathcal{T}_n^{\alpha}$. Since φ is onto and $\varepsilon_i\varphi = \varepsilon_{i\lambda}$ this means $\varepsilon_{i\lambda} \circ_\beta \eta = \varepsilon_{j\lambda} \circ_\beta \eta$ for every $\eta \in \mathcal{T}_n^{\beta}$. Hence $i\lambda\beta\eta = j\lambda\beta\eta$ for every $\eta \in \mathcal{T}_n^{\beta}$. If we chose $\eta = 1_{[n]}$ we see that $i\rho\beta = j\rho\beta$. Thus $i\rho \sim_\beta j\rho$.

Conversely assume that $i\rho \sim_\beta j\rho$ which means $i\lambda\beta = j\lambda\beta$. Then $i\lambda\beta\eta = j\lambda\beta\eta$ for every $\eta \in \mathcal{T}_n^{\beta}$. Hence $\varepsilon_i\varphi \circ_\beta \gamma\varphi = \varepsilon_j\varphi \circ_\beta \gamma\varphi$ for every $\gamma \in \mathcal{T}_n^{\alpha}$. If $\psi : \mathcal{T}_n^{\beta} \to \mathcal{T}_n^{\alpha}$ is the inverse of φ, then $\varepsilon_i\varphi\psi \circ_\beta \gamma\varphi\psi = \varepsilon_j\varphi\psi \circ_\beta \gamma\varphi\psi$ for every $\gamma \in \mathcal{T}_n^{\alpha}$. Thus $\varepsilon_i \circ_a \gamma = \varepsilon_j \circ_a \gamma$ for every $\gamma \in \mathcal{T}_n^{\alpha}$. But this implies that $i\alpha\eta = j\alpha\eta$ for every $\gamma \in \mathcal{T}_n^{\alpha}$. If we choose $\gamma = 1_{[n]}$ we see that $i\alpha = j\alpha$.

Now by the Lemma 10 we get $\mathbf{t}(\alpha) = \mathbf{t}(\beta)$. $\square$

5. NILPOTENT TRANSFORMATIONS

5.1. NILPOTENT TRANSFORMATIONS

An element $\alpha \in \mathcal{PT}_n$ is a **nilpotent** transformation if $\alpha^m = \bar{0}$ for some $m > 0$ where in $\bar{0}$ is the transformation which maps every element $k \in \langle n \rangle$ to 0. Let us denote the nilpotent elements of $\mathcal{PT}_n$ by $\mathcal{N}(\mathcal{PT}_n)$. The **degree of nilpotency** of a nilpotent element α is the smallest positive integer k with $\alpha^k = \bar{0}$.

5.2. APPLICATIONS OF PRÜFER CODE REPRESENTATION

We give two applications of interpreting a nilpotent transformation as a labelled tree and Prüfer code representation of labelled trees which can be used as a representation of corresponding nilpotent transformation. If a tree T has k vertices Prüfer code of T has $k-2$ coordinates. So, if $T \in \mathcal{L}_n = \mathcal{L}(< n >)$, then T has $n+1$ (including 0) vertices. Therefore Prüfer code of T has $n+1-2 = n-1$ coordinates. $\deg v$ denotes the degree of a vertex v of T. An application is the proof of the following theorem as suggested in [4].

Theorem 31 ([23])**.** *$\mathcal{PT}_n$ contains exactly $(n+1)^{(n-1)}$ nilpotent elements.*

Proof. A tree in $\mathcal{L}_n$ has $n+1$ vertices. Hence By Cayley's Theorem $\mathcal{L}_n$ has $(n+1)^{(n-1)}$ elements. Thus

$$|\mathcal{N}(\mathcal{PT}_n)| = |\mathcal{N}(\mathcal{PT}_n)| = |\mathcal{L}_n| = (n+1)^{(n-1)}$$

□

We may actually use this interpretation to present another proof of a theorem given in [23]. We use the notation $S(p,q)$ for the Stirling number of second kind which expresses the number of ways one can partition p elements into q subsets.

Theorem 32 ([23])**.** *The semigroup $\mathcal{PT}_n$ contains $k!\begin{pmatrix} n \\ k \end{pmatrix} S(n, k+1)$ nilpotent elements of rank k.*

Before we proceed, we recall a fact concerning Striling numbers.

Theorem 33 ([14]). $\sum_{j=k}^{n-1} \begin{pmatrix} n-1 \\ j \end{pmatrix} S(j,k) = S(n,k+1)$.

We are now ready to present our proof of Theorem 32.

Proof of 32. There are $\binom{n}{k}$ choices as the image set of a transformation in $\mathcal{N}(\mathcal{PT}_n)$ with rank k. Let $\{r_1, ..., r_k\}$ be one of the choices. Define $P(r_1, ..., r_k)$ as the set of all elements of $\mathcal{N}(\mathcal{PT}_n)$ with the image set $R = \{r_1, ..., r_k\}$. If $\alpha \in \mathcal{N}(\mathcal{PT}_n)$ has the image set R, then image set of $\overline{\alpha}$ is $\{0\} \cup R$, $\operatorname{rank} \overline{\alpha} = k+1$ and $\overline{\alpha} \in \mathcal{N}(\mathcal{PT}_n)$. If $T \in \mathcal{L}_n$ is such that $\varphi(T) = \overline{\alpha}$, then $r_1, ..., r_k$ are certainly non leaf nodes of T by Corollary 1. So we must count the number of trees in $\mathcal{L}_n$ such that $r_1, ..., r_k$ are non leaf nodes of T. Let $LT(r_1, ..., r_k)$ be the set of such trees. Naturally one node of T is labelled 0. This node may or may not be a leaf. We set $LT_i(r_1, ..., r_k)$ to be the set of all elements of $\mathcal{L}_n$ such that 0 appears in the Prüfer code i many times or $\deg 0 = i+1$. Since the prüfer code of $T \in LT_i(r_1, ..., r_k)$ has exactly $n+1-2 = n-1$ coordinates, then we must $0 \leq i \leq n-1-k$.

$LT(r_1, ..., r_k)$ is the disjoint union of $LT_i(r_1, ..., r_k)$ for $0 \leq i \leq n-1-k$. So we only have to determine the cardinality of $LT_i(r_1, ..., r_k)$. This means that we need to find the number of all Prüfer codes with i many 0's and $r_1, ..., r_k$. To do this we first choose a subset B of $n-1$ coordinates containing i elements. There are $\binom{n-1}{i}$ many choices for these coordinate sets. Next, we pick a partition of the remaining $n-1-i$ coordinates into k disjoint subsets $S_1, \ldots, S_k$. We set coordinates in S_j equal to r_j, $j = 1, \ldots, k$. There are $k!$ ways of doing this for there are $k!$ permutations of $S_1, \ldots, S_k$. So the number of elements in $LT_i(r_1, ..., r_k)$ is

$$\begin{pmatrix} n-1 \\ i \end{pmatrix} k! S(n-i-1,k)$$

It follows that

$$|LT(r_1, ..., r_k)| = k! \sum_{i=0}^{n-1-k} \begin{pmatrix} n-1 \\ i \end{pmatrix} S(n-i-1,k)$$

By setting $j = n - i - 1$ and taking into acount that

$$\binom{n-1}{n-1-j} = \binom{n-1}{j}$$

we get

$$\begin{aligned} &\sum_{i=0}^{n-1-k} \binom{n-1}{i} S(n-1-i,k) \\ = &\sum_{j=n-1}^{k} \binom{n-1}{j} S(j,k) = S(n,k+1) \end{aligned}$$

by Theorem 33. Thus

$$|LT(r_1,...,r_k)| = k!S(n,k+1)$$

Since there are $\binom{n}{k}$ choices of $(r_1,...,r_k)$, then

$$|\mathcal{N}(\mathcal{PT}_n)| = \binom{n}{k} k!S(n,k+1)$$

as required. □

Theorem 34. *[4]The number of nilpotent elements of defect k in the semigroup $\mathcal{IS}_n$ equals $\frac{n!}{k!}\binom{n-1}{k-1}$.*

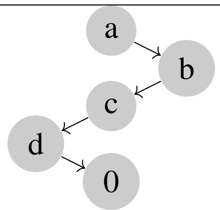
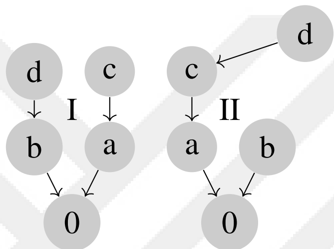
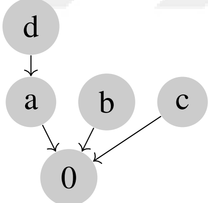
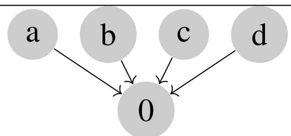
Proof. Let $\mathcal{L}_n$ be the set of labelled trees with labels in $0, 1, \ldots .n$. $\alpha \in \mathcal{L}_n$ belongs to $\mathcal{IS}_n$ if and only if for each node w of α either $w = 0$ or $w \in [n]$ and $\deg w = 1$ or $\deg w = 2$. If α has defect k, then the number of nodes of degree 1 is k. There are $\binom{n}{k}$ choices of such nodes. Suppose we have already chosen a set of k elements from $[n]$. Remaining $n - k$ elements will appear in Prüfer coding which is an $n - 1$ tuple c. The rest of the coordinates of c will be filled with $(n-1) - (n-k) = k - 1$ zeros. Out of $n - 1$ coordinates we have $\binom{n-1}{k-1}$ ways of placing 0's. There are $(n-k)!$ ways of placing the remaining $n - k$ nodes. So the total number of nilpotent elements of defect k in the semigroup $\mathcal{IS}^n$ equals

$$(n-k)!\binom{n}{k}\binom{n-1}{k-1} = \frac{n!}{k!}\binom{n-1}{k-1}$$

□

In Table 5.1 we list all nilpotent transformations in $\mathcal{IS}_4$ with defect k for $1 \leq k \leq 4$.

Table 5.1. Illustration of $\mathcal{IS}_4$ classified via defect k

$k = 1$ $\frac{4!}{1!}\binom{3}{0} = 24$		We may arrange a, b, c, d in $4! = 24$ ways.
$k = 2$ $\frac{4!}{2!}\binom{3}{1} = 36$		I) We may choose a, b in $\binom{4}{2} = 6$ ways. Given a, b, we may choose c in 2 ways. We may place c on top of one of a, b in 2 ways. A total of 24 distinct trees II) We may choose a, b in $\binom{4}{2} = 6$ ways. Given a, b we may choose c in 2 ways. A total of 12 distinct trees
$k = 3$ $\frac{4!}{3!}\binom{3}{2} = 12$		We may choose a, b, c in $\binom{4}{3} = 4$ ways. Given a, b, c we may choose one of a, b, c to place d on top in 3 ways. A total of 12 distinct trees
$k = 4$ $\frac{4!}{4!}\binom{3}{3} = 1$		

Corollary 2. *[4]The number of nilpotent elements in the semigroup $\mathcal{IS}_n$ is*

$$\sum_{k=1}^{n} \frac{n!}{k!}\binom{n-1}{k-1}$$

Proof. $\alpha \in \mathcal{IS}_n$, then α must have defect at least 1. For otherwise α will be a permutation of $[n]$ and hence will have a cycle in one of its orbits. But this would mean that α is not nilpotent. Now the claim follows from Theorem 34. □

Theorem 35. *[4]Each maximal nilpotent subsemigroup of $\mathcal{IS}^n$ has cardinality B_n, n the*

Bell number.

Proof. We know that a maximal nilpotent subsemigroup of $\mathcal{IS}_n$ corresponds to a partition on $[n]$ into singletons. Assume that $\sigma = \{\{b_1\}, \ldots, \{b_n\}\}$ is such a partition. Naturally $[n]$ is well ordered by means of σ as $b_1 <_\sigma \cdots <_\sigma b_n$. Let us be given a partition λ of $[n]$. Let us assume that $\lambda = \{V_1, \ldots, V_k\}$. Each V_i is well orders with respect to the ordering induced by σ. We define a nilpotent element α_λ corresponding to λ as follows:

$$\langle n \rangle \text{ is the vertex set of } \alpha$$

$$(x, 0) \text{ is an edge in } \alpha \text{ if } x = \min V_i \text{ for some } i = 1, \ldots, k$$

$$\text{if } y \neq 0, \text{ then } (x, y) \text{ is an edge in } \alpha \text{ if } x, y \in V_i \text{ for some } i = 1, \ldots, k \text{ and } y <_\sigma x$$

Then the set $N_\sigma = \{\alpha_\lambda : \lambda \text{ is a partition of } [n]\}$ is the maximal nilpotent subsemigroup in $\mathcal{IS}_n$. If $\lambda \neq \mu$, then $\alpha_\lambda \neq \alpha_\mu$. Thus there are exactly B_n elements in N_σ. □

Two illustrations of N_σ are given in Table 5.2 and Table 5.3; first ilusstration is for $n = 3$ and $\sigma = \{\{3\}, \{1\}, \{2\}\}$ and the latter illustration is for 5.3 an is given for $n = 4$ and $\sigma = \{\{3\}, \{4\}, \{1\}, \{2\}\}$.

Table 5.2. N_σ when $n = 3$ and $\sigma = \{\{3\}, \{1\}, \{2\}\}$

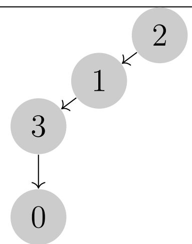
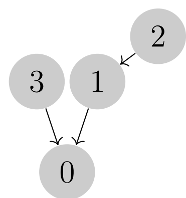
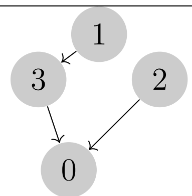
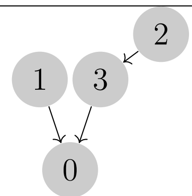
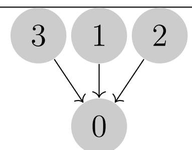
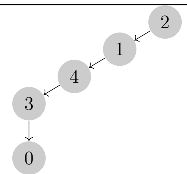
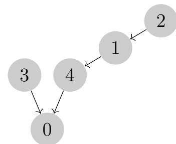
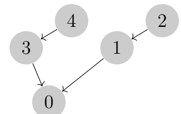
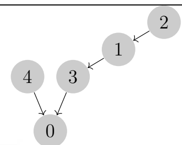
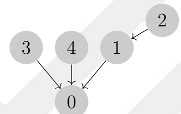
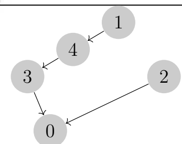
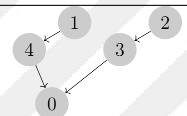
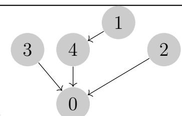
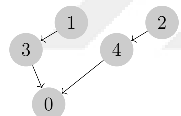
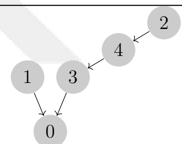
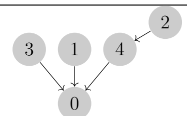
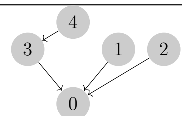
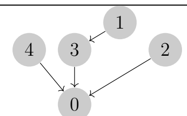
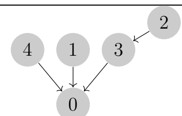
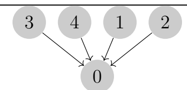
$\lambda : \{3, 1, 2\}$		$\lambda : \{3\}, \{1, 2\}$	
$\lambda : \{3, 1\}, \{2\}$		$\lambda : \{1\}, \{3, 2\}$	
$\lambda : \{3\}, \{1\}, \{2\}$			

Table 5.3. N_σ when $n = 4$ and $\sigma = \{\{3\}, \{4\}, \{1\}, \{2\}\}$

$\lambda : \{3, 4, 1, 2\}$		$\lambda : \{3\}, \{4, 1, 2\}$	
$\lambda : \{3, 4\}, \{1, 2\}$		$\lambda : \{4\}, \{3, 1, 2\}$	
$\lambda : \{3\}, \{4\}, \{1, 2\}$		$\lambda : \{3, 4, 1\}, \{2\}$	
$\lambda : \{4, 1\}, \{3, 2\}$		$\lambda : \{3\}, \{4, 1\}, \{2\}$	
$\lambda : \{3, 1\}, \{4, 2\}$		$\lambda : \{1\}, \{3, 4, 2\}$	
$\lambda : \{3\}, \{1\}, \{4, 2\}$		$\lambda : \{3, 4\}, \{1\}, \{2\}$	
$\lambda : \{4\}, \{3, 1\}, \{2\}$		$\lambda : \{4\}, \{1\}, \{3, 2\}$	
$\lambda : \{3\}, \{4\}, \{1\}, \{2\}$			

5.3. MAXIMAL NILPOTENT SEMIGROUPS

Every partition $\rho = \{M_1, M_2, \ldots, M_m\}$ of $[n]$ yields a partition $\rho^0 = \{M_0 = \{0\}, M_1, M_2, \ldots, M_m\}$ of $\langle n\rangle$. Associated with the partition ρ^0 is a partial order on $\langle n\rangle$ defined by

$$x <_\rho y \text{ if and only if there exists } i, j \in \langle m\rangle \text{ such that } x \in M_i,\ y \in M_j, \text{ and } i < j$$

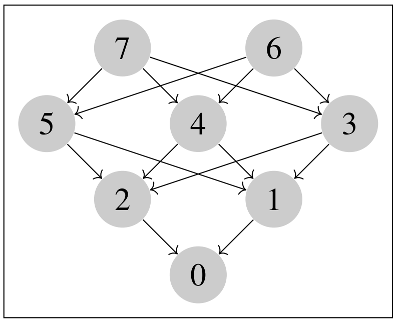


Figure 5.1. Hasse diagram of $\rho^0 = \{\{0\}, \{1,2\}, \{3,4,5\}, \{6,7\}\}$

Hasse diagram for $\rho^0 = \{\{0\}, \{1,2\}, \{3,4,5\}, \{6,7\}\}$ is given in Figure 5.1

Given a partition $\rho^0 = \{\{0\}, M_1, M_2, \ldots, M_m\}$ we define the directed graph H_ρ as follows. The vertex set is $\langle n \rangle$. (y, x) is an edge if and only if $x <_\rho y$. Let $x, y \in \langle n \rangle$ be distinct elements. Then either $x, y \in M_i$ for some $i \in \langle m \rangle$ or $x \in M_i$, $y \in M_j$ for $i \neq j$. In the first case we write $x \prec y$ if $x < y$ in the usual order of natural numbers, and in the latter case we write $x \prec y$ if $x <_\rho y$. Clearly $\prec$ is a linear order in $\langle n \rangle$. This allows us to write the adjacency matrix A_ρ by naming the rows and columns with the letters in $\langle n \rangle$, in reverse order with respect to $\prec$. This will set

$$\left[A_\rho\right]_{y,x} = \begin{cases} 1 & x \in M_i, y \in M_j \text{for } i < j \\ 0 & \text{otherwise} \end{cases}$$

We also have the diagonal matrix D_{out}

$$[D_{out}]_{x,x} = \begin{cases} n + 1 - \sum_{j=i}^{m} \left|M_j\right| & x \in M_i \text{ for some } i \in [n] \\ 0 & \text{otherwise} \end{cases}$$

where again the rows and columns are named with the letters in $\langle n \rangle$, in reverse order with respect to $\prec$.

The Laplacian matrix L_{out} is thus given by

$$[L_{out}]_{y,x} = \begin{cases} n + 1 - \sum_{j=i}^{m} \left|M_j\right| & y = x \in M_i \text{ for some } i \in [n] \\ -1 & x \in M_i, y \in M_j \text{for } i < j \\ 0 & \text{otherwise} \end{cases} \tag{5.1}$$

this is a lower square diagonal matrix.

Corresponding to such a partition is the maximal nilpotent subsemigroup defined by

$$N_\rho = \left\{\alpha : \mathcal{PT}_n : x\alpha <_\rho x \text{ for all } x \in [n] \text{ and } 0 = 0\alpha\right\}$$

Recall that for a transformation $\alpha \in \mathcal{PT}_n$, the directed graph Γ_α is defined by

$$V(\Gamma_\alpha) = \langle n\rangle \text{ and } (r,s) \in \langle n\rangle \times \langle n\rangle \text{ is an edge in if and only if } r\alpha = s$$

Naturally $0\alpha = 0$ for every $\alpha \in \mathcal{PT}_n$ which yields a loop at 0. We write T_α to be the graph obtained from Γ_α by removing this loop. We first note that H_ρ has the properties stated above for directed graphs, that is it has no multiple edges and no loops.

Then we have the following lemma.

Lemma 11. *Let $\rho^0 = \{\{0\}, M_1, M_2, \ldots, M_m\}$. $\alpha \in N_\rho$ if and only if T_α is an incoming directed spanning tree rooted at 0.*

Proof. Let us first pick $\alpha \in N_\rho$. For all $x \in M_1$ we have $x\alpha <_\rho x$. This shows that $0 = x\alpha$ for all $x \in M_1$. Let (t, s_1) and (t, s_2) be edges in T_α. Since α is a function, then $t\alpha = s_1$ and $t\alpha = s_2$ imply that $s_1 = s_2$. Thus out degree of vertex t is at most 1. But if $t \in [n]$ and $t\alpha = s$, then (t,s) is an edge in T_α. This shows that out degree of vertex t is 1. It is clear that the root vertex 0 has out-degree 0. If T_ρ has a cycle $\{i_1, i_2, \ldots, i_n\}$ such that $\left(i_j, i_{j+1}\right)$ is an edge where $i_{n+1} = i_1$, then $i_{j+1} = i_j\alpha$ and $i_1 = i_{n+1}$. But since $\alpha \in N_\rho$ this would imply $i_1 = i_{n+1} <_\rho i_{n-1} <_\rho \cdots <_\rho i_1$, a contradiction. Thus T_α is an incoming directed spanning tree rooted at 0.

Conversely suppose that T_α is an incoming directed spanning tree rooted at 0. We must show that $\alpha \in N_\rho$. Clearly we must simply show that if $x \in [n]$, then $x\alpha <_\rho x$. There exists a unique M_j such that $x \in M_j$. If $j = 1$, then there is only one edge in H_ρ with tail x and head 0. Since x has out-degree 1 in T_α, this edge must be an edge in T_α. In other words $x\alpha = 0 <_\rho x$. So we may assume that $1 < j$. x has out-degree 1 in T_α, so $(x, x\alpha)$ is an edge in T_α. Thus $(x, x\alpha)$ is an edge of H_ρ. It follows that $x\alpha <_\rho x$ for all $x \in [n]$. In other words $\alpha \in N_\rho$. □

The following theorem can be found in [4]. We restate it with our interpretation using directed spanning trees rooted at 0.

Theorem 36. *[4]Let* $\rho^0 = \{M_0 = \{0\}, M_1, M_2, \ldots, M_m\}$*. Then the number of elements in* N_ρ *is given by*

$$\begin{aligned} \left|N_\rho\right| &= \prod_{i=1}^{m}\left[(n+1) - \sum_{j=m-i+1}^{m}\left|M_j\right|\right]^{|M_{m-i+1}|} \\ &= \prod_{i=1}^{m}\left(\sum_{j=0}^{m-i}\left|M_j\right|\right)^{|M_{m-i+1}|} \end{aligned}$$

Proof. L_{out} is a lower triangular matrix given in 5.1 whose diagonal is the same as the diagonal of D_{out}. This makes the theorem clear. □

Example 37. $[n] = \{1,2,3,4,5,6,7\}$ *,* $m = 3$*,* $\rho = \{\{1,2\},\{3,4\},\{5,6,7\}\}$*. The corresponding graph is pictured in Figure 5.2.*

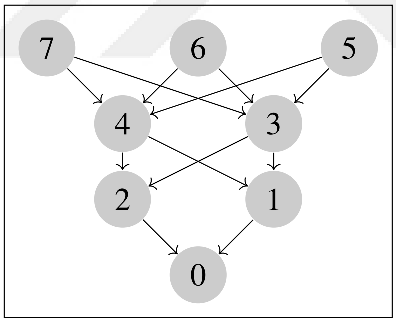


Figure 5.2. Hasse diagram for $\rho = \{\{1,2\},\{3,4\},\{5,6,7\}\}$

$$L_{out} = \begin{array}{c} \\ 7 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{array} \begin{array}{c} \begin{array}{cccccccc} 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{array} \\ \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 3 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 3 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \end{pmatrix} \end{array}$$

$$L_{out}^0 = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 3 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 3 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 1 \end{pmatrix}, \quad \det L_{out}^0 = 1125$$

Example 38. $[n] = \{1, 2, 3\}$, $m = 2, \rho = \{\{1\}, \{2, 3\}\}$. *The corresponding graph is pictured in 5.3.*

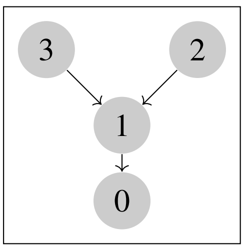
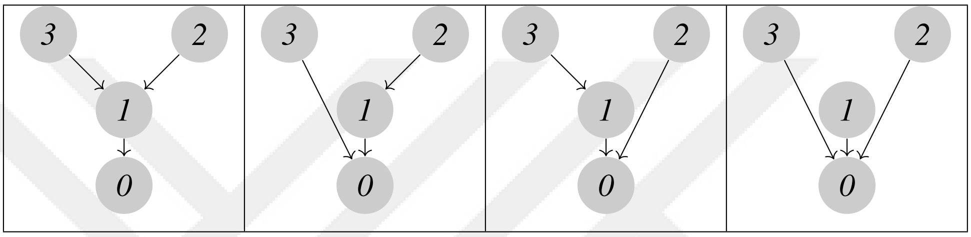


Figure 5.3. Hasse diagram for $\rho = \{\{1\}, \{2, 3\}\}$

$$L_{out} = \begin{array}{c} \\ 3 \\ 2 \\ 1 \\ 0 \end{array}\begin{array}{c} \begin{array}{cccc} 3 & 2 & 1 & 0 \end{array} \\ \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & 0 \end{pmatrix} \end{array}, \quad L_{out}^0 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{pmatrix}, \quad \det L_{out}^0 = 4$$

Table 5.4. Incoming directed spanning trees rooted at 0 for Hasse diagram 5.3



Example 39. $[n] = \{1, 2, 3\}$, $m = 2, \rho = \{\{1, 2\}, \{3\}\}$. *The corresponding graph is given in 5.4.*

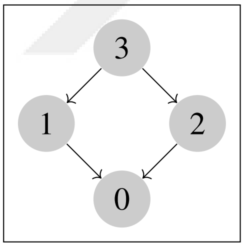
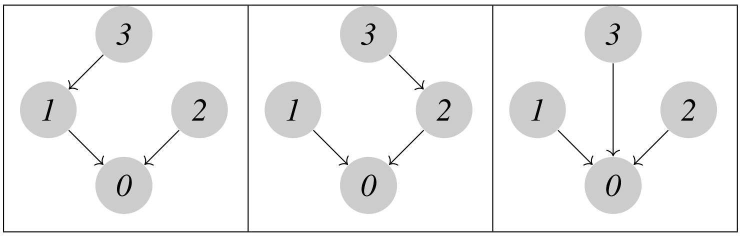


Figure 5.4. Hasse diagram for $\rho = \{\{1, 2\}, \{3\}\}$

$$L_{out} = \begin{array}{c} \\ 3 \\ 2 \\ 1 \\ 0 \end{array}\begin{array}{c} \begin{array}{cccc} 3 & 2 & 1 & 0 \end{array} \\ \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & 0 \end{pmatrix} \end{array}, \quad L_{out}^0 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}, \quad \det L_{out}^0 = 3$$

Table 5.5. Incoming directed spanning trees rooted at 0 for Hasse diagram 5.4



Example 40. $[n] = \{1,2,3,4\}$, *$m = 2$, and $\rho = \{\{1,2\},\{3,4\}\}$.The corresponding graph is given in 5.5.*

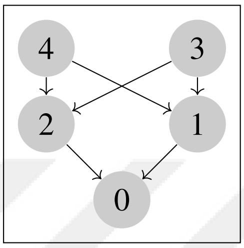
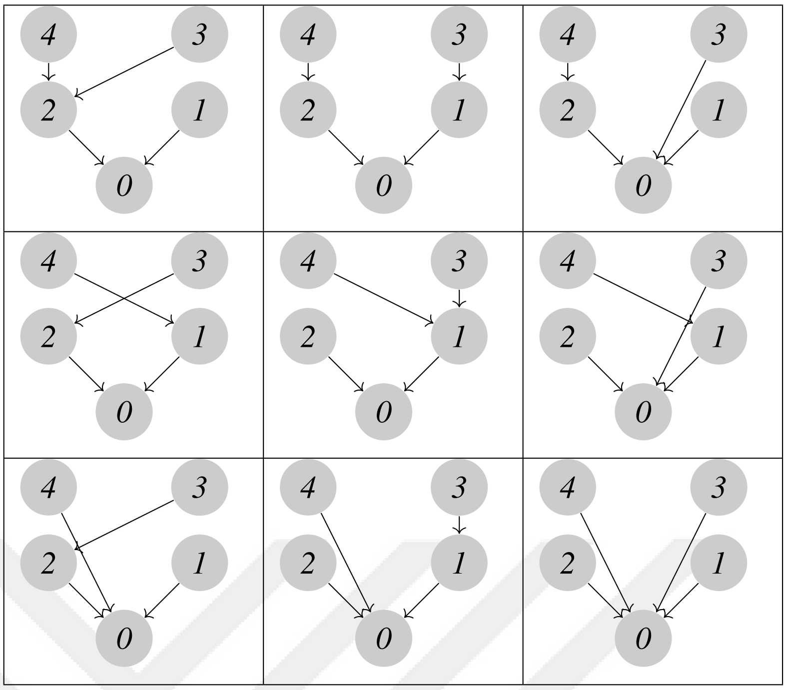


Figure 5.5. Hasse diagram for $\rho = \{\{1,2\},\{3,4\}\}$

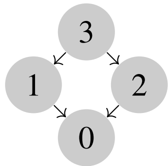
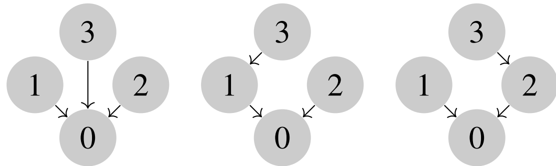
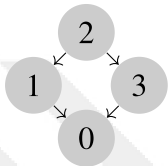
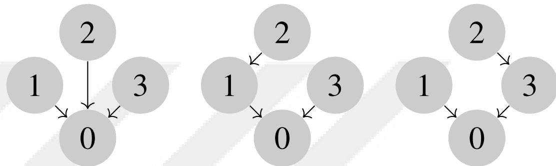
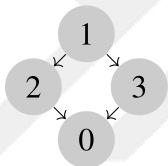
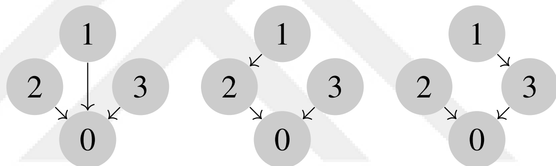
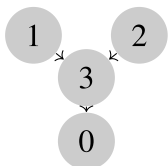
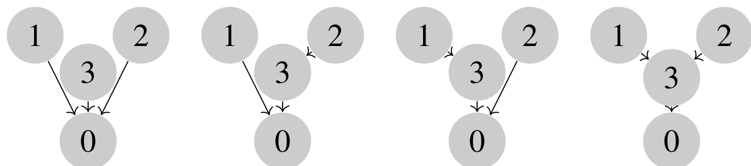
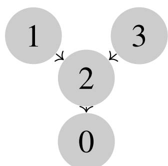
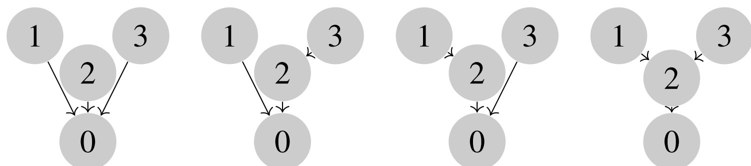
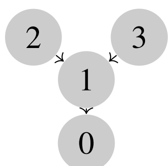
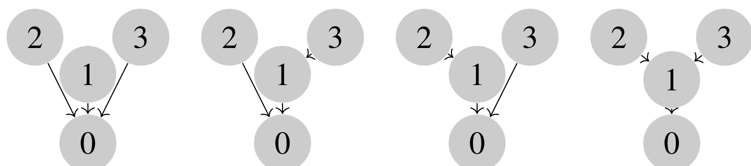
$$L_{out} = \begin{array}{c} \\ 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} 4 & 3 & 2 & 1 & 0 \end{array} \\ \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & 0 \end{pmatrix} \end{array}, \quad L^0_{out} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{pmatrix}, \quad \det L^0_{out} = 9$$

Table 5.6. Incoming directed spanning trees rooted at 0 for Hasse diagram in Figure 5.5



In the next table we list all the incoming directed spanning trees rooted at 0 corresponding to Hasse diagrams of the form $\{\{a\},\{b,c\}\}$ or $\{\{a,b\},\{c\}\}$

Table 5.7. IDSTR0 for Hasse diagrams of the form $(1,2)$ or $(2,1)$

Hasse diagram for $\{\{1,2\},\{3\}\}$ 	
Hasse diagram for $\{\{1,3\},\{2\}\}$ 	
Hasse diagram for $\{\{2,3\},\{1\}\}$ 	
Hasse diagram for $\{\{3\},\{1,2\}\}$ 	
Hasse diagram for $\{\{2\},\{1,3\}\}$ 	
Hasse diagram for $\{\{1\},\{2,3\}\}$ 	

5.4. NILPOTENT SUBSEMIGROUPS CORRESPONDING TO A PARTIAL ORDERS

We denote the set of all antireflexive partial orders on the set $[n]$ by $\mathfrak{O}_n$. We can extend any partial order $\lambda \in \mathfrak{O}_n$ to a partial order on $\langle n\rangle$ by simply defining $0 <_\lambda x$ for every $x \in [n]$. We keep denoting this partial order by λ. Given such a partial order λ and $x \in \langle n\rangle$ we define the **degree** of x, denoted by $\deg_\lambda(x)$ as follows: We let $\deg_\lambda(0) = 0$. If $x \in [n]$ then we know that $0 <_\lambda x$. Since $\langle n\rangle$ is finite there is sequence $0 = x_k <_\lambda x_{k-1} <_\lambda \cdots <_\lambda x_0 = x$ of maximum length k. We define $\deg_\lambda(x) = k$.

This partitions $\langle n\rangle$ into sets by means of the equivalence relation $x \sim y$ if and ony if $\deg_\lambda(x) = \deg_\lambda(y)$. Note that if $\deg_\lambda(x) = k \geq 1$ and $0 = x_k <_\lambda x_{k-1} <_\lambda \cdots <_\lambda x_0 = x$ is a sequence of maximum length k, then $\deg_\lambda(x_i) = i$ for every $0 \leq i \leq k$. So , once there is an element of degree k, then there are element of each degree i for every $i \leq k$. This shows that if $m_\lambda = \max_{x\in\langle n\rangle} \deg_\lambda(x)$, there are exactly m_λ equivalence classes. Let $E(\lambda) = \left\{E_0 = \{0\}, E_1, E_2, \ldots, E_{m_\lambda}\right\}$.

If $\lambda \in \mathfrak{O}_n$ associated with the partial order λ we define the directed graph H_λ as follows: The vertex set is $\langle n\rangle$. (y, x) is an edge if and only if $x <_\lambda y$. Let $x, y \in \langle n\rangle$ be distinct elements. Then either $x, y \in E_i$ for the same i, or $x \in E_i$, $y \in E_j$ for $i \neq j$. In the first case we write $x \prec y$ if $x < y$ in the usual order of natural numbers, and in the latter case we write $x \prec y$ if $x <_\lambda y$. Clearly $\prec$ is a linear order in $\langle n\rangle$. This allows us to write the adjacency matrix A_λ by naming the rows and columns with the letters in $\langle n\rangle$, in reverse order with respect to $\prec$. This will set

$$\left[A_\rho\right]_{y,x} = \begin{cases} 1 & x \in E_i, y \in E_j \text{ for } i < j \\ 0 & \text{otherwise} \end{cases}$$

We also have the diagonal matrix D_{out}

$$[D_{out}]_{x,x} = \begin{cases} \deg_\lambda x & x \in E_i \text{ for some } i \in \langle n\rangle \\ 0 & \text{otherwise} \end{cases}$$

5.5. PARTITIONS CONTAINING *M* PARTS

We now consider the partitions partitioning $[n]$ into m parts taking into account the order. The number of all partitions is given by the following theorem.

Theorem 41. *[4] The number of solutions of the equation*

$$x_1 + \cdots + x_m = n \tag{5.2}$$

where x_i's are positive integers is $\binom{n-1}{m-1}$.

For each solution $(x_1, \cdots, x_m)$ of the equation 5.2, let us denote the set of all m-tuples $(A_1, \ldots, A_m)$ consisting of subsets $A_1, \ldots, A_m$ of $[n]$ with $|A_i| = x_i$ for $i = 1, \ldots, m$ by m parts by $C_n(x_1, \cdots, x_m)$. We let

$$C_n(m) = \bigcup_{x_1+\cdots+x_m=n} C_n(x_1, \cdots, x_m)$$

Example 42. *If $n = 3$ and $m = 2$, then $\binom{n-1}{m-1} = 2$. Indeed, the only solutions of $x_1 + x_2 = 3$ are $x_1 = 1, x_2 = 2$ or $x_1 = 2$ and $x_2 = 1$. Hence*

$$C_3(1,2) = \{(\{1\}, \{2,3\}), (\{2\}, \{1,3\}), (\{3\}, \{1,2\})\}$$

and

$$C_3(2,1) = \{(\{2,3\}, \{1\}), (\{1,3\}, \{2\}), (\{1,2\}, \{3\})\}$$

Theorem 43. *[4] Given a solution $(x_1, \cdots, x_m)$ of 5.2, $C_n(x_1, \cdots, x_m)$ contains $\binom{n}{x_1, \ldots, x_m}$ elements.*

Proof. This is clear since

$$\binom{n}{x_1}\binom{n-x_1}{x_2}\cdots\binom{n-\sum_{i=1}^{m-1}}{x_m} = \binom{n}{x_1, \ldots, x_m}$$

□

Theorem 44. *If $\lambda = (A_1, \ldots, A_m)$ and $\mu = (B_1, \ldots, B_m)$ are elements of $C_n(x_1, \cdots, x_m)$, then H_λ and H_μ are isomorphic.*

Proof. $|A_i| = |B_i|$ for $i = 1, \ldots, m$. So there is a bijection $\sigma : [n] \to [n]$ such that $A_i\sigma = B_i$ for $i = 1, \ldots, m$. If we extend σ to $\langle n \rangle$ by defining $0\sigma = \sigma$, then σ yields an isomorphism between H_λ and H_μ. To see this let $x, y \in \langle n \rangle$ and (y, x) be an edge in H_λ, i.e. $x <_\lambda y$. This means that

$$x <_\lambda y \implies \text{there exists } i, j \in \langle m \rangle \text{ such that } x \in A_i,\, y \in A_j, \text{ and } i < j$$

Hence

$$i, j \in \langle m \rangle \text{ are such that } x\sigma \in A_i\sigma = B_i,\, y\sigma \in A_j\sigma = B_j, \text{ and } i < j \implies x\sigma <_\lambda y\sigma$$

This shows that $(y\sigma, x\sigma)$ is an edge in H_μ. We may similarly show that if (y, x) be an edge in H_μ, then $\left(y\sigma^{-1}, x\sigma^{-1}\right)$ is an edge in H_λ. Hence σ is an isomorphism between H_λ and H_μ. □

Corollary 3. *There are $\binom{n}{x_1,\ldots,x_m}$ distinct nilpotent subsemigroups of order*

$$s_n(x_1, \cdots, x_m) = \prod_{i=1}^{m} \left[(n+1) - \sum_{j=0}^{i-1} x_{m-j} \right]^{x_{m-i+1}}$$

Proof. By Theorem 44, we know that if $\lambda, \mu \in C_n(x_1, \cdots, x_m)$, then H_λ and H_μ are isomorphic. So by Theorem 43 there are $\binom{n}{x_1,\ldots,x_m}$ distinct elements of $C_n(x_1, \cdots, x_m)$ such that any two are isomorphic. Corresponding to each $\lambda \in C_n(x_1, \cdots, x_m)$ there are $s_n(x_1, \cdots, x_m)$ elements in nilpotent subsemigroup N_λ. This proves the claim. □

Theorem 45. *There are exactly*

$$\sum_{i=0}^{k-1} \binom{n}{i} . S(n, i+1) . i!$$

nilpotent elements of nilpotency degree $\leq k$ in $\mathcal{PT}_n$.

Theorem 46. *[4] The number of maximal elements in both $Nil_k(\mathcal{PT}_n)$ and $Nil_k(IS_n)$*

equals

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n = k!S(n,k)$$

In a directed graph $G(V,E)$ a vertex b is said to be **accessible** from another vertex a if G contains a walk from a to b. Additionally, we'll say that all vertices are accessible from themselves.

A vertex $v \in V$ in a directed graph $G(V,E)$ is a **root** if every other vertex is accessible from v.

A directed graph $T(V,E)$ is a **directed tree** or **arborescence** if

i) T contains a root.

ii) The underlying graph of T, i.e. the graph obtained by ignoring the direction of the edges is a tree.

Definition 14. *A subgraph $T(V,E)$ of a digraph $G(V,E)$ is a **spanning arborescence** if T is an arborescence that contains all the vertices of G.*

5.6. NILPOTENT SUBSEMIGROUPS OF $\mathcal{PT}_N$ AND $\mathcal{IS}_N$

Let S be a semigroup and $A \subset S$. For an integer $k \geq 1$ we define

$$\begin{aligned} A^1 &= A \\ A^k &= A^{k-1}A \text{ for } k \geq 2 \end{aligned}$$

Definition 15. *A semigroup S with zero element 0_S is **nilpotent** of **nilpotency degree** $k \geq 1$ if $S^k = \{0_S\}$ and $S^r \neq \{0_S\}$ for any $1 \leq r < k$. An element $a \in S$ is **nilpotent** of **nilpotency degree** $k \geq 1$ if $a^k = 0_S$ and $a^r \neq 0_S$ for any $1 \leq r < k$.*

We adopt some notation introduced in [4].

$\mathfrak{O}_n$: The set of antisymmetric partial order on $[n]$. Given any $\lambda \in \mathfrak{O}$ we extend it to a partial order in $\langle n \rangle$ by leting $0 <_\lambda x$ for every $x \in [n]$. We will do so in the sequel.

$\mathfrak{O}_n^{(k)}$: If $\lambda \in \mathfrak{O}_n$ and $x \in [n]$, then there is a chain $0 \ <_\lambda x_1 <_\lambda \cdots <_\lambda x_s = x$ of length s where x_i's belong to $[n]$. $\mathfrak{O}_n^{(k)}$ is the set of all elements of $\mathfrak{O}_n$ whose longest chain has length k.

0_n : The element of $\mathcal{PT}_n$ which maps each $x \in \langle n \rangle$ to 0.

$\mathrm{Nil}\,(S)$: Nilpotent subsemigroups of a semigroup S.

$\mathrm{Nil}_k\,(S)$: The elements of $\mathrm{Nil}\,(S)$ of nilpotency degree k. Notice that $\mathrm{Nil}_1\,(S) = \{0_S\}$.

$\mathrm{nd}\,(X)$: The nilpotency degree of X where X is either an element or a semigroup.

Theorem 47. *[4] A finite semigroup S with zero element 0_S is nilpotent if and only if each element in S is nilpotent.*

Proof. Let us first assume that each element in S is nilpotent. Let $|S| = n$. Pick any non-zero, not necessarily distinct , elements $b_1, \ldots, b_n \in S \backslash \{0_S\}$. We claim that $b = b_1 \ldots b_n = 0_S$. Suppose that $b_1 \ldots b_n \neq 0_S$. Let $c_i = b_1 \cdots b_i$, $i = 1, \ldots, n$. Then $c_i \neq 0$ for any $i = 1, \ldots, n$ for otherwise $b = 0_S$. But there are n non zero elements $c_1, \ldots, c_n \in S \backslash \{0_S\}$ and $|S \backslash \{0_S\}| = n - 1$. So $c_i = c_j$ for some $1 \leq i < j \leq n$. Let $x = c_i$ and $y = b_{i+1} \ldots b_j$. Then $xy = c_j$.. This gives two non-zero elements $x, y \in S$ such that $x = xy$. But y is nilpotent, and say $y^k = 0_S$. So

$$x = xy \Longrightarrow x = xy = xy^2 \Longrightarrow \ldots \Longrightarrow x = xy^k = 0_S$$

a contradiction. So $S^n = \{0_S\}$, i.e. S is nilpotent of nilpotency degree at most $n = |S|$.

Converse is clear. □

Any partial order ρ on $[n]$ can be extended to a partial order on $\langle n \rangle$ by simply defining 0 as the minimum element. Given an antisymmetric partial order ρ on $[n]$, we define the subset N_ρ of $\mathcal{PT}_n$

$$N_\rho = \left\{\alpha \in \mathcal{PT}_n : x\alpha <_\rho x \text{ for every } x \in [n]\right\} \text{ and } N'_\rho = N_\rho \cap \mathcal{IS}_n$$

Lemma 12. *N_ρ (resp. N'_ρ) is a nilpotent subsemigroup of $\mathcal{PT}_n$ (resp. $\mathcal{IS}_n$). Moreover if $\rho \in \mathfrak{D}_n^{(k)}$, then N_ρ (resp. N'_ρ) is in* $\mathrm{Nil}_k(\mathcal{PT}_n)$ *(resp.* $\mathrm{Nil}_k(\mathcal{IS}_n)$ *).*

Proof. Let $\alpha, \beta \in N_\rho$. Given $x \in [n]$

$$x(\alpha\beta) = (x\alpha)\beta <_\rho x\alpha <_\rho x$$

Thus $\alpha\beta \in N_\rho$. It follows that N_ρ is a subsemigroup of $\mathcal{PT}_n$. Consequently N'_ρ is a subsemigroup of $\mathcal{IS}_n$. Let us pick $\alpha \in N_\rho$. We adopt the notation $x\alpha^0 = x$.

Now, suppose that $\rho \in \mathfrak{D}_n^{(k)}$ and $\beta_1, \ldots, \beta_k \in N_\rho$. We must show $\beta_1 \cdots \beta_k = 0_n$. Suppose this is not true. Then there exists $x \in [n]$ such that $x(\beta_1 \cdots \beta_k) \neq 0$. Then $x\beta_1 \cdots \beta_i \in [n]$ for any $1 \leq i \leq k$. So,

$$0 <_\rho x(\beta_1 \cdots \beta_k) <_\rho \cdots <_\rho x\beta_1\beta_2 <_\rho x\beta_1 <_\rho x$$

is a chain of ρ of length at least $k+1$, a contradiction. This shows that $\mathrm{nd}\left(N_\rho\right) \leq k$. But there is a chain of ρ

$$0 <_\rho x_1 <_\rho \cdots <_\rho x_k$$

Let us set $x_0 = 0$. Define $\alpha \in \mathcal{IS}_n$ by

$$y\alpha = \begin{cases} x_{j-1} & \text{if } 1 \leq j \leq k \text{ and } y = x_j \\ 0 & \text{otherwise} \end{cases}$$

One can easily check that α is nilpotent with $\mathrm{nd}(\alpha) = k$. Moreover, $\alpha \in N'_\rho \subset N_\rho$. Thus N_ρ (resp. N'_ρ) is a nilpotent subsemigroup in $\mathrm{Nil}_k(\mathcal{PT}_n)$ (resp. $\mathrm{Nil}_k(\mathcal{IS}_n)$). ◻

Let T be a nilpotent subsemigroup of $\mathcal{PT}_n$ (resp. $\mathcal{IS}_n$). Define the relation σ_T as follows: Let $x \in \langle n \rangle$, $y \in [n]$.

$$x <_{\sigma_T} y \text{ if and only if there exists } \alpha \in T \text{ such that } x = y\alpha$$

Lemma 13. *$\sigma_T \in \mathfrak{D}_n$. Moreover if T is in* $\mathrm{Nil}_k(\mathcal{PT}_n)$ *(resp.* $\mathrm{Nil}_k(\mathcal{IS}_n)$ *), then $\sigma_T \in \mathfrak{D}_n^{(k)}$.*

Proof. Suppose that $x \in [n]$ and $x <_{\sigma_T} x$. Then there exists $\alpha \in S$ such that $x = x\alpha$. But then $0 \neq x = x\alpha^k$ for any $k \geq 1$, contradicting to the fact that α is nilpotent. So σ_T is an antisymmetric.

Let $x <_{\sigma_T} y \in [n]$ and $y <_{\sigma_T} z \in [n]$. Then there are $\alpha, \beta \in S$ such that $x = y\alpha$ and $y = z\beta$. Thus $x = z(\beta\alpha)$ and $\beta\alpha \in S$. This shows that $x <_{\sigma_T} z$. So σ_T is a transitive relation and hence a partial order on $[n]$.

Now, suppose that T is in $\mathrm{Nil}_k(\mathcal{PT}_n)$. If $k = 1$, then $T = \{0_n\}$. So, if $x \in [n]$, then $0 = x0_n <_{\sigma_T} x$ is the longest chain for σ_T. So we may assume that $k \geq 2$. There are elements $\beta_1, \beta_2, \ldots, \beta_k \in T$ such that

$$\beta_1\beta_2 \cdots \beta_k = 0_n \text{ but } \beta_1 \cdots \beta_s \neq 0_n \text{ for any } 1 \leq s < k$$

since T is in $\mathrm{Nil}_k(\mathcal{PT}_n)$. So there is an element $x \in [n]$ such that $x\beta_1 \cdots \beta_{k-1} \neq 0$. This implies that

$$x\beta_1\beta_2 \cdots \beta_k = 0 <_{\sigma_T} x(\beta_1 \cdots \beta_{k-1}) <_{\sigma_T} \cdots <_{\sigma_T} x\beta_1\beta_2 <_{\sigma_T} x\beta_1 < x$$

is a chain for σ_T of length k.

Suppose that

$$0 \ <_{\sigma_T} x_1 <_{\sigma_T} \cdots <_{\sigma_T} x_s$$

is a chain for σ_T of length s. Then there are elements $\alpha_1, \alpha_2, \ldots, \alpha_s \in T$ such that

$$x_{s-1} = x_s\alpha_s, \ldots, x_1 = x_2\alpha_2, 0 = x_1\alpha_1$$

So this chain is of the form

$$0 = x\alpha_1\alpha_2 \cdots \alpha_s <_{\sigma_T} \cdots <_{\sigma_T} x\alpha_s\alpha_{s-1} <_{\sigma_T} x\alpha_s < x$$

But since $\alpha_1\alpha_2 \cdots \alpha_k = 0$, then we must have $s \leq k$. This proves that $\sigma_T \in \mathfrak{O}_n^{(k)}$. □

Theorem 48. *a) Let T, S be nilpotent subsemigroups of $\mathcal{PT}_n$ (resp. $\mathcal{IS}_n$). If $T \subset S$,* *then* $\sigma_T \subset \sigma_S$.

b) *Let* λ, μ *be two antisymmetric partial orders on* $[n]$. *If* $\lambda \subset \mu$, *then* $N_\lambda \subset N_\mu$ *(resp.* $N'_\lambda \subset N'_\mu$ *).*

c) *If* λ *is an antisymmetric partial order on* $[n]$, $\sigma_{N_\lambda} = \lambda$.

Proof. (a) and (b) are clear.

(c) We let $T = N_\lambda$. If $x <_{\sigma_T} y$, then there exists $\alpha \in T = N_\lambda$ such that $x = y\alpha$. But $\alpha \in N_\lambda = T$ and $x = y\alpha$ implies that $x <_\lambda y$. So $\sigma_T \subset \lambda$.

Conversely let $x <_\lambda y$. Consider $\alpha \in \mathcal{IS}_n$ such that $y\alpha = x$ and $z\alpha = 0$ for $z \neq y$ and $z \in [n]$. Then α is nilpotent. We claim that $\alpha \in N_\lambda = T$. For let $z \in [n]$. If $z \neq y$, then $z\alpha = 0$. If $z = y$, then $y\alpha = x$. In each case $z\alpha <_\lambda z$. Thus $\alpha \in N_\lambda = T$. Since $y\alpha = x$, then $x <_{\sigma_T} y$. So $\lambda \subset \sigma_T$. This shows that $\lambda = \sigma_T$. □

Example 49. *One expects that if* T *is in* $\mathrm{Nil}(\mathcal{PT}_n)$, *then* $N_{\sigma_T} = T$ *as a counterpart to Theorem 48(c). It is easy to show that* $T \subset N_{\sigma_T}$. *But* T *may be chosen so that* $N_{\sigma_T} \nsubseteq T$. *We give an example of such a* $T \in \mathrm{Nil}(\mathcal{PT}_n)$.

Let

$$\alpha_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & 0 \end{pmatrix}, \theta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \end{pmatrix}$$

We have the mutiplication table

	0_3	α_1	α_2	α_3
0_3	0_3	0_3	0_3	0_3
α_1	0_3	0_3	α_3	0_3
α_2	0_3	0_3	0_3	0_3
α_3	0_3	0_3	0_3	0_3

Thus $T = \{0_3, \alpha_1, \alpha_2, \alpha_3\} \in \mathrm{Nil}(\mathcal{PT}_3)$. *We claim that* $\theta \in N_{\sigma_T}$ *which is not in* T.

$$1\theta = 2 = 1\alpha_1 <_{\sigma_T} 1,\ 2\theta = 3 = 2\alpha_2 <_{\sigma_T} 2,\ 3\theta = 0 = 3\alpha_1 <_{\sigma_T} 3$$

So $x\theta <_{\sigma_T} x$ *for every* $x \in [n]$*, consequently* $\theta \in N_{\sigma_T}$.

6. MAXIMAL NILPOTENT SUBSEMIGROUPS IN PARTIAL TRANSFORMATION SEMIGROUPS

6.1. MAXIMAL NILPOTENT SUBSEMIGROUPS

Naturally $\mathrm{Nil}_k(\mathcal{PT}_n)$ and $\mathrm{Nil}(\mathcal{PT}_n)$ are ordered by inclusion. Similarly $\mathrm{Nil}_k(\mathcal{IS}_n)$ and $\mathrm{Nil}(\mathcal{IS}_n)$ are ordered by inclusion.

Since these sets are finite, then there are maximal elements in them. $\mathfrak{O}_n$ and $\mathfrak{O}_n^{(k)}$ are also are ordered by inclusion.

Let us first list the maximal elements in $\mathrm{Nil}_k(\mathcal{PT}_n)$.

Lemma 14. *a) If $T \in \mathrm{Nil}_k(\mathcal{PT}_n)$ (resp. $T \in \mathrm{Nil}_k(\mathcal{IS}_n)$) is maximal, then σ_T is maximal in $\mathfrak{O}_n^{(k)}$ and $N_{\sigma_T} = T$ (resp. $N'_{\sigma_T} = T$).*

b) If $\rho \in \mathfrak{O}_n^{(k)}$ is maximal, then N_ρ (resp.N'_ρ) is maximal in $\mathrm{Nil}_k(\mathcal{PT}_n)$ (resp. $\mathrm{Nil}_k(\mathcal{IS}_n)$).

Proof. (a) We know that $\sigma_T \in \mathfrak{O}_n^{(k)}$. Suppose that $\lambda \in \mathfrak{O}_n^{(k)}$ and $\sigma_T \subset \lambda$. Then

$$\sigma_T \subset \lambda \Longrightarrow T \subset N_{\sigma_T} \subset N_\lambda \Longrightarrow T = N_\lambda \Longrightarrow \sigma_T = \sigma_{N_\lambda} = \lambda$$

So σ_T is maximal and $N_{\sigma_T} = N_\lambda = T$. In case when $T \in \mathrm{Nil}_k(\mathcal{IS}_n)$, then

$$\sigma_T \subset \lambda \Longrightarrow T \subset N'_{\sigma_T} \subset N'_\lambda \Longrightarrow T = N'_\lambda \Longrightarrow \sigma_T = \sigma_{N'_\lambda} = \lambda$$

So σ_T is maximal and $N'_{\sigma_T} = N'_\lambda = T$.

(b) We know $N_\rho \in \mathrm{Nil}_k(\mathcal{PT}_n)$. Let $S \in \mathrm{Nil}_k(\mathcal{PT}_n)$ and $N_\rho \subset S$. Then

$$N_\rho \subset S \Longrightarrow \rho = \sigma_{N_\rho} \subset \sigma_S \Longrightarrow \rho = \sigma_S \Longrightarrow S \subset N_{\sigma_S} = N_\rho \Longrightarrow S = N_\rho$$

So N_ρ is maximal . In case when $T \in \mathrm{Nil}_k\,(\mathcal{IS}_n)$, then

$$N'_\rho \subset S \Longrightarrow \rho = \sigma_{N_\rho} \subset \sigma_S \Longrightarrow \rho = \sigma_S \Longrightarrow S \subset N'_{\sigma_S} = N'_\rho \Longrightarrow S = N'_\rho$$

So N'_ρ is maximal . □

Let $\mathfrak{m} = (M_1, \ldots, M_k)$ be an ordered partition of $[n]$. Then we define the partial order, which we denote by $\tau_{\mathfrak{m}}$, as follows.

If $x, y \in \langle n\rangle$, then $x <_{\tau_{\mathfrak{m}}} y$ if and only if either $x = 0$ and $y \in [n]$ or $x, y \in [n]$ and $\exists i, j$ such that $1 \leq i < j \leq k$ such that $x \in M_i$ and $y \in M_j$.

We will denote the set of all ordered partitions of $[n]$ by $\mathfrak{M}_n^{(k)}$.

Definition 16. *Let $\lambda \in \mathfrak{O}_n^{(k)}$ and $x \in [n]$. A chain $0 \;<_\lambda x_1 <_\lambda \cdots <_\lambda x_s = x$ of length s where x_i's belong to $[n]$ is called a chain in λ with head x.*

Definition 17. *Let $\lambda \in \mathfrak{O}_n^{(k)}$. For each $x \in [n]$ let $\ell_\lambda(x)$ be length of longest chain in λ with head x.*

It is clear that if $\lambda \in \mathfrak{O}_n^{(k)}$, then $1 \leq \ell_\lambda(x) \leq k$ for each $x \in [n]$.

Lemma 15. *$\lambda, \mu \in \mathfrak{O}_n^{(k)}$ and $\lambda \subset \mu$. Then $\ell_\lambda(x) \leq \ell_\mu(x)$ for each $x \in [n]$.*

Proof. Let $x \in [n]$ and $r = \ell_\lambda(x)$. There is a chain $0 \;<_\lambda x_1 <_\lambda \cdots <_\lambda x_r = x$ of length r where x_i's belong to $[n]$. Thus $0 \;<_\mu x_1 <_\mu \cdots <_\mu x_r = x$ since $\lambda \subset \mu$. This shows that $\ell_\lambda(x) = r \leq \ell_\mu(x)$. □

Lemma 16. *Let $\lambda \in \mathfrak{O}_n^{(k)}$. Define*

$$M_i = \{x \in [n] : \ell_\lambda(x) = i\}$$

Then $\mathfrak{m}_\lambda = (M_1, \ldots, M_k)$ is a partition of $[n]$ and if $\tau_{\mathfrak{m}_\lambda}$ is the partial order associated with $\mathfrak{m}_\lambda$, then $\lambda \subset \tau_{\mathfrak{m}_\lambda}$.

Proof. To simplify the writing let us write $\ell(x)$ for $\ell_\lambda(x)$.

i) $M_i \neq \phi$ for each $1 \leq i \leq k$.: Since $\lambda \in \mathfrak{O}_n^{(k)}$, then there exists a chain $0 <_\lambda x_1 <_\lambda \cdots <_\lambda x_k$ of length k and k is the maximum length of all chains in λ. Then $x_i \in M_i$ for $1 \leq i \leq k$. For let $1 \leq i \leq k$. Since $0 <_\lambda x_1 <_\lambda \cdots <_\lambda x_i$ is a chain with head x_i, then $i \leq \ell(x_i) = s$. But if there is a chain $0 <_\lambda y_1 <_\lambda \cdots <_\lambda y_s = x_i$ with head x_i. Then

$$0 <_\lambda y_1 <_\lambda \cdots <_\lambda y_s = x_i <_\lambda x_{i+1} <_\lambda \cdots <_\lambda x_k$$

is a chain of length $s + k - i$. So we must have $s + k - i \leq k$, which implies that $\ell(x_i) = s \leq i$. Hence $x_i \in M_i$.

ii) If $1 \leq i < j \leq k$, then $M_i \cap M_j = \phi$.: Suppose to the contrary that $x \in M_i \cap M_j$. Then $\ell(x) = i < j = \ell(x)$, a contradiction.

iii) $[n] = M_1 \cup \cdots \cup M_k$. : If $x \in [n]$, then $x \in M_{\ell(x)}$.

iv) $\lambda \subset \tau_{\mathfrak{m}_\lambda}$. : Let $x <_\lambda y$. If $\ell(x) = s$. Then there is a chain $0 <_\lambda y_1 <_\lambda \cdots <_\lambda y_s = x$ with head x. Thus $0 <_\lambda y_1 <_\lambda \cdots <_\lambda y_s = x <_\lambda y$ is a longer chain with head y. Hence $\ell(x) < \ell(y)$. It follows that

$$x, y \in [n], \ell(x) < \ell(y) \leq k \text{ and } x \in M_{\ell(x)}, y \in M_{\ell(y)}$$

So by the defininition of $\tau_{\mathfrak{m}_\lambda}$ we have $x <_{\tau_{\mathfrak{m}_\lambda}} y$. □

Theorem 50. *a) $\lambda \in \mathfrak{O}_n^{(k)}$ is maximal if and only if $\lambda = \tau_\mathfrak{m}$ for some $\mathfrak{m} \in \mathfrak{M}_n^{(k)}$.*

b) If $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}_n^{(k)}$ and $\mathfrak{m} \neq \mathfrak{n}$, then $\tau_\mathfrak{m} \neq \tau_\mathfrak{n}$.

Proof. a) Let $\lambda \in \mathfrak{O}_n^{(k)}$ be maximal. Then $\lambda \subset \tau_{\mathfrak{m}_\lambda}$ implies that $\lambda = \tau_{\mathfrak{m}_\lambda}$.

Conversely let $\mathfrak{m} \in \mathfrak{M}_n^{(k)}$ and $\rho \in \mathfrak{O}_n^{(k)}$ be such that $\alpha = \tau_\mathfrak{m} \subset \rho$. If $\mathfrak{n} = \mathfrak{m}_\rho$, then

$$\alpha \subset \rho \subset \tau_{\mathfrak{m}_\rho} = \tau_\mathfrak{n} = \beta$$

This implies that $\alpha = \tau_\mathfrak{m} \subset \tau_\mathfrak{n} = \beta$. Let $\mathfrak{m} = (M_1, \ldots, M_k)$ and $\mathfrak{n} = (N_1, \ldots, N_k)$.

We will prove by induction on j by starting from $j = k$ down to $j = 1$ that $M_j = N_j$.

$j = k$: Let $x \in M_k$, then $1 = \ell_\alpha(x) \leq \ell_\beta(x) \leq k$. Thus $\ell_\beta(x) = k$. This shows $x \in N_k$. So $M_k \subset N_k$. Let $x \in N_k$, then $\exists i$ such that $x \in M_i$. If $i < k$, then pick any $y \in M_{i+1}$. But then

$$x <_\alpha y \Longrightarrow x <_\beta y \Longrightarrow k = \ell_\beta(x) < \ell_\beta(y)$$

a contradiciton. It follows that $x \in M_k$. We have shown that $N_k = M_k$.

$1 < j \leq k$: Suppose we have shown that $N_k = M_k, \ldots, N_j = M_j$. Let $x \in M_{j-1}$, then

$$x \notin \bigcup_{i=j}^{k} M_i = \bigcup_{i=j}^{k} N_i \tag{6.1}$$

Therefore

$$j-1 = \ell_\alpha(x) \leq \ell_\beta(x) \leq j-1 \Longrightarrow \ell_\beta(x) = j-1 \Longrightarrow x \in N_{j-1}$$

So $M_{j-1} \subset N_{j-1}$. Let $x \in N_{j-1}$, then $\exists i$ such that $x \in M_i$. By (6.1) $i \leq j-1$. If $i < j-1$, then $i+1 < j$. Pick any $y \in M_{i+1}$. We have

$$x <_\alpha y \Longrightarrow x <_\beta y$$

Thus

$$j-1 = \ell_\beta(x) <_\beta \ell_\beta(y) \Longrightarrow j \leq \ell_\beta(y) = s \Longrightarrow y \in N_s = M_s, i+1 < j \leq s$$

Thus $y \in M_s \cap M_{i+1} = \phi$, a contradiciton. It follows that , $M_{j-1} = N_{j-1}$. This completes the induction and proves that $\alpha = \beta$. So,

$$\alpha = \tau_{\mathfrak{m}} \subset \rho \subset \tau_{\mathfrak{m}_\rho} = \tau_{\mathfrak{n}} = \beta = \alpha \Longrightarrow \tau_{\mathfrak{m}} = \rho$$

Hence α is maximal.

b) It suffices to note that

$$\mathfrak{m}_{\tau_{\mathfrak{m}}} = \mathfrak{m}$$

So if $\tau_{\mathfrak{m}} = \tau_{\mathfrak{n}}$, then

$$\mathfrak{m} = \mathfrak{m}_{\tau_{\mathfrak{m}}} = \mathfrak{m}_{\tau_{\mathfrak{n}}} = \mathfrak{n}$$

□

Theorem 51. *Let k be an integer and $1 \leq k \leq n$.*

a) If $\mathfrak{m} \in \mathfrak{M}_n^{(k)}$, then $N_{\tau_{\mathfrak{m}}}$ and $N'_{\tau_{\mathfrak{m}}}$ are maximal in $\mathrm{Nil}_k\,(\mathcal{PT}_n)$ and $\mathrm{Nil}_k\,(\mathcal{IS}_n)$, respectively.

b) If N is maximal in $\mathrm{Nil}_k\,(\mathcal{PT}_n)$, then $N = N_{\tau_{\mathfrak{m}}}$ for some $\mathfrak{m} \in \mathfrak{M}_n^{(k)}$.

c) If N' is maximal in $\mathrm{Nil}_k\,(\mathcal{IS}_n)$, then $N' = N'_{\tau_{\mathfrak{m}}}$ for some $\mathfrak{m} \in \mathfrak{M}_n^{(k)}$.

d) If $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}_n^{(k)}$ and $\mathfrak{m} \neq \mathfrak{n}$, then $N_{\tau_{\mathfrak{m}}} \neq N_{\tau_{\mathfrak{n}}}$ and $N'_{\tau_{\mathfrak{m}}} \neq N'_{\tau_{\mathfrak{n}}}$.

Proof. a) By Theorem 50 $\tau_{\mathfrak{m}}$ is maximal in $\mathfrak{O}_n^{(k)}$. Thus by Lemma 14 $N_{\tau_{\mathfrak{m}}}$ is maximal in $\mathrm{Nil}_k\,(\mathcal{PT}_n)$ and $N'_{\tau_{\mathfrak{m}}}$ is maximal in $\mathrm{Nil}_k\,(\mathcal{IS}_n)$.

b) If N is maximal in $\mathrm{Nil}_k\,(\mathcal{PT}_n)$, then by Lemma 14 σ_N is maximal in $\mathfrak{O}_n^{(k)}$ and by Theorem 50 $\sigma_N = \tau_{\mathfrak{m}}$ for some $\mathfrak{m} \in \mathfrak{M}_n^{(k)}$. Thus $N = N_{\sigma_N} = N_{\tau_{\mathfrak{m}}}$.

c) If N' is maximal in $\mathrm{Nil}_k\,(\mathcal{IS}_n)$, then by Lemma 14 σ_N is maximal in $\mathfrak{O}_n^{(k)}$ and by Theorem 50 $\sigma_N = \tau_{\mathfrak{m}}$ for some $\mathfrak{m} \in \mathfrak{M}_n^{(k)}$. Thus $N\prime = N'_{\sigma_N} = N'_{\tau_{\mathfrak{m}}}$.

d) If $N_{\tau_{\mathfrak{m}}} = N_{\tau_{\mathfrak{n}}}$, then $\tau_{\mathfrak{m}} = \sigma_{N_{\tau_{\mathfrak{m}}}} = \sigma_{N_{\tau_{\mathfrak{n}}}} = \tau_{\mathfrak{n}}$. This implies $\mathfrak{m} = \mathfrak{n}$. Same proof applies when $N'_{\tau_{\mathfrak{m}}} = N'_{\tau_{\mathfrak{n}}}$.

□

6.2. MAXIMAL NILPOTENT SUBSEMIGROUPS OF $\mathcal{IS}_N$

We notice the following easy fact.

Lemma 17. *If* $\mathfrak{m} = (M_1, \ldots, M_k) \in \mathfrak{M}_n^{(k)}$, *then considering* $\tau_{\mathfrak{m}}$ *as a partial order on* $\langle n \rangle$ *is*

$$\tau_{\mathfrak{m}} = \bigcup_{0 \le i < j \le k} M_i \times M_j = \bigcup_{i=0}^{k-1} \bigcup_{j=i+1}^{k} M_i \times M_j$$

where $M_0 = \{0\}$.

Proof. This is easily follows from the definition of $\tau_{\mathfrak{m}}$. □

This lemma shows that $\tau_{\mathfrak{m}}$ is a disjoint union of sets $\{M_i \times M_j : 0 \le i < j \le k\}$.

We know that the partial order $\tau_{\mathfrak{m}}$ associated with an $\mathfrak{m} \in \mathfrak{M}_n^{(k)}$ is maximal in $\mathfrak{O}_n^{(k)}$. But if $\mathfrak{n} \in \mathfrak{M}_n^{(s)}$ where $s \neq k$, then we may have $\tau_{\mathfrak{n}} \subsetneq \tau_{\mathfrak{m}}$.

Lemma 18. *Let* $0 \le s \le t-2$ *and* $\mathfrak{m} = (P_1, \ldots P_s, P_{s+1}, P_{s+2}, \cdots, P_t) \in \mathfrak{M}_n^{(t)}$ *and* $\mathfrak{n} = (P_1, \ldots P_s, P_{s+1} \cup P_{s+2}, \cdots, P_t) \in \mathfrak{M}_n^{(t-1)}$.

a) $\tau_{\mathfrak{m}} = \tau_{\mathfrak{n}} \cup P_{s+1} \times P_{s+2}$ *and* $\tau_{\mathfrak{n}} \cap P_{s+1} \times P_{s+2} = \phi$. *In particular* $\tau_{\mathfrak{n}} \subset \tau_{\mathfrak{m}}$.

b) $N_{\tau_{\mathfrak{n}}} \subset N_{\tau_{\mathfrak{m}}}$ *and* $N'_{\tau_{\mathfrak{n}}} \subset N'_{\tau_{\mathfrak{m}}}$.

Proof. a) Suppose we are given a pair We set $P'_i = P_i$ for $i = 1, \ldots, s$, $P'_{s+1} = P_{s+1} \cup P_{s+2}$, and $P'_i = P_{i+1}$ for $i = s+2, \ldots, t$. Then by Lemma 17

$$\tau_{\mathfrak{n}} = \bigcup_{0 \le i < j \le t-1} P'_i \times P'_j \text{ and } \tau_{\mathfrak{m}} = \bigcup_{0 \le i < j \le t} P_i \times P_j$$

Let $0 \le i < j \le t-1$.

$$
\begin{array}{ll}
0 \le i \le s & \begin{cases} \text{If } j \le s, \text{ then} \\ P'_i \times P'_j = P_i \times P_j \\ \text{If } j = s+1, \text{ then} \\ P'_i \times P'_j \\ \qquad = P_i \times (P_{s+1} \cup P_{s+2}) \\ \qquad = (P_i \times P_{s+1}) \cup (P_i \times P_{s+2}) \\ \text{If } s+1 < j, \text{ then} \\ P'_i \times P'_j = P_i \times P_{j+1} \end{cases} \\
i = s+1 < i < j \le t-1 & \begin{cases} \text{In this case } s+1 < i < j \\ \text{implies that } s+2 \le i < j \text{ . Hence} \\ P'_{s+1} \times P'_j \\ \qquad = (P_{s+1} \cup P_{s+2}) \times P_{j+1} \\ \qquad = \left(P_{s+1} \times P_{j+1}\right) \cup \left(P_{s+2} \times P_{j+1}\right) \end{cases} \\
s+1 < i < j \le t-1 & \begin{cases} P'_i \times P'_j = P_i \times P_j \end{cases}
\end{array}
$$

This shows that every pair that appear in the decomposition of $\tau_{\mathfrak{m}}$ also appears in the decomposition of $\tau_{\mathfrak{n}}$ except $P_{s+1} \times P_{s+2}$. This proves the lemma. □

We first prove the following lemma which proves that the cardinality of $N'_{\tau_{\mathfrak{m}}}$ only depends on $(m_1, \ldots, m_k) \in (\mathbb{N}^+)^k$ where $\mathfrak{m} = (M_1, \ldots, M_k) \in \mathfrak{M}_n^{(k)}$ and $|M_i| = m_i$ for each $i = 1, \ldots, k$.

Lemma 19. *Let $\mathfrak{m} = (M_1, \ldots, M_k)$, $\mathfrak{n} = (N_1, \ldots, N_k)$ be such that $|M_i| = |N_i|$ for each $i = 1, \ldots, k$. Then there is an isomorphism $\varphi : N_{\tau_{\mathfrak{m}}} \to N_{\tau_{\mathfrak{m}}}$ such that $\left(N'_{\tau_{\mathfrak{m}}}\right)\varphi = N'_{\tau_{\mathfrak{n}}}$.*

Proof. Let σ be a permutation of $[n]$ such that $M_i\sigma = N_i$ for each $i = 1, \ldots, k$. We let $0\sigma = 0$. If $x, y \in \langle n \rangle$ are such that $x <_{\tau_{\mathfrak{m}}} y$, then either $x = 0$ and $y \in [n]$ or $x, y \in [n]$ and $\exists i, j$ such that $1 \le i < j \le k$ such that $x \in M_i$ and $y \in M_j$. Thus either $x\sigma = 0$ and $y\sigma \in [n]$ or $x\sigma, y\rho \in [n]$ and $\exists i, j$ such that $1 \le i < j \le k$ such that $x\sigma \in M_i\sigma = N_i$ and $y\sigma \in M_j\sigma = M_j$. Thus

$$x <_{\tau_{\mathfrak{m}}} y \Longrightarrow x\sigma <_{\tau_{\mathfrak{n}}} y\sigma$$

Since $M_i = N_i\sigma^{-1}$ for each $i = 1, \ldots, k$. $\alpha \in N_{\tau_{\mathfrak{m}}}$ we also have

$$x\sigma <_{\tau_{\mathfrak{n}}} y\sigma \Longrightarrow x <_{\tau_{\mathfrak{m}}} y$$

So

$$x <_{\tau_{\mathfrak{m}}} y \Longleftrightarrow x\sigma <_{\tau_{\mathfrak{n}}} y\sigma$$

This proves the lemma. □

Definition 18. *Given* $(m_1, \ldots, m_k) \in (\mathbb{N}^+)^k$ *where* $m_1 + \cdots + m_k = n$*, let us choose* $\mathfrak{m} = (M_1, \ldots, M_k) \in \mathfrak{M}_n^{(k)}$ *such that* $|M_i| = m_i$ *for each* $i = 1, \ldots, k$*. We define*

$$\mathbf{X}(m_1, \ldots, m_k) = \left|N'_{\tau_{\mathfrak{m}}}\right|$$

We write a few partial computations.

Theorem 52. *If* $m, n \in \mathbb{N}^+$ *then* $\mathbf{X}(m, n) = \mathbf{X}(n, m)$ *and*

$$\mathbf{X}(m, n) = \sum_{i=0}^{\min\{m,n\}} i! \binom{n}{i} \binom{m}{i}$$

Proof. Let us choose $\mathfrak{m} = (A, B) \in \mathfrak{M}_n^{(2)}$ such that $|A| = m$ and $|B| = n$. Given any $\alpha \in N'_{\tau_{\mathfrak{m}}}$, then $B\alpha \setminus \{0\}$ is a subset of $A \setminus \{0\}$ such that both sets have the same cardinality $0 \leq i \leq \min\{m, n\}$. There are $\binom{n}{i}$ ways to choose a subset from B and $\binom{m}{i}$ ways to choose a subset from A. One a choice is made there are $i!$ ways to match them up. This proves the theorem. □

Theorem 53. *If* $m, n, k \in \mathbb{N}^+$ *then* $\mathbf{X}(m, n, k) = \mathbf{X}(k, n, m)$ *and*

$$\mathbf{X}(m, n, k) = \sum_{i=0}^{m} \sum_{j=0}^{m-i} \sum_{u=0}^{k-i} u!j!i! \binom{m}{i} \binom{k}{i} \binom{m-i}{j} \binom{n}{j} \binom{n}{u} \binom{k-i}{u}$$

Proof. Let us choose $\mathfrak{m} = (A, B, C) \in \mathfrak{M}_n^{(3)}$ such that $|A| = m$, $|B| = n$, and $|C| = k$. □

Let us give some special values :

$$\mathbf{X}(m, 1) = \mathbf{X}(1, m) = m + 1$$
$$\mathbf{X}(m, 2) = \mathbf{X}(2, m) = m^2 + m + 1$$
$$\mathbf{X}(m, 3) = \mathbf{X}(3, m) = m^3 + 2m + 1$$

We first prove a reduction formula found in [4].

Lemma 20. *[4]If* $0 \leq s$, $1 \leq t$, *then*

$$\mathbf{X}(p_1, \ldots p_s, 1, r_1, \cdots, r_t) = \mathbf{X}(p_1, \ldots p_s, 1 + r_1, \cdots, r_t) + r_1 \mathbf{X}(p_1, \ldots p_s, r_1, \cdots, r_t)$$

Proof. $\mathfrak{m} = (P_1, \ldots P_s, \{a\}, \cdots, R_t) \in \mathfrak{M}_n^{(s+t+1)}$ such that $|P_i| = p_i$ for $i = 1, \ldots, s$ and $|R_j| = r_j$ for $j = 1, \ldots, t$. Let $\mathfrak{n} = (P_1, \ldots P_s, R_1 \cup \{a\}, \cdots, R_t) \in \mathfrak{M}_n^{(p+t)}$. By Lemma 18 $\tau_{\mathfrak{n}} \subset \tau_{\mathfrak{m}}$. Hence $N'_{\tau_{\mathfrak{n}}} \subset N'_{\tau_{\mathfrak{m}}}$.

Let $\beta \in N'_{\tau_{\mathfrak{m}}}$ and $\beta \notin N'_{\tau_{\mathfrak{n}}}$. By the definition of $\beta \in N'_{\tau_{\mathfrak{m}}}$whenever $w, z \in [n]$ satisfy $w <_{\tau_{\mathfrak{m}}} z$, then $z\beta = w$.But

$$\tau_{\mathfrak{m}} = \tau_{\mathfrak{n}} \cup \{a\} \times R_1$$

So if $\beta \notin N'_{\tau_{\mathfrak{n}}}$ there is a pair $w, z \in [n]$ such that $w <_{\tau_{\mathfrak{m}}} z$ but $w \nless_{\tau_{\mathfrak{n}}} z$. The only possibility for such a pair is when this pair belongs to $\{a\} \times R_1$. In other words $z\beta = a$. For each $z \in R_1$ define

$$S_z = \left\{\beta \in N'_{\tau_{\mathfrak{m}}} : a = z\beta\right\}$$

It is clear that S_z and S_v are disjoint for distinct z, v in R_1. We have just proved that $N'_{\tau_{\mathfrak{n}}}$ can be written as the disjoint union of $N'_{\tau_{\mathfrak{n}}}$ and S_z's. That is

$$N'_{\tau_{\mathfrak{m}}} = N'_{\tau_{\mathfrak{n}}} \cup \bigcup_{z \in R_1} S_z$$

We now, claim that S_z is isomorphic to $N'_{\tau_{\mathfrak{u}}}$ where $\mathfrak{u} = (P_1, \ldots P_s, \cdots, R_t)$. To see this define $\psi : S_z \longrightarrow N'_{\tau_{\mathfrak{u}}}$ as follows: Given $\beta \in S_z$ let

$$w(\beta\psi) = \begin{cases} a\beta & \text{if } w = z \\ w\beta & \text{if } w \neq z \end{cases}$$

It is easy to see that ψ is an isomorphism. Therefore

$$\left|N'_{\tau_{\mathfrak{m}}}\right| = \left|N'_{\tau_{\mathfrak{n}}}\right| + \sum_{z \in R_1} |S_z| = \left|N'_{\tau_{\mathfrak{m}}}\right| + r_1 \left|N'_{\tau_{\mathfrak{u}}}\right|$$

This proves the lemma. ◻

We may use Lemma 20 to prove the following:

Theorem 54. *For any permutation σ of $[k]$ we have* $\mathbf{X}(m_1, \ldots, m_k) = \mathbf{X}(m_{1\sigma}, \ldots, m_{k\sigma})$.

Proof. To prove this we use induction on p which is the sum of the number of m_i greater than 1. If $p = 0$, then $m_i = 1$ for all i and there is nothing to prove. Suppose $p > 0$ and $m_i > 1$ for some $1 \leq i \leq k$. By Lemma 20 we have

$$\mathbf{X}(m_1, \ldots m_{i-1}, 1, m_i - 1, \cdots, m_k) = \mathbf{X}(m_1, \ldots m_{i-1}, m_i, \cdots, m_k) + (m_i - 1)\mathbf{X}(m_1, \ldots m_{i-1}, m_i - 1, \cdots, m_k)$$

Thus

$$\begin{aligned} & \mathbf{X}(m_1, \ldots m_{i-1}, m_i, \cdots, m_k) \\ = \; & \mathbf{X}(m_1, \ldots m_{i-1}, 1, m_i - 1, \cdots, m_k) - (m_i - 1)\mathbf{X}(m_1, \ldots m_{i-1}, m_i - 1, \cdots, m_k) \end{aligned}$$

We notice that the sum of the number of m_i greater than 1 for each term on the second is at most $p - 1$. So by induction hypothesis we may write

$$\begin{aligned} & \mathbf{X}(m_1, \ldots m_{i-1}, m_i, \cdots, m_k) \\ = \; & \mathbf{X}(1, m_i - 1, \ldots m_{i-1}, m_1, \cdots, m_k) - (m_i - 1)\mathbf{X}(m_i - 1, \ldots m_{i-1}, m_1, \cdots, m_k) \\ = \; & \mathbf{X}(m_i, \ldots m_{i-1}, m_1, \cdots, m_k) \end{aligned}$$

This proves that we may exchange m_i and m_1. But the transpoistions $(1, 2)$, $(1, 3)$,. . . ,$(1, k)$ generate all permutations in $\mathcal{S}_k$. This proves the theorem. □

Definition 19. *Given a polynomial $f(x) = \sum_{i=0}^{m} a_i x^i$ we define $f(B) = \sum_{i=0}^{m} a_i B_i$ where B_i is the i th Bell number.*

It is clear that if f and g are two polynomials, then $(f + g)(B) = f(B) + g(B)$.

Definition 20. *If $(m_1, \ldots, m_k) \in (\mathbb{N}^{+})^k \in$, then we define the polynomial $F(m_1, \ldots, m_k)(x)$ as*

$$F(m_1, \ldots, m_k)(x) = x^{\underline{m_1}} x^{\underline{m_2}} \cdots x^{\underline{m_k}}$$

where $x^{\underline{m}}$ is the falling factorial $x^{\underline{m}} = x(x-1)\cdots(x-m+1)$.

Theorem 55. *[4]*$\mathbf{X}(m_1,\ldots,m_k) = F(m_1,\ldots,m_k)(B)$.

Proof. We note that if each $m_i = 1$, then $\mathbf{X}(1,\ldots,1) = B_k$. We also have $F(1,\ldots,1)(x) = x^{\underline{1}}x^{\underline{1}}\cdots x^{\underline{1}} = x^k$. Therefore $F(1,\ldots,1)(B) = B_k$ as required. We only prove the general case we use induction on p which is the sum of the number of m_i greater than 1. If $p = 0$, then $m_i = 1$ and we are done. Suppose $p > 0$ and $m_i > 1$ for some $1 \le i \le k$. Since By Theorem 54 $\mathbf{X}(m_1,\ldots m_{i-1},m_i,\cdots,m_k) = \mathbf{X}(m_i,\ldots m_{i-1},m_1,\cdots,m_k)$, we may assume that $i = 1$. By Lemma 20 we have

$$\begin{aligned}&\mathbf{X}(1,m_1-1,\ldots m_i,\cdots,m_k)\\ =\;&\mathbf{X}(m_1,\ldots,m_i,\cdots,m_k) + (m_1-1)\mathbf{X}(m_1-1,\ldots m_i,\cdots,m_k)\end{aligned}$$

Thus

$$\begin{aligned}&\mathbf{X}(m_1,\ldots,m_i,\cdots,m_k)\\ =\;&\mathbf{X}(1,m_1-1,\ldots m_i,\cdots,m_k) - (m_1-1)\mathbf{X}(m_1-1,\ldots m_i,\cdots,m_k)\end{aligned}$$

We notice that the sum of the number of m_i greater than 1 for each term on the second is at most $p-1$. So by induction hypothesis we have

$$\begin{aligned}&\mathbf{X}(m_1,\ldots,m_k)\\ =\;&F(1,m_1-1,\ldots m_i,\cdots,m_k)(B) - (m_1-1)F(m_1-1,\ldots m_i,\cdots,m_k)(B)\end{aligned}$$

But

$$\begin{aligned}&F(1,m_1-1,\ldots m_i,\cdots,m_k)(x) - (m_1-1)F(m_1-1,\ldots m_i,\cdots,m_k)(x)\\ =\;&xx^{\underline{m_1-1}}x^{\underline{m_2}}\cdots x^{\underline{m_k}} - (m_1-1)x^{\underline{m_1-1}}x^{\underline{m_2}}\cdots x^{\underline{m_k}}\\ =\;&(x-m_1+1)x^{\underline{m_1-1}}x^{\underline{m_2}}\cdots x^{\underline{m_k}}\\ =\;&x^{\underline{m_1}}x^{\underline{m_2}}\cdots x^{\underline{m_k}}\end{aligned}$$

Thus

$$\mathbf{X}(m_1,\dots,m_k) = F(m_1,\dots,m_k)(B)$$

This completes the proof. $\square$

This theorem actually implies many identities involving Bell numbers.

Example 56. *Clearly* $\mathbf{X}(n) = 1$*. So we must have* $x^{\underline{n}}(B) = 1$*. Let us prove this in a different way. We know that*

$$B_m = \sum_{i=1}^{m} S(m,i) = \sum_{i=1}^{m} S(m,i)\cdot 1$$

where $S(m,i)$ *is the Stirling number of the second kind. So by the inversion formula*

$$1 = \sum_{i=1}^{m} (-1)^{m-i} s(m,i) B_i = x^{\underline{n}}(B)$$

where $s(m,i)$ *is the Stirling number of the first kind.*

Example 57. *We have seen that*

$$\mathbf{X}(m,1) = \mathbf{X}(1,m) = m+1$$
$$\mathbf{X}(m,2) = \mathbf{X}(2,m) = m^2+m+1$$

So, using the first one we get

$$\sum_{i=1}^{m} (-1)^{m-i} s(m,i) B_{i+1} = m+1$$

Let us rewrite the second one by noting that

$$\mathbf{X}(2,m) = \left(x(x-1)x^{\underline{m}}\right)(B) = x(x-1)\sum_{i=1}^{m}(-1)^{m-i} s(m,i)$$

Now,

$$
\begin{aligned}
x(x-1)x^{\underline{m}} &= x(x-1)\sum_{i=1}^{m}(-1)^{m-i}s(m,i)x^{i} \\
&= \sum_{i=1}^{m}(-1)^{m-i}s(m,i)x^{i+2}-\sum_{i=1}^{m}(-1)^{m-i}s(m,i)x^{i+1} \\
&= \sum_{i=1}^{m}(-1)^{m-i}s(m,i)x^{i+2}-\sum_{i=0}^{m-1}(-1)^{m-i-1}s(m,i+1)x^{i+2} \\
&= \sum_{i=0}^{m}(-1)^{m-i}s(m,i)x^{i+2}+\sum_{i=0}^{m}(-1)^{m-i}s(m,i+1)x^{i+2} \\
&= \sum_{i=0}^{m}(-1)^{m-i}\left(s(m,i)+s(m,i+1)\right)x^{i+2}
\end{aligned}
$$

Here we used the fact that $s(m,0)=s(m,m+1)=0$. *This implies*

$$
\sum_{i=0}^{m}(-1)^{m-i}\left(s(m,i)+s(m,i+1)\right)B_{i+2}=m^{2}+m+1
$$

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