DO IMPOSSIBILITY RESULTS SURVIVE IN HISTORICALLY STANDARD DOMAINS?

A Master's Thesis

by
EBRU GÜRER

Department of
Economics
Bilkent University
Ankara
February 2008

DO IMPOSSIBILITY RESULTS SURVIVE IN HISTORICALLY STANDARD DOMAINS?

The Institute of Economics and Social Sciences
of
Bilkent University

by

EBRU GÜRER

In Partial Fulfillment of the Requirements For the Degree
of
MASTER OF ARTS

in

THE DEPARTMENT OF
ECONOMICS
BILKENT UNIVERSITY
ANKARA

February 2008

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Prof. Dr. Semih Koray

Supervisor

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Assoc. Prof. Dr. Ferhad Hüseyin

Examining Committee Member

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Assoc. Prof. Dr. Sinan Sertöz

Examining Committee Member

Approval of the Institute of Economics and Social Sciences

Prof. Dr. Erdal Erel

Director

ABSTRACT

DO IMPOSSIBILITY RESULTS SURVIVE IN HISTORICALLY STANDARD DOMAINS?

GÜRER, EBRU

M.A., Department of Economics

Supervisor: Prof. Semih Koray

February 2008

One of the major assumptions common to all impossibility results in social choice theory is that of ”full” or rich enough domain. Thus, a major stream of attempts has focused on how to restrict the domains of social choice functions in order to escape impossibilities, without paying much attention to the question of whether there exist actual societies with such restricted domains of preference profiles, however. The notion of an unrestricted domain is based on the assumption that the individuals form their preferences independent of each other. If one replaces this assumption by one under which individual preferences are clustered around a ”social norm” in a unipolar standard society, the question of how this kind of restricted domain restriction influences the existence of a Maskin monotonic, surjective and nondictatorial social choice function becomes important.

We employ the so-called Manhattan metric to measure the degree of how clustered a society around a social norm is. We then try to characterize what degrees of clustering around a social norm allow us to escape impossibility results, in an attempt to shed some light on the question of whether impossibilities in social choice theory arise from assuming the existence of historically

impossible societies.

Keywords: Manhattan metric, Maskin monotonicity, dictatoriality .

ÖZET

TARİHSEL OLARAK STANDART OLAN TANIM BÖLGELERİNDE İMKANSIZLIK SONUÇLARI GEÇERLİ OLMAYI SÜRDÜRÜR MÜ?

GÜRER, Ebru

Yüksek Lisans, Ekonomi Bölümü

Tez Yöneticisi: Prof. Semih Koray

Şubat 2008

Sosyal seçim kuramının bütün imkansızlık sonuçlarında ortak olan ana varsayımlarından biri de, tanım bölgesine ilişkin tam ya da yeterince zengin bölge varsayımlarıdır. Sosyal seçim kuramıda karşılık geldikleri toplumların toplumların gerçekte var olup olmadığına dikkat edilmeksizin, imkansızlık sonuçlarından kurtulabilmek için, tanım bölgelerini kısıtlama yolunda birçok teşebbüste bulunulmuştur. Kısıtlanmamış bir tanım bölgesi düşüncesi bireylerin birbirinden bağımsız olması varsayımına dayanmaktadır. Eğer bu varsayımı, bireylerin tercihlerinin tek kutuplu standart bir toplumun sosyal normu etrafında toplaşması varsayımıyla değiştirirsek, bu kısıtlamanın Maskin tekdüze, örten ve diktatorlük olmayan bir sosyal seçim fonksiyonunun varlığına etkisi önemli bir soru haline gelir.

Bir toplumun bir sosyal norm etrafında ne kadar yoğunlaşmış olduğunu ölçmek için Manhattan uzaklığı denilen bir metrik kullanıyoruz. Ardından bir sosyal norm etrafında ne kadar yoğunlaşmanın imkansızlık sonuçlarından kurtulunmasını sağlayacağını karakterize etmeye çalışıyoruz. Bu tür bir çalışmanın tarihsel olarak imkansız olan bazı toplumları varsaymanın, sosyal seçim kuramındaki imkasızlıklar üstüne ne etkide bulunduğu sorusuna ışık tutacağını düşünüyoruz.

Anahtar Kelimeler: Manhattan uzaklığı, Maskin tekdüzelik, diktatörlük

ACKNOWLEDGMENTS

I would like to express my kind and deepest thanks to my advisor Prof. Semih Koray for his helpful comments and encouragement. I owe the completion of this thesis to him.

I would like to thank my friend Ali Şamil Kavruk for his suggestions to improve this thesis.

I would like to thank Prof. Tarık Kara and my friends, especially Battal Doğan, Kemal Yıldız, Pelin Pasin, in our study group for coming and listening the preliminary version of this work and two other committee members Prof. Farhad Huseyin and Prof. Sinan Sertöz for coming to my thesis committee.

Many thanks to BWED 2007 members especially to Prof. Remzi Sanver for reading commenting, and suggesting useful ideas.

My special thanks to my family and whose support I always felt with me.

For the format and technical support I thank Ms.Meltem Sağtürk.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

We start with an alternative set $A = \{a_1, ...a_n\}$ and an individual set $I = \{1, ..., N\}$. We form the set of complete, transitive and antisymmetric binary relations $L(A) = \mathbf{P}$ on the choice set A. This forms the largest set that individuals' preferences can come from. But we will mostly be working with particular restricted domains $\mathbf{D} \subset \mathbf{P}$ in this study. Restricted domains are frequently employed notion in social choice theory in an attempt to avoid impossibility results which hold on sufficiently rich domains. In our study we also consider particular domain restrictions to see their impact upon impossibility results. The kind of restrictions that we deal with here are, however, based on entirely different reasons than those that have been introduced in the literature so far.

In history, it has been "common opinions of the public" upon which the stability of a society is based. The rules governing a society evolve in time to meet certain basic societal requirements. Unless there is some kind of "common approval" or "common belief" on the basics, however, it is difficult to imagine what will make a society to last. Thus, we expect that individual preferences cluster around "social norms" in at least some important social choice problems. In this study, we focus on social choice problems, where there is a social norm and examine how close individual preferences should be clustered around it to turn impossibilities to possibilities.

First we fix an ordering P_s on A which is meant to reflect a social norm. We then use the Manhattan metric to measure the distance of an individual preference to P_s. We thus obtain a classification of the orderings in $\mathbf{P}$ according to their distances to $\mathbf{P}$. The subdomain that we denote D_k with $k \in \{0, 1, ..., n(n-1)/2\}$, consists of elements from $\mathbf{P}$ whose distances to P_s are less than or equal to k. The elements in D_1, for example, are the orderings with distance 1 to P_s and P_s itself.

The distance of a preference ordering to the "standard social ordering P_s" in terms of the Manhattan metric is the minimal number of elementary steps needed to transform that preference into P_s. An elementary step is changing the order of two consecutively ranked alternatives, keeping the rest of the ordering unchanged. It is easy to see that the maximal distance of an ordering to P_s is $n(n-1)/2$, and it is attained when the ordering ?? starts with is the reversal of P_s if $|A| = n$. The bottom ranked alternative goes to the top in $n-1$ steps. Now the new bottom ranked alternative requires $n-2$ steps to reach the second rank. Continuing similarly, the initial ordering will be reversed in $(n-1) + ... + 2 + 1 = n(n-1)/2$ steps.

We will focus on Mueller-Satterthwaite Theorem (1977) as a representative of impossibility theorems in this study. That is we will be examining the existence of social choice functions satisfying Maskin monotonicity, surjectivity and non-dictatoriality on our domains, $D_k(k = 0, 1, ..., \frac{n(n-1)}{2})$ under the presence of at least three alternatives. Our first result is that there does not exist any Maskin monotonic and surjective SCF on D_k with $k \leq n-2$ for $n \geq 3$ alternatives. As D_0 consists of one profile only, there is no surjective SCF on D_0. When $k \in \{1, ..., n-2\}$, the impossibility arises because the bottom alternative which has to be chosen at some profile never gets top ranked by any of the individuals. Then we give an example of a Maskin monotonic, surjective and nondictatorial SCF on D_{n-1}. This also shows that D_{n-1} domain is the minimal domain allowing the existence of an SCF with the required properties. For the remaining domains D_k with $k \in \{n, ..., n(n-1)/2\}$, we prove the impossibility of a Maskin monotonic, surjective and nondictatorial SCF on D_k.

In our study, we use the notion of a linked domain Aswal et al. (2002) extensively. The notion of a linked domain is based on the notion of connected pairs of alternatives, which is being utilized in the proof of our main theorem concerning the impossibility on D_n.

The rest of the thesis is organized as follows. We introduce basic notions in Chapter 2 and our main results in Chapter 3. Chapter 4 concludes.

CHAPTER 2

PRELIMINARIES

Let $A = \{a_1, a_2, ..., a_n\}$ be an alternative set, and $I = \{1, ..., N\}$ a set of individuals. Let $\mathbf{P}$ stand for the class of linear orderings on A. Any $P \in \mathbf{P}^{\mathbf{N}}$ is referred to as a preference profile.A preference profile P includes preferences of all individuals'. In particular P_i shows the ordering of individual $i \in I$, with the usual understanding that $a_j P_i a_k$ denotes a_j is preferred to a_k by individual i. We denote $r_k(P_i)$ for the k^{th} ranked alternative in P_i for any $k \in \{1, ..., n\}$. As usual, (P_i', P_{-i}) is used to show the preference profile obtained from P by replacing P_i in P by P_i'.

Definition. For any linear ordering $P_1 \in \mathbf{P}$, we will refer to the operation of interchanging the positions of two alternatives with consecutive ranks in P_1 to obtain another linear ordering as an elementary operation applied to P_1.

Definition. For any P_1,$P_2 \in \mathbf{P}$ the *Manhattan distance* $m(P_1, P_2)$ between P_1 and P_2 is defined to be the minimal number of elementary operations needed to obtain P_2 from P_1.

Example 1. For $A = \{a, b, c, d\}$ let us fix an ordering $P_s \in \mathbf{P}$, where

$$P_s = \begin{matrix} a \\ b \\ c \\ d \end{matrix}$$

with the usual understanding that $r_1(P_s) = a, ..., r_4(P_s) = d$. It is easy to see that $Max_{P_i \in \mathbf{P}}\ m(P_s, P_i) = \frac{4.3}{2} = 6$ and is attained at

$$P_s^{'} = \begin{matrix} d \\ c \\ b \\ a \end{matrix}$$

In general, when $|A| = n$, this maximal distance according to the $Manhattanmetric$ again occurs between a given ordering P_s and its reversal $P_s^{'}$, and it is equal to $\frac{n.(n-1)}{2}$. To see this, note that one needs $n-1$ elementary operations for the bottom alternative in P_s to reach the top rank. Now the bottom alternative needs to go to the second rank, requiring $n-2$ elementary operations. Continuing in a similar fashion until we reach $P_s^{'}$, the minimal total number of steps need is seen to be $(n-1) + (n-2) + ... + 1 = \frac{n(n-1)}{2}$.

For any nonempty subset $\mathbf{D}$ of $\mathbf{F}$ $f : \mathbf{D}^N \rightarrow A$ is called a social choice function (SCF).

We will now imagine that a particular ordering $P_s \in \mathbf{P}$ represents a social norm around which our society I clusters regarding its members' preferences on A. We will use the maximal Manhattan distance of individual preferences to the social norm P_s in a society as a measure of the degree of clustering around P_s. Formally, for any $P_s \in \mathbf{P}$ and any $k \in \{0, 1, ..., \frac{n(n-1)}{2}\}$ we define $D_k(P_s) = \{P_i \in \mathbf{P} : m(P_i, P_s) \leqslant k\}$. Note that $D_0(P_s) = \{P_s\}$, while

$D_{\frac{n(n-1)}{2}}(P_s) = \mathbf{P}$

Definition. An SCF f is said to be unanimous if $f(P) = a_j$ whenever $r_1(P_i) = a_j$ for all $i \in \{1, 2, ..., N\}$.

Let $L(x, P_i) = \{y \in A : \ xP_iy\}$ denote the lower counter of x relative to P_i.

Definition. (*Maskin monotonicity*)An SCF $f : D^N \to A$ is called Maskin monotonic if for any $x \in A$, any $i \in I$, any $P \in \mathbf{D}^N$ and any $P_i^{'} \in \mathbf{D}$
$f(P) = x \ [L(x, P_i) \subset L(x, P'_i)] \Rightarrow f(P_i^{'}, P_{-i}) = x$

Definition. An SCF $f : D^N \to A$ is called *dictatorial* if there exists an individual $i \in I$ such that for all $P \in \mathbf{D}^N$, $f(P) = r_1(P_i)$.

Definition. A domain $\mathbf{D} \subset \mathbf{P}$ is said to be minimally rich if, for all $a \in A$, there exists $P_i \in \mathbf{D}$ such that $r_1(P_i) = a$.

Definition. A pair of alternatives a_j and a_k are said to be connected in $\mathbf{D}$, denoted by $a_j \sim a_k$, if there exist $P_i, P_m \in \mathbf{D}$ such that $r_1(P_i) = a_j$, $r_2(P_i) = a_k$, $r_1(P_m) = a_k$, $r_2(P_m) = a_j$.

The relation $\sim$ is symmetric, i.e. $a_j \sim a_k$ implies $a_k \sim a_j$.

Definition. A domain $\mathbf{D}$ is called linked if there exists a one to one function $\sigma \ : \{1, ..., n\} \ \to \ \{1, ..., n\}$ such that
(i)$a_{\sigma(1)} \sim a_{\sigma(2)}$
(ii)$a_{\sigma(j)}$ is linked to $\{a_{\sigma(1)}, a_{\sigma(2)}, \dots a_{\sigma(j-1)}\}$ for $j = 3, \dots, n$.

Definition. An alternative $a \in A$ is linked to a set $\{b_1, b_2, \dots b_j\} \subset A$ if there exist two distinct elements $c, d \in \{b_1, b_2, \dots, b_j\}$ such that $a \sim c$ and $a \sim d$

Definition. A domain has $\mathbf{D} \subset \mathbf{P}$is said to have unique seconds property if there exist x, $y \in A$ such that for all $P_i \in \mathbf{D}$ with $r_1(P_i) = x$, one has $r_2(P_i) = y$.

The following example is an illustration of minimal richness and the unique seconds property.

Example 2. Let $A = \{a, b, d, c\}$ be the alternative set and

$$P_s = \begin{matrix} a \\ b \\ c \\ d \end{matrix}$$

be the standard ordering. Below are the orderings grouped according to their distance to the standard ordering.

$$\underbrace{\begin{matrix} a \\ b \\ c \\ d \end{matrix}}_{of\ distance\ 0} \underbrace{\begin{matrix} b & a & a \\ a & c & b \\ c & b & d \\ d & d & c \end{matrix}}_{of\ distance\ 1} \underbrace{\begin{matrix} b & b & c & a & a \\ c & a & a & c & d \\ a & d & b & d & b \\ d & c & d & b & c \end{matrix}}_{of\ distance\ 2} \underbrace{\begin{matrix} c & d & a & b & b & c \\ b & a & d & c & d & a \\ a & b & c & d & a & d \\ d & c & b & a & c & b \end{matrix}}_{of\ distance\ 3}$$

$$\underbrace{\begin{matrix} c & c & b & d & d \\ b & d & d & b & a \\ d & a & c & a & c \\ a & b & a & c & b \end{matrix}}_{of\ distance} \underbrace{\begin{matrix} d & d & c \\ c & b & d \\ a & c & b \\ b & a & a \end{matrix}}_{of\ distance\ 5} \underbrace{\begin{matrix} d \\ c \\ b \\ a \end{matrix}}_{of\ distance\ 6}$$

Note that $\mathbf{D}_3(P_s)$ is minimally rich and has the unique seconds property property with $x = d$ and $y = a$. $\mathbf{D}_6(P_s)$ is the entire domain. Now forth, we will write D_k instead of $D_k(P_s)$ to simplify notation.

CHAPTER 3

MAIN RESULTS

In this chapter we will keep a particular ordering $P_s \in \mathbf{P}$ fixed to represent a norm ordering on A. We will write simply D_k for $D_k(P_s)$.

Theorem 1. *Let A be an alternative set with n elements and I is the individual set with N members. For $k \in \{1, ..., n-2\}$, if $f : \boldsymbol{D}_k{}^N \rightarrow A$ is surjective, then f is not Maskin monotonic.*

Proof. First observe that for $k \in \{1, ..., n-2\}$, $\mathbf{D}_k$ is not minimally rich, since we need $n-1$ steps for the bottom alternative in the norm ordering P_s to reach the top. Now assume that f is surjective. Suppose also that f is Maskin monotonic. Then there exists a preference profile P such that $f(P) = r_1(P_s) = a$. Let $P_i^{'} \in D_k$ be the linear ordering in D_k where a has the highest possible rank, i.e. $a = r_{n-k}(P_i^{'})$. Let $\tilde{P} \in D_k{}^N$ be such that $\tilde{P}_j = P_i^{'}$ for each $j \in I$. By monotonicity $f(\tilde{P}) = a$. Now since f is surjective there is also a preference profile $P^{''} \in D_k{}^N$ that is mapped to the top alternative of P_s, say $r_1(P_s) = b$. Now $\tilde{P}$ is such that $r_1(\tilde{P}_j) = b$ for all $j \in I$, so that $f(\tilde{P}) = b$ as well by monotonicity. So this contradiction implies that f i not Maskin monotonic.

□

Lemma 1. *For $|A| = n$, D_k has the unique seconds property and is minimally rich with $k \in \{1, 2, ... \frac{n(n-1)}{2}\}$ iff it is $k = n-1$.*

Proof. Obviously for all $a \in A$ there exists $P_i \in \mathbf{D}_{n-1}$ such that $r_1(P_i) = a$ which simply means that D_{n-1} is minimally rich. Furthermore, when the bottom alternative of P_s goes to the top the only alternative can come after that in D_{n-1}, is the top alternative of P_s, so that the domain has the unique seconds property with the bottom alternative of P_s as x and the top alternative of P_s as y in the definition. Now for the proof of the only if part, we know that we need at least $n-1$ steps for the bottom alternative to the top. So, minimal richness implies that $k \in \{n-1, ..., \frac{n(n-1)}{2}\}$. However, it is clear that D_{n-1} is the only one having unique seconds property among these domains.

□

Lemma 2. *Let* $\mathbf{D}$ *be a minimally rich domain with the unique seconds property. Then for any* $N \geq 2$ *there exists a non-dictatorial, surjective, Maskin monotonic SCF* $f : \mathbf{D}^N \to A$.

Proof. As D has the unique seconds property there exist $a, b \in A$.Define $f : \mathbf{D}^N \to A$ by such that $r_2(P_i) = b$ whenever $P_i \in \mathbf{D}$ with $r_1(P_i) = a$.

$$f(P_1, P_2, ..., P_n) = \begin{cases} r_1(P_1)\ if; & r_1(P_1) \neq a \\ \text{x}\ \ if; & r_1(P_1) = a \end{cases}$$

where xP_2y with $\{x, y\} = \{a, b\}$

Clearly f is non-dictatorial and surjective. We claim that it is Maskin monotonic.

Case1: Let $f(P_1 \times P_2) = \gamma$ where $\gamma \neq\ a\, or\, b$ then $r_1(P_1) = \gamma$. Assume $\widetilde{P}_1 \times \widetilde{R}_2$ be an γ-improvement of $P_1 \times P_2$. Clearly $f(\widetilde{P}_1 \times \widetilde{P}_2) = \gamma$ since γ-improvement of P_1 will continue to start with γ.

Case 2: Now let $f(P_1 \times P_2) = b$, so either $r_1(P_1) = b$ or $r_1(P_1) = a$ and bR_2a. For the first case we know that for any b-improvement of P_1, $\tilde{P}_1$ and any b-improvement of P_2, $\tilde{P}_2$ we will have $f(\tilde{P}_1 \times \tilde{P_1}2) = b$. For the second case we can observe that for b-improvement of P_2 always bP_2a, for P_1 we have that $r_1(P_1) = a$ and $r_2(P_1) = b$ from unique seconds property. So either $r_1(\tilde{P_1}) = a$, $r_2(\tilde{P_1}) = b$ or $r_1(\tilde{P_1}) = b$ where $\tilde{P_1}$ is b-improvement of P_1. In both cases we can see that b continues to be chosen.

Case 3: Now assume $f(P_1 \times P_2) = a$. There is only one state to realize this case. It is $r_1(P_1) = a$ and aP_2b. For any a-improvement $\tilde{P_1}$ of P_1, $r_1(\tilde{P_1}) = a$ and since $a\tilde{P_2}b$, $\tilde{P_2}$ a-improvement of P_2, a will continue to be chosen.

$\square$

Theorem 2. *Let $f : {D_n}^N \to A$ be a Maskin monotone and onto SCF, then f is dictatorial.*

Proof. In the proof of the this theorem we assume that $A = \{a, b, c, ...\}$ and standard ordering is

$$\begin{matrix} a \\ b \\ c \\ \vdots \end{matrix}$$

We argue by induction on N. For N=1 we know that the theorem is correct. Since there is only one person, he must be the dictator. Then we assume that the theorem is correct for $N = K$ and we will prove it for $N = K + 1$.

Let $F : {D_n}^{K+1} \to A$ be a Maskin monotone and onto SCF. We define $f : {D_n}^K \to A$ by $f(P_1, ..., P_K)$=$F(P_1, ..., P_K, P_K)$. Now we observe that f is onto. Since F is onto and F is Maskin monotone then it is Maskin monotone and unanimous. For each alternative $x \in A$ we can move x to the top position, by unanimity all $x \in A$ can be chosen. f is also Maskin monotone. Because $\forall x \in A$ such that $f(P_1, ..., P_N) = x$, consider that $Q_1, ..., Q_K$ are elementary improvements of x. Now since $F(Q_1, ..., Q_K, Q_K)$= $f(Q_1, ..., Q_K)$=x, f must be Maskin monotone. By our induction assumption f is dictatorial.

Assume that 1^{st} person of f is dictator. This implies that 1^{st} person of F is dictator. Given $x \in A$

$$f(\begin{matrix} x, & \vdots, & \cdots, & \vdots \\ \vdots & x & \ddots & x \end{matrix}) = x = F(\begin{matrix} x, & \vdots, & \cdots, & \vdots, & \vdots \\ \vdots & x & \ddots & x & x \end{matrix})$$

Here above we assume that P_N and P_{N+1} are same. Even if we allow P_N and P_{N+1} to be distinct, we have

$$F(\begin{matrix} x, & \vdots, & \cdots, & \vdots, & \vdots \\ \vdots & x & \ddots & x & x \end{matrix}) = x$$

So we prove that 1^{st} person of F is dictator.

Similarly if our case is that $2^{nd}, 3^{rd}, ..., N-1^{st}$ person is dictator of f, corresponding person of F is dictator. Now assume that N^{th} person of f is dictator. This implies that,

$f(P_1, ..., P_N) = r_1(P_N) = F(P_1, ..., P_N, P_N)$.

That is, if N^{th} and $N+1^{th}$ preferences are same then we are done. However they do not have to be same.

Consider if $r_1(P_N) = r_1(P_{N+1})$ then $F(P_1, ..., P_N, P_{N+1}) = r_1(P_N)$. Because

$$F(\begin{matrix} \cdot & \cdot & \cdots & x & x \\ \vdots & \vdots & \ddots & \vdots & \vdots \end{matrix}) = x$$

If we change one of the P_N and P_{N+1} such that x is still at the top, then it will be an x improvement. So F still gives x. So we have shown that N^{th} and $N+1^{th}$ are joint dictators in the sense that if their first preferences are same then F gives it.

Now define $S_{a,b,c} = \{P \in D_n : r_1(P), r_2(P) \in \{a,b,c\}\}$

We claim that $F|_{S_{a,b,c}^{N+1}}$ has image exactly $\{a,b,c\}$ It is clear that $F|_{S_{a,b,c}^{N+1}}$ has image $\supseteq \{a,b,c\}$ because of unanimity. Assume that $F(P_1, ..., P_{N+1}) = w$ where $P_i \in S_{a,b,c}$ for $i = 1, 2, ..., N+1$ but $w \notin \{a,b,c\}$.

So we have that

$$F(P_1, ..., P_{N-1}, \begin{array}{cc} x & \alpha \\ y & \beta \\ \vdots & \vdots \\ w & w \\ \vdots & \vdots \end{array}) = w$$

such that $x, y, \alpha, \beta \in \{a, b, c\}$ but $w \notin \{a, b, c\}$. This means that

$$F(P_1, ..., P_{N-1}, \begin{array}{cc} x & \alpha \\ y & \beta \\ w & w \\ \vdots & \vdots \end{array}) = w$$

because

$$\begin{array}{cc} x & \alpha \\ y & \beta \\ w & w \\ \vdots & \vdots \end{array} \text{are respectively w-improvement of} \begin{array}{cc} x & \alpha \\ y & \beta \\ \vdots & \vdots \\ w & w \\ \vdots & \vdots \end{array})$$

Remark: Here we allow n distance, so after

$$\begin{array}{cccccc} a & b & a & c & b & c \\ b & a & c & a & c & b \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \text{any other element of A can follow}$$

But at least two of the top alternatives are same. So we can put them to the top. This gives a contradiction to joint dictatoriality of N^{th} and $N+1^{th}$ people. Thus $F|_{S^{N+1}_{a,b,c}}$ has image $\{a, b, c\}$.

Now let

$$S = \left\{ \begin{array}{cccccc} a & b & a & c & b & c \\ b & a & c & a & c & b \\ \vdots, & \vdots, & \vdots, & \vdots, & \vdots, & \vdots \\ c & c & b & b & a & a \end{array} \right\}$$

Let $g : L(\{a,b,c\})^{N+1} \to \{a,b,c\}$ be the corresponding function. That is, for example

$$g\left(\begin{array}{cccc} a & c & & b \\ b, & b, & \ldots, & a \\ c & a & & c \end{array} \right) = F\left(\begin{array}{cccc} a & c & & b \\ b, & b, & & a \\ \vdots & \vdots & ,\ldots, & \vdots \\ c & a & & c \end{array} \right)$$

It is not difficult to see that g is Maskin monotone and onto. Since it is defined from a full domain by Mueller-Satterthwaite Theorem it must be dictatorial. We also have the following for all a, b, c.

$$g\left(\begin{array}{ccccc} a & & a & c & c \\ b, & \ldots, & b, & a, & a \\ c & & c & b & b \end{array} \right) = F\left(\begin{array}{ccccc} a & & a & c & c \\ b, & & b, & a, & a \\ \vdots & \ldots, & \vdots & \vdots & \vdots \\ c & & c & b & b \end{array} \right) = c$$

So either N^{th} or $N+1^{th}$ people of g is dictator. Without loss of generality we will assume that N^{th} person is the dictator. Now note that

$$g\left(\begin{array}{ccccc} b & & b & a & b \\ c, & \ldots, & c, & b, & c \\ a & & a & c & a \end{array} \right) = F\left(\begin{array}{ccccc} b & & b & a & b \\ c, & & c, & b, & c \\ \vdots & \ldots, & \vdots & \vdots & \vdots \\ a & & a & c & a \end{array} \right) = a$$

$$\text{That is } F\left(\begin{array}{c} \vdots, \\ \\ a \end{array} \ldots, \begin{array}{c} \vdots, \\ \\ a \end{array} \begin{array}{c} a, \\ \vdots \\ \end{array} \begin{array}{c} \vdots \\ \\ a \end{array}\right) = a$$

By Maskin monotonicity, whenever K^{th} person wants a first, F gives a. Similar argument is valid for b and c. So we have that K^{th} person is a, b, c dictator. (i.e if K^{th} person wants a, b or c first F gives it).

Fix $w \in A$ such that $w \notin \{a, b, c\}$. We define $S_{a,b,w} = \{P \in D_n : \ r_1(P), r_2(P) \in \{a, b, w\}\}$. We claim that $F|_{\{a,b,w\}}$ has image $\{a, b, w\}$. If not $\exists \ z \notin \{a, b, w\}$ such that

$$F(P_1, \ldots, P_{K-1}, \begin{array}{cc} x & \alpha \\ y & \beta \\ \vdots & \vdots \\ z & z \\ \vdots & \vdots \end{array}) = z, \ \text{ where } x, y, \alpha, \beta \in \{a, b, w\}.$$

Clearly $x \neq a, \ x \neq b$, because N^{th} person is a, b-dictator. So $x = w$ and $y = a$ or $y = b$. Hence $F|_{S_{a,b,w}{}^{K+1}}$ has image $\{a, b, w\}$. Now construct a set S with the following property:

$$S = \left(\begin{array}{c} a \\ b, \\ \vdots \\ w \end{array} \begin{array}{ccccc} b & a & w & b & w \\ a & w & a & w & b \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}\right)$$

The first element always exists in D_n. Similarly define $h : L(\{a, b, w\})^{K+1} \to \{a, b, w\}$ via the correspondence of $F|_{S^{N+1}}$. Again h is Maskin monotone and onto. So h is dictatorial. Moreover the following equation says that N^{th}

person of h must be dictator:

$$h\begin{pmatrix} b & b & & b & a & b \\ w, & w, & \cdots, & w, & b, & w \\ a & a & & a & w & a \end{pmatrix} = F\begin{pmatrix} b & b & & b & a & b \\ w, & w, & \cdots, & w & b & w \\ \vdots & \vdots & & \vdots & \vdots & \vdots \end{pmatrix} = a.$$

In particular

$$h\begin{pmatrix} a & & a & w & a \\ b, & \cdots, & b, & a, & b \\ w & & w & b & w \end{pmatrix} = w \Rightarrow F\begin{pmatrix} \vdots & & \vdots, & w, & \vdots \\ & ,\ldots, & & & \\ w & & w & \vdots & w \end{pmatrix} = w.$$

Now even when we allow different preferences, as long as K^{th} person chooses w first, then F gives w. Since w is arbitrary K^{th} person is a dictator. So F is dictatorial.

□

Definition. Assume A has n elements. Let D be a domain. D is said to be *top − bottom rich* if $\forall a \in A$, $\exists P, Q \in D$ such that $r_1(P) = a$ and $r_n(Q) = a$.

Theorem 3. *Let D be a top-bottom rich domain. Let $D \subset K$.*
$[f : D^N \to A\ Maskin\ monotone\ and\ onto \to f\ dictatorial] \Rightarrow [f : K^N \to A\ Maskin\ monotone\ and\ onto \to f\ dictatorial]$

Proof. Let $F : K^N \to A$ be a Maskin monotone and onto SCF. Then $F|_{D^N}$ is still Maskin monotone and onto. Wlog let 1^{st} person of $F|_{D^N}$ be the dictator. Then given $x \in A$ let $P, Q \in D$ such that $r_1(P) = r_n(Q) = x$. Since $F(P, Q, ..., Q) = x$ even when we allow different choices, all will be *x − improvements* of the previous profile. Since x was arbitrary 1^{st} person of F is dictator. So F is dictatorial. □

CHAPTER 4

CONCLUSIONS

In case a historically standard society is deemed as one where individual references cluster to some degree around a standard social ordering", we have covered all degrees of such clustering concerning the existence of a surjective, Maskin monotonic, non-dictatorial SCF in the presence of at least three alternatives. Our work partitions the collection of D_k, $k \in \{0, 1, ..., \frac{n(n-1)}{2}\}$ into three subsets. The domains D_k turn out to be sufficiently rich for the impossibility result to survive when $k \in \{n, ..., \frac{n(n-1)}{2}\}$ or the domains D_k with $k \in \{0, 1, ..., n-2\}$, we again have an impossibility result, but this time since D_k is not rich enough rather than sufficiently rich. In this range, the domain is too poor to allow every alternative to be top ranked at least once. This turns surjectivity to an overdemanding condition when conjoined with Maskin monotonicity. The most interesting case seems to be D_{n-1}, where the forces exerted by surjectivity and Maskin monotonicity seem to be "balanced," in the sense that they allow the existence of a nondictatorial SCF on D_{n-1}.

The construction of our domains D_k in the present study are based on a "unipolar society,". One may naturally also think of "bipolar societies," even though these may not be as standard as unipolar ones and may not last long.

Our present study leads to conjectures about bipolar societies that parallel the results obtained here. We imagine now two orderings which are reversal of each other to reflect opposite "social norms," around which two different parts of the society cluster. We conjecture that, in such a society split into two, it will be the sum of the allowed distances from the two opposite norm orderings that will be decisive concerning whether we will end up with possibility or impossibility results. In fact, we believe a similar taxonomy as as the one in the present study will be obtained if one replaces the allowed distance from the norm ordering here by the sum of the allowed distances from the two opposite norm orderings in a bipolar society.

BIBLIOGRAPHY

Aswal, N., Chatterji, S., Sen, A., Dictatorial Domains. Economic Theory, (2003), 45-62.

Mas-Colell, A., Whinston, M. D., Green J.R., Micreconomic Theory, New York: Oxford University press (1995).

Maskin, E., Nash Equilibrium and Welfare Optimality, mimeo, M.I.T. (1977).

Muller, E., Satterthwaite, M.A., The equivalence of strong positive association and strategy-proofness. Journal of Economic Theory, (1977), 412-418.