Quantum Mechanics of a Single Photon

by

Hassan Babaei

A Dissertation Submitted to the
Graduate School of Sciences and Engineering
in Partial Fulfillment of the Requirements for
the Degree of

Master of Science

in

Physics

KOÇ
UNIVERSITY

January 22, 2016

Quantum Mechanics of a Single Photon

Koç University
Graduate School of Sciences and Engineering
This is to certify that I have examined this copy of a master's thesis by

Hassan Babaei

and have found that it is complete and satisfactory in all respects,
and that any and all revisions required by the final
examining committee have been made.

Committee Members:

Supervisor: Prof. Dr. Ali Mostafazadeh

Prof. Dr. Tekin Dereli

Prof. Dr. Teoman Turgut

Date:

To my parents
Davoud Babaei
and
Delshad Rashid

ABSTRACT

The first quantized theory of a single photon has been the subject of interest since the early days of the emergence of quantum mechanics. Despite many successes and achievements, there is no systematic approach for quantum mechanics of a single photon. In fact there are still some major controversies and problems in this field including: a well-defined wave function in coordinate representation, a positive definite probability density, a position operator, and localizability of a single photon. In this dissertation we provide a systematic quantum mechanical treatment of photon fields and address all of these problems.

We first use dynamical equations of Maxwell fields to establish the physical space of photon fields. Next we obtain the most general admissible positive-definite inner product on this space. Imposing necessary conditions on this inner product not only makes it relativistically invariant, but also suggests a three-parameter family of Lorentz invariant inner products which can be used to introduce a genuine Hilbert space for the quantum mechanics of photon fields. We can derive the unitary quantum system by determining the generator of time-translation to be Hamiltonian. This represents a quantum mechanics of a single photon which is not the result of the restriction of Hilbert space to the positive-frequency fields. Next we obtain the generalized parity (P), generalized time-reversal (T), and generalized charge or chirality (C) operators for this system and offer a physical interpretation for its PT-, C-, and CPT-symmetries. We introduce a current density corresponding to relativistically invariant positive-definite inner product and observe that it is both conserved and manifestly covariant. We establish the mathematical formalism of observables and wave functions for the photon. We determine the photon wave function and prove that it is a generalization of both the Landau-Peierls well-known wave function for photon and the Riemann-Silberstein complex vector. Subsequently we obtain a positive-definite probability density for the spatial localization of photon fields and examine the conservation of total probability. We construct a photon position operator with commuting components. This position operator commutes with helicity and chirality operators and is equal to the position operator of massless Klein-Gordon fields with an additional term that corresponds to the Berry phase of propagating photon. As far as

we know, this is the first systematic study of the first-quantized theory of a single photon.

ÖZETÇE

Tek bir fotona ait birinci kuantizasyon kuramı, kuantum mekaniğinin ilk ortaya çıktığı zamanlardan beri her zaman ilgi konusu olmuştur. Bir çok başarı ve kazanımlara ragmen tek bir fotonun kuantum mekaniğine ait sistematik bir yaklaşım henuz oluşturulamamştır. Gerçekte ise bu alanda, koordinat gosteriminde iyi tanımlı bir dalga fonksiyonunun oluşturulması, pozitif tanımlı bir olasılık yoğunluğunun bulunması, bir konum operatoru ve tek bir fotonun localize edilebilirliği dahil olmak uzere hala bazı temel tartışmalar ve problemler mevcuttur. Bu tezde, foton alanlarının sistematik kuantum mekaniksel bir yaklaşımını ele alarak bu problemlere değineceğiz.

Foton alanlarının fiziksel uzay çozumlerini oluşturabilmek için lk unce Maxwell alanlarının dinamik denklemlerini kullanacağız. Daha sonra, foton alanlarına ait uzayda Kabul edilebilir en genel pozitif tanımlı iççarpımı elde edeceğiz. Bu iç çarpım uzerinde gerekli koşulların uygulanması sadece relativistic olarak değişmez yapmakla kalmaz aynı zamanda foton alanlarının kuantum mekaniğinde gerçek bir Hilbert uzayı tanımlamak için kullanılabilecek olan Lorentz değişmezi iç çarpımlarının uç parametreli familyasını da ortaya koyar. Uniter kuantum sistemini, zaman oteleme uretecinin Hamiltonian olarak belirlenmesi ile turetebiliriz. Bunun utesinde bu bize Hilbert uzayının pozitif frekans alanlarına sınırlandırılmasının sonucu olmayan tek bir fotona ait kuantum mekaniğini gosterir. Daha sonra, bu sistemin genelleştirilmiş eşlik (P), genelleştirilmiş zaman (T) ve genelleştirilmiş yuk veya kiralite (C) operatorlerini elde ederek PT-, C- ve CPT simetrilerinin fiziksel yorumlarını sunacağız. Relativistik olarak değişmez pozitif tanımlı iç çarpıma karşılık gelen bir akım yoğunluğu tanımlayarak bunun hem korunduğunu hem de açıkça eşdeğişken olduğunu gozlemleyeceğiz. Foton alanlarının fiziksel Hilbert uzayına ait gozlenebilirler ve dalga fonksiyonlarının matematiksel formalizmini oluşturacağız. Foton dalga fonksiyonunu belirleyerek bunun hem fotonun iyi bilinen dalga fonksiyonu olan Landau-Peierls'in genelleştirilmesine eşit olduğunu hem de Riemann-Silberstein karmaşık vektorunu verdiğini ispatlayacağız. Sonra foton alanlarının uzaysal lokalizasyonu için pozitif tanımlı bir olasılık yoğunluğu elde ederek toplam olasılığın korunumunu inceleyeceğiz. Kutleli parçacıklar için uniter olarak Newton-Wigner-Pryce konum operatorune eşit olan komutasyon yapan bileşenlere sahip foton konum operatorunu oluşturacağız. Bu konum operatoru,

sarmallık ve kiralite operatorleri ile komutasyon yaparlar ve yayılan fotonun Berry fazına karşılık gelen ek bir terime sahip olan kutlesiz Klein-Gordon alanlarının konum operatorunu verirler. Bilindiği kadarıyla bu, tek bir fotonun birinci kuantizasyon kuramna ait ilk sistematik çalışmadır.

ACKNOWLEDGMENTS

I would like to express my deepest gratitude for my supervisor Prof. Dr. Ali Mostafazadeh for his guidance and encouragement throughout my master thesis. His invaluable comments and observations during my research and study helped me to work very productively and enabled me to overcome all difficulties in solving this important problem. I would like to thank him not only for his endless supports, but also for his excellent advices, critics, patience and motivation as an inspiring mentor and advisor who helped me to learn and grow as a researcher.

I would like to thank Prof. Dr. Tekin Dereli for his extraordinary supports during my master studies. His dedication and willingness to teach theoretical physics courses together with his patience and helpfulness as a graduate advisor makes him one of my greatest professors. I immensely appreciate him for his generousness in providing graduate full scholarship award to help me fulfill my studies and thesis research as best as possible.

I would like to acknowledge Prof. Dr. Teoman Turgut for being one of my thesis committee members and for his time to read throughout my thesis.

I am deeply indebted to Dr. Victoria Taylor, my academic writing professor, and my friend Mr. Tyler Wolford for reading my thesis, revising, and making extremely useful comments in many parts. I would like to thank my friends Hamed Ghaemi, Mustafa Sarisaman, Armin Hodaei, and Mahan Rajaei for reading the first draft of my thesis and making useful comments on that.

Finally, I would like to thank my family whose motivations and encouragements helped me to overcome all challenges and difficulties that I faced during my studies. I especially want to thank my parents for all supports and love they have given me.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

NOMENCLATURE

Chapter 1

INTRODUCTION

1.1 Introduction

Since early decades of 20th century, many studies have been done on the first quantized theory of a single photon. Yet a systematic quantum mechanical treatment of a single photon is not fully developed. There are many controversial problems in this area including difficulties in defining the photon wave function in coordinate representation, a positive-definite probability density, the existence of a position operator, the corresponding localized states, and a genuine Hilbert space for the photon.

Because the photon momentum operator is well-defined, the concept of photon wave function in momentum representation has been widely accepted [1] and most researchers who work in quantum electrodynamics agree upon its definition. The quantum mechanics of a single photon in momentum representation is presented in some textbooks of quantum electrodynamics [2]. In contrast to the photon wave function in momentum representation, it is a very challenging problem to define an appropriate wave function for a single photon in coordinate representation.

The wave function of the photon has never been systematically studied and is a controversial concept. This difficulty arises from the fact that it does not have all features of the Schrödinger wave function for nonrelativistic quantum mechanics. This has led some people to claim that the photon wave function does not exist [1]. However this has not kept physicists doing research to find an appropriate position wave function for the photon. The first major study in the area of the first quantized theory of a single photon was carried out by Landau and Peierls [3] who determined the photon wave function and derived Schrodinger equations for an electromagnetic field and its interaction with matter in a configuration space. The same wave function has been rediscovered by Cook [4] and Inagaki [5].

The Landau-Peierls (LP) wave function has some advantages. For example it has the right dimension [1], and the scalar product and expectation values take simpler forms when they are expressed in terms of LP function. Nevertheless, it is not widely accepted because it is a nonlocal function. In [6] Pauli explained that the LP wave function has some disadvantages due to having such a nonlocal character; (1) the value of a nonlocal function at a point depends on the behavior of function in all spatial points of space and not only on a small neighborhood of the considered point [6, 1]. (2) It does not transform as a tensor field under the Lorentz transformation [6, 1]. (3) One cannot introduce the probability of interaction of a photon with localized charges by using the probability density of this nonlocal object [6, 1, 7]. According to Pauli [6], for the photon the wave function in coordinate representation does not exist.

Despite Pauli's conclusion many physicists have strong opinions supporting the existence of the photon wave function as discussed by Bialynicki-Birula [1] and Sipe [7]. According to these authors, due to the nonlocal nature of the LP function, it only can be considered as a secondary function which connects to a local wave function through a nonlocal transformation. Many local wave functions have been proposed to serve as the photon wave function. All of them are specified through Maxwell equations [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. The most famous example is the Riemann-Silberstein wave function $\mathbf{E} + ic\mathbf{B}$ which satisfies a complex form of the Maxwell equations [8, 9, 10, 11, 12, 13, 16]. The name of the Riemann-Silberstein for this complex vector stands for the earliest works in the use of the complex form of Maxwell equations by Reimann [1, 19] and Silberstein [1, 20]. The Riemann-Silberstein complex vector has been frequently used and advocated as the photon wave function by many physicists including Good [18], Sipe [7], Moliere [21], Oppenheimer [22] and Bialynicki-Birula [1, 8, 9, 10]. Despite its convenience and popularity, the absolute-value square of this wave function provides the local energy density and cannot be taken as the probability density of a single photon.

The problem of the photon wave function is linked with that of photon's position operator. In the formulation of quantum mechanics, the wave function can be identified by the projection of a state vector on eigenvectors of the corresponding position operator. Therefore, many people including Pauli [6] have made claims on the nonexistence of a position operator. Another challenging concept is the problem of photon's localized states which is closely related to a corresponding position operator. This becomes evident through the work of Newton and Wigner [23] on the localization of elementary particles in which they established a set of localized states for elementary particles including massive particles with arbitrary spin and massless particles with spin<1. For massless particles with higher but finite spin, starting with s=1, they did not find any localized states. In the context of their work [23], a single photon cannot be localized in a spatial point and its associated position operator does not exist.

The failure of Newton and Wigner's method to address the problem of localized states and the position operator of the photon did not stop people to search for them. The first attempt for finding a relativistic position operator is due to Pryce [24] who used several ways to generalize the definition of center of mass and obtained position operators for elementary particles including electrons and photons. The Pryce position operator [24] for the photon has not been faced with great interest because it does not have commuting components. Many other proposals for the photon position operator have been given that have the same common deficiency of lacking non-commuting components [25, 26, 27, 28, 29]. Recently Hawton [11] obtained a position operator for the photon with commuting components by adding an extra term to the Pryce operator. The main shortcoming of her approach is that it does not involve building a physical Hilbert space for a single photon.

Similar to the photon wave function and position operator, the localization of a single photon is a very old and important problem. After Newton and Wigner's influential work [23] on localization of elementary particles, Wightman [31] proposed a reformulated version of their physical ideas. This involved a notion of localizability in a region whose existence and uniqueness only relies on the transformation law of the physical system under the Euclidean group. He rederived all previous results of Newton and Wigner [23] and gave a mathematically rigorous proof of the nonexistence of localized states for a single photon [30]. In the context of imprimitive language proposed by Wightman [31] and Mackey [32] for the idea of localizability of elementary particles, Rosewarne and Sarkar [33] showed that for a single photon the imprimitivity of the Euclidean group is not consistent with the representation of the Poincare group.

Later, the definition of localizability is generalized in the work of Jauch and Piron [34] so that it can be applied to cases in which the conventional position operator does not exist. In their notion of generalized system of imprimitivities, the localizability of a particle can be restated as weakly localizable. Subsequently, Amrein [35] examined the consequences of the weak localizability introduced by Jauch and Piron [34] and showed that massless particles can be treated in an equal footing with massive ones and a single photon can be localized in arbitrary small volumes. More recently, the localization problem of the photon has been the subject of interest and several investigations have been made by different people [10, 12, 16, 25, 36, 37]. The most recent one is the work of Hawton [30] where she employs her position operator [11] to obtain localized states of the photon in momentum representation.

Although we do not have a well-defined quantum mechanical formulation for a single photon and each approach suffers from not being able to give a compatible and systematic way of defining wave functions and physical observables, for a long time experimental physicists have strived to establish a proper wave function for the photon by which they could verify the results of their experiments; for instance see [37, 38, 17]. Furthermore, recently a single photon state has become a favorable subject of research in optics and other related disciplines, and a single photon state has been detected in numerous experiments [37, 39]. In recent years the first quantized theory of a single photon has received more attention and many have tried to develop the concept of physical observables and the position wave function for a single photon. The most recent ones are related to Hawton [13, 40, 41] where she tries to develop the wave mechanics of the photon and the position observable by using pseudo-hermitian quantum mechanics. However like most previous investigations, her works lack a systematic quantum mechanical treatment by which one should first construct a genuine Hilbert space and then obtain a unitary quantum system for a first-quantized photon.

In this thesis we approach this problem in a systematic manner and provide a well-defined first quantized theory of a single photon. Unlike the previous works on this subject where they identify the solutions of the Maxwell equations (for instance the Riemann-Silberstein complex vector) as the photon wave function, we view the solutions of the Maxwell equations as abstract state vectors belonging to a complex vector space that we identify with the space of photon fields. By defining an appropriate inner product and endowing it to this space, we obtain the physical Hilbert space of a single photon. In this physical Hilbert space we define the position wave function associated with a single photon in a usual way, as a coefficient of a state vector in position basis of the physical Hilbert space. In a similar way we construct the physical observables of the unitary quantum system corresponding to the physical Hilbert space. In Refs [50, 51, 53], A. Mostafazadeh and F. Zamani employed this systematic methodology to provide a consistent quantum mechanical treatment of Klein-Gordon and Proca fields. These works provide both conceptual framework and motivation for the study of the first quantized theory of a single photon.

Due to a close relationship between Maxwell fields and Proca fields, we examine the methodology used for Proca fields [51] with some variations. In fact setting mass to zero in massive vector fields (Proca fields) leads to the Maxwell equations. Thus, one may attempt to apply the zero mass limit of Proca fields in order to investigate the quantum mechanics of a single photon. However, this leads to unpleasant consequences because the Hilbert space obtained by taking zero mass limit of Proca fields does not represent the Hilbert space of a single photon. Instead of that, from the beginning we start with Maxwell equations and derive a unitary quantum system for the photon. In the following we briefly explain our approach in more detail.

In recent years Mostafazadeh used his theory of Pseudo-Hermitian Quantum Mechanics (PHQM) [47, 48, 62, 63, 64, 65, 68, 69, 70] to address the Hilbert space problem in quantum cosmology [49, 70] and similarly devised a consistent quantum mechanical formulation of Klein-Gordon fields [50, 53] and Proca fields [51]. PHQM [47, 48, 62, 63, 64, 65, 66, 67] is a powerful method developed to provide a mathematically compatible formulation for PT-symmetric quantum systems [71, 72]. In this formulation [73] each quantum system is described by a diagonalizable Hamiltonian H [48] with a real spectrum which acts in a Hilbert space $\mathcal{H}$. Generally speaking, the Hamiltonian H is not necessarily Hermitian with respect to the inner product of $\mathcal{H}$ that we denote by $\langle\cdot,\cdot\rangle$. However it becomes Hermitian with respect to a positive-definite inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\eta_+}$ defined according to

$$\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\eta_+} := \langle\cdot,\eta_+\cdot\rangle \tag{1.1}$$

where η_+ is a positive-definite metric operator which is linear, hermitian and invertible [48, 47]. This in turn suggests the notion of a pseudo-Hermitian operator. A linear operator H is called η_+-pseudo-Hermitian if it satisfies [47, 62, 73],

$$H^\dagger := \eta_+ H \eta_+^{-1}. \tag{1.2}$$

The inner product (1.1) can be used to establish a physical Hilbert space and observables can be identified with Hermitian operators acting in this space [68].

In the reminder of this section we outline the structure of this thesis. In the second chapter, we review the literature on the subject of the first-quantized theory of a single photon and elaborate on the concepts of the wave function, physical observables, the position operator and localized states. Furthermore, we discuss the difficulties with the current formulations.

As discussed in Refs [50, 53] PHQM was used to provide a unitary quantum mechanical formulation with a genuine probabilistic interpretation for Klein-Gordon fields. Here the author considers massive scalar fields. In the third chapter, we employ a variation of PHQM method to devise a unitary first-quantized quantum theory for Klein-Gordon fields where we include the massless KG fields as well as massive ones. Most of results obtained in this section are identical with those of Refs [50, 53]. We have added this to help the reader become familiar with the method we use to study a first-quantized photon. Furthermore, because the dynamical equations satisfied by the massless scalar and vector fields are very similar, we expect similar features in the structure of the inner product, the position operator, and localized states.

In chapter 4, we employ the same method used in the Chapter 3 and formulate a unitary first-quantized theory for a single photon. We use the solutions of Maxwell equations to establish the space of photon fields. Next we equip the space of photon fields with an appropriate inner product which we call the inner product of a single photon and by Cauchy completing the resulting inner product space we arrive at the physical Hilbert space of a single photon. If we identify the Hamiltonian to be the generator of time-translations, it uniquely determines a unitary quantum system which we use to construct the physical observables and wave functions. Next, we obtain the position operator of a photon with commuting components and discuss its connection with the Hawton operator. We obtain the photon position wave function and show how it relates to the Landau-Peierls and the Riemann-Silberstein wave functions. We further derive a positive-definite probability density and remark on the conversation of the probability.

In chapter 5, we examine differences between our approach and previous works and discuss the advantages of our method.

1.2 Conventions

In this section we state the conventions which we frequently use throughout this thesis. First we define σ_0 and λ_0 as 2×2 and 3×3 identity matrices, respectively. Let us remember that σ_0 together with the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.3}$$

form a basis of the vector space of 2×2 complex matrices, and $\{\sigma_0, \sigma_3\}$ is a maximal set of commuting matrices. We denote their common eigenvectors by

$$e_+ := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad e_- := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{1.4}$$

where the labels stand for the eigenvalues of σ_3.

In a similar way, λ_0 together with

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \ \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \ \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \tag{1.5}$$

form a basis for the vector space of 3×3 complex matrices, and $\{\lambda_0, \lambda_3, \lambda_8\}$ is a maximal set of commuting matrices. Similarly their common eigenvectors are given as

$$e_{+1} := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad e_{-1} := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad e_0 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{1.6}$$

in which the labels 0 and ± 1 are the eigenvalues of λ_3.

Let us observe that we can employ the bases $\{\sigma_i\}$ and $\{\lambda_j\}$ to construct a basis $\{\Sigma_\mathfrak{m}\}$ for the vector space of 6×6 complex matrices namely,

$$\Sigma_\mathfrak{m} := \sigma_i \otimes \lambda_j, \quad \text{if} \quad \mathfrak{m} = i + 4j, \tag{1.7}$$

where $\mathfrak{m} \in \{0, 1, 2, \cdots, 35\}$. The matrices Σ_0, Σ_3, Σ_{12}, Σ_{15}, Σ_{32}, and Σ_{35} form a maximal set of commuting operators with common eigenvectors

$$e_{\epsilon,s} := e_\epsilon \otimes e_s, \tag{1.8}$$

where $\epsilon \in \{+, -\}$ and $s \in \{-1, 0, +1\}$.

Next, we use $\{\sigma_i\}$ and $\{\sigma_j\}$ to introduce the basis $\{\tilde{\Sigma}_\mathfrak{n}\}$ for the vector space of 4×4 complex matrices that consist of

$$\tilde{\Sigma}_\mathfrak{n} := \sigma_i \otimes \sigma_j, \quad \text{if} \quad \mathfrak{n} = i + 4j \in \{0, 1, 2, \cdots, 15\} \tag{1.9}$$

The matrices $\tilde{\Sigma}_0$, $\tilde{\Sigma}_3$, $\tilde{\Sigma}_{12}$ and $\tilde{\Sigma}_{15}$ form a maximal set of commuting operators with common eigenvectors

$$\tilde{e}_{\epsilon,h} := e_\epsilon \otimes \tilde{e}_h, \tag{1.10}$$

where $\epsilon \in \{+, -\}$ and $\mathrm{h} \in \{-1, +1\}$ and

$$\tilde{e}_{+1} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \tilde{e}_{-1} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{1.11}$$

are the eigenvectors of σ_3.

Chapter 2

REVIEW OF THE FIRST QUANTIZED THEORY OF PHOTON

2.1 Photon Wave function

The position wave function in non-relativistic quantum mechanics can be described by a projection of a state vector ψ on eigenstates of a position operator. This is not the approach taken by quantum optics researchers who have tried to introduce an appropriate wave function in an attempt to verify their experimental results.

Historically, the first influential research in this subject is due to Landau and Peierls [3, 1] who obtained the Schrödinger equation for an electromagnetic field interacting with matter and proposed the position wave function

$$\left[\mathfrak{K}^{-1/2} f\right](\mathbf{x}) = \pi \int \frac{f(\mathbf{x}')d^3\mathbf{x}'}{(2\pi|\mathbf{x}-\mathbf{x}'|)^{\frac{5}{2}}}, \tag{2.1}$$

where $[\mathfrak{K}f](\mathbf{x}) := -i\boldsymbol{\nabla} f(\mathbf{x})$ for all $f \in L^2(\mathbb{R}^3)$. This wave function has been rediscovered by Cook [4] and Inagaki [5].

Landau-Peierls (LP) wave function has some advantages. For instance its use simplifies the computation of scalar products and expectation values for the photon. Also the modulus squared of the LP wave function has the right dimensionality and it can be interpreted as the probability density of finding a single photon. Nevertheless, the LP function does not have many advocates because of its highly nonlocal character. According to Pauli [6, 1], it behaves non-causally during spontaneous emission by an excited atom, and the value at a point depends on its behavior in all points of space and not on a small neighborhood of the considered point. Also it does not transform under Lorentz transformations as a geometric object. As noted by Pauli, one may use the LP wave function as a secondary object which is associated to a local wave function $f(\mathbf{x})$ by a nonlocal transformation (2.1).

In order to define an appropriate position wave function for the photon, many candidates have been proposed. They are closely identified by the electromagnetic fields. The most famous candidate for the position wave function of a photon is the

Reimann-Silberstein (RS) wave function [1]. It is given by

$$\mathbf{F}(\mathbf{x},x^0) = \mathbf{E}(\mathbf{x},x^0) + i\mathbf{B}(\mathbf{x},x^0). \tag{2.2}$$

This is a three-components complex vector that has been frequently used as the photon wave function by many physicists [8, 9, 10, 11, 12, 13, 16]. Notice that (2.2) does not have the difficulties of the Landau-Peierls wave function, but one cannot use it to introduce a well-defined probability density.

Next, let us start with the Maxwell equations

$$\begin{array}{ll} \boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\rho, & \boldsymbol{\nabla}\cdot\mathbf{B} = 0, \\ \boldsymbol{\nabla}\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, & \boldsymbol{\nabla}\times\mathbf{B} = \frac{1}{c}(4\pi\mathbf{J} + \frac{\partial\mathbf{E}}{\partial t}). \end{array} \tag{2.3}$$

Note that here we use Gaussian units. In free space ($\rho = 0$ and $\mathbf{J} = 0$), the equation (2.3) gives

$$\begin{array}{ll} \boldsymbol{\nabla}\cdot\mathbf{E} = 0, & \boldsymbol{\nabla}\cdot\mathbf{B} = 0, \\ \boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial x^0}, & \boldsymbol{\nabla}\times\mathbf{B} = \frac{\partial\mathbf{E}}{\partial x^0}, \end{array} \tag{2.4}$$

where $x^0 := ct$ and $x := (\mathbf{x}, x^0)$. Now, if we write the Maxwell equations (2.4) using the Reimann-Silberstein wave function, it yields

$$i\frac{\partial\mathbf{F}(\mathbf{x},x^0)}{\partial x^0} = \boldsymbol{\nabla}\times\mathbf{F}(\mathbf{x},x^0), \tag{2.5}$$

$$\boldsymbol{\nabla}\cdot\mathbf{F}(\mathbf{x},x^0) = 0. \tag{2.6}$$

Equation (2.5) is considered as the wave equation for the photon [5].

According to Refs [21, 22], one can use spin-1 matrices,

$$S_x := i\lambda_7, \qquad S_y := -i\lambda_5, \qquad S_z := i\lambda_2, \tag{2.7}$$

and the vector identity $\mathbf{a}\times\mathbf{b} = -i(\mathbf{a}\cdot\mathbf{S})\mathbf{b}$ to rewrite the photon wave equation (2.5) in the form:

$$i\hbar\frac{\partial\mathbf{F}(\mathbf{x},x^0)}{\partial x^0} = (\mathbf{S}\cdot\frac{\hbar}{i}\boldsymbol{\nabla})\mathbf{F}(\mathbf{x},x^0). \tag{2.8}$$

This is the Schrödinger equation for the photon, and the Hamiltonian is given by

$$H := \mathbf{S}\cdot\frac{\hbar}{i}\boldsymbol{\nabla}. \tag{2.9}$$

Equation (2.6) can also be expressed in terms of spin-1 matrices as

$$(\mathbf{S}\cdot\boldsymbol{\nabla})S_j\mathbf{F}(\mathbf{x},x^0) = \boldsymbol{\nabla}_j\mathbf{F}(\mathbf{x},x^0), \tag{2.10}$$

or equivalently [24]

$$(\mathbf{S}\cdot\boldsymbol{\nabla})^2\mathbf{F}(\mathbf{x},x^0) = \boldsymbol{\Delta}\mathbf{F}(\mathbf{x},x^0). \tag{2.11}$$

In non-relativistic quantum mechanics, stationary states have particular importance, because any solution of the wave equation can be expressed in terms of these. In the wave equation for the photon (2.8), one can derive the stationary states by splitting time variable and solving the resulting eigenvalue equation for the photon. This yields

$$(\mathbf{S}\cdot\frac{\hbar}{i}\boldsymbol{\nabla})\mathbf{F}(\mathbf{x}) = \hbar\omega\mathbf{F}(\mathbf{x}). \tag{2.12}$$

Here we assume that the photon energy $\hbar\omega$ is positive. Changing the sign of i to $-i$ in the RS wave function leads to the negative energy. In order to consider contributions of both positive and negative energies, we should use both the Riemann-Silberstein three-components complex vectors $\mathbf{F}_+(\mathbf{x},x^0)$ and $\mathbf{F}_-(\mathbf{x},x^0)$ [1] that are defined by

$$\mathbf{F}_\pm(\mathbf{x},x^0) := \mathbf{E}(\mathbf{x},x^0) \pm i\mathbf{B}(\mathbf{x},x^0), \tag{2.13}$$

and they satisfy the following wave equation,

$$i\hbar\frac{\partial\mathbf{F}_\pm(\mathbf{x},x^0)}{\partial x^0} = \pm(\mathbf{S}\cdot\frac{\hbar}{i}\boldsymbol{\nabla})\mathbf{F}_\pm(\mathbf{x},x^0). \tag{2.14}$$

We can combine $\mathbf{F}_+$ and $\mathbf{F}_-$ in the form of a six-components wave equation which includes the information of both positive and negative energy states. This is given by

$$\mathfrak{F}(\mathbf{x},x^0) = \begin{pmatrix} \mathbf{F}_+(\mathbf{x},x^0) \\ \mathbf{F}_-(\mathbf{x},x^0) \end{pmatrix}. \tag{2.15}$$

From point of view of relativistic quantum mechanics, solutions of the wave equation with positive and negative energies are associated with particles and anti-particles, respectively. As photons are identical with their anti-particles, one may eliminate the negative energy part because the information of the negative energy solutions is the same as the information carried out by the positive energy solutions [1]. Therefore, the positive energy part of the function $\mathfrak{F}$ is given by

$$\Psi(\mathbf{x},x^0) := \mathfrak{F}_+(\mathbf{x},x^0) = \begin{pmatrix} \mathbf{F}_+(\mathbf{x},x^0) \\ \mathbf{F}_+^*(\mathbf{x},x^0) \end{pmatrix}. \tag{2.16}$$

Note that as we see from (2.9) the Hamiltonian is a Hermitian operator with continuous spectrum. In this approach the wave functions of a photon are constructed from the positive-energy solutions of the eigenvalue problem for the Hamiltonian

$$H\Psi(\mathbf{x}) = E\Psi(\mathbf{x}). \tag{2.17}$$

2.2 Photon Wave Function as Fourier Transform and Interpretation of Fourier Coefficients

Following [1], we see that the solutions of the wave equation (2.5) for the photon are the same as the solutions of

$$(\partial_0^2 - \mathbf{\Delta})\mathbf{F}_\pm(\mathbf{x}, x^0) = 0, \tag{2.18}$$

where $\partial_0 := \frac{\partial}{\partial x^0}$. These can be expressed in terms of the superpositions of basic plane wave solutions according to

$$\mathbf{F}_+(\mathbf{x}, x^0) = \sqrt{\hbar}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[\mathbf{f}_+(\mathbf{k})e^{-i\omega x^0 + i\mathbf{k}\cdot\mathbf{x}} + \mathbf{f}_-^*(\mathbf{k})e^{i\omega x^0 - i\mathbf{k}\cdot\mathbf{x}}\right] \tag{2.19}$$

$$\mathbf{F}_-(\mathbf{x}, x^0) = \sqrt{\hbar}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[\mathbf{f}_-(\mathbf{k})e^{-i\omega x^0 + i\mathbf{k}\cdot\mathbf{x}} + \mathbf{f}_+^*(\mathbf{k})e^{i\omega x^0 - i\mathbf{k}\cdot\mathbf{x}}\right], \tag{2.20}$$

where $\omega = c|\mathbf{k}|$ and the vectors $\mathbf{F}_+$ and $\mathbf{F}_-$ are complex conjugates of each other. Furthermore, notice that $\mathbf{f}_+(\mathbf{k})$ and $\mathbf{f}_-(\mathbf{k})$ are the corresponding complex vectors in the momentum space and satisfy the following equations,

$$i\mathbf{k} \times \mathbf{f}_\pm(\mathbf{k}) = \pm|\mathbf{k}|\mathbf{f}_\pm(\mathbf{k}) \tag{2.21}$$

$$\mathbf{k} \cdot \mathbf{f}_\pm(\mathbf{k}) = 0. \tag{2.22}$$

Here we have made use of (2.19)-(2.20) in (2.5) and (2.6). $\mathbf{f}_+(\mathbf{k})$ and $\mathbf{f}_-(\mathbf{k})$ are given by

$$\mathbf{f}_+(\mathbf{k}) = \mathbf{e}(\mathbf{k}, 1)f(\mathbf{k}, 1), \qquad \mathbf{f}_-(\mathbf{k}) = \mathbf{e}(\mathbf{k}, -1)f(\mathbf{k}, -1), \tag{2.23}$$

where $f(\mathbf{k}, \lambda)$ is a complex-valued function that describes independent degrees of freedom of a free electromagnetic field for both $\lambda = \pm 1$. $\mathbf{e}(\mathbf{k}, \lambda)$ are the normalized solutions of (2.21) with plus and minus signs corresponding to positive and negative helicities, i.e,

$$i\mathbf{k} \times \mathbf{e}(\mathbf{k}, \pm 1) = \pm|\mathbf{k}|\mathbf{e}(\mathbf{k}, \pm 1), \qquad \mathbf{k} \cdot \mathbf{e}(\mathbf{k}, \pm 1) = 0. \tag{2.24}$$

Also $\mathbf{e}(\mathbf{k}, \lambda)$ satisfies the following algebraic relations

$$\mathbf{e}^*(\mathbf{k}, \lambda) \cdot \mathbf{e}(\mathbf{k}, \lambda') = \delta_{\lambda,\lambda'}, \tag{2.25}$$

$$\sum_\lambda \mathbf{e}_i^*(\mathbf{k}, \lambda) \cdot \mathbf{e}_j(\mathbf{k}, \lambda) = \delta_{i,j} - \frac{k_i k_j}{|\mathbf{k}|^2}. \tag{2.26}$$

For brevity we set $\mathbf{e}(\mathbf{k}) = \mathbf{e}(\mathbf{k}, +1)$ and $\mathbf{e}^*(\mathbf{k}) = \mathbf{e}(\mathbf{k}, -1)$. For $\lambda = \pm 1$, $\mathbf{e}(\mathbf{k}, \lambda)$ are decomposed into two real normalized vectors $\mathbf{l}_i(\mathbf{k})$ which together with $\mathbf{n}(\mathbf{k}) = \frac{\mathbf{k}}{|\mathbf{k}|}$ form an orthonormal set:

$$\mathbf{e}(\mathbf{k}, \pm 1) = \frac{1}{\sqrt{2}}(\mathbf{l}_1(\mathbf{k}) \pm i\mathbf{l}_2(\mathbf{k})), \qquad \mathbf{l}_i(\mathbf{k}) \cdot \mathbf{l}_j(\mathbf{k}) = \delta_{ij} \tag{2.27}$$

$$\mathbf{n}(\mathbf{k}) \cdot \mathbf{l}_i(\mathbf{k}) = 0, \qquad \mathbf{l}_1(\mathbf{k}) \times \mathbf{l}_2(\mathbf{k}) = \mathbf{n}(\mathbf{k}). \tag{2.28}$$

The solutions of (2.24) can be specified up to a complex phase factor, thus the multiplication of a phase factor in $\mathbf{e}(\mathbf{k})$ accounts for the rotation of the vectors $\mathbf{l}_i(\mathbf{k})$ around the vector $\mathbf{n}(\mathbf{k})$. Now, in view of (2.23), we have

$$\mathbf{F}_+(\mathbf{x}, x^0) = \sqrt{\hbar} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \mathbf{e}(\mathbf{k}) \left[f(\mathbf{k}, 1)(\mathbf{k}) e^{-i\omega x^0 + i\mathbf{k}\cdot\mathbf{x}} + f^*(\mathbf{k}, -1) e^{i\omega x^0 - i\mathbf{k}\cdot\mathbf{x}} \right], \tag{2.29}$$

$$\mathbf{F}_-(\mathbf{x}, x^0) = \sqrt{\hbar} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \mathbf{e}^*(\mathbf{k}) \left[f(\mathbf{k}, -1)(\mathbf{k}) e^{-i\omega x^0 + i\mathbf{k}\cdot\mathbf{x}} + f^*(\mathbf{k}, 1) e^{i\omega x^0 - i\mathbf{k}\cdot\mathbf{x}} \right]. \tag{2.30}$$

In the remainder of this section we offer the physical interpretation of the Fourier coefficients $f(\mathbf{k}, \lambda)$. One may use these coefficients to express the energy, momentum, angular momentum, and moment of energy of an electromagnetic field according to

$$E = \sum_\lambda \int \frac{d^3\mathbf{k}}{(2\pi)^3|\mathbf{k}|} \hbar\omega f^*(\mathbf{k}, \lambda) f(\mathbf{k}, \lambda), \tag{2.31}$$

$$\mathbf{P} = \sum_\lambda \int \frac{d^3\mathbf{k}}{(2\pi)^3|\mathbf{k}|} \hbar\mathbf{k} f^*(\mathbf{k}, \lambda) f(\mathbf{k}, \lambda), \tag{2.32}$$

$$\mathbf{M} = \sum_\lambda \int \frac{d^3\mathbf{k}}{(2\pi)^3|\mathbf{k}|} f^*(\mathbf{k}, \lambda) \left[\hbar\mathbf{k} + \frac{1}{i}\mathbf{D}_\mathbf{k} + \lambda\hbar\frac{\mathbf{k}}{|\mathbf{k}|} \right] f(\mathbf{k}, \lambda), \tag{2.33}$$

$$\mathbf{N} = \sum_\lambda \int \frac{d^3\mathbf{k}}{(2\pi)^3|\mathbf{k}|} f^*(\mathbf{k}, \lambda) i\hbar\omega \mathbf{D}_\mathbf{k} f(\mathbf{k}, \lambda), \tag{2.34}$$

where

$$\mathbf{D}_\mathbf{k} := \partial_\mathbf{k} + i\lambda\boldsymbol{\alpha}(\mathbf{k}), \tag{2.35}$$

and

$$\boldsymbol{\alpha}(\mathbf{k}) := i \sum_\lambda \mathbf{e}^*(\mathbf{k}, \lambda) \boldsymbol{\nabla} \mathbf{e}(\mathbf{k}, \lambda). \tag{2.36}$$

The operation of $\mathbf{D}_\mathbf{k}$ is a natural covariant derivative on the light cone [56] and $\boldsymbol{\alpha}(\mathbf{k})$ is the analog of the vector potential. It describes Berrys phase [88] of a propagating photon. The p-space line integral of $\boldsymbol{\alpha}(\mathbf{k})$ over a closed contour yields

$$\gamma(C) = \oint \boldsymbol{\alpha} \cdot d\mathbf{p} = \pm\Omega(C), \tag{2.37}$$

where $\Omega(C)$ is the solid angle subtended by the contour C as seen from the origin of momentum space, and the signs depend on the orientation of paths [56]. The existence of the Berrys phase (2.37) for photons has been verified experimentally [57].

From the relations (2.31) and (2.32), it is clear that $f(\mathbf{k}, \lambda)$ expresses field amplitudes with energy $\hbar\omega$ and momentum $\hbar\mathbf{k}$. In view of (2.33), the coefficient functions $f(\mathbf{k}, +1)$ and $f(\mathbf{k}, -1)$ can be identified as field amplitudes with positive and negative helicities.

As discussed in Ref [1], the coefficient function $f(\mathbf{k}, \lambda)$ provide dual interpretations. In classical electrodynamics they illustrate all information of an electromagnetic field. In the context of the photon wave mechanics they are the components of the wave function in momentum representation. In order to avoid a possible confusion, we denote the components of the photon wave function in momentum representation by $\phi(\mathbf{k}, \lambda)$.

Next, we introduce the photon position wave function in terms of plane waves according to

$$\Psi(\mathbf{x}, x^0) = \sqrt{\hbar} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \begin{pmatrix} \mathbf{e}(\mathbf{k})\phi(\mathbf{k}, 1) \\ \mathbf{e}^*(\mathbf{k})\phi(\mathbf{k}, -1) \end{pmatrix} e^{-i\omega x^0 + i\mathbf{k}\cdot\mathbf{x}}, \tag{2.38}$$

One can perform the Fourier transformation to obtain the wave functions in momentum representation $\phi(\mathbf{k}, \lambda)$ in terms of the wave functions in position representation $\Psi(\mathbf{x}, t)$ according to

$$\phi(\mathbf{k}, 1) = \frac{1}{\sqrt{\hbar}} \mathbf{e}^*(\mathbf{k}) \cdot \int d^3\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} \Psi(\mathbf{x}, x^0). \tag{2.39}$$

2.3 Scalar Product and probabilistic Density

Following Ref [1], we can give an explicit expression for the number of photons contained in an electromagnetic field. To do this, we divide the energy of an electromagnetic field defined in (2.31) by $\hbar\omega$ i.e, the energy of each photon. This leads to

$$N = \sum_\lambda \int \frac{d^3\mathbf{k}}{(2\pi)^3|\mathbf{k}|} |f(\mathbf{k}, \lambda)|^2, \tag{2.40}$$

For a single photon, $N = 1$. Thus the normalized wave function in momentum representation $\phi(\mathbf{k}, \lambda)$ meets the following condition

$$\sum_\lambda \int \frac{d^3\mathbf{k}}{(2\pi)^3|\mathbf{k}|} |\phi(\mathbf{k}, \lambda)|^2 = 1. \tag{2.41}$$

This in turn implies that the probability density for a single photon in momentum representation should be

$$\rho = \frac{|\phi(\mathbf{k}, \lambda)|^2}{(2\pi)^3|\mathbf{k}|}. \tag{2.42}$$

Furthermore, the norm relation (2.41) suggests the existence of the corresponding inner product. For any two state vectors Ψ_1 and Ψ_2, one can define the corresponding inner product according to

$$\langle \Psi_1, \Psi_2 \rangle = \sum_{\lambda} \int \frac{d^3\mathbf{k}}{(2\pi)^3|\mathbf{k}|} \phi_1^*(\mathbf{k}, \lambda)\phi_2(\mathbf{k}, \lambda). \tag{2.43}$$

This is defined in terms of photon wave functions in momentum space. One can employ (2.39) to further express the inner product (2.43) in terms of position wave functions,

$$\langle \Psi_1, \Psi_2 \rangle = \frac{1}{2\pi^2\hbar} \int_{\mathbb{R}^3} d^3\mathbf{x} \int_{\mathbb{R}^3} d^3\mathbf{x}' \, \Psi_1^\dagger(\mathbf{x}) \frac{1}{|\mathbf{x} - \mathbf{x}'|^2} \Psi_2(\mathbf{x}'), \tag{2.44}$$

and the corresponding norm for the inner product (2.44) takes the form,

$$N = \|\Psi(\mathbf{x})\|^2 = \frac{1}{2\pi^2\hbar} \int_{\mathbb{R}^3} d^3\mathbf{x} \int_{\mathbb{R}^3} d^3\mathbf{x}' \, \Psi^\dagger(\mathbf{x}) \frac{1}{|\mathbf{x} - \mathbf{x}'|^2} \Psi(\mathbf{x}'). \tag{2.45}$$

By pursuing different approaches, many people have explored the same expressions for the inner product (2.44) and its corresponding norm (2.45). The norm relation (2.45) was obtained by Zeldovich to present the number of photons as an invariant of the classical electromagnetic field [58]. Lately, the norm relation (2.45) has played an important role in the formulation of wavelet electrodynamics by Kaiser [59]. Furthermore, Gross found that the inner product and norm are invariant under both the Poincaré and conformal transformations [60].

Defining the inner product (2.44) is essential to provide a probabilistic interpretation of a single photon, i.e. the probability of finding a single photon is defined to be the modulus square of the inner product (2.44). For instance, we determine the probability of measuring a single photon in the state Ψ_2 when the photon is in the state Ψ_1 according to $\|\langle \Psi_2, \Psi_1 \rangle\|^2$.

2.4 Expectation Values, Physical Observables, and Conservation of Probability

As discussed before, due to the lack of a concrete and well-defined formulation of the first-quantized theory of a single photon, one cannot determine the physical observables of this quantum system in a consistent way. However, some people have attempted to determine physical observables by identifying quantum mechanical expectation values with classical counterparts. This can be done by replacing the normalized photon wave functions $\phi(\mathbf{k},\lambda)$ with classical amplitudes $f(\mathbf{k},\lambda)$. In view of this notion, the classical observables are related to the quantum mechanical expectation values according to

$$\langle E\rangle = \langle\Psi|H|\Psi\rangle, \qquad \langle\mathbf{P}\rangle = \langle\Psi|\mathbf{P}|\Psi\rangle, \tag{2.46}$$

$$\langle\mathbf{M}\rangle = \langle\Psi|\mathbf{J}|\Psi\rangle, \qquad \langle\mathbf{N}\rangle = \langle\Psi|\mathbf{K}|\Psi\rangle, \tag{2.47}$$

where E, $\mathbf{P}$, $\mathbf{M}$ and N represent the energy, momentum, angular momentum and moment of energy of a classical electromagnetic field, respectively. In view of (2.31)-(2.34) and (2.46)-(2.47), the quantum mechanical observables in momentum representation take the form

$$H = \hbar\omega, \tag{2.48}$$

$$\mathbf{P} = \hbar\mathbf{k}, \tag{2.49}$$

$$\mathbf{J} = \mathbf{k}\times\frac{\hbar}{i}\mathbf{D}_{\mathbf{k}} + \lambda\hbar\frac{\mathbf{k}}{|\mathbf{k}|}, \tag{2.50}$$

$$\mathbf{K} = \hbar\omega\frac{\hbar}{i}\mathbf{D}_{\mathbf{k}}. \tag{2.51}$$

These operators are Hermitian with respect to the inner product (2.43). Also we observe that in the momentum representation, relations (2.48)-(2.51) are consistent with the interpretation of $\phi(\mathbf{k},\lambda)$ as the probability amplitude.

In a similar attempt, Good [18] offered a new inner product by comparing the energy of a classical electromagnetic field with quantum mechanical expectation value of the Hamiltonian, $\langle E\rangle = \langle\Psi|H|\Psi\rangle$. The resulting inner product takes the form

$$\langle\Psi_1|\Psi_2\rangle = \int_{\mathbb{R}^3} d^3\mathbf{x}\,\Psi_1^\dagger(\mathbf{x})\frac{1}{H}\Psi_2(\mathbf{x}). \tag{2.52}$$

In the context of Good's work [18], it was taken into account that wave functions are constructed only from the positive energy states which in turn assures the positive definiteness of the corresponding norm

$$\|\Psi\|^2 = \int_{\mathbb{R}^3} d^3\mathbf{x}\,\Psi^\dagger(\mathbf{x})\frac{1}{H}\Psi(\mathbf{x}). \tag{2.53}$$

In light of (2.52) and (2.53), the quantum mechanical expectation values of the Hamiltonian, momentum, angular momentum and boost operators take the form:

$$\langle E\rangle = \int_{\mathbb{R}^3} d^3\mathbf{x}\,\Psi^\dagger(\mathbf{x})H^{-1}(\mathbf{S}\cdot\frac{\hbar}{i}\boldsymbol{\nabla})\Psi(\mathbf{x}), \tag{2.54}$$

$$\langle \mathbf{P} \rangle = \int_{\mathbb{R}^3} d^3\mathbf{x}\, \Psi^\dagger(\mathbf{x}) H^{-1}\, \frac{\hbar}{i} \boldsymbol{\nabla} \Psi(\mathbf{x}), \tag{2.55}$$

$$\langle \mathbf{M} \rangle = \int_{\mathbb{R}^3} d^3\mathbf{x}\, \Psi^\dagger(\mathbf{x}) H^{-1} (\mathbf{x} \times \frac{\hbar}{i} \boldsymbol{\nabla}| + \hbar \mathbf{S}) \Psi(\mathbf{x}), \tag{2.56}$$

$$\langle \mathbf{N} \rangle = \int_{\mathbb{R}^3} d^3\mathbf{x}\, \Psi^\dagger(\mathbf{x}) H^{-1} H \mathbf{x} \Psi(\mathbf{x}). \tag{2.57}$$

Similarly, the operators H, $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}$ in coordinate representation are given by

$$H = \mathbf{S} \cdot \frac{\hbar}{i} \boldsymbol{\nabla}, \tag{2.58}$$

$$\mathbf{P} = \frac{\hbar}{i} \boldsymbol{\nabla}, \tag{2.59}$$

$$\mathbf{J} = \mathbf{x} \times \frac{\hbar}{i} \boldsymbol{\nabla} + \hbar \mathbf{S}, \tag{2.60}$$

$$\mathbf{K} = H \mathbf{x}. \tag{2.61}$$

These operators maintain the divergence condition (2.6) and are Hermitian with respect to the inner product (2.52). One advantage of this formalism is that in coordinate representation the momentum and angular momentum operators adopt the same expressions as in standard quantum mechanics.

In the remaining part of this section, we discuss the continuity relation for the photon's current density and the probability density. According to (2.45), the number of photons is given by a double integral which implies that the probability density in coordinate representation is a nonlocal object, although the expression for the energy has the form of a single integral over $|\Psi|^2$. In order to circumvent this difficulty, some physicists have presented the concept of "average photon energy in a region of space" to describe the probabilistic interpretation of the photon wave function [7, 8]. The "average photon energy in a region of space" is given by

$$P_E(\Omega) = \frac{\int_\Omega d^3\mathbf{x}\, \Psi^\dagger(\mathbf{x}) \Psi(\mathbf{x})}{\langle E \rangle}. \tag{2.62}$$

This can be interpreted as the probability of finding the energy of a localized photon in the region Ω. In fact $P_E(\Omega)$ describes the portion of the average total energy pertaining to the region Ω. After normalizing (2.45), we can introduce the probability density of finding the energy of a photon localized at point $\mathbf{x}$ according to

$$\rho_E(\mathbf{x}, x^0) = \frac{\Psi^\dagger(\mathbf{x}, x^0) \Psi(\mathbf{x}, x^0)}{\langle E \rangle}. \tag{2.63}$$

This quantity satisfies the continuity relation,

$$\frac{\partial \rho_E(\mathbf{x}, x^0)}{\partial x^0} + \boldsymbol{\nabla} \cdot \mathbf{j}_E(\mathbf{x}, x^0) = 0. \tag{2.64}$$

where $\mathbf{j}_E$ is the normalized average energy flux that is given by

$$\mathbf{j}_E(\mathbf{x}, x^0) = \frac{\Psi^\dagger(\mathbf{x}, x^0)\sigma_3 \mathbf{S} \Psi(\mathbf{x}, x^0)}{\langle E \rangle}. \tag{2.65}$$

We interpret this as the probability current density. As noted by Mandel and Wolf [61], the name "the energy wave function" stands for the explicit relationship between the wave function $\Psi(\mathbf{x}, x^0)$ and the average energy density. As photons do not have charge and rest mass, the localization of photons can be directly related to their energy.

2.5 Position Operator

By reviewing the literature on this subject, we encounter the problem of defining a proper position operator for the photon. In fact, this has been a long-lasting quest with no generally accepted solution. Here we review the known results on the position operator of a single photon and discuss drawbacks and difficulties of various proposals for it.

In the framework of non-relativistic quantum mechanics, the position wave function $\psi(\mathbf{x})$ is determined by the projection of a state vector ψ on the eigenvector of the position operator,

$$\psi(\mathbf{x}) = \langle \mathbf{x} | \psi \rangle. \tag{2.66}$$

Consequently, the problem of the nonexistence of the position operator for a single photon is closely related with the problem of defining a proper position wave function in coordinate representation [6]. Researchers have employed various methods to obtain a position operator with desirable properties such as having commuting components. Many candidates for the position operator of a photon have been proposed [24, 25, 27, 28, 29]. The first position operator for the photon is credited to Pryce who used a variety of definitions for the center of mass in the context of field theories in order to obtain the photon's position operator. In momentum representation, the Pryce operator is given by

$$\mathbf{x}_{Pryce} = i\hbar \boldsymbol{\nabla}_p - i\hbar \frac{\mathbf{p}}{2p^2} + \hbar \frac{\mathbf{p} \times \mathbf{S}}{p^2}, \tag{2.67}$$

where $\mathbf{S}$ is the spin operator for the photon whose components are three dimensional generators of the $SO(3)$ group.

In [25], Mourad shows that there is only one operator with certain minimal properties that enable it to be considered as the photon position operator. This operator turns out to be the Pryce position operator. This operator does not meet all the desired conditions that one expect of a position operator, in particular its components do not commute with each other. In addition to the Pryce operator, all other proposals presented in [24, 25, 27, 28, 29] fail to satisfy the condition of having commuting components.

In recent years Hawton [11] constructed a position operator for the photon with commuting components. This is given by

$$\mathbf{x}_{ij} = i\hbar\left[\delta_{ij}\boldsymbol{\nabla}_p - \alpha\delta_{ij}\frac{\mathbf{p}}{2p^2} - \sum_{\lambda=1}^{3}(\boldsymbol{\nabla}_{\mathbf{p}}\mathbf{e}_{\mathbf{p}\lambda i})\mathbf{e}_{\mathbf{p}\lambda j}\right], \tag{2.68}$$

where $\mathbf{x}_{ij}$ are the entries of the position operator $\mathbf{x}$ and α is an arbitrary constant. $\mathbf{e}_{\mathbf{p}1}$, $\mathbf{e}_{\mathbf{p}2}$ and $\mathbf{e}_{\mathbf{p}3} = \frac{\mathbf{p}}{|\mathbf{p}|}$ are transverse and longitudinal unit vectors, respectively. In spherical polar coordinates ($\mathbf{e}_{\mathbf{p}1} = \theta_{\mathbf{p}}$ and $\mathbf{e}_{\mathbf{p}2} = \phi_{\mathbf{p}}$), the Hawton position operator $\mathbf{x}$ takes the following form,

$$\mathbf{x} = i\hbar\boldsymbol{\nabla}_p - i\hbar\alpha\frac{\mathbf{p}}{2p^2} + \hbar\frac{\mathbf{p}\times\mathbf{S}}{p^2} - \frac{\hbar}{p}\cot\phi_{\mathbf{p}}\mathbf{e}_{\mathbf{p}2}S_{\mathbf{p}3}. \tag{2.69}$$

According to Hawton, by setting $\alpha = 1/2$, the first three terms of (2.69) coincide with Pryce's position operator for a photon (2.67). The last term in (2.69) is resulted from the contribution of the last part of (2.68) for $\lambda = 1, 2$ and is given by

$$\sum_{\lambda=1}^{2}(\boldsymbol{\nabla}_{\mathbf{p}}\mathbf{e}_{\mathbf{p}\lambda i})\mathbf{e}_{\mathbf{p}\lambda j} = \frac{\hbar}{p}\cot\phi_{\mathbf{p}}\mathbf{e}_{\mathbf{p}2}S_{\mathbf{p}3}, \tag{2.70}$$

which describes a Berry's Phase [11, 56] associated with the propagation of the photon. More recently, Hawton and Baylis have reexamined this position operator and extended the procedure for the construction of the photon position operators with commuting components to body rotations described by three Euler angles [30]. It should be noted that Hawton has obtained (2.69) by adding the extra term (2.70) to the Pryce operator to ensure that the resulting operator has commuting components. It does not involve the construction of a physical Hilbert space for a single photon.

Chapter 3

QUANTUM MECHANICS OF RELATIVISTIC SCALAR FIELDS

3.1 *Hermiticity and Relativistic Scalar Fields*

Following Ref [50], we start our discussion with the covariant form of relativistic scalar fields. For any time $x^0 \in \mathbb{R}$, the scalar equation is given by

$$[\partial_\mu \partial^\mu - \mathfrak{m}^2]\psi(x^0, \mathbf{x}) = 0, \tag{3.1}$$

where $\mathfrak{m} = \frac{mc}{\hbar}$, and $\psi(x^0, \mathbf{x})$ is a square integrable function in $L^2(\mathbb{R}^3)$. Let us define for all $x^0 \in \mathbb{R}$, the function $\psi(x^0) : L^2(\mathbb{R}^3) \to \mathbb{C}$ by $\psi(x^0)(\mathbf{x}) := \psi(x^0, \mathbf{x})$ and introduce a Hermitian operator $D : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ according to

$$[D\psi(x^0)](\mathbf{x}) := [-\nabla^2 + \mathfrak{m}^2]\psi(x^0, \mathbf{x}), \qquad \forall x^0 \in \mathbb{R}, \quad \forall \mathbf{x} \in \mathbb{R}^3.$$

Then we can write (3.1) as the following dynamical equation

$$[\partial_0^2 + D]\psi(x^0) = 0. \tag{3.2}$$

We can specify every scalar field with a solution of this equation, $\psi(x^0) : \mathbb{R} \to L(\mathbb{R}^2)$, and define a space of all scalar fields with the complex vector space:

$$\mathcal{V} = \left\{ \psi(x^0) : \mathbb{R} \to L(\mathbb{R}^2) \mid \ddot{\psi}(x^0) + D\psi(x^0) = 0 \right\}. \tag{3.3}$$

As shown in [74, 75, 52, 76, 77], Eq. (3.2) can be written as $i\hbar\dot{\Psi}(x^0) = H\Psi(x^0)$, where the two component Schrödinger equation for all $x^0 \in \mathbb{R}$,

$$\Psi(x^0) := \begin{pmatrix} \psi(x^0) + \psi_c(x^0) \\ \psi(x^0) - \psi_c(x^0) \end{pmatrix}, \tag{3.4}$$

$$H := \hbar \begin{pmatrix} D^{1/2} & 0 \\ 0 & -D^{1/2} \end{pmatrix}, \tag{3.5}$$

$$\psi_c(x^0) := iD^{-1/2}\dot{\psi}(x^0) =: C\psi(x^0). \tag{3.6}$$

Equation (3.5) gives precisely the Foldy Hamiltonian [52, 75], and C denotes a chirality operator for relativistic scalar field ψ [53]. The Hamiltonian H acts in the Hilbert space $\mathcal{H}' := L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ and $\Psi(x^0)$ belongs to $\mathcal{H}'$.

Note that the Hamiltonian H is Hermitian. This shows that the approach discussed in [50, 53] can be simplified.

Solving the eigenvalue problem for the Hamiltonian H yields

$$E_{\epsilon,k} = \epsilon\hbar\omega_k, \qquad \Psi_\epsilon(\mathbf{k}) := \frac{1}{2}\begin{pmatrix} 1+\epsilon \\ 1-\epsilon \end{pmatrix}\phi_{\mathbf{k}}, \quad \epsilon \in \{-1, 1\}, \tag{3.7}$$

where ϵ is the sign of energy, $\omega_k = \sqrt{k^2 + \mathfrak{m}^2}$, and for all $\mathbf{x} \in \mathbb{R}^3$, $\phi_{\mathbf{k}}(\mathbf{x}) := \langle \mathbf{x}|\mathbf{k}\rangle = \frac{1}{(2\pi)^{3/2}} e^{i\boldsymbol{\omega}_{\mathbf{k}}\cdot\mathbf{x}}$.

The eigenvectors $\Psi_\epsilon(\mathbf{k})$ form an orthonormal basis for the Hilbert space $\mathcal{H}'$, i.e,

$$\langle \Psi_\epsilon(\mathbf{k}), \Psi_{\epsilon'}(\mathbf{k}')\rangle = \delta_{\epsilon,\epsilon'}\,\delta^3(\mathbf{k} - \mathbf{k}'), \qquad \sum_{\epsilon=\pm}\int_{\mathbb{R}^3} d^3\mathbf{k}\,|\Psi_\epsilon(\mathbf{k})\rangle\langle\Psi_\epsilon(\mathbf{k})| = 1, \tag{3.8}$$

where $\langle\cdot,\cdot\rangle$ stands for the inner product of $\mathcal{H}'$. We can write H in terms of its spectral resolution:

$$H = \sum_{\epsilon=\pm}\int_{\mathbb{R}^3} d^3\mathbf{k}\, E_{\epsilon,k}|\Psi_\epsilon(\mathbf{k})\rangle\langle\Psi_\epsilon(\mathbf{k})|. \tag{3.9}$$

Next, for all $\xi := \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}$ and $\zeta := \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix} \in \mathcal{H}'$, we define an inner product

$$\langle \xi, \zeta\rangle := \sum_{i=1}^{2}\langle \xi^i, \zeta^i\rangle.$$

and thus after simplification we have

$$\langle \xi, \zeta\rangle := \frac{1}{2}\left[\langle \xi_+, \zeta_+\rangle + \langle \xi_-, \zeta_-\rangle\right], \tag{3.10}$$

where $\xi_\pm := \xi^1 \pm \xi^2, \zeta_\pm := \zeta^1 \pm \zeta^2 \in L^2(\mathbb{R}^3)$.

We further define a one-to-one mapping $U_{x^0} : \mathcal{V} \to \mathcal{H}'$ for all $x^0 \in \mathbb{R}$ according to

$$U_{x^0}\psi := \frac{1}{2}\,\Psi(x^0), \qquad \forall\psi \in \mathcal{V}. \tag{3.11}$$

Because $\mathcal{H}'$ is endowed with the inner product $\langle \xi, \zeta\rangle$, the map U_{x^0} induces a new inner product on $\mathcal{V}$ as

$$(\psi_1, \psi_2) := \langle U_{x^0}\psi_1, U_{x^0}\psi_2\rangle = \frac{1}{4}\langle \Psi_1(x^0), \Psi_2(x^0)\rangle. \tag{3.12}$$

By making use of (3.4) and (3.10), one can simplify (ψ_1, ψ_2). This yields

$$(\psi_1, \psi_2) = \frac{1}{2}\left[\langle\psi_1(x^0)|\psi_2(x^0)\rangle + \langle\dot{\psi}_1(x^0)|D^{-1}\dot{\psi}_2(x^0)\rangle\right], \tag{3.13}$$

where D^{-1} can be obtained by the spectral resolution of D. For all $\mu \in \mathbb{R}$, we define its powers as

$$D^{\mu} := \int_{\mathbb{R}^3} d^3\mathbf{k}\,(k^2 + \mathfrak{m}^2)^{\mu}\,|\mathbf{k}\rangle\langle\mathbf{k}|.$$

Because D^{-1} is a positive-definite operator, the inner product (3.13) is manifestly positive-definite. It is also dynamically invariant; one can use (3.2) to show that the derivative of the right-hand side of (3.13) vanishes identically. The only drawback of this inner product is that it is not relativistically invariant. Therefore, we search for another inner product which is relativistically invariant. In order to achieve this goal, we construct the most general positive-definite inner product on $\mathcal{V}$ and impose the constraint of relativistic invariance.

3.2 The Most General Admissible Inner Product and Conserved Current Density

Here we pursue the procedure of Ref [51] with some variations in order to obtain the most general admissible inner product for relativistic scalar fields. According to Refs [51, 70], the most general positive-definite inner product $\tilde{\eta}_+$ corresponds to the most general orthonormal basis consisting of the eigenvectors of H. Because H is Hermitian, this is given by $\tilde{\eta}_+ := \mathcal{A}^{\dagger}\mathcal{A}$, where we consider $\mathcal{A}$ to be an invertible linear operator which commutes with H. We can express $\mathcal{A}$ as

$$\mathcal{A} = \sum_{\epsilon=\pm}\int_{\mathbb{R}^3} d^3\mathbf{k}\,\alpha_{\epsilon}(\mathbf{k})|\Psi_{\epsilon}(\mathbf{k})\rangle\langle\Psi_{\epsilon}(\mathbf{k})|, \tag{3.14}$$

where $\alpha_{\epsilon}(\mathbf{k})$ are arbitrary nonzero complex numbers. Using $\tilde{\eta}_+ := \mathcal{A}^{\dagger}\mathcal{A}$ and (3.14), we obtain

$$\tilde{\eta}_+ = (L_+ + L_-)\eta_+ + (L_+ - L_-)\sigma_3, \tag{3.15}$$

where

$$L_{\pm} = \frac{1}{2}\int d^3\mathbf{k}\,|\alpha_{\pm}(\mathbf{k})|^2|\mathbf{k}\rangle\langle\mathbf{k}|. \tag{3.16}$$

This in turn implies that the inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\tilde{\eta}_+} := \langle\cdot,\tilde{\eta}_+\cdot\rangle$ takes the following form, for all $\xi, \zeta \in \mathcal{H}'$,

$$\langle\!\langle\xi,\zeta\rangle\!\rangle_{\tilde{\eta}_+} = 2\left[\langle\xi^1, L_+\zeta^1\rangle + \langle\xi^2, L_-\zeta^2\rangle\right]. \tag{3.17}$$

By viewing $\mathcal{H}'$ as a complex vector space and equipping it with this inner product, we can introduce a new inner product space whose Cauchy completion yields a Hilbert space $\mathcal{K}$.

Next we substitute $\Psi_\epsilon(\mathbf{k})$ in (3.14) and obtain the explicit expressions of $\mathcal{A}$ and $\mathcal{A}^{-1}$ according to

$$\mathcal{A} = \begin{pmatrix} \tilde{\alpha}_+ & 0 \\ 0 & \tilde{\alpha}_- \end{pmatrix}, \qquad \mathcal{A}^{-1} = \begin{pmatrix} \tilde{\alpha}_+^{-1} & 0 \\ 0 & \tilde{\alpha}_-^{-1} \end{pmatrix}, \tag{3.18}$$

where $\tilde{\alpha}_+ := \int_{\mathbb{R}^3} d^3\mathbf{k}\, \alpha_+(\mathbf{k})\, |\mathbf{k}\rangle\langle\mathbf{k}|$ and $\tilde{\alpha}_+^{-1} := \int_{\mathbb{R}^3} d^3\mathbf{k}\, \alpha_+^{-1}(\mathbf{k})\, |\mathbf{k}\rangle\langle\mathbf{k}|$. Let us observe that the operator $\mathcal{A}$ mapping $\mathcal{K}$ onto $\mathcal{H}'$ is a unitary operator, i.e, for all $\xi, \zeta \in \mathcal{H}'$,

$$\langle\!\langle \mathcal{A}\xi, \mathcal{A}\zeta \rangle\!\rangle = \langle \xi, \mathcal{A}^\dagger \mathcal{A}\zeta \rangle = \langle\!\langle \xi, \zeta \rangle\!\rangle_{\tilde{\eta}_+}, \tag{3.19}$$

where we have used the definition of $\tilde{\eta}_+$ in the second equality in (3.19). We also define a new linear operator according to

$$\mathcal{U}_{\{\mathfrak{a}\}} := \mathcal{A}\, U_{x_0^0}. \tag{3.20}$$

By using (3.18), (3.4) and (3.11) in (3.20), we compute the Foldy transformation $\mathcal{U}_{\{\mathfrak{a}\}}$ [52],

$$\mathcal{U}_{\{\mathfrak{a}\}}\psi = \frac{1}{2} \begin{pmatrix} \alpha_+[\psi(x_0^0) + i\, D^{-1/2}\, \dot{\psi}(x_0^0)] \\ \alpha_-[\psi(x_0^0) - i\, D^{-1/2}\, \dot{\psi}(x_0^0)] \end{pmatrix}. \tag{3.21}$$

In addition to (3.21), we might attempt to compute the explicit form of $\mathcal{U}_{\{\mathfrak{a}\}}^{-1} = U_{x_0^0}^{-1} \mathcal{A}^{-1}$. For any two-component vector $\xi \in \mathcal{H}'$, $\mathcal{U}_{\{\mathfrak{a}\}}^{-1}\xi$ is a relativistic scalar field which satisfies the following initial conditions:

$$\psi(x_0^0) = \alpha_+^{-1}\, \xi^1 + \alpha_-^{-1}\, \xi^2, \qquad \dot{\psi}(x_0^0) = -i \left(\alpha_+^{-1}\, D^{1/2}\, \xi^1 - \alpha_-^{-1}\, D^{1/2}\, \xi^2\right) \tag{3.22}$$

and for all $x^0 \in \mathbb{R}$ we have

$$\psi(x^0) = e^{-i(x^0 - x_0^0)D^{1/2}}\, \alpha_+^{-1}\, \xi^1 + e^{i(x^0 - x_0^0)D^{1/2}}\, \alpha_-^{-1}\, \xi^2, \tag{3.23}$$

$$\dot{\psi}(x_0^0) = -i \left[e^{-i(x^0 - x_0^0)D^{1/2}}\, \alpha_+^{-1}\, D^{1/2}\, \xi^1 - e^{i(x^0 - x_0^0)D^{1/2}}\, \alpha_-^{-1}\, D^{1/2}\, \xi^2 \right]. \tag{3.24}$$

Next we use the unitary operator $U_{x^0} : \mathcal{V} \to \mathcal{H}'$ to define a new inner product in $\mathcal{V}$

$$(\psi, \psi')_{\{\mathfrak{a}\}} := \langle\!\langle U_{x_0^0}\psi, U_{x_0^0}\psi' \rangle\!\rangle_{\tilde{\eta}_+}, \qquad \forall \psi, \psi' \in \mathcal{H}_{\{\mathfrak{a}\}}. \tag{3.25}$$

If we use (3.20) in (3.25), we obtain

$$(\psi,\psi')_{\{\mathfrak{a}\}} = \langle U_{x_0^0}\psi, \mathcal{A}^\dagger \mathcal{A} U_{x_0^0}\psi'\rangle = \langle \mathcal{U}_{\{\mathfrak{a}\}}\psi, \mathcal{U}_{\{\mathfrak{a}\}}\psi'\rangle, \tag{3.26}$$

which means that the Foldy transformation $\mathcal{U}_{\{\mathfrak{a}\}}$ is a unitary transformation. Also if we endow the space of $\mathcal{V}$ of the solutions of the Klein-Gordon equation (3.1) with the positive-definite inner product (3.25), we have

$$\begin{aligned}(\psi_1,\psi_2)_{\{\mathfrak{a}\}} := \frac{1}{2}\Big\{&\langle\psi_1(x^0),(L_+ + L_-)\psi_2(x^0)\rangle + \langle\dot\psi_1(x^0),(L_+ + L_-)D^{-1}\dot\psi_2(x^0)\rangle \\ &+ i\Big[\langle\psi_1(x^0),(L_+ - L_-)D^{\frac{-1}{2}}\dot\psi_2(x^0)\rangle - \langle\dot\psi_1(x^0),(L_+ - L_-)D^{\frac{-1}{2}}\psi_2(x^0)\rangle\Big]\Big\}\end{aligned} \tag{3.27}$$

This is the most general positive-definite inner product that is dynamically invariant, but not necessarily Lorentz invariant. Therefore, in the following we impose the constraint that this inner product be Lorentz invariant.

Similar to Refs [51, 70], first let us compute the inner product of the basic (free particle) solutions of relativistic scalar fields $\psi_{\epsilon,\mathbf{k}} = N_{\epsilon,\mathbf{k}} e^{-i\epsilon\omega_k x^0}\phi_{\mathbf{k}}$. It yields

$$(\psi_{\epsilon,\mathbf{k}},\psi_{\epsilon',\mathbf{k}'})_{\{\mathfrak{a}\}} = |N_{\epsilon,\mathbf{k}}|^2\, \mathfrak{a}_\epsilon\, \delta_{\epsilon,\epsilon'}\, \delta^3(\mathbf{k}-\mathbf{k}'), \tag{3.28}$$

where $N_{\epsilon,\mathbf{k}}$ are normalization constants and $\mathfrak{a}_\epsilon := |\alpha_\epsilon(\mathbf{k})|^2$. As we can see $\mathfrak{a}_\epsilon$ are positive real numbers, and the right hand side of (3.28) does not depend on x^0 which shows the dynamical invariance and positive definiteness of the inner product $(\cdot,\cdot)_{\{\mathfrak{a}\}}$.

Having computed the inner product (3.28), we can also obtain the inner product for any two solutions ψ and ψ'

$$\psi = \sum_{\epsilon=\pm}\int_{\mathbb{R}^3} d^3\mathbf{k}\, c_\epsilon(\mathbf{k})\psi_{\epsilon,\mathbf{k}}, \tag{3.29}$$

according to

$$(\psi,\psi')_{\{\mathfrak{a}\}} = \sum_{\epsilon=\pm}\int_{\mathbb{R}^3} d^3\mathbf{k}\, \mathfrak{a}_\epsilon\, |N_\epsilon(\mathbf{k})|^2\, c_\epsilon^*(\mathbf{k})\, c'_\epsilon(\mathbf{k}), \tag{3.30}$$

where $c_\epsilon(\mathbf{k})$ and $c'_\epsilon(\mathbf{k})$ are complex coefficients. Above we made use of (3.28), (3.29), and the fact that the inner product (3.27) is a Hermitian sesquilinear form. Because $d^3\mathbf{k}/\omega_{\mathbf{k}}$ is a relativistically invariant measure [43, 85] and the solutions ψ and ψ' are scalar fields imply that $c_\epsilon(\mathbf{k})$ must transform similarly to $\omega_{\mathbf{k}}^{-1}$ [70]. This in turn implies that in order for the inner product (3.30) to be a scalar, $\mathfrak{a}_\epsilon$ must also transform similarly to $\omega_{\mathbf{k}}$. Therefore,

$$\mathfrak{a}_\epsilon = a_\epsilon\, \omega_{\mathbf{k}}, \tag{3.31}$$

where $a_\pm$ are dimensionless positive real numbers. By substituting (3.31) into (3.16), we obtain

$$L_\pm = \frac{1}{2} a_\pm D^{1/2}. \tag{3.32}$$

Under this condition, $(\cdot,\cdot)$ is not only an invariant and positive-definite inner product, but also relativistically invariant. Thus the most general form of the inner product (3.27) which is relativistically invariant can be given as

$$\begin{aligned}(\psi_1,\psi_2)_{a,i.r} := \frac{1}{4}\Big\{&(a_+ + a_-)\Big[\langle\psi_1(x^0)|D^{1/2}\psi_2(x^0)\rangle + \langle\dot\psi_1(x^0)|D^{-1/2}\dot\psi_2(x^0)\rangle\Big]\\ &+i(a_+ - a_-)\Big[\langle\psi_1(x^0)|\dot\psi_2(x^0)\rangle - \langle\dot\psi_1(x^0)|\psi_2(x^0)\rangle\Big]\Big\}.\end{aligned} \tag{3.33}$$

Note that since both $a_\pm$ are positive numbers, $a_+ + a_-$ is a positive number as well. Therefore, we can absorb $a_+ + a_-$ into the definition of the field by introducing $a_+ + a_- := 2$ and $a_+ - a_- := 2a$. Thus the inner product (3.33) becomes

$$\begin{aligned}(\psi_1,\psi_2)_{a,i.r} := \frac{g}{2}\Big\{&\langle\psi_1(x^0)|D^{1/2}\psi_2(x^0)\rangle + \langle\dot\psi_1(x^0)|D^{-1/2}\dot\psi_2(x^0)\rangle\\ &+ia\Big[\langle\psi_1(x^0)|\dot\psi_2(x^0)\rangle - \langle\dot\psi_1(x^0)|\psi_2(x^0)\rangle\Big]\Big\},\end{aligned} \tag{3.34}$$

where $a \in (-1,1)$ and g are arbitrary real numbers. Note that g has the dimension of length which is added to make the inner product (3.34) dimensionless.

The inner product (3.34) is the most general Lorentz-invariant and positive-definite inner product on $\mathcal{V}$ that is originally obtained in Ref [53]. If we endow $\mathcal{V}$ with this inner product and make the Cauchy completion of the resulting inner product space, we arrive at the physical Hilbert space of relativistic scalar fields which we denoted it by $\mathcal{H}_a$.

Equation (3.34) gives a one parameter family of Lorentz-Invariant positive-definite inner products. The associated conserved four-vector current density J^μ_a takes the following form [53]

$$J^\mu_a(x) = \frac{g}{2}\left\{\psi(x)^* \overleftrightarrow{\partial^\mu} D^{-1/2}\dot\psi(x) - ia\psi(x)^* \overleftrightarrow{\partial^\mu}\psi(x)\right\}, \tag{3.35}$$

where $\xi\overleftrightarrow{\partial^\mu}\zeta := \xi\partial^\mu\zeta - (\partial^\mu\xi)\zeta$. By making use of the chirality operator (3.6), we can simplify J^μ_a according to

$$J^\mu_a(x) = -\frac{ig}{2}\left[\psi(x)^* \overleftrightarrow{\partial^\mu}\tilde\psi_a(x)\right], \tag{3.36}$$

where $\tilde\psi_a = \psi_c + a\psi$.

It is easy to check that the current density J_a^μ is a conserved quantity. Moreover, it is manifestly covariant and satisfies the continuity equation,

$$\partial_\mu J_a^\mu = 0. \tag{3.37}$$

If we use (3.31) in the definition of $\tilde{\alpha}_\epsilon$ and $\tilde{\alpha}_\epsilon^{-1}$, we obtain

$$\begin{aligned}\tilde{\alpha}_\epsilon &= \sqrt{1+\epsilon a}\, D^{1/4},\\ \tilde{\alpha}_\epsilon^{-1} &= \frac{1}{\sqrt{1+\epsilon a}}\, D^{-1/4}.\end{aligned} \tag{3.38}$$

Using (3.38) in (3.18) leads to

$$\mathcal{A} = \sqrt{g}\begin{pmatrix} \sqrt{1+a}\, D^{1/4} & 0 \\ 0 & \sqrt{1-a}\, D^{1/4}\end{pmatrix}, \quad \mathcal{A}^{-1} = \frac{1}{\sqrt{g}}\begin{pmatrix} \frac{1}{\sqrt{1+a}}\, D^{-1/4} & 0 \\ 0 & \frac{1}{\sqrt{1-a}}\, D^{-1/4}\end{pmatrix}, \tag{3.39}$$

and the Foldy transformation (3.21) takes the following form [53]

$$\mathcal{U}_a\psi = \frac{\sqrt{g}}{2}D^{1/4}\begin{pmatrix} \sqrt{1+a}\,\psi^+(x_0^0) \\ \sqrt{1-a}\,\psi^-(x_0^0)\end{pmatrix} = \frac{\sqrt{g}}{2}\begin{pmatrix} \sqrt{1+a}[D^{1/4}\,\psi_a(x_0^0) + iD^{-1/4}\,\dot{\psi}(x_0^0)] \\ \sqrt{1-a}[D^{1/4}\,\psi_a(x_0^0) - iD^{-1/4}\,\dot{\psi}(x_0^0)]\end{pmatrix}, \tag{3.40}$$

where $\psi^\epsilon := \frac{1}{2}(\psi \pm \psi_c)$.

Also it is remarkable to observe that the field ψ^ϵ in (3.40) satisfies the Foldy equation [52, 75]

$$i\partial_0\psi^\epsilon(x^0,\mathbf{x}) = \epsilon D^{1/2}\,\psi^\epsilon(x^0,\mathbf{x}). \tag{3.41}$$

It is straightforward to see that $\mathcal{U}_a^{-1}\xi$ is a relativistic scalar field $\psi \in \mathcal{H}_a$ which satisfies the following initial conditions for any $\xi \in \mathcal{H}'$

$$\psi(x_0^0) = \frac{1}{\sqrt{g}}D^{-1/4}\,[\frac{\xi^1}{\sqrt{1+a}} + \frac{\xi^2}{\sqrt{1-a}}], \quad \dot{\psi}(x_0^0) = \frac{-i}{\sqrt{g}}D^{1/4}\,[\frac{\xi^1}{\sqrt{1+a}} - \frac{\xi^2}{\sqrt{1-a}}]. \tag{3.42}$$

These in turn implies that for all $x^0 \in \mathbb{R}$,

$$\psi(x^0) = \frac{1}{\sqrt{g}}D^{-1/4}\left[e^{-i(x^0-x_0^0)D^{1/2}}\,\frac{\xi^1}{\sqrt{1+a}} + e^{i(x^0-x_0^0)D^{1/2}}\,\frac{\xi^2}{\sqrt{1-a}}\right], \tag{3.43}$$

$$\dot{\psi}(x^0) = \frac{-i}{\sqrt{g}}D^{1/4}\left[e^{-i(x^0-x_0^0)D^{1/2}}\,\frac{\xi^1}{\sqrt{1+a}} - e^{i(x^0-x_0^0)D^{1/2}}\,\frac{\xi^2}{\sqrt{1-a}}\right]. \tag{3.44}$$

By using (3.43) and the fact that $\psi(x^0) = \mathcal{U}_a^{-1}\xi$, we obtain

$$\mathcal{U}_a^{-1} = \frac{1}{\sqrt{g}}\left(\begin{array}{cc} \frac{D^{-1/4}e^{-i(x^0-x^0_0)D^{1/2}}}{\sqrt{1+a}} & \frac{D^{-1/4}e^{i(x^0-x^0_0)D^{1/2}}}{\sqrt{1-a}} \end{array}\right). \tag{3.45}$$

According to (3.26), $\mathcal{U}_a : \mathcal{H}_a \to \mathcal{H}'$ is a unitary transformation. Therefore, it can be used to define

$$\mathrm{h} := \mathcal{U}_a^{-1} H \mathcal{U}_a, \tag{3.46}$$

which is a Hermitian Hamiltonian acting in $\mathcal{H}_a$ [50]. By using (3.40) in (3.46), we can easily check that the action of h on any field $\psi \in \mathcal{H}_a$ is defined by

$$\mathrm{h}\psi = i\hbar\dot{\psi}. \tag{3.47}$$

Let us note that one must not confuse (3.47) with the time-independent Schrödinger equation giving the x^0-dependence of ψ [50]. In fact h maps each solution ψ of (3.2) to another solution $\dot{\psi}$, and generates time-evolution in $\mathcal{H}_a$ through the Schrödinger equation $i\hbar\frac{d}{dx^0}\psi_{x^0} = h\psi_{x^0}$. According to Ref [50], time-evolution generated by h coincides with time translation in the space $\mathcal{H}_a$. As h is a Hermitian Hamiltonian, time-translations correspond to unitary transformations of the physical Hilbert space $\mathcal{H}_a$.

As noted in Refs [50, 51] the representations of the relativistic scalar field, $(\mathcal{H}', H)$ and $(\mathcal{H}_a, \mathrm{h})$, are unitary equivalent. Therefore, we can formulate a first-quantized theory of relativistic scalar fields using either of $(\mathcal{H}', H)$ and $(\mathcal{H}_a, \mathrm{h})$.

3.3 Physical Observables and Position Wave Functions for Relativistic Scalar Fields

In section 3.2, we give two equivalent representations of the quantum mechanics of a relativistic scalar field, $(\mathcal{H}_a, \mathrm{h})$ and $(\mathcal{H}', H')$. We can employ any of these representations to examine the physical observables of this quantum system. Because the Foldy representation $(\mathcal{H}', H')$ is easy to manipulate, we first construct observables, the position wave function and localized states in this representation and then use the unitary map $\mathcal{U}_a : \mathcal{H}_a \to \mathcal{H}'$ to derive their expressions in $(\mathcal{H}_a, \mathrm{h})$.

We begin by introducing the set of basic observables in Foldy representation [50],

$$\mathbf{X}_\mu := \mathbf{x}_\mu \otimes \sigma_\mu, \qquad \mathbf{P}_\mu := \mathbf{p}_\mu \otimes \sigma_\mu, \qquad \mathcal{S}_\mu := 1 \otimes \sigma_\mu, \tag{3.48}$$

in which, $\mathbf{x}_\mu$, $\mathbf{p}_\mu = \hbar\,\mathfrak{K}$, and 1 are the position, momentum, and identity operators acting in $L^2(\mathbb{R}^3)$, $\mu \in \{0,1,2,3\}$. Also for $i \in \{0,1,2,3\}$, σ_i are Pauli matrices defined by (1.3).

Since $\mathbf{X}_0$ and $\mathcal{S}_3$ form a maximal commuting set of observables, we can write the corresponding basis of $\mathcal{H}'$ that consist of their common eigenvectors, i.e,

$$\xi^{\epsilon}_{\mathbf{x}} := |\mathbf{x}\rangle \otimes e_{\epsilon}, \tag{3.49}$$

where $e_{\epsilon} = \frac{1}{2}\begin{pmatrix} 1+\epsilon \\ 1-\epsilon \end{pmatrix}$ for both $\epsilon \in \{\pm\}$, and $|\mathbf{x}\rangle$ are the δ-function normalized position kets satisfying

$$\mathbf{x}|\mathbf{x}\rangle = \mathbf{x}|\mathbf{x}\rangle, \qquad \langle \mathbf{x}|\mathbf{x}'\rangle = \delta^3(\mathbf{x}-\mathbf{x}'), \qquad \int_{\mathbb{R}^3} d^3\mathbf{x}\, |\mathbf{x}\rangle\langle\mathbf{x}| = 1. \tag{3.50}$$

Clearly, $\mathbf{X}_0\, \xi^{\epsilon}_{\mathbf{x}} = \mathbf{x}\, \xi^{\epsilon}_{\mathbf{x}}$, and $\sigma_3\, \xi^{\epsilon}_{\mathbf{x}} = \epsilon\, \xi^{\epsilon}_{\mathbf{x}}$. Moreover,

$$\langle \xi^{\epsilon}_{\mathbf{x}}, \xi^{\epsilon}_{\mathbf{x}'}\rangle = \delta_{\epsilon,\epsilon'}\delta^3(\mathbf{x}-\mathbf{x}'), \qquad \sum_{\epsilon=\pm}\int_{\mathbb{R}^3} d^3\mathbf{x}\, |\xi^{\epsilon}_{\mathbf{x}}\rangle\langle\xi^{\epsilon}_{\mathbf{x}}| = I. \tag{3.51}$$

Similarly to Ref [50], we introduce the operators

$$\mathbf{x}_{\mu\{a\}} := \mathcal{U}_a^{-1}\, \mathbf{X}_{\mu}\, \mathcal{U}_a, \qquad \mathbf{p}_{\mu\{a\}} := \mathcal{U}_a^{-1}\, \mathbf{P}_{\mu}\, \mathcal{U}_a, \qquad s_{\mu\{a\}} := \mathcal{U}_a^{-1}\, \sigma_{\mu}\, \mathcal{U}_a, \tag{3.52}$$

that are acting in $\mathcal{H}_a$, and define localized scalar fields according to

$$\psi^{\epsilon}_{\{a\}\,\mathbf{x}} := \mathcal{U}_a^{-1}\, \xi^{\epsilon}_{\mathbf{x}}, \tag{3.53}$$

which form a complete orthonormal basis of $\mathcal{H}_{\{\mathfrak{a}\}}$:

$$(\psi^{\epsilon}_{\{a\}\,\mathbf{x}}, \psi^{\epsilon'}_{\{a\}\,\mathbf{x}'})_a = \delta_{\epsilon,\epsilon'}\delta^3(\mathbf{x}-\mathbf{x}'), \qquad \sum_{\epsilon=\pm}\int_{\mathbb{R}^3} d^3\mathbf{x}\, |\psi^{\epsilon}_{\{a\}\,\mathbf{x}})(\psi^{\epsilon}_{\{a\}\,\mathbf{x}'}| = \mathcal{S}_0. \tag{3.54}$$

Next we define the position wave function as

$$f(\epsilon, \mathbf{x}) := \langle \xi^{\epsilon}_{\mathbf{x}}, \Psi'\rangle, \tag{3.55}$$

where Ψ' is any two-component vector in $\mathcal{H}'$. Every relativistic scalar field $\psi \in \mathcal{H}_a$ can be expressed in the basis $\{\psi^{\epsilon}_{\{a\}\,\mathbf{x}}\}$ according to

$$\psi = \sum_{\epsilon=\pm}\int_{\mathbb{R}^3} d^3\mathbf{x}\, f(\epsilon,\mathbf{x})\, \psi^{\epsilon}_{\{a\}\,\mathbf{x}}, \tag{3.56}$$

Furthermore,

$$(\psi^{\epsilon}_{\{a\}\,\mathbf{x}}, \psi)_a = \langle \mathcal{U}_a\psi^{\epsilon}_{\{a\}\,\mathbf{x}}, \mathcal{U}_a\psi\rangle = \langle \xi^{\epsilon}_{\mathbf{x}}, \Psi'\rangle = f(\epsilon, \mathbf{x}). \tag{3.57}$$

The physical observables $o_{\{a\}} : \mathcal{H}_a \to \mathcal{H}_a$ can be uniquely determined in terms of their representation in the basis $\{\psi^{\epsilon}_{\{a\}\,\mathbf{x}}\}$ as

$$o_{\{a\}}\psi = \sum_{\epsilon=\pm}\int_{\mathbb{R}^3} d^3\mathbf{x}[\hat{O}f(\epsilon,\mathbf{x})]\psi^{\epsilon}_{\{a\}\,\mathbf{x}}, \qquad \forall\psi\in\mathcal{H}_a \tag{3.58}$$

where $\hat{O}f(\epsilon,\mathbf{x}) := \sum_{\epsilon'=\pm}\int_{\mathbb{R}^3} d^3\mathbf{x}' f(\epsilon',\mathbf{x}')\,\langle\xi^{\epsilon}_{\mathbf{x}}, O'\xi^{\epsilon'}_{\mathbf{x}'}\rangle$.

This furnishes the representation of observables in terms of linear operators acting on the square-integrable wave functions f.

Next, we evaluate the inner product of a pair of relativistic scalar fields $\psi,\psi'\in\mathcal{H}_a$ (the transition amplitudes between two states). The result is given by

$$(\psi,\psi')_a = \sum_{\epsilon=\pm}\int_{\mathbb{R}^3} d^3\mathbf{x}\, f(\epsilon,\mathbf{x})^* f'(\epsilon,\mathbf{x}). \tag{3.59}$$

For every observable $o_{\{a\}} : \mathcal{H}_a \to \mathcal{H}_a$, we have

$$(\psi, o_{\{a\}}\psi')_a = \sum_{\epsilon=\pm}\int_{\mathbb{R}^3} d^3\mathbf{x}\, f(\epsilon,\mathbf{x})^*\,\hat{O}f'(\epsilon,\mathbf{x}) = \sum_{\epsilon=\pm}\int_{\mathbb{R}^3} d^3\mathbf{x}\,[\hat{O}f(\epsilon,\mathbf{x})]^* f'(\epsilon,\mathbf{x}). \tag{3.60}$$

According to the above discussion, we may consider the wave functions f as the elements of $\mathcal{H}'$, defining observables with Hermitian operators $\hat{O}$ acting on the wave functions, and describe the quantum mechanics of relativistic scalar fields in terms of these wave functions. Note that applying $\mathbf{x}_{0\{a\}}$, $\mathbf{p}_{0\{a\}}$, and $s_{3\{a\}}$ on f agrees with the action of the operators $\hat{\mathbf{x}}_{0\{\mathfrak{a}\}}$, $\hat{\mathbf{p}}_{0\{a\}}$, and $\hat{s}_{3\{a\}}$ on f, that is,

$$\hat{\mathbf{x}}_{0\{a\}}f(\epsilon,\mathbf{x}) := \mathbf{x}f(\epsilon,\mathbf{x}), \qquad \hat{\mathbf{p}}_{0\{a\}}f(\epsilon,\mathbf{x}) := -i\hbar\boldsymbol{\nabla}f(\epsilon,\mathbf{x}), \qquad \hat{s}_{3\{a\}}f(\epsilon,\mathbf{x}) := \epsilon f(\epsilon,\mathbf{x}). \tag{3.61}$$

This shows that $f(\epsilon,\mathbf{x})$ are the position wave functions with definite chirality ϵ [53].

3.4 Position and Momentum Operators

In this section we derive explicit forms of the position $\mathbf{x}_0$, momentum $\mathbf{p}_0$, and angular momentum $\mathbf{L}$ operators for relativistic scalar fields. We present a concise explanation of the procedure and derive related results for $\mathbf{x}_0$, $\mathbf{p}_0$, and $\mathbf{L}$. These results are identical with those obtained in Refs [50, 53].

We define $\mathbf{x}_0$ as a position operator of a relativistic scalar field acting on $\psi\in\mathcal{H}_a$ and introduce the three-component field: $\boldsymbol{\chi} := \mathbf{x}_0\psi$. This is determined uniquely in terms of initial data $(\boldsymbol{\chi}(x_0^0), \dot{\boldsymbol{\chi}}(x_0^0))$ for some $x_0^0\in\mathbb{R}$, and its components satisfy the Klein-Gordon equation (3.2).

After doing the necessary algebra, we obtain

$$\boldsymbol{\chi}(x_0^0) = \boldsymbol{\mathfrak{X}}\psi_a(x_0^0), \qquad \dot{\boldsymbol{\chi}}(x_0^0) = (\boldsymbol{\mathfrak{X}} - \frac{i\hbar\mathbf{p}}{D})\dot{\psi}(x_0^0), \tag{3.62}$$

where

$$\boldsymbol{\mathfrak{X}} := \mathbf{x} + \frac{i\hbar\mathbf{p}}{2D} \tag{3.63}$$

It is not difficult to observe that for all $x^0 \in \mathbb{R}$ the position operator of relativistic scalar fields takes the form

$$\mathbf{x}_0 := \boldsymbol{\mathfrak{X}} - i(x^0 - x_0^0)\frac{\hbar\mathbf{p}}{D}\partial_0. \tag{3.64}$$

This is not only a Hermitian operator acting in the physical Hilbert space $\mathcal{H}_a$, but also has some very remarkable properties. It has commuting components by construction. It respects the charge superselection rule [54] and commutes with the chirality operator $\mathrm{C} = s_3$.

Similarly, in order to evaluate the momentum operator we should examine the action of $\mathbf{p}_0$ on ψ. In view of (3.52) and (3.40), we have

$$[\mathbf{p}_0\psi](x^0) = \mathbf{p}\psi(x^0), \qquad \forall x^0 \in \mathbb{R}. \tag{3.65}$$

Because for every differentiable element ϕ of $L^2(\mathbb{R}^3)$ the action of $[\frac{i\hbar\mathbf{p}}{2D}]\phi$ is along $\mathbf{p}\phi$, the angular momentum operator $\mathbf{L} := \mathbf{x}_0 \times \mathbf{p}_0$ acts on ψ according to

$$[\mathbf{L}\psi](x^0) = \mathbf{x} \times \mathbf{p}\psi(x^0), \qquad \forall x^0 \in \mathbb{R}. \tag{3.66}$$

Unlike the position operator $\mathbf{x}_0$, the momentum $\mathbf{p}_0$ and angular momentum L operators have the same expressions as their non-relativistic counterparts [53].

3.5 Position Wave Functions, Localized States and Probability Density

In a close analogy with [53], here we give the explicit form of position wave functions and localized relativistic scalar fields. In view of (3.43), (3.43), (3.49) and (3.53), the localized scalar fields with a definite helicity $\psi^{\epsilon}_{a,\mathbf{x}}$ take the following form,

$$\psi^{\epsilon}_{a,\mathbf{x}}(x^0) = \frac{1}{2\sqrt{g}}D^{-1/4}[\frac{(1+\epsilon)e^{-i(x^0-x_0^0)D^{1/2}}}{\sqrt{1+a}} + \frac{(1-\epsilon)e^{i(x^0-x_0^0)D^{1/2}}}{\sqrt{1-a}}]|\mathbf{x}\rangle, \tag{3.67}$$

and similarly its derivative has

$$\dot{\psi}^{\epsilon}_{a,\mathbf{x}}(x^0) = \frac{-i\epsilon}{2\sqrt{g}}D^{1/4}[\frac{(1+\epsilon)e^{-i(x^0-x_0^0)D^{1/2}}}{\sqrt{1+a}} + \frac{(1-\epsilon)e^{i(x^0-x_0^0)D^{1/2}}}{\sqrt{1-a}}]|\mathbf{x}\rangle. \tag{3.68}$$

We use localized scalar fields $\psi^{\epsilon}_{a,\mathbf{x}}$ to define localized states of relativistic scalar fields at a space-time point $(x^0, \mathbf{x})$ as

$$\psi^{\epsilon}_{\mathbf{y}}(x^0, \mathbf{x}) := \langle \mathbf{x} | \psi^{\epsilon}_{\mathbf{y}}(x^0) \rangle \qquad .$$

By using (3.67), we can compute the value of localized states of relativistic scalar fields. For massive case it is given by

$$\psi^{\epsilon}_{a,\mathbf{y}}(x^0, \mathbf{x}) = \frac{1}{2\pi^2 z\sqrt{g}} \int_0^{\infty} dk \, \frac{k \sin(zk)}{(k^2 + \mathfrak{m}^2)^{1/4}} \{ \frac{(1+\epsilon) e^{-i(x^0 - x^0_0)\sqrt{k^2 + \mathfrak{m}^2}}}{\sqrt{1+a}} + \frac{(1-\epsilon) e^{i(x^0 - x^0_0)\sqrt{k^2 + \mathfrak{m}^2}}}{\sqrt{1-a}} \}, \tag{3.69}$$

and for massless case

$$\psi^{\epsilon}_{a,\mathbf{y}}(x^0, \mathbf{x}) = \frac{1}{2\pi^2 z\sqrt{g}} \int_0^{\infty} dk \, k^{1/2} \sin(zk) \, \{ \frac{(1+\epsilon) e^{-i(x^0 - x^0_0)k}}{\sqrt{1+a}} + \frac{(1-\epsilon) e^{i(x^0 - x^0_0)k}}{\sqrt{1-a}} \}, \tag{3.70}$$

where $z := |\mathbf{x} - \mathbf{y}|$. For massive scalar fields we can take $g = \frac{1}{\mathfrak{m}}$ which is equal to the Compton wavelength.

For $x^0 \in \mathbb{R}$, integrals in (3.50) and (3.51) cannot be evaluated. But for $x^0 = x^0_0$, they result in

$$\psi^{\epsilon}_{a,\mathbf{y}}(x^0, \mathbf{x}) = \begin{cases} \gamma \, [2^{3/4} \pi^{3/2} \Gamma(\frac{1}{4})]^{-1} (\frac{\mathfrak{m}}{|\mathbf{x} - \mathbf{y}|})^{\frac{5}{4}} K_{\frac{5}{4}}(\mathfrak{m}|\mathbf{x} - \mathbf{y}|), & \text{Massive} \\ \gamma \, \frac{\pi}{(2\pi|\mathbf{x} - \mathbf{y}|)^{\frac{5}{2}}}, & \text{Massless} \end{cases} \tag{3.71}$$

where $\gamma := \frac{1}{\sqrt{g}}[\frac{(1+\epsilon)}{\sqrt{1+a}} + \frac{(1-\epsilon)}{\sqrt{1-a}}]$. Note that we have provided details of calculation of massless scalar fields in Appendix A. By making use of Foldy equation (3.61) for ψ^{ϵ}, we can obtain the localized states of relativistic scalar fields for all $x^0 \in \mathbb{R}$ as

$$\psi^{\epsilon}_{a,\mathbf{y}}(x^0, \mathbf{x}) = e^{i\epsilon(x^0 - x^0_0)D^{1/2}} \psi^{\epsilon}_{a,\mathbf{y}}(x^0_0, \mathbf{x}). \tag{3.72}$$

Let us observe that the expression for the massive case is identified to the result obtained by Newton and Wigner [23]. Taking $\mathfrak{m} \to 0$, we can show that $\psi_{\mathfrak{m}\to 0} \to \frac{\pi}{(2\pi z)^{5/2}}$ [29]. Furthermore, if we first set $\mathfrak{m} = 0$ (the massless case), the resulting localized state is $\psi_{massless} = \frac{\pi}{(2\pi z)^{5/2}}$ which is the same as the case when $\mathfrak{m} \to 0$ or $z \to 0$.

Next, we substitute (3.67) into (3.57) and carry out the necessary calculations to obtain the position wave function of relativistic scalar fields. This gives

$$\begin{aligned} f_a(\epsilon, \mathbf{x}) \; = \; \frac{\sqrt{g}}{4} \Big\{ & (1+\epsilon)\sqrt{1+a} \, e^{i(x^0 - x^0_0)D^{1/2}} [D^{1/4}\psi(x^0, \mathbf{x}) + iD^{-1/4}\dot{\psi}(x^0, \mathbf{x})] \\ & + (1-\epsilon)\sqrt{1-a} \, e^{-i(x^0 - x^0_0)D^{1/2}} [D^{1/4}\psi(x^0, \mathbf{x}) - iD^{-1/4}\dot{\psi}(x^0, \mathbf{x})] \Big\} \end{aligned} \tag{3.73}$$

We can use the relation:

$$\psi_a^{\pm} = \frac{1}{2}(\psi_a \pm iD^{-1/2}\psi_a)$$

to further simplify Eq. (3.73):

$$\begin{aligned} f_a(\epsilon,\mathbf{x}) \;=\; & \frac{\sqrt{g}}{2}D^{1/4}\Big\{(1+\epsilon)\sqrt{1+a}\,e^{i(x^0-x_0^0)D^{1/2}}\;\psi^+(x^0,\mathbf{x}) \\ & +(1-\epsilon)\sqrt{1-a}\,e^{-i(x^0-x_0^0)D^{1/2}}\;\psi^-(x^0,\mathbf{x})\Big\}. \end{aligned} \tag{3.74}$$

For $x^0 = x_0^0$, we can reduce (3.73) to

$$\begin{aligned} f_a(\epsilon,\mathbf{x}) \;=\; & \frac{\sqrt{g}}{4}\Big\{[(1+\epsilon)\sqrt{1+a}+(1-\epsilon)\sqrt{1-a}]D^{1/4}\psi(x^0,\mathbf{x}) \\ & +i[(1+\epsilon)\sqrt{1+a}-(1-\epsilon)\sqrt{1-a}]D^{-1/4}\dot{\psi}(x^0,\mathbf{x})\Big\}. \end{aligned} \tag{3.75}$$

Ref [53] explains all steps to obtain the expression for the probability density of the spatial localization of a relativistic scalar field. We only need to give the results of the probability density at initial time x_0^0,

$$\begin{aligned} \varrho_a(x_0^0,\mathbf{x}) :\; =\; & \sum_{\epsilon=\pm}|f_a(\epsilon,\mathbf{x})|^2 \\ =\; & \frac{g}{2}\Big\{|D^{1/4}\psi(x)|^2+|D^{-1/4}\dot{\psi}(x)|^2 \\ & +ia\Big[(D^{1/4}\psi(x))^*D^{-1/4}\dot{\psi}(x)-(D^{1/4}\psi(x))(D^{-1/4}\dot{\psi}(x))^*\Big]\Big\} \\ =\; & \frac{g}{2}\Big\{|D^{1/4}\psi(x)|^2+|D^{1/4}\psi_c(x)|^2 \\ & +ia\left[(D^{1/4}\psi(x))^*D^{1/4}\psi_c(x)-(D^{1/4}\psi(x))(D^{1/4}\psi_c(x))^*\right]\Big\}. \end{aligned} \tag{3.76}$$

This result is identical to the one given in Ref [53]. Note that the coincidence of our approach and that of Rosentein and Horwitz [82] has the following important consequence [53]. *"The first-quantized theory formulated by an explicit construction of the Hilbert space and a position observable reproduces a result obtained from the second-quantized theory"* .

As seen in [53], in order to derive the explicit form of the four-vector current density $\mathcal{J}_a^\mu$, we first identify $\mathcal{J}_a^0$ with ϱ_a. Next, we perform an infinitesimal Lorentz boost transformation,

$$x^0 \to x'^0 = x^0 - \boldsymbol{\beta}\cdot\mathbf{x}, \qquad\qquad \mathbf{x} \to \mathbf{x}' = \mathbf{x} - \boldsymbol{\beta}x^0,$$

where $\boldsymbol{\beta} := \mathbf{v}/c$. The corresponding transformation rule for $\mathcal{J}_a^0$ is

$$\mathcal{J}_a^0(x) \to \mathcal{J}'^0_a(x') = \mathcal{J}_a^0(x) - \boldsymbol{\beta}\cdot\boldsymbol{\mathcal{J}'}_a(x). \tag{3.77}$$

This is computed in Ref [53], and the result is given by

$$\begin{aligned}\boldsymbol{\mathcal{J}'}_a(x) = -\frac{ig}{4}\Big\{&(D^{1/4}\psi(x))^*\boldsymbol{\nabla}D^{-1/4}\psi_c(x) - D^{1/4}\psi(x)\boldsymbol{\nabla}(D^{-1/4}\psi_c(x))^*\\ &+(D^{1/4}\psi_c(x))^*\boldsymbol{\nabla}D^{-1/4}\psi(x) - D^{1/4}\psi_c(x)\boldsymbol{\nabla}(D^{-1/4}\psi(x))^*\\ &+a\left[(D^{1/4}\psi(x))^*\boldsymbol{\nabla}D^{-1/4}\psi(x) - D^{1/4}\psi(x)\boldsymbol{\nabla}(D^{-1/4}\psi(x))^*\right.\\ &\left.+(D^{1/4}\psi_c(x))^*\boldsymbol{\nabla}D^{-1/4}\psi_c(x) - D^{1/4}\psi_c(x)\boldsymbol{\nabla}(D^{-1/4}\psi_c(x))^*\right]\Big\}. \end{aligned}\tag{3.78}$$

This relation suggests the following expression for $\mathcal{J}_a^\mu$

$$\begin{aligned}\mathcal{J}_a^\mu(x) = -\frac{ig}{4}\Big\{&(D^{1/4}\psi(x))^*\partial^\mu D^{-1/4}\psi_c(x) - D^{1/4}\psi(x)\partial^\mu(D^{-1/4}\psi_c(x))^*\\ &+(D^{1/4}\psi_c(x))^*\partial^\mu D^{-1/4}\psi(x) - D^{1/4}\psi_c(x)\partial^\mu(D^{-1/4}\psi(x))^*\\ &+a\left[(D^{1/4}\psi(x))^*\partial^\mu D^{-1/4}\psi(x) - D^{1/4}\psi(x)\partial^\mu(D^{-1/4}\psi(x))^*\right.\\ &\left.+(D^{1/4}\psi_c(x))^*\partial^\mu D^{-1/4}\psi_c(x) - D^{1/4}\psi_c(x)\partial^\mu(D^{-1/4}\psi_c(x))^*\right]\Big\}, \end{aligned}\tag{3.79}$$

which can be written in a concise form as

$$\begin{aligned}\mathcal{J}_a^\mu(x) \;=\; \frac{g}{2}\Im\Big\{&(D^{1/4}\psi_a(x))^*\partial^\mu D^{-1/4}\psi_{a,c}(x) - (D^{1/4}\psi_{a,c}(x))\partial^\mu(D^{-1/4}\psi_a(x))^*\\ &+a\left[(D^{1/4}\psi_a(x))^*\partial^\mu D^{-1/4}\psi_a(x) - (D^{1/4}\psi_{a,c}(x))\partial^\mu(D^{-1/4}\psi_{a,c}(x))^*\right]\Big\}. \end{aligned}\tag{3.80}$$

Here $\Im$ stands for the imaginary part of its argument.

In view of (3.35), (3.59) and (3.76)

$$\int_{\mathbb{R}^3} d^3\mathbf{x}\,\varrho_a(x^0,\mathbf{x}) = \int_{\mathbb{R}^3} d^3\mathbf{x}\,J_a^0(x^0,\mathbf{x}),\tag{3.81}$$

and since J_a^μ is conserved and covariant, the total probability density, i.e,

$$\mathcal{P}_a := \int_{\mathbb{R}^3} d^3\mathbf{x}\,\varrho_a(x^0,\mathbf{x}) = \int_{\mathbb{R}^3} d^3\mathbf{x}\,J_a^0(x^0,\mathbf{x}),\tag{3.82}$$

is a frame-independent conserved quantity.

Note that, ϱ_a is not the zero-component of the four-vector current density J_a^μ which is a conserved quantity, but in view of (3.82) we observe that the integration of ϱ_a over the whole space yields the total probability which is conserved. Moreover, this global conservation law is caused by the local conservation law through the continuity equation for a four-vector current density J_a^μ [53].

Chapter 4

QUANTUM MECHANICS OF A SINGLE PHOTON

4.1 Hermiticity and Photon Fields

4.1.1 Maxwell fields

We begin our discussion with a free source Maxwell equation by using the second rank antisymmetric tensor $F^{\mu\nu}$ in terms of $\mathbf{E}$ and $\mathbf{B}$ fields,

$$F^{ij} =: \varepsilon^{ijk} B_k, \qquad F^{0i} =: E^i. \tag{4.1}$$

The Maxwell equation can be expressed as

$$\begin{aligned} &\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \qquad &&\dot{\mathbf{A}} = -\mathbf{E} - \nabla A^0, \\ &\boldsymbol{\nabla} \cdot \mathbf{E} = 0, &&\dot{\mathbf{E}} = \boldsymbol{\nabla} \times \mathbf{B}. \end{aligned} \tag{4.2}$$

Let us denote $A^\mu := (A^0, \mathbf{A})$ as a four-vector Maxwell field, and consider a gauge transformation $A^\mu \longrightarrow \tilde{A}^\mu = A^\mu + \partial^0 f$ where $\partial^\mu := (\partial^0, \nabla)$. The time-like and space-like components of a Maxwell field transforms as

$$\mathbf{A} \longrightarrow \tilde{\mathbf{A}} = \mathbf{A} + \nabla f, \qquad A^0 \longrightarrow \tilde{A}^0 = A^0 - \dot{f}. \tag{4.3}$$

If we assume $\nabla \tilde{A}^0 = 0$ for all $\mathbf{x} \in \mathbb{R}$, we get

$$\nabla \tilde{A}^0 = 0, \qquad \Rightarrow \tilde{A}^0 = \tilde{A}^0(x^0), \qquad \Rightarrow \tilde{A}^0(x^0) = \dot{f} + \tilde{A}^0(x^0), \tag{4.4}$$

and solving (4.4) leads to

$$f = \int A^0 dt + F(x^0) + \mathsf{h}(\mathbf{x}), \tag{4.5}$$

where $F(x^0)$ and $\mathsf{h}(\mathbf{x})$ are functions of time and space, respectively.

By using (4.3) and (4.4) we can write Maxwell equations (4.2) as

$$\begin{aligned} &\mathbf{B} = \boldsymbol{\nabla} \times \tilde{\mathbf{A}}, \qquad &&\dot{\tilde{\mathbf{A}}} = -\mathbf{E}, \\ &\boldsymbol{\nabla} \cdot \mathbf{E} = 0, &&\dot{\mathbf{E}} = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \tilde{\mathbf{A}}. \end{aligned} \tag{4.6}$$

One can easily check that f which satisfies the wave equation $\ddot{f} = \nabla^2 f$. Substituting (4.4) and (4.5) in this equation yields

$$\nabla^2 \mathsf{h}(\mathbf{x}) = -\dot{\tilde{A}}^0(x^0). \tag{4.7}$$

The left-hand side of this equation has a dependence on spatial coordinates while the right side has only a dependence on time. Therefore,

$$\nabla^2 \mathsf{h}(\mathbf{x}) = -\dot{\tilde{A}}^0(x^0) = -c_0,$$

where c_0 is a constant. It gives $\tilde{A}^0(x^0) = c_0 x^0 + c_1$, where c_1 is also a constant. Now, we set both c_0 and c_1 to zero and use (4.4). This yields

$$\tilde{A}^0(x^0) = A^0(x^0) - \dot{f} = 0, \tag{4.8}$$

and equivalently we have

$$f(\mathbf{x}, x^0) = \int_{x_0^0}^{x^0} A^0(\mathbf{x}, s) ds + f(\mathbf{x}, x_0^0).$$

By evaluation the divergence of both sides of (4.3) and making use of (4.8) and Lorentz gauge condition $\boldsymbol{\nabla} \cdot \mathbf{A}(\mathbf{x}, x^0) + \dot{A}^0(\mathbf{x}, x^0) = 0$, we obtain

$$\boldsymbol{\nabla} \cdot \tilde{\mathbf{A}}(\mathbf{x}, x_0^0) = \nabla^2 f(\mathbf{x}, x_0^0) - \dot{A}^0(\mathbf{x}, x_0^0). \tag{4.9}$$

If we apply the Lorentz gauge condition $\dot{\tilde{A}}^0(\mathbf{x}, x_0^0) + \boldsymbol{\nabla} \cdot \tilde{\mathbf{A}}(\mathbf{x}, x_0^0) = 0$ at time $x^0 = x_0^0$ on the right hand side of (4.9), we get

$$\boldsymbol{\nabla} \cdot \tilde{\mathbf{A}}(\mathbf{x}, x^0) = \boldsymbol{\nabla} \cdot \tilde{\mathbf{A}}(\mathbf{x}, x_0^0) = 0, \tag{4.10}$$

and using (4.8) and (4.10) leads to

$$\tilde{A}^0(x^0) = 0, \quad \textit{and} \quad \boldsymbol{\nabla} \cdot \tilde{\mathbf{A}}(\mathbf{x}, x^0) = 0. \tag{4.11}$$

This is called the Lorentz-Coulomb gauge condition.

4.1.2 Covariant Dynamical Field Equation and Helicity States

In this section we first give a covariant dynamical form of Maxwell equations and examine the physical polarizations and helicity states of Maxwell fields. In order to achieve this goal, we start with

$$\partial_\nu \partial^\nu \tilde{A}^\mu = 0, \tag{4.12}$$

where for all $x^0 \in \mathbb{R}$ and $\mu \in \{0,1,2,3\}$, $\int_{\mathbb{R}^3} d^3\mathbf{x}\, |\tilde{A}^\mu(x^0,\mathbf{x})|^2 < \infty$. We express (4.12) as a dynamical equation in the Hilbert space $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$,

$$\ddot{\tilde{A}}(x^0) + D\tilde{A}(x^0) = 0, \tag{4.13}$$

where a dot denotes a x^0-derivative. Also for all $\mu \in \{0,1,2,3\}$ and $x^0 \in \mathbb{R}$, we define functions $\tilde{A}^\mu(x^0) : \mathbb{R}^3 \to \mathbb{C}$ by $\tilde{A}^\mu(x^0)(\mathbf{x}) := \tilde{A}^\mu(x^0,\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^3$, and let $D : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ be the Hermitian operator,

$$[Df](\mathbf{x}) := [-\nabla^2]f(\mathbf{x}), \qquad \forall f \in L^2(\mathbb{R}^3), \tag{4.14}$$

such that for all $\mu \in \{0,1,2,3\}$ and $x^0 \in \mathbb{R}$, $\tilde{A}^\mu(x^0)$ belongs to $L^2(\mathbb{R}^3)$. D is a positive-definite operator with eigenvalues $\omega_{\mathbf{k}}^2 := k^2$ and corresponding eigenvectors

$$\phi_{\mathbf{k}}(\mathbf{x}) := (2\pi)^{-3/2} e^{i\mathbf{k}\cdot\mathbf{x}} = \langle \mathbf{x}|\mathbf{k}\rangle,$$

where $\mathbf{k} \in \mathbb{R}^3$, $k := |\mathbf{k}|$, and $\langle\cdot|\cdot\rangle$ denotes the inner product of $L^2(\mathbb{R}^3)$.

Now, let us consider the space-like components of plane wave solutions of (4.13). These have the form

$$\tilde{\mathbf{A}}_\epsilon(\mathbf{k},\sigma;x) = N_{\epsilon,\mathbf{k}}\, \mathbf{a}_\epsilon(\mathbf{k},\sigma)\, e^{-i\epsilon k x^0} \phi_{\mathbf{k}}(\mathbf{x}), \tag{4.15}$$

where $\epsilon \in \{\pm\}$, $N_{\epsilon,\mathbf{k}}$ are normalization constants, and $\mathbf{a}_\epsilon(\mathbf{k},\sigma)$ are normalized polarization vectors fulfilling,

$$a_{\epsilon\, i}(\mathbf{k},\sigma)\, a_\epsilon^i(\mathbf{k},\sigma') = \delta_{\sigma\sigma'}, \qquad k_i\, a_\epsilon^i(\mathbf{k},\sigma) = 0 \quad i \in \{1,2,3\}. \tag{4.16}$$

From (4.15) and (4.16) we observe that the photon field has two polarization states corresponding to $\mathbf{a}_\epsilon(\mathbf{k},\sigma)$ with $\sigma \in \{1,2\}$. These are called transverse polarization vectors because they are perpendicular to $\mathbf{k}$. Together with longitudinal vector $\mathbf{k}$, i.e,

$$\mathbf{a}_\epsilon(\mathbf{k},3) = \frac{\mathbf{k}}{k}, \tag{4.17}$$

they form an orthonormal basis [42, 43]. It is not difficult to show that the transverse polarization vectors satisfy the completeness relation,

$$\sum_{\sigma=1}^{2} a_\epsilon^i(\mathbf{k},\sigma)\, a_\epsilon^j(\mathbf{k},\sigma) = \delta^{ij} - \frac{k^i k^j}{k^2}. \tag{4.18}$$

Now we introduce circular polarization vectors [42, 43]:

$$\mathbf{u}_{\epsilon,\pm 1}(\mathbf{k}) := \frac{1}{\sqrt{2}}\left[\mathbf{a}_\epsilon(\mathbf{k},1) \pm i\,\mathbf{a}_\epsilon(\mathbf{k},2)\right], \qquad \mathbf{u}_{\epsilon,0}(\mathbf{k}) := \mathbf{a}_\epsilon(\mathbf{k},3). \tag{4.19}$$

These are the eigenvectors of the helicity operator

$$\mathfrak{h} = \frac{\mathfrak{K} \cdot \mathbf{S}}{|\mathfrak{K}|} = \hat{\mathfrak{K}} \cdot \mathbf{S} = \frac{i}{|\mathfrak{K}|} \begin{pmatrix} 0 & -\mathfrak{K}_3 & \mathfrak{K}_2 \\ \mathfrak{K}_3 & 0 & -\mathfrak{K}_1 \\ -\mathfrak{K}_2 & \mathfrak{K}_1 & 0 \end{pmatrix}, \tag{4.20}$$

where $\mathfrak{K} =: (\mathfrak{K}_1, \mathfrak{K}_2, \mathfrak{K}_3)$ is given by

$$[\mathfrak{K} f](\mathbf{x}) := -i\boldsymbol{\nabla} f(\mathbf{x}), \quad \forall f \in L^2(\mathbb{R}^3), \tag{4.21}$$

and

$$S_1 := \lambda_7, \qquad S_2 := -\lambda_5, \qquad S_3 := \lambda_2, \tag{4.22}$$

they are generators of the rotation group in its spin-one representations [44].

In general the eigenvectors (4.19) of the helicity operator (4.20) with nonzero eigenvalues have the form

$$\mathbf{u}_h(\mathbf{k}) = \frac{\mathbf{k} \times (\mathbf{m} \times \mathbf{k}) + ih|k|(\mathbf{k} \times \mathbf{m})}{\sqrt{2}|k||\mathbf{m} \times \mathbf{k}|}$$

where $h \in \{\pm 1\}$ and $\mathbf{m} =: (m_1, m_2, m_3)$ is an arbitrary vector. If we take $\mathbf{m} = \hat{\mathbf{i}}$, the unit vector in direction of x-axis, we obtain the explicit form of $\mathbf{u}_h(\mathbf{k})$ according to

$$\mathbf{u}_h(\mathbf{k}) = \frac{1}{|k|\sqrt{2(k_2^2 + k_3^2)}} \begin{pmatrix} k_2^2 + k_3^2 \\ -k_1 k_2 + ihk_3|k| \\ -k_1 k_3 - ihk_2|k| \end{pmatrix}.$$

In view of (4.21), we can rewrite Lorentz-Coulomb gauge condition and $\boldsymbol{\nabla} \cdot \mathbf{E}(x^0) = -\boldsymbol{\nabla} \cdot \dot{\tilde{\mathbf{A}}}(x^0) = 0$ according to

$$i\mathfrak{K} \cdot \tilde{\mathbf{A}}(x^0) = 0 \quad i\mathfrak{K} \cdot \dot{\tilde{\mathbf{A}}}(x^0) = 0. \tag{4.23}$$

where " $\cdot$ " is the usual dot product. Next, we use (4.13) and (4.23) to define the complex vector space ($\mathcal{V}$) of solutions of the Maxwell equations

$$\mathcal{V} := \Big\{ \tilde{A} := (0, \tilde{\mathbf{A}}) \mid \forall x^0 \in \mathbb{R},\ [\partial_0^2 + D]\tilde{A}^\mu(x^0) = 0, \\ \text{and } \exists x_0^0 \in \mathbb{R},\ \mathfrak{K} \cdot \tilde{\mathbf{A}}(x^0) = \mathfrak{K} \cdot \dot{\tilde{\mathbf{A}}}(x^0) = 0 \Big\}. \tag{4.24}$$

We call this vector space the space of photon fields. By using (4.23) and (4.24), we observe that in the space $\mathcal{V}$, $\mathfrak{h}^3 = \mathfrak{h}$ and $\mathfrak{h}^2$ is the identity operator, $1_{\mathcal{V}}$.

The solution of (4.13) can be expressed in terms of $(\tilde{\mathbf{A}}(x_0^0), \dot{\tilde{\mathbf{A}}}(x_0^0))$ [49, 78] according to

$$\tilde{\mathbf{A}}^\mu(x^0) = \cos[(x^0 - x_0^0)D^{1/2}]\tilde{\mathbf{A}}^\mu(x_0^0) + \sin[(x^0 - x_0^0)D^{1/2}]D^{-1/2}\dot{\tilde{\mathbf{A}}}^\mu(x_0^0). \qquad (4.25)$$

Here we use the spectral resolution of D to denote its powers, $D^\alpha := \int_{\mathbb{R}^3} d^3\mathbf{k}(k^2)^\alpha |\mathbf{k}\rangle\langle\mathbf{k}|$ for all $\alpha \in \mathbb{R}$.

Finally, we can build the basic (plane wave) solutions of (4.24) with a definite helicity. These are given by

$$\tilde{\mathbf{A}}^\mu_{\epsilon,s}(\mathbf{k}; x^0, \mathbf{x}) = N_{\epsilon,s}(\mathbf{k})\, \mathbf{u}^\mu_{\epsilon,s}(\mathbf{k})\, e^{-i\epsilon k x^0}\phi_{\mathbf{k}}(\mathbf{x}), \qquad \epsilon = \pm,\ s = -1, 1, \qquad (4.26)$$

where $N_{\epsilon,s}(\mathbf{k})$ are normalization constants.

4.2 Six-Component Formulation and Quantum Mechanics of Photon Fields

By using the identity $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \tilde{\mathbf{A}}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \tilde{\mathbf{A}}) - \nabla^2\tilde{\mathbf{A}}$ in Maxwell equations (4.6) and making use of the Lorentz-Coulomb gauge condition (4.11), we get

$$\begin{array}{ll} \mathbf{B} = \boldsymbol{\nabla} \times \tilde{\mathbf{A}}, & \dot{\tilde{\mathbf{A}}} = -\mathbf{E}, \\ \boldsymbol{\nabla} \cdot \mathbf{E} = 0, & \dot{\mathbf{E}} = -\nabla^2\tilde{\mathbf{A}}. \end{array} \qquad (4.27)$$

Let us define $\tilde{\mathcal{H}} := \{\mathbf{Y} : \mathbb{R}^3 \to \mathbb{C}^3 | \int_{\mathbb{R}^3} d^3\mathbf{x}\, \mathbf{Y}(\mathbf{x})^* \cdot \mathbf{Y}(\mathbf{x}) < \infty\}$ as the Hilbert space of vector fields. For every arbitrary $\mathbf{Y} \in \tilde{\mathcal{H}}$, we can use (4.20) and (4.21) to prove the following identities

$$\begin{array}{ll} [\mathfrak{K}^2\mathbf{Y}](\mathbf{x}) = -\nabla^2\mathbf{Y}(\mathbf{x}), & [(\mathfrak{K} \cdot \mathbf{S})\mathbf{Y}](\mathbf{x}) = \boldsymbol{\nabla} \times \mathbf{Y}(\mathbf{x}), \\ \mathfrak{K} \cdot [(\mathfrak{K} \cdot \mathbf{S})\mathbf{Y}] = 0, & \mathfrak{K} \cdot [(\mathfrak{K} \cdot \mathbf{S})^2\mathbf{Y}] = 0. \end{array} \qquad (4.28)$$

In view of (4.28), (4.21) and the Lorentz-Coulomb gauge condition (4.11), we can simplify $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \tilde{\mathbf{A}}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \tilde{\mathbf{A}}) - \nabla^2\tilde{\mathbf{A}}$ and obtain the following relation

$$\mathfrak{K}^2 = |\mathfrak{K}|^{-2}\mathfrak{h}^2, \qquad \forall\tilde{\mathbf{A}} \in \mathcal{V}, \qquad (4.29)$$

where we identify $\mathfrak{h}^2$ with the identity operator $\mathcal{V}$. By employing (4.29), we can present (4.27) as a set of dynamical equations

$$\dot{\tilde{\mathbf{A}}}(x^0) = -\mathbf{E}(x^0), \qquad \dot{\mathbf{E}}(x^0) = D\tilde{\mathbf{A}}(x^0) = |\mathfrak{K}|^{-2}\mathfrak{h}^2\tilde{\mathbf{A}}(x^0), \qquad (4.30)$$

in the Hilbert space $\tilde{\mathcal{H}}$. Further we can express the set of dynamical equations (4.30) as the Schrödinger equation [45, 46],

$$i\hbar\dot{\Psi}(x^0) = H\Psi(x^0), \qquad (4.31)$$

where for all $x^0 \in \mathbb{R}$, we define the state vector Ψ and the Hamiltonian H according to

$$\Psi(x^0) := \begin{pmatrix} \tilde{\mathbf{A}}(x^0) - iD^{-1/2}\mathbf{E}(x^0) \\ \tilde{\mathbf{A}}(x^0) + iD^{-1/2}\mathbf{E}(x^0) \end{pmatrix}, \tag{4.32}$$

$$H := \hbar D^{1/2}\sigma_0 \otimes \mathfrak{h}^2, \tag{4.33}$$

respectively. We note that H is a Hermitian operator, and we can use $\mathbf{E}(x^0) = -\dot{\tilde{\mathbf{A}}}(x^0)$ to determine the six-component state vector Ψ in terms of $\tilde{\mathbf{A}}$ and $\dot{\tilde{\mathbf{A}}}$.

Next, we define the helicity operator in the six-component representation of the photon fields according to

$$\Lambda := \sigma_3 \otimes \mathfrak{h}. \tag{4.34}$$

This generates a symmetry of the Hamiltonian H. The simultaneous eigenvectors of H and Λ take the form

$$\Psi_{\epsilon,h}(\mathbf{k}) := \frac{1}{2}\begin{pmatrix} 1+\epsilon \\ 1-\epsilon \end{pmatrix} \mathbf{u}_{\epsilon,h}(\mathbf{k})\phi_{\mathbf{k}}, \tag{4.35}$$

where $\epsilon \in \{-1, 1\}$, $h \in \{-1, +1\}$, $k := |\mathbf{k}|$, $E_\epsilon(\mathbf{k}) := \epsilon\hbar k$, and $\phi_{\mathbf{k}}$ are eigenvectors of D corresponding to the eigenvalues $|\mathbf{k}|$.

They satisfy

$$H\Psi_{\epsilon,h}(\mathbf{k}) = E_\epsilon(\mathbf{k})\Psi_{\epsilon,h}(\mathbf{k}), \qquad \Lambda\Psi_{\epsilon,h}(\mathbf{k}) = h\,\Psi_{\epsilon,h}(\mathbf{k}). \tag{4.36}$$

Note that the eigenvectors $\Psi_{\epsilon,h}(\mathbf{k})$ form an orthonormal basis for the Hilbert space $\mathcal{H}'$,

$$\langle \Psi_{\epsilon,h}(\mathbf{k}), \Psi_{\epsilon',h'}(\mathbf{k}')\rangle = \delta_{\epsilon,\epsilon'}\,\delta_{h,h'}\,\delta^3(\mathbf{k}-\mathbf{k}'), \qquad \sum_{\epsilon=\pm}\sum_{h=\pm 1}\int_{\mathbb{R}^3} d^3\mathbf{k} |\Psi_{\epsilon,h}(\mathbf{k})\rangle\langle\Psi_{\epsilon,h}(\mathbf{k})| = 1_{\mathcal{V}}. \tag{4.37}$$

Here $\langle\cdot,\cdot\rangle$ is the inner product of $\mathcal{H}'$, and for $\xi, \zeta \in \mathcal{H}'$, $|\xi\rangle\langle\zeta|$ is the operator defined by $|\xi\rangle\langle\zeta|\chi := \langle\zeta,\chi\rangle\xi$, for all $\chi \in \mathcal{H}'$. Furthermore, we can easily check that H and Λ have the following spectral resolutions

$$H = \sum_{\epsilon=\pm}\sum_{h=\pm 1}\int_{\mathbb{R}^3} d^3\mathbf{k}\, E_\epsilon(\mathbf{k})|\Psi_{\epsilon,h}(\mathbf{k})\rangle\langle\Psi_{\epsilon,h}(\mathbf{k})|, \quad \Lambda = \sum_{\epsilon=\pm}\sum_{h=\pm 1}\int_{\mathbb{R}^3} d^3\mathbf{k}\, h|\Psi_{\epsilon,h}(\mathbf{k})\rangle\langle\Psi_{\epsilon,h}(\mathbf{k})|. \tag{4.38}$$

Next, we pay attention to the fact that the six-components vector $\Psi(x^0)$ defined by (4.32) belongs to the complex vector space $\mathbb{C}^2 \otimes \tilde{\mathcal{H}}$. We endow this space with the L^2-inner product:

$$\langle \xi, \zeta \rangle := \sum_{i=1}^{2} \prec \boldsymbol{\xi}^i, \boldsymbol{\zeta}^i \succ, \tag{4.39}$$

where $\xi = \begin{pmatrix} \boldsymbol{\xi}^1 \\ \boldsymbol{\xi}^2 \end{pmatrix}$, $\zeta = \begin{pmatrix} \boldsymbol{\zeta}^1 \\ \boldsymbol{\zeta}^2 \end{pmatrix} \in \mathsf{C}^2 \otimes \tilde{\mathcal{H}}$. After simplification, equation (4.39) takes the form

$$\langle \xi, \zeta \rangle := \frac{1}{2} \left[\prec \boldsymbol{\xi}_+, \boldsymbol{\zeta}_+ \succ + \prec \boldsymbol{\xi}_-, \boldsymbol{\zeta}_- \succ \right], \tag{4.40}$$

where $\boldsymbol{\xi}_\pm := \boldsymbol{\xi}^1 \pm \boldsymbol{\xi}^2$, $\boldsymbol{\zeta}_\pm := \boldsymbol{\zeta}^1 \pm \boldsymbol{\zeta}^2 \in \tilde{\mathcal{H}}$. Endowing $\mathbb{C}^2 \otimes \tilde{\mathcal{H}}$ with the inner product (4.40) yields a new inner product space whose Cauchy completion is a Hilbert space. We denote it by $\mathcal{H}' := \tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}}$. Thus, by this observation we can view $\Psi(x^0)$ as a state vector in $\mathcal{H}'$ and the Hamiltonian H as a linear operator acting in $\mathcal{H}'$.

We further define $U_{x^0} : \mathcal{V} \to \mathcal{H}'$ for all $x^0 \in \mathbb{R}$ according to

$$U_{x^0} \tilde{A} := \frac{1}{2}\, \Psi(x^0), \qquad \forall A \in \mathcal{V}. \tag{4.41}$$

By reviewing the definition of $\mathcal{V}$, we find that U_{x^0} does not cover all of $\mathcal{H}' = \tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}}$, because every $\tilde{\mathbf{A}} \in \mathcal{V}$ must also satisfy the condition (4.23). Clearly we see that U_{x^0} is a one-to-one mapping but not onto; that is, $U_{x^0} : \mathcal{V} \to \mathcal{H}'$ is not a bijective map. In an effort to make it a bijective map, we define a new Hilbert space

$$\mathcal{H}'_{ph} := \left\{ \begin{pmatrix} \boldsymbol{\xi}^1 \\ \boldsymbol{\xi}^2 \end{pmatrix} \mid \boldsymbol{\xi}^1, \boldsymbol{\xi}^2 \in \tilde{\mathcal{H}}, \qquad \mathfrak{K} \cdot \boldsymbol{\xi}^1 = \mathfrak{K} \cdot \boldsymbol{\xi}^2 = 0 \right\}. \tag{4.42}$$

As $\mathcal{H}'_{ph} \subset \mathcal{H}'$, by restricting the range of U_{x^0} to $\mathcal{H}'_{ph}$, we observe that $U_{x^0} : \mathcal{V} \to \mathcal{H}'_{ph}$ turns to be an invertible operator. Now we use U_{x^0} to obtain a corresponding positive-definite inner product on the complex vector space $\mathcal{V}$ according to

$$\langle\!\langle \tilde{A}, \tilde{A}' \rangle\!\rangle := \langle U_{x^0} \tilde{A}, U_{x^0} \tilde{A}' \rangle = \frac{1}{4} \langle \Psi(x^0), \Psi'(x^0) \rangle. \tag{4.43}$$

The inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ stays invariant under the dynamics generated by H [48]; that is to say, the right-hand side of (4.43) is x^0-independent. In view of (4.43) and (4.32), we obtain

$$\langle\!\langle \tilde{A}, \tilde{A}' \rangle\!\rangle := \frac{1}{2} \left[\prec \tilde{\mathbf{A}}(x^0), \tilde{\mathbf{A}}'(x^0) \succ + \prec \mathbf{E}(x^0), D^{-1} \mathbf{E}'(x^0) \succ \right]. \tag{4.44}$$

This renders a dynamically invariant inner product on $\mathcal{V}$ because one can easily show that (4.44) vanishes identically by taking the x^0-derivative of the right-hand side. We can obtain a separable Hilbert space $\mathcal{H}$ by equipping the inner product (4.44) to $\mathcal{V}$ and making Cauchy completion of the resulting space. We call this separable Hilbert space the physical Hilbert space of the relativistic quantum mechanics of the photon fields. Next we examine some remarkable properties of the inner product (4.44):

1. This inner product is manifestly positive-definite, but it is not relativistically invariant.

2. If we restrict this inner product to the subspace of positive-frequency photon fields, it coincides with the restriction of the indefinite Maxwell inner product to this subspace.

In the remaining part of this section, we obtain the explicit forms of the generalized parity ($\mathcal{P}$), time reversal ($\mathcal{T}$), and charge grading or chirality ($\mathcal{C}$) operators for a free photon field. Comparing to Refs [50, 51] here we give the definitions of ($\mathcal{P}$), ($\mathcal{T}$), and ($\mathcal{C}$) operators of a photon field according to

$$\mathcal{P} := \sum_{\epsilon=\pm}\sum_{h=\pm 1}\int_{\mathbb{R}^3} d^3\mathbf{k}\,\epsilon\,|\Psi_{\epsilon,h}(\mathbf{k})\rangle\langle\Psi_{\epsilon,h}(\mathbf{k})|, \tag{4.45}$$

$$\mathcal{T} := \sum_{\epsilon=\pm}\sum_{h=\pm 1}\int_{\mathbb{R}^3} d^3\mathbf{k}\,\epsilon\,|\Psi_{\epsilon,h}(\mathbf{k})\rangle \star \langle\Psi_{\epsilon,-h}(\mathbf{k})|, \tag{4.46}$$

$$\mathcal{C} := \sum_{\epsilon=\pm}\sum_{h=\pm 1}\int_{\mathbb{R}^3} d^3\mathbf{k}\,\epsilon\,|\Psi_{\epsilon,h}(\mathbf{k})\rangle\langle\Psi_{\epsilon,h}(\mathbf{k})|. \tag{4.47}$$

By using (4.32) in (4.45)-(4.47), we arrive at

$$\mathcal{P} = \mathcal{C} = \Sigma_3, \qquad \mathcal{T} = \Sigma_3\,\star, \qquad \mathcal{PT} = \star. \tag{4.48}$$

Clearly we observe that the Hamiltonian H is a PT-symmetric operator. By using (4.33), we can write the operator $\mathcal{C}$ as

$$\mathcal{C} = \hbar^{-1}D^{-1/2}H = \frac{H}{\sqrt{H^2}}. \tag{4.49}$$

The chirality operator $\mathcal{C}$ is a $\mathbb{Z}_2$-grading operator for the Hilbert space and divides the Hilbert space $\mathcal{H}$ into the spans of the eigenvectors of H with positive and negative eigenvalues.

Next, we employ the unitary operator $U_{x^0}:\mathcal{H}\to\mathcal{H}'_{ph}$ to examine the generalized parity, time-reversal, and chirality operators for the photon fields $\tilde{A}\in\mathcal{H}$. We define P, T, and C operators according to

$$\mathrm{P} := U_{x^0}^{-1}\mathcal{P}U_{x^0}, \qquad \mathrm{T} := U_{x^0}^{-1}\mathcal{T}U_{x^0}, \qquad \mathrm{C} := U_{x^0}^{-1}\mathcal{C}U_{x^0}. \tag{4.50}$$

One may attempt to obtain explicit expressions for P, T. Here we only give the explicit expressions for PT and C. For all $\tilde{A}\in\mathcal{H}$ and $x^0\in\mathbb{R}$, we have

$$[\mathrm{PT}\tilde{A}](x^0) = (0, \tilde{\mathbf{A}}(-x^0)^*), \tag{4.51}$$

$$\left[\mathrm{C}\tilde{A}\right](x^0) = iD^{-1/2}\dot{\tilde{A}}(x^0) =: \tilde{A}_c(x^0). \tag{4.52}$$

These relations show that PT is an ordinary time-reversal operator, and the chirality operator C acting in $\mathcal{H}$ is a Lorentz scalar [53]. One can use the chirality operator to divide $\mathcal{V}$ into the subspaces of positive and negative energies of the photon fields according to

$$\mathcal{V}_\pm := \{\tilde{A}_\pm\in\mathcal{V}|\mathrm{C}\tilde{A}_\pm = \pm\tilde{A}_\pm\}$$

where $\tilde{A}_\pm := \frac{1}{2}(\tilde{A}\pm\mathrm{C}\tilde{A})\in\mathcal{V}_\pm$ for every $\tilde{A}\in\mathcal{V}$. Therefore any $\tilde{A}\in\mathcal{V}$ can be written as $\tilde{A}=\tilde{A}_+ + \tilde{A}_-$ and in a similar way the corresponding Hilbert space can be decomposed to $\mathcal{V}=\mathcal{V}_+\oplus\mathcal{V}_-$.

4.3 The Most General Admissible Inner Product on the Space of Photon Fields

In this section we closely follow the procedure of section 3.2 for scalar fields. Let us note that the most general admissible positive-definite and invariant inner product of photon fields can be identified by the most general orthonormal basis $\{\Psi_{\epsilon,h}(\mathbf{k})\}$ consisting of simultaneous eigenvectors of H and Λ. Because H and Λ are Hermitian operators and share common eigenvectors, they are among the physical observables of the system. We define the most general inner product according to

$$\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\tilde{\eta}_+} := \langle\cdot,\tilde{\eta}_+\cdot\rangle.$$

where

$$\tilde{\eta}_+ := \mathcal{B}^\dagger\mathcal{B} \tag{4.53}$$

and $\mathcal{B}$ is an invertible and linear operator which commutes with H and Λ. We define $\mathcal{B}$ according to

$$\mathcal{B} = \sum_{\epsilon=\pm}\sum_{h=\pm1}\int_{\mathbb{R}^3} d^3\mathbf{k}\,\beta_{\epsilon,h}(\mathbf{k})|\Psi_{\epsilon,h}(\mathbf{k})\rangle\langle\Psi_{\epsilon,h}(\mathbf{k})|, \tag{4.54}$$

where $\beta_{\epsilon,h}(\mathbf{k})$ are arbitrary nonzero complex numbers. Using the definition of $\mathcal{B}$ and $\mathcal{B}^\dagger$ in $\tilde{\eta}_+$ and carrying out necessary algebra leads to

$$\tilde{\eta}_+ = \left[L_+^+\sigma_0 + L_-^+\sigma_3\right]\mathfrak{h}^2 + \left[L_+^-\sigma_0 + L_-^-\sigma_3\right]\mathfrak{h}, \tag{4.55}$$

where

$$L_\pm^\epsilon := \frac{1}{4}\int d^3\mathbf{k}\left[|\beta_{+,+1}(\mathbf{k})|^2| + \epsilon|\beta_{+,-1}(\mathbf{k})|^2| \pm |\beta_{-,+1}(\mathbf{k})|^2| \pm \epsilon|\beta_{-,-1}(\mathbf{k})|^2\right]|\mathbf{k}\rangle\langle\mathbf{k}|. \tag{4.56}$$

Next, for all $\xi,\zeta\in\mathcal{H}'_{ph}$ we employ $\tilde{\eta}_+$ to compute the inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\tilde{\eta}_+} = \langle\cdot,\tilde{\eta}_+\cdot\rangle$. This yields

$$\langle\!\langle\xi,\zeta\rangle\!\rangle_{\tilde{\eta}_+} = \prec\boldsymbol{\xi}_+,\Theta_+\boldsymbol{\zeta}_+\succ + \prec\boldsymbol{\xi}_-,\Theta_+\boldsymbol{\zeta}_-\succ + \prec\boldsymbol{\xi}_+,\Theta_-\boldsymbol{\zeta}_-\succ + \prec\boldsymbol{\xi}_-,\Theta_-\boldsymbol{\zeta}_+\succ, \tag{4.57}$$

where $\Theta_p : \tilde{\mathcal{H}}_{ph}\to\tilde{\mathcal{H}}_{ph}$ is given by

$$\Theta_p := L_p^+\mathfrak{h}^2 + L_p^-\mathfrak{h}. \tag{4.58}$$

Viewing $\mathcal{H}'_{ph}$ as a complex vector space and endowing it with this inner product, we obtain a new inner product space whose Cauchy completion yields a Hilbert space which we denote by $\tilde{\mathcal{K}}$.

Next we employ $\Psi_{\epsilon,h}(\mathbf{k})$ to obtain the explicit forms of $\mathcal{B}$ and $\mathcal{B}^{-1}$,

$$\mathcal{B} = \begin{pmatrix} Z_+^+\mathfrak{h}^2 + Z_-^+\mathfrak{h} & 0 \\ 0 & Z_+^-\mathfrak{h}^2 + Z_-^-\mathfrak{h}\end{pmatrix}, \quad \mathcal{B}^{-1} = \begin{pmatrix} \tilde{Z}_+^+\mathfrak{h}^2 + \tilde{Z}_-^+\mathfrak{h} & 0 \\ 0 & \tilde{Z}_+^-\mathfrak{h}^2 + \tilde{Z}_-^-\mathfrak{h}\end{pmatrix}, \tag{4.59}$$

where

$$Z_\pm^\epsilon := \frac{1}{2}[\tilde{\beta}_{\epsilon,1} \pm \tilde{\beta}_{\epsilon,-1}], \qquad \tilde{Z}_\pm^\epsilon := \frac{1}{2}[\tilde{\beta}_{\epsilon,1}^{-1} \pm \tilde{\beta}_{\epsilon,-1}^{-1}], \tag{4.60}$$

$$\tilde{\beta}_{\epsilon,1} :=:= \int_{\mathbb{R}^3} d^3\mathbf{k}\,\beta_{\epsilon,1}(\mathbf{k})\,|\mathbf{k}\rangle\langle\mathbf{k}|, \qquad \tilde{\beta}_{\epsilon,-1}^{-1} := \int_{\mathbb{R}^3} d^3\mathbf{k}\,\beta_{\epsilon,-1}^{-1}(\mathbf{k})\,|\mathbf{k}\rangle\langle\mathbf{k}|. \tag{4.61}$$

The operator $\mathcal{B}$ mapping $\tilde{\mathcal{K}}$ onto $\mathcal{H}'_{ph}$ is unitary, i.e, for all $\xi,\zeta\in\mathcal{H}'_{ph}$,

$$\langle\!\langle\mathcal{B}\xi,\mathcal{B}\zeta\rangle\!\rangle = \langle\xi,\mathcal{B}^\dagger\mathcal{B}\zeta\rangle = \langle\!\langle\xi,\zeta\rangle\!\rangle_{\tilde{\eta}_+}. \tag{4.62}$$

We also define a new linear operator $\mathcal{U}_{\{\mathfrak{b}\}} : \mathcal{H}_{\{\mathfrak{b}\}} \to \mathcal{H}'_{ph}$ according to

$$\mathcal{U}_{\{\mathfrak{b}\}} := \mathcal{B}\, U_{x_0^0} \tag{4.63}$$

We employ (4.63), (4.59), (4.41) and (4.32) to compute the action of the Foldy transformation $\mathcal{U}_{\{\mathfrak{b}\}}$ on an arbitrary $\tilde{A} \in \mathcal{H}_{\{\mathfrak{b}\}}$. This gives

$$\mathcal{U}_{\{\mathfrak{b}\}}\tilde{A} = \frac{1}{2}\begin{pmatrix} \mathfrak{U}_+ \, [\tilde{\mathbf{A}}(x_0^0) - iD^{-1/2}\, \mathbf{E}(x_0^0)] \\ \mathfrak{U}_- \, [\tilde{\mathbf{A}}(x_0^0) + iD^{-1/2}\, \mathbf{E}(x_0^0)] \end{pmatrix}, \tag{4.64}$$

where the operators $\mathfrak{U}_\epsilon : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$, are given by

$$\mathfrak{U}_\epsilon = Z_+^\epsilon \mathfrak{h}^2 + Z_-^\epsilon \mathfrak{h}. \tag{4.65}$$

In order to keep the discussion short, we would not compute the explicit expression for $\mathcal{U}_{\{\mathfrak{b}\}}^{-1} = U_{x_0^0}^{-1}\tilde{\mathcal{B}}^{-1}$. By considering $\xi = \begin{pmatrix} \boldsymbol{\xi}^1 \\ \boldsymbol{\xi}^2 \end{pmatrix} \in \mathcal{H}'_{ph}$ as a six-component vector, we can show that $\mathcal{U}_{\{\mathfrak{b}\}}^{-1}\xi$ is a photon field $\tilde{A} \in \mathcal{H}_{\{\mathfrak{b}\}}$ satisfying

$$\dot{\tilde{\mathbf{A}}}(x_0^0) = \mathfrak{U}_+^{-1}\boldsymbol{\xi}^1 + \mathfrak{U}_-^{-1}\boldsymbol{\xi}^2, \qquad \mathbf{E}(x_0^0) = i\, D^{1/2}\, \left[\mathfrak{U}_+^{-1}\boldsymbol{\xi}^1 - \mathfrak{U}_-^{-1}\boldsymbol{\xi}^2\right], \tag{4.66}$$

where $\mathfrak{U}_\epsilon^{-1}$ is the inverse of $\mathfrak{U}_\epsilon$ given by

$$\mathfrak{U}_\epsilon^{-1} := \tilde{Z}_+^\epsilon \mathfrak{h}^2 + \tilde{Z}_-^\epsilon \mathfrak{h}. \tag{4.67}$$

We can use (4.25) to show that for all $x_0^0 \in \mathbb{R}$

$$\begin{aligned} \tilde{\mathbf{A}}^0(x^0) &= e^{-i(x^0 - x_0^0)D^{1/2}}\mathfrak{U}_+^{-1}\boldsymbol{\xi}^1 + e^{i(x^0 - x_0^0)D^{1/2}}\mathfrak{U}_-^{-1}\boldsymbol{\xi}^2, && (4.68)\\ \mathbf{E}(x^0) &= iD^{1/2}\left[e^{-i(x^0 - x_0^0)D^{1/2}}\mathfrak{U}_+^{-1}\boldsymbol{\xi}^1 - e^{i(x^0 - x_0^0)D^{1/2}}\mathfrak{U}_-^{-1}\boldsymbol{\xi}^2\right]. && (4.69)\end{aligned}$$

Because $U_{x^0} : \mathcal{V} \to \mathcal{H}'_{ph}$ is a unitary transformation, we can obtain a new inner product in $\mathcal{V}$ according to

$$(\!(\tilde{A}, \tilde{A}')\!)_{\{\mathfrak{b}\}} := \langle\!\langle U_{x^0}\tilde{A}, U_{x^0}\tilde{A}'\rangle\!\rangle_{\tilde{\eta}_+}, \qquad \forall \tilde{A}, \tilde{A}' \in \mathcal{H}_{\{\mathfrak{b}\}}. \tag{4.70}$$

Using (4.70) and (4.62), we further obtain

$$(\!(\tilde{A}, \tilde{A}')\!)_{\{\mathfrak{b}\}} = \langle U_{x_0^0}\tilde{A}, \mathcal{B}^\dagger\mathcal{B}U_{x_0^0}\tilde{A}'\rangle = \langle \mathcal{U}_{\{\mathfrak{b}\}}\tilde{A}, \mathcal{U}_{\{\mathfrak{b}\}}\tilde{A}'\rangle. \tag{4.71}$$

This clearly shows that the Foldy transformation $\mathcal{U}_{\{\mathfrak{b}\}}$ is a unitary transformation. Having defined $(\!(\tilde{A}, \tilde{A}')\!)_{\{\mathfrak{b}\}}$ by (4.70), we equip the space of $\mathcal{V}$ of the solutions of a single photon with the positive-definite inner product according to

$$\begin{aligned}(\!(\tilde{A}, \tilde{A}')\!)_{\{\mathfrak{b}\}} := &\prec \tilde{\mathbf{A}}(x^0), \Theta_+\tilde{\mathbf{A}}'(x^0) \succ + \prec \mathbf{E}(x^0), \Theta_+\, D^{-1}\mathbf{E}'(x^0) \succ \\ &+ i\left[\prec \mathbf{E}(x^0), \Theta_- D^{-\frac{1}{2}}\tilde{\mathbf{A}}'(x^0) \succ - \prec \tilde{\mathbf{A}}(x^0), \Theta_- D^{-\frac{1}{2}}\mathbf{E}'(x^0) \succ\right]\end{aligned} \tag{4.72}$$

This defines the most general positive-definite inner product on the space of photon fields which is also dynamically invariant. But it is not generally relativistically invariant. Similarly to the case of relativistic scalar fields, we can obtain an appropriate choice for $L_\pm$ so that the resulting inner product becomes relativistically invariant. This is shown in Appendix B. Here we provide the result,

$$\begin{aligned}(\!(\tilde{A},\tilde{A}')\!)_{\{\mathfrak{b}\}} \quad &:= \prec\tilde{\mathbf{A}}(x^0),\Theta'_+\,D^{1/2}\tilde{\mathbf{A}}'(x^0)\succ + \prec\mathbf{E}(x^0),\Theta'_+\,D^{-1/2}\mathbf{E}'(x^0)\succ \\ &+i\left[\prec\mathbf{E}(x^0),\Theta'_-\tilde{\mathbf{A}}'(x^0)\succ - \prec\tilde{\mathbf{A}}(x^0),\Theta'_-\mathbf{E}'(x^0)\succ\right] \qquad (4.73)\end{aligned}$$

where $\Theta'_p:\tilde{\mathcal{H}}\to\tilde{\mathcal{H}}$ is defined by

$$\Theta'_p := L'^{+}_p\mathfrak{h}^2, + L'^{-}_p\mathfrak{h}, \qquad (4.74)$$

$$L'^{\epsilon}_\pm := \frac{1}{4}\left[b_{+,1}+\epsilon b_{+,-1}\pm b_{-,1}\pm\epsilon b_{-,-1}\right], \qquad (4.75)$$

$b_{\epsilon,h}$ are real positive numbers, and so is L'^{+}_+. We can absorb L'^{+}_+ into the definition of the field. Therefore, the inner product (4.73) involves three nontrivial free parameters. Endowing $\mathcal{V}$ with this inner product and making the Cauchy completion of the resulting inner product space yields a separable Hilbert space which we denote by $\mathcal{H}_{\{\mathfrak{b}\}}$.

Let us consider a specific choice where $b_{\epsilon,h}=1$ for all $\epsilon=\{\pm\}$ and $h=\{\pm1\}$. This choice sets $\Theta'_+=1$ and $\Theta'_-=0$ and subsequently the inner product (4.73) takes the form

$$(\!(\tilde{A},\tilde{A}')\!)_{r.i} \quad := \kappa[\prec\tilde{\mathbf{A}}(x^0),D^{1/2}\tilde{\mathbf{A}}'(x^0)\succ + \prec\mathbf{E}(x^0),D^{-1/2}\mathbf{E}'(x^0)\succ], \qquad (4.76)$$

where κ is an arbitrary real parameter having the dimension of length. This has been introduced to make (4.76) dimensionless. Note that this inner product is very similar to (4.44), but unlike (4.44) it is relativistically invariant. We will use it to obtain the position wave function of the photon, localized states, and physical observables.

In relation (4.76) if we use $\mathbf{B}(x^0)=|\mathfrak{K}|\mathfrak{h}\tilde{\mathbf{A}}(x^0)$, we can further simplify the inner product (4.76) according to

$$(\!(\tilde{A},\tilde{A}')\!)_{r.i} \quad := \kappa[\prec\mathbf{B}(x^0),D^{-1/2}\mathbf{B}'(x^0)\succ + \prec\mathbf{E}(x^0),D^{-1/2}\mathbf{E}'(x^0)\succ]. \qquad (4.77)$$

It is remarkable to observe that the inner product(4.77) only depends on $\mathbf{E}$ and $\mathbf{B}$.

As shown in Appendix B, by fixing $b_{\epsilon,h}=1$ for all $\epsilon=\{\pm\}$ and $h=\{\pm1\}$, we can express the unitary operator $\mathcal{U}_{\{\mathfrak{b}\}}$ given by (4.64)

$$\mathcal{U}_{r.i}\tilde{A} = \frac{\sqrt{\kappa}}{2}\begin{pmatrix} D^{1/4}\,\tilde{\mathbf{A}}(x^0_0) - iD^{-1/4}\,\mathbf{E}(x^0_0) \\ D^{1/4}\,\tilde{\mathbf{A}}(x^0_0) + iD^{-1/4}\,\mathbf{E}(x^0_0)\end{pmatrix}. \qquad (4.78)$$

For each $\xi \in \mathcal{H}'_{ph}$, $\mathcal{U}_{r.i}^{-1}\xi$ is the photon field $\tilde{A} \in \mathcal{H}$ ($\mathcal{H}$ is the Hilbert space corresponding to a specific choice $b_{\epsilon,h} = 1$) which satisfies the following initial conditions:

$$\tilde{\mathbf{A}}(x_0^0) = \frac{1}{\sqrt{\kappa}} D^{-1/4}(\boldsymbol{\xi}^1 + \boldsymbol{\xi}^2), \qquad \mathbf{E}(x_0^0) = \frac{i}{\sqrt{\kappa}} D^{1/4}\,(\boldsymbol{\xi}^1 - \boldsymbol{\xi}^2), \tag{4.79}$$

This in turn imply that for all $x^0 \in \mathbb{R}$,

$$\tilde{\mathbf{A}}^0(x^0) = \frac{1}{\sqrt{\kappa}} D^{-1/4}\left[e^{-i(x^0-x_0^0)D^{1/2}}\boldsymbol{\xi}^1 + e^{i(x^0-x_0^0)D^{1/2}}\boldsymbol{\xi}^2\right], \tag{4.80}$$

$$\mathbf{E}(x^0) = \frac{i}{\sqrt{\kappa}} D^{1/4}\left[e^{-i(x^0-x_0^0)D^{1/2}}\boldsymbol{\xi}^1 - e^{i(x^0-x_0^0)D^{1/2}}\boldsymbol{\xi}^2\right], \tag{4.81}$$

where we have made use of (4.25) and (4.79).

In view of (4.71), we see that the operator $\mathcal{U}_{\{\mathfrak{b}\}} : \mathcal{H}_b \to \mathcal{H}'_{ph}$ is a unitary transformation. Therefore, similar to the relativistic scalar fields, this can be used to obtain the Hamiltonian acting $\mathcal{H}_b$ according to

$$\mathrm{h}_{\{\mathfrak{b}\}} := \mathcal{U}_{\{\mathfrak{b}\}}^{-1} H \mathcal{U}_{\{\mathfrak{b}\}}. \tag{4.82}$$

This is a Hermitian operator. Using (4.78) in (4.82), we observe that the action of $\mathrm{h}_{\{\mathfrak{b}\}}$ on any field $\tilde{A} \in \mathcal{H}_b$ has the form

$$\mathrm{h}_{\{\mathfrak{b}\}}\tilde{A} = i\hbar\dot{\tilde{A}}. \tag{4.83}$$

Similarly to relativistic scalar fields, we should not confuse (4.83) with the time-independent Schrödinger equation that gives the x^0-dependence of $\tilde{A}$ [50]. $\mathrm{h}_{\{\mathfrak{b}\}}$ generates time-evolution in $\mathcal{H}_b$ through the Schrödinger equation $i\hbar\frac{d}{dx^0}\tilde{A}_{x^0} = \mathrm{h}_{\{\mathfrak{b}\}}\tilde{A}_{x^0}$ and maps each solution $\tilde{A}$ of (4.13) to another solution $\dot{\tilde{A}}$, i.e, if $\tilde{A}_0 = \tilde{A}_{x_0^0}$ is the initial value for the one-parameter family of elements $\tilde{A}_{x^0}$ of $\mathcal{V}$, we have

$$\tilde{A}_{x^0}(x^{0'}) = [e^{-i(x^0-x_0^0)/\hbar}\tilde{A}_0](x^{0'}) = \tilde{A}_0(x^{0'} + x^0 - x_0^0),$$

for all $x^0, x^{0'} \in \mathbb{R}$.

Because $\mathcal{B}$ commutes with the Hamiltonian, we have $\mathrm{h}_{\{\mathfrak{b}\}} = \mathrm{h}$. Thus, the pairs $(\mathcal{H}_{\{\mathfrak{b}\}}, \mathrm{h})$ and $(\mathcal{H}'_{ph}, H)$ are unitarily equivalent and represent the same quantum system.

4.4 Conserved Current Density

Following [51], we can use the relativistically invariant inner product to introduce a conserved four-vector current density $J^\mu_{\{\mathfrak{b}\}}$. Here we first provide an explicit expression for a conserved four-vector density corresponding to the relativistically invariant inner product, then we compute $J^{\mu}_{\{\mathfrak{b}\}}$ for the case $b_{\epsilon,h} = 1$ which we denoted by J^{μ}. We also obtain a manifestly covariant expression for J^{μ}.

If we rewrite the inner product (4.73) in an integral form, we obtain the following expression:

$$\begin{aligned}(\!(\tilde{A},\tilde{A})\!)_{\{\mathfrak{b}\}} = \frac{1}{2}\int_{\mathbb{R}^3} d^3\mathbf{x}\Big\{&\tilde{\mathbf{A}}(x^0,\mathbf{x})^*\cdot\Big(\Theta'_+ D^{-1/2}\dot{\mathbf{E}}(x^0,\mathbf{x})\Big) - \mathbf{E}(x^0,\mathbf{x})^*\cdot\Big(\Theta'_+ D^{-1/2}\dot{\tilde{\mathbf{A}}}(x^0,\mathbf{x})\Big)\\ &-i\Big[\tilde{\mathbf{A}}(x^0,\mathbf{x})^*\cdot\big(\Theta'_-\mathbf{E}(x^0,\mathbf{x})\big) - \mathbf{E}(x^0,\mathbf{x})^*\cdot\Big(\Theta'_-\tilde{\mathbf{A}}(x^0,\mathbf{x})\Big)\Big]\Big\}\end{aligned} \tag{4.84}$$

Where $\tilde{A} \in \mathcal{V}$ is arbitrary. In analogy with non-relativistic quantum mechanics, we define the 0-component of the current density $J^{\mu}_{\{\mathfrak{b}\}}$ associated with $\tilde{A}$ as the integrand in (4.73), i.e.,

$$\begin{aligned}J^0_{\{\mathfrak{b}\}}(x) := \frac{1}{2}\Big\{&\tilde{\mathbf{A}}(x)^*\cdot\Big(\Theta'_+ D^{-1/2}\dot{\mathbf{E}}(x)\Big) - \mathbf{E}(x)^*\cdot\Big(\Theta'_+ D^{-1/2}\dot{\tilde{\mathbf{A}}}(x)\Big)\\ &-i\Big[\tilde{\mathbf{A}}(x)^*\cdot\big(\Theta'_-\mathbf{E}(x)\big) - \mathbf{E}(x)^*\cdot\Big(\Theta'_-\tilde{\mathbf{A}}(x)\Big)\Big]\Big\}.\end{aligned} \tag{4.85}$$

By setting $a_{\epsilon,h} = 1$ for all $\epsilon \in \{+,-\}$ and $h \in \{-1,+1\}$, this yields

$$J^0(x) := \frac{\kappa}{2}\Big\{\tilde{\mathbf{A}}(x)^*\cdot D^{-1/2}\dot{\mathbf{E}}(x) - \mathbf{E}(x)^*\cdot D^{-1/2}\dot{\tilde{\mathbf{A}}}(x)\Big\}. \tag{4.86}$$

In an attempt to obtain spatial components of J^{μ}, we make an infinitesimal Lorentz boost transformation so that it changes the reference frame to the one which moves with a velocity $\mathbf{v}$, namely $x^0 \to x'^0 = x^0 - \boldsymbol{\beta}\cdot\mathbf{x}$, $\mathbf{x} \to \mathbf{x}' = \mathbf{x} - \boldsymbol{\beta}x^0$ where $\boldsymbol{\beta} := \mathbf{v}/c$. Under this transformation, the four-vector $\tilde{A} = (0, \tilde{\mathbf{A}})$ transforms as

$$0 \to \tilde{A}'^0(x') = -\boldsymbol{\beta}\cdot\tilde{\mathbf{A}}(x), \qquad \tilde{\mathbf{A}}(x) \to \tilde{\mathbf{A}}'(x') = \tilde{\mathbf{A}}(x), \tag{4.87}$$

and similarly $\mathbf{E}$ transforms as follows

$$\mathbf{E}(x) \to \mathbf{E}'(x') = \mathbf{E}(x) + \boldsymbol{\beta}\times(\boldsymbol{\nabla}\times\tilde{\mathbf{A}}(x)). \tag{4.88}$$

Now, let us expand the four-vector field J^{μ} in powers of $\boldsymbol{\beta}$ and omit quadratic and higher order terms. This gives

$$J^0(x) \to J'^0(x') = J^0(x) - \boldsymbol{\beta}\cdot\mathbf{J}(x). \tag{4.89}$$

Next, we employ (4.86) to get

$$J'^0(x') := \frac{\kappa}{2}\Big\{\tilde{\mathbf{A}}'(x')^*\cdot D'^{-1/2}\dot{\mathbf{E}}'(x') - \mathbf{E}'(x')^*\cdot D'^{-1/2}\dot{\tilde{\mathbf{A}}}'(x')\Big\}, \tag{4.90}$$

where $x' := (x'^0, \mathbf{x}')$, $D' = -\nabla'^2$, $\mathbf{E}'(x') = -\dot{\tilde{\mathbf{A}}}'(x')$, and $\dot{\tilde{\mathbf{A}}}'(x') := \partial\tilde{\mathbf{A}}'(x')/\partial x'^0$. We can employ (4.87) and (4.88) in (4.90) and make use of (4.89) to determine spatial components of J^μ. This yields

$$\begin{aligned} J^i(x) \;=\; & \frac{\kappa}{2}\Big\{\tilde{A}^*(x)\cdot\partial^i D^{-1/2}\dot{\tilde{A}}(x) - [\partial^i\tilde{A}(x)^*]\cdot D^{-1/2}\dot{\tilde{A}}(x) - \\ & [\tilde{A}^*(x)\cdot\partial]D^{-1/2}\dot{\tilde{A}}^i(x) + [D^{-1/2}\dot{\tilde{A}}(x)\cdot\partial]\tilde{A}^i(x)^*\Big\}. \end{aligned} \tag{4.91}$$

Here we have also made use of the fact that the chirality operator $\mathrm{C} = iD^{-1/2}\partial_0$ is Lorentz-invariant [53], $D'^{-1/2}\partial_0' = D^{-1/2}\partial_0$. Note that in (4.91), $\partial := (\partial^0, \boldsymbol{\nabla})$, and for any two four-vectors v_1 and v_2, $v_1\cdot v_2 := v_1^\mu v_{2\mu}$.

Equation (4.91) suggests that

$$\begin{aligned} J^\mu(x) \;=\; & \frac{\kappa}{2}\Big\{\tilde{A}^*(x)\cdot\partial^\mu D^{-1/2}\dot{\tilde{A}}(x) - [\partial^\mu\tilde{A}(x)^*]\cdot D^{-1/2}\dot{\tilde{A}}(x) - \\ & [\tilde{A}^*(x)\cdot\partial]D^{-1/2}\dot{\tilde{A}}^\mu(x) + [D^{-1/2}\dot{\tilde{A}}(x)\cdot\partial]\tilde{A}^\mu(x)^*\Big\}. \end{aligned} \tag{4.92}$$

Using (4.92) and Maxwell equations, we can obtain the expression for J^0 which coincides with (4.86).

Next we use (4.52) to simplify (4.92) as

$$J^\mu(x) = \frac{\kappa}{2}\Big\{\tilde{A}_\nu(x)^* F_c^{\nu\mu}(x) - F^{\nu\mu}(x)^*\tilde{A}_{c\nu}(x)\Big\}, \tag{4.93}$$

where $F_c^{\mu\nu} := \partial^\mu\tilde{A}_c^\nu - \partial^\nu\tilde{A}_c^\mu$. The current density J^μ is generally complex-valued and has the following remarkable properties:

1. The expression (4.93) for J^μ is manifestly covariant; since $\tilde{A}$ and $\tilde{A}_c$ are four-vector fields, so is J^μ.

2. Using the fact that both $\tilde{A}$ and $\tilde{A}_c$ satisfy the Maxwell equation, we can easily check that J^μ satisfies the continuity equation

$$\partial_\mu J^\mu = 0. \tag{4.94}$$

Hence it is a conserved current density.

Finally, we use (4.93) to obtain a manifestly covariant expression for the inner product (4.76):

$$(\!(\tilde{A}, \tilde{A}')\!) = \frac{i\kappa}{2}\int_\sigma d\sigma(x)\; n_\mu(x)\Big\{\tilde{A}_\nu(x)^* F_c'^{\nu\mu}(x) - F^{\nu\mu}(x)^*\tilde{A}'_{c\nu}(x)\Big\}, \tag{4.95}$$

where *"σ is an arbitrary spacelike (Cauchy) hypersurface of the Minkowski space with volume element $d\sigma$ and unit (future) timelike normal four-vector n^μ"* [51]. In order to obtain (4.95) we have used the fact that any inner product is uniquely determined by its associated norm [55].

4.5 Four-Component Representation of the Photon Fields

We should note that working in $\mathcal{H}'_{ph}$ is not as easy as it seems. This is because the identity operator in this space is $1_{\mathcal{V}} = \mathfrak{h}^2$ and the basis given by (1.6) cannot be used as the basis of $\mathcal{H}'_{ph}$. Therefore, here we introduce a four-component representation for the photon physical Hilbert space which will be used to construct the physical observables in next section.

Now let us define a Hilbert space $\tilde{\mathcal{H}}_\pi := \{\mathbf{V} : \mathbb{R}^3 \to \mathbb{C}^2 | \int_{\mathbb{R}^3} d^3\mathbf{x}\, \mathbf{V}(\mathbf{x})^* \cdot \mathbf{V}(\mathbf{x}) < \infty\}$, and we introduce $\mathcal{H}'_\pi := \tilde{\mathcal{H}}_\pi \oplus \tilde{\mathcal{H}}_\pi$. Further we define $\Pi : \mathcal{H}'_{ph} \to \mathcal{H}'_\pi$ and $\Pi^{-1} : \mathcal{H}'_\pi \to \mathcal{H}'_{ph}$ according to

$$\Pi := \sigma_0 \otimes \pi \tag{4.96}$$

$$\Pi^{-1} := \sigma_0 \otimes \pi^{-1} \tag{4.97}$$

where π and π^{-1} are defined by

$$\pi := \sum_{h=\pm 1} \int_{\mathbb{R}^3} d^3\mathbf{k}\, |\tilde{e}_h\rangle\langle \mathbf{u}_h(\mathbf{k})|, \qquad \pi^{-1} := \sum_{h=\pm 1} \int_{\mathbb{R}^3} d^3\mathbf{k}\, |\mathbf{u}_h(\mathbf{k})\rangle\langle \tilde{e}_h|. \tag{4.98}$$

Here $|\tilde{e}_h\rangle = \tilde{e}_h|\mathbf{k}\rangle$, $|\mathbf{u}_h(\mathbf{k})\rangle = \mathbf{u}_h(\mathbf{k})|\mathbf{k}\rangle$, and $\mathbf{u}_h(\mathbf{k})$ and $\tilde{e}_h$ are given by (1.11) and (4.19), respectively. It is straightforward to establish the following identities

$$\pi\pi^{-1} = \sigma_0, \qquad \pi^{-1}\pi = \mathfrak{h}^2. \tag{4.99}$$

Next, we employ Π, Π^{-1} and (4.87) to introduce the Hamiltonian acting in $\mathcal{H}'_\pi$. The result is give by

$$H' := \Pi H \Pi^{-1} = \hbar D^{1/2} \sigma_0 \otimes \sigma_3. \tag{4.100}$$

This Hamiltonian together with $\mathcal{H}'_\pi$ describe the same quantum system as $(\mathcal{H}_{\{\mathfrak{b}\}}, \mathrm{h})$ and $(\mathcal{H}'_{ph}, H)$. Therefore, we can use any of these representations to characterize the observables of the system.

4.6 Physical Observables and Wave Functions for Photon Fields

Because the representations $(\mathcal{H}_{\{\mathfrak{b}\}}, \mathrm{h})$, $(\mathcal{H}'_{ph}, H)$, and $(\mathcal{H}'_\pi, H')$ are unitarily equivalent, they define the same quantum system. This means that we are free to use any of these representations in order to determine the observables of this quantum system. Based on our experience with KG and Proca fields [50, 51], we see that working in the Foldy representation $(\mathcal{H}'_\pi, H')$ is far more convenient than the others. Therefore, we initially define observables in the Foldy representation $(\mathcal{H}'_\pi, H')$, and further use the unitary maps $\Pi : \mathcal{H}'_{ph} \to \mathcal{H}'_\pi$ and $\mathcal{U}_{\{\mathfrak{b}\}} : \mathcal{H}_{\{\mathfrak{b}\}} \to \mathcal{H}'$ to obtain their explicit form in the standard representation $(\mathcal{H}_{\{\mathfrak{b}\}}, \mathrm{h})$.

In $\mathcal{H}'_\pi$, we introduce the following set of basic observables

$$\mathbf{X}'_{\mathfrak{m}} := \mathbf{x} \otimes \tilde{\Sigma}_{\mathfrak{m}}, \qquad \mathbf{P}'_{\mathfrak{m}} := \mathbf{p} \otimes \tilde{\Sigma}_{\mathfrak{m}}, \qquad \mathcal{S}'_{\mathfrak{m}} := 1 \otimes \tilde{\Sigma}_{\mathfrak{m}}, \tag{4.101}$$

where $\mathfrak{m} \in \{0, 1, 2, \cdots, 15\}$, $\tilde{\Sigma}_{\mathfrak{m}}$ are given in (1.9), and $\mathbf{x}$, $\mathbf{p} := \hbar\,\mathfrak{K}$, and 1 are respectively the position, momentum, and identity operators acting in $L^2(\mathbb{R}^2)$. Note also that for brevity we drop '$1\otimes$' and identify $\mathcal{S}'_{\mathfrak{m}}$ with $\tilde{\Sigma}_{\mathfrak{m}}$.

The operators $\mathbf{X}'_0$, $\tilde{\Sigma}_3$, $\tilde{\Sigma}_{12}$ and $\tilde{\Sigma}_{15}$ form a maximal commuting set of observables acting in $\mathcal{H}'_\pi$. We can employ their common eigenvectors as a basis of $\mathcal{H}'_\pi$. The basis elements are given by

$$\zeta_{\mathbf{x}}^{(\epsilon,h)} := |\mathbf{x}\rangle \otimes e_{\epsilon,h}, \qquad \mathbf{x} \in \mathbb{R}^3, \quad \epsilon \in \{-, +\}, \quad h \in \{-1, +1\}, \tag{4.102}$$

where $e_{\epsilon,h} := e_\epsilon \otimes \tilde{e}_h$ is defined in (1.10), and $|\mathbf{x}\rangle$ are the δ-function normalized position kets which satisfy $\mathbf{x}|\mathbf{x}\rangle = \mathbf{x}|\mathbf{x}\rangle$, $\langle \mathbf{x}|\mathbf{x}'\rangle = \delta^3(\mathbf{x} - \mathbf{x}')$, and $\int_{\mathbb{R}^3} d^3\mathbf{x}\, |\mathbf{x}\rangle\langle\mathbf{x}| = 1$. It is easy to see that

$$\mathbf{X}'_0\, \zeta_{\mathbf{x}}^{(\epsilon,h)} = \mathbf{x}\, \zeta_{\mathbf{x}}^{(\epsilon,h)}, \qquad \tilde{\Sigma}_3\, \zeta_{\mathbf{x}}^{(\epsilon,s)} = \epsilon\, \zeta_{\mathbf{x}}^{(\epsilon,h)}, \qquad \tilde{\Sigma}_{12}\, \zeta_{\mathbf{x}}^{(\epsilon,h)} = h\, \zeta_{\mathbf{x}}^{(\epsilon,h)}. \tag{4.103}$$

Let us also observe that $\zeta_{\mathbf{x}}^{(\epsilon,h)}$ form an orthnormal basis

$$\langle \zeta_{\mathbf{x}}^{(\epsilon,h)}, \zeta_{\mathbf{x}'}^{(\epsilon',h')} \rangle = \delta_{\epsilon,\epsilon'}\delta_{h,h'}\delta^3(\mathbf{x}-\mathbf{x}'), \qquad \sum_{\epsilon=\pm}\sum_{s=\pm 1}\int_{\mathbb{R}^3} d^3\mathbf{x} |\zeta_{\mathbf{x}}^{(\epsilon,h)}\rangle\langle\zeta_{\mathbf{x}}^{(\epsilon,h)}| = \tilde{\Sigma}_0. \tag{4.104}$$

Here we have used (4.102) and the properties of $|\mathbf{x}\rangle$.

Now, we define the basic observables according to

$$\mathbf{x}_{\mathfrak{m}\{\mathfrak{b}\}} := \mathcal{U}_{\{\mathfrak{b}\}}^{-1}\, \Pi^{-1} \mathbf{X}'_{\mathfrak{m}}\, \Pi\, \mathcal{U}_{\{\mathfrak{b}\}}, \qquad \mathbf{p}_{\mathfrak{m}\{\mathfrak{b}\}} := \mathcal{U}_{\{\mathfrak{b}\}}^{-1}\, \Pi^{-1} \mathbf{P}'_{\mathfrak{m}}\, \Pi\, \mathcal{U}_{\{\mathfrak{b}\}}, \qquad s_{\mathfrak{m}\{\mathfrak{b}\}} := \mathcal{U}_{\{\mathfrak{b}\}}^{-1}\, \Pi^{-1} \Sigma_{\mathfrak{m}}\, \Pi\, \mathcal{U}_{\{\mathfrak{b}\}}, \tag{4.105}$$

which act in $\mathcal{H}_{\{\mathfrak{b}\}}$, and introduce,

$$\tilde{A}_{\{\mathfrak{b}\}\,\mathbf{x}}^{(\epsilon,h)} := (0, \tilde{\mathbf{A}}_{\{\mathfrak{b}\}\,\mathbf{x}}^{(\epsilon,h)}) := \mathcal{U}_{\{\mathfrak{b}\}}^{-1}\, \Pi^{-1}\, \zeta_{\mathbf{x}}^{(\epsilon,h)}. \tag{4.106}$$

Using (4.104) and (4.106) in (4.70), we can show that $\tilde{A}_{\{\mathfrak{b}\}\,\mathbf{x}}^{(\epsilon,h)}$ form an orthonormal basis of $\mathcal{H}_{\{\mathfrak{b}\}}$:

$$(\!(\tilde{A}_{\{\mathfrak{b}\}\,\mathbf{x}}^{(\epsilon,h)}, \tilde{A}_{\{\mathfrak{b}\}\,\mathbf{x}'}^{(\epsilon',h')})\!)_{\{\mathfrak{b}\}} = \delta_{\epsilon,\epsilon'}\delta_{h,h'}\delta^3(\mathbf{x}-\mathbf{x}'), \qquad \sum_{\epsilon=\pm}\sum_{h=\pm 1}\int_{\mathbb{R}^3} d^3\mathbf{x} |\tilde{A}_{\{\mathfrak{b}\}\,\mathbf{x}}^{(\epsilon,h)})(\tilde{A}_{\{\mathfrak{b}\}\,\mathbf{x}}^{(\epsilon,h)}| = s_{0\{\mathfrak{b}\}},$$

(4.107)

where $\tilde{A}^{(\epsilon,h)}_{\{\mathfrak{b}\}\mathbf{x}}$ are the localized state vectors, and for all $\tilde{A}, \tilde{A}' \in \mathcal{H}_{\{\mathfrak{b}\}}$, the operator $|\tilde{A})(\tilde{A}'|$ is defined by $|\tilde{A})(\tilde{A}'|\tilde{A}'' := (\!(\tilde{A}', \tilde{A}'')\!)_{\{\mathfrak{b}\}}\tilde{A}$ for any $\tilde{A}'' \in \mathcal{H}_{\{\mathfrak{b}\}}$, and $s_{0\{\mathfrak{b}\}}$ identifies the identity operator for $\mathcal{H}_{\{\mathfrak{b}\}}$.

Next, we express an arbitrary six-component vector $\Phi \in \mathcal{H}'_\pi$ in the basis $\{\zeta^{(\epsilon,h)}_{\mathbf{x}}\}$ according to

$$\Psi' = \sum_{\epsilon=\pm}\sum_{h=\pm 1}\int_{\mathbb{R}^3} d^3\mathbf{x}\, f(\epsilon, h, \mathbf{x})\, \zeta^{(\epsilon,h)}_{\mathbf{x}}, \tag{4.108}$$

where $f : \{-,+\} \times \{-1,+1\} \times \mathbb{R}^3 \to \mathbb{C}$ is the position wave function associated with Φ and defined in the following way:

$$f(\epsilon, h, \mathbf{x}) := \langle \zeta^{(\epsilon,h)}_{\mathbf{x}}, \Phi\rangle. \tag{4.109}$$

In view of (4.70), (4.106) and (4.109) it is straightforward to see

$$(\!(\tilde{A}^{(\epsilon,h)}_{\{\mathfrak{b}\}\mathbf{x}}, \tilde{A})\!)_{\{\mathfrak{b}\}} = \langle \mathcal{U}_{\{\mathfrak{b}\}}\tilde{A}^{(\epsilon,h)}_{\{\mathfrak{b}\}\mathbf{x}}, \mathcal{U}_{\{\mathfrak{b}\}}\tilde{A}\rangle = \langle \zeta^{(\epsilon,h)}_{\mathbf{x}}, \Phi\rangle = f(\epsilon, h, \mathbf{x}). \tag{4.110}$$

This implies that every photon field $\tilde{A} \in \mathcal{H}_{\{\mathfrak{b}\}}$ can be written in terms of the basis $\{\tilde{A}^{(\epsilon,h)}_{\{\mathfrak{b}\}\mathbf{x}}\}$ according to

$$\tilde{A} = \sum_{\epsilon=\pm}\sum_{h=\pm 1}\int_{\mathbb{R}^3} d^3\mathbf{x}\, f(\epsilon, h, \mathbf{x})\, \tilde{A}^{(\epsilon,h)}_{\{\mathfrak{b}\}\mathbf{x}}. \tag{4.111}$$

Note that since the wave functions f do not depend on the choice of the parameter $\mathfrak{b}_{\epsilon,h}$, we avoid labeling them with the subscript $\{\mathfrak{b}\}$. The proof of this claim is given in Ref [53] for the Klein-Gordon fields.

In the remainder of this section, we examine the action of the physical observable $O : \mathcal{H}' \to \mathcal{H}'_\pi$ on the state vector $\Phi \in \mathcal{H}'_\pi$, i.e,

$$O\Phi = \sum_{\epsilon=\pm}\sum_{h=\pm 1}\int_{\mathbb{R}^3} d^3\mathbf{x}\, [\hat{O}f(\epsilon, h, \mathbf{x})]\, \zeta^{(\epsilon,s)}_{\mathbf{x}}, \tag{4.112}$$

where

$$\hat{O}f(\epsilon, h, \mathbf{x}) := \sum_{\epsilon'=\pm}\sum_{h'=\pm 1}\int_{\mathbb{R}^3} d^3\mathbf{x}'\, f(\epsilon', h', \mathbf{x}')\, \langle \zeta^{(\epsilon,h)}_{\mathbf{x}}, O\zeta^{(\epsilon',h')}_{\mathbf{x}'}\rangle. \tag{4.113}$$

This yields the representation of observables in terms of the linear operators acting on the square-integrable wave functions f.

In a close analogy with relativistic scalar fields, we can uniquely determine the physical observables $o_{\{\mathfrak{b}\}} : \mathcal{H}_{\{\mathfrak{b}\}} \to \mathcal{H}_{\{\mathfrak{b}\}}$ for all $\tilde{A} \in \mathcal{H}_{\{\mathfrak{b}\}}$ in terms of their representations in the basis $\{\tilde{A}^{(\epsilon,h)}_{\{\mathfrak{b}\}\,\mathbf{x}}\}$ according to

$$o_{\{\mathfrak{b}\}}\tilde{A} = \sum_{\epsilon=\pm}\sum_{h=\pm1}\int_{\mathbb{R}^3} d^3\mathbf{x}[\hat{O}f(\epsilon,h,\mathbf{x})]\tilde{A}^{(\epsilon,h)}_{\{\mathfrak{b}\}\,\mathbf{x}}, \tag{4.114}$$

where $\hat{O}f(\epsilon,h,\mathbf{x})$ is given by (4.113). This follows directly from $(\!(\tilde{A}^{(\epsilon,h)}_{\{\mathfrak{b}\}\,\mathbf{x}}, o_{\{\mathfrak{b}\}}\tilde{A}^{(\epsilon',h')}_{\{\mathfrak{b}\}\,\mathbf{x}'})\!)_{\{\mathfrak{b}\}} = \langle\zeta^{(\epsilon,h)}_{\mathbf{x}}, O'\zeta^{(\epsilon',h')}_{\mathbf{x}'}\rangle$.

Subsequently, we can employ the position wave functions to determine transition amplitudes of two photon fields

$$(\!(\tilde{A},\tilde{A}')\!)_{\{\mathfrak{b}\}} = \sum_{\epsilon=\pm}\sum_{h=\pm1}\int_{\mathbb{R}^3} d^3\mathbf{x}\, f(\epsilon,h,\mathbf{x})^* f'(\epsilon,h,\mathbf{x}). \tag{4.115}$$

Here we have made use of (4.110) and (4.111). Furthermore, we can express the expectation value of any observable $o_{\{\mathfrak{b}\}} : \mathcal{H}_{\{\mathfrak{b}\}} \to \mathcal{H}_{\{\mathfrak{b}\}}$ in a state vector $\tilde{A}'$ according to

$$\begin{aligned}(\!(\tilde{A}, o_{\{\mathfrak{b}\}}\tilde{A}')\!)_{\{\mathfrak{b}\}} &= \sum_{\epsilon=\pm}\sum_{h=\pm1}\int_{\mathbb{R}^3} d^3\mathbf{x}\, f(\epsilon,h,\mathbf{x})^*\,\hat{O}f'(\epsilon,h,\mathbf{x}) \\ &= \sum_{\epsilon=\pm}\sum_{h=\pm1}\int_{\mathbb{R}^3} d^3\mathbf{x}\,[\hat{O}f(\epsilon,h,\mathbf{x})]^* f'(\epsilon,h,\mathbf{x}).\end{aligned} \tag{4.116}$$

Similarly to the case of relativistic scalar fields, we can consider wave functions f as elements of $\mathcal{H}'_\pi$ and observables as the Hermitian operators $\hat{O}$ acting on these wave functions.

Next, we define the action of $\mathbf{x}_{0\{\mathfrak{b}\}}$, $\mathbf{p}_{0\{\mathfrak{b}\}}$, $s_{3\{\mathfrak{b}\}}$, and $s_{12\{\mathfrak{b}\}}$ on f according to

$$\hat{\mathbf{x}}_{0\{\mathfrak{b}\}}f(\epsilon,h,\mathbf{x}) := \mathbf{x}f(\epsilon,h,\mathbf{x}), \qquad \hat{\mathbf{p}}_{0\{\mathfrak{b}\}}f(\epsilon,h,\mathbf{x}) := -i\hbar\boldsymbol{\nabla}f(\epsilon,h,\mathbf{x}), \tag{4.117}$$

$$\hat{s}_{3\{\mathfrak{b}\}}f(\epsilon,h,\mathbf{x}) := \epsilon f(\epsilon,h,\mathbf{x}), \qquad \hat{s}_{12\{\mathfrak{b}\}}f(\epsilon,h,\mathbf{x}) := sf(\epsilon,h,\mathbf{x}). \tag{4.118}$$

These relations illustrate that $f(\epsilon,h,\mathbf{x})$ are the position wave functions with definite chirality ϵ and helicity h. Also let us observe that the operation of $\mathbf{x}_{0\{\mathfrak{b}\}}$, $\mathbf{p}_{0\{\mathfrak{b}\}}$, $s_{3\{\mathfrak{b}\}}$, and $s_{12\{\mathfrak{b}\}}$ on $\tilde{A}$ is equivalent to the action of the operators $\hat{\mathbf{x}}_{0\{\mathfrak{b}\}}$, $\hat{\mathbf{p}}_{0\{\mathfrak{b}\}}$, $\hat{s}_{3\{\mathfrak{b}\}}$, and $\hat{s}_{12\{\mathfrak{b}\}}$ on f.

Similarly the action of the Hamiltonian h on $\tilde{A}$ can be specified with the action of the operator $\hat{\mathrm{h}} := \hat{s}_{3\{\mathfrak{b}\}}\sqrt{\hat{\mathbf{p}}^2_{0\{\mathfrak{b}\}}}$ on the associated position wave function f in the following form:

$$\hat{\mathrm{h}}f(\epsilon,h,\mathbf{x}) = \epsilon\hbar\sqrt{-\nabla^2}f(\epsilon,h,\mathbf{x}). \tag{4.119}$$

This together with the Schrödinger equation for h, which we derived in section 4.3, characterize the dynamics of an evolving photon field $\tilde{A}_{x^0}$ in terms of the wave functions $f(\epsilon, h, \mathbf{x}; x^0) = (\!(\tilde{A}^{(\epsilon,h)}_{\{\mathfrak{b}\}\,\mathbf{x}}, \tilde{A}_{x^0})\!)_{\{\mathfrak{b}\}}$ in the following manner:

$$i\hbar\partial_0 f(\epsilon, h, \mathbf{x}; x^0) = \epsilon\hbar\sqrt{-\nabla^2}\, f(\epsilon, h, \mathbf{x}; x^0). \tag{4.120}$$

4.7 Position Operator

4.7.1 Explicit Form of the Photon's Position Operator

In this section we employ the unitary map $\mathcal{U}_{r.i}$ and its inverse constructed in section 4.3 to derive an explicit expression for the position operator $\mathbf{x}_0 : \mathcal{H} \to \mathcal{H}$. Let $\tilde{A} = (0, \tilde{\mathbf{A}}) \in \mathcal{H}$, and $\chi := \mathbf{x}_0 A$ to be the three-component field whose components satisfy (4.13) and (4.23). χ is uniquely identified in terms of the initial data $(\chi(x_0^0), \dot{\chi}(x_0^0))$ for some $x_0^0 \in \mathbb{R}$. If we use (4.105), (4.78), and the identities:

$$\left[\mathfrak{h}^2, \pi^{-1}\mathbf{x}\pi\right] = 0, \qquad \left[\mathfrak{h}, \pi^{-1}\mathbf{x}\pi\right] = 0, \tag{4.121}$$

and $[F(\mathfrak{K}), \mathbf{x}] = -i\boldsymbol{\nabla}_{\mathfrak{K}}F(\mathfrak{K})$, where F is a differentiable function, we obtain

$$\boldsymbol{\chi}(x_0^0) = \boldsymbol{\mathfrak{X}}\,\tilde{\mathbf{A}}(x_0^0), \tag{4.122}$$

$$\dot{\boldsymbol{\chi}}(x_0^0) = \left(\boldsymbol{\mathfrak{X}} - \frac{i\mathfrak{K}}{|\mathfrak{K}|^2}\right)\dot{\tilde{\mathbf{A}}}(x_0^0), \tag{4.123}$$

where

$$\boldsymbol{\mathfrak{X}} := \mathbf{x} + \frac{i\mathfrak{K}}{2|\mathfrak{K}|^2} - i\boldsymbol{\nabla}_{\mathfrak{K}}(\pi^{-1}(\mathfrak{K}))\pi(\mathfrak{K}), \tag{4.124}$$

and we define $\mathbf{x} := i\boldsymbol{\nabla}_{\mathfrak{K}}$. We take $\boldsymbol{\mathfrak{X}}$ as the photon position operator at x_0^0. Now, we use (4.25) to express $\chi(x^0)$ in terms of (4.122) and (4.123). For all $x^0 \in \mathbb{R}$, we have

$$\boldsymbol{\chi}(x^0) = \left\{\boldsymbol{\mathfrak{X}} - i(x^0 - x_0^0)\frac{\mathfrak{K}}{|\mathfrak{K}|^2}\partial_0\right\}\tilde{\mathbf{A}}(x^0). \tag{4.125}$$

This shows that $\mathbf{x}_0 = \boldsymbol{\mathfrak{X}} - i(x^0 - x_0^0)\frac{\mathfrak{K}}{|\mathfrak{K}|^2}\partial_0$. $\mathbf{x}_0$ is a Hermitian operator acting in the physical Hilbert space $\mathcal{H}$. It has the following notable properties:

1. $\mathbf{x}_0$ has commuting components, because it is obtained by a similarity transformation (4.105) from $\mathbf{X}_0'$ which according to (4.101) has commuting components.

2. It respects the charge superselection rule [54]. It commutes with the chirality operator $\mathrm{C} = s_3 = \frac{i}{|\mathfrak{K}|}\partial_0$ and helicity operators $[\mathfrak{h}, \pi^{-1}\mathbf{x}\pi] = 0$. This can be easily seen by noting that $\mathbf{x}_0$, s_3, and s_{12} are obtained via similarity transformation (4.105) from $\mathbf{X}'_0$, $\tilde{\Sigma}_3$, and $\tilde{\Sigma}_{12}$ where $[\mathbf{X}'_0, \tilde{\Sigma}_3] = 0$ and $[\mathbf{X}'_0, \tilde{\Sigma}_{12}] = 0$ according to (4.101) .

3. Its x^0-derivative, i.e, the velocity operator; i.e.,satisfies

$$\mathbf{v} := \frac{d\mathbf{x}_0}{dx^0} = i\,[\mathrm{h}, \mathbf{x}_0] = i\mathcal{U}^{-1}\Pi^{-1}\,[H', \mathbf{X}']\,\Pi\mathcal{U} = \frac{\mathbf{p}_0}{\sqrt{\mathbf{p}_0^2}}\,\mathrm{C}.$$

This means that the physical momentum is $\mathbf{p}_0\,\mathrm{C}$. This is similar to the position operator for the Kelin-Gordon and Proca fields [53, 51].

More interestingly if we use the definition of $\pi(\mathfrak{K})$ and $\pi^{-1}(\mathfrak{K})$ in equation (4.124), we obtain

$$\boldsymbol{\mathfrak{X}} = \mathbf{x} + \frac{i\mathfrak{K}}{2|\mathfrak{K}|^2} - i\sum_{h=\pm 1}\int_{\mathbb{R}^3} d^3\mathbf{k}\,\boldsymbol{\nabla}_{\mathbf{k}}(|\mathbf{u}_h(\mathbf{k})\rangle)\langle\mathbf{u}_h(\mathbf{k})|, \tag{4.126}$$

where $|\mathbf{u}_h(\mathbf{k})\rangle := \mathbf{u}_h(\mathbf{k})|\mathbf{k}\rangle$ and $\mathbf{u}_h(\mathbf{k})$ are circular polarization vectors. Furthermore, using the relation (4.19), we can prove the following identity,

$$\sum_{h=\pm 1}\int_{\mathbb{R}^3} d^3\mathbf{k}\,\boldsymbol{\nabla}_{\mathbf{k}}(|\mathbf{u}_h(\mathbf{k})\rangle)\langle\mathbf{u}_h(\mathbf{k})| = \sum_{\lambda=1}^{2}\int_{\mathbb{R}^3} d^3\mathbf{k}\,\boldsymbol{\nabla}_{\mathbf{k}}(|\mathbf{a}(\mathbf{k},\lambda)\rangle)\langle\mathbf{a}(\mathbf{k},\lambda)|. \tag{4.127}$$

This shows that the form of the position operator for the photon (4.126) is independent of the choice of the polarization (circular or linear) vectors.

4.7.2 The Action of the Position Operator on Electric and Magnetic Fields

If we introduce $\boldsymbol{\chi}_{\mathbf{E}} := \mathbf{x}_{0,\mathbf{E}}\,\mathbf{E}$ and $\boldsymbol{\chi}_{\mathbf{E}} := -\dot{\boldsymbol{\chi}}$, we can rewrite (4.123) as

$$\boldsymbol{\chi}_{\mathbf{E}}(x_0^0) = \left(\boldsymbol{\mathfrak{X}} - \frac{i\mathfrak{K}}{|\mathfrak{K}|^2}\right)\mathbf{E}(x_0^0) = \boldsymbol{\mathfrak{X}}_{\mathbf{E}}\,\mathbf{E}(x_0^0) \tag{4.128}$$

where we take $\boldsymbol{\mathfrak{X}}_{\mathbf{E}}$ as the position operator associated with the electric field. It is given by

$$\begin{aligned}\boldsymbol{\mathfrak{X}}_{\mathbf{E}} &= \mathbf{x} - \frac{i\mathfrak{K}}{2|\mathfrak{K}|^2}\boldsymbol{\nabla}_{\mathfrak{K}}(\pi^{-1}(\mathfrak{K}))\pi(\mathfrak{K}) \\ &= i\boldsymbol{\nabla}_{\mathfrak{K}} - \frac{i\mathfrak{K}}{2|\mathfrak{K}|^2} - i\sum_{h=\pm 1}\int_{\mathbb{R}^3} d^3\mathbf{k}\,\boldsymbol{\nabla}_{\mathbf{k}}(|\mathbf{u}_h(\mathbf{k})\rangle)\langle\mathbf{u}_h(\mathbf{k})|.\end{aligned} \tag{4.129}$$

Now, if we employ (4.25) to express $\boldsymbol{\chi}_{\mathbf{E}}(x^0)$ in terms of (4.122) and (4.123), we obtain

$$\boldsymbol{\chi}_{\mathbf{E}}(x^0) = \left\{ \boldsymbol{\mathfrak{X}}_{\mathbf{E}} - i(x^0 - x_0^0)\frac{\boldsymbol{\mathfrak{K}}}{|\boldsymbol{\mathfrak{K}}|^2}\partial_0 \right\} \mathbf{E}(x^0). \tag{4.130}$$

Therefore, the position operator associated with the electric field for all $x^0 \in \mathbb{R}$ is given by

$$\mathbf{x}_{0,\mathbf{E}} = \boldsymbol{\mathfrak{X}}_E - i(x^0 - x_0^0)\frac{\boldsymbol{\mathfrak{K}}}{|\boldsymbol{\mathfrak{K}}|^2}\partial_0.$$

In order to compare (4.129) with Hawton's position operator [12], i.e, (2.68). In our notation Hawton's operator takes the form:

$$\boldsymbol{\mathfrak{X}}_{Howton} = i\boldsymbol{\nabla}_{\boldsymbol{\mathfrak{K}}} - \frac{i\boldsymbol{\mathfrak{K}}}{2|\boldsymbol{\mathfrak{K}}|^2} - i\sum_{\lambda=1}^{3}\int_{\mathbb{R}^3} d^3\mathbf{k}\,\boldsymbol{\nabla}_{\mathbf{k}}(|\mathbf{a}(\mathbf{k},\lambda)\rangle)\langle\mathbf{a}(\mathbf{k},\lambda)|. \tag{4.131}$$

Comparing (4.129) and (4.131), we observe that the longitudinal vector $\mathbf{a}(\mathbf{k}, 3)$ does appear in (4.129). This means that Hawton's position operator for the photon does not act on elements of $\mathcal{V}$, because its eigenvectors do not belong to the photon physical Hilbert space $\mathcal{V}$. Furthermore, we should emphasize that Hawton's position operator was constructed only for a fixed time $x^0 = x_0^0$ [12], whereas (4.130) works for all $x^0 \in \mathbb{R}$.

In analogy with the electric field, we can obtain the explicit form of the position operator associated with the magnetic field $\mathbf{x}_{0,\mathbf{B}}$. This can be determined by acting the position operator on the magnetic field of a photon field. To this end, we define $\boldsymbol{\chi}_{\mathbf{B}} =: \mathbf{x}_{0,\mathbf{B}}\,\mathbf{B}$ where $\mathbf{B} = \boldsymbol{\nabla} \times \tilde{\mathbf{A}}$ and $\boldsymbol{\chi}_{\mathbf{B}} := \boldsymbol{\nabla} \times \boldsymbol{\chi}$. Similarly to $\boldsymbol{\chi}$, $\boldsymbol{\chi}_{\mathbf{B}}$ is a three-component field whose components satisfy similar equations to (4.13) and (4.23). Therefore, it is uniquely determined in terms of the initial data $(\boldsymbol{\chi}_{\mathbf{B}}(x_0^0), \dot{\boldsymbol{\chi}}_{\mathbf{B}}(x_0^0))$ for some $x_0^0 \in \mathbb{R}$. These are given by

$$\boldsymbol{\chi}_{\mathbf{B}}(x_0^0) = \boldsymbol{\mathfrak{X}}_{\mathbf{B}}\,\mathbf{B}(x_0^0), \tag{4.132}$$

$$\dot{\boldsymbol{\chi}}_{\mathbf{B}}(x_0^0) = \left(\boldsymbol{\mathfrak{X}}_{\mathbf{B}} - \frac{i\boldsymbol{\mathfrak{K}}}{|\boldsymbol{\mathfrak{K}}|^2} \right) \dot{\mathbf{B}}(x_0^0), \tag{4.133}$$

where

$$\begin{aligned} \boldsymbol{\mathfrak{X}}_{\mathbf{B}} &:= \mathbf{x} + i\frac{3\boldsymbol{\mathfrak{K}}}{2|\boldsymbol{\mathfrak{K}}|^2}\boldsymbol{\nabla}_{\boldsymbol{\mathfrak{K}}}(\pi^{-1}(\boldsymbol{\mathfrak{K}}))\pi(\boldsymbol{\mathfrak{K}}) \\ &= i\boldsymbol{\nabla}_{\boldsymbol{\mathfrak{K}}} + i\frac{3\boldsymbol{\mathfrak{K}}}{2|\boldsymbol{\mathfrak{K}}|^2} - i\sum_{h=\pm 1}\int_{\mathbb{R}^3} d^3\mathbf{k}\,\boldsymbol{\nabla}_{\mathbf{k}}(|\mathbf{u}_h(\mathbf{k})\rangle)\langle\mathbf{u}_h(\mathbf{k})|. \end{aligned} \tag{4.134}$$

This determines the action of the position operator on the magnetic field at x_0^0. It is not difficult to extend this to all $x^0 \in \mathbb{R}$:

$$\boldsymbol{\chi}_{\mathbf{B}}(x^0) = \left\{ \boldsymbol{\mathfrak{X}}_{\mathbf{B}} - i(x^0 - x_0^0)\frac{\boldsymbol{\mathfrak{K}}}{|\boldsymbol{\mathfrak{K}}|^2}\partial_0 \right\} \mathbf{B}(x^0). \tag{4.135}$$

Therefore, the position operator corresponding to the magnetic field for all $x^0 \in \mathbb{R}$ is given by

$$\mathbf{x}_{0,\mathbf{B}} = \boldsymbol{\mathfrak{X}}_{\mathbf{B}} - i(x^0 - x_0^0)\frac{\boldsymbol{\mathfrak{K}}}{|\boldsymbol{\mathfrak{K}}|^2}\partial_0.$$

4.8 Localized States of Photon and their Electric and Magnetic Fields of Photon Fields

4.8.1 Localized States of a Photon

In a search for determining the explicit form of localized states, we use (4.78), (4.102) and (4.106) to derive

$$\tilde{\mathbf{A}}_{\mathbf{y}}^{(\epsilon,h)}(x^0) = \frac{1}{\sqrt{\kappa}}|\boldsymbol{\mathfrak{K}}|^{-\frac{1}{2}}\; e^{-i\epsilon(x^0-x_0^0)|\boldsymbol{\mathfrak{K}}|}\,\pi^{-1}|\mathbf{y}\rangle \otimes e_h. \tag{4.136}$$

The localized photon fields $\tilde{\mathbf{A}}_{\mathbf{y}}^{(\epsilon,h)}$ at a space-time point $(x_0^0, \mathbf{x})$ have the form

$$\tilde{\mathbf{A}}_{\mathbf{y}}^{(\epsilon,h)}(x^0,\mathbf{x}) := \langle \mathbf{x}|\tilde{\mathbf{A}}_{\mathbf{y}}^{(\epsilon,h)}(x^0)\rangle. \tag{4.137}$$

Substituting (4.136) in (4.137) and performing necessary algebra, we obtain

$$\tilde{\mathbf{A}}_{\mathbf{y}}^{(\epsilon,h)}(x^0,\mathbf{x}) = \frac{1}{(2\pi)^3\sqrt{\kappa}}\int_{\mathbb{R}^3} d^3\mathbf{k}\, k^{-\frac{1}{2}} e^{-i\epsilon(x^0-x_0^0)k} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\mathbf{u}_h(\mathbf{k}), \tag{4.138}$$

where $\mathbf{u}_h(\mathbf{k})$ are transverse polarization vectors of the photon

$$\mathbf{u}_h(\mathbf{k}) = \frac{1}{|k|\sqrt{2(k_2^2+k_3^2)}}\begin{pmatrix} k_2^2+k_3^2 \\ -k_1k_2 + ihk_3|k| \\ -k_1k_3 - ihk_2|k| \end{pmatrix} \tag{4.139}$$

The integral on the right-hand side of (4.138) turn out to be extremely difficult and we were not able to evaluate them. However we can determine their dependence on $|\mathbf{x}-\mathbf{y}|$. The result is

$$\tilde{\mathbf{A}}_{\mathbf{y}}^{(\epsilon,h)}(x_0^0,\mathbf{x}) \propto \frac{1}{|\mathbf{x}-\mathbf{y}|^{5/2}}. \tag{4.140}$$

This relation is identical with the one obtained in section 3.5 for the massless scalar fields.

4.8.2 Electric and Magnetic Fields components for the Localized States of a Photon

By making use of (4.136), (4.30) and $\mathbf{B}(x^0) = |\mathfrak{K}|\mathfrak{h}\tilde{\mathbf{A}}(x^0)$, we obtain electric and magnetic fields of a photon localized at a point in space:

$$\mathbf{E}_{\mathbf{y}}^{(\epsilon,h)}(x^0) = \frac{i\epsilon}{\sqrt{\kappa}}|\mathfrak{K}|^{\frac{1}{2}}\ e^{-i\epsilon(x^0-x^0_0)|\mathfrak{K}|}\,\pi^{-1}|\mathbf{y}\rangle\otimes e_h, \tag{4.141}$$

$$\mathbf{B}_{\mathbf{y}}^{(\epsilon,h)}(x^0) = \frac{h}{\sqrt{\kappa}}|\mathfrak{K}|^{\frac{1}{2}}\ e^{-i\epsilon(x^0-x^0_0)|\mathfrak{K}|}\,\pi^{-1}|\mathbf{y}\rangle\otimes e_h. \tag{4.142}$$

The value of their fields at a space-time point $(x^0_0, \mathbf{x})$ has the form:

$$\mathbf{E}_{\mathbf{y}}^{(\epsilon,h)}(x^0,\mathbf{x}) := \langle\mathbf{x}|\mathbf{E}_{\mathbf{y}}^{(\epsilon,h)}(x^0)\rangle, \qquad \mathbf{B}_{\mathbf{y}}^{(\epsilon,h)}(x^0,\mathbf{x}) := \langle\mathbf{x}|\mathbf{B}_{\mathbf{y}}^{(\epsilon,h)}(x^0)\rangle. \tag{4.143}$$

Substituting (4.141) and (4.142) in (4.143) yields

$$\mathbf{E}_{\mathbf{y}}^{(\epsilon,h)}(x^0,\mathbf{x}) = \frac{i\,\epsilon}{(2\pi)^3\sqrt{\kappa}}\int_{\mathbb{R}^3} d^3\mathbf{k}\,k^{\frac{1}{2}}e^{-i\epsilon(x^0-x^0_0)k}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\mathbf{u}_h(\mathbf{k}), \tag{4.144}$$

$$\mathbf{B}_{\mathbf{y}}^{(\epsilon,h)}(x^0,\mathbf{x}) = \frac{h}{(2\pi)^3\sqrt{\kappa}}\int_{\mathbb{R}^3} d^3\mathbf{k}\,k^{\frac{1}{2}}e^{-i\epsilon(x^0-x^0_0)k}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\mathbf{u}_h(\mathbf{k}), \tag{4.145}$$

where $\mathbf{u}_h(\mathbf{k})$ are transverse polarization vectors of the photon.

The integral expressions in (4.144) and (4.145) are extremely complicated. We have not been able to evaluate them, but can show that

$$\mathbf{E}_{\mathbf{y}}^{(\epsilon,h)}(x^0_0,\mathbf{x}) \propto \frac{1}{|\mathbf{x}-\mathbf{y}|^{7/2}}, \tag{4.146}$$

$$\mathbf{E}_{\mathbf{y}}^{(\epsilon,h)}(x^0_0,\mathbf{x}) \propto \frac{1}{|\mathbf{x}-\mathbf{y}|^{7/2}}. \tag{4.147}$$

4.9 The Photon's Wave Function

We can use (4.136) and (4.40) to compute the right hand side of (4.110). This gives for the position wave function of a single photon:

$$f(\epsilon,h,\mathbf{x}) = \frac{\sqrt{\kappa}}{2}\,\tilde{\pi}^{\dagger}\{|\mathfrak{K}|^{\frac{1}{2}}\tilde{\mathbf{A}}(\mathbf{x},x^0_0) - i\epsilon|\mathfrak{K}|^{-\frac{1}{2}}\mathbf{E}(\mathbf{x},x^0_0)\}. \tag{4.148}$$

where

$$\tilde{\pi} := \int_{\mathbb{R}^3} d^3\mathbf{k}\,\mathbf{u}_h(\mathbf{k})|\mathbf{k}\rangle\langle\mathbf{k}|.$$

Next, we simplify (4.148) to show that

$$f(\epsilon, h, \mathbf{x}) = \frac{-i\epsilon\sqrt{\kappa}}{2}\,\tilde{\kappa}^{\dagger}\{\mathbf{E}(\mathbf{x}, x_0^0) + i\epsilon\mathbf{B}(\mathbf{x}, x_0^0)\}, \tag{4.149}$$

where

$$\tilde{\kappa} := \int_{\mathbb{R}^3} d^3\mathbf{k}\, k^{-\frac{1}{2}}\mathbf{u}_h(\mathbf{k})|\mathbf{k}\rangle\langle\mathbf{k}|. \tag{4.150}$$

This is a very useful representation of the position wave function of a single photon, because it shows its relation to the Riemann-Silberstein wave function

$$\mathbf{F}_{RS}(\mathbf{x}, x^0) := \mathbf{E}(\mathbf{x}, x^0) + i\epsilon\mathbf{B}(\mathbf{x}, x^0). \tag{4.151}$$

Next, we recall that the Landau-Peierls wave function (2.1) can be writhen in terms of the Riemann-Silberstein complex vector [3, 1] according to

$$\Psi_{LP}(x_0^0, \mathbf{x}) := \mathfrak{K}^{-\frac{1}{2}}F_{RS}(x_0^0, \mathbf{x}) = \frac{1}{2(2\pi)^{3/2}}\int_{\mathbb{R}^3} d^3\mathbf{x}'\,\frac{\mathbf{F}_{RS}(x_0^0, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^{5/2}}. \tag{4.152}$$

This implies

$$f(\epsilon, h, \mathbf{x}) = \frac{-i\epsilon\sqrt{\kappa}}{2}\tilde{\pi}(\mathfrak{K})^{\dagger}\Psi_{LP}(x_0^0, \mathbf{x}). \tag{4.153}$$

4.10 Probability Density for Spatial Localization of a Field

We can follow the approach of Refs [53, 51] to compute the probability density for the spatial localization of a photon field. The results is

$$\begin{aligned}
\varrho(x_0^0, \mathbf{x}) &:= \sum_{\epsilon=\pm}\sum_{h=\pm 1} |f(\epsilon, h, \mathbf{x})|^2 \\
&= \sum_{h=\pm 1} \frac{\kappa}{2}[|\mathfrak{K}||\tilde{\pi}^* \cdot \tilde{\mathbf{A}}(\mathbf{x}, x_0^0)|^2 + |\mathfrak{K}|^{-1}|\tilde{\pi}^* \cdot \mathbf{E}(\mathbf{x}, x_0^0)|^2] \\
&= \sum_{h=\pm 1} \frac{\kappa}{2}[|\tilde{\kappa}^* \cdot \mathbf{E}(\mathbf{x}, x_0^0)|^2 + |\mathfrak{K}|^2|\tilde{\kappa}^* \cdot \tilde{\mathbf{A}}(\mathbf{x}, x_0^0)|^2] \\
&= \sum_{h=\pm 1} \frac{\kappa}{2}[|\tilde{\kappa}^* \cdot \mathbf{E}(\mathbf{x}, x_0^0)|^2 + |\tilde{\kappa}^* \cdot \mathbf{B}(\mathbf{x}, x_0^0)|^2].
\end{aligned} \tag{4.154}$$

From section 4.2 we recall that for specific choice of $a_{\epsilon,h} = 1$ for all $\epsilon \in \{+, -\}$ and $h \in \{-1, +1\}$, the positive-definite Lorentz-invariant inner product takes the form

$$(\!(\tilde{A}, \tilde{A}')\!) = \int_{\mathbb{R}^3} d^3\mathbf{x}\,\left\{\frac{\kappa}{2}[\tilde{\mathbf{A}}(x)^* \cdot D^{-1/2}\dot{\mathbf{E}}(x) - \mathbf{E}(x)^* \cdot D^{-1/2}\dot{\tilde{\mathbf{A}}}(x)]\right\} \tag{4.155}$$

in which the integrand is the zero component of a conserved four-vector current density, $J^0(x^0, \mathbf{x})$.

Furthermore, by using (4.154) and (4.115) we observe that

$$(\!(\tilde{A}, \tilde{A}')\!) = \int_{\mathbb{R}^3} d^3\mathbf{x}\, \varrho(x^0, \mathbf{x}). \tag{4.156}$$

This together with (4.155) and (4.154) gives

$$\mathscr{P} = \int_{\mathbb{R}^3} d^3\mathbf{x}\, \varrho(x^0, \mathbf{x}) = \int_{\mathbb{R}^3} d^3\mathbf{x}\, J^0(x^0, \mathbf{x}) = (\!(\tilde{A}, \tilde{A}')\!). \tag{4.157}$$

This means that the probability density ϱ is not the zero-component of a conserved four-vector current density J^μ, but its integral over the whole space gives the total probability which is frame-independent and conserved.

Chapter 5

CONCLUSION AND DISCUSSION

In chapter 2 we presented a review of the previous research in the area of the first-quantized theory of a photon including the wave function in coordinate representation, the position operator, a probabilistic interpretation of the wave function and quantum mechanical expectation values of physical observables.

In chapter 3 we studied a relativistic scalar field (both with mass and without mass) and provided a well-defined quantum mechanical treatment of the system. We obtained the most general Lorentz-invariant positive-definite inner product on the space of the relativistic scalar fields and derived the position wave function and localized states for this system. We further showed that the localized states of a massless scalar field can be obtained as the zero-mass limit of that of the massive scalar fields. Similarly to the localized states of a photon, the localized states of a massless scalar field has the dependency as $\frac{1}{|\mathbf{x}-\mathbf{y}|^{5/2}}$ where $|\mathbf{x}-\mathbf{y}|$ is the distance from the center of localization. Furthermore, we examined the position, momentum, and angular momentum operators for the relativistic scalar fields and derived a positive-definite probability density for a measurement of the position observable.

In chapter 4, we provided a consistent description of a first-quantized photon. Unlike previous studies of this subject where the solution of the Maxwell equation was used as a primary definition of the position wave function (for instance the Riemann-Silberstein complex vector), we viewed the solutions of Maxwell equations as abstract state vectors in a complex vector space, and equipped this space with a Lorenz-invariant positive-definite inner product to obtain a genuine Hilbert space. We identified the Hamiltonian as the generator of time-translations and derived the chirality and helicity operators. We obtained the position operator for a photon with commuting components which also commutes with helicity and chirality operators. This position operator is the same as the position operator of a massless scalar field plus an additional term which corresponds to the Berry phase of a propagating photon. Similarly, we found the position operator associated with the electric and magnetic fields. The position operator associated with electric fields turned out to be very similar to the one which was obtained by Hawton; however, unlike the Hawton operator it does not involve the longitudinal component. Unlike the preceding studies, in our formulation the derivation of the physical observables (e.g. the position operator) relies on the construction of the physical Hilbert space of the photon.

We obtained the localized states of the photon photon, and characterized the photon wave functions as the projection of state vectors on the position basis (localized states) of this physical Hilbert space. Subsequently, we identified the physical observables of this quantum system in a usual sense (similar to the non-relativistic quantum mechanics). Our position wave function is not only a square-integrable function, but only involves both the Landau-Peierls wave function and the Riemann-Silberstein complex vector. In the framework of the photon wave mechanics, usually the negative-frequency solutions are discarded. In our approach they are maintained. Another fundamental and important issue that we have resolved is the difficulty of finding an appropriate positive-definite inner product and a positive-definite probability density.

Chapter 6

APPENDICES

Appendix A: Computation of $\psi^{\epsilon}_{\mathbf{y}}(x^0, \mathbf{x})$

Here we compute

$$\psi^{\epsilon}_{a,\mathbf{y}}(x^0_0, \mathbf{x}) = \frac{\gamma}{2\pi^2 z} \int_0^{\infty} dk\, k^{1/2} \sin(zk) \tag{6.1}$$

where $z := |\mathbf{x} - \mathbf{y}|$. This requires the use of the spherical Bessel function of the first kind that we can express as

$$j_n(x) = (-1)^n x^n \left(\frac{d}{xdx}\right)^n \frac{\sin x}{x}. \tag{6.2}$$

Setting $n = 0$ and $x = kz$, this yields

$$j_0(kz) = \frac{\sin kz}{kz}. \tag{6.3}$$

In view of (6.1) and (6.3),

$$\psi^{\epsilon}_{\mathbf{y}}(x^0_0, \mathbf{x}) = \frac{\gamma}{2\pi^2} \int_0^{\infty} dk\, k^{3/2} j_0(kz). \tag{6.4}$$

Next we use the identity [87]

$$j_\nu(kz) = \sqrt{\frac{\pi}{2kz}} J_{\nu+\frac{1}{2}}(kz) \tag{6.5}$$

where $J_{\nu+\frac{1}{2}}(kz)$ is the Bessel function of the first kind. Using (6.5) and (6.4), we obtain

$$\psi^{\epsilon}_{\mathbf{y}}(x^0_0, \mathbf{x}) = \frac{\gamma}{2\pi\sqrt{2\pi z}} \int_0^{\infty} dk\, k J_{\frac{1}{2}}(kz). \tag{6.6}$$

In order to evaluate the right-hand side of (6.6), we use the following integral identity

$$\int_0^{\infty} dx\, x^{m+1} e^{-\alpha x} J_\nu(\beta x) = (-1)^{m+1} \beta^{-\nu} \frac{d^{m+1}}{d\alpha^{m+1}} \left[\frac{(\sqrt{\alpha^2+\beta^2} - \alpha)^\nu}{\sqrt{\alpha^2+\beta^2}}\right]$$
$$[\beta > 0, \quad \Re\nu > -m-2]. \tag{6.7}$$

Setting $x = k$, $\beta = z$, $\alpha = 0$, $m = 0$ and $\nu = 1/2$ in (6.7), we obtain

$$\int_0^\infty dk\, k J_{\frac{1}{2}}(zk) = -z^{-\frac{1}{2}} \frac{d}{d\alpha}\left[\frac{(\sqrt{\alpha^2+z^2}-\alpha)^{\frac{1}{2}}}{\sqrt{\alpha^2+z^2}}\right]_{\alpha=0} = \frac{1}{2z^2} \tag{6.8}$$

Using this in (6.6) gives

$$\psi^\epsilon_{\mathbf{y}}(x_0^0, \mathbf{x}) = \gamma \frac{\pi}{(2\pi z)^{\frac{5}{2}}}. \tag{6.9}$$

Appendix B: Most general relativistically invariant positive-definite inner product

Recall that the inner product (4.72) is given by

$$\begin{aligned}(\!(\tilde{A}, \tilde{A}')\!)_{\{\mathfrak{b}\}} :=& \prec \tilde{\mathbf{A}}(x^0), \Theta_+ \tilde{\mathbf{A}}'(x^0) \succ + \prec \mathbf{E}(x^0), \Theta_+\, D^{-1}\mathbf{E}'(x^0) \succ \\ &+ i\left[\prec \mathbf{E}(x^0), \Theta_- D^{-\frac{1}{2}} \tilde{\mathbf{A}}'(x^0) \succ - \prec \tilde{\mathbf{A}}(x^0), \Theta_- D^{-\frac{1}{2}} \mathbf{E}'(x^0) \succ\right]\end{aligned} \tag{6.10}$$

In order to make this inner product Lorentz-invariant, we find appropriate values for $\Theta_\pm$ under which the resulting inner product turn to a relativistically invariant inner product. In order to accomplish our goal, we identify a basic solution of Maxwell fields according to

$$\tilde{A}^j_{\epsilon,h}(\mathbf{k}; x^0, \mathbf{x}) = N_{\epsilon,h}(\mathbf{k})\; u^j_{\epsilon,h}(\mathbf{k})\, e^{-i\epsilon k x^0} |\mathbf{k}\rangle, \qquad \forall j \in \{1,2,3\} \tag{6.11}$$

The inner product of any two basic solutions $\tilde{A}_{\epsilon,h}(\mathbf{k})$ and $\tilde{A}_{\epsilon',h'}$, is given as

$$(\!(\tilde{A}_{\epsilon,h}(\mathbf{k}), \tilde{A}_{\epsilon',h'}(\mathbf{k}'))\!)_{\{\mathfrak{b}\}} = (2\pi)^3 |N_{\epsilon,h}(\mathbf{k})|^2\, \mathfrak{b}_{\epsilon,h}\, \delta^3(\mathbf{k}-\mathbf{k}')\delta_{\epsilon,\epsilon'}\delta_{h,h'}, \tag{6.12}$$

where we introduce $\mathfrak{b}_{\epsilon,h} := |\beta_{\epsilon,h}|^2$, and $N_{\epsilon,h}(\mathbf{k})$ are normalization constants. Because $\mathfrak{b}_{\epsilon,h}$ are positive real numbers and the right-hand side of (6.12) is independent of x^0, the relation (6.12) presents the invariance and positive definiteness of the inner product (6.11).

As we have obtained the inner product of two basic solutions of the Maxwell fields in (6.12), now we are in the position to compute the inner product for any two solutions $\tilde{A}$ and $\tilde{A}'$ having the general form of

$$\tilde{A} = \sum_{\epsilon=\pm}\sum_{h=\pm 1}\int_{\mathbb{R}^3} d^3\mathbf{k}\, c_{\epsilon,h}(\mathbf{k}) \tilde{A}_{\epsilon,h}(\mathbf{k}), \tag{6.13}$$

where $c_{\epsilon,h}(\mathbf{k})$ are complex coefficients. By employing (6.12) and (6.13) and also the fact that the inner product (6.11) is a Hermitian sesquilinear form, we get

$$(\!(\tilde{A}, \tilde{A}')\!)_{\{\mathfrak{b}\}} = \sum_{\epsilon=\pm}\sum_{h=\pm 1}\int_{\mathbb{R}^3} d^3\mathbf{k}\, \mathfrak{b}_{\epsilon,h}\, |N_{\epsilon,h}(\mathbf{k})|^2\, c^*_{\epsilon,h}(\mathbf{k})\, c'_{\epsilon,h}(\mathbf{k}). \tag{6.14}$$

Because $d^3\mathbf{k}/k$ is a relativistically invariant measure [43] and $\tilde{A}$ and $\tilde{A}_{\epsilon,h}(\mathbf{k})$ are four-vectors, we conclude that $c_{\epsilon,h}(\mathbf{k})$ should transform similarly to k^{-1}. This clearly shows that in order for the inner product (6.14) to be a scalar, $\mathfrak{b}_{\epsilon,h}$ must also transform similarly to k. Therefore,

$$\mathfrak{b}_{\epsilon,h} = b_{\epsilon,h}\, k \tag{6.15}$$

where $b_{\epsilon,h}$ are dimensionless positive real numbers. In light of (6.15) and (4.55), we obtain

$$L_{\pm}^{\epsilon} = \frac{1}{4}\left[b_{+,1} + \epsilon b_{+,-1} \pm b_{-,1} \pm \epsilon b_{-,-1}\right] D^{1/2} = L_{\pm}^{\prime\epsilon}\, D^{1/2}, \tag{6.16}$$

where we denote $L_{\pm}^{\prime\epsilon} := \frac{1}{4}\left[b_{+,1} + \epsilon b_{+,-1} \pm b_{-,1} \pm \epsilon b-, -1\right]$. Next we define a new operator Θ_p' acting in $\tilde{\mathcal{H}}$ by

$$\Theta_p' := L_p^{\prime +}\mathfrak{h}^2 + L_p^{\prime -}\mathfrak{h}. \tag{6.17}$$

Using (6.17) and (4.58), we can show that

$$\Theta_p = \Theta_p'\, D^{1/2}.$$

This relation together with (6.10) gives

$$\begin{aligned}(\!(\tilde{A}, \tilde{A}')\!)_{\{\mathfrak{b}\}} \quad :=& \prec \tilde{\mathbf{A}}(x^0), \Theta_+'\, D^{1/2}\tilde{\mathbf{A}}'(x^0) \succ + \prec \mathbf{E}(x^0), \Theta_+'\, D^{-1/2}\mathbf{E}'(x^0) \succ \\ &+ i\left[\prec \mathbf{E}(x^0), \Theta_-'\tilde{\mathbf{A}}'(x^0) \succ - \prec \tilde{\mathbf{A}}(x^0), \Theta_-'\mathbf{E}'(x^0) \succ .\right]\end{aligned} \tag{6.18}$$

It is straightforward to see that $L_+^{\prime +}$ is a positive number, and we can absorb it in the definition of the field. This in turn implies that relation (6.18) defines a three-parameter family of relativistically invariant inner products.

Next, we consider the simplest choice where $b_{\epsilon,h} = 1$ for all $\epsilon = \{\pm\}$ and $h = \{\pm 1\}$. This leads to

$$\Theta_+' = 1, \qquad\qquad \Theta_-' = 0 \tag{6.19}$$

and the inner product (6.18) becomes

$$(\!(\tilde{A}, \tilde{A}')\!)_{r.i} \quad := \prec \tilde{\mathbf{A}}(x^0), D^{1/2}\tilde{\mathbf{A}}'(x^0) \succ + \prec \mathbf{E}(x^0), D^{-1/2}\mathbf{E}'(x^0) \succ . \tag{6.20}$$

In the remainder of this section we try to derive the Foldy transformation associated with the most general relativistically invariant inner product (6.18) and the inner product (6.20). We begin our discussion by computing the values of $\beta_{\epsilon,h}$ and $\beta_{\epsilon,h}^{-1}$

for $\mathfrak{b}_{\epsilon,h} = b_{\epsilon,h}\,k$ which we have imposed to derive the most general Lorentz-invariant inner product (6.18). Using this condition, we obtain $\beta_{\epsilon,h}$ and $\beta^{-1}_{\epsilon,h}$ in the following forms,

$$\tilde{\beta}_{\epsilon,h} = \sqrt{b_{\epsilon,h}}\, D^{1/4},, \qquad \tilde{\beta}^{-1}_{\epsilon,h} = \frac{1}{\sqrt{b_{\epsilon,h}}} D^{-1/4}. \tag{6.21}$$

In view of (6.18), we obtain the explicit expressions for $\mathcal{B}$ and $\mathcal{B}^{-1}$ according to

$$\mathcal{B} = \begin{pmatrix} Z'^{+}_{+}\mathfrak{h}^2 + Z'^{+}_{-}\mathfrak{h} & 0 \\ 0 & Z'^{-}_{+}\mathfrak{h}^2 + Z'^{-}_{-}\mathfrak{h} \end{pmatrix}, \quad \mathcal{B}^{-1} = \begin{pmatrix} \tilde{Z}'^{+}_{+}\mathfrak{h}^2 + \tilde{Z}'^{+}_{-}\mathfrak{h} & 0 \\ 0 & \tilde{Z}'^{-}_{+}\mathfrak{h}^2 + \tilde{Z}'^{-}_{-}\mathfrak{h} \end{pmatrix}, \tag{6.22}$$

where

$$Z'^{\epsilon}_{\pm} := \frac{1}{2}[\sqrt{b_{\epsilon,1}} \pm \sqrt{b_{\epsilon,-1}}], \qquad \tilde{Z}'^{\epsilon}_{\pm} := \frac{1}{2}[\frac{1}{\sqrt{b_{\epsilon,1}}} \pm \frac{1}{\sqrt{b_{\epsilon,-1}}}]. \tag{6.23}$$

The unitary operator $\mathcal{U}^{(r.i)}_{\{\mathfrak{b}\}}$ (Foldy transformation) corresponding to the Lorentz-invariant inner product (6.18) is given by

$$\mathcal{U}^{(r.i)}_{\{\mathfrak{b}\}}\,\tilde{A} = \frac{1}{2}\begin{pmatrix} \mathfrak{U}'_{+}\,[D^{1/4}\,\tilde{\mathbf{A}}(x^0_0) - iD^{-1/4}\,\mathbf{E}(x^0_0)] \\ \mathfrak{U}'_{-}\,[D^{1/4}\,\tilde{\mathbf{A}}(x^0_0) + iD^{-1/4}\,\mathbf{E}(x^0_0)] \end{pmatrix}, \tag{6.24}$$

where the operators $\mathfrak{U}'_{\epsilon} : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ take the form

$$\mathfrak{U}'_{\epsilon} = Z'^{\epsilon}_{+}\mathfrak{h}^2 + Z'^{\epsilon}_{-}\mathfrak{h}. \tag{6.25}$$

$\mathcal{U}^{(r.i)-1}_{\{\mathfrak{b}\}}\xi$ is the photon field $\tilde{A} \in \mathcal{H}_{\{\mathfrak{b}\}}$ which satisfies the following initial conditions:

$$\dot{\tilde{\mathbf{A}}}(x^0_0) = D^{-1/4}[\mathfrak{U}'^{-1}_{+}\boldsymbol{\xi}^1 + \mathfrak{U}'^{-1}_{-}\boldsymbol{\xi}^2], \qquad \mathbf{E}(x^0_0) = i\,D^{1/4}\,\left[\mathfrak{U}'^{-1}_{+}\boldsymbol{\xi}^1 - \mathfrak{U}'^{-1}_{-}\boldsymbol{\xi}^2\right]. \tag{6.26}$$

Here $\mathfrak{U}'^{-1}_{\epsilon}$ is the inverse of $\mathfrak{U}'_{\epsilon}$, that is given by

$$\mathfrak{U}'^{-1}_{\epsilon} := \tilde{Z}'^{\epsilon}_{+}\mathfrak{h}^2 + \tilde{Z}'^{\epsilon}_{-}\mathfrak{h}. \tag{6.27}$$

We can also determine

$$\tilde{\mathbf{A}}^0(x^0) = D^{-1/4}\left[e^{-i(x^0-x^0_0)D^{1/2}}\mathfrak{U}'^{-1}_{+}\boldsymbol{\xi}^1 + e^{i(x^0-x^0_0)D^{1/2}}\mathfrak{U}'^{-1}_{-}\boldsymbol{\xi}^2\right], \tag{6.28}$$

$$\mathbf{E}(x^0) = iD^{1/4}\left[e^{-i(x^0-x^0_0)D^{1/2}}\mathfrak{U}'^{-1}_{+}\boldsymbol{\xi}^1 - e^{i(x^0-x^0_0)D^{1/2}}\mathfrak{U}'^{-1}_{-}\boldsymbol{\xi}^2\right], \tag{6.29}$$

for all $x^0_0 \in \mathbb{R}$. For the choice, $b_{\epsilon,h} = 1$ for all $\epsilon = \{\pm\}$ and $h = \{\pm 1\}$, $Z'^{\epsilon}_{+} = 1$, $Z'^{\epsilon}_{-} = 0$, $\tilde{Z}'^{\epsilon}_{+} = 1$ and $\tilde{Z}'^{\epsilon}_{-} = 0$, and we have

$$\mathfrak{U}'_{\epsilon} = \mathfrak{h}^2 = 1_{\mathcal{V}}. \tag{6.30}$$

We also compute the corresponding expressions for $\mathcal{B}$ and $\mathcal{B}^{-1}$. The result is

$$\mathcal{B} = \mathcal{B}^{-1} = \sigma_0 \otimes \mathfrak{h}^2. \tag{6.31}$$

A similar reasoning gives the following expression for the Foldy transformation $\mathcal{U}_{r.i}$

$$\mathcal{U}_{r.i}\tilde{A} = \frac{1}{2}\left(\begin{array}{c} D^{1/4}\,\tilde{\mathbf{A}}(x_0^0) - iD^{-1/4}\,\mathbf{E}(x_0^0) \\ D^{1/4}\,\tilde{\mathbf{A}}(x_0^0) + iD^{-1/4}\,\mathbf{E}(x_0^0) \end{array}\right). \tag{6.32}$$

Notice that ξ is a six-component vector with components $\boldsymbol{\xi}^1$ and $\boldsymbol{\xi}^2$. Thus in view of (6.32), we can show that $\mathcal{U}_{r.i}^{-1}\xi$ is a photon field $\tilde{A} \in \mathcal{H}_{\{\mathfrak{b}\}}$ satisfying the initial conditions:

$$\tilde{\mathbf{A}}(x_0^0) = D^{-1/4}(\boldsymbol{\xi}^1 + \boldsymbol{\xi}^2), \qquad \mathbf{E}(x_0^0) = i\,D^{1/4}\,(\boldsymbol{\xi}^1 - \boldsymbol{\xi}^2). \tag{6.33}$$

This in turn implies that for all $x^0 \in \mathbb{R}$,

$$\tilde{\mathbf{A}}^0(x^0) \;=\; D^{-1/4}\left[e^{-i(x^0-x_0^0)D^{1/2}}\boldsymbol{\xi}^1 + e^{i(x^0-x_0^0)D^{1/2}}\boldsymbol{\xi}^2\right], \tag{6.34}$$

$$\mathbf{E}(x^0) \;=\; iD^{1/4}\left[e^{-i(x^0-x_0^0)D^{1/2}}\boldsymbol{\xi}^1 - e^{i(x^0-x_0^0)D^{1/2}}\boldsymbol{\xi}^2\right]. \tag{6.35}$$

Next, we provide an alternative form of the Foldy transformation. Using the chirality operator, we can express $\mathcal{U}_{\{\mathfrak{b}\}}$ and $\mathcal{U}_{\{\mathfrak{b}\}}^{(r.i)}$ according to

$$\mathcal{U}_{\{\mathfrak{b}\}}\tilde{A} = \left(\begin{array}{c} \mathfrak{U}_+\tilde{\mathbf{A}}_+(x_0^0) \\ \mathfrak{U}_-\tilde{\mathbf{A}}_-(x_0^0) \end{array}\right). \tag{6.36}$$

$$\mathcal{U}_{\{\mathfrak{b}\}}^{(r.i)}\tilde{A} = D^{1/4}\left(\begin{array}{c} \mathfrak{U}'_+\tilde{\mathbf{A}}_+(x_0^0) \\ \mathfrak{U}'_-\tilde{\mathbf{A}}_-(x_0^0) \end{array}\right). \tag{6.37}$$

For the case that $\beta_{\epsilon,h} = 1$ for all $\epsilon \in \{+,-\}$ and $h \in \{-1,+1\}$, we obtain

$$\mathcal{U}_{r.i}\tilde{A} = D^{1/4}\left(\begin{array}{c} \tilde{\mathbf{A}}_+(x_0^0) \\ \tilde{\mathbf{A}}_-(x_0^0) \end{array}\right), \tag{6.38}$$

where $\tilde{\mathbf{A}}_\pm$ is $\pm$-energy component of a photon field. These equations show that $\mathfrak{U}_\epsilon\tilde{\mathbf{A}}_\epsilon$ (and similarly $\mathfrak{U}'_\epsilon D^{1/4}\,\tilde{\mathbf{A}}_\epsilon$ and $D^{1/4}\,\tilde{\mathbf{A}}_\epsilon$) satisfies the Foldy equation [46, 52]

$$i\partial_0\mathfrak{U}_\epsilon\tilde{\mathbf{A}}_\epsilon := \epsilon D^{\frac{1}{2}}\mathfrak{U}_\epsilon\tilde{\mathbf{A}}_\epsilon. \tag{6.39}$$

BIBLIOGRAPHY

[1] I. Bialynicki-Birula, Progress in Optics, Vol. XXXVI, Edited by E. Wolf, Elsevier, Amsterdam (1939).

[2] A. I. Akhiezer and V. B. Berestetskii, Quantum Electrodynamics, (Interscience, New York), Ch. 1 (1965).

[3] L. D. Landau and R. Peierls, Quantenelektrodynamik im Konfigurationsraum, Z. Phys. **62**, 188 (1930)

[4] R. J. Cook, Phys. Rev. A **25**, 2164 (1982);
R. J. Cook, Phys. Rev. A **26**, 2754 (1982).

[5] T. Inagaki, Phys. Rev. A **49**, 2839 (1994).

[6] W. Pauli, General Principles of Quantum Mechanics, Springer, Berlin, (1980).

[7] J. F. Sipe, Phys. Rev. A **52**, 1875 (1995)

[8] I. Bialynicki-Birula, Acta. Phys. Pol. A **86**, 97 (1994)

[9] I. Bialynicki-Birula, Coherence and Quantum Optics VII, Edited by J. H. Eberly, L. Mandel, and E. Wolf, Plenum, New York (1996).

[10] I. Bialynicki-Birula, Phys. Rev. Let. **80**, 5247 (1998).

[11] M. Howton, Phys. Rev. A **59**, 954 (1999).

[12] M. Howton, Phys. Rev. A **59**, 3223 (1999).

[13] M. Howton, Phys. Rev. A **75**, 062107 (2007).

[14] D. Dragoman, J. Opt. Soc. Am. B **24**, 922 (2007).

[15] L. Mandel, Phys. Rev. **144**, 1071 (1966).

[16] O. Keller, Phys. Rep. **411**, 1 (2005).

[17] B. J. Smith and M. G. Raymer, New J. Phys. **9**, 414 (2007).

[18] R. H. Good, Jr. Phys. Rev. **105**, 1914 (1957).

[19] H. Weber, Die partiellen Differential-Gleichungen der mathematischen Physik nach Riemann's Vorlesungen (Friedrich Vieweg und Sohn, Braunschweig) (1901) p. 348.

[20] L. Silberstein, Ann. d. Phys. **22**, 579 (1907);
L. Silberstein, Ann. d. Phys. **24**, 783 (1907);
L. Silberstein, The Theory of Relativity, MacMillan, London (2nd enlarged edition) (1914).

[21] G. Moliere, Ann. d. Phys. **6**, 146 (1949).

[22] J. R. Oppenheimer, Phys. Rev. **38**, 735 (1931).

[23] T. D. Newton and E. P. Wigner, Rev. Mod. Phys. **21**, 400 (1949).

[24] M. H. L. Pryce, Proc. R. Soc. Lond. A **195**, 62 (1948).

[25] J. Mourad, Phys. Lett. A **182**, 319 (1993).

[26] T. F. Jordan, J. Math. Phys. **21**, 2028 (1980);
T. F. Jordan and N. Mukunda, Phys. Rev. **132**, 1842 (1963).

[27] H. Bacry,Localizability and Space in Quantum Physics, Springer-Verlag (1988).

[28] R. A. Berg, J. Math. Phys. **6**, 34 (1965).

[29] Z. K. Silagadze, SLAC-PUB-5754 Rev. (1993).

[30] M. Hawton and W. E. Baylis, Phys. Rev. A **64**, 012101 (2001).

[31] A. S. Wightman, Rev. Mod. phys. **34**, 845 (1962).

[32] G. W. Mackey, Unitary Group Representation in Physics, Probability and Number Theory, Newyork; Addison-wesley (1962).

[33] D. Rosewarne and S. Sakar, Quantum Opc. **4**, 405 (1992).

[34] J. M. Jauch and C. Piron, Helv. Phys. Acta **40**, 559 (1967).

[35] W. O. Amrein, Helv. Phys. Acta **42**, 149 (1969).

[36] C. Adlard, E. R. Pike and S. Sarkar, Phys. Rev. Lett. **79**, 1585 (1997).

[37] C. K. Hong and L. Mandel, Phys. Rev. Lett. **56**, 58 (1986).

[38] K. W. Chan, C. K. Law and J. H. Eberly, Phys. Rev. Lett. **88**, 100402 (2002);
C. K. Law and J. H. Eberly, Phys. Rev. Lett. **76**, 1055 (1996);
C. K. Law and H. J. Kimble, J. Mod. Opt. **44**,2067 (1997).

[39] S. Catani, M. Fontannaz and J. Guillet, JHEP **05**, 028 (2002);
L. Yuan, B. E. Kardynal, A. W. Sharpe and A. J. Shields, App. Phys. Lett. **91**, 041114 (2007);
F. Rieke and D. A. Baylor Rev. Mod. Phys. **70**, 1027 (1998);
A. Lacaita, F. Zappa, S. Cova and P. Lovati, App. Opt. **35**, 2986 (1996);
G. Nogues, et,al. Nature, **400**, 239 (1999);
The CMS collaboration, arXiv:1201.3093v2 [nucl-ex], 2013;
R. Ichou and D. d́Enterria, Rev. Rev. D **82**, 014015 (2010).

[40] M. Howton, The Nature of Light: What is a Photon?, SPIE, San Diego, ipaper 6664-5 (2007) and arXiv:0711.0112v1 [quant-ph], (2007).

[41] M. Howton and V. Debierre, arXiv:1512.06067v1 [quant-ph], (2015)

[42] W. Greiner and J. Reinhardt, *Field Quantization* (Springer, Berlin, (1996).

[43] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, (1995).

[44] W. Greiner and B. Müller, *Quantum Mechanics: Symmetries* (Springer, Berlin, (1994).

[45] M. Taketani and S. Sakata, Proc. Phys.-Math. Soc. Japan **22**, 757 (1940), reprinted as Prog. Theo. Phys. Suppl. **1**, 84 (1955).

[46] K. M. Case, Phys. Rev. **95**, 1323 (1954).

[47] A. Mostafazadeh, J. Math. Phys. **43**, 2814 (2002).

[48] A. Mostafazadeh, J. Math. Phys. **43**, 205 (2002).

[49] A. Mostafazadeh, Ann. Phys. (N.Y.) **309**, 1 (2004).

[50] A. Mostafazadeh, Int. J. Mod. Phys. A **21**, 2553 (2006).

[51] F. Zamani and A. Mostafazadeh, J. Math. Phys. **50**, 052302 (2009).

[52] L. L. Foldy, Phys. Rev. **102**, 568 (1956).

[53] A. Mostafazadeh and F. Zamani, Ann. Phys. (N.Y.) **321**, 2183 (2006).

[54] G. C. Wick, A. S. Wightman and E. P. Wigner, Phys. Rev. **88**, 101 (1952).

[55] T. Kato, *Perturbation Theory for Linear Operators* (Springer, Berlin, 1995).

[56] I. Bialynicki-Birula and Z. Bialynicka-Birula, Phys. Rev. D **35**, 2384 (1987).

[57] R. Y. Chiao and Y. S. Wu, Phys. Rev. lett. **57**, 933 (1986).

[58] Ya. B. Zeldovich, Dokl. Acad. Sci. USSR **163**, 1359 (1965).

[59] G. Kaiser, Phys. Lett. A **168**, 28 (1992).

[60] L. Gross, J. Math. Phys. **5**, 687 (1964).

[61] L. Mandel and E. Wolf, Optical coherence and quantum optics, (Cambridge University Press, Cambridge) (1995).

[62] A. Mostafazadeh, J. Math. Phys. **43**, 3944-3951 (2002).

[63] A. Mostafazadeh, Nucl. Phys. B **640**, 419-434 (2002).

[64] A. Mostafazadeh, Mod. Phys. Lett. A **17**, 1973-1977 (2002).

[65] A. Mostafazadeh, J. Math. Phys. **43**, 2814 (2002).

[66] A. Mostafazadeh, J. Math. Phys. **44**, 974-989 (2003).

[67] A. Mostafazadeh, J. Phys. A: Math. Gen. **36**, 7081-7091 (2003).

[68] A. Mostafazadeh and A. Batal, J. Phys. A: Math. Gen. **37**, 11645-11679 (2004).

[69] A. Mostafazadeh, J. Phys. A: Math. Gen. **38**, 3213-3234 (2005).

[70] A. Mostafazadeh, Class. Quantum Grav. **20**, 155-171 (2003).

[71] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998).

[72] C. M. Bender, S. Boettcher, and S. P. Meisenger, J. Math. Phys. **40**, 2201 (1999).

[73] A. Mostafazadeh, Czech J. Phys. **54**, 1125-1132 (2004).

[74] W. Greiner, *Relativistic Quantum Mechanics* (Springer, Berlin, 1994)

[75] K. M. Case, Phys. Rev. **95**, 1323 (1954).

[76] A. Mostafazadeh, J. Phys. A: Gen. **32**, 7827 (1998).

[77] H. Feshbach and F. Villars, Rev. Mod. Phys. **30**, 24 (1958).

[78] S.. A. Fulling, *Aspects of Quantum Field Theory in Curved Space-Time* (Cambridge University Press, Cambridge, 1989)

[79] F. G. Scholtz, H. B. Geyer, and F. J. W. Hahne, Ann. Phys. **213**, 74 (1992).

[80] P. P. Woodard, Class. Quantum. Grav. **10**, 483 (1993).

[81] J. B. Hartle and D. Marolf, Rev. D **56**, 6247 (1997).

[82] F. Embacher, Hadronic. J. **21**, 337 (1998).

[83] F. S. G. Von Zuben, J. Math. Phys. **41**, 6093 (2000).

[84] J. J. Halliwell and J. Thorwart, Phys. Rev. D **64**, 124018 (2001).

[85] F. J. Ynduráin, *Relativistic Quantum Mechanics and Introduction to Field Theory* (Springer, Berlin, 1996).

[86] B. Rosenstein and L. P. Horwitz, Phys. A: Math. Gen. **18**, 2115 (1985).

[87] A. Jeffrey and D. Zwillinger, *Gradshteyn and Ryzhik's Table of Integrals, Series, and Products, 7th Edition* (Elsevier, 2007).

[88] I. Bialynicki-Birula and Z. Bialynicka-Birula, Phys. Rev. D, **35**, 2383 (1987).