65498

# ON THE SOLUTIONS OF SINGULAR DIFFERENTIAL EQUATIONS


by

İbrahim ÇELİK



June, 1997

İZMİR

# ON THE SOLUTIONS OF SINGULAR DIFFERENTIAL EQUATIONS

A Thesis Submitted to the
Graduate School of Natural and Applied Sciences of
Dokuz Eylül University
In Partial Fulfillment of the Requirements for
the Degree of Doctor of Philosophy in .Mathematics Education

by

İbrahim ÇELİK

June, 1997
İZMİR

Ph.D. THESIS EXAMINATION RESULT FORM

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

Prof. Dr. Güzin Gökmen

(Advisor)

Prof. Dr. Turgut Öziş

(Committee Member)

Doç. Dr. Billur Kaymakçalan

(Committee Member)

Approved by the

Graduate School of Natural and Applied Sciences

Prof. Dr.Macit Toksoy

Director

# ACKNOWLEDGMENTS

I would like to express my deep gratitude to my supervisor Prof. Dr. Güzin Gökmen whose guidance and encouragement enabled me to accomplish this thesis.

I also wish to thank Prof. Dr. Turgut Öziş for his comments and advises on this work.

İbrahim ÇELİK

# ABSTRACT


The question of existence and uniqueness of solutions to singular nonlinear two-point boundary value problems is of fundamental importance. In this thesis existence and uniqueness of solutions to a mixed problem are established and a nonlinear model is developed and is solved approximately as an application to the theory.

Existence and uniqueness results are obtained for the mixed boundary value problem

$$\begin{cases} \dfrac{1}{p(t)}(p(t)y'(t))' + f(t,y(t),p(t)y'(t)) = 0, \quad 0 < t < 1 \\ \lim\limits_{t\to 0^+} p(t)y'(t) = 0, \quad y(1) = 0 \\ \lim\limits_{y\to 0^+} f(t,y,py') = \infty \end{cases}$$

with $f:[0,1]\times(0,\infty)\times(-\infty,\infty)\to(0,\infty)$ continuous and $p\in C[0,1]\cap C'(0,1]$ with $p>0$ on $(0,1)$, $p(0)=0$ and $\int_0^1 \frac{dt}{p(t)} = \infty$

A mathematical model is improved for the temperature distribution in the human head by assuming that the thermal conductivity k is not a constant but a function of temperature T. Numerical solutions are obtained for this model by the finite difference method and are compared with some others results.

# ÖZET


Doğrusal olmayan tekil iki nokta sınır değer problemlerinin çözümlerinin varlığı ve tekliği sorusu oldukça önemlidir. Bu tezde belirli tip karışık sınırdeğer probleminin çözümünün varlığı ve tekliği elde edilmiş ve kuramsal çalışmaya bir uygulama olarak da doğrusal olmayan bir fiziksel model geliştirilerek sayısal çözümü yapılmıştır.

Aşağıdaki karışık sınır değer problemlerinin çözümlerinin varlık ve tekliği ile ilgili sonuçlar elde edilmiştir.

$$\begin{cases} \dfrac{1}{p(t)}(p(t)y'(t))' + f(t,y(t),p(t)y'(t)) = 0, \quad 0 < t < 1 \\ \lim\limits_{t\to 0^+} p(t)y'(t) = 0, \quad y(1) = 0 \\ \lim\limits_{y\to 0^+} f(t,y,py') = \infty \end{cases}$$

$f : [0,1] \times (0,\infty) \times (-\infty,\infty) \to (0,\infty)$ sürekli ve $p \in C[0,1] \cap C'(0,1]$, $(0,1)$ aralığında $p > 0$, $p(0) = 0$ ve $\int_0^1 \dfrac{dt}{p(t)} = \infty$

k termal iletkenliği sabit olmayıp T sıcaklığının bir fonksiyonu olarak kabul edildiğinde insan başındaki sıcaklık dağılımını veren bir matematik model geliştirilmiştir. Elde edilen doğrusal olmayan diferansiyel denklem sonlu farklar yöntemi ile çözülerek sayısal sonuçlar elde edilmiş ve daha önceki sonuçlarla karşılaştırılmıştır.

# CONTENTS

**Chapter Four**

**APPLICATION TO THE THEORY - A NONLINEAR MODEL OF HEAT CONDUCTION IN THE HUMAN HEAD**



**Chapter Five**

**NUMERICAL COMPUTATION**

# LIST OF TABLES

# LIST OF FIGURES

# CHAPTER ONE
# INTRODUCTION

## 1.0 Introduction

In this chapter we summarize the works done on singular and nonsingular boundary value problems which are encountered in applications

Two-point boundary value problems associated with the second order equation

$$y'' = F(t, y, y'), \qquad a < t < b \tag{1.1}$$

can be singular in a variety of ways: a and/or b may be infinite, F may be unbounded near some $t_0 \in [a,b]$, or F may be unbounded near some particular value of y (or $y'$). This last possibility does not occur in linear problems.

## 1.1 Some Singular Problems

Problems on infinite intervals were studied by Granas [25], Erbe and Schmitt [19], Berestycki, Lions, and Peletier [7], and by Baxley [2]. Problems on finite intervals in which the singularity is caused by F being unbounded near a particular value of y were studied by Luning and Perry [32], Nachman and Calleari [34], Taliaferro [48], Gatica, Oliker, and Waltman [23] and Baxley [3]. Recently, Gatica, Hernandez, and Waltman [24] considered a finite interval problem which has singularities from F becoming infinite as $t \to a$ and $y \to 0$. We should also note that the problem considered by Erbe and Schmitt [19] on $0 < t < \infty$ is singular at $t = 0$.

The boundary value problem

$$y'' + \frac{2}{t} y' = f(t,y), \qquad 0 < t < 1$$

$$y'(0) = 0, \;\; ay'(1) + y(1) = b, \;\; a > 0$$

can model various physiological processes. For example if $f(t,y) = \dfrac{\alpha y}{y+k}$ then above equation arises in the study of steady-state oxygen diffusion in a cell with Michaelis Menten kinetics [30].

If $f(t,y) = -\kappa \exp(-\beta y)$, $\kappa, \beta > 0$ heat conduction in human brain is modeled [1,22]. Here f is the heat production rate per unit volume, y is the absolute temperature and t is the radial distance from the center.

In 1927, L.H. Thomas and E. Fermi independently derived a boundary value problem for determining the electrical potential in an atom. Their analysis leads to the nonlinear singular second order equation

$$y'' = t^{-1/2} y^{3/2}.$$

The neutral atom with Bohr radius b is given by

$$y(0) = 1 \qquad by'(0) - y(b) = 0,$$

the ionized atom is given by

$$y(0) = 1 \qquad y(b) = 0,$$

and the isolated neutral atom is given by

$$y(0) = 1, \qquad \lim_{t \to \infty} y(t) = 0.$$

A problem involving heat and mass transfer in a porous catalyst leads to the problem

$$y'' = \alpha\, y \exp\left[\frac{\gamma\,\beta(1-y)}{1+\beta\,(1-y)}\right], \qquad 0 < t < 1$$

$$y'(0) = 0\,, \quad y(1) = 0$$

where $\alpha$, $\beta$, $\gamma$ are positive constants. The apparent singularity at $y = 1+1/\beta$ is not the case because the solution actually satisfies $y(t) \le 1$ on $0 \le t \le 1$. Similar behavior is encountered in the following problem involving diffusion in a chemical catalytic converter,

$$y'' = \frac{1}{y+2}(y')^2, \qquad 0 < t < 1$$

$$y'(0) = A, \qquad y(1) = B$$

where $B < A < 2$. The apparent singularity at $y = -2$ does no real harm because the solution satisfies $B \le y(t) \le A$ on $0 \le t \le 1$. These two examples are discussed by Baxley [4].

In almost all of the articles referenced above, $F(t, y, y')$ is independent of $y'$. Motivation has come from interest in the problem

$$y'' + a(t)y^{-p} = 0, \qquad 0 < x < 1 \tag{1.2}$$

where $p > 0$ and $a(t) > 0$ and are continuous on $[0,1]$, and the problem

$$y'' + \frac{N-1}{t}y' + a(t)y^{-p} = 0, \qquad 0 < t < 1 \tag{1.3}$$

where again $p > 0$ and $a(t) > 0$ on $[0,1]$, which is encountered in the search for radial solution of elliptic partial differential equations in $R^N$ which have the form more general equation

$$y'' + f(t,y) = 0, \tag{1.7}$$

where $f : (0,1) \times (0,\infty) \to (0,\infty)$ is continuous and $f(t,y)$ is nonincreasing in y for fixed $t \in (0,1)$, and they dealt with the more general two point boundary conditions

$$a_0 y(0) - a_1 y'(0) = 0 \qquad b_0 y(1) + b_1 y'(1) = 0 \tag{1.8}$$

where $a_0, a_1, b_0, b_1 \geq 0$ and $a_0 b_0 + a_1 b_0 + a_0 b_1 > 0$. Assuming that $f(t,y) \to +\infty$ as $y \to 0^+$ uniformly on compact subsets of $(0,1)$, and that

$$\int_0^1 f(t, \theta\, s(t))dt < \infty \tag{1.9}$$

for each $\theta > 0$, Gatica, Oliker, and Waltman [23] proved existence of positive solution $y \in C^1[0,1] \cap C^2(0,1)$ of (1.7), (1.8). They also provide a uniqueness theorem generalizing the corresponding statement of Taliaferro. Note that for $f(t,y) = a(t)/y^p$, (1.9) holds if and only if (1.6) holds.

In contrast to the shooting method used by Taliaferro, Gatica, Oliker, and Waltman [23] first proved an interesting fixed point theorem for decreasing maps on a Banach space and then applied it to the boundary value problem.

In [3], Baxley proved existence and uniqueness of a solution of the nonlinear boundary value problem

$$y'' = -\frac{t^2}{32y^2} + \frac{\lambda^2}{8} \qquad 0 < t \leq 1$$

$$y(0) = 0, \;\; 2y'(1) + (1+\nu)y(1) = 0$$

where $\lambda > 0$, $\nu > 0$ are constants. He solved the above boundary value problem approximately.

In [57] , Zhang studied the generalized Emden-Fowler equations

$$y'' + p(t)y^{\lambda} = 0 \tag{1.10}$$

$$y(0) = y(1) = 0 \tag{1.11}$$

when $p(t) \in C[0,1]$ and $\lambda > 0$ the problem (1.10), (1.11) being nonsingular. When p(t) is not continuous at the end points of (0,1) ( including the case that p(t) is unbounded on (0,1) ), or $\lambda < 0$ , then problem (1.10), (1.11) is singular, while for the remaining cases $0 < \lambda < 1$ and $\lambda > 1$, the problem being the so-called sublinear and superlinear problems, respectively. He obtained a necessary and sufficient condition for the existence of positive solutions of the singular problem (1.10), (1.11) in sublinear case.

In [35], O'Regan studied the problems of the form

$$y'' = \phi^2 f(t, y, y'), \quad 0 < t < 1$$

with various boundary conditions and

$$y'' + f(t, y, y') = 0, \quad 0 < t < 1$$

$$y(0) = y(1) = 0$$

where $\phi^2 > 0$ is a constant usually called the Thiele modulus and nonlinear term f may be singular at $y = 0, t = 0$ and/or at $t = 1$.

In [23], Gatica, Hernandez and Waltman developed their fixed point theorem and applied to the more difficult boundary value problem

$$y'' + \frac{N-1}{t} y' + a(t) y^{-p} = 0, \qquad 0 < t < 1$$
$$y'(0) = 0 \qquad y(1) = 0. \tag{1.12}$$

Replacing $a(x)y^{-p}$ with $f : [0,1) \times (0,\infty) \to (0,\infty)$ continuous, they assumed f is decreasing in y for each t, $\int_0^1 f(t,y)dt < \infty$ for each $y > 0$, $f(t,y) \to +\infty$ as $y \to 0^+$ and $f(t,y) \to 0$ as $y \to +\infty$ both limits being uniform on compact subset of (0,1) and finally $\int_0^1 f(t, \theta(1-t))dt < \infty$ for each $\theta > 0$. They then proved existence of a positive solution $y \in C^1[0,1] \cap C^2(0,1)$ for (1.12) and provided a uniqueness theorem.

In [5], Baxley studied the boundary value problems of the form

$$y'' + g(t,y') + f(t,y) = 0 \quad a < t < b \tag{1.13}$$

$$a_0 y(a) - a_1 y'(a) = A, \; b_0 y(b) + b_1 y'(b) = B \tag{1.14}$$

where $a_0, a_1, b_0, b_1, A, B \geq 0$ and $a_0 b_0 + a_1 b_0 + a_0 b_1 > 0$. He extended the basic result of Taliaferro regarding the sufficiency of the condition (1.5) or (1.6) for existence to the equation (1.13) in which significant nonlinearity is allowed in the term $g(t,y')$. The results in [23] is also a special case. He constructed a sequence of problems which approximate (1.13), (1.14), which are not singular, but which "converge" to the singular problem. He had applied a previous result [4] to obtain existence of a positive solution $y_n$ of the nth approximating problem and used a priori estimates to justify application of Ascoli's theorem.

In [6], Baxley and Gersdorff studied the equation

$$y'' + f(t,y') = \phi^2 g(t,y), \qquad a < t < b \tag{1.15}$$

$$y'(a) = 0 \ , \ \beta y'(b) + \alpha y(b) = A, \ \ \beta \geq 0 \ , \ \alpha, A > 0 \qquad (1.16)$$

where $\phi^2 > 0$ is the ( positive ) Thiele modulus, $\lim_{y \to 0^+} g(t,y) = \infty$. They showed that (1.15), (1.16) has at least one positive solution under suitable conditions on f and g.

In [18], Dunninger and Kurtz studied singular nonlinear problems of the form

$$-\frac{1}{p(t)}(p(t)u'(t))' = f(u(t)) \qquad 0 < t < 1$$

with Dirichlet or mixed boundary conditions.

In [36], O'Regan presented the existence of the kth order nonlinear system of boundary value problems of the form

$$y^{(k)} = f(t, y, y', \ldots, y^{(k-1)}), \qquad 0 < t < 1$$

with different boundary conditions, where nonlinear term f may be singular at $t = 0$ and/or at $t = 1$.

In [38], O'Regan studied the existence of solution to boundary value problems of the form

$$\frac{1}{p(t)}(p(t)y'(t))' = \frac{f(t,y)}{\Psi(t)}$$

$$y(0) = a \ , \ \ r\, y'(b) - k\, y(b) = h \ ; \ \ r^2 + h^2 > 0$$

where $f : [0,b] \times R \to R$ , $p : [0,b] \to [0, \infty)$ , and $\psi : [0,b] \to [0,\infty)$ are continuous with $p, \psi > 0$ on $(0,b]$.

In [39], O'Regan studied the existence of solution of the equation

$$\frac{1}{p(t)}(p(t)y'(t))' = q(t)f(t,y(t),p(t)y'(t)), \quad 0 < t < 1$$

$$y(1) = \lim_{t\to 0^+} p(t)y'(t) = 0 \quad \text{or} \quad y(0) = y(1) = 0$$

where $f : [0,1] \times R^{2n} \to R^n$ is continuous and $p(t) \in C[0,1] \cap C^1(0,1]$, with $q(t)$ on $(0,1)$ and $p(t) > 0$ on $(0,1]$ and q may be singular at $t = 0$ and/or at $t = 1$.

In [40], O'Regan studied the existence of solution of second order differential equation

$$\frac{1}{p(t)}(p(t)y'(t))' = q(t)f(t,y(t),p(t)y'(t)), \quad 0 < t < 1$$

with $y(t)$ satisfying either the boundary conditions

$$y(0) = r\,y'(1) - k\,y(1) = 0\,;\ r^2 + k^2 > 0\,,\ r \geq 0 \ \text{and}\ r \neq k\,p(1)\int_0^1 \frac{ds}{p(s)}$$

or

$$\alpha\, y'(0) + \beta\, y(0) = r\,y'(1) - k\,y(1) = 0\,;\ r^2 + k^2 > 0,\ \alpha^2 + \beta^2 > 0, \alpha > 0\,, r > 0$$

where $f:[0,1]\times B^2 \to B$ is continuous, $q \in C(0,1)$ with $q > 0$ on $(0,1)$ and $p(t) \in C[0,1] \cap C^1(0,1]$ satisfying $p(t) > 0$ on $(0,1]$ if first condition is satisfied or $p(t) > 0$ on $[0,1]$ if second condition is satisfied. B is real Banach space.

In [41], O'Regan studied the existence of solution of second order boundary value problems of the form

$$\frac{1}{p(t)}(p(t)y'(t))' = q(t)f(t,y(t),p(t)y'(t)), \quad 0 < t < 1$$

$$y(1) = \lim_{t\to 0^+} p(t)y'(t) = 0$$

where $f:[0,1]\times R^2\to R$ is continuous, $q\in C(0,1)$, $p(t)\in C[0,1]\cap C^1(0,1]$, with $p(t)>0$ on $(0,1]$, $p(0)=0$ and $\int_0^1 \frac{ds}{p(s)}=\infty$

In the above models singularity appear from independent variable as $p(0)=0$

In [37], O'Regan studied the existence of solution of nonlinear boundary value problems of the form

$$\begin{gathered}\frac{1}{p(t)}(p(t)y'(t))'+q(t)f(t,y(t))=0,\quad 0<t<1\\ y(1)=\lim_{t\to 0^+}p(t)y'(t)=0\end{gathered}\tag{1.17}$$

where $p(t)\in C[0,1]\cap C^1(0,1]$, $q(t)\in C(0,1)$ together with $q(t)>0$ on $(0,1)$ and $p(t)>0$ on $(0,1]$,

$$\int_0^1 p(u)q(u)du<\infty \quad \text{and} \quad \int_0^1 \frac{1}{p(u)}\int_0^u p(s)q(s)ds\,du<\infty$$

and f is continuous on $[0,1]\times(0,\infty)$ with $\lim_{y\to 0^+} f(t,y(t))=+\infty$ uniformly on compact subset of $(0,1)$. In this problem singularities may come from independent and dependent variables. He constructed the approximate problem

$$\begin{gathered}\frac{1}{p(t)}(p(t)y'(t))'+q(t)f(t,y(t))=0,\quad 0<t<1\\ \lim_{t\to 0^+}p(t)y'(t)=0 \qquad y(1)=\frac{1}{n}\end{gathered}\tag{1.18}^n$$

and he gave the following conditions about f

$$\begin{cases} \text{f is continuous on } [0,1]\times(0,\infty) \text{ with } \lim\limits_{y\to 0^+} f(t,y)=+\infty \\ \\ \text{uniformly on compact subset of } (0,1) \end{cases}$$

$$\begin{cases} 0 < f(t,y) \le g(y)+h(y) \text{ on } (0,1)\times(0,\infty), \text{ where } g>0 \text{ is continuous} \\ \\ \textit{and} \text{ nonincreasing on } (0,\infty) \text{ and } h\ge 0 \text{ is continuous on } [0,\infty) \end{cases}$$

h/g is nondecreasing on $(0,\infty)$ and there exists a constant $N > 0$ such that for $z > 0$

$$\int_0^z \frac{du}{g(u)} \le \left\{1+\frac{h(z)}{g(z)}\right\}\int_0^1 \frac{1}{p(t)}\int_0^t p(s)q(s)\,ds\,dt + \int_0^1 \frac{du}{g(u)}$$

implies $z \le N$. Under the assumptions of above conditions, he showed that $(1.18)^n$ has a solution $y_n \in C[0,1] \cap C^1(0,1] \cap C^2(0,1)$ for each $n \in N^+$ and (1.17) has a solution $y \in C^1[0,1] \cap C^2(0,1)$.

In [43], O'Regan studied the existence of solution of nonlinear boundary value problems of the form

$$\begin{aligned} &\frac{1}{p(t)}(p(t)y'(t))' = q(t)f(t,y(t),p(t)y'(t)), \quad 0<t<1 \\ &y(0)=a>0, \quad y(1)=0 \end{aligned} \tag{1.19}$$

where nonlinear term f may be singular at $y = 0$. He assumed that f satisfies

$$\begin{cases} f>0 \text{ is continuous on } [0,1]\times(0,\infty)\times(-\infty,\infty) \text{ with } \lim\limits_{y\to 0^+} f(t,y,v)=+\infty \\ \\ \text{uniformly on compact subset of} [0,1]\times(-\infty,\infty), \text{ where } v=py' \end{cases} \tag{1.20}$$

and

$$\begin{cases} f(t,y,v) \le [g(y)+h(y)]k(|v|) \ \text{ on } [0,1]\times(0,\infty)\times(-\infty,\infty) \\ \text{where g} > 0 \text{ is continuous and nonincerasing on } (0,\infty) \\ \text{and } \ h \ge 0,\ k > 0 \text{ are continous on } [0,\infty). \end{cases} \tag{1.21}$$

He constructed the approximate problems

$$\begin{gathered} \frac{1}{p(t)}(p(t)y'(t))' + q(t)f(t,y(t),p(t)y'(t)) = 0, \quad 0 < \mathrm{t} < 1 \\ y(0) = a > 0, \quad y(1) = \frac{1}{n} \end{gathered} \tag{1.22$^{n}$}$$

First of all he assumed that

$$\begin{cases} p \in C[0,1] \cap C^1(0,1), q(t) \in C(0,1) \text{ together} \\ \text{with } \ q(t) > 0 \text{ and } p(t) > 0 \text{ on } (0,1) \end{cases} \tag{1.23}$$

$$\int_0^1 \frac{du}{p(u)} < \infty \quad \text{and} \quad \int_0^1 p(u)q(u)du < \infty \tag{1.24}$$

and

$$\begin{cases} \displaystyle\int_0^a g(u)du < \infty \text{ and that } \sup_{(0,1)} p^2(t)q(t) < \infty. \text{ Let } I(z) = \int_0^z \frac{udu}{k(u)} \ \text{ with} \\ \displaystyle\phi^2 \sup_{(0,1)} p^2(t)q(t) \int_0^a [g(u)+h(u)]du < \int_{a\sup_{[0,1]} p(t)}^{\infty} \frac{u}{k(u)} du \ \text{ and } \phi \text{ satisfying} \\ \displaystyle\int_0^1 \frac{du}{p(u)} I^{-1}\left(\phi^2 \sup_{(0,1)} p^2(t)q(t) \int_0^a [g(u)+h(u)]du\right) < a \end{cases}$$

hold. Hence he showed that (1.19) has a solution $y \in C[0,1] \cap C^2(0,1)$ with additional conditions

$$\int_0^1 p(t)q(t)g\left(\int_t^1 \frac{ds}{p(s)}\right)dt < \infty$$

and

$$\text{there exist } \tau \text{ with } \inf_{[0,1]} p(t) \ge \tau \text{ and } \tau \int_0^1 \frac{ds}{p(s)} \ge 1$$

Secondly, he assumed (1.20), (1.21), (1.23) and (1.24) hold. In addition he assumed that

$$\begin{cases} \text{there exists constant } \alpha,\ \beta > 1, \dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1 \text{ with } \displaystyle\int_0^a g^{\alpha}(u)du < \infty \\[2ex] \text{and } \displaystyle\int_0^1 [p(t)]^{\frac{(\alpha+1)\beta}{\alpha}} q^{\beta}(t)dt < \infty. \text{ Letting } J(z) = \int_0^z \frac{u^{\frac{1}{\alpha}} du}{k(u)} \text{ with} \\[2ex] \phi^2 \left\{\displaystyle\int_0^a [g(u)+h(u)]^{\alpha} du\right\}^{\frac{1}{\alpha}} \left\{\displaystyle\int_0^1 p^{\frac{(\alpha+1)\beta}{\alpha}} q^{\beta} dt\right\}^{\frac{1}{\beta}} < \displaystyle\int_{a \sup_{[0,1]} p(t)}^{\infty} \frac{u^{\frac{1}{\alpha}}}{k(u)} du \\[2ex] \text{and } \phi \text{ satisfying } \displaystyle\int_0^1 \frac{du}{p(u)} J^{-1}\left(\phi^2 \left\{\int_0^a [g(u)+h(u)]^{\alpha} du\right\}^{\frac{1}{\alpha}} \left\{\int_0^1 p^{\frac{(\alpha+1)\beta}{\alpha}} q^{\beta} dt\right\}^{\frac{1}{\beta}}\right) < a \end{cases}$$

he showed that (1.19) has a solution $y \in C[0,1] \cap C^2(0,1)$.

To obtain existence results, O'Regan used the Nonlinear Alternative of Leray - Schauder in all of his papers.

In [56], Zhang studied singular second order differential equations of the form

$$Lu \equiv \frac{1}{q(t)}(p(t)u'(t))' = f(t, u(t), p(t)u'(t)), \quad 0 < t < 1 \tag{1.25}$$

and boundary conditions

$$u(0) = u(1) = 0 \tag{1.26}$$

or

$$u(t) \text{ continuous at } t = 0, \quad u(1) = 0 \tag{1.27}$$

where $f(t,u\ v) \in C([0,1] \times R^2)$, $p,q \in C[0,1]$ with $p,q > 0$ on $(0,1]$ and

$$\int_0^1 \frac{dt}{p(t)} = \infty, \quad \int_0^1 \frac{1}{p(t)} \int_0^t q(s)p(s)dsdt < \infty$$

By the transformation of variable

$$y(x) = (1 - x)u(t)$$

he transformed the equation (1.19) into the following equivalent equation

$$y''(x) = F(x,y,y') \equiv \frac{p(t(x))q(t(x))}{(1-x)^3} f\left(t(x), \frac{y}{1-x}, [(1-x)y' + y]\right) \quad x \in [0,1) \tag{1.28}$$

and transformed the boundary conditions (1.21) into

$$y(0) = 0, \quad \lim_{x \to 1} \frac{y(x)}{1-x} \text{ exists}$$

or equivalently

$$y(0) = y(1) = 0 \tag{1.29}$$

where

$$x = x(t) \equiv \begin{cases} 1-(1+\int_t^1 \frac{ds}{p(s)})^{-1}, & t \in (0,1] \\ 1 & t = 0 \end{cases}$$

and t(x) is the inverse of the function x(t). He showed that (1.28), (1.29) has at least one solution $u(t) \in C^1[0,1] \cap C^2[0,1)$. By the following lemma he proved that (1.25), (1.27) has at least one solution $u(t) \in C[0,1] \cap C^1(0,1]$.

**Lemma** If problem (1.28), (1.29) possesses a solution $y(x) \in C^1[0,1] \cap C^2[0,1)$ then problem (1.25), (1.27) possesses a solution $u(t) \in C[0,1] \cap C^1(0,1]$ with $p(t)u'(t) \in C[0,1] \cap C^1(0,1]$.

In this thesis, in chapter three, we will study the existence and uniqueness of solutions of nonlinear boundary value problems of the form

$$\frac{1}{p(t)}(p(t)y'(t))' + f(t, y(t), p(t)y'(t)) = 0, \quad 0 < t < 1$$

$$y(1) = \lim_{t \to 0^+} p(t)y'(t) = 0$$

where f is continuous on $[0,1]\times(0,\infty)\times(-\infty,\infty)$ with $\lim_{y\to 0^+} f(t,y,py') = +\infty$ uniformly on compact subset of $[0,1]\times(-\infty,\infty)$ and $p(t) \in C[0,1] \cap C^1(0,1]$, with $p(t) > 0$ on $(0,1]$, $p(0) = 0$ and

$$\int_0^1 \frac{ds}{p(s)} = \infty$$

To show the existence of a solution of the above boundary value problem, we will extend O'Regan's ideas in [43] which he used to obtain existence of solutions of the boundary value problems of the form

$$y'' + f(t,y) = 0$$

$$-\alpha\, y(0) + \beta\, y'(0) = 0 \qquad \alpha^2 + \beta^2 > 0$$

$$ay(1) + by'(1) = 0 \qquad a^2 + b^2 > 0$$

$$\alpha, \beta, a, b \geq 0 \quad \text{with} \quad \alpha + a > 0$$

where f : $[0,1] \times (0,\infty) \to (0,\infty)$ continuous with $\lim\limits_{y \to 0^+} f(t,y) = +\infty$

To show uniqueness, we will assume that f :$[0,1]\times(0,\infty)\times(-\infty,\infty)\to(-\infty,\infty)$ is continuous and $f(t,y,py')$ is strictly decreasing in y and y′ for $t\in(0,1)$, integrable over $[0,1]$ for each fixed $y > 0$ and $\lim\limits_{t \to 0^+} \dfrac{p(t)}{p'(t)} = 0$.

In chapter four, we will develop a model for the singular boundary value problem of temperature distribution in the human head and in chapter five we will solve this problem approximately by the finite difference method.

## CHAPTER TWO

# PRELIMINARIES AND EXISTENCE PRINCIPLES

### 2.0 Introduction

In this chapter we give some definitions and existence theorems for the existence of solution to two regular and singular boundary value problems

### 2.1 Preliminaries

We define a set $S \subseteq R^n$ to be compact if every sequence of elements of S has a point of accumulation in S. This property is equivalent to the following properties, which could be taken as alternative definitions:

**a)** Every infinite subset of S has a point of accumulation in S.

**b)** Every sequence of elements of S has a convergent subsequence whose limit is in S

**Theorem 2.1**

A set is compact if and only if it is closed and bounded.

**Theorem 2.2**

Let S be a compact set and let f be a continuous function of S. Then the image of f is compact.

**Theorem 2.3**

Let S be a compact set, and let f be a continuous function on S. Then f is uniformly continuous i.e., given $\varepsilon$ there exists $\delta$ such that whenever $z,w \in S$ and $|z - w| < \delta$, then $|f(z) - f(w)| < \varepsilon$.

Let X and Y be normed linear space

**i)** A map $F: X \to Y$ is called compact if $F(X)$ is contained in a compact subset of Y

**ii)** $F: X \to Y$ is called completely continuous if the image of each bounded set in X is contained in a compact subset of Y.

**iii)** A compact map $F: X \to Y$ is called finite dimensional if $F(x)$ is contained in a finite dimensional linear subset of Y.

**Theorem 2.4 (Nonlinear Alternative)**

Let C be a convex subset of a normed linear space E and let U be an open subset of C and $\overline{U}$ and $\partial U$ the closure of U in C and the boundary of U in C respectively. Let $F:\overline{U} \to C$ be a compact map with $p^* \in U$. Then F has at least one of the following two properties:

**i)** F has a fixed point

**ii)** there is an $x \in \partial U$ with $x = \lambda F(x) + (1 - \lambda)p^*$ for some $\lambda \in (0,1)$.

Let $-\infty < a < b < \infty$. A collection of real valued functions $A = \{f_i : [a,b] \to R\}$ is said to be uniformly bounded if there exists a constant $M > 0$ with $|f_i(x)| \leq M$ for all $f_i \in A$. A is said to be equicontinuous if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $x_0$, $x_1 \in [a,b]$ then $|x_0 - x_1| < \delta$ implies $|f_i(x_0) - f_i(x_1)| < \varepsilon$ for all $f_i \in A$.

**Theorem 2.5**

Let A be a closed subset of the function space C([a,b],R). If A is uniformly bounded and equicontinuous then A is compact.

**Theorem 2.6 ( Arzela - Ascoli )**

Let A be a subset of C([a,b],R). If A is uniformly bounded and equcontinuous then $\overline{\mathrm{A}}$ is compact.

## 2.2 Singular Problems in the Independent Variable t

We consider the "mixed" two point boundary value problem

$$\begin{cases} \frac{1}{p}(py')' + f(t,y,py') = 0 & 0<t<1 \\ \lim_{t\to 0^+} p(t)y'(t) = 0, \quad y(1) = d \end{cases} \tag{2.1}$$

By a solution, we mean a function $y \in C[0,1]\cap C^1(0,1]\cap C^2(0,1)$ with $py' \in C[0,1]$ which satisfies the differential equation on (0,1) and stated boundary condition in (2.1). Associated with (2.1), we have a family of problems

$$\begin{cases} \frac{1}{p}(py')' + \lambda.f(t,y,py') = 0 & 0<t<1 \\ \lim_{t\to 0^+} p(t)y'(t) = 0, \quad y(1) = d \end{cases} \tag{2.2}_\lambda$$

where $0<\lambda<1$.

**Theorem 2.7**

Suppose (2.1) holds and assume

$$f : [0,1]\times R\times R \rightarrow R \text{ is continuous} \tag{2.3}$$

$$\begin{cases} p \in C[0,1] \cap C^1(0,1] \text{ together with } p > 0 \text{ on } (0,1) \\ p(0) = 0 \text{ and } \int_0^1 \frac{dt}{p(t)} = \infty \end{cases} \tag{2.4}$$

$$\int_0^1 p(t)dt < \infty \tag{2.5}$$

$$\int_0^1 \frac{1}{p(t)} \int_0^t p(s)ds dt < \infty \tag{2.6}$$

In addition suppose there is a constant M, independent of $\lambda$, with

$$\text{Max}\left\{ \sup_{[0,1]} |y(t)| \quad , \quad \sup_{(0,1]} |p(t)y'(t)| \right\} = \|y\|_1 \leq M$$

Then (2.1) has at least one solution y in $C[0,1] \cap C^1(0,1] \cap C^2(0,1)$ with $py' \in C[0,1]$.

**Proof**

By the integration of differential equation $(2.2)_\lambda$ from 0 to t

$$\int_0^t (py')'ds + \lambda \int_0^t p(s) f(s, y(s), p(s)y'(s))ds = 0$$

is obtained and further integration by parts gives

$$p(t)y'(t) + \lambda \int_0^t p(s) f(s, y(s), p(s)y'(s))ds = 0.$$

Hence

$$y'(t) = -\lambda \frac{1}{p(t)} \int_0^1 p(s) f(s, y(s), p(s)y'(s)) ds.$$

If we integrate the last equation from t to 1, we obtain

$$y(1) - y(t) = -\lambda \int_t^1 \frac{1}{p(u)} \int_0^u p(s) f(s, y(s), p(s)y'(s)) dsdu.$$

Hence solving $(2.2)_\lambda$ is equivalent to finding a $y \in C[0,1] \cap C^1(0,1]$ which satisfies

$$y(t) = d + \lambda \int_t^1 \frac{1}{p(u)} \int_0^u p(s) f(s, y(s), p(s)y'(s)) dsdu. \tag{2.7}$$

Let $K^1[0,1] = \{ y \in C[0,1],\ py' \in C[0,1] \text{ with } \|y\|_1 < \infty \}$ which is the Banach space and

$$K^1_{B_0}[0,1] = \left\{ y \in K^1[0,1]\text{: } \lim_{t \to 0^+} p(t)y'(t) = 0,\ \ y(1) = 0 \right\}.$$

We define the operator $M^* : K^1_{B_0}[0,1] \to K^1_{B_0}[0,1]$ by

$$M^* y(t) = \int_t^1 \frac{1}{p(u)} \int_0^u p(s) f(s, y(s), p(s)y'(s)) dsdu.$$

Consequently $(2.2)_\lambda$ is equivalent to the fixed point problem

$$y = (1-\lambda)d + \lambda(M^* y(t) + d) \equiv (1-\lambda)d + \lambda N y(t).$$

Now $M^*$ is continuous and also completely continuous by the Arzela-Ascoli theorem. To see this let $\Omega \subseteq K^1_{B_0}[0,1]$ be bounded. The boundedness of $M^*\Omega$ is trivial and the equicontinuity on [0,1] follows from assumptions (2.5) and (2.6) and following inequalities (here $y \in \Omega$ and $0 \le s \le t \le 1$);

$$\left|M^{*}y(t) - M^{*}y(s)\right| \le \int_{s}^{t} \frac{1}{p(u)} \int_{0}^{u} p(z)\left|f(z, y(z), p(z)y'(z))\right| dz du$$

and

$$\left|p(t)(M^{*}y)'(t) - p(s)(M^{*}y)'(s)\right| \le \int_{s}^{t} p(z)\left|f(z, y(z), p(z)y'(z))\right| dz .$$

$K^{1}_{B_0}[0,1]$ is a closed subset of $K^{1}[0,1]$. Consequently the Arzela-Ascoli theorem implies that $M^{*}: K^{1}_{B_0}[0,1] \to K^{1}_{B_0}[0,1]$ is completely continuous. Let

$$K^{1}_{B}[0,1] = \left\{ y \in K^{1}[0,1] : \lim_{t \to 0^{+}} p(t)y'(t) = 0,\ \ y(1) = d \right\}$$

and set

$$U = \left\{ y \in K^{1}_{B} : \|y\|_{1} < M + 1 \right\},\ \ C = K^{1}_{B}[0,1] \ \text{ and } \ E = K^{1}[0,1]$$

Then theorem ( Nonlinear Alternative ) applies with $p^{*} = d$, but by choosing U as above possibility ii) of theorem is ruled out and so we deduce that N has a fixed point, that is (2.1) has a solution $y \in C[0,1]$ with $py' \in C[0,1]$. The fact that $y \in C^{2}(0,1)$ follows from (2.7) with $\lambda = 1$.

**Corollary 2.8**

Suppose the conditions of Theorem 2.7 are satisfied. Assume p(0) = 0 and $\lim_{t \to 0^{+}} (p(t)/p'(t))$ exists. Then (2.1) has at least one solution in $C^{1}[0,1] \cap C^{2}(0,1)$. If in addition p(0) = 0 and $\lim_{t \to 0^{+}} (p(t)/p'(t)) = 0$ then $y'(o) = 0$ for any solution $y \in C^{1}[0,1] \cap C^{2}(0,1)$ of (2.1).

**Proof**

L'Hospital's rule implies

$$\lim_{t\to 0^+}\frac{y(t)-y(0)}{t}$$

$$=\lim_{t\to 0^+}\left\{-\frac{1}{t}\int_0^t\frac{1}{p(u)}\int_0^u p(s)f(s,y(s),p(s)y'(s))dsdu\right\}$$

$$=f(0,y(0),0)\lim_{t\to 0^+}\frac{\mathrm{p(t)}}{\mathrm{p'(t)}}=y'(0)$$

If $\lim_{t\to 0^+}\frac{p(t)}{p'(t)}=0$ then $y'(0)=0$ is obtained.

## 2.3 Singular Problems in the Dependent and Independent variables t and y

Many mathematical models give rise to singular boundary value problems of the form

$$\begin{cases}\frac{1}{p}(py')'+\Phi(t)y^{-\gamma}=0 & 0<t<1\\ \lim_{t\to 0^+}p(t)y'(t)=0, & y(1)=0\end{cases}$$

with $\Phi\in C[0,1]$, $\Phi\geq 0$ on $[0,1]$, $p\in C[0,1]\cap C^1(0,1]$ with $p>0$ on $[0,1]$ and $\gamma>0$.

In this thesis we present existence results for the general second order boundary value problems of the form;

$$\begin{cases}\frac{1}{p}(py')'+f(t,y,py')=0 & 0<t<1\\ \lim_{t\to 0^+}p(t)y'(t)=0, & y(1)=0\\ \lim_{y\to 0^+}f(t,y,py')=\infty\end{cases}\tag{2.8}$$

with f : $[0,1] \times (0,\infty) \times (-\infty,\infty) \to (0,\infty)$ continuous and $p \in C[0,1] \cap C^1(0,1]$ with $p > 0$ on $(0,1)$, $p(0) = 0$ and $\int_0^1 \frac{dt}{p(t)} = \infty$.

Here we give two theorems following [43] that will be used in the following chapter for proving theorems concerning the existence of solutions of (2.8).

**Theorem 2.9**

Suppose (2.4) and (2.6 ) are satisfied . Then

$$\|y\|^2 \le \|y'\|^2 \int_0^1 p(t) \int_t^1 \frac{ds}{p(s)} dt \tag{2.9}$$

for all functions $y \in D = \{ w \in C[0,1] :$ w and pw′ are absolutely continuous on [0,1] and $\frac{1}{p}(pw')' \in L_p^2\,[0.1]$ with $\lim_{t \to 0^+} p(t) w'(t) = 0$, $w(1) = 0 \}$

where $\|y\|^2 = \int_0^1 p y^2 dt$ and $\|y'\|^2 = \int_0^1 p (y')^2 dt$

**Proof**

It is clear that

$$|y(t)| \le \int_t^1 |y'(s)| ds = \int_t^1 \sqrt{p(s)}\, |y'(s)| \frac{ds}{\sqrt{p(s)}}$$

By using Hölder's integral inequality we obtain,

$$|y(t)| \le \left( \int_t^1 p(s) |y'(s)| ds \right)^{\frac{1}{2}} \left( \int_t^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}} \le \left( \int_0^1 p(s) |y'(s)| ds \right)^{\frac{1}{2}} \left( \int_t^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}}.$$

Thus for $t \in (0,1)$

$$p(t)|y(t)|^2 \le \int_0^1 p(t)|y'(t)|dt\, p(t)\int_t^1 \frac{ds}{p(s)}$$

and so

$$\int_0^1 p(t)|y(t)|^2 dt \le \left(\int_0^1 p(t)|y'(t)|^2 dt\right)\int_0^1 p(t)\int_t^1 \frac{ds}{p(s)}dt$$

Consequently

$$\|y\|^2 \le \|y'\|^2 \int_0^1 p(t)\int_t^1 \frac{ds}{p(s)}dt$$

is obtained

**Theorem 2.10**

Let $n \in N^+=\{1,2,\ldots\}$ and suppose $\lim_{t\to 0^+} p(t)y'(t)=0$, $y(1)=1/n$ and (2.4) and (2.5) are satisfied.

**i)** Let $y(t) > 1/n$ for $t \in (0,1)$ and $o \le \gamma \le 1$; then

$$\|y\|^2 \le \|y'\|^2 \int_0^1 p(t)\int_t^1 \frac{ds}{p(s)}dt + 2\int_0^1 p|y(t)|dt + \int_0^1 p(t)dt. \qquad (2.10)$$

**ii)** Let $y(t) > 1/n$ for $t \in (0,1)$ and $\gamma > 1$; then

$$|y|_* \le \frac{(\gamma+1)^2}{4}\|y\|_\gamma \int_0^1 p(t)\int_t^1 \frac{ds}{p(s)}dt + (\gamma+1)\int_0^1 p(t)y^\gamma(t)dt. \qquad (2.11)$$

Here

$$|y|_* = \int_0^1 py^{\gamma+1}dt \quad \text{and} \quad \|y\|_\gamma = \int_0^1 py^{\gamma-1}(y')^2 dt.$$

**Proof**

**i)** Let $w(t) = y(t) - \frac{1}{n}$ so $\lim_{t\to 0^+} p(t)w'(t) = 0$. Now theorem 2.9 implies

$$\int_0^1 p(t)\left(y(t)-\frac{1}{n}\right)^2 dt \le \|y'\|^2 \int_0^1 p(t)\int_t^1 \frac{ds}{p(s)}\,dt .$$

Hence we have,

$$\int_0^1 p(t)y^2(t)dt - \frac{2}{n}\int_0^1 p(t)y(t)dt + \frac{1}{n^2}\int_0^1 p(t)dt \le \|y'\|^2 \int_0^1 p(t)\int_t^1 \frac{ds}{p(s)}\,dt .$$

The estimate

$$\|y\|^2 \le \|y'\|^2 \int_0^1 p(t)\int_t^1 \frac{ds}{p(s)}dt + 2\int_0^1 p|y(t)|dt + \int_0^1 p(t)dt$$

is obtained.

**ii)** $w(t) = \left(y(t)-\frac{1}{n}\right)^{\frac{\gamma+1}{2}}$, so $w'(t) = \frac{\gamma+1}{2}\left(y(t)-\frac{1}{n}\right)^{\frac{\gamma-1}{2}} y'(t)$, $\lim_{t\to 0^+} p(t)w'(t) = 0$

$w(1)=0$ and $0 \le y(t) - 1/n \le y(t)$ on $[0,1]$. Hence

$$w'(t) = \frac{\gamma+1}{2}\left(y(t)-\frac{1}{n}\right)^{\frac{\gamma-1}{2}} y'(t) \le \frac{\gamma+1}{2} y^{\frac{\gamma-1}{2}}(t)y'(t)$$

Now theorem 2.9 implies

$$\int_0^1 p(t)\left(y(t)-\frac{1}{n}\right)^{\gamma+1} dt \le \int_0^1 p(t)\left(\frac{\gamma+1}{2} y^{\frac{\gamma-1}{2}}(t)y'(t)\right)^2 dt \int_0^1 p(t)\int_t^1 \frac{ds}{p(s)}dt$$

$$\le \frac{(\gamma+1)^2}{4}\int_0^1 p(t)y^{\gamma-1}(t)(y'(t))^2 dt \int_0^1 p(t)\int_t^1 \frac{ds}{p(s)}dt$$

In addition, since $(1-x)^{\gamma+1} \geq 1-(\gamma+1)x$ for $|x| \leq 1$, we have that

$$\left(y-\frac{1}{n}\right)^{\gamma+1} = y^{\gamma+1}\left(1-\frac{1}{ny}\right)^{\gamma+1} \geq y^{\gamma+1}\left(1-(\gamma+1)\frac{1}{ny}\right)$$

$$\geq y^{\gamma+1} - \frac{\gamma+1}{n}y^{\gamma} \geq y^{\gamma+1} - (\gamma+1)y^{\gamma}.$$

So the previous inequality becomes,

$$\int_0^1 p(t)(y^{\gamma+1}(t)-(\gamma+1)y^{\gamma}(t))dt \leq \frac{(\gamma+1)^2}{4}\int_0^1 p(t)y^{\gamma-1}(t)(y'(t))^2\,dt \int_0^1 p(t)\int_t^1 \frac{ds}{p(s)}dt,$$

therefore implying,

$$\int_0^1 p(t)y^{\gamma+1}(t)dt \leq \frac{(\gamma+1)^2}{4}\int_0^1 p(t)y^{\gamma-1}(t)(y'(t))^2\,dt \int_0^1 p(t)\int_t^1 \frac{ds}{p(s)}dt + (\gamma+1)\int_0^1 p(t)y^{\gamma}(t)dt.$$

Hence

$$|y|_* \leq \frac{(\gamma+1)^2}{4}\|y\|_{\gamma} \int_0^1 p(t)\int_t^1 \frac{ds}{p(s)}dt + (\gamma+1)\int_0^1 p(t)y^{\gamma}(t)dt$$

and so (2.11) is obtained.

# CHAPTER THREE

# EXISTENCE AND UNIQUENESS THEOREMS

## 3.0 Introduction

It will be shown that (2.8) has a $C[0,1]\cap C^1(0,1]\cap C^2(0,1)$ solution. To do this the following "approximate" problems will be first established;

$$\begin{cases} \frac{1}{p}(py')' + f(t,y,py') = 0 \qquad 0 < t < 1 \\ \lim_{t\to 0^+} p(t)y'(t) = 0, \qquad y(1) = \frac{1}{n} \qquad n \in N^+ \end{cases} \tag{3.1$^n$}$$

Then to show that (2.8) has a solution, we let $n \to \infty$; the key idea in this step is the Arzela - Ascoli theorem.

## 3.1 Existence of Solution

In this part we give two theorems for the existence of the solution of a two point boundary value problem which has singularities which appear both from independent and dependent variables.

**Theorem 3.1**

Suppose (2.4), (2.5) and (2.6) are satisfied and $f:[0,1]\times(0,\infty)\times(-\infty,\infty)\to(0,\infty)$ is continuous with $\lim_{y\to 0^+} f(t,y,py') = +\infty$ on compact subset of $[0,1]\times(-\infty,\infty)$. Suppose further that;

$$\begin{cases} 0 < f(t, y(t), p(t)y'(t)) \leq Ay + h(y) + By^{-\gamma} + g(y, y') \text{ on } (0,1) \times (0,\infty) \times (-\infty,\infty). \\ \text{Here A} \geq 0 \text{ B} \geq 0,\ \gamma \geq 0 \text{ are constants with h} \geq 0 \text{ on } (0,\infty),\ g \geq 0 \\ \text{on } (0,1) \times (0,\infty) \times (-\infty,\infty) \text{ and } \sup(p(t)) = 1 \text{ on } [0,1], \end{cases} \tag{3.2}$$

and

$$\begin{cases} \text{there exists constants C} \geq 0,\ \text{D} \geq 0,\ \text{E} \geq 0, \text{F} \geq 0,\ 0 \leq \tau < 1 \text{ with} \\ y^{\gamma} h(y) \leq Cy^{\gamma+\tau} + D \text{ and } yg(y, y') \leq E(y')^2 + \text{F}|y'| \text{ for } y > 0, \end{cases} \tag{3.3}$$

hold. Then a $C[0,1]\cap C^1(0,1]\cap C^2(0,1)$ solution of $(3.1)^n$ exists for each $n \in N^+$ in each of the following cases;

**i)** $0 \leq \gamma \leq 1$ with

$$E + A\int_0^1 p(t) \int_t^1 \frac{ds}{p(s)} dt < 1, \qquad \int_0^1 p(t) \left( \int_t^1 \frac{ds}{p(s)} \right)^{\frac{1-\gamma}{2}} dt < \infty, \qquad \int_0^1 p(t) \left( \int_t^1 \frac{ds}{p(s)} \right)^{\frac{\tau+1}{2}} dt < \infty,$$

$$\int_0^1 \frac{1}{p(t)} \int_0^t p(s) \left( \int_s^1 \frac{dz}{p(z)} \right)^{\frac{1-\gamma}{2}} ds dt < \infty, \qquad \int_0^1 \frac{1}{p(t)} \int_0^t p(s) \left( \int_s^1 \frac{dz}{p(z)} \right)^{\frac{\tau+1}{2}} ds dt < \infty,$$

**ii)** $\gamma > 1$ with

$$A\frac{(\gamma+1)^2}{4} \int_0^1 p(t) \int_t^1 \frac{ds}{p(s)} dt < \gamma, \qquad \int_0^1 p(t) \left( \int_t^1 \frac{ds}{p(s)} \right)^{\frac{\gamma+\tau}{\gamma+1}} dt < \infty, \qquad \int_0^1 p(t) \left( \int_t^1 \frac{ds}{p(s)} \right)^{\frac{\gamma-1}{\gamma+1}} dt < \infty,$$

$$\int_0^1 \frac{1}{p(t)} \int_0^t p(s) \left( \int_s^1 \frac{dz}{p(z)} \right)^{\frac{\gamma+\tau}{\gamma+1}} ds dt < \infty, \qquad \int_0^1 \frac{1}{p(t)} \left( \int_0^t p(s) \left( \int_s^1 \frac{dz}{p(z)} \right)^{\frac{\gamma-1}{\gamma+1}} ds \right)^{\frac{1}{2}} dt < \infty$$

**Proof**

Consider the family of problems

$$\begin{cases} \frac{1}{p}(py')' + \lambda f^*(t,y,py') = 0 & 0 < t < 1,\ 0 < \lambda < 1 \\ \lim_{t \to 0^+} p(t)y'(t) = 0, & y(1) = \frac{1}{n} \end{cases} \qquad (3.4)^n_\lambda$$

where $f^* > 0$ is any continuous extension of f from $y \geq 1/n$. Let y be a solution to $(3.4)^n_\lambda$. It is claimed $y \geq 1/n$ on $[0,1]$.

Since

$$(py')' = -\lambda . pf^*(t,y,py') \leq 0,$$

$py'$ is nonincreasing. Hence $y'$ is nonincreasing. Since $\lim_{t \to 0^+} p(t)y'(t) = 0$ we have that $y'(t) \leq 0$ on $[0,1]$ and so y is nonincreasing. Because $y(1) = 1/n$, we obtain $y(1) = 1/n \leq y(t) \leq y(0)$

Thus $y(t) \geq 1/n$ on $[0,1]$ and so every solution of $(3.4)^n_\lambda$ is also a solution of

$$\begin{cases} \frac{1}{p}(py')' + \lambda . f(t,y) = 0, & 0 < t < 1,\ 0 < \lambda < 1 \\ \lim_{t \to 0^+} p(t)y'(t) = 0, & y(1) = \frac{1}{n}\ . \end{cases} \qquad (3.5)^n_\lambda$$

Now, it will be shown that there exists a constant $M_0$, independent of $\lambda$, ( and in fact n ) with

$$\frac{1}{n} \leq y(t) \leq M_0,\ t \in [0,1] \qquad (3.6)$$

for each solution $y \in C[0,1] \cap C^1(0,1] \cap C^2(0,1)$ of $(3.4)^n_\lambda$. Consider the cases $0 \leq \gamma \leq 1$ and $\gamma > 1$ separately.

**i)** $0 \le \gamma \le 1$;

Assumptions (3.2) and (3.3) imply

$$-(py')'y \le (B+D)py^{1-\gamma} + Cpy^{\tau+1} + Apy^2 + Ep(y')^2 + Fp|y'|.$$

Integrating from 0 to 1 yields

$$\int_0^1 p|y'|^2\,dt \le (B+D)\int_0^1 p|y|^{1-\gamma}\,dt + C\int_0^1 p|y|^{\tau+1}\,dt + A\int_0^1 p|y|^2\,dt$$
$$+E\int_0^1 p|y'|^2\,dt + F\int_0^1 p|y'|dt + \frac{y'(1)p(1)}{n}.$$

Because $y'(t) \le 0$ on $t \in [0,1]$, $y'(1) \le 0$. Hence

$$\int_0^1 p|y'|^2\,dt \le (B+D)\int_0^1 p|y|^{1-\gamma}\,dt + C\int_0^1 p|y|^{\tau+1}\,dt + A\int_0^1 p|y|^2\,dt + E\int_0^1 p|y'|^2\,dt + F\int_0^1 p|y'|dt.$$

Thus

$$\|y'\|^2 \le (B+D)\int_0^1 p|y|^{1-\gamma}\,dt + C\int_0^1 p|y|^{\tau+1}\,dt + F\int_0^1 p|y'|dt + E\|y'\|^2 + A\|y\|^2$$

and so

$$(1-E)\|y'\|^2 \le (B+D)\int_0^1 p|y|^{1-\gamma}\,dt + C\int_0^1 p|y|^{\tau+1}\,dt + F\int_0^1 p|y'|dt + A\|y\|^2 \qquad (3.7)$$

Putting (2.10) into (3.7) we obtain,

$$\begin{aligned}(1-E-A\int_0^1 p(t)\int_t^1 \frac{ds}{p(s)}dt)\|y'\|^2 &\le (B+D)\int_0^1 p|y|^{1-\gamma}\,dt + C\int_0^1 p|y|^{\tau+1}\,dt \\ &\quad + F\int_0^1 p|y'|dt + 2A\int_0^1 p|y|dt + A\int_0^1 p(t)dt \end{aligned} \qquad (3.8)$$

Also since

$$y(t)=\frac{1}{n}+\int_t^1(-y'(s))ds=\frac{1}{n}+\int_t^1\sqrt{p(s)}(-y'(s))\frac{1}{\sqrt{p(s)}}ds \qquad t\in(0,1)$$

by using Hölder's inequality,

$$|y(t)|\le\|y'(t)\|\left(\int_t^1\frac{ds}{p(s)}\right)^{\frac{1}{2}}+1 \tag{3.9}$$

is obtained for $t\in(0,1)$. Thus we get

$$\int_0^1 p|y(t)|^{1-\gamma}dt\le\int_0^1 p\left(1+\|y'\|\left(\int_t^1\frac{ds}{p(s)}\right)^{\frac{1}{2}}\right)^{1-\gamma}dt\,.$$

Because $(a+b)^c\le 2^c(a^c+b^c)$ for $c>0$ ; $a,b\ge 0$

$$\int_0^1 p|y(t)|^{1-\gamma}dt\le 2^{1-\gamma}\int_0^1 p(t)dt+2^{1-\gamma}\|y'\|^{1-\gamma}\int_0^1 p(t)\left(\int_t^1\frac{ds}{p(s)}\right)^{\frac{1-\gamma}{2}}dt \tag{3.10}$$

$$\int_0^1 p|y(t)|^{\tau+1}dt\le 2^{\tau+1}\int_0^1 p(t)dt+2^{\tau+1}\|y'\|^{\tau+1}\int_0^1 p(t)\left(\int_t^1\frac{ds}{p(s)}\right)^{\frac{\tau+1}{2}}dt \tag{3.11}$$

$$\int_0^1 p|y(t)|dt\le\int_0^1 p(t)dt+\|y'\|\int_0^1 p(t)\left(\int_t^1\frac{ds}{p(s)}\right)^{\frac{1}{2}}dt \tag{3.12}$$

hold and,

$$\int_0^1 p|y'(t)|dt = \int_0^1 \sqrt{p}|y'(t)|\sqrt{p}dt \le \left(\int_0^1 p|y'|^2 dt\right)^{\frac{1}{2}}\left(\int_0^1 pdt\right)^{\frac{1}{2}} = \|y'\|\left(\int_0^1 pdt\right)^{\frac{1}{2}} \tag{3.13}$$

is satisfied. Substituting (3.10), (3.11), (3.12) and (3.13) into (3.8),

$$\begin{aligned}(1-E-A\int_0^1 p(t)\int_t^1 \frac{ds}{p(s)}dt)\|y'\|^2 &\le 2^{1-\gamma}(B+D)\int_0^1 pdt + 2^{1-\gamma}(B+D)\|y'\|^{1-\gamma}\int_0^1 p(t)\left(\int_t^1 \frac{ds}{p(s)}\right)^{\frac{1-\gamma}{2}} dt \\ &+ 2^{\tau+1}C\int_0^1 pdt + 2^{\tau+1}C\|y'\|^{\tau+1}\int_0^1 p(t)\left(\int_t^1 \frac{ds}{p(s)}\right)^{\frac{\tau+1}{2}} dt + F\|y'\|\left(\int_0^1 pdt\right)^{\frac{1}{2}} \\ &+ 2A\|y'\|\int_0^1 p(t)\left(\int_t^1 \frac{ds}{p(s)}\right)^{\frac{1}{2}} dt + 2A\int_0^1 pdt\end{aligned}$$

is obtained. Hencethe estimate

$$(1-E-A\int_0^1 p(t)\int_t^1 \frac{ds}{p(s)}dt)\|y'\|^2 \le Q_1\|y'\|^{1-\gamma} + Q_2\|y'\|^{\tau+1} + Q_3\|y'\| + Q_4 \tag{3.14}$$

holds, where

$$Q_1 = 2^{1-\gamma}(B+D)\int_0^1 p(t)\left(\int_t^1 \frac{ds}{p(s)}\right)^{\frac{1-\gamma}{2}} dt$$

$$Q_2 = 2^{\tau+1}C\int_0^1 p(t)\left(\int_t^1 \frac{ds}{p(s)}\right)^{\frac{\tau+1}{2}} dt$$

$$Q_3 = 2A\int_0^1 p(t)\left(\int_t^1 \frac{ds}{p(s)}\right)^{\frac{1}{2}} dt + F\left(\int_0^1 p(t)dt\right)^{\frac{1}{2}}$$

$$Q_4 = (2^{1-\gamma}(B+D) + 2^{\tau+1}C + 3A)\int_0^1 p(t)dt\,.$$

Note also if $L \geq 0$, $0 \leq k < 2$, $0 < m < 1$ are given constants, then there exists a constant $N > 0$ such that

$$Lx^k \leq \frac{m}{2}x^2 + N \qquad \text{for all } x \geq 0.$$

Thus the estimates

$$Q_1\|y'\|^{1-\gamma} \leq \frac{1}{2}\frac{1}{3}(1 - E - A\int_0^1 p(t)\int_t^1 \frac{ds}{p(s)}dt)\|y'\|^2 + N_1$$

$$Q_2\|y'\|^{\tau+1} \leq \frac{1}{2}\frac{1}{3}(1 - E - A\int_0^1 p(t)\int_t^1 \frac{ds}{p(s)}dt)\|y'\|^2 + N_2$$

$$Q_3\|y'\| \leq \frac{1}{2}\frac{1}{3}(1 - E - A\int_0^1 p(t)\int_t^1 \frac{ds}{p(s)}dt)\|y'\|^2 + N_3$$

hold. Then from (3.14),

$$\frac{1}{2}(1 - E - A\int_0^1 p(t)\int_t^1 \frac{ds}{p(s)}dt)\|y'\|^2 \leq Q_5$$

is deduced, where $Q_5 = N_1 + N_2 + N_3 + Q_4$.

Consequently there exists a constant $M_1$, independent of $\lambda$, with $\|y'\| \leq M_1$.

The differential equation and assumptions (3.2) and (3.3) imply

$$-(py')'y \leq (B + D)py^{1-\gamma} + Cpy^{\tau+1} + Apy^2 + Ep(y')^2 + Fp|y'|.$$

Integrating this last inequality from 0 to t yields

$$-pyy' + \int_0^t p|y'|^2 ds \le (B+D)\int_0^t p|y|^{1-\gamma} ds + C\int_0^t p|y|^{\tau+1} ds + A\int_0^t p|y|^2 ds$$

$$+ E\int_0^t p|y'|^2 ds + F\int_0^t p|y'| ds.$$

Hence

$$-pyy' + (1-E)\int_0^t p|y'|^2 ds \le (B+D)\int_0^t p|y|^{1-\gamma} ds + C\int_0^t p|y|^{\tau+1} ds$$

$$+A\int_0^t p|y|^2 ds + F\int_0^t p|y'| ds$$

holds. Because (1 - E) > 0 then we have that

$$-pyy' \le (B+D)\int_0^t p|y|^{1-\gamma} ds + C\int_0^t p|y|^{\tau+1} ds + A\int_0^t p|y|^2 ds + F\int_0^t p|y'| ds.$$

Thus

$$-yy' \le (B+D)\frac{1}{p}\int_0^t p|y|^{1-\gamma} ds + C\frac{1}{p}\int_0^t p|y|^{\tau+1} ds + A\frac{1}{p}\int_0^t p|y|^2 ds + F\frac{1}{p}\int_0^t p|y'| ds$$

is obtained. In this last inequality

$$\int_0^t p|y(s)|^{1-\gamma} ds \le 2^{1-\gamma}\int_0^t p(s)ds + 2^{1-\gamma}\|y'\|^{1-\gamma}\int_0^t p(s)\left(\int_s^1 \frac{dz}{p(z)}\right)^{\frac{1-\gamma}{2}} ds$$

$$\int_0^t p|y(s)|^{\tau+1} ds \le 2^{\tau+1}\int_0^t p(s)ds + 2^{\tau+1}\|y'\|^{\tau+1}\int_0^t p(s)\left(\int_s^1 \frac{dz}{p(z)}\right)^{\frac{\tau+1}{2}} ds$$

$$\int_0^t p|y(s)|^2 ds \le 2\int_0^t p(s)ds + 2\|y'\|^2\int_0^s p(s)\left(\int_s^1 \frac{dz}{p(z)}\right) ds$$

and

$$\int_0^t p|y'(t)|ds = \int_0^t \sqrt{p}|y'(s)|\sqrt{p}ds \leq \left(\int_0^1 p|y'|^2 dt\right)^{\frac{1}{2}} \left(\int_0^t pds\right)^{\frac{1}{2}} = \|y'\| \left(\int_0^t pds\right)^{\frac{1}{2}}$$

are satisfied. Hence we have,

$$-yy' \leq (2^{1-\gamma}(B+D) + 2^{\tau+1}C + 2A)\frac{1}{p}\int_0^t pds + 2^{1-\gamma}(B+D)\|y'\|^{1-\gamma}\frac{1}{p}\int_0^t p(s)\left(\int_s^1 \frac{dz}{p(z)}\right)^{\frac{1-\gamma}{2}} ds$$

$$+2^{\tau+1}C\|y'\|^{\tau+1}\frac{1}{p}\int_0^t p(s)\left(\int_s^1 \frac{dz}{p(z)}\right)^{\frac{\tau+1}{2}} ds + 2A\|y'\|^2 \frac{1}{p}\int_0^t p(s)\left(\int_s^1 \frac{dz}{p(z)}\right)ds + F\|y'\|\frac{1}{p}\left(\int_0^t p(s)ds\right)^{\frac{1}{2}}$$

Again integrating the lastly obtained inequality from 0 to 1, we get ,

$$y^2(0) \leq 2^{1-\gamma}(B+D)\|y'\|^{1-\gamma}\int_0^1 \frac{1}{p(t)}\int_0^t p(s)\left(\int_s^1 \frac{dz}{p(z)}\right)^{\frac{1-\gamma}{2}} dsdt + F\|y'\|\int_0^1 \frac{1}{p(t)}\left(\int_0^t p(s)ds\right)^{\frac{1}{2}} dt$$

$$+2^{\tau+1}C\|y'\|^{\tau+1}\int_0^1 \frac{1}{p(t)}\int_0^t p(s)\left(\int_s^1 \frac{dz}{p(z)}\right)^{\frac{\tau+1}{2}} dsdt + 2A\|y'\|^2\int_0^1 \frac{1}{p(t)}\int_0^t p(s)\left(\int_s^1 \frac{dz}{p(z)}\right)dsdt$$

$$+(2^{1-\gamma}(B+D) + 2^{\tau+1}C + 2A)\int_0^1 \frac{1}{p(t)}\int_0^t p(s)dsdt + 1 \leq M_0$$

Consequently

$$\frac{1}{n} \leq y(t) \leq M_0 \qquad t \in [0,1].$$

**ii)** $\gamma > 1$;

Assumptions (3.2) and (3.3) imply

$$-(py')'y^{\gamma} \leq (B+D)p + Cpy^{\gamma+\tau} + Apy^{1+\gamma} + Ep|y'|^2 y^{\gamma-1} + Fpy^{\gamma-1}|y'|.$$

Integrating from 0 to 1 yields

$$\gamma\int_0^1 py^{\gamma-1}|y'|^2 dt \leq (B+D)\int_0^1 pdt + C\int_0^1 py^{\gamma+\tau}dt + A\int_0^1 py^{1+\gamma}dt$$
$$+E\int_0^1 py^{\gamma-1}|y'|^2 dt + F\int_0^1 py^{\gamma-1}|y'|dt + \frac{y'(1)p(1)}{n^{\gamma}} \quad .$$

Because $y'(t) \leq 0$ on $t \in [0,1]$, $y'(1) \leq 0$, and we have,

$$\gamma\int_0^1 py^{\gamma-1}|y'|^2 dt \leq (B+D)\int_0^1 pdt + C\int_0^1 py^{\gamma+\tau}dt + A\int_0^1 py^{1+\gamma}dt$$
$$+E\int_0^1 py^{\gamma-1}|y'|^2 dt + F\int_0^1 py^{\gamma-1}|y'|dt \quad .$$

Thus

$$\gamma\ \|y\|_{\gamma} \leq (B+D)\int_0^1 pdt + C\int_0^1 py^{\gamma+\tau}dt + A|y|_* + E\|y\|_{\gamma} + F\int_0^1 py^{\gamma-1}|y'|dt$$

holds, implying

$$(\gamma - E)\|y\|_{\gamma} \leq (B+D)\int_0^1 pdt + C\int_0^1 py^{\gamma+\tau}dt + A|y|_* + F\int_0^1 py^{\gamma-1}|y'|dt. \tag{3.15}$$

Inserting (2.11) into (3.15) we obtain;

$$(\gamma - E - A\frac{(\gamma+1)^2}{4}\int_0^1 p(t)\int_t^1 \frac{ds}{p(s)}dt)\|y\|_{\gamma} \leq (B+D)\int_0^1 pdt + C\int_0^1 py^{\gamma+\tau}dt$$
$$+F\int_0^1 py^{\gamma-1}|y'|dt + A(\gamma+1)\int_0^1 py^{\gamma}dt \tag{3.16}$$

Also since it is true that,

$$y^{\frac{\gamma+1}{2}} = \left(\frac{1}{n}\right)^{\frac{\gamma+1}{2}} + \frac{(\gamma+1)}{2}\int_t^1 (-y^{\frac{\gamma-1}{2}}(s))y'(s)ds$$

$$\leq 1 + \frac{\gamma+1}{2}\int_t^1 \sqrt{p(s)}(-y^{\frac{\gamma-1}{2}}(s))y'(s)\frac{1}{\sqrt{p(s)}}ds \qquad t \in (0,1)$$

holds, by using Hölder's inequality, we obtain

$$y^{\frac{\gamma+1}{2}} \le 1+\frac{(\gamma+1)}{2}\left(\int_t^1 p(s)y^{\gamma-1}|y'|^2\,ds\right)^{\frac{1}{2}}\left(\int_t^1 \frac{ds}{p(s)}\right)^{\frac{1}{2}}.$$

Thus we arrive at the estimates;

$$\begin{cases} y^{\gamma+1} \le 2+\dfrac{(\gamma+1)^2}{2}\|y\|_\gamma \displaystyle\int_t^1 \frac{ds}{p(s)} \\ y^{\gamma} \le 2^{\frac{2\gamma}{\gamma+1}}+(\gamma+1)^{\frac{2\gamma}{\gamma+1}}\left(\|y\|_\gamma\right)^{\frac{\gamma}{\gamma+1}}\left(\displaystyle\int_t^1 \frac{ds}{p(s)}\right)^{\frac{\gamma}{\gamma+1}} \\ y^{\gamma+\tau} \le 2^{\frac{2(\gamma+\tau)}{\gamma+1}}+(\gamma+1)^{\frac{2(\gamma+\tau)}{\gamma+1}}\left(\|y\|_\gamma\right)^{\frac{\gamma+\tau}{\gamma+1}}\left(\displaystyle\int_t^1 \frac{ds}{p(s)}\right)^{\frac{\gamma+\tau}{\gamma+1}} \\ y^{\gamma-1} \le 2^{\frac{2(\gamma-1)}{\gamma+1}}+(\gamma+1)^{\frac{2(\gamma-1)}{\gamma+1}}\left(\|y\|_\gamma\right)^{\frac{\gamma-1}{\gamma+1}}\left(\displaystyle\int_t^1 \frac{ds}{p(s)}\right)^{\frac{\gamma-1}{\gamma+1}} \end{cases} \tag{3.17}$$

By using Hölder's inequality and $(a+b)^c \le 2^c(a^c+b^c)$ for $c>0$ ; $a,b \ge 0$ (3.17)gives rise to;

$$\begin{cases} \displaystyle\int_0^1 py^{\gamma-1}|y'|dt \le \left(\|y\|_\gamma\right)^{\frac{1}{2}}\left(\int_0^1 py^{\gamma-1}dt\right)^{\frac{1}{2}} \\ \displaystyle\int_0^1 py(t)^{\gamma}\,dt \le 2^{\frac{2\gamma}{\gamma+1}}\int_0^1 p(t)dt+(\gamma+1)^{\frac{2\gamma}{\gamma+1}}\left(\|y\|_\gamma\right)^{\frac{\gamma}{\gamma+1}}\int_0^1 p(t)\left(\int_t^1 \frac{ds}{p(s)}\right)^{\frac{\gamma}{\gamma+1}}dt \\ \displaystyle\int_0^1 py(t)^{\gamma+\tau}\,dt \le 2^{\frac{2(\gamma+\tau)}{\gamma+1}}\int_0^1 p(t)dt+(\gamma+1)^{\frac{2(\gamma+\tau)}{\gamma+1}}\left(\|y\|_\gamma\right)^{\frac{\gamma+\tau}{\gamma+1}}\int_0^1 p(t)\left(\int_t^1 \frac{ds}{p(s)}\right)^{\frac{\gamma+\tau}{\gamma+1}}dt \\ \left(\displaystyle\int_0^1 py^{\gamma-1}dt\right)^{\frac{1}{2}} \le 2^{\frac{(3\gamma-1)}{2(\gamma+1)}}\left(\int_0^1 p(t)dt\right)^{\frac{1}{2}}+2^{\frac{1}{2}}(\gamma+1)^{\frac{\gamma-1}{\gamma+1}}\left(\|y\|_\gamma\right)^{\frac{\gamma-1}{2(\gamma+1)}}\left(\int_0^1 p(t)\left(\int_t^1 \frac{ds}{p(s)}\right)^{\frac{\gamma-1}{\gamma+1}}dt\right)^{\frac{1}{2}} \end{cases} \tag{3.18}$$

Substituting (3.18) into (3.16),

$$(\gamma - E - A\frac{(\gamma+1)^2}{4}\int_0^1 p(t)\int_t^1 \frac{ds}{p(s)}dt)\|y\|_\gamma \le Q_1\left(\|y\|_\gamma\right)^{\frac{\gamma+\tau}{\gamma+1}} + Q_2\left(\|y\|_\gamma\right)^{\frac{\gamma}{\gamma+1}} + Q_3\left(\|y\|_\gamma\right)^{\frac{1}{2}} + Q_4$$

is obtained, where

$$Q_1 = C(\gamma+1)^{\frac{2(\gamma+\tau)}{\gamma+1}}\int_0^1 p(t)\left(\int_t^1 \frac{ds}{p(s)}\right)^{\frac{\gamma+\tau}{\gamma+1}} dt$$

$$Q_2 = A(\gamma+1)^{\frac{3\gamma+1}{\gamma+1}}\int_0^1 p(t)\left(\int_t^1 \frac{ds}{p(s)}\right)^{\frac{\gamma}{\gamma+1}} dt + F2^{\frac{1}{2}}(\gamma+1)^{\frac{\gamma-1}{\gamma+1}}\left(\int_0^1 p(t)\left(\int_t^1 \frac{ds}{p(s)}\right)^{\frac{\gamma-1}{\gamma+1}} dt\right)^{\frac{1}{2}}$$

$$Q_3 = F\,2^{\frac{3\gamma-1}{2(\gamma+1)}}\left(\int_0^1 p(t)dt\right)^{\frac{1}{2}}$$

$$Q_4 = (B + D + 2^{\frac{2(\gamma+\tau)}{\gamma+1}}C + 2^{\frac{2\gamma}{\gamma+1}}A)\int_0^1 p(t)dt\,.$$

Thus there exists a constant $Q_5$ with

$$\frac{1}{2}(\gamma - E - A\frac{(\gamma+1)^2}{4}\int_0^1 p(t)\int_t^1 \frac{ds}{p(s)}dt)\|y\|_\gamma \le Q_5\,.$$

Hence there exists a constant $M_1$, independent of $\lambda$ ,with

$$\|y\|_\gamma = \int_0^1 py^{\gamma-1}|y'|^2 dt \le M_1\ .$$

The differential equation and assumptions (3.2) and (3.3) imply

$$-(py')'y^\gamma \le (B+D)p + Cpy^{\gamma+\tau} + Apy^{1+\gamma} + Ep|y'|^2 y^{\gamma-1} + Fpy^{\gamma-1}|y'|$$

which is integrated from 0 to t in order to yields;

$$-py^{\gamma}y' + \gamma\int_0^t py^{\gamma-1}|y'|^2\,ds \le (B+D)\int_0^t pds + C\int_0^t py^{\gamma+\tau}ds + A\int_0^t py^{1+\gamma}ds$$

$$+E\int_0^t py^{\gamma-1}|y'|^2\,ds + F\int_0^t py^{\gamma-1}|y'|ds.$$

Hence

$$-py^{\gamma}y' + (\gamma-E)\int_0^t py^{\gamma-1}|y'|^2\,ds \le (B+D)\int_0^t pds + C\int_0^t py^{\gamma+\tau}ds$$

$$+A\int_0^t py^{1+\gamma}ds + F\int_0^t py^{\gamma-1}|y'|ds$$

is deduced. Because (γ - E) > 0 we have that

$$-py^{\gamma}y' \le (B+D)\int_0^t pds + C\int_0^t py^{\gamma+\tau}ds + A\int_0^t py^{1+\gamma}ds + F\int_0^t py^{\gamma-1}|y'|ds.$$

Thus

$$-y^{\gamma}y' \le (B+D)\frac{1}{p}\int_0^t pds + C\frac{1}{p}\int_0^t py^{\gamma+\tau}ds + A\frac{1}{p}\int_0^t py^{1+\gamma}ds + +F\frac{1}{p}\int_0^t py^{\gamma-1}|y'|ds$$

In this last inequality the following are satisfied;

$$\begin{cases} \displaystyle\int_0^t py(s)^{\gamma+1}ds \le 2\int_0^t p(s)ds + (\gamma+1)^2\|y\|_{\gamma}\int_0^t p(s)\left(\int_s^1 \frac{dz}{p(z)}\right)ds \\ \displaystyle\int_0^t py(s)^{\gamma+\tau}ds \le 2^{\frac{2(\gamma+\tau)}{\gamma+1}}\int_0^t p(s)ds + (\gamma+1)^{\frac{2(\gamma+\tau)}{\gamma+1}}\left(\|y\|_{\gamma}\right)^{\frac{\gamma+\tau}{\gamma+1}}\int_0^t p(s)\left(\int_s^1 \frac{dz}{p(z)}\right)^{\frac{\gamma+\tau}{\gamma+1}}ds \\ \displaystyle\int_0^t py^{\gamma-1}|y'|ds \le 2^{\frac{(3\gamma-1)}{2(\gamma+1)}}\left(\|y\|_{\gamma}\right)^{\frac{1}{2}}\left(\int_0^t p(s)ds\right)^{\frac{1}{2}} + 2^{\frac{1}{2}}(\gamma+1)^{\frac{\gamma-1}{\gamma+1}}\left(\|y\|_{\gamma}\right)^{\frac{\gamma}{(\gamma+1)}}\left(\int_0^t p(s)\left(\int_s^1 \frac{dz}{p(z)}\right)^{\frac{\gamma-1}{\gamma+1}}ds\right)^{\frac{1}{2}} \end{cases}$$

Hence it follows that

$$-y^{\gamma}y' \le ((B+D)+2A+2^{\frac{2(\gamma+\tau)}{\gamma+1}}C)\frac{1}{p}\int_0^t p(s)ds + 2^{\frac{(3\gamma-1)}{2(\gamma+1)}}F\left(\|y\|_\gamma\right)^{\frac{1}{2}}\frac{1}{p}\left(\int_0^t p(s)ds\right)^{\frac{1}{2}}$$

$$+A\frac{(\gamma+1)^2}{2}\|y\|_\gamma\frac{1}{p}\int_0^t p(s)\left(\int_s^1\frac{dz}{p(z)}\right)ds + C(\gamma+1)^{\frac{2(\gamma+\tau)}{\gamma+1}}\left(\|y\|_\gamma\right)^{\frac{\gamma+\tau}{\gamma+1}}\frac{1}{p}\int_0^t p(s)\left(\int_s^1\frac{dz}{p(z)}\right)^{\frac{\gamma+\tau}{\gamma+1}}ds$$

$$+F2^{\frac{1}{2}}(\gamma+1)^{\frac{\gamma-1}{\gamma+1}}\left(\|y\|_\gamma\right)^{\frac{\gamma}{(\gamma+1)}}\frac{1}{p}\left(\int_0^t p(s)\left(\int_s^1\frac{dz}{p(z)}\right)^{\frac{\gamma-1}{\gamma+1}}ds\right)^{\frac{1}{2}}$$

holds. Again integrating the above inequality from 0 to 1, we arrive at

$$y^{\gamma+1}(0) \le ((B+D)+2A+2^{\frac{2(\gamma+\tau)}{\gamma+1}}C)\int_0^1\frac{1}{p}\int_0^t p(s)dsdt + 2^{\frac{(3\gamma-1)}{2(\gamma+1)}}F\left(\|y\|_\gamma\right)^{\frac{1}{2}}\int_0^1\frac{1}{p}\left(\int_0^t p(s)ds\right)^{\frac{1}{2}}dt$$

$$+A\frac{(\gamma+1)^2}{2}\|y\|_\gamma\int_0^1\frac{1}{p}\int_0^t p(s)\left(\int_s^1\frac{dz}{p(z)}\right)ds\,dt + C(\gamma+1)^{\frac{2(\gamma+\tau)}{\gamma+1}}\left(\|y\|_\gamma\right)^{\frac{\gamma+\tau}{\gamma+1}}\int_0^1\frac{1}{p}\int_0^t p(s)\left(\int_s^1\frac{dz}{p(z)}\right)^{\frac{\gamma+\tau}{\gamma+1}}dsdt$$

$$+F2^{\frac{1}{2}}(\gamma+1)^{\frac{\gamma-1}{\gamma+1}}\left(\|y\|_\gamma\right)^{\frac{\gamma}{(\gamma+1)}}\int_0^1\frac{1}{p}\left(\int_0^t p(s)\left(\int_s^1\frac{dz}{p(z)}\right)^{\frac{\gamma-1}{\gamma+1}}ds\right)^{\frac{1}{2}}dt\ +1 \le M_0$$

Thus

$$\frac{1}{n} \le y(t) \le M_0,\ \ t \in [0,1].$$

Consequently in all case there exists a constant $M_0$ independent of $\lambda$ with

$$\frac{1}{n} \le y(t) \le M_0 \quad t \in [0,1].$$

Now, let $-py' > \sup(p(t))M_0$, then $-y'>M_0$. Integration form 0 to 1 we have $y(0)-y(1) > M_0$. Hence $y(0)>M_0$. This is a contradiction. Consequently $-py' \le \sup(p(t))M_0$. Hence there exists a constant $M_2$ independent of $\lambda$ and also independent of n such that $|py'| \le M_2$. Thus

$$\sup_{[0,1]}|py'| \le M_2 \tag{3.19}$$

holds.

Now (3.6), (3.19) and Theorem 2.1 imply that $(3.4)_1^n$ has a solution y for each $n \in N^+$. In addition since $y \ge 1/n$ on [0,1], $(3.5)_1^n$ has a solution $y \in C[0,1] \cap C^1(0,1] \cap C^2(0,1)$ for each $n \in N^+$.

**Theorem 3.2**

Suppose conditions of Theorem 3.1 are satisfied and $\lim_{t \to 0^+} (p(t)/p'(t))$ exists.

In addition suppose

$$\begin{cases} \text{for each constant } M > 0 \text{ and } H > 0 \text{ there exists a function } \Psi \\ \text{continuous on } [0,1] \text{ and positive on } (0,1) \text{ with} \\ f(t,y,py') \ge \Psi(t) \text{ on } (0,1) \times (0,M] \times [-H,H]. \end{cases} \tag{3.20}$$

Then a $C^1[0,1] \cap C^2(0,1)$ solution of (2.8) exists.

**Proof**

Theorem 3.1 implies that $(3.1)^n$ has a solution $y_n$ for each n and there exists constants $M_0$ and $M_2$ independent of n such that

$$|y_n| \le M_0 \ , \quad \text{for } t \in [0,1]$$

and by p(t) being positive for $t \in (0,1]$,

$$|p(t)y_n'(t)| = p(t)|y_n'(t)| \le M_2, \quad \text{for } t \in (0,1]$$

hold. Hence from Corollary 2.8 we obtain

$$|y_n'(t)| \le M_3, \text{ for } t \in [0,1]$$

where $M_3$ is constant independent of n. It follows that $\{y_n\}$ is uniformly bounded and for $t,s \in [0,1]$,

$$|y_n(t) - y_n(s)| \le \left|1 - \int_t^1 y_n'(z)dz - 1 + \int_s^1 y_n'(z)dz\right|$$

$$\le \left|\int_s^t y_n'(z)dz\right| \le M_3|t-s| \le M_3\delta = \varepsilon$$

Hence $\{y_n\}$ is equicontinuous on [0,1]. So Arzela Ascoli theorem guarantees the existence of a subsequence $\{y_{n'}\}$ converging uniformly on [0,1] to some $y \in C[0,1]$. Now $y(1) = 0$ with $y \ge 0$ on [0,1] but in fact (3.20) implies $y > 0$ on [0,1) ; to see this note that since $M_0$ is independent of $n'$ there exists $\Psi(t)$ ( independent of $n'$ ) such that

$$-(py_{n'}')' \ge p\Psi$$

and consequently

$$y_{n'}(t) \ge \frac{1}{n'} + \int_t^1 \frac{1}{p(u)} \int_0^u p(s)\Psi(s)dsdu$$

holds. Letting $n' \to \infty$ yields the result . Also $y_{n'}$ satisfies the integral equation

$$y_{n'}(t) = y_{n'}(0) + \int_t^1 \frac{1}{p(u)} \int_0^u p(s)f(s, y_{n'}(s), p(s)y_{n'}'(s))dsdu$$

For $t \in [0,1)$ and $s \in [0,t]$ , we have

$$f(s, y_{n'}(s), p(s)y_{n'}'(s)) \to f(s, y(s), p(s)y'(s))$$

uniformly since f is uniformly continuous on compact subset of $[0,1]\times(0,M_0]\times[-M_2,M_2]$. Thus letting $n' \to \infty$ yields

$$y(t) = y(0) - \int_0^t \frac{1}{p(u)} \int_0^u p(s) f(s, y(s), p(s)y'(s))\,ds\,du$$

From this integral equation, we see that

$$y \in C^1[0,1] \cap C^2(0,1), \quad \frac{1}{p}(py')' + f(t,y,py') = 0, \text{ and } y(1) = \lim_{t\to 0^+} p(t)y'(t) = 0$$

and hence the proof is completed.

### 3.2 Uniqueness of solution

In this part we give a theorem on uniqueness of a positive solution for the problem (2.8).

**Theorem 3.3**

If $f:[0,1)\times(0,\infty)\times(-\infty,\infty) \to (0,\infty)$ is continuous and $f(t,y,py')$ is strictly decreasing in $y$ and $y'$ for $t \in (0,1)$, and integrable over $[0,1]$ for each fixed $y > 0$ and $\lim_{t\to 0^+} \frac{p(t)}{p'(t)} = 0$ then (2.8) has a unique positive solution.

**Proof**

Suppose that there are at least two distinct solutions $\phi_1$, $\phi_2$. From the boundary conditions $\phi_1'(0) = \phi_2'(0) = 0$ and $\phi_1(1) = \phi_2(1) = 0$. There exists $a,b \in [0,1]$, $a < b$, such that one of the solutions, say $\phi_1$, is strictly greater than the other solution, $\phi_2$, over $[a,b]$. Define $w : [0,1] \to R$ by

$$w(t) = \phi_1(t) - \phi_2(t).$$

It is the case that $w(t) > 0$ if $t \in [a,b]$; the claim is that $w(t) \geq 0$ for $t \in [0,1]$. Otherwise $w$ must have a point of minimum in $[0,1]$, say $t_0$, with $w(t_0) < 0$.

Since $w(1) = 0$, it must be $t_0 \in [0,1)$; if $t_0 \in (0,1)$ then $w(t_0) < 0$, $w'(t_0) = 0$ and $w''(t_0) \geq 0$. But

$$w''(t_0) = f(t_0, \phi_2(t_0), p(t_0)\phi'_2(t_0)) - f(t_0, \phi_1(t_0), p(t_0)\phi'_1(t_0)) < 0$$

which is a contradiction. Thus the minimum must occur at $t = 0$ so $w(0) < 0$, $w'(0) = \phi_1'(0) - \phi_2'(0) = 0$ and $w''(0) \geq 0$. But

$$w''(0) = f(0, \phi_2(0), 0) - f(0, \phi_1(0), 0) < 0$$

which is a contradicts the fact that minimum of $w$ occurs at $t = 0$. Hence $w(t) \geq 0$ for $t \in [0,1]$

From the boundary condition $w'(0) = 0$ and $w(t) \geq 0$ for $t \in [0,1]$ and also $w(0) \geq 0$. Hence $\phi_1(0) \geq \phi_2(0)$, $\phi_1'(0) = \phi_2'(0) = 0$ and

$$w''(0) = f(0, \phi_2(0), 0) - f(0, \phi_1(0), 0) \geq 0$$

so there exist $\varepsilon > 0$ such that if $t \in [0, \varepsilon)$, then $w'(t)$ is nondecreasing. Hence $w'(t) \geq 0$ and $\phi_1'(t) \geq \phi_2'(t)$ for $t \in [0, \varepsilon)$. Also for $t \in [0, \varepsilon)$ $w(t)$ is nondecreasing and $w(t) \geq 0$ that is, $\phi_1(t) \geq \phi_2(t)$ for $t \in [0, \varepsilon)$. But for $t \in [0, \varepsilon)$

$$\phi_1(t) = \int_t^1 \frac{1}{p(u)} \int_0^u p(s) f(s, \phi_1(s), p(s)\phi_1'(s))\, ds\, du$$

$$\leq \int_t^1 \frac{1}{p(u)} \int_0^u p(s) f(s, \phi_2(s), p(s)\phi_2'(s))\, ds\, du = \phi_2(t)$$

This is a contradiction. Hence $\phi_1(t) = \phi_2(t)$ and $w(t) = 0$ for $t \in [0, \varepsilon)$. Because $w(0) = 0$ and $w(1) = 0$, from the Rolle's theorem there must be $t_0 \in (0,1)$ such that $w'(t_0) = 0$. So $w(t_0) = \phi_1(t_0) - \phi_2(t_0) \geq 0$ and $w'(t_0) = \phi_1'(t_0) - \phi_2'(t_0) = 0$.

$$w''(t_0) = f(t_0, \phi_2(t_0), p(t_0)\phi'_2(t_0)) - f(t_0, \phi_1(t_0), p(t_0)\phi'_1(t_0)) \geq 0$$

So there exist $\varepsilon > 0$ such that if $t \in (t_0, t_0 + \varepsilon)$ then $w'(t)$ is nondecreasing. Hence $w'(t) \geq 0$ and $\phi_1'(t) \geq \phi_2'(t)$ for $t \in (t_0, t_0 + \varepsilon)$. Also for $t \in (t_0, t_0 + \varepsilon)$, $w(t) \geq 0$ that is $\phi_1(t) \geq \phi_2(t)$. But for $t \in (t_0, t_0 + \varepsilon)$

$$\phi_1(t) = \int_t^1 \frac{1}{p(u)} \int_0^u p(s) f\big(s, \phi_1(s), p(s)\phi_1'(s)\big) ds\, du$$

$$\leq \int_t^1 \frac{1}{p(u)} \int_0^u p(s) f\big(s, \phi_2(s), p(s)\phi_2'(s)\big) ds\, du = \phi_2(t)$$

This is a contradiction. Hence $\phi_1(t) = \phi_2(t)$ and $w(t) = 0$ for $t \in [0,1]$.

## CHAPTER FOUR

# APPLICATION TO THE THEORY - A NONLINEAR MODEL OF HEAT CONDUCTION IN THE HUMAN HEAD

### 4.0 Introduction

The skin is an organ which covers the entire human body. This organ has many functions and one of them is to maintain the deep-body temperature at a constant level for a wide range of environmental conditions and metabolic rates. But there are certain difficulties in making measurements of skin temperature. It is also very difficult to measure heat flux at positions below the surface of the skin.

The distribution of the temperature in the human head in relation to several environment temperatures was measured and calculated by Thron [49] (1956). This mathematical model for temperature distribution in the human head can be represented by the following boundary value problem which is formulated by the differential equation of heat conduction

$$\frac{d^2T}{dr^2}+\frac{2}{r}\frac{dT}{dr}+\frac{Q}{k}=0 \tag{4.1}$$

and boundary conditions

$$\lim_{r\to 0^+}\frac{dT}{dr}=0, \qquad -k\left(\frac{dT}{dr}\right)_{r=R}=E(T_H-T_A)$$

where $T = T(r)$ is the temperature distribution in the head, r is the radial distance from the origin, E is environment cooling constant, $T_A$ is environment temperature, $T_H$ is periphery temperature, Q is the heat production per unit volume, k is the thermal conductivity inside the head. In this model Q and k are assumed to be positive constants. R is the radius of the head.

In this model, there is no singularity and the differential equation can be solved easily. So Thron obtained the solution of the above boundary value problem as

$$T(r) = T_A + \frac{QR^2}{6k}\left[1 + \frac{2k}{ER} - \left(\frac{r}{R}\right)^2\right] \tag{4.2}$$

He also calculated the distribution of the temperature assuming additional heat sources while the blood cools at the periphery by the differential equation (4.1) with

$$Q = Q_0 + Q_B \tag{4.3}$$

where $Q_B = Vs(T_1 - T)$, $Q_0$ is the heat production of tissue, V is volume of the flow of blood in unit volume and unit time, $T_1$ is the deep temperature of the human head and $s = 0.9\frac{\text{cal}}{^0\text{C cm}^3}$ . Hence differential equation (4.1) becomes

$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} - \frac{Vs}{k}T = -\frac{Q_0 + VsT_1}{k}$$

and Thorn also used the boundary conditions

$$\lim_{r\to 0^+}\frac{dT}{dr} = 0, \qquad -k\left(\frac{dT}{dr}\right)_{r=R} = E(T_H - T_A)$$

The numerical results of these two models are shown in the following figure

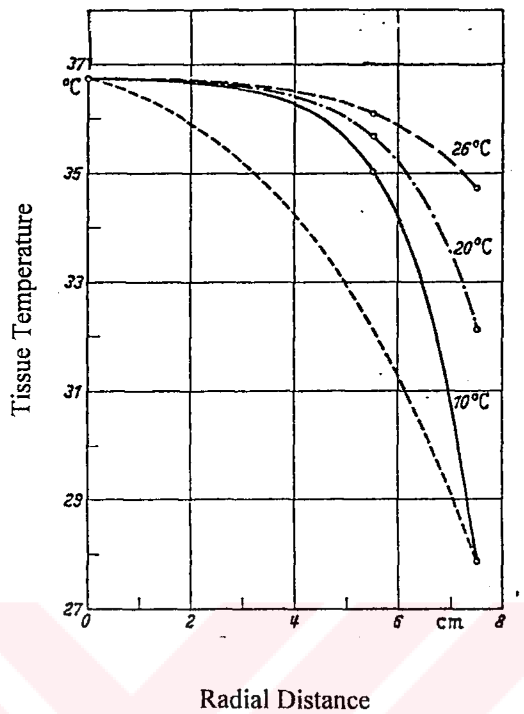


Figure 4.1 Temperature distribution given by Thron [49]. ------- obtained by (4.2) and next graphicsobtained by the solution of (4.1) with (4.3).

In the following figure we also give the experimental measurement results of Thron [49]

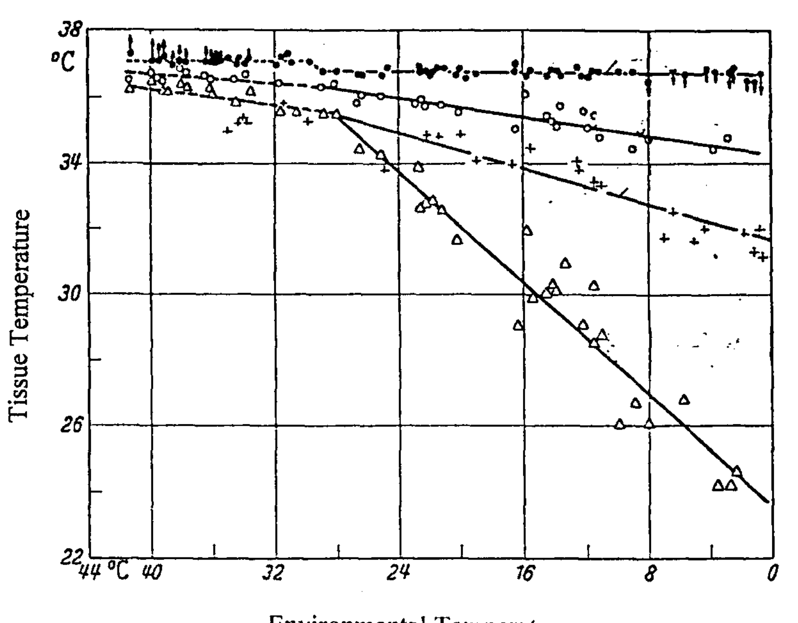


Figure 4.2 The distribution of the temperature in relation to several environment temperature was measured in the human head by Thron [49]

In figure (4.2), + is the temperature of the middle of forehead, Δ is the temperature of the outside of cheek, o is the temperature of the inside of cheek and • is the temperature of the sublingual. In the environmental temperature under the 28 °C, Thron presented the interpolation polynomial of the periphery temperatures of forehead, inside of cheek and outside of cheek.

For forehead $T_H = 33.03 + 0.14( T_A - 10 )$

For inside of cheek $T_H = 35.5 + 0.064( T_A - 16.75 )$

For outside of cheek $T_H = 30.1 + 0.42( T_A - 15.22 )$

Richardson and Whitelaw (1968) [44] calculated and measured the temperature, the heat conduction on a model and on human skin as a function of the surface temperature of the skin. They investigated whether the heat conduction is independent of the surface temperature of skin or not.

Colin, Boutelier and Houndas (1966) [16] discovered that the heat conduction is related to the temperature of the environment. They set out from the assumption, however, that the heat is only produced in the center of the body.

Flesch (1975) [22] calculated the temperature distribution by the differential equation (4.1) assuming a heat generation rate which was an explicit function of the radial distance from the center and an implicit function of the environment temperature. For the calculation of the temperature distribution in the human head for temperature of the environment of 26 °C, he assumed $Q = q_1 r^2$ where $q_1$ is positive constant and r is radial coordinate of the sphere. The solution of differential equation (4.1) is

$$T(r) = \text{const}_1 + \text{const}_2 \frac{1}{r} - \frac{1}{20}\frac{q_l}{k} r^4$$

For the condition $r = 0$ T is only finite if $\text{const}_1 = 0$. It follows:

$$T(r) = \text{const} - \frac{1}{20}\frac{q_1}{k}r^4$$

For the calculation of the temperature distribution in the human head for the temperature of the environment of 20 and 10 °C one sets $Q = q_2 r^3$ where $q_2$ is positive constant. The solution of differential equation is

$$T(r) = \text{const} - \frac{1}{30}\frac{q_2}{k}r^5$$

Gray (1980) [28] calculated the temperature distribution by the differential equation (4.1) with boundary conditions

$$T(0) \text{ is finite,} \qquad -k\left(\frac{dT}{dr}\right)_{r=R} = \beta(T - T_A)$$

where $\beta$ is heat exchange coefficient from the head to the surrounding medium and $T_A$ is the environment temperature.

His main assumption is $Q = \alpha - N\,T(r)$. Where $\alpha$ and N are constants such that N is large subject to $Q > 0$. Then the differential equation becomes

$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} + \frac{\alpha - NT}{k} = 0 \qquad 0 < r < R$$

which is easily solved to give the general solution

$$T(r) = \frac{A}{r}e^{\sqrt{(\frac{N}{k})}r} + \frac{B}{r}e^{-\sqrt{(\frac{N}{k})}r} + \frac{\alpha}{N}$$

From the boundary condition B = - A and

$$A = \frac{\beta(\alpha - NT_A)}{2N\left(\frac{k}{R^2} - \frac{\beta}{R}\right)\sinh\sqrt{\frac{N}{k}}R - 2\sqrt{N^3k}\cosh\frac{\sqrt{\frac{NR}{k}}}{R}}$$

Anderson and Arthurs (1981) [1] calculated the temperature distribution by the nonlinear boundary value problem assuming $Q = \alpha e^{-\frac{NT}{\alpha}}$. Then differential equation (4.1) takes the following form

$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} + \frac{\alpha e^{-\frac{NT}{\alpha}}}{k} = 0 \qquad 0 < r < R$$

$$T(0) \text{ is finite,} \qquad -k\left(\frac{dT}{dr}\right)_{r=R} = \beta(T - T_A)$$

where $\beta$ is heat exchange coefficient from the head to the surrounding medium, $T_A$ is the environment temperature and $\alpha, N > 0$ constant. They obtained numerical results of this model [1].

### 4.1 A Nonlinear Model of Heat Conduction in the Human Head

In the previous works done on mathematical modeling of heat conduction in the human head we have encountered with linear or nonlinear differential equations which have singularities in the independent variable and the solutions of these second order regular or singular differential equations could be obtained without any difficulties. In all these models thermal conductivity k were assumed to be constant. But it is well known that if k depends on the temperature distribution, the model will be more realistic and it can be considered as a function of T. So in this study, we assume $k = k(T) = k_0(T - T_H)^{\frac{1}{3}}$. Under this assumption we develop a mathematical model of the heat conduction in the Human head as follow.

### 4.2 The Model of Heat Conduction in the Human Head

In application of the conservation law, we first need to identify the control volume, a region of space bounded by a control surface through which energy and matter may pass

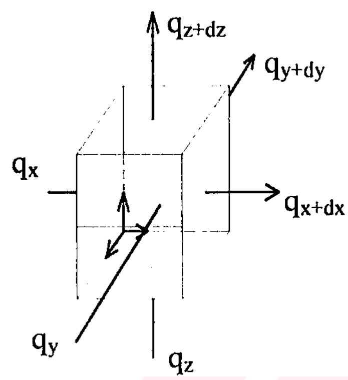


Figure 4.3 Differential Control Volume, dxdydz, for Heat Conduction Analysis

From the conservation law

$$E_{in} - E_{out} + E_g = E$$

where $E_{in} = q_x dydz + q_y dxdz + q_z dxdy$, $E_{out} = q_{x+dx} dydz + q_{y+dy} dxdz + q_{z+dz} dxdy$

$E_g = q dxdydz$ $E = \rho c \dfrac{\partial T}{\partial t} dxdydz$, $\rho$ is density and c is specific heat of the tissue.

$$\rho c \frac{\partial T}{\partial t} dxdydz = q_x dydz + q_y dxdz + q_z dxdy$$
$$- (q_{x+dx} dydz + q_{y+dy} dxdz + q_{z+dz} dxdy) + q dxdydz$$

Hence

$$\rho c \frac{\partial T}{\partial t} dxdydz = q_x dydz + q_y dxdz + q_z dxdy - q_{x+dx} dydz - \frac{\partial q_x}{\partial x} dxdydz$$
$$- q_{y+dy} dxdz - \frac{\partial q_y}{\partial y} dxdydz - q_z dxdy - \frac{\partial q_z}{\partial z} dxdydz + q dxdydz$$

and we obtain

$$\rho c\frac{\partial T}{\partial t}=-\frac{\partial q_x}{\partial x}-\frac{\partial q_y}{\partial y}-\frac{\partial q_z}{\partial z}+Q$$

where from the Fourier's law of heat conduction $q_x=-k\frac{\partial T}{\partial x}$, $q_y=-k\frac{\partial T}{\partial y}$, $q_z=-k\frac{\partial T}{\partial z}$, k is thermal conductivity $(\frac{cal}{cm\ sec\ ^{0}C}10^{-3})$, T is the temperature (°C) and Q is the heat production $(\frac{cal}{cm^3\ sec}10^{-3})$.

If we take $k=k_0(T-T_H)^{\frac{1}{3}}$ then

$$\frac{\partial q_x}{\partial x}=-\frac{k_0}{3}(T-T_H)^{-\frac{2}{3}}\frac{\partial T}{\partial x}-k_0(T-T_H)^{\frac{1}{3}}\frac{\partial^2 T}{\partial x^2}$$

$$\frac{\partial q_y}{\partial y}=-\frac{k_0}{3}(T-T_H)^{-\frac{2}{3}}\frac{\partial T}{\partial y}-k_0(T-T_H)^{\frac{1}{3}}\frac{\partial^2 T}{\partial y^2}$$

$$\frac{\partial q_z}{\partial z}=-\frac{k_0}{3}(T-T_H)^{-\frac{2}{3}}\frac{\partial T}{\partial z}-k_0(T-T_H)^{\frac{1}{3}}\frac{\partial^2 T}{\partial z^2}$$

where $T_H$ is the temperature of periphery and $k_0$ is a positive constant. Hence we obtain

$$\rho c\frac{\partial T}{\partial t}=k_0(T-T_H)^{\frac{1}{3}}(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2})+\frac{1}{3}k_0(T-T_H)^{-\frac{2}{3}}(\frac{\partial T}{\partial x}+\frac{\partial T}{\partial y}+\frac{\partial T}{\partial z})+Q$$

which is the usual equation of the heat conduction.

In the steady state heat conduction case,

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{1}{3}\frac{1}{(T-T_H)}(\frac{\partial T}{\partial x}+\frac{\partial T}{\partial y}+\frac{\partial T}{\partial z})+\frac{Q}{k_0(T-T_H)^{\frac{1}{3}}}=0$$

From the radial symmetry, above equation becomes

$$\frac{d^2T}{dr^2}+\frac{2}{r}\frac{dT}{dr}+\frac{1}{3}\frac{1}{(T-T_H)}\left(\frac{dT}{dr}\right)^2+\frac{Q}{k_0(T-T_H)^{\frac{1}{3}}}=0$$

Since heat generation is a function of temperature as Q = Q(T), we take $Q = q_1(37 - T)$ where $q_1$ is a positive constant, then

$$\frac{d^2T}{dr^2}+\frac{2}{r}\frac{dT}{dr}+\frac{1}{3}\frac{1}{(T-T_H)}\left(\frac{dT}{dr}\right)^2+\frac{q_1(37-T)}{k_0(T-T_H)^{\frac{1}{3}}}=0$$

Boundary conditions are given as

$$\left.\frac{dT}{dr}\right|_{r=0}=0, \qquad T(R)=T_H$$

where R is radial radius, then we obtain the following singular boundary value problem

$$\begin{cases} \dfrac{d^2T}{dr^2}+\dfrac{2}{r}\dfrac{dT}{dr}+\dfrac{1}{3}\dfrac{1}{(T-T_H)}\left(\dfrac{dT}{dr}\right)^2+\dfrac{q_1(37-T)}{k_0(T-T_H)^{\frac{1}{3}}}=0 \qquad 0<r<R \\ \\ \left.\dfrac{dT}{dr}\right|_{r=0}=0, \qquad T(R)=T_H \end{cases}$$

This is the boundary value problem of the heat conduction which human head is assumed as a sphere. R is radius of the sphere.

By transformation $y = T - T_H$,

$$\frac{dT}{dr}=\frac{dy}{dr}, \qquad \frac{d^2T}{dr^2}=\frac{d^2y}{dr^2}$$

we obtain

$$\begin{cases} \dfrac{d^2y}{dr^2}+\dfrac{2}{r}\dfrac{dy}{dr}+\dfrac{1}{3}\dfrac{1}{y}\left(\dfrac{dy}{dr}\right)^2+\dfrac{q_1(37-y-T_H)}{k_0 y^{\frac{1}{3}}}=0 \qquad 0<\mathrm{r}<\mathrm{R} \\ \\ \left.\dfrac{dy}{dr}\right|_{r=0}=0, \qquad \mathrm{y(R)}=0 \end{cases} \tag{4.4}$$

By $t = r/R$ then $r = Rt$, $0 < t < 1$ and

$$\frac{dy}{dr}=\frac{dy}{dt}\frac{dt}{dr}=\frac{1}{R}\frac{dy}{dt}, \qquad \frac{d^2y}{dr^2}=\frac{1}{R^2}\frac{d^2y}{dt^2}$$

we have dimensionless form of the equation (4.4)

$$\begin{cases} \dfrac{d^2y}{dt^2}+\dfrac{2}{t}\dfrac{dy}{dt}+\dfrac{1}{3}\dfrac{1}{y}\left(\dfrac{dy}{dt}\right)^2+\dfrac{q_1(37-y-T_H)R^2}{k_0 y^{\frac{1}{3}}}=0 \qquad 0<\mathrm{t}<1 \\ \\ \left.\dfrac{dy}{dt}\right|_{t=0}=0, \qquad \mathrm{y(1)}=0 \end{cases} \tag{4.5}$$

In (2.8) if we take $p(t) = t^2$ and

$$f(t,y,py')=\frac{1}{3}\frac{1}{y}\left(\frac{dy}{dt}\right)^2+\frac{q_1(37-y-T_H)R^2}{k_0 y^{\frac{1}{3}}}$$

then all properties of $p(t)$ and $f(t, y, py')$ in the theorems in chapter 3 are satisfied for (4.5). Hence the solution of the above singular boundary value problem exists and it is unique. Since we know that the solution exists and it is unique, a finite difference solution can be obtained. In chapter 5 finite difference solution is obtained and the numerical results have been compared with some previous results.

# CHAPTER FIVE

# NUMERICAL COMPUTATION

## 5.0 Introduction

In chapter four we developed a nonlinear mathematical model for the heat conduction in the human head which is given by (4.5). In this chapter this model will be solved numerically.

## 5.1 Finite Difference Formulation of the Problem

We will solve the singular boundary value problem, distribution of the temperature in the human head,

$$\begin{cases} \dfrac{d^2y}{dt^2}+\dfrac{2}{t}\dfrac{dy}{dt}+\dfrac{1}{3y}\left(\dfrac{dy}{dt}\right)^2+\dfrac{q_1(37-y-T_H)R^2}{k_0 y^{\frac{1}{3}}}=0 \qquad 0<\mathrm{t}<1 \\ \\ y'(0)=y(1)=0 \end{cases} \tag{5.1}$$

or

$$\begin{cases} \dfrac{1}{t^2}(t^2y')'+\dfrac{1}{3}\dfrac{(y')^2}{y}+\dfrac{q_1(37-y-T_H)R^2}{k_0 y^{\frac{1}{3}}}=0 \qquad 0<\mathrm{t}<1 \\ \\ \mathrm{y}'(0)=\mathrm{y}(1)=0 \end{cases}$$

by the method of finite difference.

If we divided the interval (0,1) into subinterval n with h = 1/n then by the central differences

$$y'' \cong \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \quad \text{and} \quad y' \cong \frac{y_{i+1} - y_{i-1}}{2h}$$

Then equation (5.1) reduces to

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \frac{2}{t_i}\frac{y_{i+1} - y_{i-1}}{2h} + \frac{1}{3}\frac{(y_{i+1} - y_{i-1})^2}{4h^2 y_i} + \frac{q_1(37 - y_i - T_H)R^2}{k_0 y_i^{\frac{1}{3}}} = 0$$

where $t_i = ih$. From the boundary condition

$$y'(0) = \frac{y_1 - y_{-1}}{2h} = 0 \quad \text{then} \quad y_{-1} = y_1$$

Hence for i = 0

$$2y_1 + \frac{(37 - y_0 - T_H)R^2 h^2 q_1}{k_0 y_0^{\frac{1}{3}}} - 2y_0 = 0$$

and for i = 1, 2, 3, ..., n-1

$$(1 + \frac{1}{i})y_{i+1} + \frac{(y_{i+1} - y_{i-1})^2}{12 y_i} + \frac{(37 - y_i - T_H)R^2 h^2 q_1}{k_0 y_i^{\frac{1}{3}}} - 2y_i + (1 - \frac{1}{i})y_{i-1} = 0$$

where $q_1 = 0.000002T_H^2$, for forehead $T_H = 33.03 + 0.14(T_A - 10)$, $k_0 = \dfrac{0.00009T_H}{(37 - T_H)^{\frac{1}{3}}}$

n = 100 and $T_A$ is environmental temperature.

This is a nonlinear system of equations. We can use Newton Raphson's method to solve the nonlinear system of equations. If we solve this system numerically then we obtain the following temperature distribution in forehead of human head in the various environment temperature.

## 5.2 Numerical Results and Graphs

Table 5.1 Temperature distribution in the tissue of human head relative to radial distance from the origin for various temperature of environment

| Envir. / r(cm) | 0 °C | 5 °C | 10 °C | 15 °C | 20 °C | 25 °C | 26 °C | 28 °C |
|---|---|---|---|---|---|---|---|---|
| 0.000 | 36.9111 | 36.9270 | 36.9414 | 36.9544 | 36.9662 | 36.9767 | 36.9787 | 36.9825 |
| 0.750 | 36.9050 | 36.9219 | 36.9372 | 36.9511 | 36.9636 | 36.9749 | 36.9771 | 36.9812 |
| 1.500 | 36.8854 | 36.9055 | 36.9237 | 36.9404 | 36.9555 | 36.9692 | 36.9718 | 36.9768 |
| 2.250 | 36.8474 | 36.8735 | 36.8974 | 36.9193 | 36.9394 | 36.9579 | 36.9614 | 36.9682 |
| 3.000 | 36.7810 | 36.8173 | 36.8509 | 36.8820 | 36.9109 | 36.9377 | 36.9428 | 36.9528 |
| 3.750 | 36.6681 | 36.7212 | 36.7710 | 36.8177 | 36.8614 | 36.9025 | 36.9104 | 36.9259 |
| 4.500 | 36.4758 | 36.5567 | 36.6335 | 36.7063 | 36.7753 | 36.8408 | 36.8535 | 36.8785 |
| 5.250 | 36.1435 | 36.2708 | 36.3931 | 36.5104 | 36.6231 | 36.7313 | 36.7524 | 36.7941 |
| 6.000 | 35.5524 | 35.7595 | 35.9606 | 36.1560 | 36.3460 | 36.5308 | 36.5671 | 36.6392 |
| 6.750 | 34.4375 | 34.7890 | 35.1351 | 35.4757 | 35.8111 | 36.1414 | 36.2069 | 36.3372 |
| 7.500 | 31.6301 | 32.3301 | 33.0301 | 33.7301 | 34.4301 | 35.1301 | 35.2701 | 35.5501 |

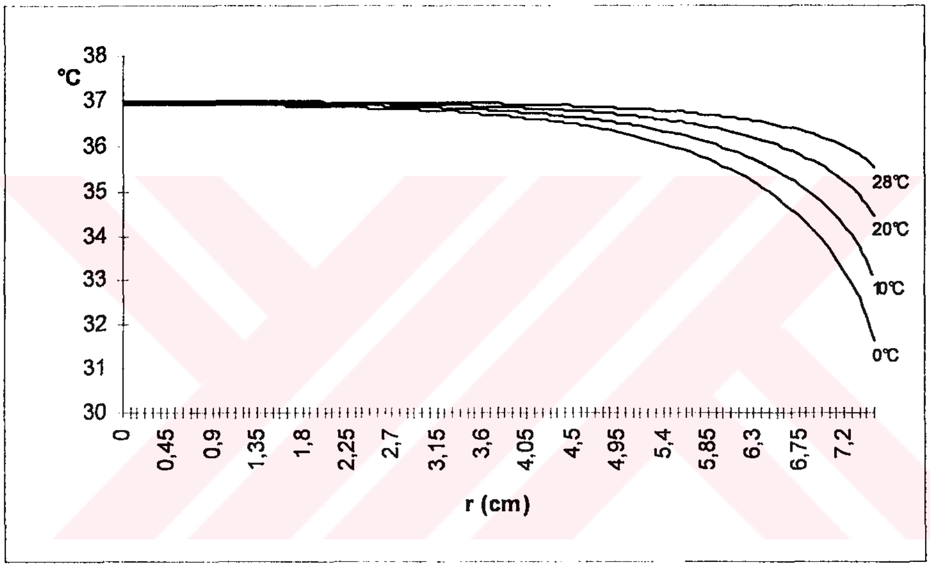


Figure 5.1 Temperature distribution in the tissue of human head relative to radial distance from the origin for several environmental temperature.

Table 5.2 Heat generation of tissue of human head relative to radial distance from the origin for various temperature of environment

| Envir. / r(cm) | 0 °C | 5 °C | 10 °C | 15 °C | 20 °C | 25 °C | 26 °C | 28 °C |
|---|---|---|---|---|---|---|---|---|
| 0.000 | 0.000178 | 0.000153 | 0.000128 | 0.000104 | 0.000080 | 0.000057 | 0.000053 | 0.000044 |
| 0.750 | 0.000190 | 0.000163 | 0.000137 | 0.000111 | 0.000086 | 0.000062 | 0.000057 | 0.000048 |
| 1.500 | 0.000229 | 0.000198 | 0.000166 | 0.000136 | 0.000106 | 0.000076 | 0.000070 | 0.000059 |
| 2.250 | 0.000305 | 0.000265 | 0.000224 | 0.000184 | 0.000144 | 0.000104 | 0.000096 | 0.000080 |
| 3.000 | 0.000438 | 0.000382 | 0.000325 | 0.000268 | 0.000211 | 0.000154 | 0.000142 | 0.000119 |
| 3.750 | 0.000664 | 0.000583 | 0.000500 | 0.000415 | 0.000329 | 0.000241 | 0.000223 | 0.000187 |
| 4.500 | 0.001049 | 0.000927 | 0.000800 | 0.000668 | 0.000533 | 0.000393 | 0.000364 | 0.000307 |
| 5.250 | 0.001714 | 0.001524 | 0.001324 | 0.001114 | 0.000894 | 0.000663 | 0.000616 | 0.000520 |
| 6.000 | 0.002896 | 0.002593 | 0.002268 | 0.001920 | 0.001550 | 0.001158 | 0.001077 | 0.000912 |
| 6.750 | 0.005127 | 0.004622 | 0.004069 | 0.003468 | 0.002819 | 0.002119 | 0.001973 | 0.001675 |
| 7.500 | 0.010745 | 0.009762 | 0.008662 | 0.007440 | 0.006093 | 0.004615 | 0.004304 | 0.003665 |

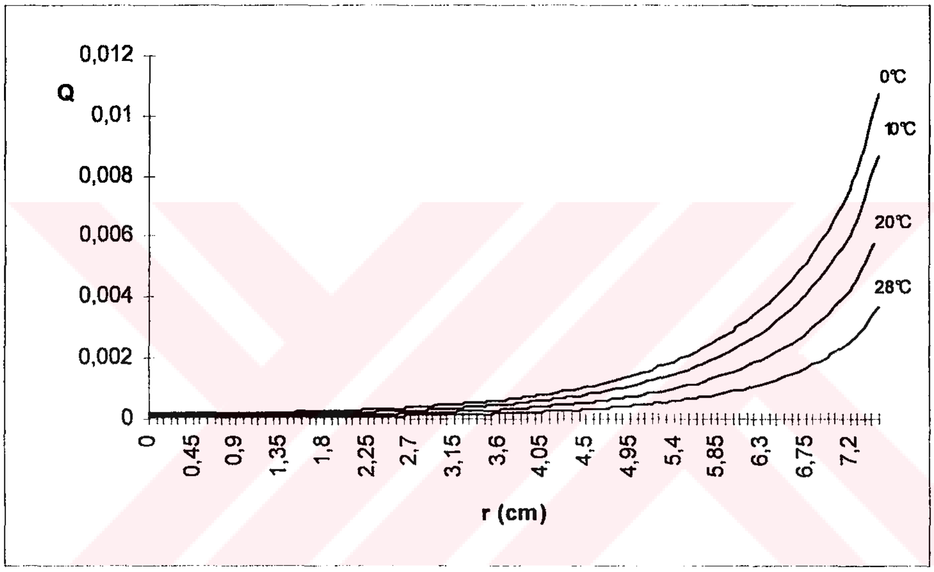


Figure 5.2 Heat generation of tissue of the human head relative to radial distance from the origin for several environmental temperature.

Table 5.3 Thermal conductivity of tissue of human head relative to radial distance from the origin for various temperature of environment

| Envir. / r(cm) | 0 °C | 5 °C | 10 °C | 15 °C | 20 °C | 25 °C | 26 °C | 28 °C |
|---|---|---|---|---|---|---|---|---|
| 0.000 | 0.002831 | 0.002894 | 0.002958 | 0.003022 | 0.003085 | 0.003149 | 0.003161 | 0.003187 |
| 0.750 | 0.002830 | 0.002893 | 0.002957 | 0.003020 | 0.003084 | 0.003148 | 0.003160 | 0.003186 |
| 1.500 | 0.002826 | 0.002890 | 0.002954 | 0.003017 | 0.003081 | 0.003144 | 0.003157 | 0.003182 |
| 2.250 | 0.002819 | 0.002883 | 0.002947 | 0.003011 | 0.003074 | 0.003138 | 0.003151 | 0.003176 |
| 3.000 | 0.002807 | 0.002871 | 0.002935 | 0.002999 | 0.003062 | 0.003126 | 0.003139 | 0.003164 |
| 3.750 | 0.002787 | 0.002851 | 0.002914 | 0.002978 | 0.003042 | 0.003106 | 0.003118 | 0.003144 |
| 4.500 | 0.002751 | 0.002815 | 0.002878 | 0.002942 | 0.003006 | 0.003069 | 0.003082 | 0.003108 |
| 5.250 | 0.002687 | 0.002750 | 0.002813 | 0.002876 | 0.002939 | 0.003002 | 0.003015 | 0.003040 |
| 6.000 | 0.002564 | 0.002625 | 0.002687 | 0.002748 | 0.002810 | 0.002871 | 0.002884 | 0.002908 |
| 6.750 | 0.002293 | 0.002350 | 0.002406 | 0.002463 | 0.002519 | 0.002576 | 0.002587 | 0.002610 |
| 7.500 | 0.000075 | 0.000081 | 0.000087 | 0.000095 | 0.000105 | 0.000119 | 0.000123 | 0.000131 |

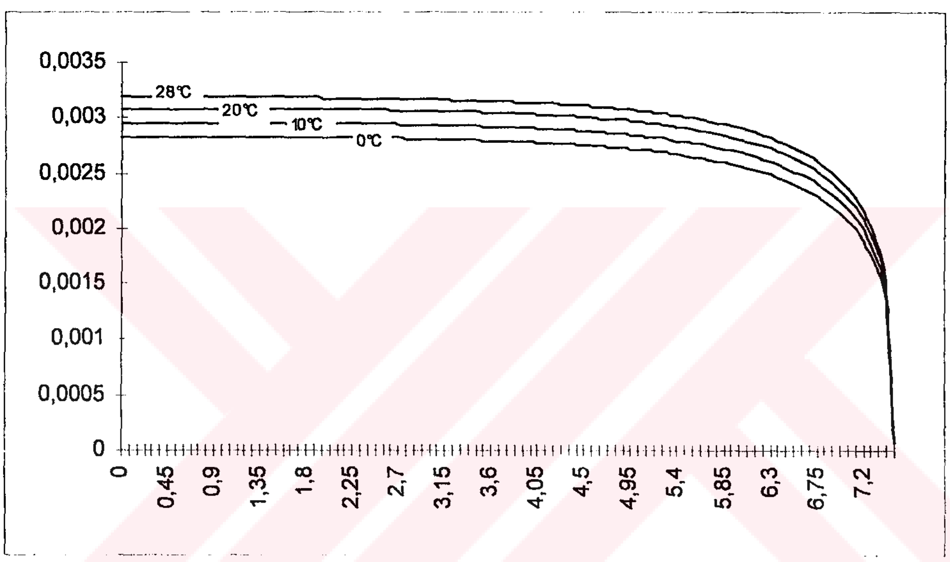


Figure 5.3 Thermal conductivity of tissue of the human head relative to radial distance from the origin for several environmental temperature.

# CONCLUSIONS

In this study existence and uniqueness results have been established for non-negative solutions to singular nonlinear two-point boundary value problems formulated by (2.8). We have followed the idea suggested in [43] to prove the existence theorem and have followed the way given in [21] for uniqueness of the solution.

As an application to this nonlinear problem we have developed a mathematical model for the heat conduction in the human head. Temperature distribution in the human head have been considered by some authors [1, 22, 28, 49]. In their models thermal conductivity k is assumed to be a constant. But experiments have shown that k is not a constant but depends on temperature T. So to be more realistic we have considered k as a function of T and developed a nonlinear singular differential equation with suitable boundary conditions of the type (2.8)

Numerical solutions have been obtained by the finite difference method and are represented graphically.

These results have been well in agreement with experience. It has been found that the generation of head is greater in the periphery of the human head then in the center and the generation of heat increases if the environment temperature decreases. Experiments have also shown that the temperature at the center of the head is not effected by the environment temperature. Our numerical solutions have given the same results.

# REFERENCES


1 ANDERSON, N. & ARTHURS, A.M. (1981). Complementary Exremum Principles for a Nonlinear Model of Heat Conduction in the Human Head, Bulletin of Math. Biol. Vol. 43 No. 3, pp. 341-346,

2 BAXLEY, J.V. (1984). Nonlinear Second Order Boundary Value Problems on $[0,\infty)$, in Qualitative Properties of Differential Equations, W. Allegratto and G. J. Butler, eds., Proce. Edmonton Conference, University of Alberta Press, Edmonton, Alberta, Canada, pp. 50-58

3 BAXLEY, J.V. (1988). A Singular Nonlinear Boundary Value Problem: Membrane Response of a Spherical Cap. Siam J. Appl. Math., 48 pp. 497-505

4 BAXLEY, J.V. (1990). Existence Theorems for Second Order Nonlinear Boundary Value Problems. J. Differential Equations, 85, pp. 125-150

5 BAXLEY, J.V. (1991). Some Singular Nonlinear Boundary Value Problems. Siam J. Math. Anal. Vol. 22, pp. 463-479

6 BAXLEY, J. V. & GERSDORFF, G. S. (1995). Singular Reaction - Diffusion Boundary Value Problems. J. Differential Equations, 115, pp. 441-457

7 BERESTYCKI, H., LIONS, H.P. & PELETIER, L.A. (1981). An Approach to Existence of Positive Solutions for Semilinear Problems in $R^N$. Indiana Univ. Math. J. Vol. 30, No.1, pp. 141-157

8 BIRKHOFF, G. & ROTA, G. (1962). Ordinary Differential Equations. Ginn and Company 1962

9 BOBISUD, L.E. (1990). Existence of Solution of Nonlinear Singular Boundary Value Problems. Applicable Anal. Vol. 35, pp. 43-57

10 BOBISUD, L.E. (1990) Asymptotic Dead Cores for Reaction - Diffusion Equations. J. Math. Appl. 147, pp. 249-262

11 BOBISUD, L.E. & LEE, Y.S. (1990). Existence for a Class of Nonlinear Singular Boundary Value Problems. Applicable Anal. Vol. 38, pp. 45-67

12 BOBISUD, L.E. & O'Regan, D. (1994). Positive Solutions for a Class of Nonlinear Singular Boundary Value Problems at Resonance. J. Math. Anal. Appl. 184, pp. 263- 284

13 CALLEGARY, A. & NACHMAN, A. (1978). Some Singular Nonlinear Differential Equations Arising in Boundary Layer Theory. J. Math. Appl. 64, pp. 95-105

14 CASH, J.R. & SILVA, H.H.M. (1993). On the Numerical Solution of a Class of Singular two Point Boundary Value Problems. J. Comput. App. Math. 45, pp. 91 - 102

15 COEWONO, E., VARJRAVELU, K. & MOHAPATRA, R.N. (1991). Existence and Nonuniqueness of Solutions of a Singular Nonlinear Boundary Layer Problem. J. Math. Anal. Appl. 159, pp. 251-270

16 COLIN, J., BOUTELIER, C. & HOUDAS, Y. (1966). Variations de la conductance thermique des tissus périphériques de l'Homme en fonction de la température ambiante. J. Physiol. Paris 58, p. 500.

17 DIAZ, J.I., MOREL, J.M. & OSWALD, L. (1987). An Elliptic Equation with Singular Nonlinearity. Comm. Partial Differential Equations, 12, pp. 1333-1344

18 DUNNINGER, D.R. & KURTZ, J.C. (1986). A priori Bounds and Existence of Positive Solutions for Singular Nonlinear Boundary Value Problems. Siam J. Math. Anal. Vol.17, No. 3, pp. 595-609

19 ERBE, L. & SCHMITT, K. (1988). On Radial Solutions of Some Semilinear Elliptic Equations. Differential and Integral Equations, 1, pp. 71-78

20 EUGENE, C. & GARTLAND, J.R. (1986). High Order Finite Difference Methods for Two Point Boundary Value Problems with Singular Sources. J. Comput. Physics 64, pp. 473-489

21 FINK, A. M., GATICA, J. A., HERNANDEZ, G. E & WALTMAN, P. (1991). Approximation of Solutions of Singular Second Order Boundary Value Problems. Siam J. Math. Anal. Vol. 22, No. 2, pp. 440-462

22 FLESCH, U. (1975). The Distribution of Heat Sources in the Human Head: A Theoretical Consideration. J. Theor. Biol., 54, pp. 285-287

23 GATICA, J. A., OLIKER, V. & WALTMAN, P. (1989). Singular Nonlinear Boundary Value Problems for Second Order Ordinary Differential Equations. J. Differential Equations 79, pp. 62-78

24 GATICA, J. A., HERNANDEZ, G. E. & WALTMAN, P. (1990). Radially Symmetric Solutions of Class of Singular Elliptic Equations. Proc. Edinburgh Math. Soc., 33, pp. 169-180

25 GRANAS, A., GUENTHER, R. B., LEE, J. & O'REGAN, D. (1986). Boundary Value Problems on Infinite Intervals and Semiconductor Devices. J. Math. Anal. Appl., 116, pp. 335-348

26 GRAY, B. F., GRAY, P. & KIRWAN, N.A. (1972). Applications of Combustion Theory to Biological Systems. Comb. Flame 18, pp. 439-459

27 GRAY, P. & SHERRINGTON, M. E. (1977). Self-heating, Chemical Kinetics, and Spontaneously Unstable Systems. Gas Kinetics and Energy Transfer Vol. 2, pp. 331-383

28 GRAY, B. F. (1980). The Distribution of Heat Sources in the Human Head - Theoretical Considerations. J. Theor. Biol. 82, pp. 473-476

29 HILDEBRAN, F.B. (1976). Advanced Calculus for Applications. Prentice Hall, Inc. Englewood Cliffs New Jersey

30 LIN, S. H. (1976). Oxygen Diffusion in a Spherical Cell with Nonlinear Oxygen Uptake Kinetics. J. Theor. Biol. 60, pp. 449-457

31 LIN, S. S. & PAI, F. M. (1991). Existence and Multiplicity of Positive Radial Solutions for Semilinear Elliptic Equations in Annular Domains. Siam J. Math. Anal. Vol. 22, No. 6, pp. 1500-1515

32 LUNING, C. & PERRY, W. (1981). Positive Solutions of Negative Exponent Generalized Emden - Fowler Boundary Value Problems. Siam J. Appl. Math. 12, pp. 874-879

33 MAWHIN, J. & OMANA, W. (1989). Two Point Boundary Value Problems for Nonlinear Perturbations of Some Singular Linear Differential Equations at Resonance. Comment. Math. Univ. Caroline 30, 3, pp. 537-550

34 NACHMAN, A. & CALLEGARI, A. (1980). A Nonlinear Singular Boundary Value Problem in the Theory of Pseudoplastic Fluids. Siam J. Appl. Math. Vol. 38, pp. 275-281

35 O'REGAN, D. (1990). Some New Results for Second Order Boundary Value Problems. J. Math. Anal. Appl. 148, pp. 548-570

36 O'REGAN, D. (1991). Second and Higher Order Systems of Boundary Value Problems. J. Math. Anal. Appl. 156, pp. 120-149

37 O'REGAN, D. (1992). Some Existence Principles and Some General Results for Singular Nonlinear Two Point Boundary Value Problems. J. Math. Anal. Appl. Vol.166, No.1, pp. 24-40

38 O'REGAN, D. (1992). Existence Theorems for Certain Classes of Singular Boundary Value Problems. J. Math. Anal. Appl. Vol. 168, No. 2, pp. 523-539

39 O'REGAN, D. (1993). Nonlinear Systems of Two Point Boundary Value Problems. Nonlinear Anal., Theory, Math. Appl., Vol. 20, No. 8, pp. 979-991

40 O'REGAN, D. (1993). Compact Self Adjoint Operators and Existence of Solutions to Singular Nonlinear Boundary Value Problems in a Real Hilbert Space. Libertas Math., Vol. 13, pp. 47-68

41 O'REGAN, D. (1993). Singular Sturm Liouville Problems and Existence of Solutions to Singular Nonlinear Boundary Value Problems. Nonlinear Anal., Theory, Math. Appl., Vol. 20, No. 7, pp. 767-779

42 O'REGAN, D. (1993). Singular Nonlinear Differential Equations Utilitas Mathematica 44, pp. 165-176

43 O'REGAN, D. (1994). Theory of Singular Boundary Value Problems. World Scientific Singapore

44 RICHARDSON, P. D. & WHITELAW, J. H. (1968). Transient Heat Transfer in Human Skin. J. Franklin Inst. 286, pp. 169-181

45 ROSS, S.L. (1974). Differential Equations. Wiley, New York

46 SACHDEV, P. L. (1991). Nonlinear O.D.E. and Their Application. Marcel Dekker, Inc.

47 STAKGOLD, I. (1979). Greens Functions and Boundary Value Problems. Wiley, New York

48 TALIAFERRO, S. D. (1979). A Nonlinear Singular Boundary Value Problem. Nonlinear Anal. T. M. A., Vol. 3, No. 6, pp. 897-904

49 THRON, H. L. (1956). Der Temperaturgradient in der menschlichen Gesichtshaut bei Einwirkung verschiedener Umgebungstemperaturen. Pflügers Arch. geo. Physiol. 263, pp. 109-126,

50 TINEO, A. (1994). On a Class of Singular Boundary Value Problems Which Contains the Boundary Conditions $X'(0) = X(1) = 0$. J. Differential Equations 113, pp. 1-16

51 TINEO, A. (1995). A Comparison Theorem for Second Order O.D.E.s and Applications to Singular Problems. J. Differential Equations 116, pp. 16-30

52 WANG, J. (1995). Solvability of Singular Nonlinear Two Point Boundary Value Problems. Nonlinear Anal., T. M. A., Vol. 24, No. 4, pp. 555-561

53 WANG, J. & JIANG, J. (1993). The Existence of Positive Solutions to a Singular Nonlinear Boundary Value Problem. J. Math. Anal. Appl., 176, pp. 322-329

54 WONG, F. H. (1993). Existence of Positive Solutions of Singular Boundary Value Problems. Nonlinear Anal., T. M. A., Vol. 21, No. 5, pp. 397-406

55 ZENGQIN, Z. (1994). Uniqueness of Positive Solutions for Singular Nonlinear Second Order Boundary Value Problems. Nonlinear Anal., T. M. A., Vol. 23, No. 6, pp. 755-765

56 ZHANG, Y. (1993). Existence of Solutions of Kind of Singular Boundary Value Problem. Nonlinear Anal., T. M. A., Vol. 21, No. 2, pp. 153-159

57 ZHANG, Y. (1994). Positive Solutions of Singular Sublinear Emden - Fowler Boundary Value Problem. J. Math. Anal. Appl, 185, pp. 215-222