DEFORMATION CLASSES OF SINGULAR QUARTIC SURFACES

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Çisem Güneş Aktaş
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We certify that we have read this dissertation and that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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ABSTRACT

DEFORMATION CLASSES OF SINGULAR QUARTIC SURFACES

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We study complex spatial quartic surfaces with simple singularities and give their classification up to equisingular deformation. Simple quartics are $K3$-surfaces and as such they can be studied by means of the global Torelli theorem and the surjectivity of the period map combined with Nikulin's theory of discriminant forms. We reduce the classification problem to a certain arithmetical problem concerning lattice extensions. Then, based on Nikulin's existence criterion, we list all sets of simple singularities realized by non-special quartics; the result is stated in terms of perturbations of a few extremal sets. For each realizable set of singularities, we use Miranda–Morrison's theory to give a complete description of the connected components of the corresponding equisingular stratum.

Keywords: Complex quartic, $K3$-surface, simple singularity, singular quartic.

ÖZET

TEKİL KUARTİK YÜZEYLERİN DEFORMASYON SINIFLARI

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Basit tekillikleri olan kompleks uzay kuartik yüzeyleri çalıştık ve onların eştekil deformasyonlar altında sınıflandırmasını verdik. Basit kuartikler $K3$-yüzeyleridir ve bu sayede Nikulin'in diskriminant formları kuramı ile kombine edilmiş global Torelli kuramı ve periyot fonksiyonunun örtenliği kullanılarak çalışılabilir. Sınıflandırma problemini kafes genişletmelerine ilişkin belli bir aritmetik probleme indirgedik. Daha sonra, Nikulin'in varolma kriterine dayanarak, özel olmayan kuartikler tarafından gerçeklenen bütün basit tekillik kümelerini listeledik; sonuç bir kaç uç kümenin pertürbasyonları cinsinden ifade edilmiştir. Her gerçeklenir tekillik kümesine karşılık gelen eştekil tabakanın bağlantılı parçalarının tam bir tarifini vermek için Miranda–Morrison kuramını kullandık.

Anahtar sözcükler: Kompleks kuartik, $K3$-yüzeyi, basit tekillik, tekil kuartik.

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Chapter 1

Introduction

1.1 Principal results

Throughout this thesis, all algebraic varieties are over the field $\mathbb{C}$ of complex numbers. A *quartic* is a surface in $\mathbb{P}^3$ of degree 4. We confine ourselves to *simple* quartics only, *i.e.*, those with **A**–**D**–**E** type singularities (see [1]). Two such quartics are said to be *equisingular deformation equivalent* if they belong to the same deformation family in which the total Milnor number stays constant.

Four seems to be the last degree where one can hope to obtain a complete equisingular deformation classification. Quartics with a non-simple singular point (which are typically rational or ruled as abstract algebraic) have been treated by A. Degtyarev in [2, 3], where a complete classification is obtained and described in terms of lattice embeddings. An alternative description of some non-simple sets of singularities in terms of Dynkin diagrams is found in T. Urabe [4, 5]. On the other hand, *simple* quartics are $K3$-surfaces, and as such they can be studied by using the global Torelli theorem [6] and the surjectivity of the period map [7], combined with V. V. Nikulin's theory of discriminant forms [8]. This approach was used by Urabe [9, 10], who showed that the total Milnor number μ of a simple quartic does not exceed 19 and listed (in terms of perturbations of Dynkin graphs) all realizable sets of singularities with the total Milnor number $\mu \leqslant 17$. From a slightly different perspective, also worth mentioning is the study of simple *real* quartics up to *equivariant* equisingular deformation, see, e.g., the classification of nonsingular real quartics by V. Kharlamov [11] or the recent classification of the arrangements of the ten nodes of a real determinantal quartic by A. Degtyarev and I. Itenberg [12].

In this thesis, we start a systematic study of the equisingular stratification of the space of simple quartics. The principal difficulty here is the great number of strata (about 12 thousands); thus, as a first step, we confine ourselves to the so-called *non-special* quartics. The precise definition is rather technical, and we postpone it till §3.4. In a sense, the non-special quartics are an analogue of irreducible plane sextic curves admitting no dihedral coverings, *cf.* [13]; in a similar vein, we have the following geometric characterization, which is proved in §3.4.

Theorem 1.1.1. *A simple quartic $X \subset \mathbb{P}^3$ is non-special if and only if*

$$H_1(X \smallsetminus (\operatorname{Sing} X \cup H)) = 0,$$

where $\operatorname{Sing} X$ *is the set of the singular points of X and H is a generic hyperplane section of X.*

Still, the number of strata is too large to be listed explicitly, hence, in the existence part we adopt the approach of [13] and describe only the strata extremal with respect to degeneration. (Note that, unlike [4, 5, 9, 10], we do not introduce any artificial Dynkin graphs: all sets of singularities mentioned below and all perturbations thereof are indeed realized by quartics.) Recall that a set of simple singularities can be identified with a root system, *i.e.*, a negative definite lattice generated by vectors of square -2 (see Dufree [1] and §3.1); the rank of this lattice is the total Milnor number of the quartic. By a *perturbation* of a set of simple singularities $\mathbf{S}$ we mean any set of simple singularities $\mathbf{S}'$ whose Dynkin graph is an induced subgraph of that of $\mathbf{S}$ (see §4.2.1). Recall, further, that for a simple quartic $X \subset \mathbb{P}^3$, one has $\mu(X) \leq 19$ (see *e.g.*, [9]); X is called *maximizing* if $\mu(X) = 19$.

Denote by $\mathcal{M}(\mathbf{S})$ the equisingular stratum of simple quartics with a given set of singularities $\mathbf{S}$, and let $\mathcal{M}_1(\mathbf{S}) \subset \mathcal{M}(\mathbf{S})$ be the equisingular strata of non-special simple quartics. Since the non-special property is obviously deformation invariant, $\mathcal{M}_1(\mathbf{S})$ consists of whole connected components of $\mathcal{M}(\mathbf{S})$. A connected component $\mathcal{D} \subset \mathcal{M}(\mathbf{S})$ is called *real* if it is preserved as a set under the complex conjugation map conj : $\mathbb{P}^3 \to \mathbb{P}^3$. Clearly, this property is independent of the choice of the coordinates in $\mathbb{P}^3$, and all components of $\mathcal{M}(\mathbf{S})$ split into real and pairs of complex conjugate ones.

In this thesis we completed the equisingular deformation classification of non-special simple quartics. Our principal result is a complete description of the equisingular strata $\mathcal{M}_1(\mathbf{S})$ of non-special simple quartics.

Theorem 1.1.2. *A set of singularities* $\mathbf{S}$ *is realizable as the set of singularities of a non-special simple quartic if and only if* $\mathbf{S}$ *can be obtained by a perturbation from one of the sets of singularities listed in Tables* 1.1 *and* 1.2*. The numbers* (r, c) *of, respectively, real and pairs of complex conjugate components of the strata* $\mathcal{M}_1(\mathbf{S})$ *with* $\mu(\mathbf{S}) = 19$ *are shown in Table* 1*. If* $\mathbf{S}$ *is one of*

$$\mathbf{D}_6 \oplus 2\mathbf{A}_6, \quad \mathbf{D}_5 \oplus 2\mathbf{A}_6 \oplus \mathbf{A}_1, \quad 2\mathbf{A}_7 \oplus 2\mathbf{A}_2, \quad 3\mathbf{A}_6, \quad 2\mathbf{A}_6 \oplus 2\mathbf{A}_3$$

then $\mathcal{M}_1(\mathbf{S})$ *consists of two complex conjugate components; in all other cases, the stratum* $\mathcal{M}_1(\mathbf{S})$ *is connected.*

In Table 1.1, we list the maximizing sets of singularities, which are all extremal. In most cases the transcendental lattice (the orthogonal complement of the Néron Severi lattice in $H_2(X)$) is unique in its genus; in the six cases marked with ^{2} in the table the genus consists of two lattices. Recall that each maximizing quartic surface is defined over an algebraic number field and this number of transcendental lattices is a lower bound for the degree of this field.

A complete list of all possible combinations of simple singularities realized by a complex quartic surface (not-necessarily non-special) was previously found by Yang [14]. His technique is also based on Nikulin's criterion of existence of lattices [8] and he also represents the result in terms of perturbations of certain extremal sets. We restrict our attention to non-special quartics and our extremal sets of singularities are extremal in this restricted class. For this reason, we can also assert that any perturbation of a set S of simple singularities is realized by a perturbation of simple quartics.

The minimal resolution of a simple quartic gives us a $K3$-surfaces. Hence, the classification problem can be reduced to a certain arithmetic problem by

Table 1.1: The spaces $\mathcal{M}_1(\mathbf{S})$ with $\mu(\mathbf{S}) = 19$

Singularities	(r, c)
$2\mathbf{E}_8 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1,0)$
$\mathbf{E}_8 \oplus \mathbf{E}_7 \oplus \mathbf{A}_4$	$(1,0)$
$\mathbf{E}_8 \oplus \mathbf{E}_6 \oplus \mathbf{D}_5$	$(1,0)$
$\mathbf{E}_8 \oplus \mathbf{E}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	$(1,0)$
$\mathbf{E}_8 \oplus \mathbf{D}_7 \oplus 2\mathbf{A}_2$	$(1,0)$
$\mathbf{E}_8 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_1$	$(1,0)$
$\mathbf{E}_8 \oplus \mathbf{A}_9 \oplus \mathbf{A}_2$	$(1,0)$
$\mathbf{E}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5$	$(1,0)$
$\mathbf{E}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	$(1,0)$
$\mathbf{E}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$	$(0,1)$
$\mathbf{E}_8 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1,0)$
$\mathbf{E}_7 \oplus \mathbf{E}_6 \oplus \mathbf{A}_6$	$(1,0)$
$\mathbf{E}_7 \oplus \mathbf{A}_{12}$	$(1,1)^2$
$\mathbf{E}_7 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_2$	$(0,1)$
$\mathbf{E}_7 \oplus \mathbf{A}_8 \oplus \mathbf{A}_4$	$(2,0)^2$
$\mathbf{E}_7 \oplus 2\mathbf{A}_6$	$(0,1)$
$\mathbf{E}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(1,0)$
$2\mathbf{E}_6 \oplus \mathbf{D}_7$	$(1,0)$
$\mathbf{E}_6 \oplus \mathbf{D}_{13}$	$(1,0)$
$\mathbf{E}_6 \oplus \mathbf{D}_9 \oplus \mathbf{A}_4$	$(1,0)$
$\mathbf{E}_6 \oplus \mathbf{A}_{13}$	$(1,0)$
$\mathbf{E}_6 \oplus \mathbf{A}_{12} \oplus \mathbf{A}_1$	$(1,0)$
$\mathbf{D}_{15} \oplus 2\mathbf{A}_2$	$(1,0)$
$\mathbf{D}_{11} \oplus \mathbf{A}_6 \oplus \mathbf{A}_2$	$(0,1)$
$\mathbf{D}_9 \oplus \mathbf{A}_6 \oplus 2\mathbf{A}_2$	$(1,0)$
$\mathbf{D}_7 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_2$	$(0,1)$
$\mathbf{D}_7 \oplus 2\mathbf{A}_6$	$(0,1)$
$\mathbf{D}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(0,1)$
$\mathbf{D}_7 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2$	$(1,0)$

Singularities	(r, c)
$\mathbf{A}_{18} \oplus \mathbf{A}_1$	$(1,1)^2$
$\mathbf{A}_{17} \oplus \mathbf{A}_2$	$(1,1)$
$\mathbf{A}_{16} \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1,0)$
$\mathbf{A}_{15} \oplus 2\mathbf{A}_2$	$(0,1)$
$\mathbf{A}_{14} \oplus \mathbf{A}_5$	$(0,2)$
$\mathbf{A}_{14} \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$	$(0,2)$
$\mathbf{A}_{13} \oplus \mathbf{A}_6$	$(0,2)$
$\mathbf{A}_{13} \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(1,0)$
$\mathbf{A}_{12} \oplus \mathbf{A}_6 \oplus \mathbf{A}_1$	$(1,1)^2$
$\mathbf{A}_{12} \oplus \mathbf{A}_5 \oplus \mathbf{A}_2$	$(1,1)^2$
$\mathbf{A}_{12} \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(0,1)$
$\mathbf{A}_{12} \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$	$(2,0)$
$\mathbf{A}_{11} \oplus \mathbf{A}_6 \oplus \mathbf{A}_2$	$(0,2)$
$\mathbf{A}_{10} \oplus \mathbf{A}_9$	$(1,1)^2$
$\mathbf{A}_{10} \oplus \mathbf{A}_8 \oplus \mathbf{A}_1$	$(0,1)$
$\mathbf{A}_{10} \oplus \mathbf{A}_7 \oplus \mathbf{A}_2$	$(0,2)$
$\mathbf{A}_{10} \oplus \mathbf{A}_6 \oplus \mathbf{A}_3$	$(0,2)$
$\mathbf{A}_{10} \oplus \mathbf{A}_6 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1,0)$
$\mathbf{A}_{10} \oplus \mathbf{A}_5 \oplus \mathbf{A}_4$	$(1,0)$
$\mathbf{A}_{10} \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$	$(0,1)$
$\mathbf{A}_9 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2$	$(1,1)$
$\mathbf{A}_9 \oplus \mathbf{A}_6 \oplus 2\mathbf{A}_2$	$(1,0)$
$\mathbf{A}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5$	$(1,1)$
$\mathbf{A}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	$(0,1)$
$\mathbf{A}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$	$(0,3)$
$\mathbf{A}_7 \oplus 2\mathbf{A}_6$	$(0,2)$
$\mathbf{A}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(0,1)$
$2\mathbf{A}_6 \oplus \mathbf{A}_5 \oplus \mathbf{A}_2$	$(2,0)$
$2\mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(0,1)$
$\mathbf{A}_6 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$	$(2,0)$

Table 1.2: Extremal sets of singularities with $\mu(\mathbf{S}) = 18$

$\mathbf{E}_8 \oplus \mathbf{D}_{10}$	$\mathbf{D}_{14} \oplus \mathbf{A}_4$	$\mathbf{D}_6 \oplus 3\mathbf{A}_4$
$\mathbf{E}_8 \oplus \mathbf{D}_9 \oplus \mathbf{A}_1$	$\mathbf{D}_{10} \oplus \mathbf{A}_8$	$2\mathbf{D}_5 \oplus \mathbf{A}_8$
$2\mathbf{E}_7 \oplus 2\mathbf{A}_2$	$\mathbf{D}_{10} \oplus 2\mathbf{A}_4$	$2\mathbf{D}_5 \oplus 2\mathbf{A}_4$
$\mathbf{E}_7 \oplus \mathbf{D}_{11}$	$2\mathbf{D}_9$	$\mathbf{D}_5 \oplus \mathbf{A}_9 \oplus \mathbf{A}_4$
$\mathbf{E}_7 \oplus \mathbf{D}_9 \oplus \mathbf{A}_2$	$\mathbf{D}_9 \oplus \mathbf{A}_9$	$\mathbf{D}_5 \oplus \mathbf{A}_8 \oplus \mathbf{A}_5$
$\mathbf{D}_{18}$	$\mathbf{D}_9 \oplus \mathbf{A}_8 \oplus \mathbf{A}_1$	$\mathbf{D}_5 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_4$
$\mathbf{D}_{17} \oplus \mathbf{A}_1$		

using the global Torelli theorem for $K3$-surfaces [6] and the surjectivity of the period map [7]. Our principal result, Theorem 1.1.2, is proved by a reduction to an arithmetical problem [12] (*cf.* also [15]), concerning enumeration of abstract homological types. The proof depends mainly on Nikulin‘s theory of lattice extensions and discriminant forms. By applying Nikulins's existence theorem [8] (the algorithm given in this theorem can easily be implemented), we list all possible combinations of simple singularities realized by a non-special quartic and instead of compiling a long computer aided table, we express this list in terms of perturbations of certain sets of singularities given in Tables 1.1 and 1.2. For the uniqueness part, unfortunately Nikulin's sufficient uniqueness conditions do not suffice our purposes. Instead, to obtain the desired uniqueness results, our principal novelty was a systematic usage of the Miranda–Morrison theory [16, 17, 18] computing the genus groups and a few other bits missing in [8] in the case of indefinite lattices. Our computations relies on the stronger uniqueness criteria of Miranda–Morrison developed in [16, 17, 18]. By using their results, for each realizable set of singularties, we describe the connected components of the corresponding equisingular strata.

The rest of this thesis is organized as follows:

In Chapter 2, based on Nikulin's work [8], we introduce the necessary terminology about integral lattices, discriminant forms and lattice extensions; then, we outline the fundamentals of Miranda-Morison's theory [16, 17, 18] which are used in §4.2.

In Chapter 3, we discuss the relation between simple quartics and $K3$-surfaces and construct the concept of abstract homological type. Next, we recall the reduction of the classification problem to the arithmetical classification of abstract homological types. Then we restrict our attention to non-special quartics and give a geometric characterization (Theorem 1.1.1) of them which motivates our study of non-special simple quartic surfaces. We prove Theorem 1.1.1 in this chapter by using is purely homotopy theoretical arguments. We also introduce the notion of strata to identify different families of quartics.

Finally, Chapter 4 is devoted to the proof of our principal result, namely Theorem 1.1.2 which depends essentially on the auxiliary material presented in §2 and §3. The arithmetical reduction obtained in chapter §3 is used to prove

Theorem 1.1.2. The list of realizable sets of singularities is obtained by applying Nikulin's existence theorem [8]. Then, the enumeration of each of the connected components of each stratum $\mathcal{M}_1(\mathbf{S})/\operatorname{conj}$ or $\mathcal{M}_1(\mathbf{S})$ is based on the uniqueness criterion due to Miranda and Morrison [16, 17, 18].

1.2 Notations

- The letter p always denotes an ordinary prime number
- $\mathbb{F}_p$: the field with p elements
- $\mathbb{Z}_p$: p-adic integers
- $\mathbb{Q}_p$: p-adic rationals
- $\mathbb{P}$: the set of prime numbers
- $\left(\frac{n}{p}\right)$: the Legendre symbol
- $R^\times$: the group of units of a ring R
In particular, the structure of $R^\times/(R^\times)^2$ for $R = \mathbb{Q}$, $\mathbb{Q}_p$ and $\mathbb{Z}_p$ is given as follows:

  (a) $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$: a free $\mathbb{F}_2$-module with basis $\{-1\} \cup \mathbb{P}$, *i.e.*, the set of all square free integers

  (b) $\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$: if p is an odd prime then $\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2 = \{\pm 1\}$ and if $p = 2$ then $\mathbb{Z}_2^\times/(\mathbb{Z}_2^\times)^2 = (\mathbb{Z}/8)^\times = \{1, 3, 5, 7\} \cong \{\pm 1\} \times \{\pm 1\}$

  (c) $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$: any element of $\mathbb{Q}_p^\times$ can be written uniquely as $p^k u$ for some $k \in \mathbb{Z}$ and a unit $u \in \mathbb{Z}_p^\times$. Thus we have,

$$\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2 = \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2 \times \{\pm 1\}$$

- $\mathbf{A}_n, n \geq 1$, $\mathbf{D}_m, m \geq 4$ and $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$: the irreducible root lattices (see §2.2.3)
- $\mathbf{U} = \mathbb{Z}u_1 \oplus \mathbb{Z}u_2$ where $u_1^2 = u_2^2 = 0$ and $u_1 \cdot u_2 = 1$ (see §3.1)
- $\mathbf{L} = 2\mathbf{E}_8 \oplus 3\mathbf{U}$ (see §3.1)
- $\Gamma_p = \{\pm 1\} \times \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$ (see §2.3.1)
- $\Gamma_0 = \{\pm 1\} \times \{\pm 1\}$ (see §2.3.1)
- $\Gamma_{\mathbb{A}}$: the adelic constructtion for Γ_p groups (see §2.3.1)
- $g(L)$: the genus group of the integral quadratic form L (see §2.2.1)

Chapter 2

Integral Lattices

In this chapter, we outline the basic definitions and facts about Nikulin's theory of integral lattices, discriminant forms and lattice extensions which are the main tools used throughout this thesis; then we recast the principal results of Miranda-Morrison's theory on which most computations are based. For additional details about the materials presented in this chapter the reader is referred to [8], [16], [17] and [18] .

2.1 Finite quadratic forms

A *finite quadratic form* is a finite abelian group $\mathcal{L}$ equipped with a map $q\colon \mathcal{L} \to \mathbb{Q}/2\mathbb{Z}$. satisfying $q(x+y) = q(x) + q(y) + 2b(x,y)$ and $q(nx) = n^2x$ for all $x, y \in \mathcal{L}$, where $b\colon \mathcal{L}\otimes\mathcal{L} \to \mathbb{Q}/\mathbb{Z}$ is a symmetric bilinear form (which is determined by q). To reduce the notation we write x^2 for $q(x)$ and $x \cdot y$ for $b(x,y)$. A finite quadratic form $\mathcal{L}$ is called *non-degenerate* if the associated homomorphism $\mathcal{L} \to \operatorname{Hom}(\mathcal{L}, \mathbb{Q}/\mathbb{Z})$ given by $x \mapsto (y \mapsto x \cdot y)$ is an isomorphism. For a prime p, let $\mathcal{L}_p := \mathcal{L} \otimes \mathbb{Z}_p$, which is called the *p-primary part* of $\mathcal{L}$. Any finite quadratic form $\mathcal{L}$ can be written as an orthogonal sum of its p-primary components $\mathcal{L}_p := \mathcal{L}\otimes\mathbb{Z}_p$, *i.e.*, $\mathcal{L} = \bigoplus_p \mathcal{L}_p$ where the summation runs over all primes p. Denote by $\ell(\mathcal{L})$ the minimal number of generators of $\mathcal{L}$ and define $\ell_p(\mathcal{L}) := \ell(\mathcal{L}_p)$. The *scale* of $\mathcal{L}$ is the smallest order which appears in any decomposition of $\mathcal{L}$ into cyclic

$\mathbb{Z}$-modules.

Consider a fraction $\frac{m}{n} \in \mathbb{Q}/2\mathbb{Z}$ with $g.c.d(m,n) = 1$ and $mn = 0 \bmod 2$. By $\langle \frac{m}{n} \rangle$, we denote the finite non-degenerate quadratic form on $\mathbb{Z}/n\mathbb{Z}$ generated by an element of square $\frac{m}{n}$ and of order n. In particular, if $k \geq 1$ and p is prime, the isomorphism class of $\langle \frac{m}{p^k} \rangle$ depens only on $\alpha = m \bmod (\mathbb{Z}_p^{\times})^2$; this class is denoted by $\mathcal{W}_{p,k}^{\alpha}$. There are also two families of indecomposable forms of length 2, namely, the quadratic forms $\mathcal{U}_k$ and $\mathcal{V}_k$ on $\mathbb{Z}/2^k\mathbb{Z} \times \mathbb{Z}/2^k\mathbb{Z}$ defined by the matrices

$$\mathcal{U}_k = \begin{pmatrix} 0 & \frac{1}{2^k} \\ \frac{1}{2^k} & 0 \end{pmatrix} \qquad \text{and} \qquad \mathcal{V}_k = \begin{pmatrix} \frac{1}{2^{k-1}} & \frac{1}{2^k} \\ \frac{1}{2^k} & \frac{1}{2^{k-1}} \end{pmatrix}.$$

Nikulin [8] showed that any non-degenerate finite quadratic form $\mathcal{L}$ can be written as an orthogonal sum of cyclic summands of the form $\langle \frac{m}{n} \rangle$ and copies of $\mathcal{U}_k$ and $\mathcal{V}_k$. In particular, for the 2-primary part $\mathcal{L}_2$, one can find many ways of decomposing $\mathcal{L}_2$ into rank one and two summands. The following Lemma establishes a special kind of decomposition which will be used in the following chapters.

Lemma 2.1.1 (Miranda–Morrison [18]). *Every nondegenrate quadratic form $\mathcal{L}$ on a finite abelian 2-group has an orthogonal direct sum decomposition*

$$\mathcal{L} \cong \bigoplus_{k \geq 1} (\mathcal{U}_k^{\oplus n(k)} \oplus \mathcal{V}_k^{\oplus m(k)} \oplus \mathcal{W}(k)) \tag{2.1.1}$$

where $m(k) \leq 1$, $\operatorname{rk}(\mathcal{W}(k)) \leq 2$ and $\mathcal{W}(k)$ is a sum of forms of type $\mathcal{W}_{2,k}^{\alpha}$. Moreover the quantities $n(k) + m(k)$ and $\operatorname{rk}(\mathcal{W}(k))$ are invariants of the form $\mathcal{L}$.

A decomposition of a quadratic form as in Lemma 2.1.1 is called a *partial normal form.*

The *Brown invariant* of a finite quadratic form $\mathcal{L}$ is the residue $\operatorname{Br} \mathcal{L} \in \mathbb{Z}/8\mathbb{Z}$ defined by the Gauss sum

$$\exp\left(\frac{1}{4} i\pi \operatorname{Br} \mathcal{L}\right) = |\mathcal{L}|^{-\frac{1}{2}} \sum_{x \in \mathcal{L}} \exp(i\pi x^2).$$

The Brown invariants of indecomposable p-primary blocks are as follows:

$$\begin{aligned}
&\operatorname{Br}\left\langle \frac{2a}{p^{2s-1}} \right\rangle = 2\Big(\frac{a}{p}\Big) - \Big(\frac{-1}{p}\Big) - 1, \\
&\operatorname{Br}\left\langle \frac{2a}{p^{2s}} \right\rangle = 0 \quad (\text{for } p \text{ odd, } s \geq 1 \text{ and g. c. d.}(a,p) = 1), \\
&\operatorname{Br}\left\langle \frac{a}{2^k} \right\rangle = a + \frac{1}{2}k(a^2-1) \bmod 8 \quad (\text{for } k \geq 1 \text{ and odd } a \in \mathbb{Z}), \\
&\operatorname{Br}\mathcal{U}_{2^k} = 0, \\
&\operatorname{Br}\mathcal{V}_{2^k} = 4k \bmod 8 \quad (\text{for all } k \geq 1).
\end{aligned}$$

Given a prime p, we define the *determinant* $\det_{\mathrm{p}}(\mathcal{L})$ as the determinant of the matrix of the quadratic from on $\mathcal{L}_p$ in an appropriate basis (see [19] and [8] for details). For a non-degenerate $\mathcal{L}_p$, one has $\det_p(\mathcal{L}) = u/|\mathcal{L}_p|$ where u is a well defined element of $u \in \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$. If $p = 2$, the determinant $\det_2(\mathcal{L})$ is well defined only if $\mathcal{L}_2$ is even.

A finite quadratic form is called *even* if x^2 is an integer for all elements $x \in \mathcal{L}$ of order 2; otherwise it is called *odd*. This definition implies that a quadratic form is odd if and only if it contains $\langle \pm\frac{1}{2} \rangle$ as an orthogonal summand.

2.2 Integral lattices and discriminant forms

2.2.1 Integral lattices

An *(integral) lattice* is a free abelian group L of finite rank with a symmetric bilinear form $b\colon L \otimes L \to \mathbb{Z}$. For short, we use the multiplicative notation $x \cdot y$ for $b(x, y)$ and x^2 for $b(x, x)$. A lattice L is called *even* if a^2 is an even integer for all $a \in L$. It is called *odd* otherwise. The determinant $\det(L)$ is defined to be the determinant of the Gram matrix of b in any basis of L. Since the transition matrix between any two integral bases has determinant ± 1, $\det(L) \in \mathbb{Z}$ is well defined. A lattice L is called *non-degenerate* if $\det(L) \neq 0$; it is called *unimodular* if $\det(L) = \pm 1$.

From now on all lattices considered are even and non-degenerate.

Given a lattice L, the form $b : L \otimes L \to \mathbb{Z}$ can be extended by linearity to a form $(L \otimes \mathbb{Q}) \otimes_{\mathbb{Q}} (L \otimes \mathbb{Q}) \to \mathbb{Q}$. If L is non-degenerate, the dual group $L^* := \mathrm{Hom}(L, \mathbb{Z})$ can be identified with the subgroup

$$\left\{x \in L \otimes \mathbb{Q} \;\middle|\; x \cdot y \in \mathbb{Z} \text{ for all } y \in L\right\}$$

Since the original bilinear form b on L is integer valued, L is a finite index subgroup of its dual. The quotient L^*/L is called the *discriminant group* of L and is denoted by $\mathcal{L}$ or $\operatorname{disc} L$. If $\{e_1, e_2, \ldots e_n\}$ is a basis set for L and $\{e_1^*, e_2^*, \ldots, e_n^*\}$ is the dual basis for L^*, then the Gram matrix $[e_i \cdot e_j]$ is exactly the matrix of the homomorphism $\varphi : L \to L^*$, $x \mapsto [y \mapsto x \cdot y]$. Hence one has $|\mathcal{L}| = |\det(L)|$. Note that $x \cdot y \in \mathbb{Z}$ whenever $x \in L$ or $y \in L$. Thus, $\mathcal{L}$ inherits from $L \otimes \mathbb{Q}$ a symmetric bilinear form $b_{\mathcal{L}} : \mathcal{L} \otimes \mathcal{L} \to \mathbb{Q}/\mathbb{Z}$; it is called the *discriminant form.* If L is even, this form $b_{\mathcal{L}}$ can be promoted to the quadratic extension $q_{\mathcal{L}} : \mathcal{L} \to \mathbb{Q}/2\mathbb{Z}$, $x \bmod L \mapsto x^2 \bmod 2\mathbb{Z}$. Hence, the discriminant form of an even lattice is a finite quadratic form. Given a prime p, we use the notation $\operatorname{disc}_p L$ or $\mathcal{L}_p$ for the p-primary part of $\mathcal{L}$, *i.e.*, $\mathcal{L}_p = \mathcal{L} \otimes \mathbb{Z}_p$. The *signature* of a non-degenerate lattice L is the pair (σ_+, σ_-) of its positive and negative inertia indices. Two non-degenerate integral lattices are said to have the same *genus* if their localizations over $\mathbb{R}$ and over $\mathbb{Z}_p$ are isomorphic. The following few statements give the relation between the genus of an even integral lattice and its discriminant form.

Theorem 2.2.1 (Nikulin [8])**.** *The genus of an even integral lattice L is determined by its signature $(\sigma_+ L, \sigma_- L)$ and discriminant form* $\operatorname{disc} L$*.*

Theorem 2.2.2 (van der Blij [20])**.** *For any non-degenerate even integral lattice L one has* $\operatorname{Br} \mathcal{L} = \sigma_+ - \sigma_- \mod 8$.

We denote by $g(L)$ the set of all isomorphism classes of all non-degenerate even integral lattices with the same genus as L. Each set $g(L)$ is known to contain finitely many isomorphism classes.

Theorem 2.2.3 (Nikulin [8])**.** *Let $\mathcal{L}$ be a finite quadratic form and let $\sigma_\pm$ be a pair of integers. Then, the following four conditions are necessary and sufficient for the existence of a non-degenerate even integral lattice L whose signature is (σ_+, σ_-) and whose discriminant form is $\mathcal{L}$:*

1. *$\sigma_\pm \geq 0$ and $\sigma_+ + \sigma_- \geq \ell(\mathcal{L})$;*

2. $\sigma_+ - \sigma_- = \operatorname{Br} \mathcal{L} \mod 8$*;*

3. for each $p \neq 2$*, either* $\sigma_+ + \sigma_- > \ell_p(\mathcal{L})$ *or* $\det_p(\mathcal{L}) \equiv (-1)^{\sigma_-} \cdot |\mathcal{L}| \mod (\mathbb{Z}_p^\times)^2$*;*

4. either $\sigma_+ + \sigma_- > \ell_2(\mathcal{L})$*, or* $\mathcal{L}_2$ *is odd, or* $\det_2(\mathcal{L}) \equiv \pm|\mathcal{L}| \mod (\mathbb{Z}_2^\times)^2$*.*

2.2.2 Automorphisms of lattices

An *isometry* of integral lattices is a homomorphism of abelian groups preserving the forms. The group of auto-isometries of L is denoted by $O(L)$. There is a natural homomorphism $d\colon O(L) \to \operatorname{Aut}(\mathcal{L})$, where $\operatorname{Aut}(\mathcal{L})$ denotes the group of automorphisms of $\mathcal{L}$ preserving the discriminant form q on $\mathcal{L}$. Obviously, one has $\operatorname{Aut}(\mathcal{L}) = \prod_p \operatorname{Aut}(\mathcal{L}_p)$, where the product runs over all primes. The restrictions of d to the p-primary components are denoted by $d_p : O(L) \to \operatorname{Aut}(\mathcal{L}_p)$.

Given a vector u in L with $u \neq 0$, the *reflection* against its orthogonal hyperplane is the automorphism

$$\begin{aligned} r_u :& L \to L \\ & x \mapsto x - 2\frac{(x \cdot u)}{u^2} u \end{aligned}$$

The reflection r_u is well-defined whenever $u \in (\frac{u^2}{2})L^*$, in particular $u^2 = \pm 1$ or $u^2 = \pm 2$. Note that $r_u^2 = \mathrm{id}$, *i.e.*, r_u is an involution. Each image $d_p(r_u) \in \operatorname{Aut}(\mathcal{L}_p)$ is also a reflection (see §2.3.2). If $u^2 = \pm 1$ or $u^2 = \pm 2$, then r_u is always well-defined and the induced automorphism $d(r_u)$ is the identity.

2.2.3 Root systems

A *root* in a lattice L is an element $v \in L$ of square -2. A *root system* is a negative definite lattice generated by its roots. Each root system splits uniquely into orthogonal direct sum of its irreducible components. As explained in [21], the irreducible root systems are $\mathbf{A}_n, n \geq 1$, $\mathbf{D}_m, m \geq 4$ and $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$. The corresponding discriminant forms are as follows:

$$\operatorname{disc}\mathbf{A}_n = \Big\langle -\frac{n}{n+1}\Big\rangle, \quad \operatorname{disc}\mathbf{D}_{2k+1} = \Big\langle -\frac{2k+1}{4}\Big\rangle,$$
$$\operatorname{disc}\mathbf{D}_{8k\pm 2} = 2\Big\langle \mp\frac{1}{2}\Big\rangle, \quad \operatorname{disc}\mathbf{D}_{8k} = \mathcal{U}_1, \quad \operatorname{disc}\mathbf{D}_{8k+4} = \mathcal{V}_1,$$
$$\operatorname{disc}\mathbf{E}_6 = \Big\langle \frac{2}{3}\Big\rangle, \quad \operatorname{disc}\mathbf{E}_7 = \Big\langle \frac{1}{2}\Big\rangle, \quad \operatorname{disc}\mathbf{E}_7 = 0.$$

Given a root system S, the group generated by reflections (defined by the roots of S) acts simply transitively on the set of Weyl chambers of S. The roots constituting a single Weyl chamber form a *standard basis* for S; these roots are naturally identified with the vertices of the Dynkin graph $\Gamma := \Gamma_S$. Thus, one has

$$O(S) = \operatorname{Ref}(S) \rtimes \operatorname{Sym}(\Gamma),$$

where $\operatorname{Sym}(\Gamma)$ denotes the group of symmetries of Γ and $\operatorname{Ref}(S)$ is the subgroup of $O(S)$ generated by reflections. Since $\operatorname{Ref}(S)$ acts identically on $\operatorname{disc} S$, we have $\operatorname{Im} d = d(\operatorname{Sym}(\Gamma))$. Irreducible root systems correspond to connected Dynkin graphs. The following statement follows immediately from the classification of connected Dynkin graphs (see N. Bourbaki [21]).

Lemma 2.2.1. *Let $\Gamma = \Gamma_S$ be the connected Dynkin graph of an irreducible root system S. Then,*

1. *if S is $\mathbf{A}_1$, $\mathbf{E}_7$ or $\mathbf{E}_6$, then* $\operatorname{Sym}(\Gamma) = 1$
2. *if S is $\mathbf{D}_4$, then* $\operatorname{Sym}(\Gamma) = \mathbb{S}_3$
3. *for all other types,* $\operatorname{Sym}(\Gamma) = \mathbb{Z}_2$

If S is $\mathbf{A}_p$, $p \geq 2$, $\mathbf{D}_{2k+1}$ or $\mathbf{E}_8$, then the only nontrivial symmetry of Γ induces $-\operatorname{id}$ on $\mathcal{S}$. If S is $\mathbf{E}_8$ then $\mathcal{S} = 0$ and if S is $\mathbf{A}_1$, $\mathbf{A}_7$ of $\mathbf{D}_{2k}$, the groups $\mathcal{S}$ are $\mathbb{F}_2$ modules and $-\operatorname{id} = \operatorname{id}$ on $\operatorname{Aut}\mathcal{S}$.

Additional details on irreducible root systems can be found in N. Bourbaki [21].

2.2.4 Lattice extensions

A non-degenerate even integral lattice L containing even lattice S is called an *extension* of S. An *isomorphism* between two extensions $L_1 \supset S$ and $L_2 \supset S$ is an isometry between L_1 and L_2 taking S to S. In particular, if the isomorphism $L_1 \to L_2$ restricts to id on S, the extensions L_1 and L_2 are called *strictly isomorphic.* For a given subgroup A of $O(S)$, we define A*-isomorphisms* of extensions of S as those which restrict to an element of A on S.

Recall that S is assumed to be non-degenerate, hence given a *finite index* extension $L \supset S$, one has $L \subset S^*$. Thus there are inclusions $S \subset L \subset L^* \subset S^*$ which imply $L/S \subset S^*/S = \mathcal{S}$. The subgroup $\mathcal{K} = L/S$ of $\mathcal{S}$ is called the *kernel* of the finite index extension $L \supset S$. Since L is an even integral lattice, the discriminant quadratic form on $\mathcal{S}$ restricts to zero on $\mathcal{K}$, *i.e.,* $\mathcal{K}$ is isotropic.

Proposition 2.2.1 (Nikulin [8])**.** *Let S be a non-degenerate even lattice. The map $L \mapsto \mathcal{K} = L/S$ establishes a one-to-one correspondence between the set of isomorphism classes of finite index extensions $L \supset S$ and the set of isomorphism classes of isotropic subgroup $\mathcal{K} \subset \mathcal{S}$. Under this correspondence one has $\mathcal{L} = \mathcal{K}^\perp/\mathcal{K}$.*

Proposition 2.2.2 (Nikulin [8])**.** *Let $L \supset S$ be a finite index extension of a lattice S and let $\mathcal{K} \subset \mathcal{S}$ be its kernel. Then an auto-isometry $S \to S$ extends to L if and only if the induced automorphism of $\mathcal{S}$ preserves $\mathcal{K}$.*

An extension $L \supset S$ is called *primitive* if L/S is torsion free. Following Nikulin [8], we confine ourselves to the special case where L is unimodular and S is a primitive non-degenerate sublattice of a unimodular lattice L. Clearly, $S^\perp$ is also primitive in L and L is a finite index extension of $S \oplus S^\perp$. Furthermore, since $\operatorname{disc} L = 0$, the kernel $\mathcal{K} \subset \mathcal{S} \oplus \mathcal{S}^\perp$ is the graph of an anti-isometry $\psi\colon \mathcal{S} \to \operatorname{disc} S^\perp$. Hence the genus $g(S^\perp)$ is determined by the genera $g(S)$ and $g(L)$. Conversely, given a lattice $N \in g(S^\perp)$ and an anti-isometry $\psi\colon \mathcal{S} \to \mathcal{N}$ where $\mathcal{N}$ is the discriminant group of N, the graph of ψ is an isotropic subgroup $\mathcal{K} \subset \mathcal{S} \oplus \mathcal{S}^\perp$ and the corresponding finite index extension $S \oplus N \hookrightarrow L$ is a unimodular primitive extension of S with $S^\perp \cong N$. Any anti-isometry $\psi\colon \mathcal{S} \to \operatorname{disc} S^\perp$ induces an isomorphism $\theta\colon \operatorname{Aut}(\mathcal{S}) \cong \operatorname{Aut}(\mathcal{N})$ which gives rise to the homomorphism $d^\psi\colon O(S) \to \operatorname{Aut}(\mathcal{N})$ given by $d^\psi = \theta \circ d$. Recall that there is a natural homomorphism $d\colon O(N) \to \mathrm{Aut}(\mathcal{N})$. Thus, since also an indefinite unimodular lattice is unique in its genus(see [8]), we have the following theorem.

Theorem 2.2.4 (Nikulin [8])**.** *Let L be an indefinite unimodular even lattice and $S \subset L$ a non-degenerate primitive sublattice. Fix a subgroup $A \subset O(S)$. Then the A-isomorphism class of a primitive extension $S \subset L$ is determined by*

1. *a choice of a lattice $N \in g(S^\perp)$ and*
2. *a choice of a double coset $c_N \in d^\psi(A)\backslash Aut(\mathcal{N})/\operatorname{Im} d$ (for a given N and some anti-isometry $\psi : \mathcal{S} \to \mathcal{N}$ inducing d^ψ).*

Theorem 2.2.5 (Nikulin [8])**.** *Let L be an indefinite unimodular even lattice, $S \subset L$ a non-degenerate primitive sublattice and $\psi : \mathcal{S} \to \mathcal{N}$ be the anti-isometry corresponding to the extension $S \subset L$ where $N = S^\perp$. Then a pair of isometries $\alpha_S \in O(S)$ and $\alpha_N \in O(N)$ extends to L if and only if $d^\psi(\alpha_S) = d(\alpha_N)$.*

2.3 Miranda–Morrison's theory

2.3.1 Miranda–Morrison's results

Nikulin [8] gave a sufficient condition for a lattice N to be unique in its genus and for the natural homomorphism $O(N) \to \mathrm{Aut}(\mathcal{N})$ to be onto by he following theorem:

Theorem 2.3.1 (see Theorem 1.14.2 in [8])**.** *Let L be an indefinite even integral lattice, $\operatorname{rk} L \geq 3$. The following two conditions are sufficient for L to be unique in its genus and for the natural homomorphism $O(L) \to \mathrm{Aut}(\mathcal{L})$ to be surjective:*

1. *for each $p \neq 2$, $\operatorname{rk} L \geq \ell_p(\mathcal{L}) + 2$*
2. *either $\operatorname{rk} L \geq \ell_2(\mathcal{L}) + 2$ or $\mathcal{L}_2$ contains a subform isomorphic to $\mathcal{U}_k$, $\mathcal{V}_k$ as an orthogonal summand.*

However this result is not enough to prove what we need, instead we follow the stronger uniqueness criteria developed in Miranda-Morrison [16, 17, 18].

Warning: The notation that we adopt following Nikulin [8] in this thesis is slightly different than that of Miranda - Morrison [16, 17, 18] at the point how we relate quadratic and bilinear forms: We are using the convention that $q(x+y)-q(x)-q(y)=2b(x,y)$ while Miranda - Morrison has $q(x+y)-q(x)-q(y)=b(x,y)$ (without multiplication by 2 on the right hand side). Also the finite quadratic forms in [16, 17, 18] take values in $\mathbb{Q}/\mathbb{Z}$. Therefore, the values of all quadratic forms in [16, 17, 18] should be multiplied by 2.

Recall that,

- $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ is the free $\mathbb{F}_2$-module with basis $\{-1\}\cup\mathbb{P}$, *i.e.*, the set of all square free integers,
- $\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2=\{\pm1\}$ if p is odd and $\mathbb{Z}_2^\times/(\mathbb{Z}_2^\times)^2=(\mathbb{Z}/8)^\times=\{1,3,5,7\}\cong\{\pm1\}\times\{\pm1\}$ if $p=2$,
- $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2=\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2\times\{\pm1\}$.

Let p be a prime. Define

$$\Gamma_p:=\{\pm1\}\times\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2,$$
$$\Gamma_0:=\{\pm1\}\times\{\pm1\}\subset\{\pm1\}\times\mathbb{Q}^\times/(\mathbb{Q}^\times)^2.$$

It is convenient to introduce the following subgroups related to Γ_p :

- $\Gamma_{p,0}:=\{\pm1\}\times\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2\subset\Gamma_p$.
  If $p\neq2$, then $\Gamma_{p,0}=\{(1,1),(1,u_p),(-1,1),(-1,u_p)\}$; where u_p is the only nontrivial element of $\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$.
  If $p=2$, then $\Gamma_{2,0}=\{(1,1),(1,3),(1,5),(1,7),(-1,1),(-1,3),(-1,5),(-1,7)\}$.
- $\Gamma_p^{++}:=\{1\}\times\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2\subset\Gamma_{p,0}$.
- $\Gamma_{2,2}:=\{(1,1),(1,5)\}\subset\Gamma_2^{++}$.
- $\Gamma'_{2,0}:=\Gamma_{2,0}/\Gamma_{2,2}$ (and $\Gamma'_{p,0}:=\Gamma_{p,0}$ for $p\neq2$).
- $\Gamma_0^{--}:=\{(1,1),(-1,-1)\}\subset\Gamma_0$.

Let, further,

$$\Gamma_{\mathbb{A},0} := \prod_p \Gamma_{p,0} \subset \Gamma_{\mathbb{A}} := \Gamma_{\mathbb{A},0} \cdot \sum_p \Gamma_p,$$

where " $\cdot$ " denotes the sum of the subgroups. Note that

$$\Gamma_{\mathbb{A}} = \{(d_p, s_p) \in \prod_p \Gamma_p \,|(d_p, s_p) \in \Gamma_{p,0} \text{ for almost all } p\}.$$

The natural map $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2 \to \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$ induces canonical maps

$$\varphi_p : \Gamma_0 \to \Gamma_{p,0}. \tag{2.3.1}$$

which is defined by,

p	$\varphi_p(1,1)$	$\varphi_p(1,-1)$	$\varphi_p(-1,1)$	$\varphi_p(-1,-1)$
2	$(1,1)$	$(1,7)$	$(-1,1)$	$(-1,7)$
1 mod 4	$(1,1)$	$(1,1)$	$(-1,1)$	$(-1,1)$
3 mod 4	$(1,1)$	$(1,u_p)$	$(-1,1)$	$(-1,u_p)$

Let N be an indefinite lattice with $\mathrm{rk}(N) \geq 3$. We will use certain subgroups $\Sigma_p^\sharp(N) \subset \Gamma_{p,0}$. In the notation of [18] (which slightly differs from the notation in [16, 17]), one has $\Sigma_p^\sharp(N) := \Sigma^\sharp(N \otimes \mathbb{Z}_p)$; we refer the reader to [18] (see chapter 7, section 4) for the precise definitions. We put $\tilde{\Sigma}_p(N) := \varphi_p^{-1}(\Sigma_p^\sharp(N)) \subset \Gamma_0$. The subgroups $\Sigma_p^\sharp(N)$ (and hence $\tilde{\Sigma}_p(N)$) are computed explicitly in [18] (see Theorems 12.1, 12.2, 12.3 and 12.4 in chapter 7) as follows.

Theorem 2.3.2 (Theorem 12.1 in Miranda–Morrison [18])**.** *Let N be a non-degenerate indefinite even lattice with $\mathrm{rk}(N) \geq 3$ and $p \neq 2$.*

1. *If $\mathrm{rk}(N) = \ell(\mathcal{N}_p)$ then $\Sigma_p^\sharp(N) = \{(1,1)\}$*
2. *If $\mathrm{rk}(N) = \ell(\mathcal{N}_p) + 1$ then $\Sigma_p^\sharp(N) = \{(1,1), (-1, \left(\frac{2\Delta}{p}\right))\}$ where $\Delta := \det(N). \det_p(\mathcal{N}_p)$*
3. *If $\mathrm{rk}(N) \geq \ell(\mathcal{N}_p) + 2$ then $\Sigma_p^\sharp(N) = \Gamma_{p,0}$.*

Theorem 2.3.3 (Theorem 12.2 in Miranda–Morrison [18]). *Let N be a non-degenerate indefinite even lattice with* $\mathrm{rk}(N) \geq 3$ *and* $p = 2$*. Assume the partial normal form of* $\mathcal{N}_2$ *is given as*

$$\mathcal{N}_2 = \mathcal{U}_1^{n(1)} \oplus \mathcal{V}_1^{m(1)} \oplus \mathcal{W}(1) \oplus \mathcal{U}_2^{n(2)} \oplus \mathcal{V}_2^{m(2)} \oplus \mathcal{W}(2) \oplus \mathcal{N}_2'$$

where $\mathrm{scale}(\mathcal{N}_2') \geq 3$*. Then the groups* $\Sigma_2^\sharp(N)$ *are given by Table 2.1 and by Theorems 2.3.4 and 2.3.5 below.*

Table 2.1: The subgroups $\Sigma_2^\sharp(N)$

$\mathrm{rk}(N) - \ell(\mathcal{N}_p)$	$n(1) + m(1)$	$\ell(\mathcal{W}(1))$	$n(2) + m(2)$	$\ell(\mathcal{W}(2))$	$\Sigma_2^\sharp(N)$
> 0					$\Gamma_{2,0}$
0	> 0	> 0			$\Gamma_{2,0}$
0	> 0	0			$\Gamma_{2,1}$
0	0	2			*see Thm. 2.3.4*
0	0	1			*see Thm. 2.3.5*
0	0	0	> 0		$\Gamma_{2,2}$
0	0	0	0	2	$\Gamma_{2,2}$
0	0	0	0	≤ 1	$\{(1,1)\}$

Theorem 2.3.4 (Theorem 12.3 in Miranda–Morrison [18]). *Let N be a non-degenerate indefinite even lattice with* $\mathrm{rk}(N) \geq 3$*,* $p = 2$ *and* $\mathrm{rk}(N) = \ell(\mathcal{N}_2)$*. Assume the partial normal form of* $\mathcal{N}_2$ *is given as*

$$\mathcal{N}_2 = \mathcal{W}_{2,1}^{\theta} \oplus \mathcal{W}_{2,1}^{\eta} \oplus \mathcal{U}_2^{n(2)} \oplus \mathcal{V}_2^{m(2)} \oplus \mathcal{W}(2) \oplus \mathcal{N}_2'$$

where $\mathrm{scale}(\mathcal{N}_2') \geq 3$*. Then the groups* $\Sigma_2^\sharp(N)$ *are given by Table 2.2.*

Table 2.2: The subgroups $\Sigma_2^\sharp(N)$

$\theta\eta \bmod 4$	$\ell(\mathcal{W}(2))$	$\Sigma_2^\sharp(N)$	$\theta \bmod 4$
3		$\Gamma_{2,0}$	
1	> 0	$\Gamma_{2,0}$	
1	0	$\{(1,1),(1,5),(-1,\theta),(-1,5\theta)\}$	1
			3

Theorem 2.3.5 (Theorem 12.4 in Miranda–Morrison [18]). *Let N be a non-degenerate indefinite even lattice with* $\mathrm{rk}(N) \geq 3$*,* $p = 2$ *and* $\mathrm{rk}(N) = \ell(\mathcal{N}_2)$*. Assume the partial normal form of* $\mathcal{N}_2$ *is given as*

$$\mathcal{N}_2 = \mathcal{W}_{2,1}^{\theta} \oplus \underbrace{\mathcal{U}_2^{n(2)} \oplus \mathcal{V}_2^{m(2)} \oplus \mathcal{W}(2) \oplus \mathcal{U}_3^{n(3)} \oplus \mathcal{V}_3^{m(3)} \oplus \mathcal{W}(3) \oplus \mathcal{N}_2''}_{\mathcal{N}_2'}$$

where $\mathrm{scale}(\mathcal{N}_2'') \geq 4$*. Moreover let* $\Delta = (2\det_2(\mathcal{N}_2')\det(N)) \bmod 8$*. Then the groups* $\Sigma_2^\sharp(N)$ *are given by Table 2.3.*

Table 2.3: The subgroups $\Sigma_2^\sharp(N)$

$n(2)+m(2)+\ell(\mathcal{W}(3))$	$\mathcal{W}(2)$	$\Sigma_2^\sharp(N)$
>0	$\neq 0$	$\Gamma_{2,0}$
>0	0	$\langle(1,5),(-1,\Delta)\rangle$
0	rank 2	$\Gamma_{2,0}$
0	$\mathcal{W}_{2,2}^{\eta}$, $\theta\eta \equiv 3 \bmod 4$	$\langle(1,7),(-1,\Delta)\rangle$
0	$\mathcal{W}_{2,2}^{\eta}$, $\theta\eta \equiv 1 \bmod 4$	$\langle(1,3),(-1,\Delta)\rangle$
0	0	$\{(1,1),(-1,\Delta)\}$

Also defined in [18] (see chapter 8, sections 5, 6 and 7) is the $\mathbb{F}_2$-module

$$E(N) := \Gamma_{\mathbb{A},0} / \prod_p \Sigma_p^\sharp(N) \cdot \Gamma_0. \tag{2.3.2}$$

This module is finite. Indeed, following [18] (see Definition 7.4 in chapter 8), we call a prime p *regular* with respect to N if $\Sigma_p^\sharp(N) = \Gamma_{p,0}$. Crucial is the fact that a prime p is regular unless $p \mid \det(N)$; thus, (2.3.2) reduces to finitely many primes p:

$$E(N) = \prod_{p \mid \det(N)} \Gamma_{p,0} / \prod_{p \mid \det(N)} \Sigma_p^\sharp(N) \cdot \Gamma_0. \tag{2.3.3}$$

Theorem 2.3.6 (Miranda–Morrison [18]). *Let N be a non-degenerate indefinite even lattice with* $\mathrm{rk}(N) \geq 3$*. Then there is an exact sequence*

$$O(N) \xrightarrow{d} \mathrm{Aut}(\mathcal{N}) \xrightarrow{\mathrm{e}} E(N) \to g(N) \to 1, \tag{2.3.4}$$

where $g(N)$ is the genus group of N.

A simplified version of (2.3.3) computing the numeric invariants

$$e_p(N) := [\Gamma_{p,0} : \Sigma_p^\sharp(N)] \text{ and } \tilde{\Sigma}_p(N) = \varphi_p^{-1}(\Sigma_p^\sharp(N)) \subset \Gamma_0,$$

is found in [16, 17]. This gives us the size of the group $E(N)$: One has

$$|E(N)| = \frac{e(N)}{[\Gamma_0 : \tilde{\Sigma}(N)]} \tag{2.3.5}$$

where

$$e(N) := \prod_p e_p(N), \quad \tilde{\Sigma}(N) := \bigcap_p \tilde{\Sigma}_p(N),$$

and the product and intersection run over all primes p or, equivalently, over all primes $p \mid \det(N)$.

The following theorem can be deduced from Theorems 2.2.4 and 2.3.6.

Theorem 2.3.7 (Miranda–Morrison [16, 17])**.** *Let S be a primitive sublattice of an even unimodular lattice L such that $N := S^\perp$ is a non-degenerate indefinite even lattice with* $\mathrm{rk}(N) \geq 3$. *Then the strict isomorphism classes of primitive extensions $S \hookrightarrow L$ are in a canonical one-to-one correspondence with the group $E(N)$.*

As explained §2.2.4, given a unimodular lattice L and a primitive sublattice $S \subset L$, one has an anti-isometry $\psi\colon \mathcal{S} \to \mathcal{N}$ (where $N = S^\perp$), which induces a homomorphism $d^\psi\colon O(S) \to \mathrm{Aut}(\mathcal{N})$. If N is indefinite and $\mathrm{rk}(N) \geq 3$, then $d(O(N)) \subset \mathrm{Aut}(\mathcal{N})$ is a normal subgroup with abelian quotient (see (2.3.4)) and we have a homomorphism $d^\perp\colon O(S) \to \mathrm{Aut}(\mathcal{N}) \xrightarrow{\mathrm{e}} E(N)$ independent of the choice of an anti-isometry ψ. The next statement follows from Theorems 2.3.6 and 2.2.4.

Corollary 2.3.1. *Let S be a primitive sublattice of an even unimodular lattice L such that $N := S^\perp$ is a non-degenerate indefinite even lattice with* $\mathrm{rk}(N) \geq 3$ *and let $A \subset O(S)$ be a subgroup. Then, the A-isomorphism classes of primitive extensions $S \hookrightarrow L$ are in a one-to-one correspondence with the $\mathbb{F}_2$-module $E(N)/d^\perp(A)$.*

2.3.2 Reflections

Recall that $\mathrm{Aut}(\mathcal{N}) = \prod_p \mathrm{Aut}(\mathcal{N}_p)$ where p runs over all primes. Let s be a prime and $\alpha \in \mathcal{N}_s$ such that

$$s^k\alpha = 0 \text{ and } \alpha^2 = \frac{2u}{s^k} \bmod 2\mathbb{Z},\ \text{g.c.d}(u,s) = 1, \text{ for some } k \in \mathbb{N}. \tag{2.3.6}$$

We denote by $\mathcal{N}_s^\dagger$ the set of all elements $\alpha \in \mathcal{N}_s$ satisfying (2.3.6) and let $\mathcal{N}^\dagger = \bigcup_s \mathcal{N}_s^\dagger$. Given $\alpha \in \mathcal{N}_s^\dagger$ one can define a map,

$$\mathcal{N}_s \to \mathbb{Z}/s^k, \quad x \mapsto \frac{2(x\cdot\alpha)}{\alpha^2} \bmod s^k$$

Thus, there is a reflection $r_\alpha \in \mathrm{Aut}\,\mathcal{N}_s$ given by

$$r_\alpha\colon x \mapsto x - \frac{2(x\cdot\alpha)}{\alpha^2}\alpha.$$

If $\alpha^2 = \frac{1}{2} \bmod \mathbb{Z}$ and $2\alpha = 0$ then $r_\alpha = \mathrm{id}$.

Let p be a prime and consider the homomorphism

$$\mathrm{Aut}(\mathcal{N}) = \prod_p \mathrm{Aut}(\mathcal{N}_p) \xrightarrow{\phi} \prod_p \Sigma_p(N)/\Sigma_p^\sharp(N)$$

which is the product of the epimorphisms

$$\phi_p\colon\ \mathrm{Aut}(\mathcal{N}_p) \twoheadrightarrow \Sigma_p(N)/\Sigma_p^\sharp(N)$$

introduced in Miranda–Morrison [18] (see chapter 8, section 7). The images of the homomorphism ϕ_p can be computed on reflections as follows: For a prime s and an element $\alpha \in \mathcal{N}_s^\dagger$, the image of the reflection $r_\alpha \in \mathrm{Aut}(\mathcal{N}_s)$ under ϕ_s is given by $\phi_s(r_\alpha) = (-1, us^k)$, see (2.3.6). If $s = 2$ and $\alpha^2 = 0 \bmod \mathbb{Z}$, then $\phi_s(r_\alpha)$ is only well-defined $\bmod \Gamma_2^{++}$. If $s = 2$ and $\alpha^2 = \frac{1}{2} \bmod \mathbb{Z}$, then $\phi_s(r_\alpha)$ is well-defined $\bmod \Gamma_{2,2}$. In these cases to determine the value of $\phi_s(r_\alpha)$, we need more information about α and N.

Given another prime p, we define the *p-norm* $|\alpha|_p \in \{\pm 1\}$ of $\alpha \in \mathcal{N}_s^\dagger$ by

$$|\alpha|_p := \begin{cases} \chi_p(s^k) & \text{if } s \neq p, \\ \chi_p(u) & \text{if } s = p, \end{cases} \tag{2.3.7}$$

where the homomorphism $\chi_p : \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2 \to \{\pm 1\}$ is defined as

$$\chi_p(u) := \begin{cases} \left(\frac{u}{p}\right) & \text{if } p \neq 2, \\ u \bmod 4 & \text{if } p = 2. \end{cases}$$

Note that $|\alpha|_2$ is undefined when $p = 2$ and $\alpha^2 = 0 \mod \mathbb{Z}$. Following [13], given primes p and s and a vector $\alpha \in \mathcal{N}_s^\dagger$, we introduce the group

$$E_p(N) := \begin{cases} \{\pm 1\} & \text{if } p = 1 \bmod 4 \text{ and } e_p(N) \cdot |\tilde{\Sigma}_p(N)| = 8, \\ 1 & \text{otherwise,} \end{cases}$$

the map $\bar{\phi}_p : \mathcal{N}_s^\dagger \to E_p(N)$,

$$\bar{\phi}_p(\alpha) := \begin{cases} 1 & \text{if } E_p(N) = 1, \\ |\alpha|_p & \text{otherwise,} \end{cases}$$

and the map $\bar{\beta}_p : \mathcal{N}_s^\dagger \to \Gamma_0$,

$$\bar{\beta}_p(\alpha) := \begin{cases} (\delta_p(\alpha) \cdot |\alpha|_p, 1) & \text{if } p = 1 \bmod 4, \\ \delta_p(\alpha) \times |\alpha|_p & \text{otherwise,} \end{cases}$$

where the map

$$\delta_p(\alpha) := (-1)^{\delta_{p,s}} \tag{2.3.8}$$

(here $\delta_{p,s}$ is the conventional Kronecker symbol). Note that we have the assignment

$$r_\alpha \mapsto (\delta_p(\alpha), |\alpha|_p) \in \Gamma'_{p,0}.$$

The following lemmas provide an explicit description for the group $E(N)$ and compute the image of the homomorphism e on the reflections r_α for the special case when N has one or two irregular primes.

Lemma 2.3.1 (Akyol–Degtyarev [13])**.** *Let N be a non-degenerate indefinite even lattice with* $\mathrm{rk}(N) \geq 3$, $\Sigma_2^\sharp(N) \supset \Gamma_{2,2}$, *and assume that N has one irregular prime p. Then $E(N) = E_p(N)$ and $\mathrm{e}(r_\alpha) = \bar{\phi}_p(\alpha)$ for any $\alpha \in \mathcal{N}^\dagger$.*

Lemma 2.3.2 (Akyol–Degtyarev [13])**.** *Let N be a non-degenerate indefinite even lattice with* $\mathrm{rk}(N) \geq 3$, $\Sigma_2^\sharp(N) \supset \Gamma_{2,2}$, *and assume that N has two irregular primes p, q. Then*

$$E(N) = E_p(N) \times E_q(N) \times (\Gamma_0/\tilde{\Sigma}_p(N) \cdot \tilde{\Sigma}_q(N)),$$
$$\mathrm{e}(r_\alpha) = \bar{\phi}_p(\alpha) \times \bar{\phi}_q(\alpha) \times (\bar{\beta}_p(\alpha) \cdot \bar{\beta}_q(\alpha)),$$

for any $\alpha \in \mathcal{N}^\dagger$ such that $\alpha^2 \neq 0 \bmod \mathbb{Z}$ if $p = 2$ or $q = 2$.

Corollary 2.3.2 (Akyol–Degtyarev [13]). *Under the hypothesis of Lemma 2.3.2, assume, in addition, that* $|E(N)| = |E_p(N)| = 2$. *Then* $E(N) = E_p(N)$ *and* $\mathrm{e}(r_\alpha) = |\alpha|_p$ *for any* $\alpha \in \mathcal{N}^\dagger$.

2.3.3 Positive sign structure

Let N be a non-degenerate lattice. The orthogonal projection of any maximal positive definite subspace in $N \otimes \mathbb{R}$ to any other such subspace is an isomorphism of vector spaces. Thus a choice of an orientation of one maximal positive definite subspace in $N \otimes \mathbb{R}$ defines a coherent orientation of any other. A choice of an orientation of a maximal positive definite subspace of $N \otimes \mathbb{R}$ is called a *positive sign structure*. We denote by $O^+(N)$ the subgroup of $O(N)$ consisting of the isometries preserving a positive sign structure. Obviously either $O^+(N) = O(N)$ or $O^+(N) \subset O(N)$ is a subgroup of index 2. In the latter case, each element of $O(N) \smallsetminus O^+(N)$ is called a $+$-*disorienting* isometry of N. Following [18], we define the map $\det_+ : O(N) \to \{\pm 1\}$ as

$$\det_+(a) := \begin{cases} +1 & \text{if } a \text{ preserves the positive sign structure,} \\ -1 & \text{if } a \text{ reserves the positive sign structure.} \end{cases}$$

Note that $\operatorname{Ker}(\det_+) = O^+(N)$.

Proposition 2.3.1 (Miranda–Morrison [18]). *Let N be a non-degenerate indefinite even lattice with* $\operatorname{rk}(N) \geq 3$. *Then one has* $\tilde{\Sigma}(N) \subset \Gamma_0^{--}$ *if and only if* $\det_+(a) = 1$ *for all* $a \in \operatorname{Ker}[d \colon O(N) \to \operatorname{Aut}(\mathcal{N})]$.

Hence, if $\tilde{\Sigma} \subset \Gamma_0^{--}$, the map $\det_+$ has a well-defined descent $\det_+ \colon \operatorname{Im} d \to \{\pm 1\}$. The following lemma computes the images of the function $\det_+$ on reflections.

Lemma 2.3.3 (Akyol–Degtyarev [13]). *Let N be a non-degenerate indefinite even lattice with* $\operatorname{rk}(N) \geq 3$, $\Sigma_2^\sharp(N) \supset \Gamma_{2,2}$, *and assume that there is a prime p such that* $\tilde{\Sigma}_p(N) \subset \Gamma_0^{--}$. *Then, for an element* $\alpha \in \mathcal{N}^\dagger$ *such that* $r_\alpha \in \operatorname{Im} d$ *and* $\alpha^2 \neq 0 \bmod \mathbb{Z}$ *if* $p = 2$, *one has* $\det_+(r_\alpha) = \delta_p(\alpha) \cdot |\alpha|_p$.

Defined in [17], we introduce the group

$$E^+(N) := \Gamma_{\mathbb{A},0} / \prod_p \Sigma_p^\sharp(N) \cdot \Gamma_0^{--}. \tag{2.3.9}$$

(Similar to (2.3.2) and (2.3.3) the actual computation reduces to finitely many primes $p \mid \det(N)$.) As in Theorem 2.3.6 there is an exact sequence

$$O^+(N) \xrightarrow{d} \operatorname{Aut}(\mathcal{N}) \xrightarrow{\mathrm{e}^+} E^+(N) \to g(N) \to 1.$$

The size of the group $E^+(N)$ is also computed in [17]: one replaces $[\Gamma_0 : \tilde{\Sigma}(N)]$ in (2.3.5) with $[\Gamma_0^{--} : \tilde{\Sigma}(N) \cap \Gamma_0^{--}]$. For an irregular prime p, we denote $\tilde{\Sigma}_p^+(N) := \tilde{\Sigma}_p(N) \cap \Gamma_0^{--}$.

Given a unimodular even lattice L and a primitive sublattice $S \subset L$ such that $N := S^\perp$ is a non-degenerate indefinite lattice with $\operatorname{rk}(N) \geq 3$, then $d(O(S)) \subset \operatorname{Aut}(\mathcal{N})$ is a normal subgroup with abelian quotient (see (2.3.4)) and we have a well-defined homomorphism $d_+^\perp \colon O(S) \to \operatorname{Aut}(\mathcal{N}) \xrightarrow{\mathrm{e}^+} E^+(N)$ independent of the choice of an anti-isometry ψ, *cf.* the definition of $d^\perp$ in §2.3.

Let p and s be two irregular primes and choose an element $\alpha \in \mathcal{N}_s^\dagger$ as in (2.3.6), we introduce the group

$$E_p^+(N) := \begin{cases} E_p(N) & \text{if } p = 1 \bmod 4, \\ \Gamma_0/\tilde{\Sigma}_p(N) \cdot \Gamma_0^{--} & \text{otherwise,} \end{cases}$$

the map $\bar{\phi}_p^+ \colon \mathcal{N}_s^\dagger \to E_p^+(N)$,

$$\bar{\phi}_p^+(\alpha) := \begin{cases} \bar{\phi}_p(\alpha) & \text{if } p = 1 \bmod 4, \\ \delta_p(\alpha) \cdot |\alpha|_p & \text{if } p \neq 1 \bmod 4 \text{ and } E_p^+(N) \neq 1, \\ 1 & \text{if } p \neq 1 \bmod 4 \text{ and } E_p^+(N) = 1, \end{cases}$$

and the map $\bar{\beta}_p^+ \colon \mathcal{N}_s^\dagger \to \Gamma_0^{--}$,

$$\bar{\beta}_p^+(\alpha) := \begin{cases} \delta_p(\alpha) \cdot |\alpha|_p & \text{if } p = 1 \bmod 4, \\ |\alpha|_p & \text{if } p \neq 1 \bmod 4 \text{ and } E_p^+(N) \neq 1, \\ \operatorname{proj}(\bar{\beta}_p(\alpha)) & \text{if } p \neq 1 \bmod 4 \text{ and } E_p^+(N) = 1, \end{cases}$$

where $\operatorname{proj} \colon \Gamma_0 \to \Gamma_0/\tilde{\Sigma}_p(N) = \Gamma_0^{--}$ is the projection map. Next lemma computes the group $E^+(N)$ and the values of the homomorphism e^+ on the reflections r_α

Lemma 2.3.4 (Akyol–Degtyarev [13]). *Let N be a non-degenerate indefinite even lattice with $\operatorname{rk}(N) \geq 3$, $\Sigma_2^\sharp(N) \supset \Gamma_{2,2}$ and assume that N has two irregular primes p, q. Then*

$$E^+(N) = E_p^+(N) \times E_q^+(N) \times (\Gamma_0^{--}/\tilde{\Sigma}_p^+(N) \cdot \tilde{\Sigma}_q^+(N))$$
$$\mathrm{e}^+(r_\alpha) = \bar{\phi}_p^+(\alpha) \times \bar{\phi}_q^+(\alpha) \times (\bar{\beta}_p^+(\alpha) \cdot \bar{\beta}_q^+(\alpha))$$

for any $\alpha \in \mathcal{N}^\dagger$ *such that* $\alpha^2 \neq 0 \bmod \mathbb{Z}$ *if* $p = 2$ *or* $q = 2$.

Chapter 3

Simple Quartics

The main objective of this chapter is to present the arithmetical reduction of the classification problem. We start with the discussion of the relation between simple quartics and $K3$-surfaces which enables us to study the classification problem by means of the global Torelli theorem and the surjectivity of the period map. Then we introduce the notion of abstract homological type. The classification problem can be reduced to an arithmetical problem concerning enumeration of the abstract homological types. In the last part, we confine ourselves to the non-special quartics and prove Theorem 1.1.1 which is one of our principal results.

3.1 Quartics and $K3$-surfaces

A *quartic* is a surface $X \subset \mathbb{P}^3$ of degree four. A quartic is *simple* if all its singular points are simple, i.e., those of type $\mathbf{A}, \mathbf{D}, \mathbf{E}$. Isomorphism classes of simple singularities are known to be in a one-to-one correspondence with those of irreducible root systems (see Dufree [1] for details). Hence, a set of simple singularities can be identified with a root system, the irreducible summands of the latter (see §2.2.3) corresponding to the individual singularity points.

Let $X \subset \mathbb{P}^3$ be a simple quartic and consider its minimal resolution of singularities $\tilde{X}$. It is well known that $\tilde{X}$ is a $K3$-surface; hence, $H_2(\tilde{X}) \cong 2\mathbf{E}_8 \oplus 3\mathbf{U}$, where

$\mathbf{U}$ is the hyperbolic plane defined as $\mathbf{U} := \mathbb{Z}u_1 \oplus \mathbb{Z}u_2$, $u_1^2 = u_2^2 = 0$ and $u_1 \cdot u_2 = 1$, its signature is $(1, 1)$. The root lattice $\mathbf{E}_8$ is the even unimodular, negative definite lattice of signature $(0, 8)$. Note that $2\mathbf{E}_8 \oplus 3\mathbf{U}$ is the only even unimodular lattice of signature $(\sigma_+, \sigma_-) = (3, 19)$. We fix the notation $\mathbf{L}_X := H_2(\tilde{X})$ and $\mathbf{L} := 2\mathbf{E}_8 \oplus 3\mathbf{U}$.

For each simple singular point p of X the components of the exceptional divisor over p are smooth rational (-2)-curves spanning an irreducible root lattice in $\mathbf{L}_X$. These sublattices are obviously orthogonal and their orthogonal sum, identified with the set of singularities of X, is denoted by $\mathbf{S}_X$. The rank $\operatorname{rk}(\mathbf{S}_X)$ equals the total Milnor number $\mu(X)$. Since $\sigma_-(\mathbf{L}) = 19$ and $\mathbf{S}_X \subset \mathbf{L}$ is negative definite, one has $\mu(X) \leq 19$ (see [9], *cf.*, [22]). If $\mu(X) = 19$, the quartic is called *maximizing*. We introduce the following objects:

- $\mathbf{S}_X \subset \mathbf{L}_X$: the sublattice generated the set of classes of exceptional divisors contracted by the blow-down map $\tilde{X} \to X$;
- $h_X \in \mathbf{L}_X$: the class of the pull-back of a generic plane section of X;
- $\mathbf{S}_{X,h} = \mathbf{S}_X \oplus \mathbb{Z}h_X \subset \mathbf{L}_X$;
- $\tilde{\mathbf{S}}_X \subset \tilde{\mathbf{S}}_{X,h} \subset \mathbf{L}_X$: the primitive hulls of $\mathbf{S}_X$ and $\mathbf{S}_{X,h}$, respectively, *i.e*, $\tilde{\mathbf{S}}_X := (\mathbf{S}_X \otimes \mathbb{Q}) \cap \mathbf{L}_X$ and $\tilde{\mathbf{S}}_{X,h} := (\mathbf{S}_{X,h} \otimes \mathbb{Q}) \cap \mathbf{L}_X$, .
- $\omega_X \subset \mathbf{L}_X \otimes \mathbb{R}$: the oriented 2-subspace spanned by the real and imaginary parts of the class of a holomorphic 2-form on $\tilde{X}$ (the *period* of $\tilde{X}$).

The triple $(\mathbf{S}_X, h_X, \mathbf{L}_X)$ is called the *homological type of* X.

3.2 Abstract homological types

The set of singularities of a quartic $X \subset \mathbb{P}^3$ can be viewed as a root lattice $\mathbf{S} \subset \mathbf{L}$.

Definition 3.2.1. *A configuration (extending a given set of singularities* $\mathbf{S}$*) is a finite index extension* $\tilde{\mathbf{S}}_h \supset \mathbf{S}_h := \mathbf{S} \oplus \mathbb{Z}h$*,* $h^2 = 4$*, satisfying the following conditions:*

1. *each root* $r \in (\mathbf{S} \otimes \mathbb{Q}) \cap \tilde{\mathbf{S}}_h$ *with* $r^2 = -2$ *is in* $\mathbf{S}$,

2. $\tilde{\mathbf{S}}_h$ *does not contain an element* v *with* $v^2 = 0$ *and* $v \cdot h = 2$.

An automorphism *of a configuration* $\tilde{\mathbf{S}}_h$ *is an auto-isometry of* $\tilde{\mathbf{S}}_h$ *preserving* h. *The group of automorphisms of* $\tilde{\mathbf{S}}_h$ *is denoted by* $\mathrm{Aut}_h(\tilde{\mathbf{S}}_h)$. *One has the obvious inclusions* $\mathrm{Aut}_h(\tilde{\mathbf{S}}_h) \subset O(\tilde{\mathbf{S}}) \subset O(\mathbf{S})$, *the latter is due to* (1) *in Definition 3.2.1, since* $\mathbf{S}$ *is recovered as the sublattice in* $h^\perp \subset \tilde{\mathbf{S}}_h$ *generated by roots.*

Definition 3.2.2. *An abstract homological type extending a fixed set of singularities* $\mathbf{S}$ *is an extension of* $\mathbf{S}_h := \mathbf{S} \oplus \mathbb{Z}h$, $h^2 = 4$, *to a lattice* L *isomorphic to* $2\mathbf{E}_8 \oplus 3\mathbf{U}$, *such that the primitive hull* $\tilde{\mathbf{S}}_h$ *of* $\mathbf{S}_h$ *in* L *is a configuration.*

An abstract homological type is uniquely determined by the triple $\mathcal{H} = (\mathbf{S}, h, \mathbf{L})$. An *isomorphism* between two abstract homological types $\mathcal{H}_i = (\mathbf{S}_i, h_i, \mathbf{L}_i)$, $i = 1, 2$, is an isometry $\mathbf{L}_1 \to \mathbf{L}_2$, taking h_1 and $\mathbf{S}_1$ to h_2 and $\mathbf{S}_2$, respectively (as a set).

Given an abstract homological type $\mathcal{H} = (\mathbf{S}, h, \mathbf{L})$, we let $\tilde{\mathbf{S}} := (\mathbf{S} \otimes \mathbb{Q}) \cap \mathbf{L}$ and $\tilde{\mathbf{S}}_h := (\mathbf{S}_h \otimes \mathbb{Q}) \cap \mathbf{L}$ be the primitive hulls of $\mathbf{S}$ and $\mathbf{S}_h$, respectively. Note that $\tilde{\mathbf{S}} = h^\perp_{\tilde{\mathbf{S}}_h}$, *i.e.*, $\tilde{\mathbf{S}}$ is also the primitive hull of $h^\perp$. The orthogonal complement $\mathbf{S}_h^\perp$ is a non-degenerate lattice with $\sigma_+ \mathbf{S}_h^\perp = 2$. It follows that all positive definite 2-subspaces in $\mathbf{S}_h^\perp \otimes \mathbb{R}$ can be oriented in a coherent way (see §2.3.3).

Definition 3.2.3. *An orientation of an abstract homological type* $\mathcal{H} = (\mathbf{S}, h, \mathbf{L})$ *is a choice* θ *of one of the coherent orientations of positive definite* 2*-subspaces of* $\mathbf{S}_h^\perp \otimes \mathbb{R}$

An *isomorphism* between two oriented singular homological type $(\mathcal{H}_i, \theta_i)$, $i = 1, 2$, is an isomorphism $\mathcal{H}_1 \to \mathcal{H}_2$, taking θ_1 to θ_2. A singular homological type is called *symmetric* if $(\mathcal{H}, \theta)$ is isomorphic to $(\mathcal{H}, -\theta)$ for some orientation θ of $\mathcal{H}$, i.e., $\mathcal{H}$ admits an automorphism reversing the orientation.

3.3 Deformation classification of simple quartics

Due to Saint-Donat [23] and Urabe [9], a triple $\mathcal{H} = (\mathbf{S}, h, \mathbf{L})$ is isomorphic to the homological type $(\mathbf{S}_X, h_X, \mathbf{L}_X)$ of a simple quartic $X \subset \mathbb{P}^3$ if and only if $\mathcal{H}$ is an abstract homological type in the sense of Definition 3.2.2. In this case, the oriented 2-subspace ω_X introduced in §3.1 defines an orientation of $\mathcal{H}$.

The following statement is a consequence of the global Torelli theorem and the surjectivity of the period map.

Theorem 3.3.1 (see Theorem 2.3.1 in [12])**.** *The map sending a simple quartic surface $X \subset \mathbb{P}^3$ to its oriented homological type establishes a one to one correspondence between the set of equisingular deformation classes of quartics with a given set of simple singularities* $\mathbf{S}$ *and the set of isomorphism classes of oriented abstract homological types extending* $\mathbf{S}$*. Complex conjugate quartics have isomorphic homological types that differ by the orientations.*

3.4 Non-special quartics

Definition 3.4.1. *A quartic X is called* non-special *if its homological type is primitive, i.e.,* $\mathbf{S}_h \subset \mathbf{L}$ *is a primitive sublattice.*

Note that the homological type $\mathcal{H} = (\mathbf{S}, h, \mathbf{L})$ is primitive if and only if $\tilde{\mathbf{S}}_h = \mathbf{S}_h$, in this case, one has $\operatorname{disc}\tilde{\mathbf{S}}_h = \mathcal{S} \oplus \langle \frac{1}{4} \rangle$ and $\operatorname{Aut}_h(\tilde{\mathbf{S}}_h) = O(\mathbf{S})$.

For a given set of simple singularities $\mathbf{S}$, the corresponding equisingular stratum of quartics is denoted by $\mathcal{M}(\mathbf{S})$. Our primary interest is the family $\mathcal{M}_1(\mathbf{S}) \subset \mathcal{M}(\mathbf{S})$ constituted by the non-special quartics with the set of singularities $\mathbf{S}$. More generally, since the kernel $\mathcal{K}$ of the finite index extension $\mathbf{S}_h \subset \tilde{\mathbf{S}}_h$ is obviously invariant under equisingular deformations, one can consider the strata $\mathcal{M}_*(\mathbf{S}) \subset \mathcal{M}(\mathbf{S})$ where the subscript $*$ is the sequence of invariant factors of the kernel $\mathcal{K}$.

Our study of non-special quartics $\mathcal{M}_1(\mathbf{S})$ is motivated by the Theorem 1.1.1 which is proved below.

Proof of the Theorem 1.1.1. Let $X \subset \mathbb{P}^3$ be a simple quartic and consider its minimal resolution of singularities $\tilde{X}$. We denote by E the exceptional divisor of the blow up $\tilde{X} \to X$. Note that $X \smallsetminus (\operatorname{Sing} X \cup H) \cong \tilde{X} \smallsetminus (E \cup H)$, where $\operatorname{Sing} X$ is the set of the singular points of X and H is a generic hyperplane section. Recall that $\mathbf{S}_X$ is the sublattice in $\mathbf{L}_X = H_2(\tilde{X})$ generated by the components of E (see §3.1). Thus, one has $H_2(E \cup H) = \mathbf{S}_X \oplus \mathbb{Z}h_X \subset \mathbf{L}_X = H_2(\tilde{X})$.

We have the following cohomology exact sequence of pair $(\tilde{X}, E \cup H)$:

$$\cdots \xrightarrow{j^*} H^2(\tilde{X}) \xrightarrow{i^*} H^2(E \cup H) \xrightarrow{\delta} H^3(\tilde{X}, E \cup H) \xrightarrow{j^*} \underbrace{H^3(\tilde{X})}_{0} \to \cdots .$$

Hence, $H^3(\tilde{X}, E \cup H) = \operatorname{coker} i^*$. By universal coefficients, for $Y = \tilde{X}$ or $E \cup H$ we have a natural exact sequence

$$0 \to \operatorname{Ext}(H_1(Y)) \to H^2(Y) \to \operatorname{Hom}(H_2(Y)) \to 0$$

Since in both cases $H_1(Y)$ is torsion free, we have $H^2(Y) = \operatorname{Hom}(H, \mathbb{Z})$ and i^* is the adjoint of the map

$$i_*\colon H_2(E \cup H) \to H_2(\tilde{X}),$$

which is the inclusion $\mathbf{S}_{X,h} \hookrightarrow \mathbf{L}_X$. By definition of the primitive hull $\tilde{\mathbf{S}}_{X,h}$ and of the other groups involved, we have an exact sequence

$$0 \to H_2(E \cup H) \xrightarrow{i_*} H_2(\tilde{X}) \to \tilde{\mathbf{S}}_{X,h}/\mathbf{S}_{X,h} \oplus \mathrm{F} \to 0,$$

where F is a finitely generated free abelian group and $\tilde{\mathbf{S}}_{X,h}/\mathbf{S}_{X,h}$ is a torsion group. This sequence can be regarded as a free resolution of $\tilde{\mathbf{S}}_{X,h}/\mathbf{S}_{X,h} \oplus \mathrm{F}$ and, by the definition of derived functor, we have the following isomorphisms

$$\operatorname{coker} i^* = \operatorname{Ext}\left(\tilde{\mathbf{S}}_{X,h}/\mathbf{S}_{X,h} \oplus \mathrm{F}, \mathbb{Z}\right) = \operatorname{Ext}\left(\tilde{\mathbf{S}}_{X,h}/\mathbf{S}_{X,h}, \mathbb{Z}\right).$$

Combining these observations with Poincaré–Lefschetz duality $H_1(\tilde{X} \smallsetminus (E \cup H)) = H^3(\tilde{X}, E \cup H)$, we conclude that

$$H_1(\tilde{X} \smallsetminus (E \cup H)) = \operatorname{Ext}\left(\tilde{\mathbf{S}}_{X,h}/\mathbf{S}_{X,h}, \mathbb{Z}\right) \cong \tilde{\mathbf{S}}_{X,h}/\mathbf{S}_{X,h}$$

(the last isomorphism being not natural). In particular $H_1(\tilde{X} \smallsetminus (E \cup H)) = 0$ if and only if $\mathbf{S}_{X,h} = \tilde{\mathbf{S}}_{X,h}$ *i.e*, if and only if X is non-special. $\square$

Chapter 4

Proof of The Principal Result

This chapter is devoted to the proof of the main result of this thesis. To prove our principal result Theorem 1.1.2, following the classical approach, we reduce the problem of classifying complex non-special quartics up to equisingular deformation classification to an arithmetical problem about lattices. On this arithmetical side, after applying Nikulin's existence theorem [8], the principal novelty is the systematic usage of the lattice theoretical results of Miranda-Morrison theory [16, 17, 18].

4.1 Statement

We studied sets of simple singularities $\mathbf{S}$ realized by non-special quartics and we found that there 2872 such singularities realized by non-maximal, non-special quartics and there are 59 maximizing sets (see Table 1.1) of simple singularities realized by non-special quartics. The main difficulty here is the great number of strata. Hence, in the existence part we follow the degeneration approach given in [13] and describe only those strata that are extremal with respect to degeneration. The following theorem is our principal result which gives a complete description of the equisingular strata of non-special quartics. For the readers' convenience, we restate Theorem 1.1.2 which is already stated in the introduction as our principal result.

Theorem 4.1.1. *A set of singularities* $\mathbf{S}$ *is realizable as the set of singularities of a non-special simple quartic if and only if* $\mathbf{S}$ *can be obtained by a perturbation from one of the sets of singularities listed in Tables* 1 *and* 2*. The numbers* (r, c) *of, respectively, real and pairs of complex conjugate components of the strata* $\mathcal{M}_1(\mathbf{S})$ *with* $\mu(\mathbf{S}) = 19$ *are shown in Table* 1*. If* $\mathbf{S}$ *is one of*

$$\mathbf{D}_6 \oplus 2\mathbf{A}_6, \quad \mathbf{D}_5 \oplus 2\mathbf{A}_6 \oplus \mathbf{A}_1, \quad 2\mathbf{A}_7 \oplus 2\mathbf{A}_2, \quad 3\mathbf{A}_6, \quad 2\mathbf{A}_6 \oplus 2\mathbf{A}_3$$

then $\mathcal{M}_1(\mathbf{S})$ *consists of two complex conjugate components; in all other cases, the stratum* $\mathcal{M}_1(\mathbf{S})$ *is connected.*

4.2 Proof of Theorem 4.1.1

For the reader's convenience, we divide the proof into three propositions, of which Theorem 4.1.1 is an immediate consequence.

4.2.1 The existence

First we prove the existence part of the Theorem 4.1.1 by checking all possible sets of singularities one by one for realizability, we obtain a list which is too big to state explicitly. Instead, we restate this list in terms of perturbations as follows.

Proposition 4.2.1. *Realizable are all sets of singularities that can be obtained by a perturbation from either the* 59 *maximizing sets of singularities listed in Table* 1.1 *or* 19 *sets of singularities with the Milnor number* 18 *listed in Table* 1.2.

Proof. According to Theorems 3.3.1 and Definition 3.4.1, a set of singularities $\mathbf{S}$ is realized by a non-special quartic if and only if $\mathbf{S}$ extends to a primitive homological type. Thus, we are interested in primitive extensions $\mathbf{S}_h \hookrightarrow \mathbf{L} = 3\mathbf{U} \oplus 3\mathbf{E}_8$. Since the homological type is primitive, one has disc $\tilde{\mathbf{S}}_h = \mathcal{S} \oplus \langle \frac{1}{4} \rangle$, and the realizable sets are easily found by using Nikulin's Existence Theorem (Theorem 2.2.3) applied to the genus of the transcendental lattice $\mathbf{T} := \mathbf{S}^{\perp}$, which is determined by $\mathbf{S}$, see §2.2.4. Implementing the algorithm in GAP [24], we found that 2872 sets of simple singularities are realized by non-maximal non-special quartics and 59 sets of simple singularities are realized by maximal non-special quartics. According to E. Looijenga [25], deformation classes of perturbations of an individual simple singularity of type $\mathbf{S}$ are in a one-to-one correspondence with the isomorphism classes of primitive extensions $\mathbf{S}' \hookrightarrow \mathbf{S}$ of root lattices, see §2.2.3 and §2.2.4. As shown in [26], the latter is the case if and only if the Dynkin graph of $\mathbf{S}'$ is an induced subgraph of that of $\mathbf{S}$. Hence, given a simple quartic X, any perturbation X to a simple quartic X' gives rises to a perturbation of the set of singularities $\mathbf{S}$ of X to the set of singularities $\mathbf{S}'$ of X'. Conversely, any induced subgraph of the Dynkin graph of a simple quartic X is that of an appropriate small perturbation X' of X. Proof of this statement repeats, almost literally, the proof of a similar theorem for plane sextic curves (see Proposition 5.1.1 in [27]). Accordingly, the list of 2872 sets of simple singularities realized by non-maximal non-special quartics is compared against the list of all perturbations of the 59 maximizing sets of singularities given in Table 1.1 and 19 sets of singularities with Milnor number 18 given in Table 1.2 which can not be obtained by a perturbation from any of 59 sets of singularities given in Table 1.1. The two lists coincide. $\square$

Let $\mathbf{S}$ be one of the realizable sets of singularities and $\mathbf{T}$ a representative of the genus $g(\mathbf{S}_h^\perp)$. By Theorem 3.3.1, the connected components of the space $\mathcal{M}_1(\mathbf{S})$ modulo complex conjugation $\operatorname{conj} : \mathbb{P}^3 \to \mathbb{P}^3$ are enumerated by the isomorphism classes of primitive homological types extending $\mathbf{S}$. We investigate these isomorphism classes separately for the maximizing case, *i.e.*, $\mu(\mathbf{S}) = 19$, and non-maximizing case, *i.e*, $\mu(\mathbf{S}) \leq 18$.

If $\mu(\mathbf{S}) = 19$, the transcendental lattice $\mathbf{T}$ is a positive definite sublattice of rank 2, and the numbers (r, c) of connected components of the space $\mathcal{M}_1(\mathbf{S})$ listed in Table 1.1 can easily be computed by Gauss theory of binary quadratic forms [28] (A. Degtyarev, private communication); details will appear elsewhere. Thus, throughout the rest of the proof we assume $\mu(\mathbf{S}) \leq 18$.

4.2.2 The strata in $\mathcal{M}_1/\operatorname{conj}$

We classify non-special simple quartics up to equisingular deformation and complex conjugation.

Proposition 4.2.2. *For each realizable set of singularities* $\mathbf{S}$ *with* $\mu(\mathbf{S}) \leq 18$*, the space* $\mathcal{M}_1(\mathbf{S})/\operatorname{conj}$ *is connected.*

Proof. If $\mu(\mathbf{S}) \leq 18$, then $\mathbf{T}$ is an indefinite lattice with $\operatorname{rk}\mathbf{T} \geq 3$ and we can apply Mirranda–Morison's theory. We try to enumerate primitive homological types $\mathcal{H} = (\mathbf{S}, h, \mathbf{L})$ extending $\mathbf{S}$, *i.e.*, the primitive extensions $\mathbf{S}_h \hookrightarrow L$. Since the extension is primitive, $\tilde{\mathbf{S}}_h = \mathbf{S}_h$, one has $\operatorname{disc}\tilde{\mathbf{S}}_h = \mathcal{S} \oplus \langle \frac{1}{4} \rangle$ and $\operatorname{Aut}_h(\mathbf{S}_h) \cong O(\mathbf{S})$. Then we have a well-defined homomorphism $d^{\perp}\colon O(\mathbf{S}) \to E(\mathbf{T})$, and by Corollary 2.3.1,

$$\pi_0(\mathcal{M}_1(\mathbf{S})/\operatorname{conj}) \cong \operatorname{Coker}\big(d^{\perp} : O(\mathbf{S}) \to E(\mathbf{T})\big). \tag{4.2.1}$$

Thus, the space $\mathcal{M}_1(\mathbf{S})/\operatorname{conj}$ is connected (equivalently, the primitive homological type extending $\mathbf{S}$ is unique up to isomorphism) if and only if the map $d^{\perp}$ is surjective, and it is this latter statement that we prove below.

Out of the 2872 sets of singularities realized by non-special non-maximizing quartics, for 2830 sets of singularities one gets $E(\mathbf{T}) = 1$ by using (2.3.5), and the assertion follows automatically.

One has $|E(\mathbf{T})| > 1$ for the remaining 42 cases listed in Table 4.1 in which for each set of singularities $\mathbf{S}$, we list

- the order of the group $|E(\mathbf{T})|$,
- the irregular primes
- the map $d^{\perp} : O(\mathbf{S}) \to \operatorname{Aut}(\mathcal{T}) \xrightarrow{\mathrm{e}} E(\mathbf{T})$ in terms of the images $\mathrm{e}(r_\alpha)$ computed by applying Lemma 2.3.1, Lemma 2.3.2 and Corollary 2.3.2 (see Remark 4.2.1).
- elements of $O(\mathbf{S})$ generating $E(\mathbf{T})$ (see Remark 4.2.2)

Remark 4.2.1. *For each set of singularity in Table 4.1, the map* $\mathrm{e}\colon \operatorname{Aut}(\mathcal{T}) \to E(\mathbf{T})$ *is given in terms of the images of reflections* $r_\alpha \in \operatorname{Aut}(\mathcal{T})$*, where* α *is an element of* $\mathcal{T}$ *satisfying 2.3.6 and the functions* $\delta_p(\alpha)$ *and* $|\alpha|_p$ *(for some prime p)are given by* (2.3.7) *and* (2.3.8)

Table 4.1: The spaces $\mathcal{M}_1(\mathbf{S})/\operatorname{conj}$ with $|E(\mathbf{T})| > 1$

Singularities	$\lvert E(T)\rvert$	primes	e: $\mathrm{Aut}(\mathcal{T}) \to E(\mathbf{T})$	generators of $E(\mathbf{T})$
$2\mathbf{E}_6 \oplus \mathbf{D}_4 \oplus \mathbf{A}_2$	2	$2,3$	$\delta_2(\alpha)\cdot\delta_3(\alpha)$	$\mathbf{A}_2^{\circlearrowleft}$
$\mathbf{E}_6 \oplus 2\mathbf{D}_5 \oplus \mathbf{A}_2$	2	$2,3$	$\lvert\alpha\rvert_2\cdot\lvert\alpha\rvert_3$	$\mathbf{A}_2^{\circlearrowleft}$
$2\mathbf{D}_7 \oplus 2\mathbf{A}_2$	2	$2,3$	$\delta_2(\alpha)\cdot\delta_3(\alpha)\cdot\lvert\alpha\rvert_2\cdot\lvert\alpha\rvert_3$	$\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$2\mathbf{E}_6 \oplus 2\mathbf{A}_3$	2	$2,3$	$\delta_2(\alpha)\cdot\delta_3(\alpha)\cdot\lvert\alpha\rvert_2\cdot\lvert\alpha\rvert_3$	$\mathbf{A}_3^{\circlearrowleft}$
$\mathbf{E}_8 \oplus \mathbf{D}_4 \oplus 3\mathbf{A}_2$	2	$2,3$	$\delta_2(\alpha)\cdot\delta_3(\alpha)$	$\mathbf{A}_2^{\circlearrowleft}$
$\mathbf{D}_7 \oplus \mathbf{D}_5 \oplus 3\mathbf{A}_2$	4	$2,3$	$(\delta_2(\alpha)\cdot\delta_3(\alpha), \lvert\alpha\rvert_2\cdot\lvert\alpha\rvert_3)$	$\mathbf{A}_2^{\circlearrowleft}$ and $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$\mathbf{D}_{12} \oplus 3\mathbf{A}_2$	2	$2,3$	$\delta_2(\alpha)\cdot\delta_3(\alpha)$	$\mathbf{A}_2^{\circlearrowleft}$
$\mathbf{D}_{11} \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$	2	$2,3$	$\delta_2(\alpha)\cdot\delta_3(\alpha)\cdot\lvert\alpha\rvert_2\cdot\lvert\alpha\rvert_3$	$\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$\mathbf{E}_6 \oplus \mathbf{D}_4 \oplus 2\mathbf{A}_4$	2	$2,5$	$\lvert\alpha\rvert_5$	$\mathbf{A}_4^{\circlearrowleft}$
$\mathbf{D}_5 \oplus \mathbf{D}_4 \oplus 4\mathbf{A}_2$	2	$2,3$	$\delta_2(\alpha)\cdot\delta_3(\alpha)$	$\mathbf{A}_2^{\circlearrowleft}$
$2\mathbf{D}_5 \oplus 2\mathbf{A}_4$	2	$2,5$	$\delta_2(\alpha)\cdot\delta_5(\alpha)\cdot\lvert\alpha\rvert_5$	$\mathbf{A}_4 \leftrightarrow \mathbf{A}_4$
$\mathbf{D}_9 \oplus \mathbf{A}_3 \oplus 3\mathbf{A}_2$	4	$2,3$	$(\delta_2(\alpha)\cdot\delta_3(\alpha), \lvert\alpha\rvert_2\cdot\lvert\alpha\rvert_3)$	$\mathbf{A}_2^{\circlearrowleft}$ and $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$\mathbf{D}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_1$	2	5	$\lvert\alpha\rvert_5$	$\mathbf{A}_4^{\circlearrowleft}$
$\mathbf{E}_8 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_2$	2	$2,3$	$\delta_2(\alpha)\cdot\delta_3(\alpha)\cdot\lvert\alpha\rvert_2\cdot\lvert\alpha\rvert_3$	$\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$\mathbf{E}_8 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2$	2	5	$\lvert\alpha\rvert_5$	$\mathbf{A}_2^{\circlearrowleft}$ or $\mathbf{A}_4^{\circlearrowleft}$
$\mathbf{E}_7 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	2	5	$\lvert\alpha\rvert_5$	$\mathbf{A}_2^{\circlearrowleft}$ or $\mathbf{A}_4^{\circlearrowleft}$
$\mathbf{E}_7 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_3$	2	5	$\lvert\alpha\rvert_5$	$\mathbf{A}_4^{\circlearrowleft}$
$\mathbf{D}_7 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$	2	$2,3$	$\lvert\alpha\rvert_2\cdot\lvert\alpha\rvert_3$	$\mathbf{A}_2^{\circlearrowleft}$
$\mathbf{D}_7 \oplus \mathbf{A}_7 \oplus 2\mathbf{A}_2$	2	$2,3$	$\lvert\alpha\rvert_2\cdot\lvert\alpha\rvert_3$	$\mathbf{A}_2^{\circlearrowleft}$
$\mathbf{D}_5 \oplus 2\mathbf{A}_3 \oplus 3\mathbf{A}_2$	2	$2,3$	$\lvert\alpha\rvert_2\cdot\lvert\alpha\rvert_3$	$\mathbf{A}_2^{\circlearrowleft}$
$\mathbf{D}_6 \oplus 3\mathbf{A}_4$	2	5	$\lvert\alpha\rvert_5$	$\mathbf{A}_4^{\circlearrowleft}$
$\mathbf{D}_4 \oplus 3\mathbf{A}_4 \oplus \mathbf{A}_1$	2	5	$\lvert\alpha\rvert_5$	$\mathbf{A}_4^{\circlearrowleft}$
$\mathbf{D}_5 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$	2	$3,5$	$\lvert\alpha\rvert_5$	$\mathbf{A}_2^{\circlearrowleft}$ or $\mathbf{A}_4^{\circlearrowleft}$
$\mathbf{D}_5 \oplus \mathbf{A}_6 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$	2	$2,3$	$\delta_2(\alpha)\cdot\delta_3(\alpha)\cdot\lvert\alpha\rvert_2\cdot\lvert\alpha\rvert_3$	$\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$\mathbf{D}_4 \oplus 2\mathbf{A}_4 \oplus 3\mathbf{A}_2$	4	$2,3,5$	see Table 4.2	
$3\mathbf{A}_{10} \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$	2	5	$\lvert\alpha\rvert_5$	$\mathbf{A}_2^{\circlearrowleft}$ or $\mathbf{A}_4^{\circlearrowleft}$
$3\mathbf{A}_4 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$	2	5	$\lvert\alpha\rvert_5$	$\mathbf{A}_2^{\circlearrowleft}$ or $\mathbf{A}_4^{\circlearrowleft}$
$2\mathbf{A}_4 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_2$	4	$2,3,5$	see Table 4.3	
$3\mathbf{A}_4 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$	2	$3,5$	$\lvert\alpha\rvert_5$	$\mathbf{A}_2^{\circlearrowleft}$ or $\mathbf{A}_4^{\circlearrowleft}$
$3\mathbf{A}_4 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	2	5	$\lvert\alpha\rvert_5$	$\mathbf{A}_2^{\circlearrowleft}$ or $\mathbf{A}_4^{\circlearrowleft}$
$\mathbf{A}_5 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$	2	$3,5$	$\lvert\alpha\rvert_5$	$\mathbf{A}_2^{\circlearrowleft}$ or $\mathbf{A}_4^{\circlearrowleft}$
$\mathbf{A}_6 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2$	2	$3,5$	$\lvert\alpha\rvert_5$	$\mathbf{A}_2^{\circlearrowleft}$ or $\mathbf{A}_4^{\circlearrowleft}$
$2\mathbf{A}_6 \oplus 3\mathbf{A}_2$	2	$3,7$	$\delta_3(\alpha)\cdot\delta_7(\alpha)\cdot\lvert\alpha\rvert_3\cdot\lvert\alpha\rvert_7$	$\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$2\mathbf{A}_6 \oplus 2\mathbf{A}_3$	2	$2,7$	$\lvert\alpha\rvert_2\cdot\lvert\alpha\rvert_7$	$\mathbf{A}_3^{\circlearrowleft}$
$\mathbf{A}_7 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$	2	$2,3$	$\delta_2(\alpha)\cdot\delta_3(\alpha)\cdot\lvert\alpha\rvert_2\cdot\lvert\alpha\rvert_3$	$\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$\mathbf{A}_7 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	2	5	$\lvert\alpha\rvert_5$	$\mathbf{A}_2^{\circlearrowleft}$ or $\mathbf{A}_4^{\circlearrowleft}$
$2\mathbf{A}_7 \oplus 2\mathbf{A}_2$	4	$2,3$	see Table 4.4	
$\mathbf{A}_8 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_2$	4	$2,3$	$(\delta_2(\alpha)\cdot\delta_3(\alpha), \lvert\alpha\rvert_2\cdot\lvert\alpha\rvert_3)$	$\mathbf{A}_2^{\circlearrowleft}\,\mathit{and}\,\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$\mathbf{A}_8 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2$	2	$3,5$	$\lvert\alpha\rvert_5$	$\mathbf{A}_2^{\circlearrowleft}$ or $\mathbf{A}_4^{\circlearrowleft}$
$\mathbf{A}_9 \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$	2	$3,5$	$\lvert\alpha\rvert_5$	$\mathbf{A}_2^{\circlearrowleft}$ or $\mathbf{A}_4^{\circlearrowleft}$
$\mathbf{A}_9 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$	2	5	$\lvert\alpha\rvert_5$	$\mathbf{A}_2^{\circlearrowleft}$ or $\mathbf{A}_4^{\circlearrowleft}$
$\mathbf{A}_{11} \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$	4	$2,3$	$(\delta_2(\alpha)\cdot\delta_3(\alpha), \lvert\alpha\rvert_2\cdot\lvert\alpha\rvert_3)$	$\mathbf{A}_2^{\circlearrowleft}$ and $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$

Remark 4.2.2. *The symmetry (denoted by* $\mathbf{A}_n^{\circlearrowleft}$ *in Table 4.1) of the Dynkin graph* $\mathbf{A}_n$ *induce the reflection* $r_\beta \in \operatorname{Aut}(\mathcal{S})$ *where* $\beta \in \operatorname{disc}\mathbf{S}$ *is such that* $\beta^2 = -\frac{m}{m+1}$*. Likewise interchanging two copies of* $\mathbf{A}_n$ *(denoted by* $\mathbf{A}_n \leftrightarrow \mathbf{A}_n$ *in Table 4.1) induce the reflection* $r_{\beta'} \in \operatorname{Aut}(\mathcal{S})$ *where* $\beta' \in \operatorname{disc}\mathbf{S}$ *is such that* $(\beta')^2 = -\frac{2m}{m+1}$*. Similarly the automorphisms* $\mathbf{E}_6^{\circlearrowleft}$ *and* $\mathbf{E}_6 \leftrightarrow \mathbf{E}_6$ *is denoted by the elements* $\sigma, \sigma' \in \operatorname{disc}\mathbf{S}$ *such that* $\sigma^2 = \frac{2}{3}$ *and* $(\sigma')^2 = \frac{4}{3}$*, respectively. Since* $\operatorname{disc}\mathbf{S}^\perp \cong -\operatorname{disc}\mathbf{S}$*, one can identify the elements* $\beta, \beta', \sigma, \sigma' \in \mathcal{S}$ *with the elements* $\alpha, \alpha', \gamma, \gamma' \in \mathcal{T}$ *such that* $\alpha^2 = \frac{m}{m+1}$*,* $(\alpha')^2 = \frac{2m}{m+1}$*,* $\gamma^2 = -\frac{2}{3}$ *and* $(\gamma')^2 = -\frac{4}{3}$ *. As explained in §2.3.2, any element* $\alpha \in \mathcal{T}$ *give rise to a reflection* $r_\alpha \operatorname{Aut}(\mathcal{T})$ *on which the images of the map* $\mathrm{e}\colon \operatorname{Aut}(\mathcal{T}) \to E(\mathbf{T})$ *are calculated.*

A few details about the table are explained below.

There are 18 set of singularities $\mathbf{S}$ containing a point of type $\mathbf{A}_4$ and satisfying the hypothesis of Lemma 2.3.1 or Corollary 2.3.2 with $p = 5$. For these set of singularities one has $|E(\mathbf{T})| = 2$ and a nontrivial symmetry of any type $\mathbf{A}_4$ point maps to the generator $-1 \in E(\mathbf{T})$.

There are 8 sets of singularities containing a point of type $\mathbf{A}_2$ and satisfying the hypothesis of Lemma 2.3.2 with $p = 2$, $q = 3$. For these 8 cases, one has $|E(\mathbf{T})| = 2$ and a nontrivial symmetry of any type $\mathbf{A}_2$ point maps to the generator $-1 \in E(\mathbf{T})$. As listed in the Table 4.1, for the following 4 sets of singularities,

$$\mathbf{D}_9 \oplus \mathbf{A}_3 \oplus 3\mathbf{A}_2, \quad \mathbf{D}_7 \oplus \mathbf{D}_5 \oplus 3\mathbf{A}_2,$$
$$\mathbf{A}_{11} \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2, \quad \mathbf{A}_8 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_2,$$

one has $|E(\mathbf{T})| = 4$. Each of these 4 sets has two irregular primes $p = 2$, $q = 3$, and for all of them the homomorphism given by Lemma 2.3.2 is

$$\mathrm{e}(r_\alpha) = (\delta_2(\alpha) \cdot \delta_3(\alpha), |\alpha|_2 \cdot |\alpha|_3) \in \{\pm 1\} \times \{\pm 1\}.$$

A symmetry of any type $\mathbf{A}_2$ point and a transposition $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$ give rise to reflections $r_\alpha, r_\sigma \in \operatorname{Aut}\mathcal{T}$ with $\alpha^2 = \frac{2}{3}$ and $\sigma^2 = \frac{4}{3}$. The images $\mathrm{e}(r_\alpha) = (-1, -1)$ and $\mathrm{e}(r_\sigma) = (-1, 1)$ are linearly independent, thus generating the group $E(\mathbf{T})$.

The remaining 9 sets of singularities do not fit in the previous cases, however they still satisfy the assumptions of Lemma 2.3.2, which yields $|E(\mathbf{T})| = 2$.

Finally, what remains are the three sets of singularities

$$\mathbf{D}_4 \oplus 2\mathbf{A}_4 \oplus 3\mathbf{A}_2, \quad 2\mathbf{A}_7 \oplus 2\mathbf{A}_2 \quad 2\mathbf{A}_4 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_2,$$

to which Lemmas 2.3.1, 2.3.2 or Corollary 2.3.2 do not apply: the two former have three irregular primes and for the latter, the group $\Sigma_2^\sharp(\mathbf{T})$ does not contain $\Gamma_{2,2}$.

For them, we compute the group $E(\mathbf{T})$ directly from the definition (2.3.2) which can be restated as

$$E(T) = \prod_{p|\det(T)} \Gamma_{p,0} \Big/ \prod_{p|\det(T)} \Sigma_p^\sharp(T) \cdot \varphi(\Gamma_0),$$

where we identify the inclusion $\Gamma_0 \hookrightarrow \Gamma_{\mathbb{A},0}$ with the product $\varphi := \prod_p \varphi_p$ (see 2.3.1). For the case

$$2\mathbf{A}_4 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_2,$$

the computation can be summarized in the Table 4.2:

Table 4.2: The set of singularities $\mathbf{S} = 2\mathbf{A}_4 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_2$

	$\Gamma_{5,0}$		$\Gamma_{3,0}$		$\Gamma'_{2,0}$	
generator of $\Sigma_5^\sharp(T)$	-1	-1	1	1	1	1
generator of $\Sigma_3^\sharp(T)$	1	1	-1	1	1	1
generator of $\Sigma_2^\sharp(T)$	1	1	1	1	1	1
$\varphi(-1,1)$	-1	1	-1	1	-1	1
$\varphi(1,-1)$	1	1	1	-1	1	-1
	δ_5	$\lvert\cdot\rvert_5$	δ_3	$\lvert\cdot\rvert_3$	δ_2	$\lvert\cdot\rvert_2$
a symmetry of $\mathbf{A}_4$	-1	-1	1	-1	1	1
a transposition $\mathbf{A}_4 \leftrightarrow \mathbf{A}_4$	-1	1	1	-1	1	1
a symmetry of $\mathbf{A}_2$	1	-1	-1	1	1	-1
a transposition $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$	1	-1	-1	-1	1	-1

Here, in the first part of the table, the entries of the first three rows indicate the generators of the groups $\Sigma_p^\sharp(T)$ inside the groups $\Gamma_{p,0}$ or $\Gamma'_{2,0}$(see §2.3), and the entries in the fourth and fifth row indicate the images $\varphi_p(-1,1)$ and $\varphi_p(1,-1)$; the second part contains the images under $d^\perp$ of certain isometries of $\mathbf{S}$.

The rank of the matrix composed by the 9 rows of the table (see, Remark 4.2.3) is $6 = \dim \Gamma'_{2,0} + \dim \Gamma_{3,0} + \dim \Gamma_{5,0}$, which implies that $d^{\perp}$ is surjective. For the case

$$\mathbf{D}_4 \oplus 2\mathbf{A}_4 \oplus 3\mathbf{A}_2,$$

the computation is given the Table 4.3. Similar to the previous case, the rank

Table 4.3: The set of singularities $\mathbf{S} = \mathbf{D}_4 \oplus 2\mathbf{A}_4 \oplus 3\mathbf{A}_2$

	$\Gamma_{5,0}$		$\Gamma_{3,0}$		$\Gamma'_{2,0}$	
generator of $\Sigma_5^\sharp(T)$	-1	1	1	1	1	1
generator of $\Sigma_3^\sharp(T)$	1	1	1	1	1	1
generators of $\Sigma_2^\sharp(T)$	1	1	1	1	1	1
	1	1	1	1	1	-1
$\varphi(-1,1)$	-1	1	-1	1	-1	1
$\varphi(1,-1)$	1	1	1	-1	1	-1
	δ_5	$\lvert\cdot\rvert_5$	δ_3	$\lvert\cdot\rvert_3$	δ_2	$\lvert\cdot\rvert_2$
a symmetry of $\mathbf{A}_4$	-1	-1	1	-1	1	1
a transposition $\mathbf{A}_4 \leftrightarrow \mathbf{A}_4$	-1	1	1	-1	1	1
a symmetry of $\mathbf{A}_2$	1	-1	-1	1	1	-1
a transposition $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$	1	-1	-1	-1	1	-1

of the matrix composed by the 10 rows of the table (see, Remark 4.2.3) is $6 = \dim \Gamma'_{2,0} + \dim \Gamma_{3,0} + \dim \Gamma_{5,0}$, which proves that $d^{\perp}$ is surjective.

For $2\mathbf{A}_7 \oplus 2\mathbf{A}_2$, where $\Sigma_2^\sharp(\mathbf{T}) \not\supset \Gamma_{2,2}$, we have to modify $|\cdot|_2$ by replacing χ_2 with $\chi_2(u) = u \mod 8 \in \{1,3,5,7\} = \mathbb{Z}_2^\times/(\mathbb{Z}_2^\times)^2$ and consider the full group $\Gamma_{2,0}$ instead of $\Gamma'_{2.0}$ (see §2.3). Accordingly, instead of the notation $\{1,3,5,7\}$ we have to change to the notation $\{(1,1),(-1,-1),(1,-1),(-1,1)\}$. Then the computation is as in the Table 4.4. The rank of the matrix composed by the 8 rows of the table (see, Remark 4.2.3) is $5 = \dim \Gamma_{2,0} + \dim \Gamma_{3,0} + \dim \Gamma_{5,0}$, which shows that $d^{\perp}$ is surjective. □

Remark 4.2.3. *Here and below, when speaking about ranks and dimensions, we regard all groups Γ_*, E, E^+, etc. as $\mathbb{F}_2$-vector spaces. In particular, when computing the rank of a matrix, we need to switch from the multiplicative notation $\{1,-1\}$ to the additive $\{0,1\}$.*

Corollary 4.2.1 (of the proof)**.** *For all sets of singularities $\mathbf{S}$ with $\mu(\mathbf{S}) \leq 18$, the corresponding transcendental lattice $\mathbf{T}$ is unique in its genus, i.e., $g(\mathbf{T}) = 1$.*

Table 4.4: The set of singularities $\mathbf{S} = 2\mathbf{A}_7 \oplus 2\mathbf{A}_2$

	$\Gamma_{3,0}$		$\Gamma_{2,0}$		
generator of $\Sigma_3^\sharp(T)$	-1	-1	1	1	1
$\Sigma_2^\sharp(T) = \{1\}$	1	1	1	1	1
$\varphi(-1,1)$	-1	1	-1	1	1
$\varphi(1,-1)$	1	-1	1	-1	1
	δ_3	$\lvert\cdot\rvert_3$	δ_2	$\lvert\cdot\rvert_2$	
a symmetry of $\mathbf{A}_2$	-1	1	1	-1	-1
a transposition $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$	-1	-1	1	-1	-1
a symmetry of $\mathbf{A}_7$	1	1	-1	-1	1
a transposition $\mathbf{A}_7 \leftrightarrow \mathbf{A}_7$	1	-1	-1	-1	1

4.2.3 Deformation Classification

Finally, we give a complete description of the equisingular strata of non-special simple quartics.

Proposition 4.2.3. *If* $\mathbf{S}$ *is one of*

$$\mathbf{D}_6 \oplus 2\mathbf{A}_6, \quad \mathbf{D}_5 \oplus 2\mathbf{A}_6 \oplus \mathbf{A}_1, \quad 2\mathbf{A}_7 \oplus 2\mathbf{A}_2, \quad 3\mathbf{A}_6, \quad 2\mathbf{A}_6 \oplus 2\mathbf{A}_3$$

then $\mathcal{M}_1(\mathbf{S})$ *consists of two complex conjugate components; in all other cases with* $\mu(\mathbf{S}) \leq 18$*, the stratum* $\mathcal{M}_1(\mathbf{S})$ *is connected.*

Proof. By Proposition 4.2.2, $\mathcal{M}_1(\mathbf{S})$ is connected if and only if the (unique) homological type extending $\mathbf{S}$ is symmetric; otherwise $\mathcal{M}_1(\mathbf{S})$ consists of two complex conjugate components. By Theorem 2.2.5, homological type is symmetric if and only if there is an isometry $a \in O(\mathbf{T})$ with $\det_+(a) = -1$ (see §2.3.3) satisfying $d(a) \in d^\psi(O(\mathbf{S}))$, where d^ψ is the map induced by any anti-isometry $\psi\colon \mathcal{S} \oplus \langle \frac{1}{4} \rangle \to \mathcal{T}$. We consider separately the cases $|E(\mathbf{T})| = |E^+(\mathbf{T})|$ and $|E(\mathbf{T})| < |E^+(\mathbf{T})|$.

Lemma 4.2.1. *If* $|E(\mathbf{T})| = |E^+(\mathbf{T})|$*, then* $\mathcal{M}_1(\mathbf{S})$ *is connected.*

Proof. By definition, we have an exact sequence

$$0 \to \Gamma_0/\Gamma_0^{--} \cdot \tilde{\Sigma}(\mathbf{T}) \to E^+(\mathbf{T}) \to E(\mathbf{T}) \to 0. \qquad (4.2.2)$$

Hence, $|E(\mathbf{T})| = |E^+(\mathbf{T})|$ if and only if $\tilde{\Sigma}(\mathbf{T}) \not\subset \Gamma_0^{--}$. Then, by Proposition 2.3.1, there exist a +-disorienting isometry of $\mathbf{T}$ inducing the identity on $\operatorname{disc}\mathbf{T}$, and Theorem 2.2.5 applies. □

Lemma 4.2.2. *If $|E(\mathbf{T})| < |E^+(\mathbf{T})|$, then $\mathcal{M}_1(\mathbf{S})$ is connected if and only if $d_+^\perp \colon O(S) \to E^+(\mathbf{T})$ is an epimorphism (see §2.3.3).*

Proof. The non-trivial element of the kernel $K := \Gamma_0/\Gamma_0^{--} \cdot \tilde{\Sigma}(\mathbf{T}) \cong \{\pm 1\}$ in (4.2.2) is the image under the composed map $O(\mathbf{T}) \to \operatorname{Aut}(\mathcal{T}) \to E^+(\mathbf{T})$ of any element $a \in O(\mathbf{T})$ with $\det_+(a) = -1$. Thus, $\mathcal{M}_1(\mathbf{S})$ is connected if and only if $\operatorname{Im}(d_+^\perp) \cap K \neq 0$, *i.,e.*, $\operatorname{rank} d_+^\perp > \operatorname{rank} d^\perp$ (see Remark 4.2.3). On the other hand, Proposition 4.2.2 can be recast in the form $\operatorname{rank} d^\perp = \dim E(\mathbf{T})$. Since $\dim E_+(\mathbf{T}) = \dim E(\mathbf{T}) + 1$, the statement follows. □

Lemma 4.2.1 implies the connectedness of $\mathcal{M}_1(\mathbf{S})$ for 2721 sets of singularities. As one has $|E^+(\mathbf{T})| = |E(\mathbf{T})| = 1$ for 2705 of them and $|E^+(\mathbf{T})| = |E(\mathbf{T})| = 2$ for 16 of them. For the remaining 151 sets of singularities, one has $|E(\mathbf{T})| < |E^+(\mathbf{T})|$. For 118 of them, listed in Table 4.5, one has $|E(\mathbf{T})| = 1$ and $\tilde{\Sigma}_p(\mathbf{T}) \subset \Gamma_0^{--}$ for some prime p which is indicated with a $[\,\cdot\,]_p$ pattern in the table. Since $|E(\mathbf{T})| = 1$, the map $d : O(\mathbf{T}) \to \operatorname{Aut}(\mathcal{T})$ is surjective and the isomorphism

$$\operatorname{Aut}(\mathcal{T})/O^+(\mathbf{T}) = \Gamma_0/\Gamma_0^{--} \cdot \tilde{\Sigma}(\mathbf{T}) = E^+(\mathbf{T}) = \{\pm 1\}$$

(see(4.2.2)) is the descent of $\det_+$, which is well-defined due to Proposition 2.3.1. Except for the three cases

$$\mathbf{D}_6 \oplus 2\mathbf{A}_6, \quad \mathbf{D}_5 \oplus 2\mathbf{A}_6 \oplus \mathbf{A}_1, \quad 3\mathbf{A}_6$$

which are marked with * in Table 4.5, one can use Lemma 2.3.3 to show that there exists an element $a \in O(\mathbf{S})$ such that $\det_+(d^\psi(a)) = -1$. Namely,

- if $p = 2$, take for a a nontrivial symmetry of $\mathbf{A}_2$, $\mathbf{D}_5$, $\mathbf{E}_6$ or $\mathbf{D}_9$,
- if $p = 3$, take for a a nontrivial symmetry of $\mathbf{A}_2$, $\mathbf{D}_5$ or $\mathbf{A}_8$,
- if $p = 7$, take for a a nontrivial symmetry of $\mathbf{A}_2$.

Table 4.5: Exceptional sets of singularities

Singularities	Singularities	Singularities
$\mathbf{E}_8 \oplus [2\mathbf{D}_5]_2$	$\mathbf{D}_9 \oplus [\mathbf{A}_5 \oplus 2\mathbf{A}_2]_3$	$\mathbf{A}_3 \oplus [6\mathbf{A}_2]_3$
$\mathbf{E}_6 \oplus [\mathbf{D}_7 \oplus \mathbf{D}_5]_2$	$[\mathbf{E}_6 \oplus 3\mathbf{A}_2]_3 \oplus 4\mathbf{A}_1$	$[\mathbf{A}_5 \oplus 5\mathbf{A}_2]_3$
$[\mathbf{D}_7 \oplus 2\mathbf{D}_5]_2$	$\mathbf{E}_8 \oplus \mathbf{A}_4 \oplus [2\mathbf{A}_2]_3 \oplus 2\mathbf{A}_1$	$2\mathbf{A}_3 \oplus [5\mathbf{A}_2]_3$
$[2\mathbf{D}_9]_2$	$\mathbf{D}_6 \oplus [5\mathbf{A}_2]_3$	$[4\mathbf{A}_3]_2 \oplus 2\mathbf{A}_2$
$[\mathbf{D}_{13} \oplus \mathbf{D}_5]_2$	$\mathbf{D}_7 \oplus [4\mathbf{A}_2]_3 \oplus 2\mathbf{A}_1$	$\mathbf{A}_4 \oplus [4\mathbf{A}_2]_3 \oplus 4\mathbf{A}_1$
$[2\mathbf{E}_6]_3 \oplus \mathbf{D}_5 \oplus \mathbf{A}_1$	$\mathbf{D}_7 \oplus \mathbf{A}_3 \oplus [3\mathbf{A}_2]_3 \oplus \mathbf{A}_1$	$[\mathbf{A}_5 \oplus 4\mathbf{A}_2]_3 \oplus \mathbf{A}_3$
$[\mathbf{E}_6 \oplus \mathbf{A}_2]_3 \oplus \mathbf{D}_{10}$	$[\mathbf{D}_7 \oplus 2\mathbf{A}_3]_2 \oplus 2\mathbf{A}_2$	$[2\mathbf{A}_5 \oplus 3\mathbf{A}_2]_3$
$[3\mathbf{D}_5]_2 \oplus \mathbf{A}_2$	$\mathbf{D}_7 \oplus \mathbf{A}_4 \oplus [3\mathbf{A}_2]_3$	$\mathbf{A}_6 \oplus [5\mathbf{A}_2]_3$
$[\mathbf{D}_9 \oplus \mathbf{A}_3]_2 \oplus \mathbf{E}_6$	$\mathbf{D}_7 \oplus [\mathbf{A}_5 \oplus 2\mathbf{A}_2]_3 \oplus \mathbf{A}_1$	$[\mathbf{A}_8 \oplus 4\mathbf{A}_2]_3$
$[\mathbf{D}_9 \oplus \mathbf{D}_5 \oplus \mathbf{A}_3]_2$	$[\mathbf{E}_6 \oplus 2\mathbf{A}_2]_3 \oplus \mathbf{A}_4 \oplus 3\mathbf{A}_1$	$\mathbf{A}_4 \oplus [3\mathbf{A}_3]_2 \oplus 2\mathbf{A}_2$
$\mathbf{E}_8 \oplus \mathbf{E}_6 \oplus \mathbf{A}_2 \oplus [2\mathbf{A}_1]_2$	$\mathbf{D}_4 \oplus [5\mathbf{A}_2]_3 \oplus \mathbf{A}_1$	$2\mathbf{A}_4 \oplus [3\mathbf{A}_2]_3 \oplus 3\mathbf{A}_1$
$\mathbf{E}_6 \oplus [\mathbf{D}_7]_2 \oplus 2\mathbf{A}_2$	$\mathbf{D}_5 \oplus [4\mathbf{A}_2]_3 \oplus 3\mathbf{A}_1$	$[\mathbf{A}_5 \oplus 3\mathbf{A}_2]_3 \oplus 2\mathbf{A}_3$
$[\mathbf{E}_7 \oplus \mathbf{D}_7]_2 \oplus 2\mathbf{A}_2$	$\mathbf{D}_6 \oplus \mathbf{A}_3 \oplus [4\mathbf{A}_2]_3$	$[2\mathbf{A}_5 \oplus 2\mathbf{A}_2]_3 \oplus \mathbf{A}_3$
$[\mathbf{D}_9 \oplus \mathbf{D}_5]_2 \oplus \mathbf{A}_4$	$\mathbf{D}_6 \oplus [\mathbf{A}_5 \oplus 3\mathbf{A}_2]_3$	$[3\mathbf{A}_5 \oplus \mathbf{A}_2]_3$
$2\mathbf{E}_6 \oplus \mathbf{A}_2 \oplus [3\mathbf{A}_1]_2$	$\mathbf{D}_7 \oplus \mathbf{A}_4 \oplus [3\mathbf{A}_2]_3 \oplus \mathbf{A}_1$	$\mathbf{A}_6 \oplus \mathbf{A}_3 \oplus [4\mathbf{A}_2]_3$
$\mathbf{E}_8 \oplus [\mathbf{D}_5 \oplus \mathbf{A}_1]_2 \oplus 2\mathbf{A}_2$	$\mathbf{D}_7 \oplus [\mathbf{A}_5 \oplus \mathbf{A}_2]_3 \oplus \mathbf{A}_4$	$\mathbf{A}_6 \oplus [3\mathbf{A}_3]_2 \oplus \mathbf{A}_2$
$[\mathbf{E}_6 \oplus 2\mathbf{A}_2]_3 \oplus \mathbf{D}_7 \oplus \mathbf{A}_1$	$\mathbf{D}_7 \oplus \mathbf{A}_6 \oplus [2\mathbf{A}_2]_3 \oplus \mathbf{A}_1$	$\mathbf{A}_6 \oplus [\mathbf{A}_5 \oplus 3\mathbf{A}_2]_3$
$\mathbf{E}_6 \oplus [\mathbf{D}_7 \oplus \mathbf{A}_5]_2$	$[\mathbf{D}_7 \oplus \mathbf{A}_3]_2 \oplus \mathbf{A}_6 \oplus \mathbf{A}_2$	$[\mathbf{A}_7 \oplus 2\mathbf{A}_3]_2 \oplus 2\mathbf{A}_2$
$[\mathbf{D}_{13} \oplus \mathbf{A}_1]_2 \oplus 2\mathbf{A}_2$	$\mathbf{D}_7 \oplus [\mathbf{A}_8 \oplus \mathbf{A}_2]_3 \oplus \mathbf{A}_1$	$[\mathbf{A}_8 \oplus 3\mathbf{A}_2]_3 \oplus \mathbf{A}_3$
$2\mathbf{E}_6 \oplus \mathbf{A}_4 \oplus [2\mathbf{A}_1]_2$	$[\mathbf{E}_6 \oplus \mathbf{A}_2]_3 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_1$	$[\mathbf{A}_8 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_2]_3$
$\mathbf{E}_6 \oplus [\mathbf{D}_4]_2 \oplus 3\mathbf{A}_2$	$\mathbf{D}_4 \oplus [5\mathbf{A}_2]_3 \oplus 2\mathbf{A}_1$	$\mathbf{A}_9 \oplus [4\mathbf{A}_2]_3$
$[\mathbf{E}_7 \oplus \mathbf{D}_4]_2 \oplus 3\mathbf{A}_2$	$\mathbf{D}_4 \oplus \mathbf{A}_3 \oplus [4\mathbf{A}_2]_3 \oplus \mathbf{A}_1$	$[\mathbf{A}_{11} \oplus 3\mathbf{A}_2]_3$
$\mathbf{E}_6 \oplus [\mathbf{D}_5 \oplus 2\mathbf{A}_1]_2 \oplus 2\mathbf{A}_2$	$\mathbf{D}_4 \oplus \mathbf{A}_4 \oplus [4\mathbf{A}_2]_3$	$\mathbf{A}_6 \oplus \mathbf{A}_4 \oplus [2\mathbf{A}_3]_2 \oplus \mathbf{A}_2$
$\mathbf{E}_6 \oplus [\mathbf{D}_5 \oplus 2\mathbf{A}_3]_2$	$\mathbf{D}_4 \oplus [\mathbf{A}_5 \oplus 3\mathbf{A}_2]_3 \oplus \mathbf{A}_1$	$\mathbf{A}_6 \oplus [\mathbf{A}_5 \oplus 2\mathbf{A}_2]_3 \oplus \mathbf{A}_3$
$[2\mathbf{D}_5 \oplus 2\mathbf{A}_3]_2$	$\mathbf{D}_5 \oplus \mathbf{A}_4 \oplus [3\mathbf{A}_2]_3 \oplus 2\mathbf{A}_1$	$\mathbf{A}_6 \oplus [2\mathbf{A}_5 \oplus \mathbf{A}_2]_3$
$[\mathbf{E}_6 \oplus 3\mathbf{A}_2]_3 \oplus \mathbf{D}_4 \oplus \mathbf{A}_1$	$\mathbf{D}_6 \oplus \mathbf{A}_6 \oplus [3\mathbf{A}_2]_3$	$[2\mathbf{A}_6]_7 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$
$[\mathbf{E}_6 \oplus \mathbf{A}_5 \oplus \mathbf{A}_2]_3 \oplus \mathbf{D}_4$	$^*\mathbf{D}_6 \oplus [2\mathbf{A}_6]_7$	$[2\mathbf{A}_6]_7 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$
$[\mathbf{E}_6 \oplus \mathbf{A}_2]_3 \oplus \mathbf{D}_5 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	$\mathbf{D}_6 \oplus [\mathbf{A}_8 \oplus 2\mathbf{A}_2]_3$	$^*[3\mathbf{A}_6]_7$
$\mathbf{E}_6 \oplus [\mathbf{D}_5 \oplus \mathbf{A}_3]_2 \oplus \mathbf{A}_4$	$\mathbf{D}_4 \oplus \mathbf{A}_4 \oplus [4\mathbf{A}_2]_3 \oplus \mathbf{A}_1$	$[\mathbf{A}_7 \oplus \mathbf{A}_3]_2 \oplus \mathbf{A}_6 \oplus \mathbf{A}_2$
$\mathbf{E}_6 \oplus [\mathbf{D}_5 \oplus \mathbf{A}_7]_2$	$\mathbf{D}_4 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus [3\mathbf{A}_2]_3$	$[\mathbf{A}_8 \oplus \mathbf{A}_5 \oplus \mathbf{A}_2]_3 \oplus \mathbf{A}_3$
$[2\mathbf{D}_5 \oplus \mathbf{A}_1]_2 \oplus 3\mathbf{A}_2$	$\mathbf{D}_4 \oplus [\mathbf{A}_5 \oplus 2\mathbf{A}_2]_3 \oplus \mathbf{A}_4$	$[\mathbf{A}_8 \oplus 2\mathbf{A}_5]_3$
$[2\mathbf{D}_5 \oplus \mathbf{A}_3]_2 \oplus \mathbf{A}_4$	$\mathbf{D}_4 \oplus \mathbf{A}_6 \oplus [3\mathbf{A}_2]_3 \oplus \mathbf{A}_1$	$[\mathbf{A}_8 \oplus 2\mathbf{A}_2]_3 \oplus \mathbf{A}_6$
$[2\mathbf{D}_5 \oplus \mathbf{A}_7]_2$	$\mathbf{D}_4 \oplus \mathbf{A}_7 \oplus [3\mathbf{A}_2]_3$	$[2\mathbf{A}_8 \oplus \mathbf{A}_2]_3$
$\mathbf{D}_{10} \oplus [3\mathbf{A}_2]_3 \oplus \mathbf{A}_1$	$\mathbf{D}_4 \oplus [\mathbf{A}_8 \oplus 2\mathbf{A}_2]_3 \oplus \mathbf{A}_1$	$\mathbf{A}_9 \oplus \mathbf{A}_3 \oplus [3\mathbf{A}_2]_3$
$[\mathbf{E}_6 \oplus \mathbf{A}_8]_3 \oplus \mathbf{D}_4$	$^*\mathbf{D}_5 \oplus [2\mathbf{A}_6]_7 \oplus \mathbf{A}_1$	$\mathbf{A}_9 \oplus [\mathbf{A}_5 \oplus 2\mathbf{A}_2]_3$
$\mathbf{D}_9 \oplus [4\mathbf{A}_2]_3$	$[7\mathbf{A}_2]_3$	$\mathbf{A}_{10} \oplus [2\mathbf{A}_3]_2 \oplus \mathbf{A}_2$
$[2\mathbf{D}_5]_2 \oplus \mathbf{A}_8$	$\mathbf{D}_4 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus [2\mathbf{A}_2]_3$	$[\mathbf{A}_{11} \oplus \mathbf{A}_5 \oplus \mathbf{A}_2]_3$
$\mathbf{D}_{10} \oplus \mathbf{A}_4 \oplus [2\mathbf{A}_2]_3$	$\mathbf{D}_4 \oplus \mathbf{A}_{10} \oplus [2\mathbf{A}_2]_3$	$\mathbf{A}_{12} \oplus [3\mathbf{A}_2]_3$
$\mathbf{E}_8 \oplus [3\mathbf{A}_2]_3 \oplus 3\mathbf{A}_1$	$[5\mathbf{A}_2]_3 \oplus 5\mathbf{A}_1$	$[\mathbf{A}_{14} \oplus 2\mathbf{A}_2]_3$
$\mathbf{D}_7 \oplus [4\mathbf{A}_2]_3 \oplus \mathbf{A}_1$		

Hence for those 115 cases listed in Table 4.5, the homological type extending $\mathbf{S}$ is symmetric which shows that $\mathcal{M}_1(\mathbf{S})$ is connected. The remaining three sets of singularities (marked with * in Table 4.5)

$$\mathbf{D}_6 \oplus 2\mathbf{A}_6, \quad \mathbf{D}_5 \oplus 2\mathbf{A}_6 \oplus \mathbf{A}_1, \quad 3\mathbf{A}_6$$

with $p = 7$ are exceptional (and are listed as such in the statement), as Lemma 2.3.3 implies $(\det_+) \circ d^\psi \equiv 1$. Indeed, the image of d^ψ is generated by the reflections r_α with either

- $\alpha \in \mathcal{T}_7$, $\alpha^2 = \frac{6}{7}$ (a nontrivial symmetry of $\mathbf{A}_6$) or,
- $\alpha \in \mathcal{T}_7$, $\alpha^2 = \frac{12}{7}$ (a transposition $\mathbf{A}_6 \leftrightarrow \mathbf{A}_6$) or,
- $\alpha \in \mathcal{T}_2$ (a nontrivial symmetry of $\mathbf{D}_6$ or $\mathbf{D}_5$).

One has $\delta_7(\alpha) = |\alpha|_7 = -1$ in the first two cases and $\delta_7(\alpha) = |\alpha|_7 = 1$ in the last one. Hence, these sets of singularities extend to an asymmetric homological type, which implies that $\mathcal{M}_1(\mathbf{S})$ consists of two complex conjugate components.

There are 30 other sets of singularities still satisfying the condition $\Gamma_{2,2} \subset \Sigma_2^\sharp(\mathbf{T})$ and having two irregular primes (p, q), so that we can apply Lemma 2.3.4. Among them, 13 sets of singularities contain two type $\mathbf{A}_2$ points and have $(p, q) = (2, 3)$ and $|E^+(\mathbf{T})| = 4$. A nontrivial symmetry of any type $\mathbf{A}_2$ point and a transposition $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$ map to two linearly independent elements generating $E^+(\mathbf{T})$. The remaining 17 sets of singularities are listed in Table 4.6, where we indicate the irregular primes (p, q), order of $E^+ := E^+(\mathbf{T})$ and a collection of isometries of $\mathbf{S}$ whose images generate $E^+(\mathbf{T})$. The set of singularities $\mathbf{S} = 2\mathbf{A_6}\oplus 2\mathbf{A}_3$ marked as exceptional is one of the special cases listed in the statement. We have $|E^+(\mathbf{T})| = 4$ and the group $O(\mathbf{S})$ is generated by

- a nontrivial symmetry of $\mathbf{A}_3$, mapped to $(1, 1) \in E^+(\mathbf{T})$, or
- the transposition $\mathbf{A}_3 \leftrightarrow \mathbf{A}_3$, mapped to $(1, 1) \in E^+(\mathbf{T})$, or
- a nontrivial symmetry of $\mathbf{A}_6$, mapped to $(-1, 1) \in E^+(\mathbf{T})$, or
- the transposition $\mathbf{A}_6 \leftrightarrow \mathbf{A}_6$, mapped to $(-1, 1) \in E^+(\mathbf{T})$.

Table 4.6: Extremal singularities

Singularities	(p,q)	$\lvert E^+\rvert$	isometries of $\mathbf{S}$ generating $E^+(\mathbf{T})$
$2\mathbf{E}_6 \oplus 2\mathbf{A}_3$	$(2,3)$	4	$\mathbf{E}_6^{\circlearrowleft}$ and $\mathbf{A}_3 \leftrightarrow \mathbf{A}_3$
$\mathbf{D}_9 \oplus \mathbf{A}_3 \oplus 3\mathbf{A}_2$	$(2,3)$	8	$\mathbf{A}_2^{\circlearrowleft}$, $\mathbf{A}_3^{\circlearrowleft}$and $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$\mathbf{D}_8 \oplus \mathbf{A}_6 \oplus 2\mathbf{A}_2$	$(2,3)$	2	$\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$\mathbf{D}_8 \oplus \mathbf{A}_3 \oplus 3\mathbf{A}_2$	$(2,3)$	2	$\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$\mathbf{D}_7 \oplus \mathbf{D}_5 \oplus 3\mathbf{A}_2$	$(2,3)$	8	$\mathbf{A}_2^{\circlearrowleft}$, $\mathbf{D}_5^{\circlearrowleft}$and $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$\mathbf{D}_7 \oplus \mathbf{D}_4 \oplus 3\mathbf{A}_2$	$(2,3)$	2	$\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$\mathbf{D}_6 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2$	$(3,5)$	2	$\mathbf{A}_2^{\circlearrowleft}$
$2\mathbf{D}_5 \oplus 2\mathbf{A}_4$	$(2,5)$	4	$\mathbf{D}_5^{\circlearrowleft}$and $\mathbf{A}_4 \leftrightarrow \mathbf{A}_4$
$\mathbf{D}_5 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$	$(3,5)$	4	$\mathbf{A}_2^{\circlearrowleft}$and $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$\mathbf{D}_4 \oplus 2\mathbf{A}_6 \oplus \mathbf{A}_2$	$(2,7)$	2	$\mathbf{A}_6^{\circlearrowleft}$
$\mathbf{A}_{11} \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$	$(2,3)$	8	$\mathbf{A}_2^{\circlearrowleft}$, $\mathbf{A}_3^{\circlearrowleft}$and $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$\mathbf{A}_8 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_2$	$(2,3)$	8	$\mathbf{A}_2^{\circlearrowleft}$, $\mathbf{A}_3^{\circlearrowleft}$and $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$2\mathbf{A}_6 \oplus 2\mathbf{A}_3$	$(2,7)$	4	exceptional
$2\mathbf{A}_6 \oplus 3\mathbf{A}_2$	$(3,7)$	4	$\mathbf{A}_2^{\circlearrowleft}$and $\mathbf{A}_6^{\circlearrowleft}$
$2\mathbf{A}_5 \oplus 2\mathbf{A}_4$	$(3,5)$	2	$\mathbf{A}_4^{\circlearrowleft}$
$3\mathbf{A}_4 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$	$(3,5)$	4	$\mathbf{A}_2^{\circlearrowleft}$and $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$2\mathbf{A}_4 \oplus 4\mathbf{A}_2$	$(2,3)$	2	$\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$

It follows that $d_+^{\perp}$ is not surjective.

Finally, what remains are the 3 sets of singularities

$$\mathbf{D}_4 \oplus 2\mathbf{A}_4 \oplus 3\mathbf{A}_2, \quad 2\mathbf{A}_7 \oplus 2\mathbf{A}_2, \quad 2\mathbf{A}_4 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_2$$

to which Lemma 2.3.4 does not apply as for the first two of them we have three irregular primes and for the last one we have $\Sigma_2^\sharp \not\supset \Gamma_{2,2}$. For these three cases we need to compute the groups $E^+(\mathbf{T})$ directly from the definition (2.3.9) which can be restated as

$$E^+(T) = \prod_{p|\det(T)} \Gamma_{p,0} \Big/ \prod_{p|\det(T)} \Sigma_p^\sharp(T) \cdot \varphi(\Gamma_0^{--})$$

where we identify the inclusion $\Gamma_0^{--} \hookrightarrow \Gamma_{\mathbb{A},0}$ with the product $\varphi := \prod_p \varphi_p$ (see 2.3.1).

For the case

$$\mathbf{D}_4 \oplus 2\mathbf{A}_4 \oplus 3\mathbf{A}_2,$$

the computation can be summarized by the Table 4.7. The rank of the matrix

Table 4.7: The set of singularities $\mathbf{S} = \mathbf{D}_4 \oplus 2\mathbf{A}_4 \oplus 3\mathbf{A}_2$

	$\Gamma_{5,0}$		$\Gamma_{3,0}$		$\Gamma'_{2,0}$	
generator of $\Sigma_5^\sharp(T)$	-1	1	1	1	1	1
generator of $\Sigma_3^\sharp(T)$	1	1	1	1	1	1
generators of $\Sigma_2^\sharp(T)$	1	1	1	1	1	1
	1	1	1	1	1	-1
$\varphi(-1,-1)$	-1	1	-1	- 1	-1	-1
	δ_5	$\lvert\cdot\rvert_5$	δ_3	$\lvert\cdot\rvert_3$	δ_2	$\lvert\cdot\rvert_2$
a symmetry of $\mathbf{A}_4$	-1	-1	1	-1	1	1
a transposition $\mathbf{A}_4 \leftrightarrow \mathbf{A}_4$	-1	1	1	-1	1	1
a symmetry of $\mathbf{A}_2$	1	-1	-1	1	1	-1
a transposition $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$	1	-1	-1	-1	1	-1

composed by the 9 rows of the table (see, Remark 4.2.3) is $6 = \dim \Gamma'_{2,0} + \dim \Gamma_{3,0} + \dim \Gamma_{5,0}$, which proves that $d_+^\perp$ is surjective.

For

$$2\mathbf{A}_4 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_2,$$

the computation is given by the Table 4.8. The rank of the matrix composed by

Table 4.8: The set of singularities $\mathbf{S} = 2\mathbf{A}_4 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_2$

	$\Gamma_{5,0}$		$\Gamma_{3,0}$		$\Gamma'_{2,0}$	
generator of $\Sigma_5^\sharp(T)$	-1	-1	1	1	1	1
generator of $\Sigma_3^\sharp(T)$	1	1	-1	1	1	1
generator of $\Sigma_2^\sharp(T)$	1	1	1	1	1	1
$\varphi(-1,-1)$	-1	1	-1	-1	-1	-1
	δ_5	$\lvert\cdot\rvert_5$	δ_3	$\lvert\cdot\rvert_3$	δ_2	$\lvert\cdot\rvert_2$
a symmetry of $\mathbf{A}_4$	-1	-1	1	-1	1	1
a transposition $\mathbf{A}_4 \leftrightarrow \mathbf{A}_4$	-1	1	1	-1	1	1
a symmetry of $\mathbf{A}_2$	1	-1	-1	1	1	-1
a transposition $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$	1	-1	-1	-1	1	-1

the 8 rows of the table (see, Remark 4.2.3) is $6 = \dim \Gamma'_{2,0} + \dim \Gamma_{3,0} + \dim \Gamma_{5,0}$, which implies that $d^\perp$ is surjective.

For

$$\mathbf{S} = 2\mathbf{A}_7 \oplus 2\mathbf{A}_2,$$

which is the last exceptional case listed in statement, we have $\Sigma_2^\sharp(\mathbf{T}) \not\supset \Gamma_{2,2}$ and $|\cdot|_2$ needs to be modified by replacing χ_2 with $\chi_2(u) = u \bmod 8 \in \{1,3,5,7\} = \mathbb{Z}_2^\times/(\mathbb{Z}_2^\times)^2$ and we have to consider the full group $\Gamma_{2,0}$ instead of $\Gamma'_{2,0}$ (see §2.3). This group is a $\mathbb{F}_2$-vector space generated by 5 and 7, instead of $\{1,3,5,7\}$ we use the notation $\{(1,1),(-1,-1),(1,-1),(-1,1)\}$. The computation can be summarized as in Table 4.9. The rank of the matrix composed by the 7 rows of the

Table 4.9: The set of singularities $\mathbf{S} = 2\mathbf{A}_7 \oplus 2\mathbf{A}_2$

	$\Gamma_{3,0}$		$\Gamma_{2,0}$		
generator of $\Sigma_3^\sharp(T)$	-1	-1	1	1	1
$\Sigma_2^\sharp(T) = \{1\}$	1	1	1	1	1
$\varphi(-1,-1)$	-1	-1	-1	-1	1
	δ_3	$\lvert\cdot\rvert_3$	δ_2	$\lvert\cdot\rvert_2$	
a symmetry of $\mathbf{A}_2$	-1	1	1	-1	-1
a transposition $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$	-1	-1	1	-1	-1
a symmetry of $\mathbf{A}_7$	1	1	-1	-1	1
a transposition $\mathbf{A}_7 \leftrightarrow \mathbf{A}_7$	1	-1	-1	-1	1

table (see Remark 4.2.3) is $4 < \dim \Gamma_{3,0} + \dim \Gamma_{2,0}$, which implies that $d_+^\perp$ is not surjective. $\square$

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