

PERFORMANCE EVALUATION OF
TELECOMMUNICATION NETWORKS BASED ON
MULTI-REGIME FLUID QUEUES

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DOCTOR OF PHILOSOPHY

By

Hüseyin Emre Kankaya

July 2009

I certify that I have read this thesis and that in my opinion it is fully adequate,
in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

Assoc. Prof. Dr. Nail Akar(Supervisor)

I certify that I have read this thesis and that in my opinion it is fully adequate,
in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

Prof. Dr. Ülkü Gürler

I certify that I have read this thesis and that in my opinion it is fully adequate,
in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

Assoc. Prof. Dr. Ezhan Karahan

I certify that I have read this thesis and that in my opinion it is fully adequate,
in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

Assoc. Prof. Dr. Elif Uysal-Bıyıkoglu

I certify that I have read this thesis and that in my opinion it is fully adequate,
in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

Asst. Prof. Dr. Sinan Gezici

Approved for the Institute of Engineering and Sciences:

Prof. Dr. Mehmet Baray
Director of Institute of Engineering and Sciences

ABSTRACT

PERFORMANCE EVALUATION OF TELECOMMUNICATION NETWORKS BASED ON MULTI-REGIME FLUID QUEUES

Hüseyin Emre Kankaya

Ph. D. in Electrical and Electronics Engineering

Supervisor: Assoc. Prof. Dr. Nail Akar

July 2009

In this thesis, a stochastic transformation method is proposed that reduces (by using sample path arguments) a certain set of performance evaluation problems to those that are described by the so-called first-order multi-regime feedback Markov fluid queues (FMFFQ). In addition, we provide differential equations and boundary conditions for the solvability of such fluid queues and we propose computationally efficient and stable numerical algorithms to find their steady-state performance measures. We apply this method to a number of scenarios, the queueing analysis of which are already available in the literature, with the goal of demonstrating the generality of this approach. The real value we add is to use this method to solve exactly the queueing systems for which exact analytical methods were not available and we present three such problems. In the first problem, we explore the steady-state behavior of the unfinished work (also called the horizon parameter) in a multi-class optical burst switching (OBS) network using different offset times for each traffic class for the purpose of service differentiation. We find the exact distribution of the horizon parameter and per-class blocking probabilities in the steady-state by transforming the horizon process

into an FMFFQ. In the second problem, we study the steady-state behavior of an optical buffer using fiber delay lines (FDLs). Assuming a Markovian optical packet arrival process with phase-type distributed service times, we propose an exact FMFFQ-based solution for the steady-state distribution of the scheduling horizon and hence various other performance measures including packet blocking probability and expected delay. In the third problem, we find the steady-state joint probability density functions of FMFFQs with hysteretic control (FMFFQ-H in short). Furthermore, we show that it is possible to find the steady-state performance measures of a number of queueing system with hysteretic control by transforming these suitably to FMFFQ-Hs.

Keywords: Performance Evaluation, Optical Burst Switching, First-order Multi-regime Feedback Fluid Queues, Horizon-based advance reservation, Fiber Delay Lines, Hysteretic Control

ÖZET

TELEKOMÜNİKASYON AĞLARININ ÇOK BÖLGELİ AKIŞKAN KUYRUK TABANLI BAŞARIM DEĞERLENDİRMESİ

Hüseyin Emre Kankaya

Elektrik ve Elektronik Mühendisliği Bölümü Doktora

Tez Yöneticisi: Doç. Dr. Nail Akar

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Bu tez, bir kısım başarımların değerlendirilmesi problemini birinci derece çok bölgeli geribeslemeli akışkan kuyruk (FMFFQ) olarak adlandırılan probleme (örneklem argümanları kullanılarak) indirgeyen bir belirsiz dönüşüm yöntemini sunmaktadır. Buna ek olarak, bu tarz kuyrukların çözümü için difransiyel denklemleri ve sınır şartlarını verip, kararlı durum başarımların ölçütlerinin bulunabilmesi için verimli ve kararlı hesaplama yöntemlerini de sağlamaktayız. Bu yöntemi, yaklaşımın genelliğini göstermek için literatürde kuyruk analizi olan problemlere uygulamaktayız. Gerçek anlamdaki katkımız analitik çözümü bulunmayan kuyruk sistemlerini tam olarak çözmek için kullanmamızdır ve bu çeşit üç problem sunmaktayız. Birinci problemde, hizmet farklılaştırılması için her trafik sınıfının farklı zaman farkı kullandığı bir çok sınıflı optik kümelenme anahtarlamalı (OBS) kanaldaki bitmemiş iş yükününün (görüş değişkeni olarak da adlandırılır) kararlı durum davranışını incelemekteyiz. Kararlı durumda görüş parametresinin tam dağılımını ve her sınıf için kayıp olasılıklarını, görüş sürecini FMFFQ'ya dönüştürerek buluyoruz. İkinci problemde fiber gecikme yolu (FDL) kullanan bir optik ara belleğin davranışları üzerine çalışmaktayız. Faz tipi dağılımlı hizmet

zamanına sahip paketlerin Markov varış süreci ile ulaştığını varsayarak, sıralama görüsünün kararlı durum dağılımının, ve dolayısıyla paket kayıp olasılıklarını ve ortalama gecikme zamanının, bulunması için FMFFQ-tabanlı bir tam çözüm yöntemi sunmaktayız. Üçüncü problemde, geç tepki denetimli bir FMFFQ'nun (kısaca FMFFQ-H) bileşik olasılık yoğunluk fonksiyonlarını bulmaktayız. Daha da ötesi, FMFFQ-H'a uygun dönüşüm ile geç tepki denetimine sahip bir çok kuyruk sisteminin de kararlı başarımlar ölçütlerinin bulunabildiğini gösteriyoruz.

Anahtar Kelimeler: Başarım değerlendirme, OBS, Birinci Derece Çok Bölgeli Akışkan Kuyruklar, Görü-tabanlı ileri rezervasyon, Fiber gecikme yolları, Geç tepkili denetim

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Dedicated to my father, Ahmet Hamdi Kankaya...

Chapter 1

INTRODUCTION

Queueing systems are known to play a crucial role in the performance evaluation of computer and telecommunication networks and systems [33], [32], [51]. In [23, page 3] six basic characteristics of queueing systems are listed as

1. Arrival pattern of customers,
2. Service patterns,
3. Queue discipline,
4. System capacity,
5. Number of service channels,
6. Stages of service.

As indicated in [23], most of the queueing systems can perfectly be described by specifying these six characteristics. For example, the M/M/1 queue, probably the most basic system in queueing theory is described by a Poisson arrival stream (1st characteristic), exponentially distributed independent service times (2nd characteristic), first-come first-serve queue discipline (3rd characteristic), an

infinite system capacity (4th characteristic), single server (5th characteristic), and a single stage of service (6th characteristic). In a telecommunication network performance evaluation problem, the general practice (if possible) is to reduce the problem to the analysis of a queueing system whose analysis boils down to finding one or more of the performance measures in the steady-state such as the workload distribution, the distribution of number of customers (or packets) in the system, the waiting time distribution, the blocking probability (in a loss system), etc. The advantages of being able to analyze a queueing system model in performance evaluation problems can be stated as follows:

- In general, queueing analysis provides performance measures far faster than a straightforward simulation-based study,
- Analysis can provide insight and guidelines for how to engineer a network or a system,
- In cases where an optimization method is unavailable, repeated analysis on problem parameters can help optimize a system.

Queueing systems are under consideration for exactly one hundred years commencing with the study of Erlang [20] in 1909. However, a demand still exists for analytical methods to solve queueing systems that are arising due to new technologies, protocols, systems. There is also an interest for computationally efficient and stable algorithms for queueing systems for which analysis is already possible. In this thesis, a stochastic transformation method is proposed that reduces (by using sample path arguments to be described later) a certain set of performance evaluation problems to those that are described by the so-called *feedback Markov fluid queues* that are well-known in the literature. In addition, we propose a computationally efficient and stable numerical algorithm to find the steady-state performance measures of these queues. In this thesis, we apply this method to a number of scenarios, the analysis of some of which is already

available in the literature and we provide alternative solutions with the underlying reason being the demonstration of the generality of the approach. On the other hand, the real value we add is to use this method to solve queueing systems for which exact analysis was not possible, for example the work presented in Chapters 4, 5 and 6 are in this class of problems.

In our solutions, we heavily utilize the properties and the solutions of fluid queues. The first characteristic (the arrival pattern) distinguishes fluid queues from traditional queueing systems fed with point processes such as the M/M/1 queue. In fluid queues, the workload does not arrive at distinct time points, rather it arrives continuously, resembling to the flow of a liquid to a reservoir (fluid queue). If the properties of the flow possess a Markovian nature, then this queueing system is referred to as *Markovian fluid queue*. Markovian fluid queues are characterized by two processes, namely the *content* and the *background* (or so called the *modulating*) processes. The content process has a continuous domain and it indicates the amount of workload (or fluid) in the queue (or reservoir). The state-space of the background process is finite and that process determines the rate of net flow of workload into the queue. If the rate of that flow is deterministic at each state of the background process, then this Markovian fluid queue is called *first-order fluid queue*. If the rates do not depend on the queue occupancy, then these queues are categorized as *first-order single-regime fluid queue*, in short FSFQ. However, if the flow rate is a deterministic piecewise-constant function of the content process then this process is referred to as *first-order multi-regime feedback fluid queue*, or in short FMFFQ. Second-order Markov fluid queues allow drifts to be described by Brownian motion but we leave such queues outside the scope of this thesis. We now provide a rigorous description of FSFQs and FMFFQs.

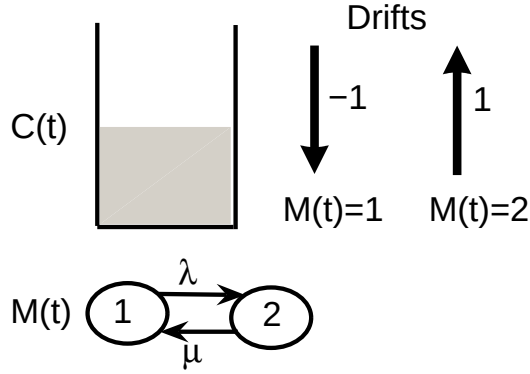


Figure 1.1: A two-state first-order single-regime fluid queue.

1.2 First-order single-regime fluid queues

An FSFQ is a double process $\{C(t), M(t); t \geq 0\}$ where $\{C(t); t \geq 0\}$ denotes the fluid content (or queue occupancy) process and $\{M(t); t \geq 0\}$ is the background (or so called modulating) process. $M(t)$ is a continuous time Markov chain (CTMC) with a finite state space $I = \{1, 2, \dots, M\}$ and with infinitesimal generator Q . When $M(t)$ is visiting state $i \in I$, $C(t)$ changes with a constant drift r_i as long as $C(t)$ stays in the interval $(0, T)$. The boundary T (queue size) can be finite or infinite. Assuming infinite T , the queue dynamics can be summarized as

$$\frac{dC(t)}{dt} = \begin{cases} 0, & C(t) = 0, r_{M(t)} \leq 0, \\ r_{M(t)}, & \text{Otherwise.} \end{cases}.$$

Fig. 1.1 demonstrates a two-state FSFQ with an infinite queue size. Here, $M(t)$ comprises two states. When $M(t)$ is at state 1 then $C(t)$ decreases with a drift of -1 and at state 2, it increases with drift 1. A sample path of $C(t)$ for this example is illustrated in Fig. 1.2 in which solid and dotted lines represent the queue content when the background process is at states 1 and 2, respectively.

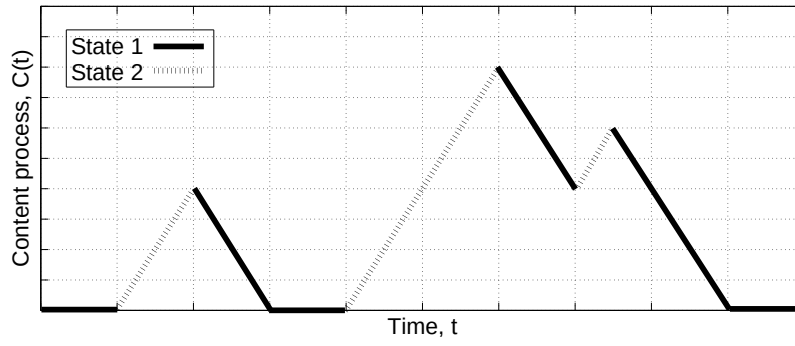


Figure 1.2: Evolution of $C(t)$ in the example depicted in Fig. 1.1.

1.3 First-order multi-regime feedback fluid queues

An FMFFQ is a double process $\{C(t), M(t); t \geq 0\}$, as well. Again $\{C(t); t \geq 0\}$ is the content process and $\{M(t); t \geq 0\}$ is the background process. We divide the domain of $C(t)$ into K regimes by defining $K + 1$ boundary points $0 = T^{(0)} < T^{(1)} < \dots < T^{(K)} \leq \infty$. $C(t)$ is said to be in regime k (at boundary point k) if $T^{(k-1)} < C(t) < T^{(k)}$ ($C(t) = T^{(k)}$). Contrary to single regime fluid queues, the drift of $C(t)$ depends not only on the state of the background process but also the instantaneous queue content in multi-regime feedback fluid queues. The latter dependency can be described by a piece-wise constant function of $C(t)$. In other words, when the background process is visiting state $i \in I$, $C(t)$ changes with drift $r_i^{(k)}$ ($\tilde{r}_i^{(k)}$) if the content is in regime k (at boundary point k). In multi-regime case, $M(t)$ is not a CTMC anymore, however it behaves as a CTMC with a finite state space I and infinitesimal generator $Q^{(k)}$ ($\tilde{Q}^{(k)}$) within each regime k (and at each boundary point k). A two-state two regime feedback fluid queue is presented in Fig. 1.3. In this figure, the state diagrams of the modulating process in the two regimes and the two boundaries and the corresponding drifts at each state are depicted. A sample path of the content process is shown in Fig. 1.4 for this configuration. Since the drifts in the first regime are twice as much as the

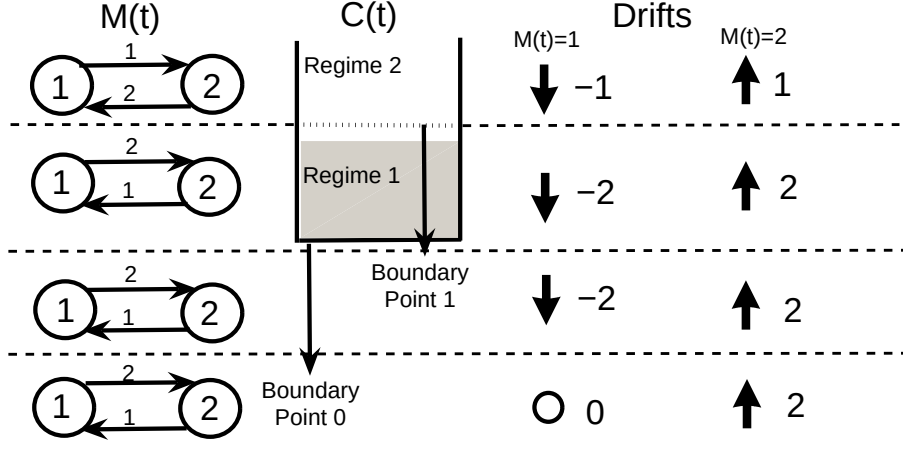


Figure 1.3: A first-order two-state two-regime feedback fluid queue.

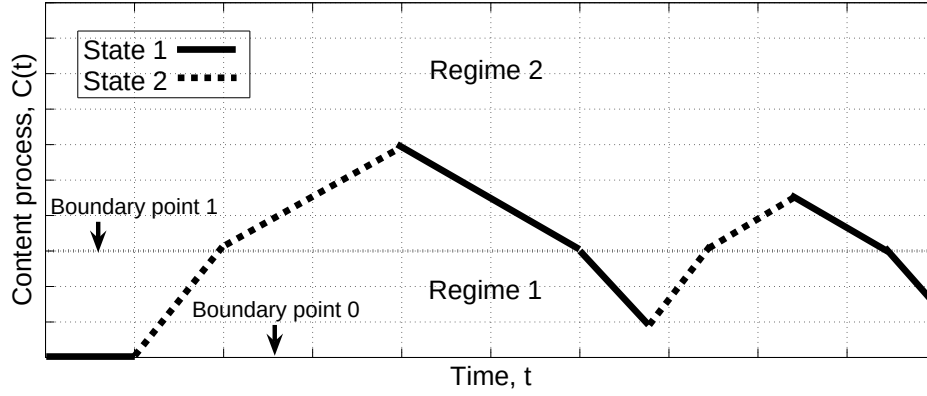


Figure 1.4: Evolution of $C(t)$ in the example depicted in Fig. 1.3.

drifts in the second regime, the queue content changes twice as faster in the first regime.

It is easy to see that FSFQs are special cases of FMFFQs. The steady-state joint probability distributions of FMFFQs are studied in [54] and [29]. Although FSFQs or FMFFQs may appear at a first glance very special structures, there are a wide range of problems that can be reduced to either one of the two structures by sample-path-based transformations. We next give a motivating example for this transformation.

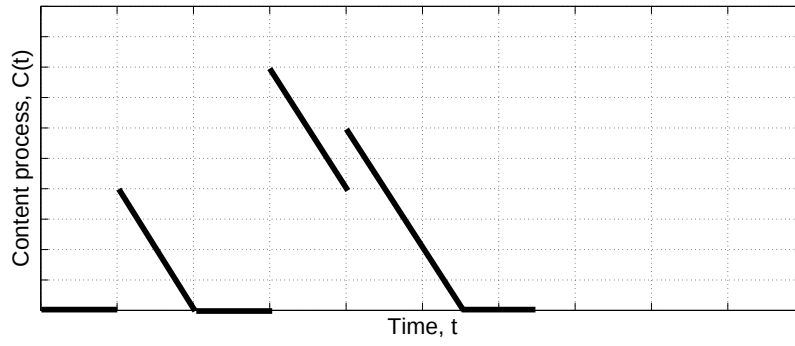


Figure 1.5: Reconstruction of Fig. 1.2 after we delete dotted lines.

1.4 Motivation

In Fig. 1.2 we illustrated the sample path of a two-state first-order single-regime fluid queue with an infinitesimal generator

$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

and the drift matrix

$$R = \text{diag}(-1, 1).$$

In this example, the queue size is assumed to be infinite. Since there is a single regime, the background process is a CTMC. Therefore, the duration of time at each state is an exponentially distributed random variable. The drift is 1 at state 2, so the rise in the queue level is exponentially distributed with mean $1/\mu$ when the background is at state 2.

Next, we delete the dotted parts of Fig. 1.2 and obtain the plot in Fig. 1.5. In Fig. 1.5, after each exponentially distributed time with mean $1/\lambda$ the plot makes an upward jump with an exponentially distributed amount with mean $1/\mu$. In between two successive jumps the content decreases with drift -1. Therefore the plot in Fig. 1.5 is obviously a sample path of the workload process in an M/M/1 queue with an arrival rate λ and service rate μ . It is easy to see that the steady-state distribution of the workload process of that M/M/1 queue is equal to the steady-state conditional distribution of the fluid queue, in Fig. 1.1, conditioned

on being at state 1. This fact is illustrated in [6] and [16]. Moreover, it is shown in [16] that this observation is valid not only for M/M/1 but also for a more general queueing system, the so-called MAP/PH/1 queue with MAP (Markovian Arrival Process) arrivals and PH-type (Phase Type) service times. Furthermore, [64] obtains the Laplace-Stieljes transform of the steady-state distribution of the workload in GI/PH/1 queue using that method. A similar method is used in [65] as well. These solutions are based on a transformation of the stochastic process under consideration to an FSFQ so we refer to this solution method as *FSFQ-based solution*. In this thesis, our motivation is to improve this basic transformation idea in such a way that it provides steady-state solutions for certain queueing systems of our interest and for which an exact analysis is not available. Contrary to previous FSFQ-based solutions, our solutions are *FMFFQ-based* and we therefore transform the stochastic processes of interest to FMFFQs. This is particularly useful for state-dependent queueing systems. Moreover, the transformations we propose are more general than the ones proposed in the literature. We believe that understanding FMFFQ-based solutions offers an important avenue to solving unsolved difficult analysis problems in the fields of performance evaluation, queueing theory, etc.

In this thesis, three separate queueing analysis problems are exactly analyzed using FMFFQ-based methods. The first two of these problems are related and are faced in a recently proposed optical switching paradigm, the so-called Optical Burst Switching (OBS) [46]. The third problem focuses on hysteresis-based queue control and its analysis and has no direct linkage with a specific technology as the first two. Before moving forward, we next briefly present the basics of optical networks and the OBS paradigm in the following section.

1.5 An Overview of Optical Networks and Optical Burst Switching

This section provides an overview of optical networks with particular emphasis on optical burst switching (OBS). We refer to [68] and [14] for more details.

With advances in Wavelength Division Multiplexing (WDM) technology that can carry multiple signals on multiple wavelengths on a single fiber, the capacity of a single fiber has increased tremendously in the recent years. On the other hand, rates of electronic processing dictated by Moore's Law have lagged the rate of increase of traffic demands due to emerging high-speed applications such as video streaming, peer to peer applications, distributed computing, etc. Optical switching has emerged as an alternative switching paradigm as a result of this mismatch. Three types of optical switching architectures are proposed: Optical Circuit Switching (OCS), Optical Packet Switching (OPS) and Optical Burst Switching (OBS).

OCS is the optical counterpart of circuit switching in electronic networks. OCS comprises three phases; lightpath set-up, data transmission and lightpath tear down. In OCS, a separate lightpath, i.e., a concatenation of fiber/wavelength pairs toward a destination, is set up between each source-destination pair and all the traffic is sent through this lightpath. Switch configurations are carried out in the lightpath set-up phase and the data stays in the optical domain end-to-end and therefore OCS is transparent in terms of bit rates and modulation types. Furthermore, no buffering is needed in the optical domain. However, since the lightpath is not shared between different traffic demands and set-up time overheads are typically large, OCS is not considered to be a bandwidth-efficient solution for bursty data traffic as in the Internet [47]. OCS is a widely used switching technique used in large backbones today. On the other hand, Optical Packet Switching (OPS) is the optical equivalent of electronic packet

switching. It is suitable for supporting bursty traffic since it allows statistical sharing of the wavelengths among different traffic demands with different origins and/or destinations. In OPS, control header is carried with its payload and it is processed electronically or optically at each intermediate node whereas the payload remains in the optical domain while its header is being processed. OPS appears to be more of a longer-term optical networking paradigm due to the limits in current optical technologies [47].

Optical burst switching provides a granularity between OPS and OBS. At the edge of an OBS network, client protocol (such as Internet Protocol) packets from different sources destined for the same egress edge node are assembled into an optical burst at the ingress edge node. When the optical burst arrives at the egress node, it is de-assembled back into IP packets and IP packets are electronically routed to their destinations. When a burst is formed, a Burst Control Packet (BCP) is associated with it and sent to the network in advance on a separate control channel. The control packet is then processed electronically at each core node. According to the information carried in the BCP, the resources are reserved for the burst and switch configurations are made before the actual payload arrives.

The JET protocol is the most widely adopted reservation protocol for OBS networks which does not require any kind of optical buffering at the intermediate nodes [70]. In JET-based OBS, a control packet is first sent toward the network over the control channel to make a reservation for the burst. After an offset time, the burst is sent to the core network without waiting for an acknowledgment for the connection establishment, hence JET-based OBS uses the one-way reservation mechanism contrary to OCS. Since switch settings are done before the burst arrives, the burst passes through the network without requiring O/E/O conversion and buffering. Fig. 1.6 depicts how the JET protocol operates. As shown in this figure, first the control packet is sent toward the network. The control

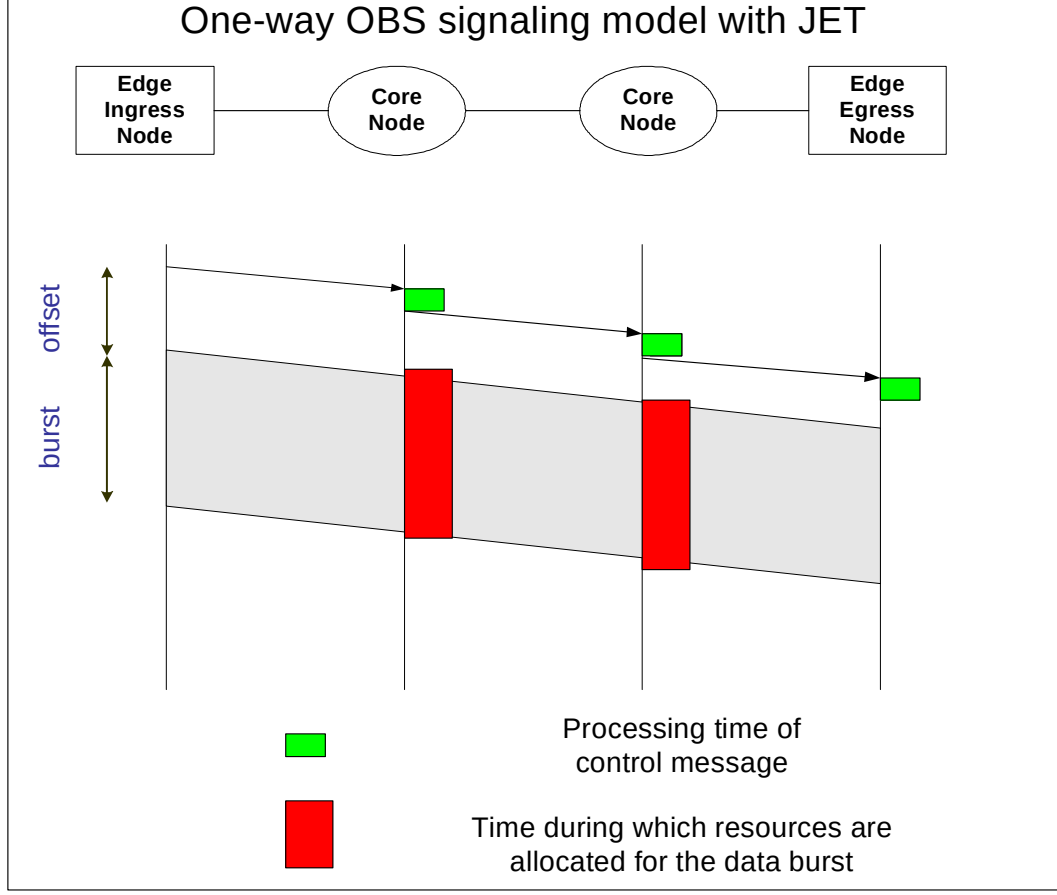


Figure 1.6: OBS with JET protocol.

packet is then processed at the intermediate nodes and wavelength reservation and switch settings are done. During this time, the burst is buffered at the ingress node. After an offset time T_{off} , the burst is sent to the network over the optical data channel. Suppose that the time required to process the control packet and to configure the switch is Δ and the number of the hops to the destination is H . Then, the offset time, T_{off} , should be chosen such that $T_{off} \geq H\Delta$ to ensure that there is enough time for each intermediate node to complete the processing of the control packet before the burst arrives. Suppose we set $T_{off} = 3\Delta$. Then, the total delay experienced by the burst in Fig. 1.6 is $P + 3\Delta$ where P is the propagation delay.

When multiple bursts contend for the same outgoing wavelength, contention is said to occur. Basically there are three ways of resolving contention [2]:

- *Wavelength Domain:* The contending bursts can be sent on a different wavelength channel of the designated output link by using wavelength converters. The switch may have Full Wavelength Conversion (FWC) capability or Partial Wavelength Conversion (PWC) capability. In FWC, a burst arriving at a certain wavelength can be switched onto any other wavelength toward its destination. In Partial Wavelength Conversion (PWC), there is a limited number of converters which are shared among all wavelengths, and consequently some bursts cannot be switched toward their destination (and therefore blocked) when all converters are busy despite the availability of free channels on wavelengths different from the incoming wavelength [2].
- *Time Domain:* Contention may be resolved in time domain by delaying the contending bursts by using coils of fiber lines which are called *Fiber Delay Lines* (FDL). FDLs provide only a fixed amount of delay as opposed to electronic buffers.
- *Space Domain:* In the space domain contention resolution scheme, one of the contending bursts is sent over an alternative route toward the destination. This scheme is called “deflection routing”. In OBS, the deflection routes should be determined in advance and the offset time should be chosen to compensate for the extra processing time encountered by the BCP.

Client packets belong to different traffic classes with different QoS (Quality of Service) requirements in terms of performance parameters such as loss, delay, delay jitter, etc. QoS provisioning refers to methods that are employed to satisfy the QoS requirements of different classes. One basic QoS management mechanism used in electronically switched networks is priority queueing in which we have mainly two traffic classes, namely high priority and low priority classes, and the mechanism ensures that high priority traffic is not adversely impacted by low priority packets. Due to lack of optical buffers, this mechanism cannot be

directly implemented in OBS networks. In the offset-based differentiation mechanism, an extra offset time is assigned to bursts having high priority packets resulting in an earlier reservation for such bursts. The offset time-based differentiation mechanism allows one to achieve absolute isolation between high priority and low priority classes. However, to achieve almost perfect isolation, the length of this extra offset time has to be as long, at least, as a few average low priority burst durations.

1.6 Thesis Overview

This thesis is logically divided into three parts 1) Theory and methods (Chapter 2) 2) Transformation ideas (Chapter 3) and 3) Applications (Chapters 4, 5, and 6). A detailed description of these three parts and our contributions in this thesis are described below.

In Chapter 2, we provide the differential equations and the boundary conditions for FMFFQs. The steady-state behavior of FMFFQs was first studied in [54]. We study the same problem for a wider range of boundary types and zero-drift regimes in Chapter 2. We use the joint probability density functions and the probability mass accumulations at the boundary points rather than the joint cumulative distribution functions in the description of the steady-state behavior. This choice of ours simplifies the notation and presentation. Moreover, we propose a numerically stable and computationally efficient algorithm (whose main engine turns out to be the so-called ordered Schur factorization) to solve the steady-state probability measures of FMFFQs. Our findings in this chapter have appeared in [29].

Chapter 3 gives a set of examples illustrating the ideas behind FSFQ-based and FMFFQ-based solutions to certain queueing analysis problems upon which the rest of the thesis is built. In these problems, the original stochastic model

is not of FSFQ- or FMFFQ-type but we propose certain transformations on the original problem to convert them to either FSFQs or FMFFQs and use the algorithms of Chapter 2 to solve them. FSFQ-based solutions are previously studied in [6], [16], [16], [64] and [65]. To the best of our knowledge, we use FMFFQs for the first time for queueing analysis problems for which the original process of interest is not of FMFFQ-type but we use certain transformations to reduce them to FMFFQ. Five basic examples are presented in this chapter. The first three of them present FSFQ-based solutions. The process in the first of the FMFFQ-based examples has a continuous state space and the transformation used can be considered as a primitive form of the transformations in the applications part of this thesis. In our final example, the stochastic process has a discrete state space and this example is presented to show the strength of the FMFFQ-based solutions.

In the applications part (Chapters 4, 5, and 6), we study three performance evaluation problems. The first problem studied in Chapter 4 is a method to find the steady-state blocking probabilities in a multi-class single-channel optical burst switching (OBS) link using offset-based service differentiation by assigning different offset times to different traffic classes. The traffic classes with larger offset times are known to have relatively lower blocking probabilities. In this work, we exactly solve for the blocking probabilities in case the incoming traffic to the link is Poisson and per-class packet length distributions are of PH-type. The unfinished work in this system can not directly be modeled by an FMFFQ but with certain sample path-based transformations, we reduce the problem to the solution of an FMFFQ. To the best of our knowledge, an exact algorithm to this problem did not exist in the literature. Our findings have appeared in [30].

In Chapter 5, we propose a solution to the steady-state distribution of the unfinished work for an optical buffer using a finite number (say K) of fiber delay lines (FDLs) for a single-channel optical switching system. Such buffers can be

used in OBS or OPS networks for contention resolution purposes. The work in [49] assumes generic distributions for packet interarrivals and services and the authors present a Markov chain model for solving the optical buffer. The entries of the Markov chain can be found by calculating certain definite integrals for which computational procedures are not elaborated. Moreover, depending on the distributions used in the model, the number of the states of the Markov chain can be excessive which then calls for an approximate state reduction method for computational feasibility. In two special cases, the authors [49] have shown that their solution is exact: i) exponential interarrivals and generic packet lengths ii) generic interarrivals and exponential packet lengths. Again, following similar traffic assumptions, the case of $K = \infty$ is studied by [53] to develop heuristics for the case of $K < \infty$ through formulas involving infinite sums. Three special cases corresponding to (i) exponentially distributed, (ii) deterministic, (iii) and mixtures of deterministic burst lengths, are given for which the resulting formula does not involve an infinite sum and therefore the method turns out to be computationally effective. An approximate method based on the assumption $K = \infty$ is proposed in [35] based on the generating function approach for the case of Poisson arrivals and generally distributed packet lengths. Again, the unfinished work in this system is not of FMFFQ-type but with similar sample path-based transformations, we reduce the problem to an FMFFQ and exactly solve for the associated blocking probabilities in case the arrival stream is a MAP (Markovian Arrival Process) and packet lengths are of PH-type. Contrary to previous solutions, our solution allows auto-correlation between successive interarrival times which is important for modeling bursty Internet traffic. To the best of our knowledge, this is the first exact result obtained in the literature in this general setting. Our findings in this chapter have recently been submitted for potential publication [28].

In Chapter 6, we study the steady-state joint probability density function for the queue occupancy of FMFFQs with hysteretic control (FMFFQ-H in short).

Unlike FMFFQs, we replace each static threshold (or boundary point) with forward and backward thresholds between which hysteretic control is applied. To our best knowledge, there has not been a previous attempt to solve any of the performance measures of that stochastic model neither approximately nor exactly in the literature. In this chapter, we solve two performance measures of FMFFQ-Hs exactly in the steady-state: i) joint probability density function of the queue occupancy and ii) the expected number of crossings of the thresholds in unit time. Furthermore, in the same chapter, we show that it is possible to find the steady-state performance measures of basic queueing systems, such as M/M/1, possessing hysteretic control with similar transformation ideas we used in the other chapters.

Finally, Chapter 7 concludes the thesis and lists possible future research areas.

Chapter 2

FIRST-ORDER MULTI-REGIME FEEDBACK FLUID QUEUES

In this chapter, we study first-order multi-regime feedback fluid queues (FMFFQs in short). The boundary conditions are derived in terms of joint densities and for a wide range of state types based on [54], [29] and [28]. The ordered Schur factorization is used as a numerical engine to find the steady-state distribution of the FMFFQs.

2.1 Introduction

FMFFQs are described by a joint Markovian process $\{C(t), M(t); t \geq 0\}$ where $\{C(t); t \geq 0\}$ refers to the buffer content at time t and $\{M(t); t \geq 0\}$ is an underlying continuous-time Markov chain that determines the net rate (entry rate minus exit rate or drift) at which the buffer content $C(t)$ changes. The

latter process $\{M(t); t \geq 0\}$ is often called the background or the modulating process of the Markov fluid queue.

A key reference on Markov fluid queues is the spectral approach in [4] for infinite buffer capacities. [62] extends this analysis to finite fluid queues using the spectral approach. [34] gives a more recent and extensive overview of Markov fluid queues and the spectral approach.

More general models, known as feedback fluid queues, were introduced in [1] and [54] where both the rate of change of the buffer content and the background process are allowed to depend on the instantaneous queue occupancy. The solution is extended via the addition of new boundary conditions for a set of new drift configuration schemes in [29] and [28]. We note that feedback fluid queues have proven useful for modeling controlled telecommunication networks. In one of earlier works, [17] allow the drifts to change with respect to only the queue occupancy as a sub-case of feedback fluid queues and in a piecewise constant fashion, in the context of modeling a networking system with loss priorities. A similar system with drifts being piece-wise continuous functions of the buffer content is analyzed in [31]. The more general feedback fluid queues in which the background process is also allowed to vary with respect to buffer content are studied in [55] and [37] (see also [38]); the former study allows continuous feedback whereas the latter assumes piecewise constant feedback to model a network access system. [11] studies a model which can be viewed as a continuous feedback fluid queue with two background states and unlimited buffer content. Although continuous feedback fluid queues are powerful for modeling purposes, it is generally hard to obtain numerical results except for special cases; see for example the two-state example given in [55]. The goal of this chapter is to present the steady-state solution of feedback fluid queues in case drifts and the background process are piecewise constant functions of the buffer content.

Ramaswami [48] provides a systematic approach to Markov fluid queues using the matrix geometric approach. A similar method was proposed by Soares and Latouche [56] for finite buffered fluid queues. Other alternative techniques to solve Markov fluid queues include the probabilistic analysis of Asmussen [5] using an iterative technique, the Wiener-Hopf factorization method of Rogers [52], and the forward-backward decomposition approach of Akar and Sohraby [3] using the matrix sign function. For the piecewise constant feedback case, Mandjes et al. [37] propose a spectral expansion approach to solve the fluid queue of interest.

It is not only hard to obtain numerical results for large state spaces as stated in [37] but also the spectral expansion approach for fluid queues turns out to be ill-conditioned as stated by Akar and Sohraby [3]. The goal of this chapter is to obtain a computationally stable numerical algorithm for the steady-state solution of feedback fluid queues in case drifts and the background process are piecewise constant functions of the buffer content.

The organization of the chapter is as follows. We describe the FMFFQ in Section 2.2 and provide in compact form the boundary conditions for the steady-state distribution of the underlying system. In Section 2.3, we describe the spectral expansion approach. A brief summary of the ordered Schur form is presented in Section 2.4. The numerical procedure for the solution of FMFFQs based on ordered Schur factorization is given in Section 2.5. We present numerical examples to justify the stability and validate the effectiveness of the proposed approach in Section 2.6.

2.2 Stochastic Model

We describe the general FMFFQ model under consideration in this section. We let $C(t)$ denote the fluid level in the queue and $M(t)$ denote the state of the background process at time t . We identify thresholds $0 = T^{(0)} < T^{(1)} < \dots <$

$T^{(K)} < \infty$ such that the buffer is said to be in regime k if $T^{(k-1)} < C(t) < T^{(k)}$. The buffer is said to be at threshold k when $C(t) = T^{(k)}$. $T^{(0)}$ and $T^{(K)}$ are terminal boundary points whereas the other thresholds correspond to intermediate boundary points. We assume that the background process $\{M(t); t \geq 0\}$ has the state space $\{1, 2, \dots, M\}$. When the system is in regime k (at threshold $T^{(k)}$) then the background process $M(t)$ behaves according to a Markov process with generator $Q^{(k)}$ ($\tilde{Q}^{(k)}$). The drift while at state m , $1 \leq m \leq M$, in regime k (at threshold $T^{(k)}$) is denoted by $r_m^{(k)}$ ($\tilde{r}_m^{(k)}$). We let $R^{(k)}$ ($\tilde{R}^{(k)}$) to be the diagonal matrix of drifts in regime k (at threshold $T^{(k)}$). The dynamics of the buffer content is given by:

$$\frac{dC(t)}{dt} = \begin{cases} \max(0, \tilde{r}_{M(t)}^{(0)}) & \text{if } C(t) = 0, \\ r_{M(t)}^{(k)} & \text{if } T^{(k-1)} < C(t) < T^{(k)}, 1 \leq k \leq K, \\ \tilde{r}_{M(t)}^{(k)} & \text{if } C(t) = T^{(k)}, 1 \leq k \leq K-1, \\ \min(0, \tilde{r}_{M(t)}^{(K)}) & \text{if } C(t) = T^{(K)}. \end{cases} \quad (2.1)$$

Let $f^{(k)}(y, t)$ denote the row vector of transient joint probability density functions at time t in regime k for $1 \leq k \leq K$, i.e., $f^{(k)}(y, t) = [f_1^{(k)}(y, t), f_2^{(k)}(y, t), \dots, f_M^{(k)}(y, t)]$, where

$$f_m^{(k)}(y, t) = \frac{\partial F_m^{(k)}(y, t)}{\partial y}, \quad T^{(k-1)} < y < T^{(k)}, \quad 1 \leq k \leq K, \quad 1 \leq m \leq M, \quad (2.2)$$

and

$$F_m^{(k)}(y, t) = P(C(t) \leq y, M(t) = m), \quad T^{(k-1)} < y < T^{(k)}, \quad 1 \leq k \leq K, \quad 1 \leq m \leq M. \quad (2.3)$$

The steady-state joint density function can then be defined via taking the limit of (2.2) as $t \rightarrow \infty$, i.e., $f_m^{(k)}(y) = \lim_{t \rightarrow \infty} f_m^{(k)}(y, t)$. We then define the steady-state joint density vector

$$f^{(k)}(y) = \begin{bmatrix} f_1^{(k)}(y) & f_2^{(k)}(y) & \cdots & f_M^{(k)}(y) \end{bmatrix}. \quad (2.4)$$

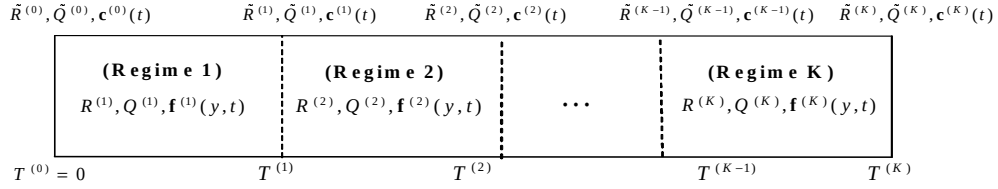


Figure 2.1: The first order multi-regime feedback fluid queue.

Similarly, we define $F_m^{(k)}(y) = \lim_{t \rightarrow \infty} F_m^{(k)}(y, t)$ and $F^{(k)}(y) = [F_1^{(k)}(y), F_2^{(k)}(y), \dots, F_M^{(k)}(y)]$.

Moreover, we define $c^{(k)}(t)$ to be the row vector of transient probability mass accumulations at the boundary point $T^{(k)}$ at time t :

$$c^{(k)}(t) = \begin{bmatrix} c_1^{(k)}(t) & c_2^{(k)}(t) & \cdots & c_M^{(k)}(t) \end{bmatrix}, \quad 0 \leq k \leq K, \quad (2.5)$$

where

$$c_m^{(k)}(t) = P(C(t) = T^{(k)}, M(t) = m), \quad 0 \leq k \leq K, \quad 1 \leq m \leq M. \quad (2.6)$$

The steady-state steady probability mass accumulations at the boundary points are defined by means of taking the limit of (2.6) as $t \rightarrow \infty$, i.e., $c_m^{(k)} = \lim_{t \rightarrow \infty} c_m^{(k)}(t)$, $0 \leq k \leq K$, $1 \leq m \leq M$. We also define $c^{(k)} = [c_1^{(k)}, c_2^{(k)}, \dots, c_M^{(k)}]$. Finally, we define the joint CDF $F_m(y)$ for all y such that

$$F_m(y) = F_m^{(k)}(y), \quad 1 \leq m \leq M, \quad \text{when } T^{(k-1)} < y < T^{(k)},$$

and the buffer occupancy CDF $F(y) = [F_1(y), F_2(y), \dots, F_M(y)]e$ where e is a column vector of ones of appropriate size. Note that by definition, $F_m(y)$ and $F(y)$ are right-continuous at the boundary points. The joint PDF $f_m(y)$ and the buffer occupancy PDF $f(y)$ are defined accordingly. The steady solution to the feedback fluid queue involves the calculation of $f^{(k)}(\cdot)$ for $1 \leq k \leq K$ and $c^{(k)}$ for $0 \leq k \leq K$ which is the scope of the current paper. The FMFFQ is illustrated in Fig. 2.1.

Let $S_+^{(k)}$, $S_0^{(k)}$, and $S_-^{(k)}$ denote the set of states with positive, zero, and negative drifts, respectively, in regime k . Similarly, let $\tilde{S}_+^{(k)}$, $\tilde{S}_0^{(k)}$, and $\tilde{S}_-^{(k)}$ denote the set of states with positive, zero, and negative drifts, respectively, at the

boundary point $T^{(k)}$. Moreover, $S = S_+^{(k)} \cup S_-^{(k)} \cup S_0^{(k)}$, $\forall k$ denotes the set of all states. Finally, we define for $1 \leq k \leq K$ the mean drifts at regime k :

$$\mu^{(k)} = \pi^{(k)} R^{(k)} e,$$

where $\pi^{(k)}$ denotes the steady-state vector of $Q^{(k)}$, i.e.,

$$\pi^{(k)} Q^{(k)} = 0, \quad \pi^{(k)} e = 1.$$

Given the notation above, we make the following assumptions on the drift configuration of the system by some of which we rule out anomalous situations.

- A1)** $S_+^{(k)} \neq \emptyset, S_-^{(k)} \neq \emptyset, 1 \leq k \leq K, \tilde{S}_+^{(k)} \neq \emptyset, \tilde{S}_-^{(k)} \neq \emptyset, 0 \leq k \leq K$.
- A2)** If $m \in S_+^{(k)} \cap S_-^{(k+1)}$ then $m \in \tilde{S}_0^{(k)}, 1 \leq k \leq K - 1$.
- A3)** If $r_m^{(k)} = 0$ then $\tilde{r}_m^{(k-1)} = \tilde{r}_m^{(k)} = 0$.
- A4)** The drifts at other states are either zero, or left or right continuous at the corresponding boundary points.
- A5)** In all regimes, the mean drifts are nonzero, i.e., $\mu^{(k)} \neq 0, 1 \leq k \leq K$.

Next, we justify the assumptions. Violation of **A1** restricts the buffer content process to stay in a certain portion of the buffer or leads the content process to stick to a boundary point. It is also natural to select the drift at an intermediate boundary point $T^{(k)}$ as either left continuous ($\tilde{r}_m^{(k)} = r_m^{(k)}$) or right continuous ($\tilde{r}_m^{(k)} = r_m^{(k+1)}$) or simply zero ($\tilde{r}_m^{(k)} = 0$). For any m and k satisfying the condition $m \in S_+^{(k)} \cap S_-^{(k+1)}$, the buffer content would make infinitesimal oscillations around the boundary point $T^{(k)}$ at an infinite rate under the assumption of left and right continuities of drifts. We use the assumption **A2** to avoid this situation. The assumption **A3** removes the possibility of having more than one probability mass accumulation within arbitrarily small neighborhoods of boundary points. Additionally, we assume **A4** to avoid a situation as the one described for **A3**.

Finally, we avoid double eigenvalues at zero by ruling out zero mean drifts in **A5**.

We next provide the main theorem of this chapter that characterizes the solution to the steady-state joint density and the probability mass accumulations at the boundary points.

Theorem 1. *The steady-state joint density vector $f^{(k)}(\cdot)$ of the feedback Markov fluid queue satisfies the differential equation*

$$\frac{d}{dy} f^{(k)}(y) R^{(k)} = f^{(k)}(y) Q^{(k)}, \quad T^{(k-1)} < y < T^{(k)}, \quad 1 \leq k \leq K, \quad (2.7)$$

along with the boundary conditions

$$c_m^{(k)} = 0, \quad \forall m \notin \tilde{S}_0^{(k)}, \quad 0 \leq k \leq K, \quad (2.8)$$

$$f^{(0)}(0+) R^{(1)} = c^{(0)} \tilde{Q}^{(0)}, \quad (2.9)$$

$$f^{(k+1)}(T^{(k)}+) R^{(k+1)} - f^{(k)}(T^{(k)}-) R^{(k)} = c^{(k)} \tilde{Q}^{(k)}, \quad 1 \leq k \leq K-1, \quad (2.10)$$

$$f_m^{(k)}(T^{(k)}-) = 0, \quad \forall m \in S_-^{(k)} \cap (\tilde{S}_0^{(k)} \cup \tilde{S}_+^{(k)}), \quad 1 \leq k \leq K-1, \quad (2.11)$$

$$f_m^{(k+1)}(T^{(k)}+) = 0, \quad \forall m \in (\tilde{S}_0^{(k)} \cup \tilde{S}_-^{(k)}) \cap S_+^{(k+1)}, \quad 1 \leq k \leq K-1, \quad (2.12)$$

$$f^{(K)}(T^{(K)}-) R^{(K)} = -c^{(K)} \tilde{Q}^{(K)}, \quad (2.13)$$

$$\sum_{m=1}^M \left(\sum_{k=1}^K \int_{T^{(k-1)}+}^{T^{(k)}-} f_m^{(k)}(x) dx + \sum_{k=0}^K c_m^{(k)} \right) = 1. \quad (2.14)$$

Proof. The differential equation (2.7) and the boundary conditions (2.9), (2.10), (2.13) and (2.14) can be driven by using the results of [37]. [37] find the following ordinary differential equations of the joint distribution functions for FMFFQs:

$$\begin{aligned} \frac{d}{dy} F^{(k)}(y) R^{(k)} &= (F^{(k)}(y) - F^{(k)}(T^{(k-1)})) Q^{(k)} \\ &+ (F^{(k)}(T^{(k-1)}) - F^{(k-1)}(T^{(k-1)}-)) \tilde{Q}^{(k-1)} \\ &+ (F^{(k-1)}(T^{(k-1)}-) - F^{(k-1)}(T^{(k-2)})) Q^{(k-1)} \\ &\vdots \end{aligned}$$

$$\begin{aligned}
& + \left(F^{(1)}(T^{(1)}-) - F^{(1)}(T^{(0)}) \right) Q^{(1)} \\
& + F^{(1)}(T^{(0)}) \tilde{Q}^{(0)}, \quad T^{(k-1)} < y < T^{(k)}, \quad 1 \leq k \leq K \quad (2.15)
\end{aligned}$$

Equation (2.7) can be obtained by just differentiating both sides of (2.15) with respect to y . The boundary condition (2.8) is an immediate consequence of the fact that if nothing forces the content to stay at a point then the probability mass accumulation will be zero at that point. These conditions are equivalent to the continuity conditions in the cumulative distribution function formulation of [37]. Equation (2.9) can be obtained by just writing (2.15) for the first regime and equating the variable y to $T^{(0)}+$ in that equation. Equation (2.10) can be obtained by writing (2.15) for regime k and equating the variable y to $T^{(k)}-$ in this equation and then writing (2.15) for regime $k+1$ and equating the variable y to $T^{(k)}+$, and then subtracting the former one from the latter. Equation (2.13) can be obtained by writing (2.15) for the K^{th} regime and equating the variable y to $T^{(K)}-$ in this equation and then subtracting it side by side from the following equation obtained by [37]:

$$\begin{aligned}
0 &= \pi \tilde{Q}^{(K)} + F^{(K)}(T^{(K)}-) \left(Q^{(K)} - \tilde{Q}^{(K)} \right) \\
&+ F^{(K)}(T^{(K-1)}) \left(\tilde{Q}^{(K-1)} - Q^{(K)} \right) \\
&+ F^{(K-1)}(T^{(K-1)}-) \left(Q^{(K-1)} - \tilde{Q}^{(K-1)} \right) \\
&\vdots \\
&+ F^{(1)}(T^{(1)}-) \left(Q^{(1)} - \tilde{Q}^{(1)} \right) + F^{(1)}(T^{(0)}) \left(\tilde{Q}^{(0)} - Q^{(1)} \right), \quad (2.16)
\end{aligned}$$

where π is the vector of steady-state probabilities at the uppermost boundary point of the queue. The equation (2.14) is the normalization condition, i.e., the sum of all probabilities accumulated at the boundary points and integral of the probability densities over all regimes is equal to 1.

Next, we derive the boundary condition (2.11). For this purpose, we write the forward Kolmogorov equations for the following function defined for $T^{(k-1)} <$

$y < T^{(k)}$ just below the boundary point $T^{(k)}$:

$$\overleftarrow{P}_m^{(k)}(x, y, t) = P(x \leq C(t) < y, M(t) = m), m \in S_-^{(k)} \cap (\tilde{S}_0^{(k)} \cup \tilde{S}_+^{(k)}). \quad (2.17)$$

We first write the left hand side of the Kolmogorov equation:

$$\overleftarrow{P}_m^{(k)}(T^{(k)} + r_m^{(k)} \Delta t, T^{(k)}, t + \Delta t) = \quad (2.18)$$

$$P(T^{(k)} + r_m^{(k)} \Delta t \leq C(t + \Delta t) < T^{(k)}, M(t + \Delta t) = m). \quad (2.19)$$

We assume that Δt is sufficiently small and the probability of more than one transition in the background process within time interval $[t, t + \Delta t]$ is $o(\Delta t)$. Since $m \in S_-^{(k)} \cap (\tilde{S}_0^{(k)} \cup \tilde{S}_+^{(k)})$, according to the assumptions made in Section 2, $r_m^{(k+1)}$ is either zero or positive. For both of these cases, $C(t + \Delta t)$ can not be in the interval $[T^{(k)} + r_m^{(k)} \Delta t, T^{(k)})$ without any state transitions in the background process in the time interval $[t, t + \Delta t]$. For the case of one transition in the background process, the interval of the content process at time t does not constitute any probability mass accumulation. The corresponding transition from any other state n can occur only in regime k with probability $O(\Delta t)$. Therefore, if we divide both sides of the forward Kolmogorov equation by Δt and take the limit of both sides as $\Delta t \rightarrow 0$ we obtain zero on the right hand side. However, the left hand side becomes

$$\lim_{\Delta t \rightarrow 0} \frac{\overleftarrow{P}_m^{(k)}(T^{(k)} + r_m^{(k)} \Delta t, T^{(k)}, t + \Delta t)}{\Delta t} = -r_m^{(k)} f_m^{(k)}(T^{(k)-}, t), \quad (2.20)$$

$$m \in S_-^{(k)} \cap (\tilde{S}_0^{(k)} \cup \tilde{S}_+^{(k)}). \quad (2.21)$$

Therefore by equating (2.21) to zero, (2.11) can be obtained. Derivation of (2.12) is quite similar to the derivation of (2.11). For this case we need to define for $T^{(k)} < y < T^{(k+1)}$ the following function:

$$\vec{P}_m^{(k+1)}(x, y, t) = P(x < C(t) \leq y, M(t) = m), m \in (\tilde{S}_0^{(k)} \cup \tilde{S}_-^{(k)}) \cap S_+^{(k+1)}, \quad (2.22)$$

and write the forward Kolmogorov equations for $x = T^{(k)}$ and $y = T^{(k)} + r_m^{(k)} \Delta t$.

□

Remark 1. When compared with [37] that uses cumulative distribution functions for the formulation of the boundary conditions, we use in this chapter probability density functions which shorten the representation of these boundary conditions.

Remark 2. The extension of Theorem 1 to the infinite buffer case is straightforward. If the length of regime K is infinite, i.e., $T^{(K)} = \infty$, and if the stability condition

$$\pi^{(K)} R^{(K)} e < 0$$

holds, then instead of the boundary conditions (2.13) and (2.8) for $T^{(K)}$, we have to use the boundary condition:

$$a_i^{(K)} = 0, \forall i : \Re\{\lambda_i\} > 0 \quad (2.23)$$

where $a_i^{(K)}$ are the coefficients in the general solution of (2.7) given in (2.24).

2.3 Spectral solution

In this section, we present the spectral solution to FMFFQs. For this purpose, we note that in the special case of $S_0^{(k)} = \emptyset$, the general solution to (2.7) is given by:

$$f^{(k)}(y) = \sum_{i \neq j} a_i^{(k)} \exp(\lambda_i^{(k)} y) \phi_i^{(k)} + a_j^{(k)} \phi_j^{(k)}, T^{(k-1)} < y < T^{(k)}, 1 \leq k \leq K \quad (2.24)$$

where $(\lambda_i^{(k)}, \phi_i^{(k)})$ is the i th eigenvalue- left eigenvector pair of the matrix $Q^{(k)} (R^{(k)})^{-1}$ and $\phi_j^{(k)}$ denotes the eigenvector corresponding to the zero eigenvalue. Here we assume simple eigenvalues as in [37]. In this solution, there are MK unknown “ a ” coefficients and $MK + M$ unknown “ c ” coefficients (probability mass accumulations at the boundary points) that we need to find.

The general form of the solution when $S_0^{(k)} \neq \emptyset$ is the same as (2.24) whereas in this case we have $KM - \sum_k |S_0^{(k)}|$ unknown “ a ” coefficients and $MK + M$

unknown “ c ” coefficients, totalling $2KM + M - \sum_k |S_0^{(k)}|$ unknowns to be calculated.

In the spectral solution of the system of ordinary differential equations of Theorem 1, the general methodology is as following: First, the general form of the solution (2.24) is directly substituted into the boundary conditions listed in this theorem; Second, an $xA = b$ vector-matrix equation, in which x is the vector of unknown coefficients, is formed using those substitutions. Finally, the values of unknown coefficients are found and substituted into the general solution.

2.4 Ordered Schur Form

We start by stating without proof the following theorem based on Golub and Van Loan[22, Ch. 7].

Theorem 2. *Let A be a real square matrix. Then, there exists an orthogonal matrix Z such that*

$$Z^T A Z = D = \begin{bmatrix} D_{11} & D_{12} \\ 0 & D_{22} \end{bmatrix}, \quad (2.25)$$

where D is upper block triangular with either 1-by-1 or 2-by-2 diagonal blocks (hence the term quasi-triangular), respectively, comprising real and complex conjugate eigenvalues of A in any desired order. The matrix D is said to be in ordered real Schur form, and the columns of matrix Z are referred to as Schur vectors.

Remark 3. *In the ordered Schur form, any ordering policy can be used; for example eigenvalues with negative real parts can be placed at the upper-left corner (D_{11} has all its eigenvalues in the left half plane) and those with positive real parts at the lower-right corner (D_{22} has all its eigenvalues in the right half plane) .*

Given the matrix A and its ordered Schur form as in (2.25), suppose that we want to find a nonsingular Y such that

$$Y^{-1}AY = \begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix} \quad (2.26)$$

by eliminating the upper-right partition D_{12} of matrix D . To find Y , we block diagonalize the matrix D of (2.25) as follows:

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} \\ 0 & D_{22} \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} = \begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix}, \quad (2.27)$$

where X should be found by solving the Sylvester matrix equation (Golub and Van Loan [22, Ch. 7])

$$D_{11}X - XD_{22} = D_{12}, \quad (2.28)$$

for quasi-triangular D_{11} and D_{22} . The existence of a unique solution X stems from the spectrum disjointness of D_{11} and D_{22} . Then, Y follows from (2.25), (2.26), and (2.27) as

$$Y = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} - Z_{11}X \\ Z_{21} & Z_{22} - Z_{21}X \end{bmatrix}, \quad (2.29)$$

where Z is partitioned accordingly. For more information on ordered real Schur factorization and block diagonalization, we refer the reader to [22, Ch. 7].

We also note that appropriate computational routines are readily available in LAPACK [57] and MATLAB [40] both for ordered real Schur factorization and for solving the Sylvester matrix equation (2.28) with upper quasi-triangular coefficient matrices D_{11} and D_{22} . We note that obtaining the real Schur form is known to be backward stable and has a complexity of an^3 , where a accounts for the iteration in the algorithm and may vary between 10 and 25; see Van Dooren [66]. On the other hand, for the computation of the Jordan form that requires calculation of all eigenvectors, there are no results of guaranteed backward stability and the complexity of this decomposition is much higher than that of the

Schur decomposition; see Van Dooren [66]. For this reason, Van Dooren [66] strongly recommends not to use the Jordan decomposition whenever one can use the more reliable Schur form as we do in the current paper. We note that this view is also shared by Bai et al. [8] in which the standard serial algorithm for the eigen decomposition problem is the ordered Schur factorization due to its well-established numerical stability. We also note the alternative matrix sign function method [8] which is not as numerically stable for the case of close to zero drifts.

Assume that the square matrix A has eigenvalues at the origin. It is also clear that one can use the ordered Schur form to first place the eigenvalues at zero at the upper-left corner followed by eigenvalues with negative and positive real parts, respectively. Then solving two Sylvester equations, one can obtain a matrix Y to block-diagonalize A as

$$Y^{-1}AY = \begin{bmatrix} A_0 & 0 & 0 \\ 0 & A_- & 0 \\ 0 & 0 & A_+ \end{bmatrix}, \quad (2.30)$$

where the real parts of the eigenvalues of A_0 , A_- , and A_+ are zero, negative, and positive, respectively. The spectral decomposition given in (2.30) is crucial to our numerical algorithm for solving feedback fluid queues. The advantage of this decomposition is that the cases of multiple eigenvalues and/or complex eigenvalues are treated naturally without complex arithmetic and moreover we stay away from the potentially ill-conditioned eigenvector problem. We now present a numerical example to demonstrate the algorithm to obtain the decomposition (2.30) using Matlab 7.0. For this purpose, we fix

$$A = \begin{bmatrix} 4 & 1 & 1/3 & -1 & 1/2 \\ -1 & -4 & 1/3 & -1 & 1/2 \\ -1 & 1 & -4/3 & -1 & 1/2 \\ -1 & 1 & 1/3 & 4 & 1/2 \\ -1 & 1 & 1/3 & -1 & -2 \end{bmatrix},$$

which corresponds to $A = A^{(2)} = Q^{(2)} (R^{(2)})^{-1}$ for the second regime of the five state fluid queue of Example 1. We first use the matlab function `schur.m` to obtain the Schur form of the matrix A by issuing the command

$$[Z_1, D_1] = \text{schur}(A)$$

which generates an orthogonal matrix Z_1 and a quasi upper-triangular matrix D_1 (simply upper-triangular for this example) such that $Z_1^T A Z_1 = D_1$. We obtain at this step

$$D_1 = \begin{bmatrix} 5.0000 & -0.0000 & -0.0000 & 0.0000 & -0.0000 \\ 0 & 1.9206 & 0.0000 & 3.5515 & -1.1180 \\ 0 & 0 & -4.1758 & -1.5042 & 0.4735 \\ 0 & 0 & 0 & -0.0000 & -0.6542 \\ 0 & 0 & 0 & 0 & -2.0782 \end{bmatrix}.$$

We then use the matlab function `ordschur.m` to obtain the ordered Schur form of the matrix A by issuing the command

$$[Z, D] = \text{ordschur}(Z_1, D_1, [1 \ 1 \ 2 \ 3 \ 2])$$

to place the eigenvalue at the origin (4th diagonal entry of D_1) to the upper-left corner and then the ones with negative real parts (3rd and 5th entries) and finally those with positive real parts (1st and 2nd). We obtain in this second step an orthogonal matrix Z such that

$$Z^T A Z = D = \begin{bmatrix} 0 & -0.7155 & -0.3130 & -0.0000 & -3.3137 \\ 0 & -4.1758 & -0.0511 & -0.0000 & 1.9897 \\ 0 & 0 & -2.0782 & 0.0000 & 1.1073 \\ 0 & 0 & 0 & 5.0000 & -0.0000 \\ 0 & 0 & 0 & 0 & 1.9206 \end{bmatrix}.$$

For zeroing the off-diagonal entries of the first row of D , we solve the Sylvester matrix equation $D(1, 1)X_1 - X_1 D(2 : 5, 2 : 5) = D(1, 2 : 5)$ or equivalently we issue the Matlab command

$$X_1 = -D(1,2:5)/D(2:5,2:5)$$

since $D(1,1) = 0$. We also zero the block off-diagonal entries in the 2nd and 3rd rows by solving the Sylvester matrix equation $D(2:3,2:3)X_2 - X_2D(4:5,4:5) = D(2:3,4:5)$ via the Matlab function `lyap.m` by issuing the command

$$X_2 = \text{lyap}(-D(2:3,2:3), D(4:5,4:5), D(2:3,4:5))$$

We finally set

$$Y = Z \begin{bmatrix} I & -X_1 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & -X_2 \\ 0 & 0 & I \end{bmatrix}$$

which satisfies the form of (2.30) with

$$Y^{-1}AY = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -4.1758 & -0.0511 & 0 & 0 \\ 0 & 0 & -2.0782 & 0 & 0 \\ 0 & 0 & 0 & 5.0000 & -0.0000 \\ 0 & 0 & 0 & 0 & 1.9206 \end{bmatrix}.$$

2.5 Solving FMFFQs

The spectral approach has a number of drawbacks:

- i) The approach fails when the eigenvalues are not simple,
- ii) Finding the eigenvectors of a general non-symmetric matrix is known to be ill-conditioned especially for eigenvalues close to each other,
- iii) The linear equation to solve the unknowns of the spectral expansion in (2.24) is known to be ill-conditioned; see the numerical experimentation of Akar and Sohraby [3].

Based on these observations, we use in this chapter the so-called ordered Schur form as opposed to the Jordan form, the latter required for finding all eigenvectors.

Throughout this section, we will focus on a particular regime k . In Theorem 1, we have shown that the probability density function satisfies for regime k the following differential equation:

$$\frac{d}{dy} f^{(k)}(y) R^{(k)} = f^{(k)}(y) Q^{(k)}, \quad T^{(k-1)} < y < T^{(k)}. \quad (2.31)$$

Assume that $|S_0^{(k)}| = 0$. Then under the assumption $\mu^{(k)} \neq 0$, we have one eigenvalue at the origin, $|S_+^{(k)}| - 1$ eigenvalues with negative real parts and $|S_-^{(k)}|$ eigenvalues with positive real parts in case $\mu^{(k)} < 0$. When $\mu^{(k)} > 0$, we again have one eigenvalue at the origin but $|S_+^{(k)}|$ eigenvalues with negative real parts and $|S_-^{(k)}| - 1$ eigenvalues with positive real parts. From continuity arguments, when $\mu^{(k)} = 0$, we have two eigenvalues at the origin and $|S_+^{(k)}| - 1$ eigenvalues with negative real parts and $|S_-^{(k)}| - 1$ eigenvalues with positive real parts. These arguments also hold for the case $|S_0^{(k)}| \neq 0$ but in terms of the number of generalized eigenvalues since $R^{(k)}$ in (2.31) is not invertible; see Akar and Sohraby [3] for a description of generalized eigenvalues and eigenvectors. As stated before, we assume in this paper that $\mu^{(k)} \neq 0$ to avoid double eigenvalues at the origin. Let us also assume without loss of generality that the states are ordered in such a way in regime k that states with zero drifts are numbered after the ones with nonzero drifts. If not, one can employ a similarity transformation to do so. Under this assumption, (2.7) can then be rewritten as:

$$\frac{df_n^{(k)}(y)}{dy} R_n^{(k)} = f_n^{(k)}(y) \underbrace{(Q_{nn}^{(k)} - Q_{nz}^{(k)} (Q_{zz}^{(k)})^{-1} Q_{zn}^{(k)})}_{Q_n^{(k)}}, \quad (2.32)$$

$$f_z^{(k)}(y) = -f_n^{(k)}(y) Q_{nz}^{(k)} (Q_{zz}^{(k)})^{-1}, \quad (2.33)$$

where we have

$$f^{(k)}(y) = \begin{bmatrix} f_n^{(k)}(y) & f_z^{(k)}(y) \end{bmatrix}, \quad (2.34)$$

and

$$R^{(k)} = \begin{bmatrix} R_n^{(k)} & 0 \\ 0 & 0 \end{bmatrix}, \quad Q^{(k)} = \begin{bmatrix} Q_{nn}^{(k)} & Q_{nz}^{(k)} \\ Q_{zn}^{(k)} & Q_{zz}^{(k)} \end{bmatrix}.$$

Here $f_n^{(k)}$ and $f_z^{(k)}$ refer to the steady state joint probability density vectors corresponding to $n^{(k)} = |S_+^{(k)}| + |S_-^{(k)}|$ states with nonzero drifts and $z^{(k)} = |S_0^{(k)}|$ states with zero drifts, respectively. The differential equation (2.32) can then be rewritten as

$$\frac{d}{dy} f_n^{(k)}(y) = f_n^{(k)}(y) Q_n^{(k)} (R_n^{(k)})^{-1} = f_n^{(k)}(y) A^{(k)}.$$

Using the procedure described in Section 2.4, one can find a matrix $Y^{(k)}$ such that

$$(Y^{(k)})^{-1} A^{(k)} Y^{(k)} = \begin{bmatrix} 0 & & \\ & A_-^{(k)} & \\ & & A_+^{(k)} \end{bmatrix}, \quad (2.35)$$

where the real parts of the eigenvalues of $A_-^{(k)}$ and $A_+^{(k)}$ are negative and positive, respectively. This decomposition leads us to

$$\frac{d}{dy} f_n^{(k)}(y) Y^{(k)} = f_n^{(k)}(y) Y^{(k)} (Y^{(k)})^{-1} A^{(k)} Y^{(k)}. \quad (2.36)$$

We then define

$$z^{(k)}(y) = f_n^{(k)}(y) Y^{(k)},$$

where

$$z^{(k)}(y) = \begin{bmatrix} z_0^{(k)}(y) & z_-^{(k)}(y) & z_+^{(k)}(y) \end{bmatrix}$$

with appropriate sizes for $z_0^{(k)}(y)$, $z_-^{(k)}(y)$ and $z_+^{(k)}(y)$. We then obtain the following solutions

$$z_0^{(k)}(y) = a_0^{(k)}, \quad (2.37)$$

$$z_-^{(k)}(y) = z_-^{(k)}(T^{(k-1)}) \exp\{A_-^{(k)}(y - T^{(k-1)})\}, \quad (2.38)$$

$$z_+^{(k)}(y) = z_+^{(k)}(T^{(k)}) \exp\{-A_+^{(k)}(T^{(k)} - y)\}, \quad (2.39)$$

where $a_0^{(k)}$ is a constant. A suitable decomposition of matrix $(Y^{(k)})^{-1}$ is

$$(Y^{(k)})^{-1} = \begin{bmatrix} \tilde{Y}_0^{(k)} \\ \tilde{Y}_-^{(k)} \\ \tilde{Y}_+^{(k)} \end{bmatrix},$$

and the relation $f_n^{(k)}(y) = z^{(k)}(y)(Y^{(k)})^{-1}$ yields

$$\begin{aligned} f_n^{(k)}(y) &= a_0^{(k)} \tilde{Y}_0^{(k)} \\ &+ z_-^{(k)}(T^{(k-1)}) \exp\{A_-^{(k)}(y - T^{(k-1)})\} \tilde{Y}_-^{(k)} \\ &+ z_+^{(k)}(T^{(k)}) \exp\{-A_+^{(k)}(T^{(k)} - y)\} \tilde{Y}_+^{(k)}. \end{aligned}$$

Using the identities (2.33) and (2.34), we finally write

$$\begin{aligned} f^{(k)}(y) &= a_0^{(k)} L_0^{(k)} \\ &+ a_-^{(k)} \exp\{A_-^{(k)}(y - T^{(k-1)})\} L_-^{(k)} \\ &+ a_+^{(k)} \exp\{-A_+^{(k)}(T^{(k)} - y)\} L_+^{(k)}, \end{aligned} \tag{2.40}$$

where

$$\begin{aligned} a_-^{(k)} &= z_-^{(k)}(T^{(k-1)}), \\ a_+^{(k)} &= z_+^{(k)}(T^{(k)}), \\ L_i^{(k)} &= \left[\tilde{Y}_i^{(k)}, -\tilde{Y}_i^{(k)} Q_{nz}^{(k)} (Q_{zz}^{(k)})^{-1} \right], i = 0, -, +. \end{aligned}$$

The unknown coefficients of the expression (2.40) are $a_0^{(k)}$, $a_-^{(k)}$, and $a_+^{(k)}$. In order to calculate these coefficients, we suggest to feed the expression for $f^{(k)}(y)$ in (2.40) into the boundary conditions of Theorem 1 as opposed to the spectral expansion given in (2.24). With this approach, we not only eliminate the numerical stability problems arising in the process of finding eigenvectors but also the linear equation to solve the coefficients in the expression (2.40) becomes stable since all the involved matrices, e.g., $A_-^{(k)}$ and $-A_+^{(k)}$, are stability matrices, i.e., their eigenvalues have negative real parts.

2.6 Numerical Examples

2.6.1 Example 1.

We consider the following five state example with a finite buffer of size two. The buffer consists of two regimes 1 and 2, corresponding to the intervals $(0, 1)$ and $(1, 2)$, respectively. We let

$$R^{(1)} = \text{diag}(1, -2, 2, -4, 0), \quad R^{(2)} = \text{diag}(-1, 1, 3, -1, 2),$$

and

$$\tilde{Q}^{(0)} = Q^{(1)} = \tilde{Q}^{(1)} = Q^{(2)} = \tilde{Q}^{(2)} = \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix}.$$

The drifts at the boundary point 1 in the first and the fifth states are assumed to be zero. The drifts at the same boundary point in the remaining states can be either zero or left or right continuous. We let the drift to be equal to zero for the intermediate boundary point in the second state. Finally, the drift at the same point in the third and the fourth states are selected as left continuous. The functions $F_m(y)$ are plotted for $m = 1, \dots, 5$ with respect to the buffer occupancy y in Fig. 2.2. We observe that the analytical results agree with the simulation results confirming the boundary conditions given in Theorem 1 for various types of states as stated in Theorem 1. Note that the derivative of $F_2(y)$ is zero on both sides of the boundary point 1. Furthermore, the derivative of $F_5(y)$ on the right side of boundary point 1 approaches to zero.

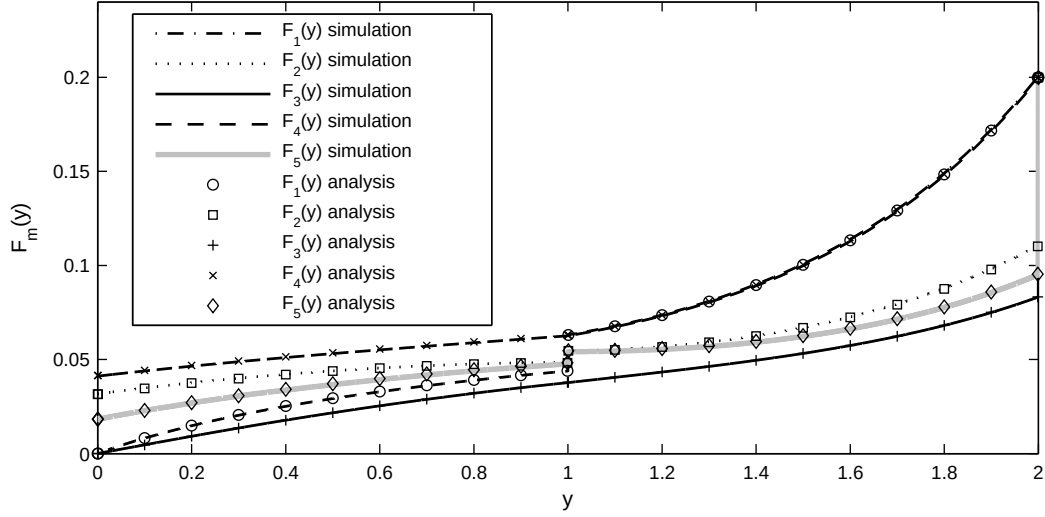


Figure 2.2: The quantity $F_m(y)$ plotted for $m = 1, \dots, 5$ as a function of the buffer occupancy y in Example 1.

2.6.2 Example 2

In this example, we consider a telecommunications link applying an adaptive admission control policy to the link buffer with capacity c and buffer size B . We assume there are N registered users. A user is modeled as an ON-OFF source which generates a workload at rate r when the source is ON. When the source becomes ON, the link buffer is to make a decision on whether to accept this workload as a whole to the link buffer. In the OFF state, the source does not generate any workload and no decisions are made. The alternating ON times and the OFF times are exponentially distributed with means $1/\mu$ and $1/\lambda$, respectively, for each source. The workload admission policy uses the buffer occupancy as the admission criterion; all new workloads are accepted when the buffer occupancy is below a predetermined threshold T and rejected otherwise. In this way, the adaptive system prevents random losses due to buffer overflow. It is clear that this problem fits into a two-regime FMFFQ. The modulating process is governed by the number of admitted users. The state transition diagrams of this Markov chain are illustrated in Fig. 2.3 depending on whether the buffer occupancy is below or above T . An aggressive admission policy, i.e., $T \rightarrow B$ reduces

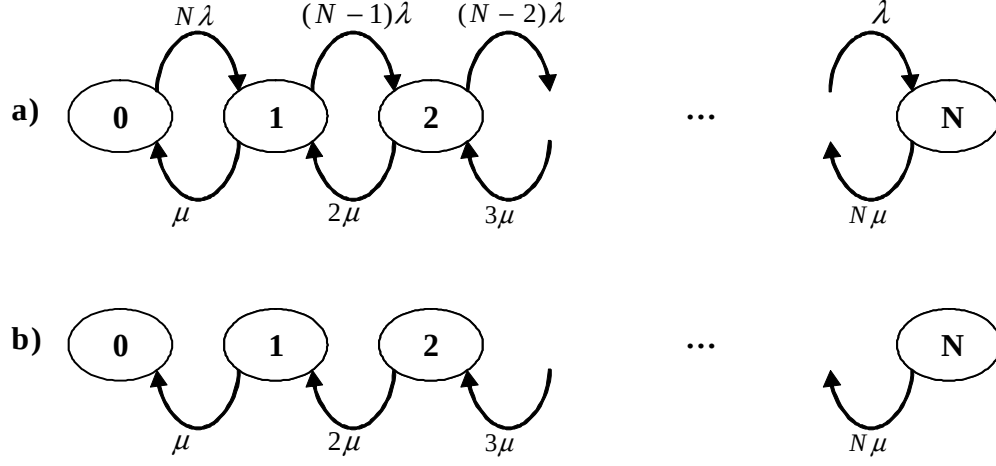


Figure 2.3: State transition diagram for Example 2 when the buffer occupancy is a) below T b) above T .

workload blocking probability but introduces buffer overflow which leads to random losses among all admitted workloads. On the other hand, for a conservative policy, i.e., $T \rightarrow 0$, buffer overflow probabilities will be reduced at the expense of increased workload blocking probabilities. In order to quantify this tradeoff, we define T_{max} to be the maximum value of the threshold parameter T at which the buffer overflow probability is kept below an acceptable value say 10^{-3} . As an example, we fix $r = 32$ Kb/sec, $c = 500$ Kb/sec, and $B = 50$ ms. and we plot T_{max} (in ms.) as a function of the number of registered users in Fig. 2.4 for two scenarios i) $\lambda = 20, \mu = 10$ ii) $\lambda = 10, \mu = 5$, the first scenario being for a more dynamic scenario where the transition rates are relatively higher. It is clear that one can employ a more aggressive admission policy with fewer registered users and more dynamic environment.

2.6.3 Example 3

In this example, we compare the stability of the conventional spectral expansion approach and the ordered Schur factorization approach while the drift in a certain state for a given regime approaches to zero. For this purpose, we construct a

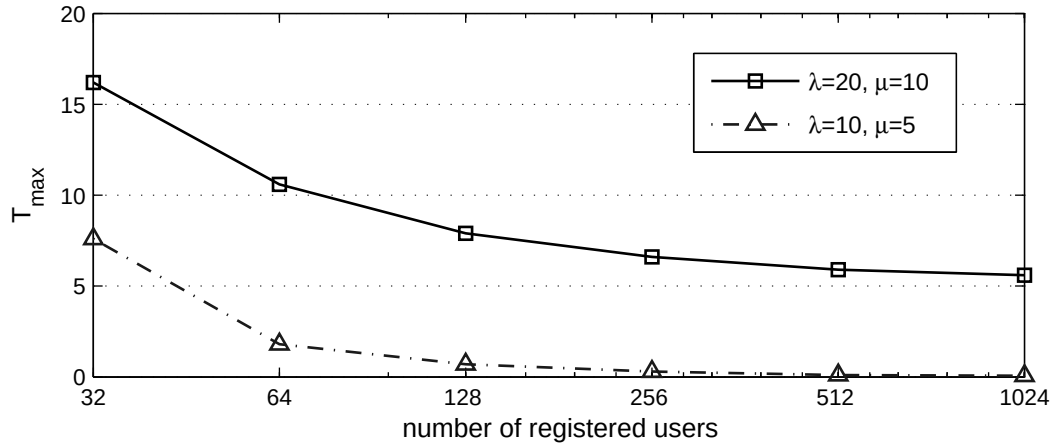


Figure 2.4: The maximum values for the parameter T for Example 2 at which the loss probability is below 10^{-3} .

simple three-state two-regime example for a buffer of size three consisting of two regimes 1 and 2, corresponding to the intervals $(0, 1)$ and $(1, 3)$, respectively. We let

$$\tilde{Q}^{(0)} = Q^{(1)} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}, \tilde{Q}^{(1)} = Q^{(2)} = \tilde{Q}^{(2)} = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix},$$

$$R^{(1)} = \text{diag}(r, -1, 2), R^{(2)} = \text{diag}(-0.5, -1, 1),$$

and

$$\tilde{R}^{(0)} = \text{diag}(0, 0, 2), \tilde{R}^{(1)} = \text{diag}(r, -1, 2), \tilde{R}^{(2)} = \text{diag}(-0.5, -1, 0),$$

where the variable $r < 0$.

In Fig. 2.5, the CDF of the buffer occupancy $F(y)$ is obtained using the ordered Schur factorization and is compared against simulations for three different values of $r = -0.05, -0.005, 0$. We show in this figure that we get accurate results even for very small values of the drift parameter r . Furthermore, Fig. 2.5 demonstrates that the solution of the fluid queue with $r < 0$ as $r \rightarrow 0$ converges to the solution of the system for $r = 0$ that is solved by using the boundary conditions given in Theorem 1.

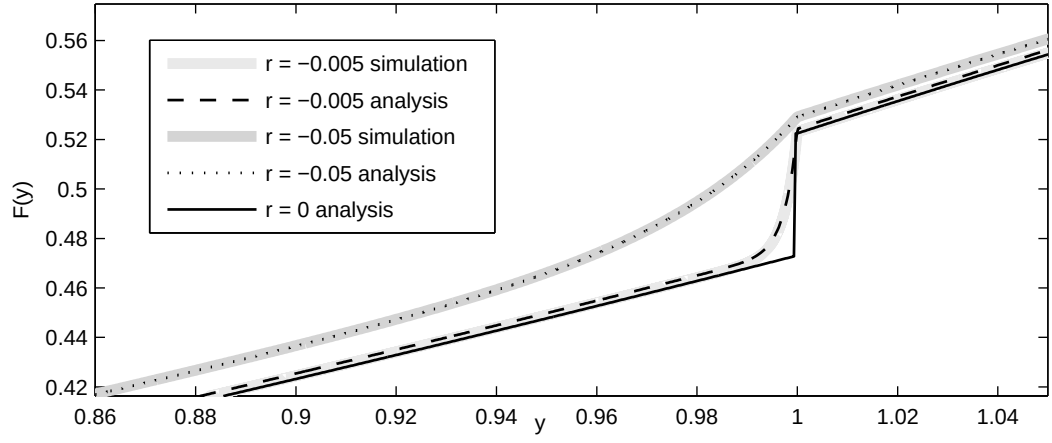


Figure 2.5: The quantity $F(y)$ plotted as a function of the buffer occupancy y in the vicinity of the boundary point $y = 1$ in Example 1.

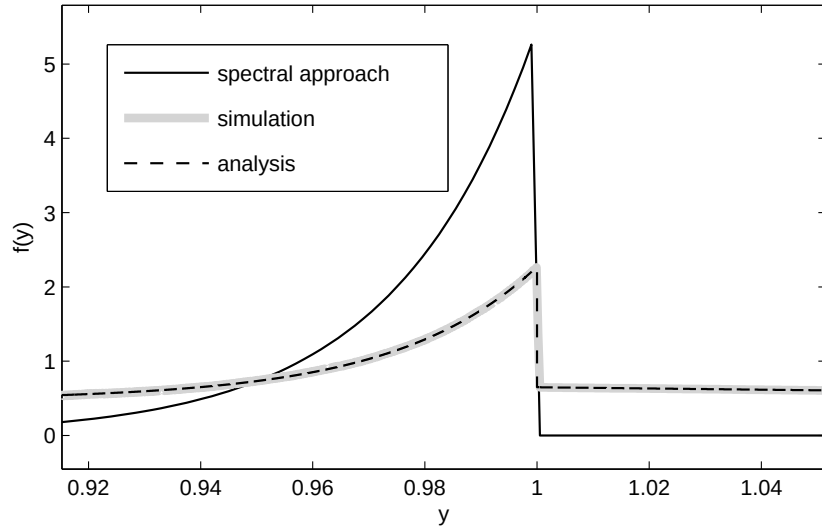


Figure 2.6: Comparison of the spectral approach and the ordered Schur factorization methods for $r = -0.05$ in Example 1.

In Fig. 2.6, the buffer occupancy PDF is obtained using the ordered Schur factorization, together with the spectral approach and simulations for the case $r = -0.05$. It is clear from Fig. 2.6 that the spectral approach does not provide accurate results in this case whereas our proposed method continues to give correct results even for very small absolute values of r .

2.6.4 Example 4

In this example, we vary the r parameter of the previous example in the neighborhood of $r = -1$ at which the mean drift in the first regime is exactly zero. In Fig. 2.7, we plot $F(y)$ for three values of r , namely for $r = -0.8$, $r = -0.9$ and $r = -0.99$, so as to compare with the simulation result for $r = -1$. Fig. 2.7 clearly demonstrates that the analysis results converge to the simulation result as $r \rightarrow -1$ without having any computational difficulty in the vicinity of a zero mean drift regime.

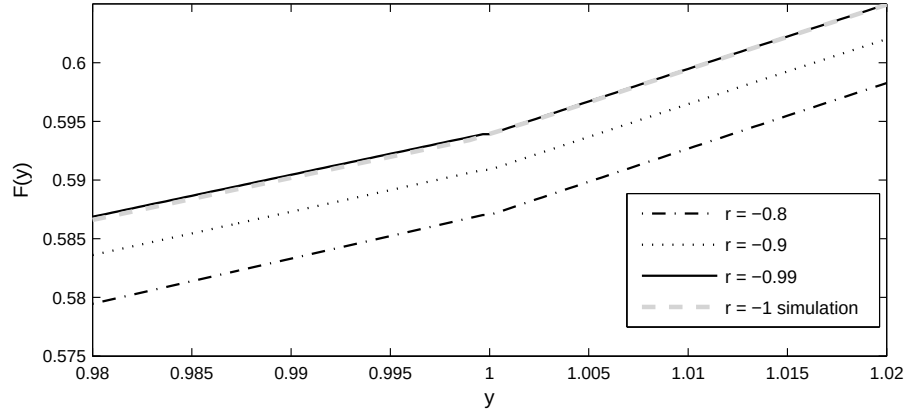


Figure 2.7: The quantity $F(y)$ plotted as a function of the buffer occupancy y in the vicinity of the boundary point $y = 1$ in Example 3.

Chapter 3

FLUID QUEUE-BASED ANALYSIS: MOTIVATION AND EXAMPLES

3.1 Introduction

This chapter introduces a collection of transformation methods which can be used to convert a wide range of classical queueing analysis problems to those that can be analysed using the framework of Markov fluid queues. Later in this thesis, we will present three particular case studies in Chapters 4, 5, and 6 that study three practical and (to the best of our knowledge) unsolved queueing analysis problems. In this chapter, we present the general methodology underlying the transformations we propose and provide a number of examples to validate our approach. In these examples, we provide FMFFQ-based solutions of well-known queueing analysis problems, such as the waiting time distribution in an M/M/1 queue as well as problems with relatively more complex structures such as the unfinished work distribution for an M/M/1 queue with a dynamic admission

policy. The methods we use in our solutions are similar to each other but each problem appears to have its own transformation of some sort so that the problem at hand can be converted to a FMFFQ problem. Unfortunately, this observation prevents us from writing a unified algorithmic procedure for general queueing analysis problems. We believe that the current chapter will help the reader understand the general transformation ideas and make use of these ideas in solving her own queueing analysis problem.

In Chapter 2, we presented the differential equations satisfied by an FMFFQ and the complete list of boundary conditions. Rather than the differential equations, boundary conditions and their solution techniques, the methods described in this chapter study FMFFQs from a different perspective. Here, we deal with the sample path structure of an FMFFQ and we study the possibility of finding exact solutions for probability measures in other queueing systems (not necessarily of FMFFQ-type) by suitably transforming them (using sample path arguments) to FMFFQs. The basic variant of such methods have been proposed previously in [6], [7], [50], [65] and [16]. In these studies, transformations are used to convert problems to only those that are described by first-order single-regime fluid queues (FSFQ). Moreover, the considered problems are classical problems of queueing theory, such as M/M/1 waiting time in [6], or M/PH/1 waiting time in [16]. To the best of our knowledge, we believe that the transformation idea is used the first time for FMFFQs in this thesis and in this way we have been able to solve some unsolved queueing analysis problems.

The rest of this chapter is organized as follows. Section 3.2 presents the basic properties of the sample path of Markovian fluid queues. In Section 3.3 we give the basic ideas used in the solutions with the use of a collection of examples. In Section 3.4, we generalize the solution for systems possessing more than one regime. Section 3.5 briefly describes further available generalizations in the

methodology. Section 5 provides an algorithmic guideline for the transformation of simple processes. Section 6 discusses the limitations of the transformation.

3.2 Sample path structure of the Markovian Fluid Queues

Many continuous-time continuous-space processes can be defined by their sample path structures rather than using complex mathematical formulations. For example the unfinished work in an M/M/1 queue can be defined by exponentially distributed jumps with mean $1/\mu$ and linearly decreasing parts with unit drift for an exponentially distributed amount of time with mean $1/\lambda$ in between successive jumps. This type of description may be mathematically informal but it facilitates presentation and visualization. This subsection presents the generic structure of the sample paths of the content process in Markovian fluid queues.

In first-order single regime feedback fluid queues (FSFQs), the behavior of the content process is dependent on the state of the background process. In FMFFQs, the content itself is a factor on this behavior, as well. During the time in which the the background process is visiting a certain state, the content moves linearly up or down (depending on the sign of the drift). The amount of the movement in the content is obviously a random variable. In this subsection, we explore the distributions of these random variables in various cases. This exploration is important for our solution techniques because the processes that we study in this thesis are defined by linear increases, decreases, and jumps whose amounts can be captured by some combinations of the random variables which can be observed in FMFFQs. We discuss the availability of a solution using the methods in conclusion of this chapter, however one should conclude from the previous statement that the applicability of the methods strongly depends

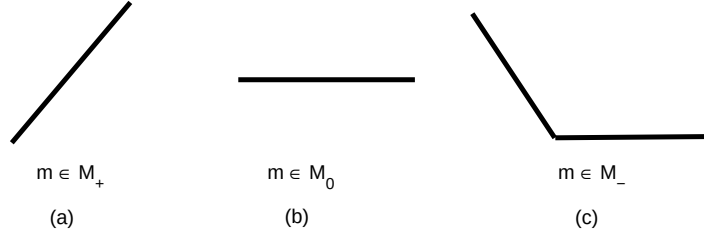


Figure 3.1: Evolution of the content process of an FSFQ with infinite queue size when the background process is at a constant state.

on the similarity (or sometimes, more strongly, equality) of the distributions of increases, decreases and also jumps in the process under consideration.

FSFQs are special cases of FMFFQs and due to their relatively simpler construction the FSFQs are easier to analyze and visualize. So, we commence with examining the sample path of an FSFQ with infinite queue size. The background process of an FSFQ is a continuous-time Markov chain (CTMC). The content process moves in a continuous domain $[0, \infty)$. Assume that S_+ , S_- and S_0 are the subsets of the state space of the background process with positive, negative and zero drifts, respectively. If we investigate the sample path of an infinite size FSFQ, we observe three types of behaviors as depicted in Fig. 3.1. The background process stays in any state $m \in M$ for an exponentially distributed amount of time with mean $-1/q_{mm}$. During this time, if the process is at a state $m_1 \in S_+$ then the content process moves upward linearly with drift $r_{m_1} > 0$, see Fig. 3.1 (a). The amount of increase in the content process is also an exponentially distributed random variable. At any state $m_2 \in S_0$, the content process stays constant, see Fig. 3.1 (b). As shown in Fig. 3.1 (c), while the background process is at $m_3 \in S_-$, the amount of decrease in the content process is a random variable with truncated exponential distribution in which the truncation level depends on the starting level of the content process and the drift at that state. It is important to note that when the starting level of the content is equal to zero, regardless of the drift parameter, the truncated exponential distribution

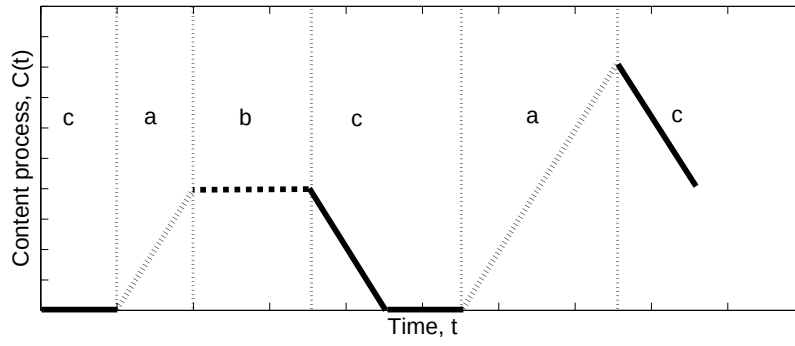


Figure 3.2: Evolution of the content process of an FSFQ with infinite queue size.

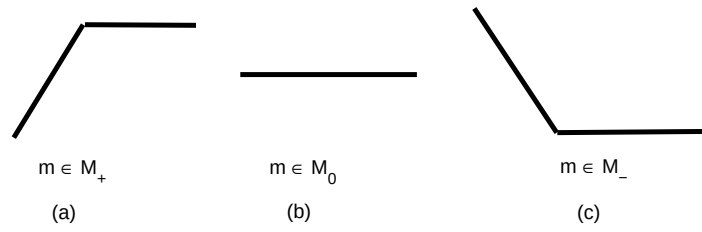


Figure 3.3: Evolution of the content process of an FSFQ with finite queue size when the background process is at a constant state.

becomes the degenerate distribution at zero. Any sample path of an FSFQ is composed of appropriate concatenations of these three basic movement types.

In Fig. 3.2 we show a sample path of an FSFQ with infinite queue size. In this figure all three basic movement types can be seen.

In Fig. 3.3, we illustrate possible types of movements of the content process in an FSFQ with finite queue size. In this model, the existence of the upper limit truncates the upward linear increase. Therefore, at states with positive drifts, the amount of increase is a random variable with truncated exponential distribution, see Fig. 3.3 (a). The last two movement types of the previous model are valid here, as well (Compare Fig. 3.1 (b) and (c) with Fig. 3.3. (b) and (c)). Assuming that T is the queue size, all possible sorts of changes can be observed in the sample path illustrated in Fig. 3.4.

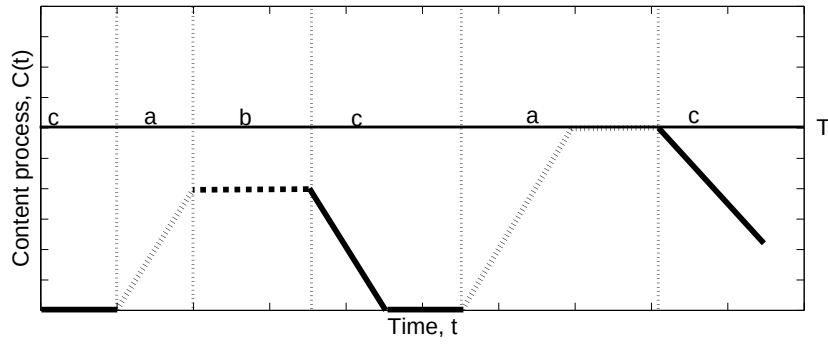


Figure 3.4: Evolution of the content process of an FSFQ with finite queue size.

Due to its relatively more complex structure, the number of possible movement types in FMFFQs is much more than the ones in FSFQs. Contrary to FSFQs, we can assign distinct drift and infinitesimal generator parameters to each regime and boundary point. This freedom allows us to capture some additional behaviors in addition to the ones described in Figs. 3.1 and 3.2. For example, if we assign zeros to one of the rows of the infinitesimal generator in a regime then once the background process enters the corresponding state in that regime, it can not make any transition until the content process leaves that regime. Therefore the distribution of the amount of change at such a state is a function of the steady-state joint distribution of the FMFFQ. This property is one of building blocks of the solutions in the successive chapters. In addition to that, we can also stop the content at a boundary point for an exponentially distributed amount of time once the content reaches there, by assigning zero drift at that boundary point. Moreover, by applying different movement types in different regimes we can obtain a large collection of possibilities. For example, let the FMFFQ make a transition to a state $m \in S_+^{(K-1)}$ at time t_0 when the content is x , and the corresponding row in the infinitesimal generator is zero for this state in the current regime. However, assume that the same row is non-zero and the drift of that state is positive in the uppermost infinite size regime K . Then, the amount of continuous increase in this state starting from t_0 is the summation of two values: $T^{(K-1)} - x$ and an exponentially distributed random variable with

mean $1/q_{mm}^{(K)}$. Finally, it is important to note that the overall background process of FMFFQs doesn't have to be a CTMC, unlike FSFQs.

3.3 Basic Ideas

The basic solution idea used to solve probabilistic measures of numerous stochastic processes in the steady-state can be given as follows. If we can insert certain sample path portions to the sample path of a stochastic process, so that it becomes a sample path of a FMFFQ, then we can find the steady-state distribution of the original stochastic process just using normalization. The point here is to apply the right insertions at the right places.

The solutions in the subsequent examples are previously explored in [6] and [16]. Related to the theme of this section, we begin with the steady-state waiting time distribution in the simplest Markovian queueing system, namely the M/M/1 queue. First, we demonstrate the standard solution and then we solve the same problem using FSFQs.

3.3.1 Example 1: Waiting time distribution in an M/M/1 queue - Standard solution

Our first example demonstrates the basic idea for a simple M/M/1 queue. An M/M/1 system is a queueing system with exponentially distributed service times with mean $1/\mu$ and the arrival process is a Poisson stream with rate λ . Hence $\rho = \lambda/\mu$. In this example, we illustrate the standard steady-state solution for the density function of the waiting time in the queue, T_q . This solution can be found in any introductory queueing theory textbook. For example see [23] and

[33].

$$P\{T_q \leq x\} = P\{T_q = 0\} + \sum_{n=1}^{\infty} P\{n \text{ service completions in } (0, x] | \text{arrival finds } n \text{ customers in the system}\} \cdot p_n \quad (3.1)$$

$$= 1 - \rho + (1 - \rho) \sum_{n=1}^{\infty} \rho^n \int_0^x \frac{\mu(\mu y)^{n-1}}{(n-1)!} e^{-\mu y} dy \quad (3.2)$$

$$= 1 - \rho + (1 - \rho) \rho \int_0^x \sum_{n=0}^{\infty} \frac{\mu(\mu y \rho)^{n-1}}{(n-1)!} e^{-\mu y} dy \quad (3.3)$$

$$= 1 - \rho + (1 - \rho) \rho \int_0^x \mu e^{-\mu y(1-\rho)} dy \quad (3.4)$$

$$= 1 - \rho e^{-x(\mu-\lambda)} \quad (3.5)$$

In (3.1), p_n is the steady-state probability that there are n customers in the system.

3.3.2 Example 1 continued: Waiting time distribution in an M/M/1 queue - FSFQ-based solution

Next, we find the steady-state waiting time distribution of an arbitrary packet arrival to an M/M/1 queueing system by using the FSFQ-based solution method. Well-known Poisson Arrivals See Time Averages (PASTA) property tells us that if the arrival stream is Poisson (as in the M/M/1 case) then the time average statistics and the ensemble average statistics are identical. Using this property, we conclude that the workload distribution and the waiting time distribution in an M/M/1 queue are the same. In Fig. 3.5, we illustrate a sample path of the workload process. In this figure, the workload steadily decrease with a constant drift -1 between two successive arrivals and at the arrival epochs it makes an upward jump with an amount equal to the service time of the arrival.

In order to solve the steady-state distribution of the workload, we first modify the sample path in such a way that the modified sample path can be viewed as

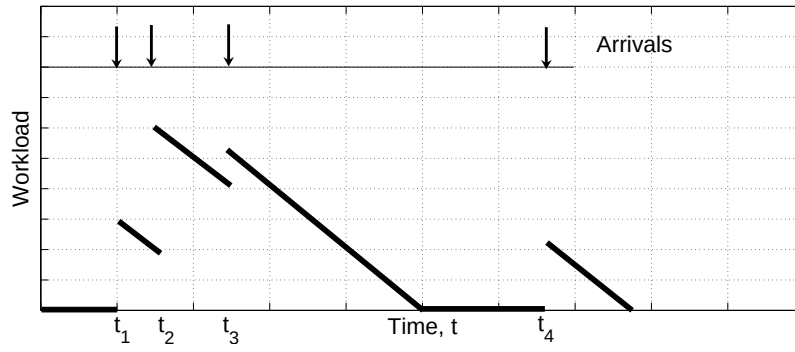


Figure 3.5: Evolution of the workload process in M/M/1 queueing system.

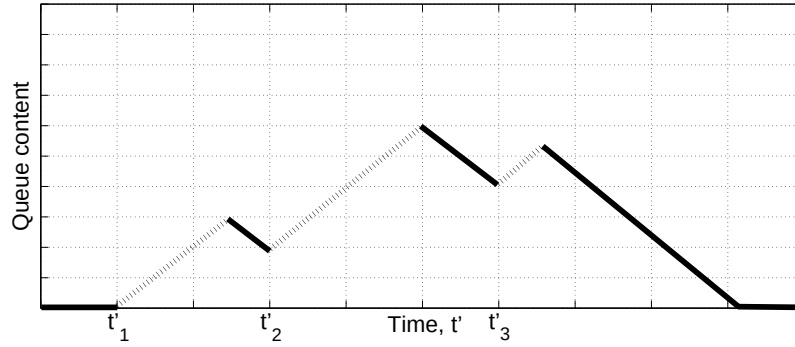


Figure 3.6: Modification of the sample path in Fig. 3.5.

a sample path of a FSFQ. It is easy to see that, the sample path of this process is composed of the parts illustrated in Fig. 3.1 (a) and exponentially distributed jumps in between. We left the decreasing parts as they are, because they already possess the sample path properties of an FSFQ. However, jumps aren't present in the sample path of an FSFQ. Hence we have to modify these parts. To achieve this aim, we insert increasing sample path parts with constant drifts to the jump epochs so as to obtain a continuous sample path. The slope of these parts do not play a role on the results and therefore we set the slopes to unity and obtain the plot in Fig. 3.6 which clearly gives the sample path of an FSFQ.

It is easy to see if we delete dashed lines in Fig. 3.6 and concatenate the bold lines we obtain the plot in Fig. 3.5. The amounts of times spent in increasing (dashed lines) and decreasing (bold lines) parts of Fig. 3.6 are exponentially distributed random variables with means $1/\mu$ and $1/\lambda$, respectively. These two

types of behaviors (increase and decreases) can be described by two background states of FSFQ. In the former one we assume that the queue content increases with drift 1 and in the latter it decreases with drift -1. Then the drift matrices are

$$\tilde{R}^{(0)} = \text{diag}(1, 0) \text{ and } R^{(1)} = \text{diag}(1, -1).$$

A transition in the background process from state 2 to 1 happens if a new arrival occurs. Moreover, a transition from state 1 to 2 occurs after an exponentially distributed amount of time with mean $1/\mu$. Therefore the infinitesimal generators of the background process are

$$\tilde{Q}^{(0)} = Q^{(1)} = \begin{bmatrix} -\mu & \mu \\ \lambda & -\lambda \end{bmatrix}.$$

Next, we are going to find the density function of the corresponding FSFQ whose parameters are given above. Using the theory constructed in the mathematical background part (see Chapter 2) the steady-state joint density functions of this FSFQ satisfies

$$\frac{df_1^{(1)}(x)}{dx} = -\mu f_1^{(1)}(x) + \lambda f_2^{(1)}(x) \text{ and} \quad (3.6)$$

$$-\frac{df_2^{(1)}(x)}{dx} = \mu f_1^{(1)}(x) - \lambda f_2^{(1)}(x) \quad (3.7)$$

along with the boundary conditions

$$f_1^{(1)}(0+) = -\mu c_1^{(0)} + \lambda c_2^{(0)}, \quad (3.8)$$

$$-f_2^{(1)}(0+) = \mu c_1^{(0)} - \lambda c_2^{(0)}, \quad (3.9)$$

$$c_1^{(0)} = 0 \text{ and} \quad (3.10)$$

$$\int_{0+}^{\infty} (f_1^{(1)}(x) + f_2^{(1)}(x))dx + c_1^{(0)} + c_2^{(0)} = 1. \quad (3.11)$$

The general solution of the set of differential equations in (3.6) and (3.7) is

$$\begin{bmatrix} f_1^{(1)}(x) & f_2^{(1)}(x) \end{bmatrix} = a_1^{(1)} e^{\lambda_1^{(1)} x} \phi_1^{(1)} + a_2^{(1)} e^{\lambda_2^{(1)} x} \phi_2^{(1)} \quad (3.12)$$

where $(\lambda_i^{(1)}, \phi_i^{(1)})$, $i = 1, 2$, are the left eigenvalue-eigenvector pairs of the matrix

$$A = Q^{(1)}(R^{(1)})^{-1} = \begin{bmatrix} -\mu & -\mu \\ \lambda & \lambda \end{bmatrix}. \quad (3.13)$$

This matrix has two eigenvalues $\lambda_1^{(1)} = \lambda - \mu$ and $\lambda_2^{(1)} = 0$. Since the size of the queue is infinite, the coefficient of the second term in (3.12) is zero. Normalized eigenvector corresponding to the first eigenvalue is $[1, 1]$. Therefore the solution is

$$\begin{bmatrix} f_1^{(1)}(x) & f_2^{(1)}(x) \end{bmatrix} = a_1^{(1)} e^{(\lambda-\mu)x} \begin{bmatrix} 1 & 1 \end{bmatrix}. \quad (3.14)$$

Now, substituting (3.14) into (3.8) and (3.9) we obtain

$$c_2^{(0)} = a_1^{(1)} / \lambda. \quad (3.15)$$

And finally if we substitute (3.10), (3.14) and (3.15) into (3.11) we conclude that

$$a_1^{(1)} = \lambda \frac{\mu - \lambda}{\mu + \lambda}. \quad (3.16)$$

Therefore,

$$f_1^{(1)}(x) = f_2^{(1)}(x) = \lambda \frac{\mu - \lambda}{\mu + \lambda} e^{(\lambda-\mu)x} \text{ and } c_2^{(0)} = \frac{\mu - \lambda}{\mu + \lambda}.$$

Next we normalize the density function over the second state of the background process:

$$G(x) = \frac{c_2^{(0)} + \int_{0+}^x f_2^{(1)}(x) dx}{c_2^{(0)} + \int_{0+}^{\infty} f_2^{(1)}(x) dx} = \frac{\frac{\mu-\lambda}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda} (1 - e^{(\lambda-\mu)x})}{\frac{\mu-\lambda}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda}} = \frac{\frac{\mu}{\mu+\lambda} - \frac{\lambda}{\mu+\lambda} e^{(\lambda-\mu)x}}{\frac{\mu}{\mu+\lambda}} = 1 - \rho e^{-x(\mu-\lambda)}.$$

Therefore, we obtain exactly the same result as in the standard analysis.

The last step, the so-called normalization step, amounts to the deletion of the dashed lines and concatenating the bold lines in Fig. 3.6. This relationship can be intuitively described by using the idea behind simulation of those two processes. Let us assume that we constructed a sample path of an FSFQ for a

long duration of time via simulations and our next step is to find the steady-state density functions of this process. In order to do that, we take equidistant samples from the constructed sample path and then plot the corresponding histogram. In other words, we first define the histogram intervals (e.g., $[0, \Delta x)$, $[\Delta x, 2\Delta x)$, $[2\Delta x, 3\Delta x)$, ...) and then take equidistant samples from the sample path (e.g., $f(\Delta t)$, $f(2\Delta t)$, $f(3\Delta t)$, ... for $\Delta t \ll \Delta x$). Next we count the number of samples in each histogram interval and then plot the histogram. If our aim is to find the simulation results for joint density functions, we have to take the state of the background process into account. For the two state example given in Fig. 3.6, instead of $f(\Delta t)$, $f(2\Delta t)$, ... we need to define $f_i(\Delta t)$, $f_i(2\Delta t)$, ... where the subscript $i = 1, 2$ is an indicator of the state of the background process. Then we count the number of samples for each state ($i = 1, 2$) and obtain two histograms for the joint probability density functions. Histogram for the first state includes only the samples when the background process is at that state. Therefore sampling of the process when the background state is 1 and the sampling of the concatenated process result in the same number of samples in each histogram interval. Hence we can find the steady-state probability density function of the concatenated process by just using the normalization.

Continuing to examine applications of this sample path method, we now turn to a bit more difficult example, steady-state waiting time distribution in an M/PH/1 queue that is addressed in [16], yet this time without even mentioning about the standard solution methodology.

3.3.3 Example 2: Waiting time distribution in M/PH/1 queue- FSFQ-based solution

Let the arrival process be Poisson with rate λ and the service times are distributed according to a PH-type distribution with p transient states and 1 absorbing state

[42]. Let the governing transition matrix be

$$Q_{PH} = \begin{bmatrix} S & S^0 \\ 0 & 0 \end{bmatrix}$$

where S is an $s \times s$ nonsingular matrix, S^0 is $s \times 1$, $Se + S^0 = 0$, and e is a column vector of ones of appropriate size. Also assume that the initial probability vector is given by

$$v = [v_1 \ v_2 \ \cdots \ v_p].$$

The distribution function of PH-distribution is written as $F_L(x) = 1 - ve^{Sx}e$, $x \geq 0$. It is well-known that the time required to reach the absorbing state given that we are initially in one of the p transient states is a PH-type random variable.

The sample path of the workload in this queueing system is pretty much similar to the sample path illustrated in Fig. 3.5. The only difference lies in the fact that the upward jumps are distributed according to a PH-type distribution for this example. The transformation we propose is similar to the previous example. Instead of PH-type distributed upward jumps, we substitute steady increases with a duration governed by the PH distribution. For this purpose, we define $p + 1$ states, the first p of which are defined for the increasing parts and the last state is defined for the decreasing parts of the sample path of the transformed process. Hence, the drift configuration can be set as follows:

$$R^{(1)} = \text{diag}(1, 1, \dots, 1, -1) \text{ and } \tilde{R}^{(0)} = \text{diag}(1, 1, \dots, 1, 0).$$

An arrival occurs whenever the background process makes a transition from $p + 1$ to any of the first p states. Provided that a transition occurs, the next state is determined by the initial probability vector. The background process escapes from the first p states when an absorption occurs in the underlying PH-type distribution. Therefore the infinitesimal generator of the FSFQ should be

$$\tilde{Q}^{(0)} = Q^{(1)} = \begin{bmatrix} S & S^0 \\ \lambda v & -\lambda \end{bmatrix}.$$

One can easily find the probability density functions and the probability mass accumulations of this FSFQ utilizing the solution given in Chapter 2. Once we find these probability measures, the following normalization should be applied in order to find the steady-state waiting time density function, denoted by $g(x)$, in the associated M/PH/1 queue.

$$g(x) = \frac{f_{p+1}^{(1)}(x)}{c_{p+1}^{(0)} + \int_0^{\infty} f_{p+1}^{(1)}(x)dx}.$$

Next, we extend the arrival process to a Markovian arrival process (abbreviated as MAP).

3.3.4 Example 3: Waiting time distribution in MAP/M/1 queue - FSFQ-based solution

The MAP is described by two processes; the count process $N(t)$ that counts the number of arrivals up to time t and the phase process $J(t)$, assuming values in $\{0, 1, \dots\}$ and $\{1, 2, \dots, d\}$, respectively. The two-dimensional Markov process $\{N(t), J(t)\}$ is then modelled as a Markov process on the state-space $\{(i, j) : i \in \mathcal{Z}, 1 \leq j \leq d\}$ whose infinitesimal generator matrix Q can be represented in block form as

$$Q_{MAP} = \begin{bmatrix} D_0 & D_1 & 0 & 0 & 0 & \cdots \\ 0 & D_0 & D_1 & 0 & 0 & \cdots \\ 0 & 0 & D_0 & D_1 & 0 & \cdots \\ 0 & 0 & 0 & D_0 & D_1 & \cdots \\ 0 & 0 & 0 & 0 & D_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (3.17)$$

where D_0 and D_1 are square matrices of the same size satisfying the relation $(D_0 + D_1)e = 0$ where e is a column vector of ones with appropriate size.

At every transition epoch from D_0 to D_1 an arrival occurs. In this system, it is easy to see that each interarrival time depends on the first state at the beginning of the interarrival. Therefore an arrival stream constructed by MAP is correlated. Some other well-known arrival processes such as the Poisson process and the Markov modulated Poisson process are special cases of MAP. Our next example presents a FSFQ-based solution for the waiting time distribution in an MAP/M/1 queue.

We assume that service times are exponentially distributed with mean $1/\mu$ and the system possesses a single server. The idea of the extension to PH-distributed service times in Example 2 can be applied to reach the results of MAP/PH/1 queue, however for the sake of brevity here we only consider exponentially distributed service times. Unlike Example 1, here we set MAP as the arrival process. Making use of Kendall's notation, this system is expressed as MAP/M/1, in short. Let the infinitesimal generator of the arrival stream is equal to the one represented in (3.17). In addition to that let D_0 and D_1 are $d \times d$ square matrices. In this example, our aim is to find the steady-state waiting time distribution of an arrival to the queue. The general methodology is the similar to the examples above. We insert continuous increases at arrival epochs and then obtain a sample path of an FSFQ. Then we solve the steady-state distribution of this FSFQ using the methodology described in Chapter 2. By using normalization it is easy to obtain the steady-state distribution of workload for a MAP/M/1 queue. However, contrary to the previous examples here we need one more step. In the examples above, the arrival process were Poisson; hence the time averages and the ensemble averages are the same. ASTA (Arrival See Poisson Averages) property is not valid for MAP. But, luckily for MAP streams the ensemble averages are functions of the time averages and they can be found easily once the time averages are obtained. One more difference between using MAP and Poisson streams lies in the fact that: so as to conserve the correlation between successive arrivals we have to define d auxiliary states in

the background process of the FSFQ for MAP/M/1 queue whereas the required number of auxiliary states was 1 in the M/M/1 queue. The last difference is based on the conservation of possibly non-zero correlation between successive interarrival times. Once an arrival occurs, the FSFQ start to evolve on a dashed line of Fig. 3.6. Over that dashed line, we stop the arrival process until the content process starts to decrease again and then we continue to the arrival process MAP. But, over those time intervals to preserve the information about the last state of MAP we define d auxiliary states. Each of these states corresponds to a state of MAP. So, totally there are $2d$ states in the background process of FSFQ with the drift and variance parameters:

$$R^{(1)} = \text{diag}(I_d, -I_d), \tilde{R}^{(0)} = \text{diag}(I_d, 0_d) \text{ and}$$

$$Q^{(1)} = \tilde{Q}^{(0)} = \begin{bmatrix} -\mu I & \mu I \\ D_1 & D_0 \end{bmatrix}$$

where I_d and 0_d are identity and zero matrices of size $d \times d$, respectively. Let $W(t)$ and $M_{MAP}(t)$ denote the workload process and the current state of the MAP, respectively, in MAP/M/1. After we calculate the steady-state joint probability density functions of that FSFQ, $f_i^{(1)}$, $i = 1, \dots, 2d$, we make normalization to find the joint probability density functions of the workload process in MAP/M/1, $g_i(x) = \frac{d}{dx} \lim_{t \rightarrow \infty} P\{W(t) \leq x, M_{MAP}(t) = i\}$, $i = d+1, \dots, 2d$ as

$$g_i(x) = \frac{f_i^{(1)}(x)}{\sum_{i=d+1}^{2d} \left(c_i^{(0)} + \int_0^\infty f_i^{(1)}(x) dx \right)}.$$

Let $g(x)$ be the steady-state probability density function of the workload seen by the arrivals. This function can be found as

$$g(x) = \frac{\begin{bmatrix} g_{d+1}(x) & g_{d+2}(x) & \cdots & g_{2d}(x) \end{bmatrix} D_1 e}{\pi D_1 e}$$

where e is the column vector of ones with length d and π is the steady-state vector of $D_0 + D_1$.

3.4 Generalization of the idea to first-order multi-regime feedback fluid queues

Examples and descriptions until this subsection solely require the solution of the FSFQs. However, the method itself is applicable to more general systems which can require the solution of multi-regime feedback fluid queues. Actually the solutions in Chapters 4, 5 and 6 fall into this category. The basic ideas mentioned in Section 3.3 are valid for FMFFQs as well. However having multiple regimes and the freedom in the selection of parameters lead us to a capability set by which we can attack a larger set of problems. These problems include the workload distribution in Markovian queues applying admission policies depending on the unfinished work; mass functions in discrete-valued Markovian processes with limited queue sizes, some advance reservation systems etc. As our next example, we present an FMFFQ-based solution to the steady-state unfinished work and the blocking probability in an M/M/1 queue with workload-dependent admission policy.

3.4.1 Example 4: M/M/1 queueing system with workload dependent admission policy

In this example, the queue rejects any arrival if the current workload is greater than a predetermined value, otherwise the arrival is accepted. Let the arrival process be Poisson with rate λ and the service times of packets be distributed according to an exponentially distributed random variable with mean $1/\mu$. An arrival is admitted into the queue if the current workload is less than T and it is rejected otherwise. First, we find the steady-state probability density of the workload in the system and then by using this result, we derive the blocking (or rejection) probability for an arbitrary arrival.

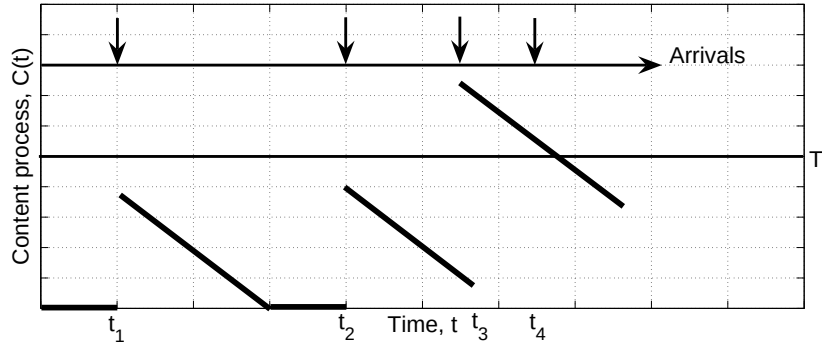


Figure 3.7: Evolution of the workload process in Example 4.

A sample path of the workload is illustrated in Fig. 3.7. In this figure, four arrivals occur at times t_1 , t_2 , t_3 and t_4 . The first three arrivals are admitted to the system but the final arrival at t_4 is rejected because the workload is greater than T at t_4 . Here we apply the same methodology as in the previous examples. However, here there are two different regimes (or policies). Hence we first define the regimes $(0, T)$ and (T, ∞) in the FMFFQ. There are two types of behaviors in Fig. 3.7: jumps and decreases with constant drift. Jumps are exponentially distributed. We insert continuously increasing sample path parts at jump epochs and present these parts by defining a single state (state 1) in the background process of the FMFFQ. The remaining parts can be described by another state (state 2). According to the definition of the process, we define the drift parameters of this FMFFQ as follows:

$$R^{(1)} = \tilde{R}^{(1)} = R^{(2)} = \text{diag}(1, -1) \text{ and } \tilde{R}^{(0)} = \text{diag}(1, 0).$$

The admission policy can be easily described by assigning the following infinitesimal generators for the background process:

$$\tilde{Q}^{(0)} = Q^{(1)} = \begin{bmatrix} -\mu & \mu \\ \lambda & -\lambda \end{bmatrix} \text{ and } \tilde{Q}^{(1)} = Q^{(2)} = \begin{bmatrix} -\mu & \mu \\ 0 & 0 \end{bmatrix}.$$

$\tilde{Q}^{(0)}$ and $Q^{(1)}$ are identical to the infinitesimal generator of FSFQ used to solve the workload distribution of an M/M/1 system of Example 1 because the

queue behaves the same as an ordinary M/M/1 if the workload is less than the threshold T . However, the second rows of $\tilde{Q}^{(1)}$ and $Q^{(2)}$ are zero, indicating the fact that once the background state is in state 2 there is not going to be transition to the first state until the lower regime $(0, T)$ is reached. This is the result of the rejection policy in the region.

Once we solve the steady-state joint probability density functions of the FMFFQ whose parameters are defined above, we should apply the final step (normalization step) so as to find out the steady-state workload density of the M/M/1 queueing system with workload dependent admission policy. Denoting the density and the mass accumulation of this process by $g(x)$ and c , respectively, the following relations are obvious.

$$g(x) = \begin{cases} \frac{f_2^{(1)}(x)}{c_2^{(0)} + \int_0^T f_2^{(1)}(x)dx + \int_T^\infty f_2^{(2)}(x)dx}, & 0 < x < T \\ \frac{f_2^{(2)}(x)}{c_2^{(0)} + \int_0^T f_2^{(1)}(x)dx + \int_T^\infty f_2^{(2)}(x)dx}, & T < x < \infty \end{cases}$$

and

$$c = \frac{c_2^{(0)}}{c_2^{(0)} + \int_0^T f_2^{(1)}(x)dx + \int_T^\infty f_2^{(2)}(x)dx}.$$

Using the well-known PASTA property, one can easily derive the blocking probability P_b as

$$P_b = \int_T^\infty g(x)dx.$$

The next example demonstrates that we can even sometimes find the steady-state measures for discrete-valued queueing processes.

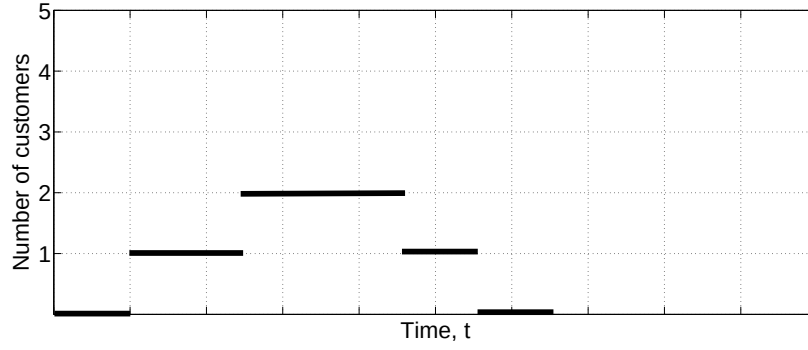


Figure 3.8: Evolution of the number of customers in an M/M/1/K queue

3.4.2 Example 5: Distribution of the number of customers in an M/M/1/K queueing system

This example explains an interesting solution to the steady-state distribution of the number of customers in an M/M/1/K queue. The closed-form solution of this problem is well-known and we give this example just to illustrate the fact that our method is also applicable to discrete-valued queueing processes. Let the arrival process be Poisson with rate λ and let the service times of packets be distributed according to an exponentially distributed random variable with mean μ . Also assume that new arrivals are blocked if there are K customers in the system. Without loss of generality, assume that K is an even number. We present the FMFFQ-based solution of this problem.

We study the queue occupancy in terms of the number of customers in the system which is a discrete-valued process for which one of its sample paths is illustrated in Fig. 3.8. At any point of the state space $(0, 1, \dots, K)$ the process stays for an exponential amount of time, then makes a jump to one of the neighboring states. As in the previous examples if we insert continuous increases with constant drifts instead of the jumps, we can again achieve a sample path of an associated FMFFQ.

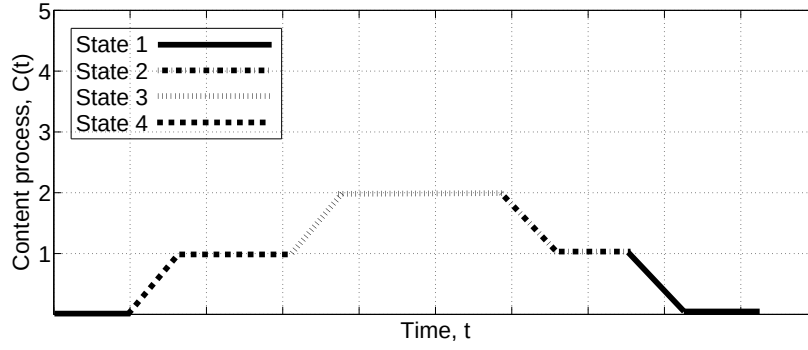


Figure 3.9: Evolution of the transformed process in Example 5.

Define $K + 1$ boundary points (including 0) in an FMFFQ. Each boundary point represents the number of customers in the system, i.e., k^{th} boundary point means that there are k customers in the system. We represent the lines in Fig. 3.9 with four states in the background process of the FMFFQ:

- *State 1*: the last change in the number of customers is due to a service completion and the current number of customers is even,
- *State 2*: the last change in the number of customers is due to a service completion and the current number of customers is odd,
- *State 3*: the last change in the number of customers is due to an arrival and the current number of customers is even,
- *State 4*: the last change in the number of customers is due to an arrival and the current number of customers is odd.

Fig. 3.9 shows each state in a different line style. From the descriptions of the states, the following parameters of the corresponding FMMFQ are obvious:

$$r_1^{(k)} = -1, \tilde{r}_1^{(k)} = -1, \tilde{r}_1^{(k-1)} = 0 \text{ if } k \text{ is odd,}$$

$$r_2^{(k)} = -1, \tilde{r}_2^{(k)} = -1, \tilde{r}_2^{(k-1)} = 0 \text{ if } k \text{ is even,}$$

$$r_3^{(k)} = 1, \tilde{r}_3^{(k-1)} = 1, \tilde{r}_3^{(k)} = 0 \text{ if } k \text{ is even,}$$

$$r_4^{(k)} = 1, \tilde{r}_4^{(k-1)} = 1, \tilde{r}_4^{(k)} = 0 \text{ if } k \text{ is odd.}$$

If the content is equal to $T^{(k)}$ (i.e., there are k customers), for $k = 2, 4, \dots, K-2$ and the current state is 1 or 3, after an exponentially distributed time with mean $1/(\lambda + \mu)$ the number of customers change: increase by 1 with probability $\lambda/(\lambda + \mu)$ or decrease by one. This statement is also valid if the current state is 2 or 4 and the content is equal to $T^{(k)}$ (i.e., there are k customers), for $k = 1, 3, \dots, K-1$. Mathematically,

$$\tilde{Q}^{(k)} = \begin{cases} \begin{pmatrix} -(\lambda + \mu) & \mu & 0 & \lambda \\ X & X & X & X \\ 0 & \mu & -(\lambda + \mu) & \lambda \\ X & X & X & X \end{pmatrix}, & k = 2, 4, \dots, K-2 \\ \begin{pmatrix} X & X & X & X \\ \mu & -(\lambda + \mu) & \lambda & 0 \\ X & X & X & X \\ \mu & 0 & \lambda & -(\lambda + \mu) \end{pmatrix}, & k = 1, 3, \dots, K-1 \end{cases}$$

where X denotes unspecified parameters (don't care situation).

If the system is empty and the state is 1 then after an exponentially amount of time with mean $1/\lambda$, the number of customers will increase by one thus the background state will make a transition to state 4, thus

$$\tilde{Q}^{(0)} = \begin{bmatrix} -\lambda & 0 & 0 & \lambda \\ X & X & X & X \\ X & X & X & X \\ X & X & X & X \end{bmatrix}.$$

If the system is full and the state is 3 then after an exponentially amount of time with mean $1/\mu$ the number of customers will decrease by one thus the

background state will make a transition to state 2, therefore

$$\tilde{Q}^{(K)} = \begin{bmatrix} X & X & X & X \\ X & X & X & X \\ 0 & \mu & -\mu & 0 \\ X & X & X & X \end{bmatrix}.$$

Not only in the infinitesimal generators but also in drift matrices some of the parameters aren't defined. Any appropriate selection of these parameters doesn't affect the results provided that the drift configuration doesn't violate the assumptions of Chapter 2 and there is a single recurrent class in the state-space of the transformed process.

Let p_n be the probability that the number of customers is n in steady-state. This probability can be found by using the following normalization over thresholds

$$p_n = \begin{cases} \frac{c_1^{(n)} + c_3^{(n)}}{\sum_{\text{even } k} (c_1^{(k)} + c_3^{(k)}) + \sum_{\text{odd } k} (c_2^{(k)} + c_4^{(k)})}, & n \text{ is even,} \\ \frac{c_2^{(n)} + c_4^{(n)}}{\sum_{\text{even } k} (c_1^{(k)} + c_3^{(k)}) + \sum_{\text{odd } k} (c_2^{(k)} + c_4^{(k)})}, & n \text{ is odd.} \end{cases}.$$

3.5 Further Developments

The method is not just restricted to the solutions obtained by inserting continuous parts instead of jumps in the original process. Improving it one further step, we can insert some auxiliary regimes to the first order multi-regime feedback fluid queues, especially to solve systems involving dynamic arrangement of parameters. An obvious example of this idea is given in the analysis of hysteretic control in Chapter 6. The solution of this problem, even with simplifying assumptions, is relatively more complex than any other solution of this thesis. So we refer to Chapter 6 for a detailed analysis.

3.6 An Algorithmic Guideline for Simple Transformations

As mentioned before, it is very difficult to give a generic algorithmic transformation method for all types of available solutions. However, in this section we will give an algorithmic guideline for the transformation of simple processes. This algorithm includes five steps. In the first step, we determine the boundary points and the regimes; in the second step we explore a way of modeling jumps if there exist any and define some of the drift parameters; in the third step we identify the remaining drift and the infinitesimal generator parameters; in the fourth step we solve the FMFFQ; and finally we present the backward transformation in measure domain.

Boundary Points and Regimes :

- Select 0 as the lowermost boundary point.
- If the process is unbounded, then the maximal or the uppermost boundary point is ∞ ; otherwise determine the maximum reachable point by the process and set this value as the uppermost boundary point.
- Set all of the points where the queue behavior changes or the queue content stays for a while as the intermediate boundary points.
- Define each region between two successive boundary points as a regime.

Jumps :

- i) If there exists any jump in the process and the amount of that jump can be described by a state transition diagram with finite number of exponential random variables, then replace each jump with a linearly increasing sample path part with drift 1.

- ii) If there exists any jump in the process and if the process reaches a certain intermediate boundary point at each jump epoch with probability 1, then replace this jump with a linearly increasing sample path part with drift 1.
- iii) If there exists any jump in the process and if the process bypasses a certain intermediate boundary point at each jump epoch with probability 1, then divide the jump into two parts: The first part is the space interval from the last point before the jump and the higher most boundary point that is passes with probability 1; and set the second jump as the rest of the jump. If the second part of the jump can be described by a state transition diagram, then it means that we can model the second part as well as the first part of the jump using FMFFQs

Remaining Parameter Set :

- If (ii) is valid for a jump then set the corresponding row of the infinitesimal generator for the state describing the jump as all zeros.
- If there exists an (i)-type jump, then define the jump with the same number of states as in the state transition diagram. Set the same transition among those states.
- If there exists and (iii)-type jump then combine the above two idea in the same process.
- If the evolution of the process between jump epochs can be described by a state transition diagram with a finite number of states, then define those states in the background process of the FMFFQs. Assign the slope of the movement between jump epochs as the drift parameters of those states.
- Define the transition rates to jump epochs and apply in the parameter set.

FMFFQ Solution :

- Solve the joint probability density functions and the joint probability mass accumulations of the FMFFQ using the results in Chapter 2.

Backward Transformation :

- Draw a sample path of the process including all possible scenarios and the sample path of the corresponding FMFFQ following the same evolution in between jump epochs.
- Determine which background states and the spaces should be deleted on the sample path of the FMFFQ to obtain the sample path of the original process.
- Make a normalization over the integration of remaining joint probability density functions and joint probability mass accumulations.
- After normalization, add those remaining probabilistic measures to obtain the final results.

3.7 Discussion on the Limits of the Transformation

In this section we make a discussion on whether a solution is valid or not via the transformation proposed in this thesis. First of all, if one knows the general solution form of the problem and this solution form can be described by a finite number of exponential modes then one can find a transformation to model this process as an FMFFQ since FMFFQs also satisfies this solution form. But this type of approach is only realizable if we are trying to find an alternative solution to a problem whose standard solution is already known.

Secondly, there are some restrictions on the slopes of the process which is aimed to be transformed into an FMFFQ. One should not expect a transformation into FMFFQ if the original process is composed of sample path parts whose slopes are continuously changing depending on the queue parameter. The slope should be constant over finite time intervals. Not only that but also the slope should not change depending on the time, continuously.

Thirdly, the state space of the boundary points should be discrete and finite. The solution of FMFFQ is not a valid for continuously moving boundary points.

Finally, the random nature of FMFFQs emerges from stochastic nature of the background process. Given that the regime is constant, the background process is a CTMC with a finite number of states as mentioned before. Therefore, the number of states should be finite. Furthermore, the background process is described by exponential random variables with some predetermined transition rates. If the transition mechanism is a general distribution as in semi-Markov chain, then those types of processes can not be transformed into an FMFFQ, as well.

3.8 Conclusions

To sum up, if the sample path of a process can be transformed into the sample path of a Markovian fluid queue via addition of regimes, states etc. then the steady-state measures of this process can be derived from the joint steady-state probability density functions of the corresponding Markovian fluid queue. These modifications strongly rely on the creativity of the analyst. From the examples of this section and the results of the successive chapters one can readily see that this FMFFQ-based solution is capable of handling a large class of problems related to queueing theory and by the way performance evaluation of communication networks. Honestly, sometimes our solution method is much more complex than

the standard solution as in the case of Examples 1 and 5. However, inverse of this statement is true for some other examples, see successive chapters.

Availability of our solution is dependent on the stochastic structure of the concerned process. Even though the method itself doesn't promise the solution of the measures in general systems such as the waiting time in a G/G/1 queue at the moment, it is still valid for quite a range of applications (see Chapters 4, 5 and 6; and the examples of this chapter). Availability of the solution of a problem using the mentioned method can be guessed if the result or the general mathematical form of the result is known or derived by using some other method. The general form of the steady-state joint distribution of an FMFFQ is the summation of finite number of exponentials with vector coefficients (see the results in Chapter 2). Therefore, if the general form of the solution of a problem is known to be in this form then it means that the mentioned method is most possibly applicable. However, it doesn't mean that any problem with that solution form can be interpreted by this method, easily. The difficulty arises due to the fact that the exponents and the vector formed coefficient of exponentials in that general form are the left eigenvalue-eigenvector pairs of a matrix.

Chapter 4

EXACT ANALYSIS OF OFFSET-BASED SERVICE DIFFERENTIATION IN SINGLE-CHANNEL MULTI-CLASS OBS

In this chapter and [30], we study a multi-class optical burst switching (OBS) node using the horizon reservation scheme. Multiple traffic classes are differentiated using different offset times per class. Assuming Poisson burst arrivals and phase-type distributed burst lengths, we exactly solve for per-class blocking probabilities using the well-known theory of first-order multi-regime feedback fluid queues (FMFFQs).

4.1 Introduction

Three main switching paradigms appear in optical networks: Optical Circuit Switching (OCS), Optical Packet Switching (OPS) and Optical Burst Switching (OBS). In OCS, a lightpath through which the data transmission is realized is established prior to each connection statically or dynamically. In OPS, each packet is routed independent of each other. OBS is a candidate architecture for the future optical Internet that is based on aggregation of client packets into so-called bursts at the edge of the OBS domain. When a burst is formed, its reservation request is signalled out of band using a burst control packet (BCP). The burst is transmitted after an offset time and is transported in the optical domain. When a BCP arrives at the OBS node, the switch configuration is initiated for the corresponding burst. Different reservation models are proposed for the timing and duration of the reservation. Delayed reservation schemes, such as horizon [63], that do not perform any void filling, are simpler to implement than void filling-based JET (Just Enough Time) schemes [70]. In the horizon scheme, we only keep track of the channel horizon that is the earliest time after which there is no planned use of the channel.

In OBS, service differentiation among traffic classes can be achieved by assigning additional offset values to classes that require preferential treatment. There are analytical models of multi-class OBS systems that rely on the assumption of Poisson arrivals and exponentially distributed burst lengths. In [71], an approximative model is proposed that assumes complete isolation between classes for multi-channel OBS. Per-class loss probabilities for multi-class single-channel JET are approximately calculated in [9] under low offered load assumption and for generally distributed burst lengths. Focusing on Horizon-based single-channel OBS networks in this study, we improve upon the existing literature by proposing an exact solution while allowing more general phase-type distributed burst lengths. Our solution is based on the well-known theory of FMFFQs.

We provide the stochastic model in Section 2. Our proposed solution is presented in Section 3. Numerical examples are given in Section 4 to verify the method. Finally, we conclude.

4.2 Stochastic Model

To model burst lengths, we use phase-type distributions (PH-type). Consider a Markov process on the states $\{1, 2, \dots, m, m+1\}$ with initial probability vector $[v, 0]$, $v = [v_1, v_2, \dots, v_m]$ and infinitesimal generator

$$Q = \begin{bmatrix} S & S^0 \\ 0 & 0 \end{bmatrix},$$

where S is an $m \times m$ nonsingular matrix, S^0 is $m \times 1$, $Se + S^0 = 0$, and e is a column vector of ones of appropriate size. The time till absorption into the absorbing state $m+1$ is a random variable X which is said to have a PH-type distribution with representation (v, S) whose distribution function is written as $F_X(x) = 1 - ve^{Sx}e$, $x \geq 0$.

We focus on a tagged output port of an OBS node comprised of a single wavelength channel with the channel capacity normalized to unity. In our system, bursts are assumed to belong to one of the traffic classes in the class-set $\mathcal{I} = \{1, 2, \dots, I\}$. We assume that burst (or BCP) arrivals destined to the tagged output port are Poisson with rate λ_i for class- i whose lengths, $1 \leq i \leq I$, are assumed to be modeled by a PH-type distribution characterized with the pair (v_i, S_i) with m_i transient states. Let $S_i^0 = -S_i e = [S_{i,1}^0, S_{i,2}^0, \dots, S_{i,m_i}^0]^T$ and $v_i = [v_{i,1}, v_{i,2}, \dots, v_{i,m_i}]$. Also let the $(k, l)^{th}$ entry of S_i be denoted by $\{S_i\}_{k,l}$.

A class- i BCP arrives at the OBS node on behalf of a class- i burst δ_i seconds prior to the arrival of the corresponding burst. The class- i offset δ_i is assumed to be deterministic. If the channel horizon is less than or equal to δ_i at the

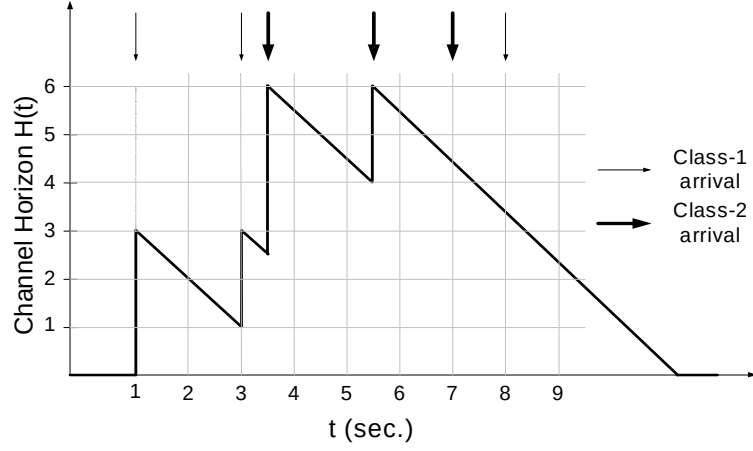


Figure 4.1: Evolution of the channel horizon $H(t)$ for an example of a two-class OBS node.

arrival epoch of a BCP belonging to a class- i burst, then the burst is admitted by reserving the channel for that burst and the channel horizon increases to a value equal to the sum of the burst length (in seconds) and the offset value δ_i . Otherwise, the corresponding burst is blocked (dropped). The channel horizon decreases at a unity rate between burst arrival epochs with a boundary at the origin. Moreover, without loss of generality we assume that inequality $\delta_i < \delta_j$ holds for all $i, j \in \mathcal{I}$ satisfying $i < j$.

To illustrate the operational policy for a two class OBS Horizon system, we depict a sample path of the channel horizon process in Fig 4.1. We take $\delta_1 = 2$ and $\delta_2 = 4$. Burst lengths for class-1 and class-2 are 1 and 2, respectively. Class-1 BCP arrivals occur at $t = 1, 3, 8$ whereas class-2 arrivals at $t = 3.5, 5.5$, and 7. The BCPs arriving at $t = 7, 8$ are blocked since the channel horizon (denoted by $H(t)$) is strictly larger than the corresponding offset times at the arrival instants. Note that an accepted burst leads to an immediate jump in $H(t)$ whereas a blocked burst does not have any effect.

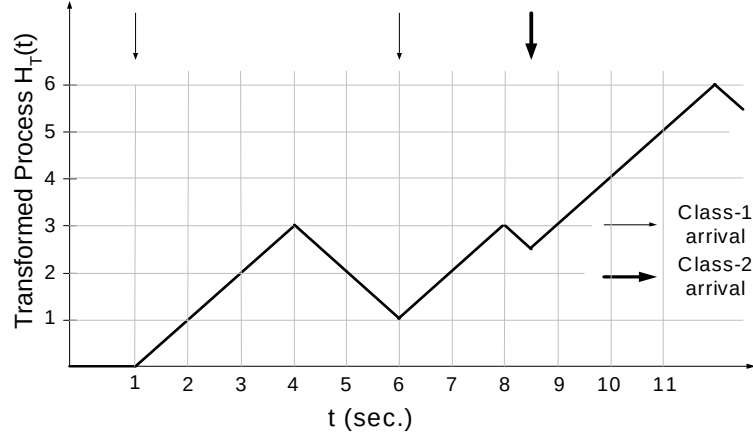


Figure 4.2: Sample path for the transformed process $H_T(t)$ for the example of Fig. 5.1.

4.3 Proposed Solution

The basic idea is as follows. We like to use FMFFQs to model multi-class OBS nodes. However, the channel horizon $H(t)$ can not be modeled directly by an FMFFQ due to the jumps involved at arrival epochs. However, consider the transformed process $H_T(t)$ obtained through $H(t)$ by simply replacing the jumps in the sample path by a linear increase (say with unity rate) which makes it possible for us to use an FMFFQ to describe the evolution of $H_T(t)$; see Fig. 4.2 for an evolution of $H_T(t)$ for the same example given in Fig. 4.1. Note that we do not allow new arrivals when $H_T(t)$ is increasing. We will then show that the steady-state behavior of the original process can be derived from that of the transformed process.

For this purpose, we first define $I + 1$ regimes by defining $I + 2$ thresholds $0 = T^{(0)} < T^{(1)} = \delta_1 < T^{(2)} = \delta_2 < \dots < T^{(I)} = \delta_I < T^{(I+1)} = \infty$. We denote the state corresponding to a decrease in $H_T(t)$ by P . Moreover, for each class- i , we define m_i dummy states which we denote by D_i^j , for $1 \leq j \leq m_i$. We inherit the notation we used for FMFFQs in Section 2 including the queue content $C(t)$, the background process $M(t)$, and the matrices $Q^{(k)}$, $\tilde{Q}^{(k)}$, $R^{(k)}$, and $\tilde{R}^{(k)}$. Moreover, we assume that the states P and D_i^j are the p^{th} and d_i^{jth}

states of the background process $M(t)$, respectively. Let $q_{i,j}^{(k)}$ and $\tilde{q}_{i,j}^{(k)}$ denote the $(i,j)^{th}$ entry of $Q^{(k)}$ and $\tilde{Q}^{(k)}$. Also note that the cardinality of the state space of this FMFFQ is $M = 1 + \sum_{i=1}^I m_i$.

We now find the parameters of the FMFFQ corresponding to the process $H_T(t)$. Note that $C(t)$ is to track the same path as $H_T(t)$ and therefore when the background process is in state P , we define the drift at that state as

$$r_p^{(k)} = -1, \quad 1 \leq k \leq I+1, \quad (4.1)$$

$$\tilde{r}_p^{(k)} = \begin{cases} -1 & \text{if } k > 0, \\ 0 & \text{if } k = 0. \end{cases} \quad (4.2)$$

When a new class- i burst is admitted, then a transition occurs from state P to D_i^j with probability $v_{i,j}$. However, for a class- i burst to be admitted, the queue content should be less than δ_i . Therefore, for $1 \leq k \leq I$

$$\tilde{q}_{p,l}^{(k)} = q_{p,l}^{(k)} = \begin{cases} -\sum_{i=k}^I \lambda_i & \text{if } l = p, \\ \lambda_i v_{i,j} & \text{if } l = d_i^j \text{ and } i \geq k, \\ 0 & \text{otherwise,} \end{cases} \quad (4.3)$$

Moreover,

$$q_{p,l}^{(I+1)} = 0, \quad 1 \leq l \leq M, \quad (4.4)$$

$$\tilde{q}_{p,l}^{(0)} = q_{p,l}^{(1)}, \quad 1 \leq l \leq M. \quad (4.5)$$

If the background process is at state D_i^j , $C(t)$ increases with unity rate up to δ_i without any state transitions. On the other hand, $C(t)$ holds on to its unity rate increase until the corresponding phase type distribution reaches its absorbing state. Therefore,

$$\tilde{r}_{d_i^j}^{(k-1)} = r_{d_i^j}^{(k)} = 1, \quad 1 \leq k \leq I+1, \quad (4.6)$$

$$\tilde{q}_{d_i^j,l}^{(k-1)} = q_{d_i^j,l}^{(k)} = 0, \quad 1 \leq l \leq M, \quad 1 \leq k \leq i, \quad (4.7)$$

$$\tilde{q}_{d_i^j,l}^{(k-1)} = q_{d_i^j,l}^{(k)} = \begin{cases} S_{i,j}^0 & \text{if } l = p, \\ \{S_i\}_{j,n} & \text{if } l = d_i^n, \quad 1 \leq n \leq m_i \\ 0 & \text{otherwise.} \end{cases} \quad (4.8)$$

This concludes the characterization of the FMFFQ describing the process $H_T(t)$. We can then solve for the steady-state joint distribution function $F_p(x)$. If we condition only on the P state then the steady-state distribution of the original process can be found. This is due to the observation that if we delete the intervals in Fig. 4.2 during which the queue is increasing, then we obtain the sample path given in Fig. 4.1. Mathematically,

$$F(x) = \lim_{t \rightarrow \infty} Pr\{H(t) \leq x\} = F_p(x)/F_p(\infty). \quad (4.9)$$

Let $f(x)$ denote the corresponding density function such that $F(x) = \int_{0-}^x f(y)dy$. Moreover, due to Poisson arrivals and from the PASTA property, a class- i burst reservation request is blocked with probability $P_{Loss,i} = 1 - F(\delta_i)$. Now assume $\delta_1 = 0$. $F(x)$ for this problem is then related to the distribution function $\bar{F}(x)$ of an auxiliary scenario when we shift the offsets δ_i by a fixed value $\varepsilon > 0$. It is not difficult to show that $F(x) = \bar{F}(x + \varepsilon)$. With this, the $\delta_1 = 0$ case can also be handled using the method described above.

4.4 Numerical Examples

4.4.1 Example 1

Let m_X and c_X denote the mean and the coefficient of variation (CoV) of a random variable X . For given $m_X > 0$ and $c_X > 1$ based on empirical data, one can find a 2-phase PH-type distribution, i.e., hyper-exponential distribution, whose mean and CoV match to m_X and c_X , respectively [61]. Note that, for an exponential random variable X , $c_X = 1$. We use the fitting procedure of [61] in the numerical examples to follow. We first assume a two-class OBS system with parameters $\delta_1 = 1, \delta_2 = 2$. Moreover, burst lengths for both classes have a mean of 1 and CoV of 2. In Fig. 4.3, we compare the steady-state horizon pdf obtained using the method of this letter and simulations for light, moderate, and heavy

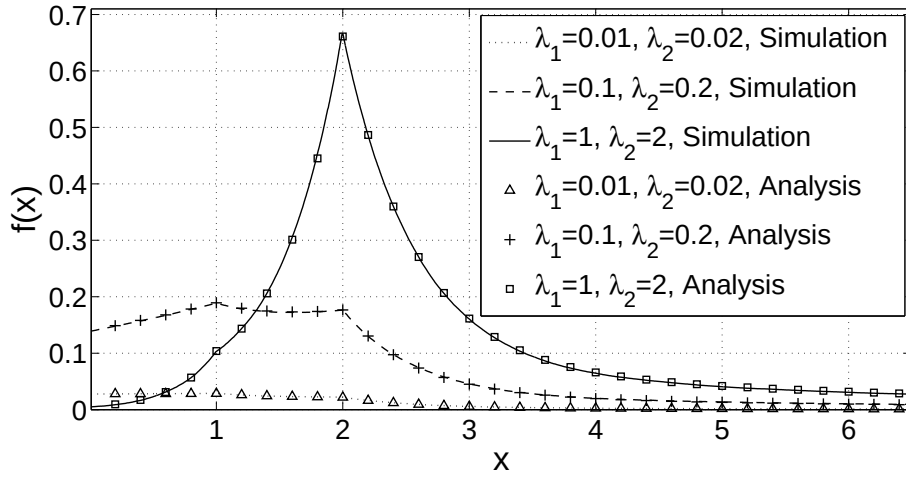


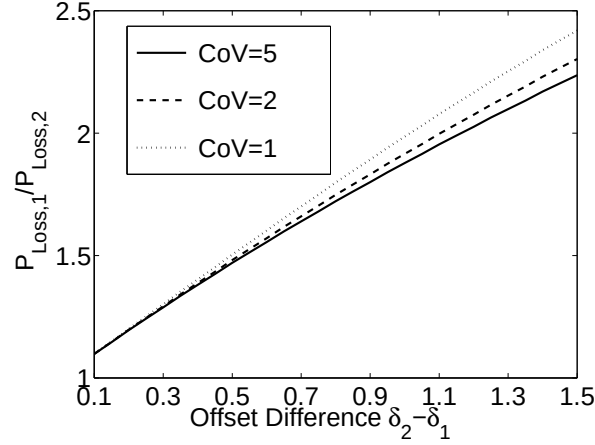
Figure 4.3: The steady-state horizon pdf $f(x)$ of the two class OBS example in Example 1.

loaded systems. We show that the results exactly match irrespective of the load.

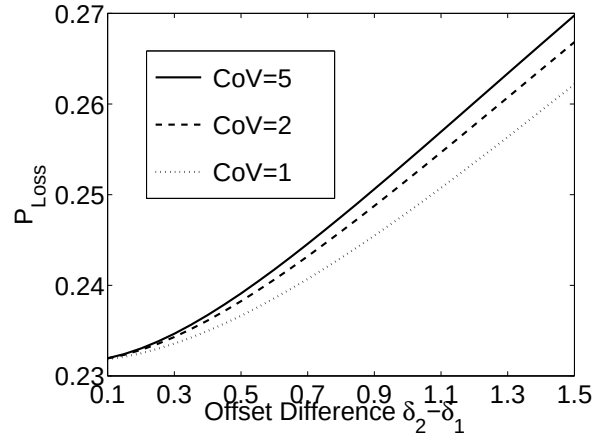
4.4.2 Example 2

In this example, we study the effect of the second order statistics of the distribution of burst lengths on loss probabilities. The system under study has $\lambda_1 = 0.1$, $\lambda_2 = 0.2$, and $\delta_1 = 1$. Burst lengths of both classes possess the same distribution with unity mean but varying CoV = 1, 2, and 5. In Fig. 4.4(a), we plot the loss probability ratio $P_{Loss,1}/P_{Loss,2}$ as a function of $\delta_2 - \delta_1$. We observe that the degree of loss differentiation increases with increased offset difference as expected but decreases as a function of the CoV of the burst lengths. In Fig. 4.4(b), we plot the overall loss probability P_{Loss} as a function of $\delta_2 - \delta_1$. It is clear that P_{Loss} increases with the offset difference due to the lack of a void filling mechanism. However, the increase in P_{Loss} is much higher for larger burst length CoV. We therefore conclude that second order statistics of the burst length is crucial in multi-class OBS systems. This result is important because several approximate

methods in the literature use the Erlang-B loss model as a basis which is insensitive to the higher order statistics of the burst lengths and in particular the CoV [15], [60], [69].



(a) $\frac{P_{Loss,1}}{P_{Loss,2}}$ as a function of $\delta_2 - \delta_1$

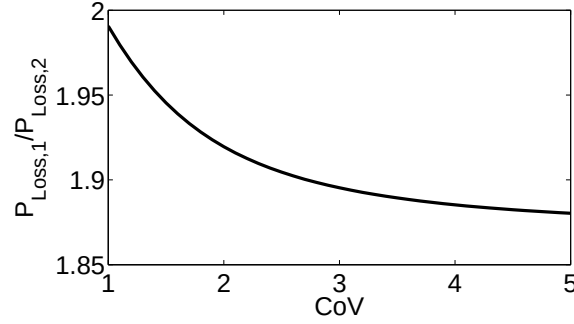


(b) P_{Loss} as a function of $\delta_2 - \delta_1$

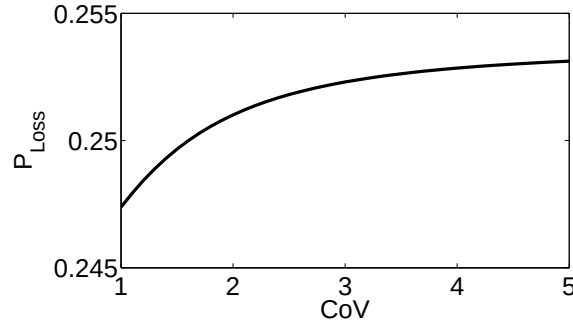
Figure 4.4: Performance of a two-class OBS system with respect to offset difference in Example 2.

In Fig. 4.5, we plot the loss probability ratio and the overall loss probability for $\delta_2 = 2$ and varying values of CoV. If CoV=1, then packet lengths are exponentially distributed with mean 1. For higher CoV values two types of packets appear in the arrival stream. The first (second) type arrives with probability p_1 (p_2) and has an exponentially distributed length with mean μ_1 (μ_2). As the CoV value increases μ_1 and μ_2 approaches 0.5 and ∞ , respectively. Moreover,

p_2 becomes closer to 1. Fig. 4.5 shows that, as CoV increases, the overall loss probability increases and the differentiations gets worse for this example.



(a) $\frac{P_{Loss,1}}{P_{Loss,2}}$ as a function of CoV



(b) P_{Loss} as a function of CoV

Figure 4.5: Performance of a two-class OBS system with respect to CoV in Example 2.

4.4.3 Example 3

Let λ_1 and λ_2 be the arrivals rates of the Poisson streams of class-1 and class-2 packets, respectively, in a two-class OBS system. Also, assume that the packet lengths of both classes are distributed according to the same exponential distribution with mean $1/\mu$. Since there is a non-zero probability that the packet length of the low priority packets (class-1) overflows the offset difference, $\delta_2 - \delta_1$, some portion of the losses of high priority packets (class-2) incurs due to the admitted low priority packets. Other losses in class-2 are caused by the packets of the same class. We call isolation to the proportion of losses in the high priority

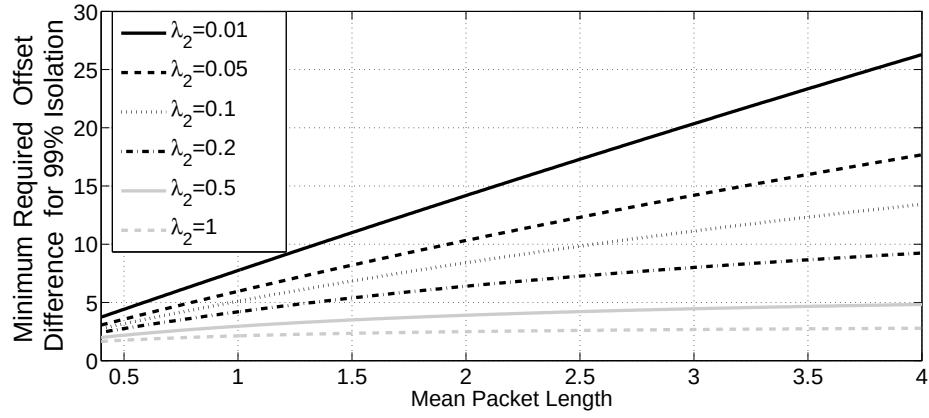


Figure 4.6: Minimum required offset difference for a 99% isolation for varying values of arrival rates.

packets due to previously admitted high priority packets to the overall loss. If the packet lengths are deterministic and lower than the offset difference, then isolation is 100%. In that case, one can find the blocking probability of the high priority packets using Erlang-B formula, exactly. 100% isolation isn't possible in the case of exponentially distributed packet lengths. However, as the offset difference increases the isolation percent increases monotonically. In Fig. 4.6, we fix $\lambda_1 = 0.1$ and plot the minimum required offset difference for a 99% isolation for varying values of $1/\mu$ and λ_2 . As opposed to deterministic packet lengths, for a high isolation value we need to set the offset difference to a few multiples of the mean packet length. Moreover, as can be seen from the plots, the minimum required offset difference has a logarithmic relation with $1/\mu$. However, for relatively high (low) values of λ_2 the minimum required offset difference converges to a constant value (linear function of $1/\mu$).

4.5 Conclusions

We exactly derive the steady-state distribution for the channel horizon and per-class blocking probabilities for a multi-class single-channel OBS node with offset-based service differentiation among traffic classes. For this purpose, we use the

theory of feedback Markov fluid queues. We validate the proposed method by numerical examples.

Chapter 5

EXACT ANALYSIS OF SINGLE-WAVELENGTH OPTICAL BUFFERS

Optical buffering via fiber delay lines is used for contention resolution in optical packet and optical burst switching nodes. This chapter addresses the problem of exactly finding the loss probabilities in an asynchronous single-wavelength optical buffer. Packet lengths are assumed to be variable and modeled with phase type distributions whereas the packet arrival process is modeled with a Markovian arrival process which can capture auto-correlations in interarrival times. The exact solution is based on the theory of feedback fluid queues for which we propose numerically efficient and stable algorithms. In this chapter and [28], we not only find the packet blocking probabilities but also the entire distribution of the unfinished work in this system from which all performance measures of interest can be derived.

5.1 Introduction

Optical packet switching-based paradigms are beginning to mature as they attempt to more efficiently utilize the fiber capacity as opposed to their optical circuit switching counterparts. Two such paradigms have recently been proposed: Optical Packet Switching (OPS) [21] and Optical Burst Switching (OBS) [46]. In this chapter, we analytically study the performance of an optical packet (or burst) switch that uses fiber delay lines. We note that the version of the problem we study is insensitive to the underlying switching paradigm whether it be OPS or OBS.

OPS can either be synchronous (time-slotted) or asynchronous (unslotted). Optical packets are of fixed-length in synchronous systems leading to a need for costly synchronization equipment. Synchronous switching is known to be performance efficient due to the alignment of packet arrivals. On the other hand, optical packets are of variable-length in asynchronous systems and therefore packet arrivals need not be aligned. Asynchronous OPS appears to be a better fit as a transport technology for the next generation Internet due to the variable lengths of IP packets in the Internet. In this chapter, we focus on asynchronous OPS.

In OPS networks, contention arises when two or more incoming optical packets contend for the same output wavelength. Methods for contention resolution include wavelength conversion [10], deflection routing [46], or optical buffering [13] which is the scope of the current chapter. In electronic buffering, contending packets are stored in RAMs (Random Access Memory) to be transmitted at a later time. However, optical RAMs are not feasible today. Instead, a well-known technique is to use FDLs (Fiber Delay Lines) in which a contending packet is sent over a coil of fiber that gives the packet a fixed delay that is enough to resolve the contention. We call this paradigm optical FDL buffering, or simply optical buffering.

There are different architectures proposed for optical buffers based on their position in the switch, namely input buffering, output buffering, and shared buffering [72]. Input buffering is known to have relatively low performance than the other two. In output buffering, a set of FDLs is dedicated for each output port (or fiber). On the other hand, in shared buffering, FDLs can be shared by all output ports for improved performance. We focus on output buffering in this chapter. In particular, we focus on the case where a set of FDLs is dedicated for each wavelength (link) of an output port. This situation arises when

- Each output port carries a single wavelength,
- We have multiple wavelengths per output port but we do not use wavelength converters that would allow buffer sharing across different wavelengths.

Hybrid use of wavelength converters and FDL sharing among multiple wavelengths of the same output port is left for future research. Optical buffers can be configured as degenerate or non-degenerate. In degenerate buffering, the lengths of FDLs are consecutive multiples of a parameter D , which is called the granularity parameter. The buffer size is denoted by K and the i th FDL, $i = 1, 2, \dots, K$, introduces a delay of iD to an incoming optical packet finding the output wavelength occupied. In non-degenerate buffering, the lengths of individual FDLs can be arbitrary and the i th FDL exerts a delay of D_i , $i = 1, 2, \dots, K$ where for $i > j$, $D_i > D_j$. Although we present an analytical method for the case of degenerate buffering only, the method can directly be applied to the non-degenerate case. The focus of this chapter is the analysis of an asynchronous single-wavelength optical buffer comprising K FDLs in degenerate configuration.

We now present an example to demonstrate how this optical buffer works. For this purpose, we define $H(t)$ as the unfinished work (or channel horizon) at time t . This means with all the admitted packets into the buffer at time t , we

need to wait for at least $H(t)$ for the link to become idle. Let us now assume an optical packet that arrives at time t . If the link is idle at this arrival epoch, then $H(t) = 0$ and the packet can immediately be forwarded over the link making $H(t^+) = B$ where B is the length of the packet (in time units). If $H(t) > KD$, then the delays introduced by each delay line are not enough to resolve the contention and the packet will be dropped without a need to modify $H(t)$. The most interesting scenario is when $H(t) \leq KD$ in which case the arriving packet of length B will be directed to the FDL that delays it by

$$D \left\lceil \frac{H(t)}{D} \right\rceil,$$

where $\lceil x \rceil$ is the smallest integer greater than or equal to x . Obviously, in this case, the unfinished work is modified as

$$H(t^+) = D \left\lceil \frac{H(t)}{D} \right\rceil + B. \quad (5.1)$$

At other times, the unfinished work $H(t)$ decreases at a unity rate until it hits zero and then stays at this level, i.e., unfinished work can not be negative.

To illustrate the operation of a single-wavelength degenerate optical buffer, we depict a sample path of the channel horizon process in Fig. 5.1 for an example when $D = K = 2$. Packet arrivals occur at time epochs $t = 1, 3, 4, 5$, and 7 with corresponding sizes of $3, 2, 2, 2$, and 1 , respectively. The packet arriving at $t = 5$ is blocked since $H(5) = 5 > KD = 4$. All other four packets are accepted into the buffer. Note that an accepted optical packet leads to an immediate jump in $H(t)$ whereas a blocked packet does not have any effect.

Various analytical methods have been proposed to analytically evaluate the performance of an asynchronous single-wavelength optical buffer which differ from each other on the basis of (i) stochastic model used for packet arrivals and packet lengths, (ii) whether K is finite or infinite (iii) whether the analytical method is exact or approximate. One of the earlier works in [12] studies an optical buffer with general K and with Poisson packet arrivals and exponential packet

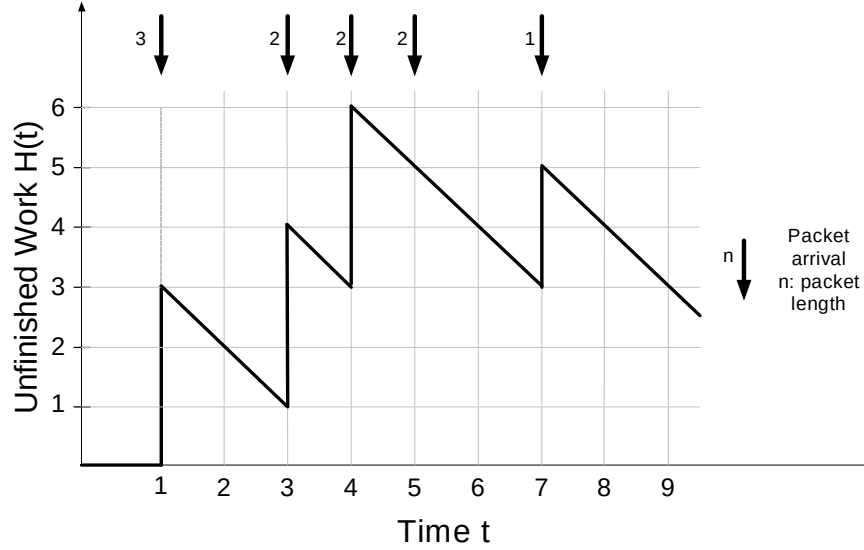


Figure 5.1: Evolution of the channel horizon $H(t)$ for a single-wavelength degenerate optical buffer with $D = K = 2$.

lengths and presents an approximate method using an iterative procedure which is simple to implement. For the same buffer, the work in [49] assumes generic distributions for packet interarrivals and services and the authors present a Markov chain model for solving the optical buffer. The entries of the Markov chain can be found by calculating certain definite integrals for which computational procedures are not elaborated. Moreover, depending on the distributions used in the model, the number of the states of the Markov chain can be excessive which then calls for an approximate state reduction method for computational feasibility. In two special cases, the authors [49] have shown that their solution is exact: i) exponential interarrivals and generic packet lengths ii) generic interarrivals and exponential packet lengths. Again, following similar traffic assumptions, the case of $K = \infty$ is studied by [53] to develop heuristics for the case of $K < \infty$ through formulas involving infinite sums. Three special cases corresponding to (i) exponentially distributed, (ii) deterministic, (iii) and mixtures of deterministic burst lengths, are given for which the resulting formula does not involve an infinite sum and therefore the method turns out to be computationally effective. An approximate method based on the assumption $K = \infty$ is proposed in [35] based

on the generating function approach for the case of Poisson arrivals and generally distributed packet lengths.

In this chapter, we find the exact steady-state distribution of the unfinished work in asynchronous single-wavelength optical buffers from which all steady-state performance measures such as packet blocking probability, expected delay, etc., can be calculated. Our assumptions and contributions are listed below.

Packet interarrival times are modeled with a versatile Markovian Arrival Process (MAP) by which one can capture dependence between successive interarrivals [43], [36]. The MAP generalizes the Poisson process by allowing non-exponential interarrival times while maintaining the underlying Markovian structure. The MAP is described by two processes; the count process $N(t)$ that counts the number of arrivals up to time t and the phase process $J(t)$, assuming values in $\{0, 1, \dots\}$ and $\{1, 2, \dots, d\}$, respectively. The two-dimensional Markov process $\{N(t), J(t)\}$ is then modeled as a Markov process on the state-space $\{(i, j) : i \in \mathbb{Z}, 1 \leq j \leq d\}$ whose infinitesimal generator matrix Q can be represented in block form as

$$Q_{MAP} = \begin{bmatrix} D_0 & D_1 & 0 & \dots \\ 0 & D_0 & D_1 & \dots \\ 0 & 0 & D_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (5.2)$$

In the above generator, D_0 and D_1 are $d \times d$ matrices, D_0 has negative diagonal elements and non-negative off-diagonal elements, D_1 is non-negative, and $\tilde{D} = D_0 + D_1$ is an irreducible infinitesimal generator. When the generator is of the form (5.2) then the underlying arrival process is called a MAP and this MAP is said to be characterized by the matrix pair (D_0, D_1) .

Let π be the stationary probability vector of the phase process with generator D so that π satisfies

$$\pi \tilde{D} = 0, \quad \pi e = 1, \quad (5.3)$$

where e is a column vector of ones of appropriate size. The mean arrival rate λ for a MAP is given by [36]:

$$\lambda = \pi D_1 e. \quad (5.4)$$

The MAP arrival model is more general than the stochastic models used for the analysis of optical buffers and it can especially be useful to demonstrate the impact of auto-correlations (in interarrival times) on system performance. Note that auto-correlations in packet interarrival times and their non-Poisson nature has been well-known for Internet traffic [44]. MMPP (Markov Modulated Poisson Process) is a sub-case of MAP and for MMPP modeling of Internet traffic, we refer the reader to [41].

Packet lengths L are modeled by Phase Type (PH-type) distributions [42]. Consider a Markov process on the states $\{1, 2, \dots, s, s+1\}$ with initial probability vector $[v, 0]$, $v = [v_1, v_2, \dots, v_s]$ and infinitesimal generator

$$Q_{PH} = \begin{bmatrix} S & S^0 \\ 0 & 0 \end{bmatrix},$$

where S is an $s \times s$ nonsingular matrix, S^0 is $s \times 1$, $Se + S^0 = 0$, and e is a column vector of ones of appropriate size. The time till absorption into the absorbing state $s+1$ is a random variable L which is said to have a PH-type distribution with representation (v, S) whose distribution function is written as $F_L(x) = 1 - ve^{Sx}e$, $x \geq 0$. PH-type distributions are known to be dense in the set of all distributions and this feature of PH-type distributions makes it a very suitable tool to model arbitrarily sized optical packets.

Moreover,

- We do not make any assumptions on the number of FDLs K . In particular, we do not need the assumption $K = \infty$ as pursued in some earlier work. With this, we can also analyze situations where one only has a few FDLs.

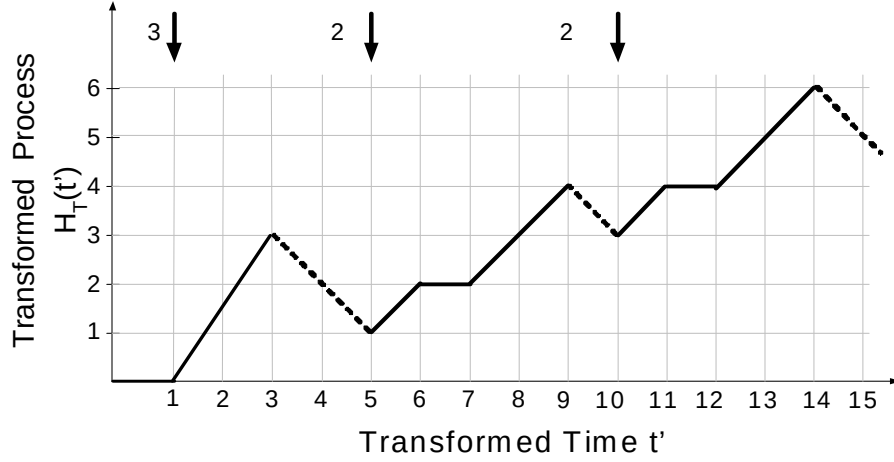


Figure 5.2: Evolution of the transformed process $H_T(t')$ for a single-wavelength degenerate optical buffer with $D = K = 2$.

- The method we use is based on the well-known theory of feedback fluid queues.
- The method we propose is exact for MAP packet arrivals and PH-type distributed packet lengths.

The remainder of the chapter is organized as follows. In Section 5.2, we provide the analytical model. Numerical results are presented in Section 5.3. We conclude in the final section.

5.2 Stochastic Analysis of Optical Buffers

Recall from Section 5.1 that the optical packet arrival process is a MAP characterized with a pair of two $d \times d$ matrices (D_0, D_1) and the packet lengths are PH-type distributed which is characterized with the pair (v, S) where v is a $1 \times s$ vector and S is a nonsingular $s \times s$ matrix and $S^0 = Se$. In Fig. 5.1, a sample path for the horizon process $H(t)$ for the asynchronous optical buffer described in Section 5.1 is given for a particular example of $K = D = 2$. Note that this sample path consists of continuous decreases with fixed drifts and upward jumps

in between. An upward jump appears in the sample path whenever an optical packet is admitted to the optical buffer. The goal of this chapter is to model the optical buffer with FMFFQs but upward jumps can not be directly modeled with FMFFQs. For this purpose, we introduce a transformed process $H_T(t')$ in which an individual jump for $H(t)$ at time t to its new value $H(t^+) = D \left\lceil \frac{H(t)}{D} \right\rceil + B$ in Eq. 5.1 is replaced with the sequence of three curves described in Table 5.1.

1. If $H_T(t'-) = 0$ then go to step 3; otherwise $H_T(t')$ increases steadily with unity drift until it hits the particular value $D \left\lceil \frac{H(t)}{D} \right\rceil$, 2. $H_T(t')$ sticks to this value for an exponentially distributed amount of time, 3. $H_T(t')$ again increases steadily with unity drift until it reaches $D \left\lceil \frac{H(t)}{D} \right\rceil + B$.

Table 5.1: The three-step procedure for obtaining the transformed process $H_T(t')$.

The transformed process $H_T(t')$ for the same example is given in Fig. 5.2 assuming that the Step 2 of the above procedure is to last one time unit for all admitted packets. The solid lines (dotted lines) are for the epochs for which H_T is increasing or staying constant (decreasing). We note that if one only concentrates on the epochs during which H_T is decreasing, then we obtain back the original sample path of $H(t)$. Equivalently, if one can find the steady-state distribution of the process $H_T(t')$ then one can recover that of the process $H(t)$ which is the scope of this chapter.

We have $M = d(s + 2)$ states in the background process. For the degenerate case, we define $K+1$ regimes and set boundary points as $T^{(k)} = kD$ for $0 \leq k \leq K$ and $T^{(K+1)} = \infty$. One could choose $T^{(k)} = D_k$ for the non-degenerate case. The drift parameters of the corresponding FMFFQ that characterizes the transformed process H_T are

$$R^{(K+1)} = \text{diag}(-I_d, -I_d, I_{sd}), \quad (5.5)$$

$$R^{(k)} = \text{diag}(-I_d, I_d, I_{sd}), 1 \leq k \leq K, \quad (5.6)$$

$$\tilde{R}^{(0)} = \text{diag}(0_d, I_d, I_{sd}), \quad (5.7)$$

$$\tilde{R}^{(k)} = \text{diag}(-I_d, 0_d, I_{sd}), 1 \leq k \leq K, \quad (5.8)$$

where I_i and 0_i denote the identity and the zero matrices of size $i \times i$, respectively.

We now provide the per-regime infinitesimal generators $Q^{(k)}, 1 \leq k \leq K+1$:

$$Q^{(k)} = \begin{bmatrix} D_0 & D_1 & 0 \\ 0 & 0 & 0 \\ \tilde{S}^0 & 0 & \tilde{S} \end{bmatrix}, k \leq K, \quad (5.9)$$

$$Q^{(K+1)} = \begin{bmatrix} \tilde{D} & 0 & 0 \\ 0 & 0 & 0 \\ \tilde{S}^0 & 0 & \tilde{S} \end{bmatrix}, \quad (5.10)$$

where $\tilde{S}^0 = I_d \otimes S^0$, $\tilde{S} = I_d \otimes S$, and $\tilde{D} = D_0 + D_1$. Now, we provide the infinitesimal generators when $C(t)$ is at a boundary point, i.e., $C(t) = T^{(k)}$, $0 \leq k \leq K$ as follows:

$$\tilde{Q}^{(k)} = \begin{bmatrix} D_0 & D_1 & 0 \\ 0 & -I_d & \tilde{V} \\ \tilde{S}^0 & 0 & \tilde{S} \end{bmatrix}, k > 0, \quad (5.11)$$

$$\tilde{Q}^{(0)} = \begin{bmatrix} D_0 & 0 & \tilde{D}_1 \\ 0 & 0 & 0 \\ \tilde{S}^0 & 0 & \tilde{S} \end{bmatrix}, \quad (5.12)$$

where $\tilde{V} = I_d \otimes v$ and $\tilde{D}_1 = D_1 \otimes v$.

The expressions (5.5), (5.6), (5.7), (5.8), (5.9), (5.10), (5.11), and (5.12), completely characterize a FMFFQ that models the evolution of the transformed process H_T . To see this, note that we have three sets of states, namely **A**, **B**, and **C**. The set **A** = $\{\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_d\}$ comprises the first d states of the FMFFQ, i.e., $\{1, 2, \dots, d\}$. On the other hand, the set **B** = $\{\mathbf{B}_1, \mathbf{B}_2, \dots, \mathbf{B}_d\}$ comprises the set of states $\{d+1, d+2, \dots, 2d\}$ of the FMFFQ. Finally, the set **C** = $\{\mathbf{C}_{11}, \mathbf{C}_{12}, \dots, \mathbf{C}_{1s}, \mathbf{C}_{21}, \dots, \mathbf{C}_{ds}\}$ consists of the states $\{2d+1, 2d+$

$2, \dots, 2d + s, 2d + s + 1, \dots, 2d + ds\}$ of the FMFFQ. When the FMFFQ is in state $\mathbf{A}_i$, then H_T is decreasing at a unity rate and the arrival MAP is in state i . On the other hand, when the FMFFQ is in state $\mathbf{B}_i$, then H_T is increasing at a unity rate, the next state of the arrival process after the last arrival is i , and we note that this increase corresponds to the increase written in Step 1 of the procedure in Table 5.1. Finally, when the FMFFQ is visiting state $\mathbf{C}_{ij}$, then H_T is increasing with type described in Step 3 of Table 5.1 and the service PH-type process describing packet lengths is in state j and the next state of the arrival process after the last arrival is i .

Assume we are at regime $k, k \leq K$, a packet arrival has just occurred and the next state of the arrival process, MAP, is i . Then the packet gets accepted to the optical buffer and the FMFFQ process transitions to state $\mathbf{B}_i$. Depending on the procedure described in Step 1 of Table 5.1, if H_T is greater than zero it should increase at a unity rate until it hits the first boundary point without changing states as dictated in the second block rows of (5.6) and (5.9). The process H_T then stays for an exponentially distributed time with mean 1 (governed by Step 2 of Table 5.1) at this boundary point eventually escaping to state $\mathbf{C}_{ij}$ with probability v_j . This behavior is indicated in the second block rows of (5.8) and (5.11). According to Step 3 of Table 5.1 and as long as the FMFFQ is in state $\mathbf{C}_{ij}$ for some j , H_T should continue to increase at a unity rate until the corresponding PH-type distribution reaches the absorbing state which occurs with rate S^0 after which we transition to $\mathbf{A}_i$. This behavior is captured by the last block rows of (5.9), (5.10), and (5.6). This increase in the queue length should not be paused if any boundary points are crossed when the queue is increasing; this behavior is captured by a drift choice of unity for all $\mathbf{C}$ states as in (5.7). When in state $\mathbf{A}_i$, the queue length starts to decrease at a unity rate and the arrival process starts to evolve according to the MAP parameters D_0 and D_1 . If an arrival occurs when in regime $K + 1$, this packet does not get accepted and H_T continues to decrease as the MAP continues to evolve. This behavior is captured in the first

block rows of (5.5) and (5.10). If a new packet arrival occurs when the FMFFQ is in regime k , $k \leq K$, and the next state of the MAP process is i , then the FMFFQ transitions to state $\mathbf{B}_i$; we capture this behavior by the first block row choice of (5.9). On the other hand, if an arrival occurs at boundary point $T^0 = 0$ and when the next state of the MAP process is i , then the arriving packet is forwarded over the direct link bypassing the Steps 1 and 2 of the procedure of Table 5.1 and therefore the corresponding FMFFQ transitions to state $\mathbf{C}_{ij}$ with probability v_j . We capture this behavior with the choice of the first block row of (5.12). There are also some situations that need to be avoided. For example, the FMFFQ cannot be at a $\mathbf{B}$ state at regime $K + 1$ during normal operation so precautions should be taken to escape from that situation which leads us to the choices of the second block rows of (5.5) and (5.10). Similarly, one cannot be at a $\mathbf{B}$ state at threshold $T^0 = 0$ so policies to escape from this situation need to be in place. The choices of the second block rows of (5.7) and (5.12) stem from this observation. At this point, we complete the characterization of the FMFFQ for the process H_T while noting that there might be other characterizations for the same FMFFQ.

One can find the steady-state joint CDF $F_m(y) = \lim_{t \rightarrow \infty} P(H_T(t) \leq y, M(t) = m)$, $1 \leq m \leq M = d(s + 2)$ for the transformed process H_T . By conditioning on the $\mathbf{A}$ states during which the queue is decreasing, we can find the steady-state CDF $G_j(y) = \lim_{t \rightarrow \infty} P(H(t) \leq y, J(t) = j)$ ($1 \leq j \leq d$) for the original process H as follows:

$$G_j(y) = \frac{F_j(y)}{\sum_{j=1}^d F_j(\infty)}, \quad (5.13)$$

where $J(t)$ is the background process for the arrival MAP. Let $G(y)$ denote the steady-state joint CDF vector $G(y) = [G_1(y), \dots, G_d(y)]$. Let $\tilde{G}(y)$ denote the steady-state joint complementary CDF vector $\tilde{G}(y) = [\tilde{G}_1(y), \dots, \tilde{G}_d(y)]$, where $\tilde{G}_m(y) = \lim_{t \rightarrow \infty} P(H(t) > y, J(t) = m)$, $1 \leq j \leq d$, and therefore

$$\tilde{G}_j(y) = G_j(\infty) - G_j(y), \quad 1 \leq j \leq d. \quad (5.14)$$

Packet losses occur when $H(t)$ exceeds $T^{(K)} = KD$. Therefore, the packet blocking probability P_b is given by

$$P_b = \frac{\tilde{G}(KD)D_1e}{\pi D_1e}. \quad (5.15)$$

The expression for the expected delay of a packet that is admitted into the optical buffer is then given by:

$$E[Delay] = \frac{\sum_{k=1}^K kD \left(\hat{G}(kD) - \hat{G}((k-1)D) \right)}{1 - P_b} \quad (5.16)$$

where

$$\hat{G}(y) = \frac{G(y)D_1e}{\pi D_1e} \quad (5.17)$$

denotes the steady-state probability that an arbitrary arriving packet finds less than y amount of unfinished work in the system. The steady-state probability that the unfinished work in the system is less than y at an arbitrary epoch is obviously different for MAP arrivals and it is equal to $G(y)e$ where e is a column vector of ones.

5.3 Numerical Results

In this section, we will provide several numerical examples to verify and validate the approach proposed in this chapter.

5.3.1 Example 1

As the first example, we study a degenerate optical buffer under Poisson input with intensity λ and exponential packet lengths which is the same problem studied in [12]. For the sake of convenience, the mean packet lengths are normalized to unity and λ is set to 0.8. For three different values of K , we find the packet

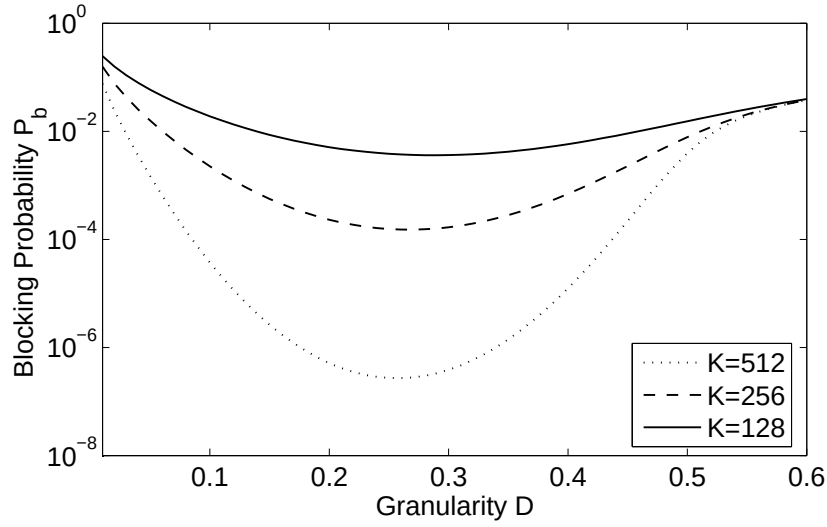


Figure 5.3: Packet blocking probability P_b as a function of the granularity D for varying values of the buffer length K .

blocking probabilities for different values of the granularity parameter D . The results are given in Fig. 5.3. The results are in accordance with the simulation results given in [12] and we show that we can obtain very accurate results even for very large K demonstrating the numerical stability of the proposed approach. It is also clear that the optimal granularity parameter under which the loss probability is maximized is sensitive to the buffer length K ; minimal blocking probabilities are achieved when the granularity parameter is set to $D = 0.29$, 0.27 , and 0.26 , corresponding to the cases of $K = 128$, 256 , and 512 , respectively.

5.3.2 Example 2

In this example, we find the steady-state probability density function (PDF) of the unfinished work in the system, denoted by $\hat{g}(y) = d\hat{G}(y)/dy$, as a function of the unfinished work y . Note that, all performance measures of interest can be derived from this PDF. We consider a degenerate optical buffer comprising $K = 2$ FDLs, with $D = 1$. The packet lengths are of PH-type distribution that

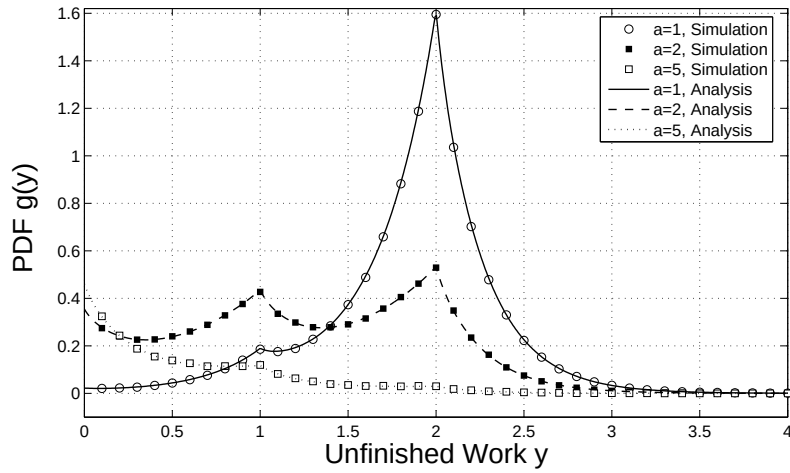


Figure 5.4: Unfinished work PDF $\hat{g}(y)$ for Example 2 for varying values of a .

is characterized with the pair (v, S) :

$$S = \begin{bmatrix} -20 & 10 \\ 10 & -10 \end{bmatrix}, \quad v = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix},$$

for which the mean packet length is 0.25. Moreover, the optical packet arrival process is assumed to be a MAP characterized with the pair (D_0, D_1)

$$D_0 = \frac{1}{a} \begin{bmatrix} -5 & 1 & 1 \\ 1 & -5 & 1 \\ 1 & 1 & -5 \end{bmatrix}, \quad D_1 = \frac{1}{a} \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

where a is a varying parameter. The mean packet interarrival time is written as $3/a$ which amounts to a load of $\rho = a/12$. Fig. 5.4 illustrates the unfinished work PDF $\hat{g}(y)$ as a function of y for varying values of a and we show that the analytical results are in perfect accordance with the simulation results. This example demonstrates that we not only find the blocking probabilities but also the entire PDF by means of the proposed approach for general PH-distributed packet lengths and MAP packet arrivals. In the next examples, we would like to draw conclusions on how packet length distributions and packet interarrival distributions impact the performance of the system and see if we can build provisioning guidelines for optical buffers.

5.3.3 Example 3

In this example, we study the impact of the Squared Coefficient of Variation (SCV) of packet lengths on blocking performance. The SCV of a random variable is the variance divided by the squared mean of that random variable and is indicative of its variability. For Poisson arrivals, we have $\text{SCV}=1$. On the other hand, the Erlang- k distribution with k phases with $\text{SCV} = 1/k$ can be used to model scenarios when $\text{SCV} < 1$. The scenario $\text{SCV} > 1$ can be modeled by using an appropriate two-phase hyperexponential distribution (denoted by H_2) with balanced means [24, pages 58–59]. Both the Erlang- k and hyperexponential distributions are of PH-type and can be addressed in the framework of the current chapter. We fix $K = 20$ and we plot the packet blocking probability as a function of the granularity D for different values of SCV under three different loads $\rho = 0.8, 0.6, 0.4$. The results are also in line with the results obtained by [53]. We have the following observations.

- As the packet length SCV increases, then the packet loss probabilities tend to increase for a wide range of the granularity parameter D which leads us to believe that packetization policies attempting to minimize the SCV are needed to address this performance impact. This relationship appears to be reversed for very large values of D but this particular regime of large granularities needs to be avoided due to high blocking probabilities and delays arising in this regime.
- The optimal granularity parameter that leads to minimal blocking probabilities, say D^* , decreases with increased load and increased packet length SCV.

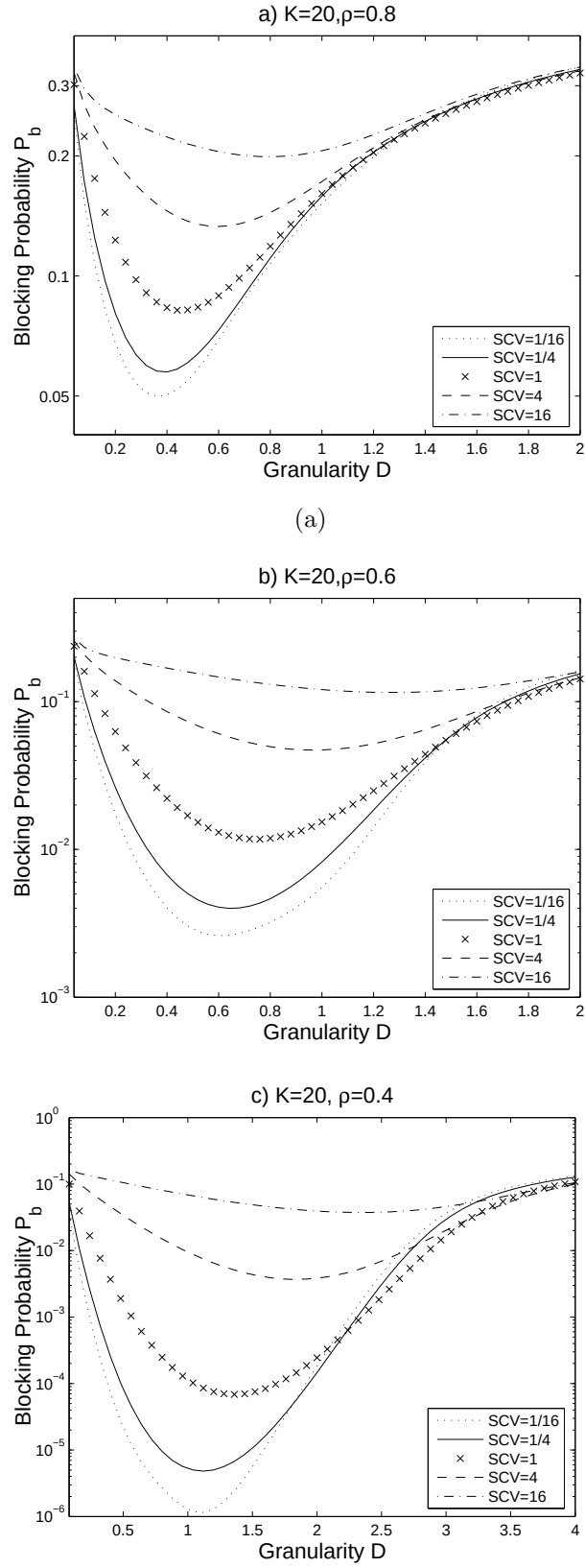


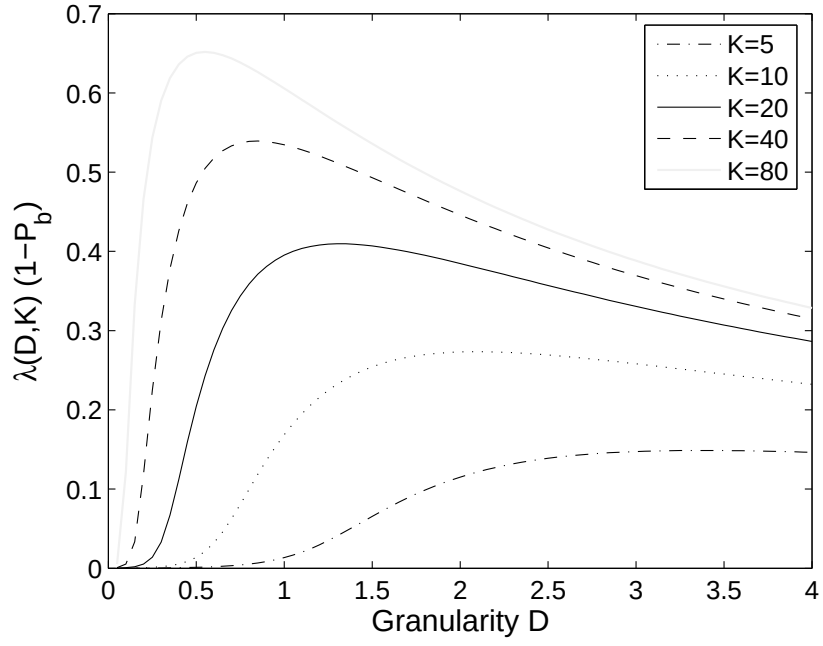
Figure 5.5: Blocking probability P_b for $K = 20$ as a function of the granularity parameter D for different values of packet length SCV a) $\rho = 0.8$ b) $\rho = 0.6$ c) $\rho = 0.4$.

5.3.4 Example 4

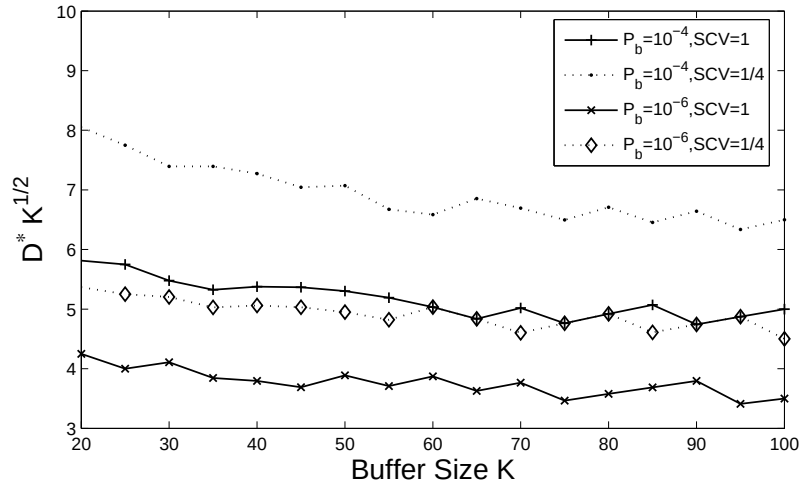
In this example, we focus on Poisson packet arrivals and exponential packet lengths and we then study the choice of the optimal granularity parameter D^* as a function of the buffer size K that maximizes the throughput of the system under a desired blocking probability of P_b . For this purpose, for a given K and for varying D , we gradually change the Poisson arrival intensity such that the desired blocking probability P_b is achieved and we denote this particular intensity by $\lambda(D, K, P_b)$. The throughput of this system is then given by $\lambda(D, K, P_b)(1 - P_b)$. The optimal granularity D^* is the value of D that maximizes $\lambda(D, K, P_b)(1 - P_b)$ for a given K and desired blocking probability P_b . We fix $P_b = 10^{-4}$ and we plot $\lambda(D, K, P_b)(1 - P_b)$ for exponential packet lengths in Fig. 5.6a as a function of D and for various values of K . It is clear that D^* decreases with increasing K . For this problem, we also seek a provisioning guideline for properly choosing the granularity parameter D . For this purpose, we also plot $D^*\sqrt{K}$ as a function of the buffer size K in Fig. 5.6b for various values of P_b and packet length SCV. Our results show that the function $D^*\sqrt{K}$ has a relatively narrow dynamic range for varying K and the optimal granularity parameter D^* approximately behaves as

$$D^* \propto K^{-1/2} \quad (5.18)$$

for the four scenarios we studied. The proportionality constant in the above equation appears to decrease with decreasing P_b and with increasing packet length SCV. To give an example, let us fix $P_b = 10^{-4}$ and SCV=1. From Fig. 5.7b, K^* behaves approximately $5/\sqrt{K}$ for relatively large values of K and when $K = 80$, the estimate of the granularity parameter equals $5/\sqrt{80} = 0.56$ which is very close to the actual optimal value as depicted in Fig. 5.6a.



(a)



(b)

Figure 5.6: The plot of a) $\lambda(D, K, P_b)(1 - P_b)$ as a function of D for various K b) $D * \sqrt{K}$ as a function of the buffer size K , both for a desired blocking probability of $P_b = 10^{-4}$.

5.3.5 Example 5

This example is similar to the previous example except that we have a constraint on the total fiber size L to be used for FDLs in a way that $L = DK(K + 1)/2$. Again we focus on Poisson packet arrivals and exponential packet lengths and we then study the choice of the optimal granularity parameter D^* this time as a function of the fiber size L that maximizes the throughput of the system for a desired blocking probability P_b . For this purpose, for a given L and for varying K , we vary the Poisson arrival intensity until a desired blocking probability P_b is achieved and we denote this particular intensity by $\lambda^0(L, K, P_b)$. The throughput of this system is then given by $\lambda^0(L, K, P_b)(1 - P_b)$. The optimal buffer size K^* is the value of K that maximizes $\lambda^0(L, K, P_b)(1 - P_b)$ for a given L and desired blocking probability P_b . We fix $P_b = 10^{-4}$ and we plot $\lambda^0(L, K, P_b)(1 - P_b)$ for exponential packet lengths in Fig. 5.7a as a function of K and for various values of L . It is clear that K^* increases with increasing fiber size L . It is interesting to note that there are cases when the achievable throughput drops as a result of an increase in the total fiber size L if the buffer size stays K fixed. This observation shows that the buffer size selection is very critical. One of the goals of this work is to find a provisioning guideline for buffer size selection. For this purpose, we also plot $(K^*)^{-3/2}L$ as a function of the fiber size L in Fig. 5.7b for various values of P_b and packet length SCV. Our results show that the function $(K^*)^{-3/2}L$ has a relatively narrow dynamic range for varying L and the optimal buffer size parameter K^* behaves as

$$K^* \propto L^{2/3} \quad (5.19)$$

for the scenarios we studied. The proportionality constant in the above equation appears to decrease with decreasing P_b and with increasing SCV. To give an example, let us fix $P_b = 10^{-4}$ and SCV=1. From Fig. 5.7b, K^* behaves approximately $2^{-2/3}L^{2/3}$ for relatively large values of L and when $L = 200$, the estimate

of the optimal buffer size becomes $2^{-2/3}200^{2/3} = 21.54$ which is very close to the actual optimal value as depicted in Fig. 5.7a.

5.3.6 Example 6

In this example, we study the impact of autocorrelation on packet blocking probability. The interarrival distribution is hyperexponential with balanced means but the autocovariance function $C(k)$ of the interarrival times is of the form $A\sigma^k$. Such an arrival process can be obtained by a two-phase MAP using the procedure of [27]. The larger the parameter σ , the more dominant the autocorrelation is. We assume exponential packet lengths again with unity mean. We fix $\rho = 0.25$ and plot the blocking probability P_b against D for various values of σ in Fig. 5.8. It is clear that autocorrelation adversely impacts the blocking rates but the optimal granularity parameter also decreases with increased σ . We leave a more detailed discussion of auto-correlated traffic modeling in optical packet switched networks for future research.

5.4 Conclusion

We provide an exact analytical procedure to solve single-wavelength optical buffers. The procedure reduces to the solution of a linear matrix equation of size $K(s+3)d + 2d$ which further reduces to the solution of an equation of size $4K + 2$ when arrivals are Poisson and packet lengths are exponential. We provide numerical examples to give insight for the operation of the optical buffer and furthermore we provide provisioning guidelines for optical buffers for improved performance.

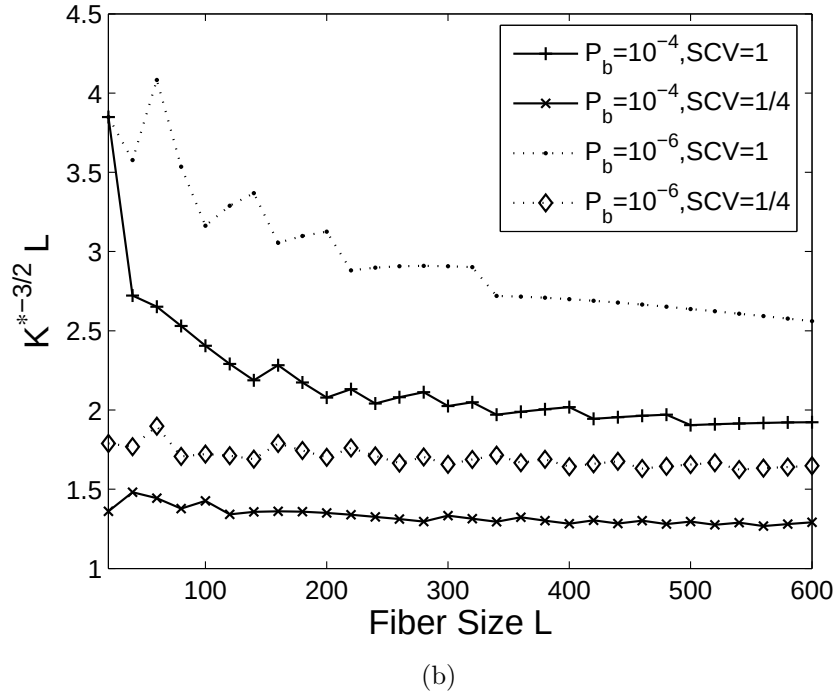
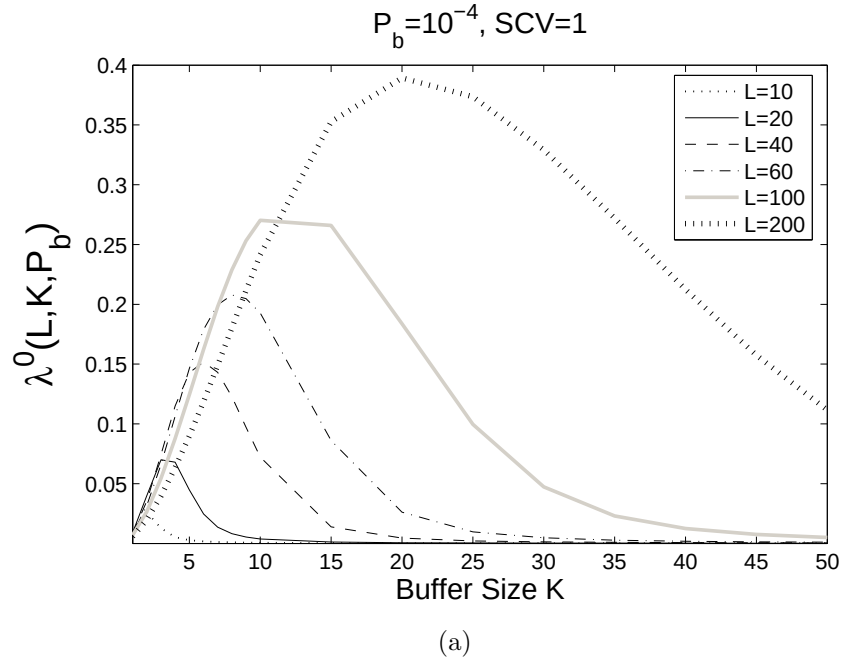


Figure 5.7: The plot of a) $\lambda^0(L, K, P_b)(1 - P_b)$ as a function of K for various values of the total fiber size L when $P_b = 10^{-4}$ and $\text{SCV} = 1$ b) $(K^*)^{-3/2}L$ as a function of the total fiber size L for two different values of P_b and packet length SCV .

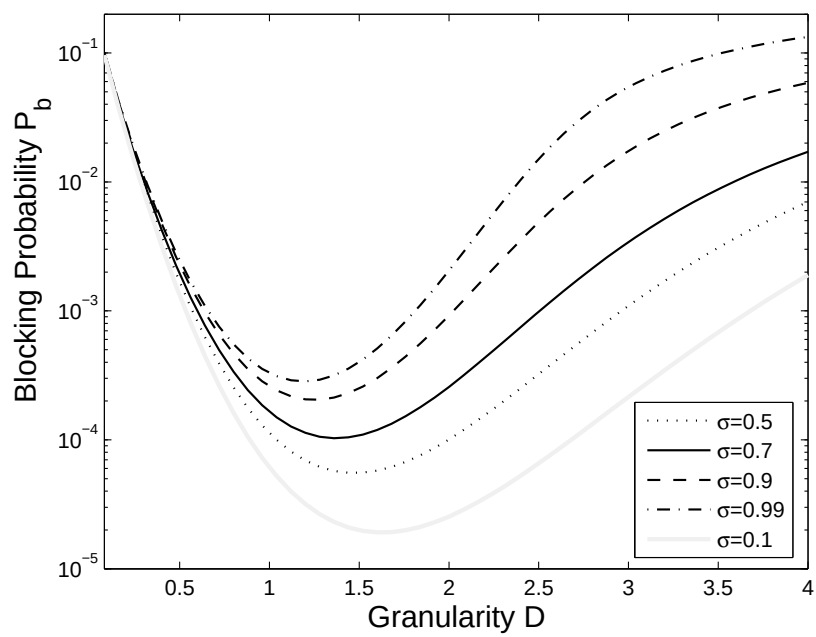


Figure 5.8: Blocking probability P_b against D for various values of σ when $\lambda = 0.25$.

Chapter 6

HYSTERETIC CONTROL IN FIRST-ORDER MULTI-REGIME FEEDBACK FLUID QUEUES

6.1 Introduction

This chapter studies the steady-state behavior of first-order multi-regime feedback fluid queues with hysteretic control (FMFFQ-H). Recall that in FMFFQs, thresholds or the so called boundary points partition the queue level into several regimes and distinct sets of parameters (i.e., drift values and infinitesimal generators) are assigned to each regime. Unlike FMFFQs, each regime in FMFFQ-Hs is separated by two thresholds called *forward* and *reverse thresholds*. If the queue content exceeds the forward threshold, then the upper neighboring regime is reached. If we want to go back to the previous regime, the queue content should reach the reverse threshold. Clearly, hysteresis methods reduce the frequency of

policy switchings employed for a specific problem. In this chapter, we study two performance measures of FMFFQ-H in the steady-state: 1) joint probability density functions of the queue occupancy and 2) the expected number of crossings of the thresholds in unit time (indicative of policy switching frequency).

The analysis of FMFFQ-H is important because of two reasons: First, FMFFQ is a practical model in the performance evaluation of communication network- for example see the references [18], [19], [67] and [39]- and our results makes the analysis of systems containing hysteretic control available, see [25], [26]. Second, our studies in Chapter 3 show that the steady-state unfinished work in some basic queueing systems can be derived from the solution of FMFFQ with the use of a simple transformation and it is not difficult to see that the solution of FMFFQ-H yields the solution of the unfinished work in the same basic queueing systems with hysteretic control by the same transformation idea. For example, we solve the steady-state probability density function and the expected number of policy switchings in unit time of an M/M/1 system with a hysteretic admission control as a case study at the end of this chapter.

To our best knowledge, there has not been any previous attempt to solve any of the two performance measures that we deal with, neither approximately nor exactly in the literature. However, the interested reader might see the references [45] and [59] for problems related to the case study.

The organization of this chapter is as follows. Section 6.2 describes the stochastic model, in detail. Section 6.3 presents a transformation of this model to an FMFFQ and gives all the necessary parameters of the transformed process. Moreover, this section derives performance measures of FMFFQ-H described above. Section 6.4 is devoted to a case study which requires two types of transformations, one of which is the one we will propose in this chapter, and the other has already described in the previous chapters. Finally, in Section 6.5 we present our conclusions.

6.2 Stochastic Model

Let $0 = T_{up}^{(0)} < T_{down}^{(1)} < T_{up}^{(1)} < T_{down}^{(2)} < T_{up}^{(2)} < \dots < T_{down}^{(K)}$ denote the boundary points in the content process. The infinitesimal generators and the drifts are constants in the regimes $(T_{up}^{(k-1)}, T_{down}^{(k)})$ for $k = 1, 2, \dots, K$, let us call them *static regimes* (SR). $Q^{(k)}$ and $R^{(k)}$ denote the infinitesimal generators and the drift matrices in the k^{th} static regime, $(T_{up}^{(k-1)}, T_{down}^{(k)})$. The remaining regimes, i.e., $(T_{down}^{(k)}, T_{up}^{(k)})$ for $k = 1, 2, \dots, K - 1$, are referred to as *hysteresis regimes* (HR). Hence, each SR is followed by an HR and vice versa; and there are K SRs and $K - 1$ HRs in the queue. In an HR $(T_{down}^{(k)}, T_{up}^{(k)})$, the parameter set is the same as that of the last visited neighboring SR. Therefore, the parameters of these regimes alter over time depending on the last visited SR.

In FMFFQs intermediate boundary points are used to describe the behavior at the transition point between neighboring regimes. For example, recall that at a background state of an FMFFQ, if the drift is negative in a regime but it is positive in the lower neighboring regime and the content touches the boundary point in between these regimes, we use the assumption that the content process stays at the intermediate boundary point until a state transition occurs in the background process. Unlike FMFFQs, in FMFFQ-Hs, the location of the transition point between the regimes is not fixed. The location of that point switches between $T_{up}^{(k)}$ and $T_{down}^{(k)}$ for HR $(T_{down}^{(k)}, T_{up}^{(k)})$, depending on history of the content process. For simplicity, we impose the assumption that the content can not stay at that boundary point, in other words, we do not allow any absorption or repulsion behavior at an intermediate boundary point. Let the infinitesimal generator and the drifts be denoted by $\tilde{Q}^{(k)}$ and $\tilde{R}^{(k)}$, respectively, at that boundary point. Finally, the background process as usual has a finite state-space $\{1, 2, \dots, M\}$.

Consider FMFFQ-H in Fig. 6.1. There are two SRs and an HR in between them. The background process has two switching states. At states 1 and 2, the

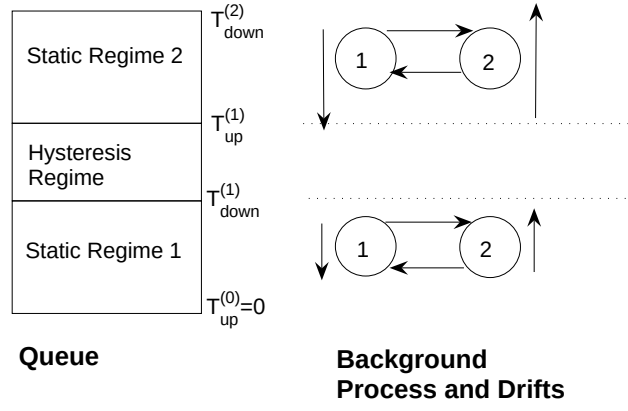


Figure 6.1: A first-order multi-regime feedback fluid queue with hysteretic control.

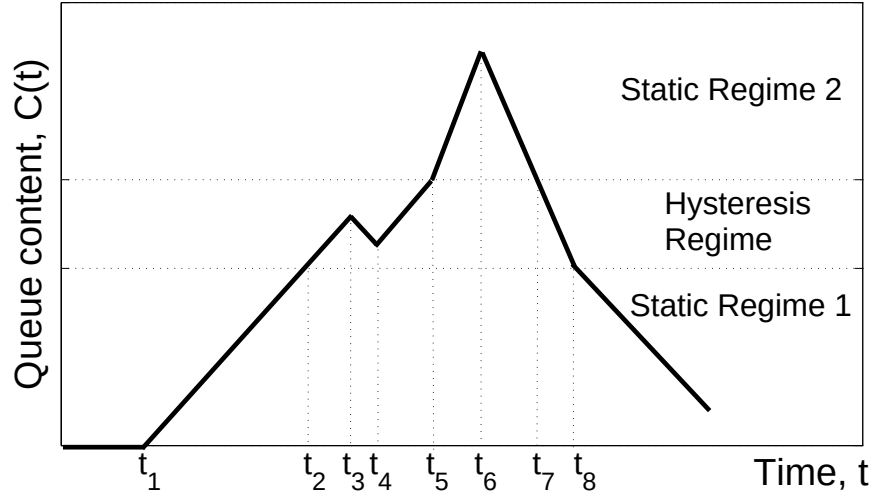


Figure 6.2: Evolution of the content process of FMFFQ-H in Fig. 6.1.

content process moves downward and upward, respectively. The change of the content in regime 2 is twice as fast as regime 1.

In Fig. 6.2, we show the evolution of the sample path of the content process, say $C(t)$, in the FMFFQ-H in Fig. 6.1. As can be seen from the figure, before t_2 the content is in SR 1, therefore between t_2 and t_5 HR behaves the same as SR 1 and the movement of the content is slow. However, the queue enters SR 2 at t_5 and it goes back to HR at t_7 . During the interval (t_7, t_8) since the last visited regime is SR 2, HR becomes the same as SR 2 and the movement is fast in this interval.

6.3 Proposed solution

As mentioned above, our first aim is to find the steady-state joint distribution functions of FMFFQ-Hs. To achieve this goal, similar to the previous solutions in this thesis, we modify the sample path of the content process, $C(t)$ and obtain the sample path of the content process of an FMFFQ denoted by $C_T(t)$. Yet, unlike previously analyzed processes, there is not any jump in this process which is consistent with the sample paths of usual fluid queues. Rather one can see dynamic regimes which have to be transformed. The modification procedure consists of two steps. In the first step, we replace HR with two regimes of the same length. In the second step, we define a new collection of states in the background process. With the use of these states, we assure that the content process of the transformed queue follows the same dynamic behavior as FMFFQ-H. Next, we elaborate on these two steps.

In the first step, we replace each dynamic HR with two static regimes, as shown in Fig. 6.3. We refer to these regimes as *upper* and *lower replacement regimes* (URRs and LRRs). We name the other regimes as *static regimes* (SRs), as in the case of FMFFQ-Hs.

As a second step, in the background process we define four classes of states: S_{odd} , S_{even} , T_{URR} and, T_{LRR} each with a cardinality of M . There are two types of functions assigned to those states: i) transmission of the content through replacement regimes; and ii) the usual movements of the content process. T_{URR} and T_{LRR} are used to execute the first function through URRs and LRRs, respectively. S_{odd} and S_{even} are defined to fulfill the second function in the odd and even numbered SRs, respectively, and their neighboring replacement regimes, as well. URRs and LRRs are going to be the replicas of the lower and the upper neighboring SRs of HRs, respectively, in terms of the drifts and the infinitesimal generators. Let us assume that while the content is moving through an SR, k ,

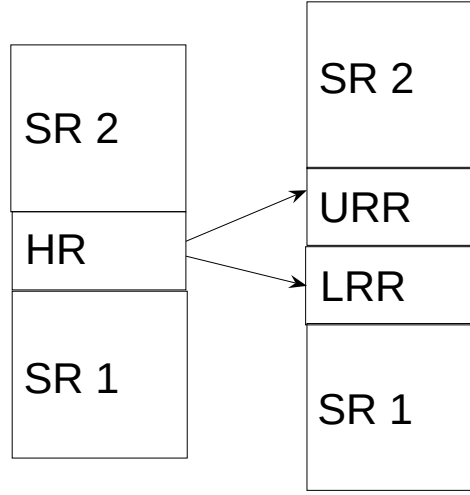


Figure 6.3: First step in the solution.

at a state in S_{odd} , it arrives at the upper boundary point of that regime. In FMFFQ-H, this action is equal to arriving at HR from below; thus we expect that regime to follow the same parameters of the regime from which the content arrives. In the transformed process we let these parameters to be operated in URR. Therefore, in the transformed process we force the content to bypass LRR and arrive at URR and continue to move with the same characteristics of the regime k . The operation is inverted when the content arrives at URR from above, SR $k + 1$. In this scenario we want the queue to follow the same characteristics of SR $k + 1$ within a regime whose length is equal to the hysteresis regime. Here, LRR plays the key role to accomplish this mission. However, before that the content has to reach LRR. T_{URR} is used to bypass the content through URR and arrive at LRR. In LRR the queue behavior is the same as SR $k + 1$. Now, assume that while the content is moving in LRR somehow a transition occurs to a state with a positive drift and the content goes up to the boundary point between URR and LRR. In this case, we want the queue to bypass URR and continue its movement in SR $k + 1$. Again the bypassing is implemented by using T_{URR}

$S_{even,1}$. During time intervals (t_2, t_5) and (t_7, t_8) $C(t)$ is in the hysteresis regime and our modification plays the key role in this regime. Before t_2 (t_7), the last visited regime is SR 1 (2), therefore the hysteresis regime behaves as if it was in SR 1 (2) during the interval (t_2, t_5) ((t_7, t_8)). We carry the content process to URR (LRR) once it touches LRR (URR) from below (above) utilizing the states $T_{LRR,2}$ ($T_{URR,1}$). Now, focus on Fig. 6.4. At time t'_2 when the background state is $S_{odd,2}$, $C_T(t)$ touches $T_{down}^{(1)}$, hence, we set the drift of this state as zero at $T_{down}^{(1)}$ and allow transition to only state $T_{LRR,2}$ in order to force the background state to make a transition that state at $T_{down}^{(1)}$. At that state, we set the drift equal to 1 and the corresponding row of the infinitesimal generator to all zeros when $C_T(t)$ is between $T_{down}^{(1)}$ and $T_{up}^{(1)}$. This parameter configuration forces $C_T(t)$ to increase up to $T_{up}^{(1)}$ without any state transition in the background process. Furthermore, for that state we set the drift as zero at $T_{up}^{(1)}$ and allow transition to only state $S_{even,2}$. In URR (LRR) the queue behaves the same as SR 1 (2). Hence, in URR (LRR) the states $S_{even,1}$ and $S_{even,2}$ ($S_{odd,1}$ and $S_{odd,2}$) follow the same transition properties and the drift parameters as $S_{odd,1}$ and $S_{odd,2}$ ($S_{even,1}$ and $S_{even,2}$) in SR 1 (2). At t'_{10} when the background process is at state $S_{even,1}$, the content touches URR from above. Now we follow a similar procedure to carry the content from $2T_{up}^{(1)} - T_{down}^{(1)}$ down to $T_{up}^{(1)}$. First, we set the drift of state $S_{even,1}$ at $2T_{up}^{(1)} - T_{down}^{(1)}$ as zero and we allow transition to only state $T_{URR,1}$ at this point. Then we set the drift of state $T_{URR,1}$ as -1 and we don't allow any transition from this state while the content is in URR. And finally we assign zero drift to that state at $T_{up}^{(1)}$ and we allow a transition only to state $S_{odd,1}$. Therefore after a finite random amount of time we will surely observe the content level at $S_{odd,2}$ at state $S_{odd,1}$.

In the example illustrated in Figs. 6.2 and 6.4 we used states $T_{LRR,2}$ and $T_{URR,1}$ to carry the content process. In Figs. 6.5 and 6.6, we consider another scenario to describe the role of $T_{LRR,1}$. The role of state $T_{URR,2}$ can be easily figured out after this description. In Fig. 6.5, the content increases at state 2 and

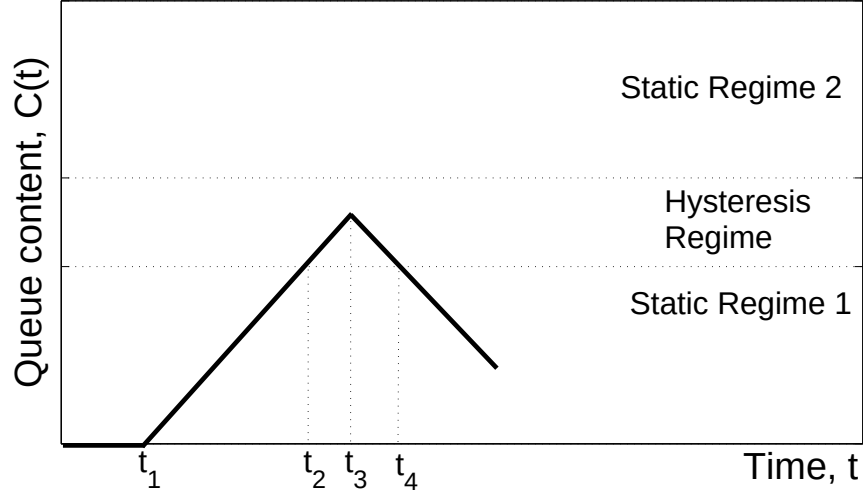


Figure 6.5: Evolution of the content process of FMFFQ-H in Fig. 6.1.

then enters the hysteresis regime at time t_2 . But, after a short amount of time the background process makes a transition to state 1, thus the content starts to decrease, and then it returns back to SR 1 without entering SR 2. Hence during this time, the parameters of SR 2 don't play a role. In Fig. 6.6, we illustrate the transformed process. The evolution is the same as Fig. 6.4 during the interval $(0, t'_6)$. At t'_6 there is a transition to state $S_{even,1}$ and the content starts to decrease until it reaches $T_{up}^{(1)}$. At this point, we assign the drift of state $S_{even,1}$ as 0 and allow transition to only state $T_{LRR,1}$. We assume that the drift of state $T_{LRR,1}$ between $T_{up}^{(1)}$ and $T_{down}^{(1)}$ is -1 and transition is not allowed. Therefore, the content moves from $T_{up}^{(1)}$ to $T_{down}^{(1)}$ steadily. At $T_{down}^{(1)}$ we set the drift of state $T_{LRR,1}$ equal to zero and we allow a transition to only state $S_{odd,1}$. By this way we carry the content from URR to SR 1.

In the sample paths illustrated in Figs. 6.4 and 6.6 we show the transformed content in dashed lines when the background process is in $T_{LRR,1}$, $T_{LRR,2}$, $T_{URR,1}$ or $T_{URR,2}$; and when the drift is zero at an intermediate boundary point. Backward transformation can be achieved in two steps: 1) drawing off the dashed lines; 2) subtracting $T_{up}^{(1)} - T_{down}^{(1)}$ from the transformed content $C_T(t)$, when it is in either URR or SR 2.

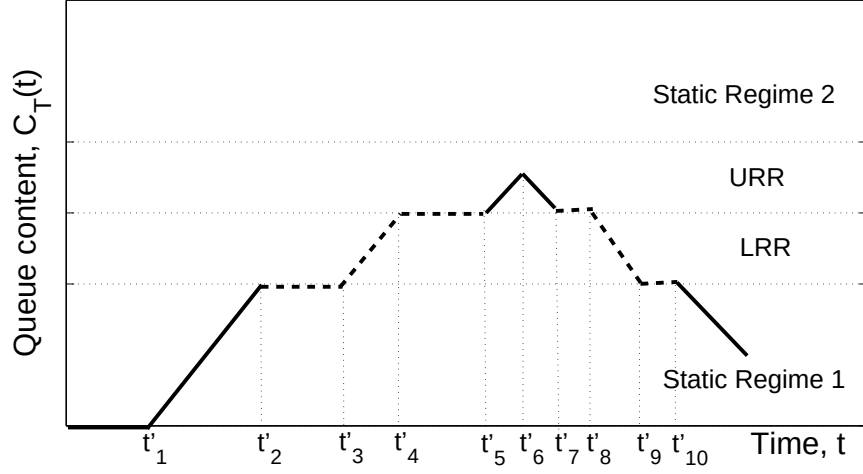


Figure 6.6: Evolution of the transformed process for the sample path in Fig. 6.5.

Next, we determine the parameters of the transformed process. In an FMFFQ-H we defined SR k as the open interval between $T_{up}^{(k-1)}$ and $T_{down}^{(k)}$ and similarly we defined HR k as the open interval between the boundary points $T_{down}^{(k)}$ and $T_{up}^{(k)}$. Recall that based on the first step of our solution procedure, we replace each hysteresis regime with two regimes which we referred to as URR and LRR. Each of these replacement regimes has the same queue length as the hysteresis regime which is replaced with. Therefore, assuming that there are K SRs and $K - 1$ HRs in the queue, the transformed process is going to have K SRs, $K - 1$ URRs and $K - 1$ LRRs. The intervals corresponding to each regime is given below.

- Regime $3k - 2$ for $k = 1, \dots, K$ (SRs):

$$\left(T_{up}^{(k-1)} + \sum_{i=1}^{k-1} (T_{up}^{(i)} - T_{down}^{(i)}), T_{down}^{(k)} + \sum_{i=1}^{k-1} (T_{up}^{(i)} - T_{down}^{(i)}) \right),$$

- Regime $3k - 1$ for $k = 1, \dots, K - 1$ (LRRs):

$$\left(T_{down}^{(k)} + \sum_{i=1}^{k-1} (T_{up}^{(i)} - T_{down}^{(i)}), T_{down}^{(k)} + \sum_{i=1}^k (T_{up}^{(i)} - T_{down}^{(i)}) \right),$$

- Regime $3k$ for $k = 1, \dots, K - 1$ (URRs):

$$\left(T_{down}^{(k)} + \sum_{i=1}^k (T_{up}^{(i)} - T_{down}^{(i)}), T_{up}^{(k)} + \sum_{i=1}^k (T_{up}^{(i)} - T_{down}^{(i)}) \right).$$

Without loss of generality we assume that K is an even number.

In the background process of FMFFQ-H we assume that there are M states. In the second step of our solution, as mentioned before, we define $2M$ *auxiliary states* to carry the content process through upper or lower replacement regimes and $2M$ states to follow the usual changes in the content. So, in the transformed process there are totally $4M$ states: $S_{odd} \cup S_{even} \cup T_{URR} \cup T_{LRR}$. We represent the new set of states in the background process of the transformed queue as $M_T = \{S_{odd}, S_{even}, T_{URR}, T_{LRR}\}$. The role of the first $2M$ states is the same as the states of FMFFQ-H. The states belonging to T_{LRR} (T_{URR}) are responsible for the two-way transitions of the content process between SR k ($k+1$) and URR (LRR) k . For the sake of accessibility, we do the following partitions on the infinitesimal generator and the drifts of the transformed process.

$$R_T^{(k)} = \text{diag} \left(R_{S_{odd}}^{(k)}, R_{S_{even}}^{(k)}, R_{T_{URR}}^{(k)}, R_{T_{LRR}}^{(k)} \right), \quad (6.1)$$

$$Q_T^{(k)} = \begin{bmatrix} Q_{S_{odd}S_{odd}}^{(k)} & Q_{S_{odd}S_{even}}^{(k)} & Q_{S_{odd}T_{URR}}^{(k)} & Q_{S_{odd}T_{LRR}}^{(k)} \\ Q_{S_{even}S_{odd}}^{(k)} & Q_{S_{even}S_{even}}^{(k)} & Q_{S_{even}T_{URR}}^{(k)} & Q_{S_{even}T_{LRR}}^{(k)} \\ Q_{T_{URR}S_{odd}}^{(k)} & Q_{T_{URR}S_{even}}^{(k)} & Q_{T_{URR}T_{URR}}^{(k)} & Q_{T_{URR}T_{LRR}}^{(k)} \\ Q_{T_{LRR}S_{odd}}^{(k)} & Q_{T_{LRR}S_{even}}^{(k)} & Q_{T_{LRR}T_{URR}}^{(k)} & Q_{T_{LRR}T_{LRR}}^{(k)} \end{bmatrix}, \quad (6.2)$$

$$\tilde{R}_T^{(k)} = \text{diag} \left(\tilde{R}_{S_{odd}}^{(k)}, \tilde{R}_{S_{even}}^{(k)}, \tilde{R}_{T_{URR}}^{(k)}, \tilde{R}_{T_{LRR}}^{(k)} \right) \text{ and} \quad (6.3)$$

$$\tilde{Q}_T^{(k)} = \begin{bmatrix} \tilde{Q}_{S_{odd}S_{odd}}^{(k)} & \tilde{Q}_{S_{odd}S_{even}}^{(k)} & \tilde{Q}_{S_{odd}T_{URR}}^{(k)} & \tilde{Q}_{S_{odd}T_{LRR}}^{(k)} \\ \tilde{Q}_{S_{even}S_{odd}}^{(k)} & \tilde{Q}_{S_{even}S_{even}}^{(k)} & \tilde{Q}_{S_{even}T_{URR}}^{(k)} & \tilde{Q}_{S_{even}T_{LRR}}^{(k)} \\ \tilde{Q}_{T_{URR}S_{odd}}^{(k)} & \tilde{Q}_{T_{URR}S_{even}}^{(k)} & \tilde{Q}_{T_{URR}T_{URR}}^{(k)} & \tilde{Q}_{T_{URR}T_{LRR}}^{(k)} \\ \tilde{Q}_{T_{LRR}S_{odd}}^{(k)} & \tilde{Q}_{T_{LRR}S_{even}}^{(k)} & \tilde{Q}_{T_{LRR}T_{URR}}^{(k)} & \tilde{Q}_{T_{LRR}T_{LRR}}^{(k)} \end{bmatrix}. \quad (6.4)$$

We specify the parameters in (6.1) through (6.4) for each regime and boundary point below.

6.3.1 Parameter set in SR

In the transformed process, we dictate the drifts and the infinitesimal generators in SRs at states belong to S_{odd} and S_{even} be the same as the ones in the odd and

even numbered regimes of the original process, respectively. Hence

$$R_{S_{odd}}^{(k)} = R^{(l)} \text{ and } Q_{S_{odd}S_{odd}}^{(k)} = Q^{(l)}, \quad k = 3l - 2, \quad l = 1, 3, \dots, K - 1, \quad (6.5)$$

$$R_{S_{even}}^{(k)} = R^{(l)} \text{ and } Q_{S_{even}S_{even}}^{(k)} = Q^{(l)}, \quad k = 3l - 2, \quad l = 2, 4, \dots, K. \quad (6.6)$$

Note that we assumed an even K in (6.5) and (6.6).

We don't allow any transition from the states in S_{odd} and S_{even} to any other class. Therefore

$$\begin{aligned} Q_{S_{odd}S_{even}}^{(k)} &= Q_{S_{odd}T_{URR}}^{(k)} = Q_{S_{odd}T_{LRR}}^{(k)} = Q_{S_{even}S_{odd}}^{(k)} = Q_{S_{even}T_{URR}}^{(k)} = \\ Q_{S_{even}T_{LRR}}^{(k)} &= 0_M, \quad k = 3l - 2, \quad l = 1, 2, \dots, K \end{aligned} \quad (6.7)$$

where 0_M is zero matrix of size $M \times M$.

For the remaining set of parameters we assume that SR below the specified LRR is an odd numbered one. For the other case, the equations from (6.8) to (6.73) are valid if we replace $l = 1, 3, \dots, K - 1$ with $l = 2, 4, \dots, K - 2$; and exchange S_{odd} and S_{even} .

6.3.2 Parameter set at the boundary point between LRR and SR

Once that boundary point is reached from below at states belong to S_{odd} with positive drifts, according to the construction of the solution, we want to carry the content from that boundary point to the one between lower and upper replacement regimes. The states of T_{LRR} with positive drifts are used to realize this transmission. So, the background process should make a transition to those states without any change in the content at that boundary point. Not to lose the state information during transmission we allow that transition from the m^{th} state of S_{odd} to the m^{th} state of T_{LRR} for any $1 \leq m \leq M$. This behavior can be captured by assigning the following parameter values.

For $m : \left(R_{S_{odd}}^{(k)}\right)_{m,m} > 0$, $k = 3l - 2$ and $l = 1, 3, \dots, K - 1$

$$\left(\tilde{R}_{S_{odd}}^{(k)}\right)_{m,m} = 0, \quad (6.8)$$

$$\left(\tilde{Q}_{S_{odd}S_{odd}}^{(k)}\right)_{m,n} = \begin{cases} -1, & n = m \\ 0, & \text{otherwise} \end{cases}, \quad n = 1, \dots, M, \quad (6.9)$$

$$\left(\tilde{Q}_{S_{odd}T_{LRR}}^{(k)}\right)_{m,n} = \begin{cases} 1, & n = m \\ 0, & \text{otherwise} \end{cases}, \quad n = 1, \dots, M, \quad (6.10)$$

$$\left(\tilde{Q}_{S_{odd}T_{URR}}^{(k)}\right)_{m,n} = \left(\tilde{Q}_{S_{odd}S_{even}}^{(k)}\right)_{m,n} = 0, \quad n = 1, \dots, M. \quad (6.11)$$

In equations (6.8) through (6.11) and in the remaining of this chapter we assume that, $(A)_{m,n}$ is the element of the matrix A in the m^{th} row and the n^{th} column. In equations (6.9) and (6.10) the values for $m = n$ don't affect the result provided the first one is a negative number and the second one is the absolute value of the first one. We select the first value as -1 without loss of generality.

The transition in the background state should be followed by a transmission in the content in upward direction, so for $m : \left(R_{S_{odd}}^{(k)}\right)_{m,m} > 0$, $k = 3l - 2$ and $l = 1, 3, \dots, K - 1$

$$\left(\tilde{R}_{T_{LRR}}^{(k)}\right)_{m,m} = 1, \quad (6.12)$$

$$\begin{aligned} \left(\tilde{Q}_{T_{LRR}S_{odd}}^{(k)}\right)_{m,n} &= \left(\tilde{Q}_{T_{LRR}S_{even}}^{(k)}\right)_{m,n} = \left(\tilde{Q}_{T_{LRR}T_{URR}}^{(k)}\right)_{m,n} = \\ \left(\tilde{Q}_{T_{LRR}T_{LRR}}^{(k)}\right)_{m,n} &= 0, \quad n = 1, \dots, M. \end{aligned} \quad (6.13)$$

In (6.12) the values of the drifts don't matter provided that they are positive numbers. Without loss of generality we set them equal to 1.

If the content reaches the boundary point between LRR and SR at a state in T_{LRR} with negative drift, then it means that the transmission of the content from URR to that boundary point is finished. During the transmissions through LRR so as to keep the state information if the background process was in the m^{th} state of S_{even} before the transmission, then the transmission is accomplished with the use of the m^{th} state of T_{LRR} . Furthermore, if the transmission ends, we

force the background process to make a transition to the m^{th} state of S_{odd} . For $m : \left(R_{S_{\text{odd}}}^{(k)}\right)_{m,m} < 0$, $k = 3l - 2$ and $l = 1, 3, \dots, K - 1$ this behavior can be realized with the selection of parameters:

$$\left(\tilde{R}_{T_{LRR}}^{(k)}\right)_{m,m} = 0, \quad (6.14)$$

$$\left(\tilde{Q}_{T_{LRR}T_{LRR}}^{(k)}\right)_{m,n} = \begin{cases} -1, & m = n \\ 0, & \text{otherwise} \end{cases}, \quad n = 1, \dots, M, \quad (6.15)$$

$$\left(\tilde{Q}_{T_{LRR}S_{\text{odd}}}^{(k)}\right)_{m,n} = \begin{cases} 1, & m = n \\ 0, & \text{otherwise} \end{cases}, \quad n = 1, \dots, M, \quad (6.16)$$

$$\left(\tilde{Q}_{T_{LRR}T_{URR}}^{(k)}\right)_{m,n} = \left(\tilde{Q}_{T_{LRR}S_{\text{even}}}^{(k)}\right)_{m,n} = 0, \quad n = 1, \dots, M. \quad (6.17)$$

At a state in S_{odd} with negative drift if the content reaches that boundary point from above then it behaves the same as the boundary point with dynamic location in $C(t)$. For $m : \left(R_{S_{\text{odd}}}^{(k)}\right)_{m,m} < 0$, $k = 3l - 2$ and $l = 1, 3, \dots, K - 1$

$$\left(\tilde{R}_{S_{\text{odd}}}^{(k)}\right)_{m,m} = \left(\tilde{R}^{(l)}\right)_{m,m}, \quad (6.18)$$

$$\left(\tilde{Q}_{S_{\text{odd}}S_{\text{odd}}}^{(k)}\right)_{m,n} = \left(\tilde{Q}^{(l)}\right)_{m,n}, \quad n = 1, \dots, M, \quad (6.19)$$

$$\left(\tilde{Q}_{S_{\text{odd}}T_{URR}}^{(k)}\right)_{m,n} = \left(\tilde{Q}_{S_{\text{odd}}T_{LRR}}^{(k)}\right)_{m,n} = \left(\tilde{Q}_{S_{\text{odd}}S_{\text{even}}}^{(k)}\right)_{m,n} = 0, \quad n = 1, \dots, M. \quad (6.20)$$

6.3.3 Parameter set in LRR

In this regime two class of states, namely S_{odd} and T_{LRR} , are active.

If the content reaches the lower boundary point of that regime from below (in this case the drift is positive in SR at the current state), recall that a transition occurs from the m^{th} state of S_{odd} to the m^{th} state of T_{LRR} . Then that new state transmits the content from that point to the boundary point between the this LRR and the neighboring URR. This transmission can be clearly realized

using the parameter set below. For $m : \left(R_{S_{odd}}^{(k)}\right)_{m,m} > 0$, $k = 3l - 1$ and $l = 1, 3, \dots, K - 1$

$$\left(R_{T_{LRR}}^{(k)}\right)_{m,m} = 1, \quad (6.21)$$

$$\begin{aligned} \left(Q_{T_{LRR}S_{odd}}^{(k)}\right)_{m,n} &= \left(Q_{T_{LRR}S_{even}}^{(k)}\right)_{m,n} = \left(Q_{T_{LRR}T_{LRR}}^{(k)}\right)_{m,n} = \\ \left(Q_{T_{LRR}T_{URR}}^{(k)}\right)_{m,n} &= 0, \quad n = 1, \dots, M. \end{aligned} \quad (6.22)$$

If the content has been carried to this regime from URR using the states in T_{URR} with a negative drift, a transition occurs to the states belonging to S_{odd} at the boundary point between the lower and the upper replacement regimes. In this scenario, the states in S_{odd} behave as if the content was in SR just above URR. Hence, for $k = 3l - 1$ and $l = 1, 3, \dots, K - 1$

$$R_{S_{odd}}^{(k)} = R^{(l+1)}, \quad (6.23)$$

$$Q_{S_{odd}S_{odd}}^{(k)} = Q^{(l+1)}, \quad (6.24)$$

$$Q_{S_{odd}S_{even}}^{(k)} = Q_{S_{odd}T_{URR}}^{(k)} = Q_{S_{odd}T_{LRR}}^{(k)} = 0_M. \quad (6.25)$$

If the content process reaches the boundary point between LRR and URR while it is moving in URR at a state in S_{even} with negative drift, then after a state transition to T_{LRR} it is transmitted to SR under LRR. So, for $m : \left(R_{S_{odd}}^{(k)}\right)_{m,m} < 0$, $k = 3l - 1$ and $l = 1, 3, \dots, K - 1$

$$\left(R_{T_{LRR}}^{(k)}\right)_{m,m} = -1, \quad (6.26)$$

$$\begin{aligned} \left(Q_{T_{LRR}S_{odd}}^{(k)}\right)_{m,n} &= \left(Q_{T_{LRR}S_{even}}^{(k)}\right)_{m,n} = \left(Q_{T_{LRR}T_{LRR}}^{(k)}\right)_{m,n} = \\ \left(Q_{T_{LRR}T_{URR}}^{(k)}\right)_{m,n} &= 0, \quad n = 1, \dots, M. \end{aligned} \quad (6.27)$$

6.3.4 Parameter set at the boundary point between URR and LRR

There are four ways to reach that point:

1) From below when the background process is at a state in T_{LRR} . In this way, the boundary point between LRR and SR is reached and therefore the content is transmitted through LRR, so it arrives at that point. In this scenario, we want the content process to adhere to that point until the background process makes a transition from the m^{th} state of T_{LRR} to the m^{th} state of S_{even} . Therefore, for $m : \left(R_{S_{\text{odd}}}^{(k)}\right)_{m, m} > 0, k = 3l - 1$ and $l = 1, 3, \dots, K - 1$

$$\left(\tilde{R}_{T_{LRR}}^{(k)}\right)_{m, m} = 0, \quad (6.28)$$

$$\left(\tilde{R}_{S_{\text{even}}}^{(k)}\right)_{m, m} = \left(R^{(l)}\right)_{m, m}, \quad (6.29)$$

$$\left(\tilde{Q}_{T_{LRR}T_{LRR}}^{(k)}\right)_{m, n} = \begin{cases} -1, & m = n \\ 0, & \text{otherwise} \end{cases}, \quad n = 1, \dots, M, \quad (6.30)$$

$$\left(\tilde{Q}_{T_{LRR}S_{\text{even}}}^{(k)}\right)_{m, n} = \begin{cases} 1, & m = n \\ 0, & \text{otherwise} \end{cases}, \quad n = 1, \dots, M, \quad (6.31)$$

$$\left(\tilde{Q}_{T_{LRR}S_{\text{odd}}}^{(k)}\right)_{m, n} = \left(\tilde{Q}_{T_{LRR}T_{URR}}^{(k)}\right)_{m, n} = 0, \quad n = 1, \dots, M, \quad (6.32)$$

$$\left(\tilde{Q}_{S_{\text{even}}S_{\text{even}}}^{(k)}\right)_{m, n} = \left(Q^{(l)}\right)_{m, n}, \quad n = 1, \dots, M, \quad (6.33)$$

$$\left(\tilde{Q}_{S_{\text{even}}S_{\text{odd}}}^{(k)}\right)_{m, n} = \left(\tilde{Q}_{S_{\text{even}}T_{URR}}^{(k)}\right)_{m, n} = \left(\tilde{Q}_{S_{\text{even}}T_{LRR}}^{(k)}\right)_{m, n} = 0, \quad n = 1, \dots, M. \quad (6.34)$$

2) From above when the background process is at a state in T_{URR} . The boundary point between URR and SR is reached from above when the content process is moving in downward direction and therefore the content is transmitted through URR, so it arrives at that point. In this scenario, we want the content process to adhere to that point until the background process makes a transition from the m^{th} state of T_{URR} to the m^{th} state of S_{odd} . Therefore, for $m : \left(R_{S_{\text{odd}}}^{(k)}\right)_{m, m} < 0, k = 3l - 1$ and $l = 1, 3, \dots, K - 1$

$$\left(\tilde{R}_{T_{URR}}^{(k)}\right)_{m, m} = 0, \quad (6.35)$$

$$\left(\tilde{R}_{S_{\text{odd}}}^{(k)}\right)_{m, m} = \left(R^{(l+1)}\right)_{m, m}, \quad (6.36)$$

$$\left(\tilde{Q}_{T_{URR}T_{URR}}^{(k)}\right)_{m,n} = \begin{cases} -1, & m = n \\ 0, & \text{otherwise} \end{cases}, \quad n = 1, \dots, M, \quad (6.37)$$

$$\left(\tilde{Q}_{T_{URR}S_{odd}}^{(k)}\right)_{m,n} = \begin{cases} 1, & m = n \\ 0, & \text{otherwise} \end{cases}, \quad n = 1, \dots, M, \quad (6.38)$$

$$\left(\tilde{Q}_{T_{URR}S_{even}}^{(k)}\right)_{m,n} = \left(\tilde{Q}_{T_{URR}T_{LRR}}^{(k)}\right)_{m,n} = 0, \quad n = 1, \dots, M, \quad (6.39)$$

$$\left(\tilde{Q}_{S_{odd}S_{odd}}^{(k)}\right)_{m,n} = \left(Q^{(l+1)}\right)_{m,n}, \quad n = 1, \dots, M, \quad (6.40)$$

$$\left(\tilde{Q}_{S_{odd}S_{even}}^{(k)}\right)_{m,n} = \left(\tilde{Q}_{S_{odd}T_{URR}}^{(k)}\right)_{m,n} = \left(\tilde{Q}_{S_{odd}T_{LRR}}^{(k)}\right)_{m,n} = 0, \quad n = 1, \dots, M. \quad (6.41)$$

3) From above when the background process is at a state in S_{even} . Here, the boundary point between URR and LRR is reached from above when the background process is at a state in S_{even} and therefore the content is going to be transmitted through LRR. In this scenario, we want the content process to adhere to that point until the background process makes a transition from the m^{th} state of S_{even} to the m^{th} state of T_{LRR} . Therefore, for $m : \left(R_{S_{odd}}^{(k)}\right)_{m,m} < 0$, $k = 3l - 1$ and $l = 1, 3, \dots, K - 1$

$$\left(\tilde{R}_{S_{even}}^{(k)}\right)_{m,m} = 0, \quad (6.42)$$

$$\left(\tilde{R}_{T_{LRR}}^{(k)}\right)_{m,m} = -1, \quad (6.43)$$

$$\left(\tilde{Q}_{S_{even}S_{even}}^{(k)}\right)_{m,n} = \begin{cases} -1, & m = n \\ 0, & \text{otherwise} \end{cases}, \quad n = 1, \dots, M, \quad (6.44)$$

$$\left(\tilde{Q}_{S_{even}T_{LRR}}^{(k)}\right)_{m,n} = \begin{cases} 1, & m = n \\ 0, & \text{otherwise} \end{cases}, \quad n = 1, \dots, M, \quad (6.45)$$

$$\left(\tilde{Q}_{S_{even}S_{odd}}^{(k)}\right)_{m,n} = \left(\tilde{Q}_{S_{even}T_{URR}}^{(k)}\right)_{m,n} = 0, \quad n = 1, \dots, M, \quad (6.46)$$

$$\begin{aligned} \left(\tilde{Q}_{T_{LRR}S_{odd}}^{(k)}\right)_{m,n} &= \left(\tilde{Q}_{T_{LRR}S_{even}}^{(k)}\right)_{m,n} = \left(\tilde{Q}_{T_{LRR}T_{LRR}}^{(k)}\right)_{m,n} = \\ \left(\tilde{Q}_{T_{LRR}T_{URR}}^{(k)}\right)_{m,n} &= 0, \quad n = 1, \dots, M. \end{aligned} \quad (6.47)$$

4) From below when the background process is at a state in S_{odd} . In this final way, the boundary point between URR and LRR is reached from below when

the background process is at a state in S_{odd} and therefore the content is going to be transmitted through URR. In this scenario, we want the content process to stick to that point until the background process makes a transition from the m^{th} state of S_{odd} to the m^{th} state of T_{URR} . Therefore, for $m : \left(R_{S_{odd}}^{(k)}\right)_{m,m} > 0$, $k = 3l - 1$ and $l = 1, 3, \dots, K - 1$

$$\left(\tilde{R}_{S_{odd}}^{(k)}\right)_{m,m} = 0, \quad (6.48)$$

$$\left(\tilde{R}_{T_{URR}}^{(k)}\right)_{m,m} = 1, \quad (6.49)$$

$$\left(\tilde{Q}_{S_{odd}S_{odd}}^{(k)}\right)_{m,n} = \begin{cases} -1, & m = n \\ 0, & \text{otherwise} \end{cases}, \quad n = 1, \dots, M, \quad (6.50)$$

$$\left(\tilde{Q}_{S_{odd}T_{URR}}^{(k)}\right)_{m,n} = \begin{cases} 1, & m = n \\ 0, & \text{otherwise} \end{cases}, \quad n = 1, \dots, M, \quad (6.51)$$

$$\left(\tilde{Q}_{S_{odd}S_{even}}^{(k)}\right)_{m,n} = \left(\tilde{Q}_{S_{odd}T_{LRR}}^{(k)}\right)_{m,n} = 0, \quad n = 1, \dots, M, \quad (6.52)$$

$$\begin{aligned} \left(\tilde{Q}_{T_{URR}S_{odd}}^{(k)}\right)_{m,n} &= \left(\tilde{Q}_{T_{URR}S_{even}}^{(k)}\right)_{m,n} = \left(\tilde{Q}_{T_{URR}T_{LRR}}^{(k)}\right)_{m,n} = \\ \left(\tilde{Q}_{T_{URR}T_{URR}}^{(k)}\right)_{m,n} &= 0, \quad n = 1, \dots, M. \end{aligned} \quad (6.53)$$

6.3.5 Parameter set in URR

Here, the active states are the ones belonging to the sets S_{even} and T_{URR} . The content process visits this regime in three different cases:

1) Assume that while the content is moving in SR under LRR it arrives at the boundary point between this regime and LRR at a state with positive drift, then it is transmitted through LRR and reaches the boundary point between LRR and URR at a state in T_{URR} . At that boundary point the background state makes a transition to the corresponding state in S_{even} . In this case, the parameters of the subset S_{even} in URR are the replicas of SR below LRR. So the parameters for $k = 3l$ and $l = 1, 3, \dots, K - 1$

$$R_{S_{even}}^{(k)} = R^{(l)}, \quad (6.54)$$

$$Q_{S_{even}S_{even}}^{(k)} = Q^{(l)}, \quad (6.55)$$

$$Q_{S_{even}S_{odd}}^{(k)} = Q_{S_{even}T_{URR}}^{(k)} = Q_{S_{even}T_{LRR}}^{(k)} = 0_M. \quad (6.56)$$

2) If the content arrives at the boundary point between LRR and URR while it is moving in LRR at a state in S_{odd} with positive drift then the content is transmitted upwards through URR. This behavior is captured with the following parameters for $m : \left(R_{S_{even}}^{(k+1)}\right)_{m,m} > 0$, $k = 3l$ and $l = 1, 3, \dots, K-1$

$$\left(R_{T_{URR}}^{(k)}\right)_{m,m} = 1, \quad (6.57)$$

$$\begin{aligned} \left(Q_{T_{URR}S_{odd}}^{(k)}\right)_{m,n} &= \left(Q_{T_{URR}S_{even}}^{(k)}\right)_{m,n} = \left(Q_{T_{URR}T_{URR}}^{(k)}\right)_{m,n} = \\ \left(Q_{T_{URR}T_{LRR}}^{(k)}\right)_{m,n} &= 0, \quad n = 1, \dots, M. \end{aligned} \quad (6.58)$$

3) If the content reaches the boundary point between SR and URR at a state in S_{even} with negative drift, then it is transmitted through URR down to LRR. Thus, for $m : \left(R_{S_{even}}^{(k+1)}\right)_{m,m} < 0$, $k = 3l$ and $l = 1, 3, \dots, K-1$

$$\left(R_{T_{URR}}^{(k)}\right)_{m,m} = -1, \quad (6.59)$$

$$\begin{aligned} \left(Q_{T_{URR}S_{odd}}^{(k)}\right)_{m,n} &= \left(Q_{T_{URR}S_{even}}^{(k)}\right)_{m,n} = \left(Q_{T_{URR}T_{URR}}^{(k)}\right)_{m,n} = \\ \left(Q_{T_{URR}T_{LRR}}^{(k)}\right)_{m,n} &= 0, \quad n = 1, \dots, M. \end{aligned} \quad (6.60)$$

6.3.6 Parameter set at the boundary point between URR and SR

There are three possible scenarios in which the content arrives at this point:

1) The content process can reach to that point from URR when the background process is at a state of S_{even} with positive drift. In this case, that point behaves the same as the boundary point with dynamic location between regimes. Therefore the parameters for $m : \left(R_{S_{even}}^{(k+1)}\right)_{m,m} > 0$, $k = 3l$ and $l = 1, 3, \dots, K-1$

$$\left(\tilde{R}_{S_{even}}^{(k)}\right)_{m,m} = \left(\tilde{R}^{(l)}\right)_{m,m}, \quad (6.61)$$

$$\left(\tilde{Q}_{S_{even}S_{even}}^{(k)}\right)_{m,n} = \left(\tilde{Q}^{(l)}\right)_{m,n}, \quad n = 1, \dots, M, \quad (6.62)$$

$$\left(\tilde{Q}_{S_{even}T_{URR}}^{(k)}\right)_{m,n} = \left(\tilde{Q}_{S_{even}T_{LRR}}^{(k)}\right)_{m,n} = \left(\tilde{Q}_{S_{even}S_{odd}}^{(k)}\right)_{m,n} = 0, \quad n = 1, \dots, M. \quad (6.63)$$

2) The content can arrive at that point from URR while it is being transmitted at a state in T_{URR} . In this case we set the parameters such that the content sticks to that point until a transition occurs to the set S_{even} . Hence, for $m :$

$$\left(R_{S_{even}}^{(k+1)}\right)_{m,m} > 0, \quad k = 3l \text{ and } l = 1, 3, \dots, K-1$$

$$\left(\tilde{R}_{T_{URR}}^{(k)}\right)_{m,m} = 0, \quad (6.64)$$

$$\left(\tilde{Q}_{T_{URR}T_{URR}}^{(k)}\right)_{m,n} = \begin{cases} -1, & m = n \\ 0, & \text{otherwise} \end{cases}, \quad n = 1, \dots, M, \quad (6.65)$$

$$\left(\tilde{Q}_{T_{URR}S_{even}}^{(k)}\right)_{m,n} = \begin{cases} 1, & m = n \\ 0, & \text{otherwise} \end{cases}, \quad n = 1, \dots, M, \quad (6.66)$$

$$\left(\tilde{Q}_{T_{URR}T_{LRR}}^{(k)}\right)_{m,n} = \left(\tilde{Q}_{T_{URR}S_{odd}}^{(k)}\right)_{m,n} = 0, \quad n = 1, \dots, M. \quad (6.67)$$

3) Finally, the content can touch to that point at a state in S_{even} from above. In this case, the content stays at the boundary point until a transition occurs to the set T_{URR} . Therefore, for $m : \left(R_{S_{even}}^{(k+1)}\right)_{m,m} < 0, k = 3l \text{ and } l = 1, 3, \dots, K-1$

$$\left(\tilde{R}_{S_{even}}^{(k)}\right)_{m,m} = 0, \quad (6.68)$$

$$\left(\tilde{Q}_{S_{even}S_{even}}^{(k)}\right)_{m,n} = \begin{cases} -1, & n = m \\ 0, & \text{otherwise} \end{cases}, \quad n = 1, \dots, M, \quad (6.69)$$

$$\left(\tilde{Q}_{S_{even}T_{URR}}^{(k)}\right)_{m,n} = \begin{cases} 1, & n = m \\ 0, & \text{otherwise} \end{cases}, \quad n = 1, \dots, M, \quad (6.70)$$

$$\left(\tilde{Q}_{S_{even}T_{LRR}}^{(k)}\right)_{m,n} = \left(\tilde{Q}_{S_{even}S_{odd}}^{(k)}\right)_{m,n} = 0, \quad n = 1, \dots, M, \quad (6.71)$$

$$\left(\tilde{R}_{T_{URR}}^{(k)}\right)_{m,m} = -1, \quad (6.72)$$

$$\begin{aligned} \left(\tilde{Q}_{T_{URR}S_{odd}}^{(k)}\right)_{m,n} &= \left(\tilde{Q}_{T_{URR}S_{even}}^{(k)}\right)_{m,n} = \left(\tilde{Q}_{T_{URR}T_{URR}}^{(k)}\right)_{m,n} = \\ \left(\tilde{Q}_{T_{URR}T_{LRR}}^{(k)}\right)_{m,n} &= 0, \quad n = 1, \dots, M. \end{aligned} \quad (6.73)$$

Remark 4. *One can easily see that some of the parameters aren't specified above, for example $R_{T_{URR}}^{(k)}$, $R_{T_{LRR}}^{(k)}$, $Q_{T_{LRR}S_{odd}}^{(k)}$, $Q_{T_{LRR}S_{even}}^{(k)}$, $Q_{T_{LRR}T_{LRR}}^{(k)}$, $Q_{T_{LRR}T_{URR}}^{(k)}$, $Q_{T_{URR}S_{odd}}^{(k)}$, $Q_{T_{URR}S_{even}}^{(k)}$, $Q_{T_{URR}T_{URR}}^{(k)}$ and $Q_{T_{URR}T_{LRR}}^{(k)}$ for $k = 3l-2$, $l = 1, 3, \dots, K-1$. The auxiliary states T_{URR} and T_{LRR} are not supposed to play any role when the content process is any of SRs since in these open intervals they aren't reachable from the states in S_{odd} or S_{even} . Furthermore, the states in S_{odd} (S_{even}) aren't reachable from the states in S_{even} (S_{odd}) in an even (odd) numbered SR. Hence, for those states in the corresponding regimes any appropriate selection of parameters doesn't affect the results. However, we should clarify what is appropriate and what is not, at this point. Let's assume that one selects zeros for the last two blocks of the infinitesimal generators and the drift matrices in SRs. Then, if the process starts with any of the auxiliary states at any location in an SR, then the queue content stays constant at that point forever and furthermore, there won't be any transition in the background process. So, considering the cross product of the domain of the content process and the state space of the background process as the state space of the double (or overall) process, a careless selection of parameters can lead us to divide the whole state space into two or more recurrent subspaces. In this case, obviously steady-state solution strongly depends on the initial probability distribution of the transformed process. But, the domain of that distribution includes some extra regimes and states that we defined artificially. One can easily see that the dependence on the initial probability distribution of the transformed process prevents us to reach a unique solution. In order to tackle this type of obstruct, we select such sub-matrices for the unspecified parameters that the whole state space has a single recurrent subspace. So any appropriate parameter set is the one that forces the process to have a single recurrent class in its domain. In fact there are infinitely many appropriate parameter sets, but selecting $R^{(1)}$ and zeros for all unspecified drift and infinitesimal generator elements, respectively, at every intermediate point of the queue and forcing the*

state transition to S_{even} (S_{odd}) at the upper (lower) terminal boundary point allow us to reach a single recurrent class. If the queue is of infinite size, i.e., the upper terminal boundary point is ∞ then that forcing has to be achieved at the uppermost regime instead of the terminal boundary point.

6.3.7 Performance measures

We explore two performance measures for a first-order multi regime feedback fluid queue with hysteresis: steady-state joint probability density functions and the steady-state expected number of crossings in a unit time. These two parameters are important to determine the optimum boundary points of the hysteresis regimes if the system is lossy and the system encounters with a cost at every time the policy changes. First, we find the steady-state joint probability density functions of FMFFQ-H. The second performance measure can be derived from these density functions. We use the same notation as Chapter 2 for the performance measures of the transformed process. For the static regimes of FMFFQ-H we add a subscript S to the casual notation, e.g., $f_m^{(k)}(x)$ is the steady-state joint probability density function of the transformed process when the content is in regime k and the background process is at state m ; so we use $f_{m,S}^{(k)}(x)$ to denote the steady-state joint probability density function of FMFFQ-H when the content is in static regime k and the background process is at state m . In addition to that we add a subscript H to the casual notation for the hysteresis regimes, e.g., we use $f_{m,H}^{(k)}(x)$ to denote the steady-state joint probability density function of FMFFQ-H when the content is in HR k (HR between SRs k and $k + 1$) and the background process is at state m . Assume that the steady-state probability density functions and the probability mass accumulations of the transformed process are obtained by using the theory constructed in Chapter 2. We use them to derive the steady-state probability density functions of FMFFQ-H using the

relations for $m = 1, 2, \dots, M$:

$$f_{m,S}^{(k)}(x) = f_m^{(3k-2)} \left(x + \sum_{l=1}^{k-1} (T_{up}^{(l)} - T_{down}^{(l)}) \right) / a, \quad (6.74)$$

$$k = 1, 3, \dots, K-1, T_{up}^{(k-1)} < x < T_{down}^{(k)},$$

$$f_{m,S}^{(k)}(x) = f_{m+M}^{(3k-2)} \left(x + \sum_{l=1}^{k-1} (T_{up}^{(l)} - T_{down}^{(l)}) \right) / a, \quad (6.75)$$

$$k = 2, 4, \dots, K, T_{up}^{(k-1)} < x < T_{down}^{(k)},$$

$$f_{m,H}^{(k)}(x) = \left(\begin{array}{c} f_m^{(3k-1)} \left(x + \sum_{l=1}^{k-1} (T_{up}^{(l)} - T_{down}^{(l)}) \right) + \\ f_{m+M}^{(3k)} \left(x + \sum_{l=1}^k (T_{up}^{(l)} - T_{down}^{(l)}) \right) \end{array} \right) / a, \quad (6.76)$$

$$k = 1, 3, \dots, K-1, T_{down}^{(k)} < x < T_{up}^{(k)},$$

$$f_{m,H}^{(k)}(x) = \left(\begin{array}{c} f_{m+M}^{(3k-1)} \left(x + \sum_{l=1}^{k-1} (T_{up}^{(l)} - T_{down}^{(l)}) \right) + \\ f_m^{(3k)} \left(x + \sum_{l=1}^k (T_{up}^{(l)} - T_{down}^{(l)}) \right) \end{array} \right) / a, \quad (6.77)$$

$$k = 2, 4, \dots, K, T_{down}^{(k)} < x < T_{up}^{(k)}$$

where the normalization constant is

$$a = 1 - \sum_{l=1}^{K-1} \sum_{m=2M+1}^{4M} \left(\int_{T^{(3l-2)}}^{T^{(3l-1)}} f_m^{(3l-1)}(x) dx + \int_{T^{(3l-1)}}^{T^{(3l)}} f_m^{(3l)}(x) dx \right) - \sum_{k=1}^{3K-1} \sum_{m=1}^{4M} c_m^{(k)} \quad (6.78)$$

Let $g_{m,H}^{(k)}$ denotes the expected number of policy changes in unit time -between regimes k and $k+1$ and at a state m - for an FMFFQ-H. Also, assume that $g_m^{(k)}$ is the expected number of crossings at queue level $T^{(k)}$ at a state m in the transformed process. It is easy to see the following relations between $g_{m,H}^{(k)}$ and $g_m^{(k)}$.

$$g_{m,H}^{(k)} = \begin{cases} g_{m+M}^{(3k)}/a, (R^{(k)})_{m,m} > 0, k = 1, 3, \dots, K-1, \\ g_m^{(3k-2)}/a, (R^{(k)})_{m,m} < 0, k = 1, 3, \dots, K-1, \\ g_m^{(3k)}/a, (R^{(k)})_{m,m} > 0, k = 2, 4, \dots, K-2, \\ g_{m+M}^{(3k-2)}/a, (R^{(k)})_{m,m} < 0, k = 2, 4, \dots, K-2. \end{cases}, m = 1, 2, \dots, M. \quad (6.79)$$

The total expected number of policy changes in unit time can be found by summing $g_{m,H}^{(k)}$ over m and k .

For a first-order single-regime fluid queue, the steady-state expected number of crossings per unit time at a buffer level x and at a state m is driven in [58] as $|r_m| f_m(x)$. Using similar steps of [58] one can easily derive the following equalities for $m = 1, 2, \dots, M$:

$$g_{m+M}^{(3k)} = f_{m+M}^{(3k)}(T^{(3k)}-) (R^{(k)})_{m,m}, (R^{(k)})_{m,m} > 0, k = 1, 3, \dots, K-1, \quad (6.80)$$

$$g_m^{(3k-2)} = -f_m^{(3k-1)}(T^{(3k-2)}+) (R^{(k+1)})_{m,m}, (R^{(k)})_{m,m} < 0, k = 1, 3, \dots, K-1, \quad (6.81)$$

$$g_m^{(3k)} = f_m^{(3k)}(T^{(3k)}-) (R^{(k)})_{m,m}, (R^{(k)})_{m,m} > 0, k = 2, 4, \dots, K-2, \quad (6.82)$$

$$g_{m+M}^{(3k-2)} = -f_{m+M}^{(3k-1)}(T^{(3k-2)}+) (R^{(k+1)})_{m,m}, (R^{(k)})_{m,m} < 0, k = 2, 4, \dots, K-2, \quad (6.83)$$

Here, we derive only the equality (6.82) by following similar steps applied in [58]. The other equations, (6.80), (6.81) and (6.83), can be obtained using similar steps. $g_m^{(3k)}$ is the expected number of upward crossings of the boundary point $T^{(3k)}$. To find this expectation we define the event, $G(m, t) = \{T^{(3k)} - h < C(t) < T^{(3k)}, M(t) = m\}$. In steady-state the probability of this event is

$$\mathbb{P} \left\{ \lim_{t \rightarrow \infty} G(m, t) \right\} = \int_{T^{(3k)}-h}^{T^{(3k)}} f_m^{(3k)}(x) dx = f_m^{(3k)}(T^{(3k)}-)h + o(h). \quad (6.84)$$

This probability is equal to the steady-state fraction of time that the queue spends in the interval $(T^{(3k)} - h, T^{(3k)})$. Hence by defining $L(m, T)$ as the expected number of upward crossings at $T^{(3k)}$ and at state m within T seconds, one can obtain the equality

$$f_m^{(3k)}(T^{(3k)}-)h + o(h) = \lim_{T \rightarrow \infty} L(m, T) \frac{h}{T (R^{(k)})_{m,m}} \quad (6.85)$$

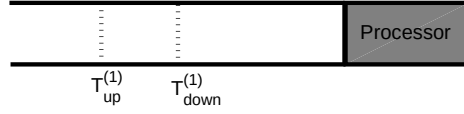


Figure 6.7: M/M/1 queue with a hysteretic control of admission policy.

where $h / (R^{(k)})_{m, m}$ represents the duration of time in the event $G(m, t)$ in steady-state. This equation yields

$$f_m^{(3k)}(T^{(3k)}-) (R^{(k)})_{m, m} + \frac{o(h)}{h} = \lim_{T \rightarrow \infty} \frac{L(m, T)}{T}. \quad (6.86)$$

If we take the limit of (6.86) as h goes to zero, obviously we obtain (6.82).

6.4 Example

This example presents the steady-state solution to a M/M/1 queue with a hysteretic control of admission policy, illustrated in Fig. 6.7. When the workload is between 0 and $T_{down}^{(1)}$ the queue admits the arrivals. Let us name this behavior as *policy 1*. But, if the workload is greater than $T_{up}^{(1)}$ it rejects the arrivals and, moreover, the server increases its processing speed in the region $(T_{up}^{(1)}, \infty)$ so as to decrease the queue occupancy as fast as possible. We call the behavior of the queue in $(T_{up}^{(1)}, \infty)$ *policy 2*. The region between $T_{down}^{(1)}$ and $T_{up}^{(1)}$ applies a hysteretic control on the admission policy. If the last visited regime is $(0, T_{down}^{(1)})$ then the arrivals are admitted in the hysteresis regime, as well. However, if the last visited regime is $(T_{up}^{(1)}, \infty)$ then the hysteresis regime applies policy 2. As time evolves the queue switches between policies 1 and 2. We assume that the queue incurs fixed costs, C_1 and C_2 , when the policy is switched and when a packet is lost due to the rejection, respectively.

We seek for a steady-state solution to density function of the queue occupancy, $C(t)$, and the expected number of policy changes in unit time when the queue

is in steady-state. By using these results, we fix the location of $T_{up}^{(1)}$ and explore the optimal location of $T_{down}^{(1)}$ for constant, unity, values of costs.

A sample path of the queue occupancy is depicted in Fig. 6.8. In this illustration, totally five packets arrive. First two of them are accepted to the queue since the workload is less than $T_{down}^{(1)}$ at the arrival epochs. Also, the third arrival is accepted because the queue is in the hysteresis regime at the arrival epoch and the last visited regime is $(0, T_{down}^{(1)})$. The fourth packet is rejected due to the fact that the workload overflows $T_{up}^{(1)}$ at the arrival epoch. And finally, the fifth packet is rejected because the queue is in the hysteresis regime at that time and the last visited regime is $(T_{up}^{(1)}, \infty)$.

Obviously, that process is not an FMFFQ-H since there are instant jumps in the sample path. In fact we use two successive transformations in order to reach the performance measures of that process. The first transformation is applied to obtain an FMFFQ-H and then we follow the transformation idea given in this chapter to obtain an FMFFQ. So, after we obtain the parameters of the resulting FMFFQ, we will find its probability density functions using the results in Chapter 2. Then we will use the solution in Section 6.3 of this chapter, to reach the performance measures of FMFFQ-H. And finally a simple normalization will yield the density function of the queue occupancy and the expected number of policy changes in unit time when the queue is in steady-state. We denote the content in the corresponding FMFFQ-H and FMFFQ by $C_{T,1}(t)$ and $C_{T,2}(t)$, respectively.

As the first transformation, we insert continuously increasing sample path parts instead of the jumps in the process $C(t)$. Fig. 6.9 illustrates the general idea. If we delete the dashed lines in this figure we basically obtain the sample path in Fig. 6.8. The sample path of $C_{T,1}(t)$ in Fig. 6.9 is a sample path of an FMFFQ-H with two static regimes in $(0, T_{down}^{(1)})$ and $(T_{up}^{(1)}, \infty)$ and a hysteresis regime in $(T_{down}^{(1)}, T_{up}^{(1)})$. There are two states in the background process. We

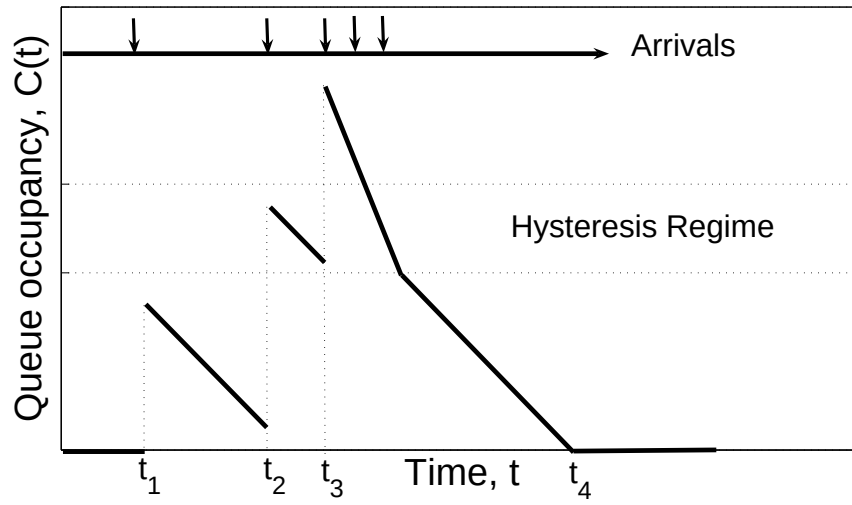


Figure 6.8: Evolution of the queue in Fig. 6.7.

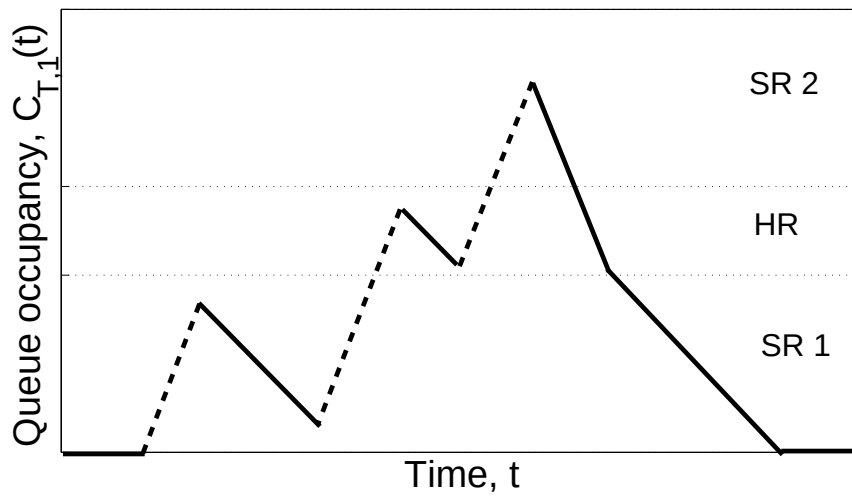


Figure 6.9: Evolution of the content after the first transformation.

assign the first and the second states to realize the steady increases and decreases of the sample path, respectively, in Fig. 6.9. So we select the drift values as 1 and -1 at states 1 and 2, respectively in SR 1. However, as stated above the speed of the processor fastens if the content is greater than $T_{up}^{(1)}$. Assume that the drift is equal to -2 at state 2 in SR 2. A state transition occurs from state 1 to 2 (2 to 1) after a random time whose distribution is equal to the packet length of an arrival (interarrival time) in SR 1. Assume that the mean of the exponentially distributed packet lengths (interarrival time) is denoted by μ (λ). Note that new arrivals are rejected in $(T_{up}^{(1)}, \infty)$, so we don't allow any transition in this region if the current state is 2. Utilizing the notation of Section 6.2, the drifts and the infinitesimal generators of FMFFQ-H are

$$\begin{aligned}\tilde{R}^{(0)} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, R^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tilde{R}^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, R^{(2)} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \\ \tilde{Q}^{(0)} &= Q^{(1)} = \tilde{Q}^{(1)} = \begin{bmatrix} -\mu & \mu \\ \lambda & -\lambda \end{bmatrix}, Q^{(2)} = \begin{bmatrix} -\mu & \mu \\ 0 & 0 \end{bmatrix}.\end{aligned}$$

Next we accomplish the second transformation, FMFFQ-H to FMFFQ, that is described in Section 6.3 of this chapter. In the first step, we replace HR with LRR and URR with lengths of $T_{up}^{(1)} - T_{down}^{(1)}$. As a second step we define $4M = 8$ states in the background process. Then we use Remark 4 together with the equations (6.5) to (6.73) to find the following sets of parameters for the FMFFQ:

$$\begin{aligned}\tilde{R}_T^{(0)} &= diag(1, 0, 1, 0, 1, 0, 1, 0), R_T^{(1)} = diag(1, -1, 1, -1, 1, -1, 1, -1), \\ \tilde{R}_T^{(1)} &= diag(0, -1, 1, -1, 1, -1, 1, 0), R_T^{(2)} = diag(1, -2, 1, -1, 1, -1, 1, -1), \\ \tilde{R}_T^{(2)} &= diag(0, -2, 1, 0, 1, 0, 0, -1), R_T^{(3)} = diag(1, -1, 1, -1, 1, -1, 1, -1), \\ \tilde{R}_T^{(3)} &= diag(1, -1, 1, 0, 0, -1, 1, -1), R_T^{(4)} = diag(1, -1, 1, -2, 1, -1, 1, -1),\end{aligned}$$

$$\tilde{Q}_T^{(0)} = \begin{bmatrix} -\mu & \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & -\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$Q_T^{(1)} = \begin{bmatrix} -\mu & \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & -\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{Q}_T^{(1)} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \lambda & -\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$Q_T^{(2)} = \begin{bmatrix} -\mu & \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{Q}_T^{(2)} = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu & \mu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Q_T^{(3)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu & \mu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & -\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

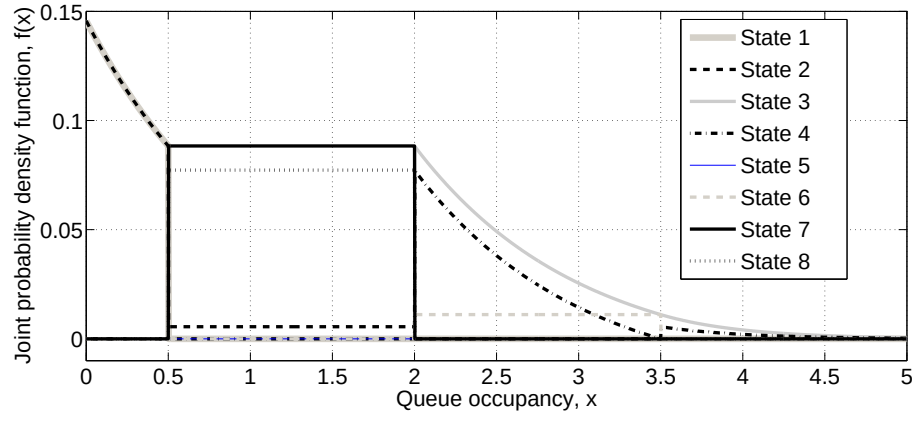


Figure 6.10: Joint probability density functions of the FMFFQ in steady-state.

$$\tilde{Q}_T^{(3)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu & \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Q_T^{(4)} = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu & \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Fig. 6.10 depicts the steady-state probability density functions of the FMFFQ for $\lambda = 1$, $\mu = 2$, $T_{down}^{(1)} = 0.5$ and $T_{up}^{(1)} = 2$. After we find those functions, in order to reach the steady-state joint probability densities of FMFFQ-H we need to do the following arrangements whose general forms are given in equations

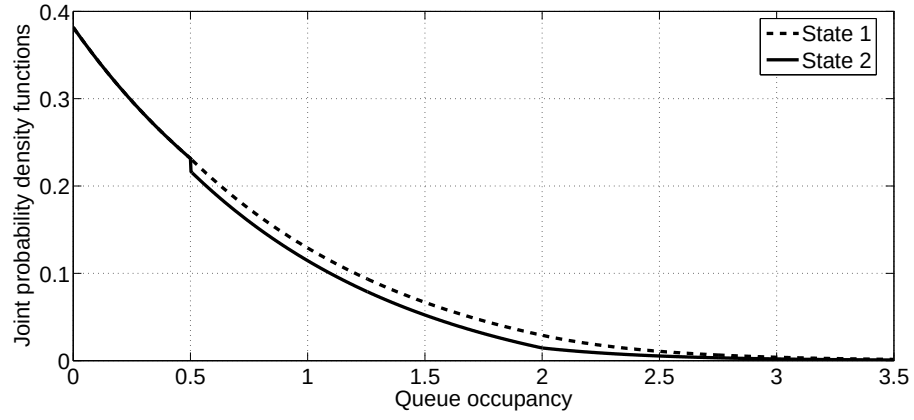


Figure 6.11: Joint probability density functions of FMFFQ-H in steady-state.

(6.74), (6.75), (6.76) and (6.77).

$$f_{m,S}^{(1)}(x) = f_m^{(1)}(x)/a, \quad 0 < x < T_{down}^{(1)}, \quad m = 1, 2,$$

$$f_{m,H}^{(1)}(x) = \left(f_m^{(2)}(x) + f_{m+M}^{(3)}\left(x + T_{up}^{(1)} - T_{down}^{(1)}\right) \right) / a, \quad T_{down}^{(1)} < x < T_{up}^{(1)}, \quad m = 1, 2,$$

$$f_{m,S}^{(2)}(x) = f_{m+M}^{(4)}\left(x + T_{up}^{(1)} - T_{down}^{(1)}\right) / a, \quad T_{up}^{(1)} < x < \infty, \quad m = 1, 2.$$

Joint probability density functions of FMFFQ-H are shown in Fig. 6.11 for $\lambda = 1$, $\mu = 2$, $T_{down}^{(1)} = 0.5$ and $T_{up}^{(1)} = 2$. The steady-state probability density function of the queue occupancy of M/M/1 queue of this example, $g(x)$, can be obtained with another normalization of the probability measures of FMFFQ-H.

$$g(x) = \begin{cases} f_{2,S}^{(1)}(x)/F_{2,S}^{(2)}(\infty), & 0 < x < T_{down}^{(1)} \\ f_{2,H}^{(1)}(x)/F_{2,S}^{(2)}(\infty), & T_{down}^{(1)} < x < T_{up}^{(1)} \\ f_{2,S}^{(2)}(x)/F_{2,S}^{(2)}(\infty), & T_{up}^{(1)} < x < \infty \end{cases}$$

In that equation, $F_{2,S}^{(2)}(x)$ denotes the steady-state joint distribution function of FMFFQ-H in SR 2 at state 2. Fig. 6.12 depicts that backward transformation for the example in Fig. 6.11.

Arrivals are Poisson, therefore the blocking probability P_b is equal to the sum of the long run proportions of time in which the queue level is above $T_{up}^{(1)}$; and it is in the hysteresis regime provided that the last visited regime is $(T_{up}^{(1)}, \infty)$.

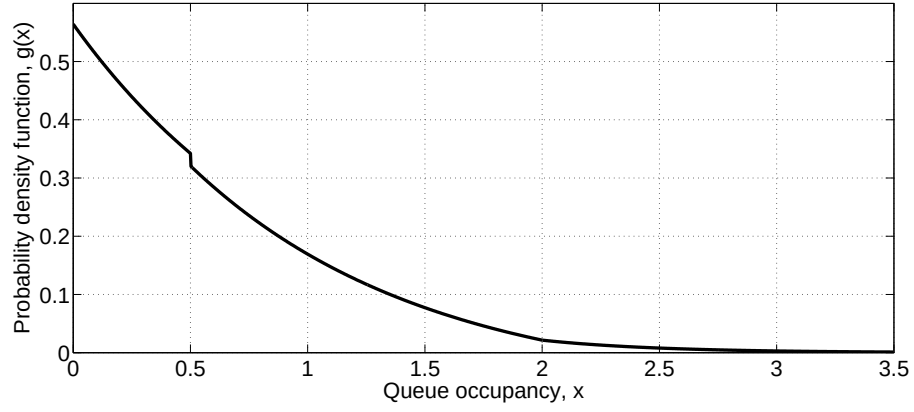


Figure 6.12: Steady-state probability density function of the M/M/1 queue in case study.

This probability can be found as

$$P_b = \int_{T_{up}^{(1)}}^{\infty} g(x) dx + \frac{\int_{T_{down}^{(1)}}^{T_{up}^{(1)}} f_2^{(2)}(x) dx}{aF_{2,S}^{(2)}(\infty)}.$$

Next we find the long run expected number of policy changes in unit time. Equations (6.80) and (6.81) yield

$$g_3^{(3)} = f_3^{(3)}(2T_{up}^{(1)} - T_{down}^{(1)}) \text{ and } g_2^{(1)} = 2f_2^{(2)}(T_{down}^{(1)}).$$

Hence, the expected numbers of crossings of the upper and the lower limits of the hysteresis regime in unit time in FMFFQ-H are $f_3^{(3)}(2T_{up}^{(1)} - T_{down}^{(1)})/a$ and $2f_2^{(2)}(T_{down}^{(1)})/a$, respectively. Finally the long run expected number of changes from policy 2 to policy 1 and from policy 2 to policy 1 in unit time are $2f_2^{(2)}(T_{down}^{(1)})/(aF_{2,S}^{(2)}(\infty))$ and $f_3^{(3)}(2T_{up}^{(1)} - T_{down}^{(1)})/(aF_{2,S}^{(2)}(\infty))$, respectively. Finally, note that these numbers should be equal to each other.

Now, we assume that $\lambda = 1$, $\mu = 2$ and $T_{up}^{(1)} = 2$. Fig. 6.13 illustrates the blocking probability and the expected number of policy changes in unit time in steady-state for different values of $T_{down}^{(1)}$. As the boundary points becomes closer, the hysteresis regime shrinks and the expected number of policy changes in unit time increases. However, we see from this figure that the blocking probability

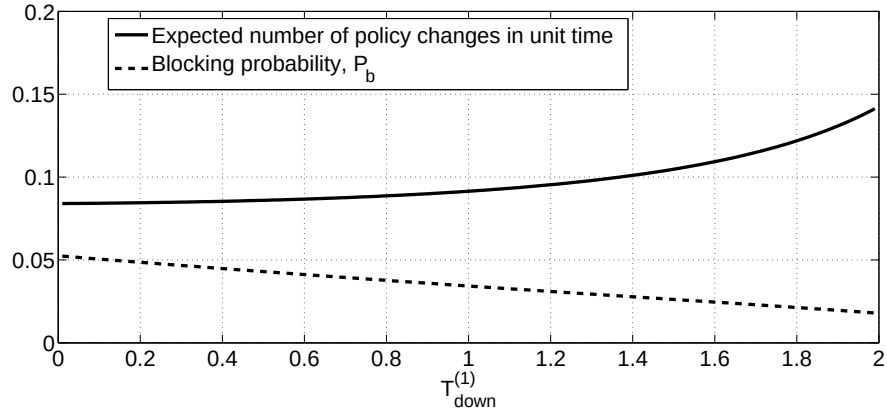


Figure 6.13: Performance measures over different values of $T_{down}^{(1)}$.

decreases. If we use unity cost assignment, the optimal point is reached for $T_{down}^{(1)} = 1.01$ with a cost of 0.1257.

6.5 Conclusions

In this chapter we analyzed two important performance measures of FMFFQ-H model: steady-state joint probability density functions and the expected number of regime switchings in unit time. Our solution basically includes a transformation from FMFFQ-H into FMFFQ. Using the transformation idea twice lead us to reach the solution to the performance measures of the basic Markovian queues with hysteretic control.

Chapter 7

CONCLUSIONS

In conclusion, this thesis improves fluid queue-based solutions and finds the exact solutions to the steady state performance measures of three applications in the performance evaluation of communication networks. These applications include the PDFs of: horizon in a multi-class OBS channel with horizon based advance reservation system; scheduling horizon in an optical buffer using FDLs; and first-order multi-regime feedback fluid queues with hysteretic control. In Chapter 2 we propose the solutions of steady-state joint probability density functions and steady-state joint probability mass accumulations via a numerically stable procedure. By deploying a backward transformation to those results in measure domain we find the steady-state probability density functions of the processes that we aim to analyze. Using steady-state probability density functions, we can derive the blocking probabilities and the delay distributions in steady-state. The delay and the blocking probability are important performance measures for the first and the second applications presented in Chapters 4 and 5, respectively. Important performance measures for the last application, in Chapter 6, are the queue occupancy, blocking probability and the expected number of policy changes in unit time in steady state. By extending the analysis in [58] a little, we can find the expected number of crossing of a certain level of the content process

in unit time in steady-state. Using that result, we find the expected number of policy changes in unit time in steady-state for a system performing a number of policies depending on the current value of the queue occupancy.

It is possible to enhance our studies through different directions. Firstly, we believe that solutions to many other performance evaluation problems, which aren't considered in this thesis, can be found with FMFFQ-based approaches, as well. So, in the future we can improve our methods and provide FMFFQ-based solutions to other performance evaluation problems. Secondly, as mentioned in Chapter 3 our FMFFQ-based solution methods conceptually resemble the Markov chain-based solutions of some discrete processes. We can study on some other candidate base models which can provide solutions to unsolved performance evaluation problems. For example, one of the candidate base models is First-order multi-dimensional multi-regime first-order feedback fluid queue (FMMFFQ). We analyze the horizon processes of Chapters 4 and 5 in a single dimension, since we look at a single output port of a node and neglect the affect of the correlation of the process in a single output port with the other ports. Analyzing the correlated horizon processes in multiple parallel nodes all together is a more realistic approach. However, that approach requires the steady-state analysis of FMMFFQs, which is not currently available. One other candidate model is second-order multi-regime feedback fluid queue (SMFFQ). SMFFQ is a general process and it reduces to FMFFQ when variance parameter is selected as zero throughout the queue at all states of the background process. Stochastic models moving as a Brownian motion in between successive jumps can be analyzed via the extension of the transformation to SMFFQs. Thirdly, our FMFFQ-based solutions include transformations of processes to FMFFQs as the first step. The correct transformations are found intuitively and unfortunately it is difficult to propose an algorithmic way to find the correct transformations. We can study on this subject after we explore all limitations and capabilities of FMFFQ-based solutions in long term.

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