ÇUKUROVA UNIVERSITY
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PhD Thesis

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BASIC WEITZENBÖCK DERIVATIONS

DEPARTMENT OF MATHEMATICS

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ABSTRACT

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Consider a field K of characteristic 0 and $X_n = \{x_1, \ldots, x_n\}$ as a finite set of variables. Now let F_n be the free metabelian Lie algebra and P_n be the free metabelian Poisson algebra of rank n generated by X_n over the field K. In this dissertation, initially, a set generator of the algebra F_5^δ of constants was given for a linear nilpotent derivation $\delta: x_5 \longrightarrow x_4 \longrightarrow 0,\ \delta: x_3 \longrightarrow x_2 \longrightarrow x_1 \longrightarrow 0$. Secondly, the algebra P_{2n}^δ of constants in free metabelian Poisson algebra was determined, where $\delta: x_{2i} \longrightarrow x_{2i-1} \longrightarrow 0,\ 1 \leq i \leq n$. Finally, the algebra $P_n^{S_n}$ of invariants of the symmetric group S_n was determined.

Keywords: Lie algebras, Poisson algebras, Weitzenböck derivations.

ÖZ

DOKTORA TEZİ

TEMEL WEİTZENBÖCK TÜREVLERİ

Andre DUSHIMIRIMANA

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K karakteristiği sıfır olan bir cisim ve $X_n = \{x_1, \ldots, x_n\}$ değişkenlerin sonlu bir kümesi olsun. Rankı n olan, X_n tarafından üretilen K cismi üzerindeki F_n serbest metabelyen Lie cebirini ve P_n serbest metabelyen Poisson cebirini ele alalım. Bu tezde ilk olarak, $\delta: x_5 \longrightarrow x_4 \longrightarrow 0$, $\delta: x_3 \longrightarrow x_2 \longrightarrow x_1 \longrightarrow 0$ bir lineer nilpotent derivasyon olmak üzere F_5^δ sabitler cebirinin bir üreteç kümesi verilmiştir. İkinci olarak, $\delta: x_{2i} \longrightarrow x_{2i-1} \longrightarrow 0$, $1 \leq i \leq n$ olmak üzere, Poisson cebiri içerisindeki P_{2n}^δ sabitler cebiri belirlenmiştir. Son olarak S_n simetrik grup olmak üzere, $P_n^{S_n}$ invaryant cebiri belirlenmiştir.

Anahtar Kelimeler: Lie cebirleri, Poisson cebirleri, Weitzenböck derivasyonları.

EXTENDED ABSTRACT

Let K be a field of characteristic 0 and let $X_n = \{x_1, \dots, x_n\}$ be a finite set of variables. Let $K[X_n]$ be the polynomial algebra over a field K of characteristic 0. A nonzero locally nilpotent linear derivation δ of the polynomial algebra $K[X_n]$ is called Weitzenböck derivation. The canonical action of the general linear group $GL_n(K)$ on the polynomial algebra $K[X_n]$ defines the set

$$K[X_n]^G = \{p \in K[X_n] \mid g \cdot p = p, \qquad \forall\, g \in G\}$$

which is the algebra of invariants of the subgroup $G \leq GL_n(K)$. The fourteenth Hilbert problem asked if the algebra $K[X_n]^G$ is finitely generated as an algebra for any $G \leq GL_n(K)$. Nagata in 1959 constructed a counterexample showing the fact that the conjecture above is not correct in general. Partial positive results had been found, as well, until then. One of the most effective results is given in 1916 by Noether where she confirmed the question affirmatively for finite groups. Another study by Weitzenböck (1932) states that the algebra

$$\mathrm{Ker}\delta = K[X_n]^\delta = \{u \in K[X_n] \mid \delta(u) = 0\}$$

of constants is finitely generated.

Now we replace the group G by the symmetric group S_n. In this case, the algebra $K[X_n]^{S_n}$ consists of elements called symmetric polynomials. One may consider the free associative algebra $K\langle X_n\rangle$ generated by X_n over K. The algebra $K\langle X_n\rangle^G$ of invariants of the group G is finitely generated if and only if G is a cyclic group. The nonassociative concept of this problem arises other approaches. We compile some of them: The algebra L_n^G cannot be finitely generated for finite groups G where L_n is the free Lie algebra of rank n over K. On the other hand, F_n^G is not finitely generated as a Lie algebra for finite groups G, provided F_n is the free metabelian Lie algebra of rank n. Although not being an infinitely generated Lie algebra, the ideal $(F_n')^G$ is finitely generated as a $K[X_n]^G$-module.

The action of a linear nilpotent operator δ on the algebra $K[X_n]$ is specified via its action on vector space KX_n with basis X_n. Up to a linear change of the basis of KX_n, the derivation δ is determined by its Jordan normal form consisting of Jordan cells with zero diagonals. Such linear nilpotent operators are called Weitzenböck.

Let δ_n be the Weitzenböck derivation of $K[X_n]$ which corresponds to a single $n \times n$ Jordan block of size n, then,

$\delta_n(x_1) = 0, \quad \delta_n(x_2) = x_1, \quad \delta_n(x_3) = x_2 \quad , \quad \dots \quad , \quad \delta_n(x_n) = x_{n-1}.$

and Weitzenböck derivations of the form δ_n are called basic. If we only have one Jordan cell, then the derivation δ_n is written as

$$\delta_n : x_n \longrightarrow x_{n-1} \longrightarrow \cdots \longrightarrow x_1 \longrightarrow 0.$$

By extending the results of Dangovski et al. (2013) we give the explicit set of generators of the algebra of constants of the $K[X_n]^{\delta_n}$-module $\left({L'_n}/{L''_n} \right)^{\delta_n}$ for $n = 5$ as a nonbasic Weitzenböck derivation, i.e., cells of size 3×2 . As fact this result is coupled in Theorem 4.1.1 of this thesis where we proved that if $\delta_{3\times 2}$ is a nonbasic Weitzenböck derivation in which $\delta_{3\times 2} : x_3 \longrightarrow x_2 \longrightarrow x_1 \longrightarrow 0$ and $\delta_{3\times 2} : x_5 \longrightarrow x_4 \longrightarrow 0$, then $K[X_5]^{\delta_{3\times 2}}$-module $\left({L'_5}/{L''_5} \right)^{\delta_{3\times 2}}$ is generated by

$$[x_2, x_1],$$
$$[x_4, x_1],$$
$$[x_5, x_1] - [x_4, x_2],$$
$$[x_5, x_4],$$
$$[x_2, x_1, x_5] + [x_4, x_2, x_2] - 2[x_4, x_1, x_3],$$
$$[x_2, x_1, x_5] - [x_3, x_1, x_4],$$
$$[x_3, x_1, x_1] - [x_2, x_1, x_2],$$
$$[x_5, x_1, x_5] - [x_5, x_2, x_4] - [x_4, x_2, x_5] + 2[x_4, x_3, x_4],$$

$$[x_2, x_1, x_5, x_5] - 2[x_3, x_1, x_4, x_5] + 2[x_3, x_2, x_4, x_4],$$

$$[x_3, x_1, x_1, x_5] - [x_2, x_1, x_2, x_5] - [x_3, x_1, x_2, x_4] + 2[x_2, x_1, x_3, x_4] \text{ and}$$

$$\begin{pmatrix} [x_5, x_2, x_2, x_2, x_4] + [x_4, x_2, x_2, x_2, x_5] + 2[x_5, x_1, x_1, x_3, x_5] \\ -[x_3, x_1, x_1, x_5, x_5] - [x_5, x_1, x_2, x_2, x_5] + [x_2, x_1, x_2, x_5, x_5] \\ -2[x_4, x_2, x_2, x_3, x_4] + 4[x_4, x_1, x_3, x_3, x_4] - 2[x_3, x_1, x_3, x_4, x_4] \\ -2[x_5, x_1, x_2, x_3, x_4] - 2[x_4, x_1, x_2, x_3, x_5] + 2[x_3, x_1, x_2, x_4, x_5] \end{pmatrix}.$$

This is considered as the first results of this work.

Let $n = 2m,\ m \geq 1$, and consider the Weitzenböck derivation δ_{2m} of $K[X_n, Y_n]$ where the Jordan form of δ_{2m} has only single Jordan cells of size 2×2 and act as it is in Nowicki conjecture (Nowicki, 1994).

Let P_n be the free metabelian Poisson algebra generated by $X_n = \{x_1, \dots, x_n\}$, and let

$\mathcal{B}_{1n} = X_n \cup \{[x_{i_1}, \dots, x_{i_k}] \mid i_1 > i_2 \leq i_3 \leq \cdots \leq i_k, \quad k \geq 2\}$

$\mathcal{B}_{2n} = \{x_i \cdot x_j, x_i \cdot x_j \cdot x_k \mid i \leq j \leq k\}$

$\mathcal{B}_{3n} = \{[x_i, x_j] \cdot x_k \mid i > j\}$

$\mathcal{B}_{4n} = \{[x_i, x_j, x_k] \cdot x_l \mid j < i \leq l, j \leq k < l\}$.

Then the direct sum of $K\mathcal{B}_{1n} \oplus K\mathcal{B}_{2n} \oplus K\mathcal{B}_{3n} \oplus K\mathcal{B}_{4n}$ provides a linear basis for the free metabelian Poisson algebra P_n by (Zhang, Z., et al 2019).

Let P_{2m} be the free metabelian Poisson algebra generated by $X_m \cup Y_m = \{x_1, \dots, x_m, y_1, \dots, y_m\}$. The Weitzenböck derivation of free metabelian Poisson algebra is defined as δ_{2m} such that

$$\delta_{2m}(y_i) = x_i, \quad \delta_{2m}(x_i) = 0$$

As second result of this thesis, we determine the K-basis and finite set of generators spanned in the algebra $P_{2m}^{\delta_{2m}}$ of free metabelian Poisson algebra P_{2m}.

Now, let P_n be the free metabelian Poisson algebra of rank n generated by X_n over K. Being preserved under the action of each permutation in S_n we can say that an element in P_n is symmetric. It is shown in this thesis that the subspaces $K\mathcal{B}_{jn},\ j = 1, 2, 3, 4$ are S_n-invariant spaces. Hence, the set $P_n^{S_n}$ of symmetric polynomials is the algebra of invariants of the symmetric group S_n in the free metabelian Poisson algebra P_n. Therefore,

$$P_n^{S_n} = (K\mathcal{B}_{1n})^{S_n} \oplus (K\mathcal{B}_{2n})^{S_n} \oplus (K\mathcal{B}_{3n})^{S_n} \oplus (K\mathcal{B}_{4n})^{S_n}.$$

Since $K\mathcal{B}_{1n} = F_n$, the free metabelian Lie algebra, then $(K\mathcal{B}_{1n})^{S_n}$ is generated by single linear symmetric polynomial $g^{(n)} = x_1 + \cdots + x_n$ and $(F_n')^{S_n}$ as a $K[X_n]^{S_n}$-module. The remaining subspace $(K\mathcal{B}_{2n})^{S_n} \oplus (K\mathcal{B}_{3n})^{S_n} \oplus (K\mathcal{B}_{4n})^{S_n}$ is of a basis

$$\mathcal{B}_n = \left\{p_1^{(n)}, p_2^{(n)}, p_3^{(n)}, p_4^{(n)}, p_5^{(n)}, p_6^{(n)}, p_7^{(n)}, p_8^{(n)}\right\}$$

Therefore, as third result of this thesis, we proved that the algebra $P_n^{S_n}$ of symmetric polynomials in P_n is spanned on the set

$$P_n = \left\{g^{(n)}\right\} \cup (F_n')^{S_n} K[X_n]^{S_n} \cup \mathcal{B}_n$$

as a K-vector space.

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1. INTRODUCTION

Consider the polynomial algebra $K[X_n]$ over a field K of characteristic 0 freely generated by the set of variables $X_n = \{x_1, \dots, x_n\}$ and that let KX_n be a vector space with basis X_n. A linear operator $\delta : K[X_n] \longrightarrow K[X_n]$ is a derivation if

$$\delta(uv) = \delta(u)v + u\delta(v), \qquad \text{for every } u, v \in K[X_n]$$

The algebraic actions of the additive group $(K, +)$ on affine K-variety X are in one-to-one correspondence with locally nilpotent derivations of its coordinate ring $K[X_n]$, that is, K-derivation δ is called locally nilpotent if $\forall\ u \in K[X_n]$, $\exists\ n = n(u) \in \mathbb{Z}^+$(a non-negative integer n depending on u) such that $\delta^n(u) = 0$. $K[X_n]$ is equivalent to the increasing union of the kernels Ker δ^n of the iterates of δ. In this case, the derivation δ acts as a nonzero nilpotent linear operator of the vector space KX_n with basis X_n, and it is known as a Weitzenböck derivation.

In the structure of commutative algebra and algebraic geometry, locally nilpotent derivations of polynomial algebra $K[X_n]$ has an essential role. In particular, it is used to determine the action of an additive group $(K, +)$ on K-algebra R and to investigate whether the map

$$\begin{aligned} \exp(\delta) :\ & K[X_n] \longrightarrow K[X_n] \\ & u \longmapsto \sum_{n=0}^{\infty} \frac{1}{n!}\delta^n(u) \end{aligned}$$

is an automorphism of $K[X_n]$ as K-algebra. Another approach is the Jacobian conjecture, which states that some special K-derivations of $K[X_n] = K[x_1, \dots, x_n]$ are locally nilpotent and that every δ_i is locally nilpotent for a commutative basis

$\{\delta_1, \dots, \delta_n\}$ of the K-derivation. One can see the book of Nowicki (1994) for further information.

Rentschler (1968) proved a well-known theorem says that if u is an element of a polynomial ring in 2 variables $K[x, y]$ over a field of characteristic 0, $f(x)$ is a polynomial in $K[x]$ and $\delta : K[x, y] \longrightarrow K[x, y]$ is a locally nilpotent derivation, then $\exists$ a tame automorphism such that $u \circ \delta \circ u^{-1} = f(x)\frac{\partial}{\partial y}$. Later, Smith (1989) proved that some K-automorphisms of $K[X_n]$ are stably tame. Recall that an automorphism which is not tame is called wild.

Let $GL_n(K)$ be the general linear group and G be a subgroup of $GL_n(K)$. If $GL_n(K)$ is acting canonically on the algebra $K[X_n]$, then the set

$$K[X_n]^G = \{p(X_n) \in K[X_n] \mid g \cdot p(X_n) = p(X_n) \; \forall \, g \in G\}$$

is the algebra of invariants of $G \leq GL_n(K)$. The fourteenth of twenty-three unsolved Hilbert problems by Hilbert (1900) asked whether the algebra $K[X_n]^G$ of invariant is finitely generated for any subgroup $G \leq GL_n(K)$. Since then, the significant works on linear actions of unipotent algebraic groups have been done including the famous counter-example of Nagata (1959) which consists of 13 commuting locally nilpotent derivations of a polynomial ring in 32 variables for which the joint kernel is not finitely generated. Although being incorrect in general, some constructive concepts on the subject have been made until then. An outstanding result was given by Noether (1916), who confirmed the case for finite groups G, and the results of Shephard and Todd (1954) and Chevalley (1955) proved that if G is generated by reflections, then the algebra $K[X_n]^G$ is generated by n algebraically independent homogeneous elements. One can see another partial approach on the problem in the results of Weitzenböck (1932) which states that a finitely generated algebra of constants of a given linear nilpotent derivation of polynomial algebra $K[X_n]$ is equivalent to the algebra of invariants of a certain group associated with the given derivation and is written in this form:

$$\mathrm{Ker}\delta = K[X_n]^{\delta} = \{u \in K[X_n] \mid \delta(u) = 0\}.$$

Seshadri (1962) gave an actual proof of the Weitzenböck theorem and Tyc (1998) gave the same proof with further minor changes.

The special case on the algebra of invariants is defined by symmetric polynomials. When the group G is now replaced by symmetric group S_n of degree n, then the elements of the algebra $K[X_n]^{S_n}$ of invariants are known as symmetric polynomials and its description is generated by algebraically independent elementary symmetric polynomials

$$x_1 + \cdots + x_n,\ \ x_1x_2 + \cdots + x_1x_n + \cdots + x_{n-1}x_n,\ \ldots, x_1 \cdots x_n.$$

Let $K\langle X_n\rangle$, $n \geq 2$, be the free (unitary or nonunitary) associative polynomial algebra generated by X_n over a field K. The concept of algebra $K\langle X_n\rangle^{S_n}$ of invariants was first discussed by Wolf (1936). One can see (Bergeron, N., et al. 2008) and (Gelfand, I., et al 1995) for more details. The algebra $K\langle X_n\rangle^{G}$ is finitely generated if and only if G is a cyclic group which acts by scalar multiplication via (Dicks and Formanek, 1982) and Kharchenko (1978). Koryukin (1984) considered a nonnatural action for reductive groups G then he showed that the algebra $K\langle X_n\rangle^{G}$ is finitely generated.

Another approach in noncommutative invariant theory is when the group acts on relatively free algebras in a variety of associative algebras. In this way, Domakos (1996) proved that for a nonzero commutator ideal T, $K\langle X_n\rangle/T = K\langle X_n\rangle$, the algebra $(K\langle X_n\rangle/T)^{G}$ is relatively free if and only if G is a pseudo-reflexion group and T contains the polynomial $\big[[x_1, x_2], x_1\big]$. On other hand for T-

ideal whose radical does not contain $[[x_1, x_2], x_1]$ or (T is the ideal of identity of $n \times n$ matrices where $n \geq 2$), the algebra $(K\langle X_n \rangle / T)^G$ is not relatively free algebra.

Let L_n be the free Lie algebra and F_n be free metabelian Lie algebra of rank n. The action of a nontrivial group G can be uniquely extended to the entire of the L_n and transform L_n into a KG-module in which every element of G acts as a Lie algebra automorphism. Bryant (1991) proved that for nontrivial finite groups G, the algebra L_n^G is never finitely generated. Afterwards, Drensky (1994) proved that F_n^G is never finitely generated for nontrivial finite groups G, where F_n is the free metabelian Lie algebra of rank n. Besides not being finitely generated, the algebra F_n^G, the commutator ideal F_n' of F_n is finitely generated $K[X_n]^G$-module under an appropriate action of $K[X_n]$ on F_n'. One may see the papers (Fındık and Öğüşlü, 2019) and (Drensky, V., et al. 2020) for generating sets of $F_n^{S_n}$, $n \geq 2$ and (Fındık and Özkurt, 2020) for symmetric polynomials of the free metabelian Leibniz algebras of rank n as a generalization of Lie algebras. Another analogue of the problem was held for the free algebra E_3 of rank 3 generated by X_3 in the variety G of Grassmann algebras. In the recent work (Akdogan and Findik, 2021) they obtained a finite free generating set for the $K[X_3]^{S_3}$-module $(E_3')^{S_3}$, where E_3' is the commutator ideal of E_3.

The action of a linear nilpotent operator δ on the algebra $K[X_n]$ is specified via its action on vector space KX_n with basis X_n. Up to a linear change of the basis of KX_n, the derivation δ is determined by its Jordan normal form

$$J(\delta) = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_s \end{pmatrix},$$

consisting of Jordan cells with zero diagonals

$$J_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

where $i = 1, \dots, s$. Such linear nilpotent operators are called Weitzenböck. For each dimension n, there are only finite number of essentially different Weitzenböck derivations. Up to a linear change of the coordinates, the Weitzenböck derivations δ have a one-to-one correspondence with the partition $(m_1 + 1, m_2 + 1, \dots, m_s + 1)$ of n, where $m_1 \geq \cdots \geq m_s \geq 0$, $(m_1 + 1) + \cdots + (m_s + 1) = n$, and the correspondence is given in terms of the size $(m_i + 1) \times (m_i + 1)$ of the Jordan cells J_i of $J(\delta)$, $i = 1, \dots, s$.

Let δ_n be the Weitzenböck derivation of $K[X_n] = K[x_1, \dots, x_n]$ which corresponds to a single $n \times n$ Jordan block of size n, then,

$\delta_n(x_1) = 0, \quad \delta_n(x_2) = x_1, \quad \delta_n(x_3) = x_2 \ , \quad \dots \quad , \quad \delta_n(x_n) = x_{n-1}.$

Every Weitzenböck derivation of the form δ_n is called basic. If we only have one Jordan cell, then the derivation δ will be denoted by

$$\delta_n\colon x_n \longrightarrow x_{n-1} \longrightarrow \cdots \longrightarrow x_1 \longrightarrow 0.$$

Let $n = 2m$, $m \geq 1$ and let P_{2m} be the free metabelian Poisson algebra generated by $X_m \cup Y_m = \{x_1, \dots, x_m, y_1, \dots, y_m\}$. The Weitzenböck derivation of free metabelian Poisson algebra is defined as δ_{2m} such that

$$\delta_{2m}(y_i) = x_i, \quad \delta_{2m}(x_i) = 0$$

here the Jordan form of δ_{2m} has only single Jordan cells of size 2×2. Such derivations are known as nonbasic Weitzenböck derivations.

Nowicki (1994) in his book conjectured that the algebra $K[X_{2m}]^{\delta_{2m}}$ is generated by X_{2m} and the determinant $U_{pq} = \begin{vmatrix} x_{2p-1} & x_{2p} \\ x_{2q-1} & x_{2q} \end{vmatrix}$, $1 \leq p < q \leq m$. Since then, many researchers have proven this conjecture with various approaches. In his Ph.D. thesis Khoury (2001) followed by his paper in (2008), he used the Gröbner basis technique to prove the conjecture. The work of Kuroda (2009), Bedratyuk (2009) and Derksen's unpublished work show how to prove the Nowicki conjecture by using the classical invariant theory. Drensky and Makar-Limanov (2009) found a system of defining equations of the algebra $K[X_{2m}, Y_{2m}]^{\delta_{2m}}$ that corresponds to the reduced Gröbner basis of a certain ideal associated with this algebra and a suitable monomial order, then they present explicit basis of $K[X_{2m}, Y_{2m}]^{\delta_{2m}}$ as a vector space and they confirmed the conjecture in elementary way without involving any invariant theory.

The free commutative associative algebras are very closely connected to the free Lie algebras via Poisson algebras. The basics of the Poisson algebra date back to the beginning of 19th century. The well-known Mathematician Siméon-Denis Poisson was studied this algebra as a pioneer linked to the three-body problem in mechanics. These algebras have also widely been used in geometry (Vaisman, 1994), quantum groups (Drinfel'd, 1987) and (Chari and Pressley, 1995), and some structural properties of commutative associative algebras (Gerstenhaber, 1964).

Quantization of Poisson superalgebras were used in order to introduce the free Poisson algebra by Shestakov (1994). Umirbaev and Shestakov (2002, 2003, 2004a and 2004b) showed that the automorphism group (Nagata, 1972) of the polynomial ring with three variables is wild as an application of the free Poisson algebra of rank three. Consider the symmetric algebra $S(L)$ of a Lie algebra L. Farkas (1998, 1999) proved that L contains an abelian subalgebra of finite codimension in the case of characteristic zero iff $S(L)$ satisfies a nontrivial Poisson identity. Afterwards, (Giambruno and Petrogradsky, 2006) obtained the results for any positive characteristic. Agore and Militaru (2015) and Zhang et al. (2019) classified the metabelian Poisson algebras of small dimensions and provided linear bases for the free metabelian Poisson algebra defined over an arbitrary field.

In this dissertation, we handle three different results. First, we calculated the generators of the algebra of constants of the $K[X_5]^{\delta_5}$-module $\left(L'_5/L''_5\right)^{\delta_{3\times 2}}$ for such that $\delta_{3\times 2}$ is a nonbasic Weitzenböck derivation, this gives an explicit (infinite) set of generators of the algebra $\left(L_5/L''_5\right)^{\delta_{3\times 2}}$ and this result follows the work of Dangovski et al (2013). Second, we assume that δ_{2m} is a nonbasic Weitzenböck derivation which acts similarly as it is in the book of the Nowicki (1994), then we find the spanning elements of the algebra $P_{2m}^{\delta_{2m}}$ of free metabelian Poisson algebras P_{2m} which is considered as an additional work of (Drensky and Fındık, 2020) and (Centrone and Fındık, 2020). Finally, we determine the algebra $P_n^{S_n}$ of symmetric polynomials in the free metabelian Poisson algebra P_n of rank n as another generalization of the results obtained in the setting of Lie algebras (Drensky, V., et al. 2020), in a similar direction to the work (Fındık and Özkurt, 2020).

2. MATERIAL AND METHODS

In this chapter, we are dealing with some basic facts and properties concerning our work and most of the contents can be found in any textbook regarding the subjects.

2.1. Definitions

Definition 2.1.1: A field is a set K together with two binary operation addition and multiplication such that the following conditions holds:

i. $(K,+)$ is abelian group
ii. $(K\backslash\{0\},\cdot)$ is abelian group
iii. Distribution low hold $\forall\, a, b, c \in K$

$$a \cdot (b + c) = a \cdot b + a \cdot c; (b + c) \cdot a = b \cdot a + c \cdot a$$

Definition 2.1.2: Let K be a field. A nonempty set V over a field K is called a vector space if the following conditions holds:

$(V,\oplus)$ is an abelian group

$\odot : K \times V \longrightarrow V$ with the following conditions

i) $\alpha \odot v \in V,\ \forall\ \alpha \in K, v \in V$
ii) $(\alpha + \beta) \odot v = (\alpha \odot v) \oplus (\beta \odot v)\ \ \forall\ \alpha, \beta \in K, v \in V$
iii) $(\alpha \cdot \beta) \odot v = \alpha \odot (\beta \odot v)\ \ \forall\ \alpha, \beta \in K, v \in V$
iv) $\alpha \odot (u \oplus v) = (\alpha \odot u) \oplus (\alpha \odot v)\ \ \forall\ \alpha \in K, u, v \in V$
v) $1 \odot v = v$

Here W is subspace of vector space V over a field K if W itself is a vector space over K with the same operation of V.

Definition 2.1.3: If $v_1, \dots, v_n \in V$, then every vector of the form $c_1 v_1 + \cdots + c_n v_n$ with $c_1, \dots, c_n \in K$ is called a linear combination of $v_1, \dots, v_n$. The set of all linear combination of subset S of V is called linear span of S and is denoted by $\langle S \rangle$ or $\mathrm{Span(S)}$ in other word

$$\mathrm{Span(S)} = \left\{ \sum_{i=1}^{n} c_i v_i \;\mid\; c_i \in K , v_i \in S \right\}.$$

If every vector of V is linear combination of element of S $(V = \langle S \rangle)$ then V is spanned or generated by S, so S is called a set of generators or spanning set of V.

Definition 2.1.4: A subset S of V is said to be linearly dependent if there exist distinct vectors $v_1, \dots, v_n \in S$ and scalars $c_1, \dots, c_n \in K$ not all are zero such that $c_1 v_1 + \cdots + c_n v_n = 0$. On the other hand, is said to be linearly independent if for any distinct vectors $v_1, \dots, v_n$ of S, $c_1 v_1 + \cdots + c_n v_n = 0$ and this implies $c_i = 0$.

Definition 2.1.5: A basis is a subset S of V over a field K which is linearly independent set and Span V while a dimension is the number of elements inside the basis.

Definition 2.1.6: Let V and W be the vector space with the same scalars. A function $T : V \longrightarrow W$ is called a linear transformation if the following two properties holds

1. $T(cu) = cT(u)$ (Homogeneity Property)
2. $T(u + v) = T(u) + T(v)$ (Additivity Property)

for all vectors $u, v \in V$ and all scalars c.

Definition 2.1.7 (Rings)**:** A set R with two binary operations $(+,\cdot)$ is called a ring if the following conditions are satisfied:

i) $(R, +)$ is an abelian group with identity 0

ii) $(R, \cdot)$ is an associative with identity 1

iii) $\forall\, r_1, r_2, r_3 \in R$

$(r_1 + r_2) \cdot r_3 = r_1 \cdot r_3 + r_2 \cdot r_3$

$r_1 \cdot (r_2 + r_3) = r_1 \cdot r_2 + r_1 \cdot r_3$

A subset $S \subseteq R$ is a subring if S itself is a ring with respect to the same binary operations on R.

Definition 2.1.8 (Unit)**:** An element $r_1 \in R$ is a unit if there exist an inverse $r_1^{-1} \in R$ such that $r_1 \cdot r_1^{-1} = r_1^{-1} \cdot r_1 = 1$. A set of all unit in R is denoted as $R^{\times}$.

Definition 2.1.9 (Ring Homomorphism)**:** Let R_1, R_2 be two rings, then a ring homomorphism is a map $\varphi: R_1 \longrightarrow R_2$ satisfying the following conditions

i) $\varphi(r_1 + r_2) = \varphi(r_1) + \varphi(r_2)$

ii) $\varphi(r_1 r_2) = \varphi(r_1)\varphi(r_2)$

for all $r_1, r_2 \in R_1$. Here φ is called isomorphism if φ is a 1-1 and onto homomorphism. If $\varphi: R_1 \longrightarrow R_2$ is a ring homomorphism, then we can define the kernel of a ring homomorphism as a set

$$Ker\ \varphi = \{r_1 \in R_1, \varphi(r_1) = 0\}.$$

Definition 2.1.10: Let R_1 be a ring. A pair of (R_2, f) is called an R_1-algebra if R_2 is a ring and $f: R_1 \longrightarrow R_2$ is a ring homomorphism.

Definition 2.1.11 (Integral Domain)**:** A nonzero commutative ring R with identity is said to be integral domain if $\forall\, r_1, r_2 \in R$, the condition $r_1 \cdot r_2 = 0$ implies $r_1 = 0$ or $r_2 = 0$.

Here we can define a field as a nonzero ring R such that each element of $R \setminus \{0\}$ is a unit.

Definition 2.1.12 (Ideal)**:** A subring $I \subseteq R$ is an ideal of R if $(I, +)$ is an abelian group and for every $r_1 \in R$ and $x \in I$, $r_1 x \in I$ and $x r_1 \in I$.

Lemma 2.1.13: If x is any element in a commutative ring R with identity, then the set $\langle x \rangle = \{x r_1 \mid r_1 \in R\}$ is an ideal and such ideal is called Principal ideal.

Proof: The set $\langle x \rangle$ is nonempty so, $0x = 0$ and $1x = x$ are in $\langle x \rangle$. The summation $x r_1 + x r_2 = x(r_1 + r_2)$ and inverse $-x r_1 = x(-r_1)$ also are in $\langle x \rangle$. Let take an arbitrary element $s_1 \in R$, then we obtain $s_1(x r_1) = x(s_1 r_1)$. Therefore, by definition 2.1.11 we conclude that $\langle x \rangle = \{x r_1 \mid r_1 \in R\}$ is an ideal.

Definition 2.1.14: Let R be a commutative ring

i) Let $p \in R$, then p is called **irreducible** if the following condition holds
   - $p \neq 0$ is not invertible and
   - $p = xy$, then either x or y is a unit.

ii) Let $P \subseteq R$ be an ideal, then P is called a **prime ideal** if
   - $P \neq R$ and
   - $xy \in P$, then $x \in P$ or $y \in P$ for all $x, y \in R$.

iii) Let M be an ideal of R. M is called a **maximal ideal** of R if I is an ideal of R such that $M \subseteq I \subseteq R$, then either $I = M$ or $I = R$.

iv) Two elements $x, y \in R$ is said to be **associate** if $x = uy$ for some unit u.

Definition 2.1.15: Let R_1 be a subring of R_2, then any element $r_1 \in R_2$ is called an algebraic over R_1 if there exist a nonzero $p \in R_1[x]$ such that $p(r_1) = 0$. Otherwise r_1 is said to be transcendental over R_1. Not that one can say that R_2 is algebraic over R_1 if each element of R_2 is algebraic over R_1 and R_1 is algebraically closed in R_2 if each element of $R_2 \backslash R_1$ is transcendental over R_1.

2.2. Polynomial Rings

Definition 2.2.1: Let R be a commutative ring, let x be a variable with coefficients in R. A polynomial over R with variable x is a set of all finite ordered system $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ such that $a_0, a_1, \dots, a_n \in R,\ a_n \neq 0,\ n \geq 0$. Here a_n and n are considered as leading coefficient and degree of polynomial over R respectively.

Remark 2.2.2: i) If a polynomial $f(x) = 0$, then the polynomial is called zero polynomial.

ii) If R is a ring with identity and $a_n = 1$, then the polynomial is called monic polynomial.

iii) If the polynomial consists only a single term a_0, then is called a constant polynomial.

A set of all polynomials over a ring R is denoted as $R[x]$. Now, since the polynomial over R preserves the properties of addition and multiplication, we can define two polynomials in $R[x]$

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = \sum_{i=0}^{n} a_i\, x^i \text{ and}$$

$$g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m = \sum_{i=0}^{m} b_i\, x^i$$

such that their sum and product are

$$\begin{aligned} f(x) + g(x) &= (a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) \\ &\quad + (b_0 + b_1x + b_2x^2 + \cdots + b_mx^m) \\ &= (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n \\ &= \sum_{i=0}^{\max(n,m)} (a_i + b_i)\, x^i \end{aligned}$$

Note that if $n > m$, then $b_{m+1} = \cdots = b_n = 0$

And

$$\begin{aligned} f(x)g(x) &= (a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)(b_0 + b_1x + b_2x^2 + \cdots + b_mx^m) \\ &= a_0b_0 + (a_0b_1 + a_1b_0)x + \cdots + a_nb_mx^{n+m} \\ &= \sum_{j=0}^{n+m}\left(\sum_{i=0}^{j} a_ib_{j-i}\right)x^j \text{ and } \begin{matrix} b_k = 0, & k \geq m+1 \\ a_k = 0, & k \geq n+1 \end{matrix} \end{aligned}$$

Theorem 2.2.3: $i)$ If R is a commutative ring with identity then $R[x]$ is too.

$ii)$ If R is an integral domain then $R[x]$ is too.

Proof: $i)$ let $f(x)$ and $g(x)$ be two polynomials in $R[x]$. The product is

$$f(x)g(x) = \sum_{j=0}^{n+m}\left(\sum_{i=0}^{j} a_ib_{j-i}\right)x^j \text{ and } \begin{matrix} b_k = 0, & k \geq m+1 \\ a_k = 0, & k \geq n+1 \end{matrix}$$

Since R is a commutative ring

$$\begin{aligned} \sum_{i=0}^{j} a_ib_{j-i} &= a_0b_j + a_1b_{j-1} + \cdots + a_jb_0 \\ &= b_0a_j + b_1a_{j-1} + \cdots + b_ja_0 \\ &= \sum_{i=0}^{j} b_ia_{j-i} \end{aligned}$$

This means that

$$f(x)g(x) = \sum_{j=0}^{n+m}\left(\sum_{i=0}^{j} a_ib_{j-i}\right)x^j = \sum_{j=0}^{n+m}\left(\sum_{i=0}^{j} b_ia_{j-i}\right)x^j = g(x)f(x)$$

Hence, $R[x]$ is commutative.

If $1 \in R$, $1 \in R[x]$ and $1f(x) = f(x)1 = f(x)$ $\forall$ $f(x) \in R[x]$. Therefore, $R[x]$ is a commutative ring with identity.

$ii)$ let R be an integral domain, we know from $i)$ $R[x]$ is a commutative ring with identity. The products of two nonzero elements in $R[x]$ has a_nb_m as a largest coefficient. Clearly $a_nb_m \neq 0$ and this implies that $f(x)g(x) \neq 0$. Therefore $R[x]$ is an integral domain too. If $a_nb_m = 0$, then its degree is zero which means that both $f(x)$ and $g(x)$ are constant polynomials and their product in $R[x]$ is equal to their product in R.

Theorem 2.2.4: Let $f(x)$, $g(x)$ be two nonzero polynomials in $R[x]$, then

i) If $f(x)+g(x) \neq 0$, then $\deg\big(f(x)+g(x)\big) \leq \max\{\deg f(x), \deg g(x)\}$

ii) If $f(x)g(x) \neq 0$, then $\deg\big(f(x)g(x)\big) \leq \deg f(x) + \deg g(x)$

Proof: Let suppose that $\deg f(x) = n$ and $\deg g(x) = m$

i) It's known that the sum of two polynomials in $R[x]$ is $\sum_{i=0}^{\max(n,m)}(a_i + b_i)\, x^i$ so,

If $n = m$, then the $\deg\big(f(x)+g(x)\big) = 0$.

If $n > m$, then $\deg\big(f(x)+g(x)\big) = \max\{\deg f(x), \deg g(x)\}$

Therefore, $\deg\big(f(x)+g(x)\big) \leq \max\{\deg f(x), \deg g(x)\}$.

ii) The product of two polynomials is $\sum_{j=0}^{n+m}\big(\sum_{i=0}^{j} a_i b_{j-i}\big)\, x^j$ by Theorem 2.2.1 $ii)$ $R[x]$ has nonzero divisor here $f(x)g(x) \neq 0$, this has $a_n b_m$ as a highest nonzero coefficient and $n+m$ as the largest degree of the product. Therefore,

$\deg\big(f(x)g(x)\big) = n + m = \deg f(x) + \deg g(x)$.

If $R[x]$ has zero divisor then $f(x)g(x) = 0$, this means that $a_n b_m = 0$ so,

$\deg\big(f(x)g(x)\big) < \deg f(x) + \deg g(x)$. Therefore,

$\deg\big(f(x)g(x)\big) \leq \deg f(x) + \deg g(x)$.

Now let consider K as a field and $K[x_1, \dots, x_n]$ as a polynomial ring in n variables over a field K. The formal expression is

$$K[x_1, \dots, x_n] = \sum_{(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n=\mathbb{N}\times\cdots\times\mathbb{N}} c_{\alpha_1,\dots,\alpha_n}\, x_1^{\alpha_1} \cdots x_n^{\alpha_n} \text{ such that } c_{\alpha_1,\dots,\alpha_n} \in K, \forall\, \alpha_1, \dots, \alpha_n \in \mathbb{N}^n.$$

The polynomial $K[x_1, \dots, x_n]$ can be defined in inductive way such that $K[x_1, \dots, x_n] = K[x_1, \dots, x_{n-1}][x_n]$. For the sake of simplicity, we can write that $K[x_1, \dots, x_n] = K[X_n]$. Since K is a ring and $n \in \mathbb{Z}_+$ one can consider the polynomial ring $K[X_n]$ as a K-algebra with the inclusion homomorphism $i: K \longrightarrow K[X_n]$.

Definition 2.2.5: Let K be a ring and S be a K-algebra

i) Let $p \in K[X_n]$ such that $p = \sum_{(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n} c_{\alpha_1,\dots,\alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and let $(s_1,\dots,s_n) \in S^n$ then we define $p(s_1,\dots,s_n) \in S$ as

$$p(s_1,\dots,s_n) = \sum_{(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n} c_{\alpha_1,\dots,\alpha_n} s_1^{\alpha_1} \cdots s_n^{\alpha_n}.$$

ii) Since $s = (s_1,\dots,s_n) \in S^n$, we define the map

$\mathrm{ev}_s : K[X_n] \longrightarrow S$

$p(x_1,\dots,x_n) \longmapsto p(s_1,\dots,s_n)$

And is called the evaluation at s.

The following theorem is known as the universal property of polynomial algebra.

Theorem 2.2.6: Let K be a ring, $n \in \mathbb{Z}_+$ and $K[X_n]$ be the polynomial ring in n variables over K. If $\varphi: K \longrightarrow S$ is a ring homomorphism and $s = (s_1,\dots,s_n) \in S^n$, then there exist a unique K-homomorphism $\psi: K[X_n] \longrightarrow S$ satisfying $\varphi(x_i) = s_i$ for all $i = 1,\dots,n$ and the following diagram commute

$$\begin{tikzcd}
K \arrow[r, "\varphi"] \arrow[d, "i"'] & S \\
K[X_n] \arrow[ur, dashed, "\exists!\,\psi"', "x_i \mapsto s_i"] &
\end{tikzcd}$$

Consequently, the following conditions holds

i) $\psi = \mathrm{ev}_s$

ii) $\mathrm{Im}\,\psi = K[s_1,\dots,s_n] = p(s_1,\dots,s_n) \mid p \in K[X_n]$

Proof: The K-homomorphism $\psi: K[X_n] \longrightarrow S$ must be the map which sends X_i to s_i and acts exactly as φ on the coefficients. Here polynomial $s_0 + s_1x + s_2x^2 + \cdots + s_nx^n$ can be sent to $\varphi(s_0) + \varphi(s_1)x + \varphi(s_2)x^2 + \cdots + \varphi(s_n)x^n$ therefore, the rest of the proof is to check whether the given map is a ring homomorphism.

Definition 2.2.7: Let S be an algebra over a ring K. An element $s = (s_1, \dots, s_n) \in S^n$ is called an algebraic dependent over K if there exist $p \in K[X_n] \setminus \{0\}$ such that $p(s_1, \dots, s_n) = 0$. Otherwise, such element is called algebraically independent over K.

Definition 2.2.8: Let R be a commutative ring with identity, then R is called Noetherian ring (or satisfies the ascending chain condition (a.c.c)) if there is no infinite increasing chain of ideals. In other words, if $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ is a chain of prime ideal then $\exists\ N \in \mathbb{N}$ such that $I_k = I_N\ \forall\ k > N$.

Theorem 2.2.9: (Hilbert's Basis Theorem). If R is Noetherian then $R[X]$ is Noetherian.

Proof: The proof can be found in Reid (1988).

Theorem 2.2.10: Let R be a ring. Then the following are equivalent.

i) R is Noetherian
ii) Every ideal of R is finitely generated
iii) Any nonempty set of ideals of R has a maximal element.

Proof: $\boldsymbol{i}) \Longrightarrow \boldsymbol{ii})$ Let I be an ideal of R. Suppose that I is not finitely generated. Let $a_1 \in I$, let $I_1 = a_1R = \langle a_1 \rangle \subseteq I$ so, $I \neq I_1$ since I is not finitely generated. Let $a_2 \in I \backslash I_1$. Let $I_2 = \langle a_1, a_2 \rangle \subseteq I$ so, $I \neq I_2$ since I is not finitely generated.

$\boldsymbol{ii}) \Longrightarrow \boldsymbol{i})$ Conversely, assume that every ideal of R is finitely generated. Suppose that $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ is a chain of ideals. Take $I = \bigcup_{n=1}^{\infty} I_n$, then I is an ideal of R. By our assumption I is finitely generated then there are elements $a_1, \dots, a_m$ of R such that $I = \langle a_1, \dots, a_m \rangle$. For $i = 1, \dots, m$, $a_i \in I$ there is n_i such that $a_i \in I_{n_i}$. Let

$\max\{n_1, \dots, n_m\} = N$. Since $n_i \leq N$, then $a_i \in I_{n_i} \subseteq I_N$ for $i = 1, \dots, m$ so $I \subseteq I_N$. Since $k \geq N$, then $I_k \subseteq I \subseteq I_N \subseteq I_k$ so, $I_k = I_N$. Therefore, the chain is constant (stable) and R is Noetherian.

$\boldsymbol{i}) \Longrightarrow \boldsymbol{iii})$ (By contradiction)

Let I be a nonempty set of ideals of R which has no maximal element. Let $I_1 \in I$. Since is not maximal element of I, then there is $I_2 \in I$ such that $I_1 \subsetneq I_2$. By induction we construct an infinite increasing chain of ideals of R such that $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$ so, R is not Noetherian.

$\boldsymbol{iii}) \Longrightarrow \boldsymbol{i})$

Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ be an increasing chain of ideals of R. Let $I = \{I_n : n \in \mathbb{N}\}$. If I is nonempty set of ideals then by our assumption I has a maximal element, say I_{n_0}.

For $k > n_0$, we have $I_{n_0} \subseteq I_k$. Since I_{n_0} is a maximal element of I and $I_k \in I$ we cannot have $I_{n_0} \subsetneq I_k$. So, $I_k = I_{n_0}$. Therefore, the chain become stable (constant) from n_0 onward.

2.3. Free Metabelian Lie Algebras

Recall that a Lie algebra L is a vector space over K endowed with a bilinear map $[.,.] : L \times L \to L, (x_1, x_2) \to [x_1, x_2]$, satisfying the following conditions:

$$L(1): [x_1, x_1] = 0$$

$$L(2): \big[[x_1, x_2], x_3\big] + \big[[x_2, x_3], x_1\big] + \big[[x_3, x_1], x_2\big] = 0$$

where $x_1, x_2, x_3 \in L$. The Lie bracket $[x_1, x_2]$ is often referred to as the commutator of x_1 and x_2. Condition $(L2)$ is known as the Jacobi identity. As the Lie bracket $[.,.]$ is bilinear, we have

$$\begin{aligned} 0 = [x_1 + x_2, x_1 + x_2] &= [x_1, x_1] + [x_1, x_2] + [x_2, x_1] + [x_2, x_2] \\ &= [x_1, x_2] + [x_2, x_1]. \end{aligned}$$

Hence condition $(L1)$ implies

$[x_1, x_2] = -[x_2, x_1]$ for all $x_1, x_2 \in L$.

One can rewrite The Jacobi identity as

$$\left[[x_1,x_2],x_3\right] = \left[x_1,[x_2,x_3]\right] + \left[[x_1,x_3],x_2\right]$$

Definition 2.3.1: For every element $x_1 \in L$ we define the map ad_{x_1}, called adjoint action, as follows:

$$ad_{x_1}: L \longrightarrow L, x_2 \longmapsto [x_1,x_2]$$

Here the linear map

$$ad: x_1 \longmapsto ad_{x_1},$$

is called the adjoint representation which is one of the well-known examples of Lie homomorphisms.

Recall that $f: L \longrightarrow L$ is a Lie homomorphism if $f([x_1,x_2]) = [f(x_1), f(x_2)]$. Let us show that ad is a homomorphism. For all $x_3 \in L$ we have that

$$\begin{aligned}[ad(x_1), ad(x_2)](x_3) &= \left[ad_{x_1}, ad_{x_2}\right](x_3) = \left(ad_{x_1}ad_{x_2} - ad_{x_2}ad_{x_1}\right)(x_3) \\ &= \left(ad_{x_1}ad_{x_2}\right)(x_3) - \left(ad_{x_2}ad_{x_1}\right)(x_3) \\ &= ad_{x_1}\left(ad_{x_2}(x_3)\right) - ad_{x_2}\left(ad_{x_1}(x_3)\right) \\ &= ad_{x_1}([x_2,x_3]) - ad_{x_2}([x_1,x_3]) = \left[x_1,[x_2,x_3]\right] - \left[x_2,[x_1,x_3]\right] \\ &= \left[[x_3,x_2],x_1\right] + \left[[x_1,x_3],x_2\right] = -\left[[x_2,x_1],x_3\right] = \left[[x_1,x_2],x_3\right] \\ &= ad_{[x_1,x_2]}(x_3) = ad([x_1,x_2])(x_3)\end{aligned}$$

Thus

$$ad([x_1,x_2]) = [ad(x_1), ad(x_2)]$$

and this implies that ad is a homomorphism.

Definition 2.3.2: For all non-negative integer m, the ideal $L^{(n)}$ of L is defined as

$L^{(0)} = L,$

$L^{(1)} = [L,L] = L' = \{[x_1,x_2] : x_1, x_2 \in L\},$

$L^{(2)} = \left[L^{(1)}, L^{(1)}\right],$

$\vdots$

$L^{(n+1)} = \left[L^{(n)}, L^{(n)}\right]$

Hence, we can iterate to define the derived sequence of L as follows:

$$L = L^{(0)} \supseteq L^{(1)} \supseteq \cdots \supseteq L^{(n)} \supseteq \cdots$$

The first derived $L^{(1)}$ and second derived $L^{(2)}$ can be denoted by L' and L'', respectively.

Definition 2.3.3: L is solvable if for some $n \geq 1$, $L^{(n)} = 0$.

Definition 2.3.4: The Lie algebra L is said to be nilpotent if for some positive integer n, $L^n = 0$.

Lemma 2.3.5: Any nilpotent Lie algebra L is solvable.

Proof: Let's use induction of k to show that $L^{(k)} \subseteq L^k$

For $k = 1$, $L^{(1)} = L' = [L, L] = L^1$

For $k - 1$, let's consider $L^{(k-1)} \subseteq L^{k-1} = [L, \dots, L]$.

$L^{(k)} = \left[L^{(k-1)}, L^{(k-1)}\right] \subseteq [L^{k-1}, L] = L^k$.

For $n \geq 1$, we have $L^n = \{0\}$, then $L^{(n)} \subseteq L^n = \{0\}$. Hence $L^{(n)} = \{0\}$. Therefore L is solvable.

Let L_n be the free Lie algebra over the field K of characteristic zero freely generated by a finite set $X_n = \{x_1, \dots, x_n\}$ with $n \geq 2$. We assume that the elements of X_n are Lie monomials of length 1. If x_1 and x_2 are Lie monomials of any length, then length of $[x_1, x_2]$ is the sum of lengths x_1 and x_2. The Lie monomials has the form of

$$[x_1, x_2, x_3] = [x_1, x_2]ad x_3 = \left[[x_1, x_2], x_3\right]$$

for the sake of simplicity. Hence, we may extent this inductively as

$[x_1, x_2, x_3, \dots, x_n] = \left[\cdots\left[[x_1, x_2], x_3\right], \dots, x_n\right], n \geq 3,$

for all $x_1, x_2, \dots, x_n \in L_n$. In this way, every element of L_n can be written as a linear combination of the left normed monomials, which means that, the set of all left normed Lie monomials spans the whole free Lie algebra L_n on X_n.

The quotient algebra

$$F_n = {}^{L_n}\!/_{L_n''} = {}^{L_n}\!/_{[[L_n, L_n],[L_n, L_n]]}$$

is called the free metabelian Lie algebra of rank n defined by metabelian identity $\left[[x_1, x_2], [x_3, x_4]\right] = 0$. The second derived ideal of F_n is equal to zero. F_n is generated by free generators $y_1 = x_1 + L_n'', \cdots, y_n = x_n + L_n''$ and this algebra is of a basis consisting of $y_1, \dots, y_n$ together with all left normed monomials of the form:

$\left[y_{i_1}, y_{i_2}, \dots, y_{i_n}\right] = \left[\cdots\left[\left[y_{i_1}, y_{i_2}\right], y_{i_3}\right], \cdots, y_{i_n}\right], i_1 > i_2 \leq i_3 \leq \cdots \leq i_n \leq n.$

Let $w \in F_n'$ and let $x_1, x_2 \in F_n$ be arbitrary elements. Then as a consequence of Jacobi identity we have

$[w, x_1, x_2] + [x_1, x_2, w] + [x_2, w, x_1] = 0.$

Since $0 = [x_1, x_2, w] \in F_n'' = \{0\}$ we get

$$[w, x_1, x_2] = -[x_2, w, x_1] = [w, x_2, x_1].$$

Thus, F_n' is furnished with a natural structure of module of $K[y_1, \cdots, y_n]$. For more information see (Bahturin, 1987).

2.4. Free Metabelian Poisson Algebras

Let K be a field. A Poisson algebra P over a field K is a vector space equipped with two bilinear operations $(P, \cdot, [\,,])$ where $(P, \cdot)$ is a commutative associative K-algebra, and $(P, [\,,])$ is a Lie algebra such that the Leibniz identity holds; i.e.,

$$[x_1, x_2 \cdot x_3] = [x_1, x_2] \cdot x_3 + x_2 \cdot [x_1, x_3] = [x_1, x_2] \cdot x_3 + [x_1, x_3] \cdot x_2$$

for each $x_1, x_2, x_3 \in P$. Hence the commutator can be considered as a K-derivation. We use the left normed commutators for simplicity:

$$\left[[x_1, x_2], x_3\right] = [x_1, x_2, x_3],$$

and extend it inductively to longer commutators. Agore and Militaru (2015) set the basic definitions in the concept of Poisson algebras. A Poisson algebra is abelian if the operations are trivial; i.e., $x_1 \cdot x_2 = [x_1, x_2] = 0$ for each pair (x_1, x_2) of elements, and any extension of an abelian Poisson algebra by another abelian Poisson algebra is called metabelian. Therefore, the free metabelian Poisson algebra exists because metabelian Poisson algebra form a variety.

Let us consider the free metabelian Poisson algebra P_n of rank $n \geq 2$ generated by the set $X_n = \{x_1, \dots, x_n\}$ over a field K of characteristic zero, and denote the K-vector spaces

$$P_n^2 = P_n \cdot P_n = K\{x_1 \cdot x_2 \mid x_1, x_2 \in P_n\},$$
$$P_n' = [P_n, P_n] = K\{[x_1, x_2] \mid x_1, x_2 \in P_n\}.$$

The following lemma can be found in (Zhang, Y et al 2019)

Lemma 2.4.1. Let P_n be a Poisson algebra. We have the following equality

$$P_n^2 \cdot P_n^2 + [P_n^2, P_n^2] = 0$$

if and only if P_n is metabelian P_n^2 is defined as the vector space

$$P_n \cdot P_n + [P_n, P_n] = 0.$$

Proof:$\Leftarrow$) Suppose that $0 \to A \to P_n \to B \to 0$ is exact sequence of Poisson algebras, where A and B are abelian. Since B is abelian Poisson algebra, we deduce that $P_n^2 + A = A$ and thus we obtain $P_n^2 \subseteq A$. Since A is also abelian we have

$$P_n^2 \cdot P_n^2 + [P_n^2, P_n^2] = 0$$

$\Longrightarrow$) Suppose that our equality $P_n^2 \cdot P_n^2 + [P_n^2, P_n^2] = 0$ is true then P_n^2 and P_n/P_n^2 are abelian. Therefore, there is an obvious exact sequence $0 \to P_n^2 \to P_n \to P_n/P_n^2 \to 0$. So, the equality $P_n \cdot P_n + [P_n, P_n] = 0$ where, $P_n^2 = P_n \cdot P_n + [P_n, P_n]$ becomes

$$(P_n \cdot P_n + [P_n, P_n]) \cdot (P_n \cdot P_n + [P_n, P_n]) + \left[P_n \cdot P_n + [P_n, P_n], P_n \cdot P_n + [P_n, P_n]\right] = 0$$

Let denote that $P_n \cdot P_n = P_n^2$ and $[P_n, P_n] = P_n'$ so, we have

$$\begin{aligned} 0 &= (P_n^2 + P_n') \cdot (P_n^2 + P_n') + [P_n^2 + P_n', P_n^2 + P_n'] \\ &= P_n^2 \cdot P_n^2 + P_n^2 \cdot P_n' + P_n' \cdot P_n^2 + P_n' \cdot P_n' + [P_n^2, P_n^2] + [P_n^2, P_n'] + [P_n', P_n^2] \\ &\quad + [P_n', P_n'] \\ &= P_n^2 \cdot P_n^2 + P_n' \cdot P_n^2 + P_n' \cdot P_n' + [P_n^2, P_n^2] + [P_n', P_n^2] + [P_n', P_n']. \end{aligned}$$

Remark 2.4.2. For two vector space V and W, if the intersection $V \cap W$ and the direct sum $V \oplus W$ are empty then $V = W = \{0\}$.

Form the above remark our equality is equivalently to

$$P_n^4 = [P_n^2, P_n^2] = [P_n', P_n^2] = P_n' \cdot P_n^2 = P_n' \cdot P_n' = [P_n', P_n'] = 0$$

as vector spaces. Therefore $[P_n', P_n^2] = 0$ is considered as Leibniz identity while $[P_n', P_n'] = 0$ is free metabelian Lie algebra.

As a results, for all $x_1, x_2, x_3, x_4 \in P_n$, we have

$$\begin{aligned} 0 &= x_1 \cdot x_2 \cdot x_3 \cdot x_4 = [x_1 \cdot x_2, x_3 \cdot x_4] = [[x_1, x_2], x_3 \cdot x_4] \\ &= [x_1, x_2] \cdot x_3 \cdot x_4 = [x_1, x_2] \cdot [x_3, x_4] = \left[[x_1, x_2], [x_3, x_4]\right]. \end{aligned}$$

Remark 2.4.3. Let $x_1, x_2, x_3, x_4 \in P_n$ from the equality above we have the following identity holds

(1) Jacobi Identity: $[x_1, x_2, x_3] + [x_2, x_3, x_1] + [x_3, x_1, x_2] = 0$,

(2) Skew symmetry: $[x_1, x_2] + [x_2, x_1] = 0$,

(3) Leibniz Identity: $[x_1, x_2 \cdot x_3] = [x_1, x_2] \cdot x_3 + [x_1, x_3] \cdot x_2$

(4) $[x_1, x_2, x_3, x_4] = [x_1, x_2, x_4, x_3]$

(5) $[x_1, x_2, x_3] \cdot x_4 = -[x_1, x_2, x_4] \cdot x_3$

(6) $[x_1, x_2, x_4] \cdot x_4 = [x_1, x_2, x_4] \cdot x_4 + [x_1, x_2, x_4] \cdot x_4 = 0$

$0 = 2[x_1, x_2, x_4] \cdot x_4 \Longrightarrow [x_1, x_2, x_4] \cdot x_4 = 0$, since char $K \neq 2$.

(7) $[x_1, x_2, x_2] \cdot x_3 = [x_1, x_2, x_2] \cdot x_3 + [x_1, x_2, x_3] \cdot x_2 = 0$

$[x_3, x_2, x_2] \cdot x_1 = [x_3, x_2, x_2] \cdot x_1 + [x_3, x_2, x_1] \cdot x_2 = 0$

to show that $[x_1, x_2, x_2] \cdot x_3 = [x_3, x_2, x_2] \cdot x_1$ we need to prove that $[x_1, x_2, x_3] \cdot x_2 = [x_3, x_2, x_1] \cdot x_2$ so,

$[x_1, x_2, x_3] = -[x_2, x_3, x_1] - [x_3, x_1, x_2] = [x_3, x_2, x_1] + [x_1, x_3, x_2]$

$[x_1, x_2, x_3] \cdot x_2 = [x_3, x_2, x_1] \cdot x_2 + [x_1, x_3, x_2] \cdot x_2 = [x_3, x_2, x_1] \cdot x_2.$

Hence,

$[x_1, x_2, x_2] \cdot x_3 = [x_3, x_2, x_2] \cdot x_1$

(8) $[x_1, x_2, x_2] \cdot x_1 = [x_1, x_2, x_2] \cdot x_1 + [x_1, x_2, x_1] \cdot x_2 = 0$

$[x_1, x_2, x_2] \cdot x_1 = -[x_1, x_2, x_1] \cdot x_2$. Hence,

$[x_1, x_2, x_2] \cdot x_1 = [x_2, x_1, x_1] \cdot x_2$

(9) $[x_1, x_2, x_3] \cdot x_4 = -[x_2, x_3, x_1] \cdot x_4 - [x_3, x_1, x_2] \cdot x_4$

$= [x_3, x_2, x_1] \cdot x_4 + [x_1, x_3, x_2] \cdot x_4$

(10) $2[x_1, x_2, x_3] \cdot x_4 = [x_1, x_2, x_3] \cdot x_4 - [x_1, x_2, x_4] \cdot x_3$

$= [x_1, x_3, x_2] \cdot x_4 - [x_2, x_3, x_1] \cdot x_4 - ([x_1, x_4, x_2] \cdot x_3 - [x_2, x_4, x_1] \cdot x_3)$

$= -[x_1, x_3, x_4] \cdot x_2 + [x_2, x_3, x_4] \cdot x_1 + [x_1, x_4, x_3] \cdot x_2 - [x_2, x_4, x_3] \cdot x_1$

$= -[x_1, x_3, x_4] \cdot x_2 + [x_1, x_4, x_3] \cdot x_2 + [x_2, x_3, x_4] \cdot x_1 - [x_2, x_4, x_3] \cdot x_1$

$= -[x_4, x_3, x_1] \cdot x_2 + [x_4, x_3, x_2] \cdot x_1 = 2[x_4, x_3, x_2] \cdot x_1$

$= 2([x_4, x_2, x_3] \cdot x_1 - [x_3, x_2, x_4] \cdot x_1)$ since char $K \neq 2$ we deduce

$$[x_1, x_2, x_3] \cdot x_4 = [x_4, x_2, x_3] \cdot x_1 - [x_3, x_2, x_4] \cdot x_1$$

for $x_2 < x_1 > x_4,\ x_2 \leq x_3 < x_4$.

3. COMPUTATIONAL FACTS

3.1. Hilbert Series

A vector space V over a field K of characteristic zero is called graded K-vector space if it has a direct sum

$$V = \sum_{k \geq 0} V^{(k)}$$

where $V^{(k)}$ is subspace and $k \geq 0$. In particular, we define a multigrading on V if

$$V = \sum_{i=1}^{m} \sum_{k_i \geq 0} V^{(k_1,\dots,k_m)},$$

and $V^{(k_1,\dots,k_m)}$ is called homogenous component of degree $(k_1, \dots, k_m)$.

Since $V = \bigoplus_{k \geq 0} V^{(k)}$ is a finitely generated graded K-algebra, then we define the Hilbert function of V as

$$H(V, k) = \dim_K\left(V^{(k)}\right)$$

where $\dim_K(V^{(k)})$ is the dimension of the vector space $V^{(k)}$ over K.

Let F_n be a relatively free algebra of the variety $\mathfrak{B}$ of (not necessarily associative or Lie) algebra, then is graded vector space with finite dimension of homogenous component of degree k of F_n. Hence, we can define formal power series also known as the Hilbert or Poincaré series

$$H(F_n, t) = \sum_{k \geq 0} dim_K(V^{(k)})\, t^k$$

Since F_n is finitely generated for each non-negative integer j, then there are many finitely monomials of degree j in generators of F_n. For multigrated of algebra F_n with a $\mathbb{Z}^k$-grading; the Hilbert series of F_n is

$$H(F_n, t_1, \cdots, t_m) = \sum_{k\geq 0} dim_K(V^{(k_1,\cdots,k_m)})\, t_1^{k_1} \cdots t_m^{k_m}.$$

where $F_n^{(k_1,\cdots,k_m)}$ is the multi-homogeneous component of degree $(k_1, \cdots, k_m)$. Equivalently, for a Weitzenböck derivation δ of F_n, the algebra of constants F_n^{δ} is also a graded and eventually its Hilbert series is

$$H\left(F_n^{\delta}, t\right) = \sum_{n\geq 0} dim_K(F_n^{\delta})^{(k)}\, t^k$$

3.2. Algebras of Constants in Polynomial Algebras

In the following examples, we use direct computations from Maple in order to get Hilbert series of the algebra of constants of polynomial algebra which will help us to get the generators of the algebra of constants of polynomial algebras.

Example 3.2.1: The algebra of constants $K[X_2]^{\delta_2}$ is generated by x_1.

Let $K[X_2]$ be the polynomial algebra generated by $X_2 = \{x_1, x_2\}$. Then $K[X_2]^{\delta_2}$ is the algebras of constants of polynomial algebras where the basic Weitzenböck derivations is of the form of $\delta_2 : x_2 \longrightarrow x_1 \longrightarrow 0$. From the Maple computation the Hilbert Series of polynomial algebra is

$$H_{G_{L_2}}(K[X_2], t_1, t_2, z) = \frac{1}{1 - t_1 z}$$

Since the numerator is a constant, this means that there is no relation

- $f_1 = \alpha x_1, \alpha \in K$

$\delta_2(f_1) = \delta_2(\alpha x_1) = \alpha\delta_2(x_1) = 0$

Example 3.2.2: The algebra of constants $K[X_3]^{\delta_3}$ is generated by

$$x_1$$
$$-2x_1x_3 + x_2^2.$$

Let $K[X_3]$ be the polynomial algebra generated by $X_3 = \{x_1, x_2, x_3\}$. Then $K[X_3]^{\delta_3}$ is the algebras of constants of polynomial algebras where the basic Weitzenböck derivations is of the form of $\delta_3 : x_3 \longrightarrow x_2 \longrightarrow x_1 \longrightarrow 0$. From the Maple computation, the Hilbert Series of polynomial algebra is

$$H_{GL_2}(K[X_3], t_1, t_2, z) = \frac{1}{(t_1^2z - 1)(t_1t_2z - 1)(t_1t_2z + 1)}$$
$$= \frac{1}{(1 - t_1^2z)(1 - t_1^2t_2^2z^2)}$$

This means that we have two generators of bidegree (2,0) with length 1 and bidegree (2,2) with length 2. The following table summarize the bidegree of x_i for $i = 1,2,3$.

x_i	Bidegree
x_1	(2,0)
x_2	(1,1)
x_3	(0,2)

Let $f_1 = 1 - t_1^2z = (2,0)1$ and $f_2 = 1 - t_1^2t_2^2z^2 = (2,2)2$

- $f_1 = \alpha x_1, \alpha \in K$

$\delta_3(f_1) = \delta_3(\alpha x_1) = \alpha\delta_3(x_1) = 0$

- $f_2 = \alpha x_1x_3 + \beta x_2^2$

$\delta_3(f_2) = 0$

$\delta_3(\alpha x_1 x_3 + \beta x_2^2) = 0$

$\delta_3(\alpha x_1 x_3) + \delta_3(\beta x_2^2) = \alpha\big(\delta_3(x_1)x_3 + x_1\delta_3(x_3)\big) + 2\beta x_2\delta_3(x_2) = 0$

$\alpha x_1 x_2 + 2\beta x_1 x_2 = (\alpha + 2\beta)x_1 x_2 = 0$

$(\alpha + 2\beta) = 0$

$\alpha = -2\beta$

Let $\beta = 1$ then $\alpha = -2$ therefore

$$f_2 = -2x_1x_3 + x_2^2$$

Example 3.2.3: The algebra of constants $K[X_4]^{\delta_4}$ is generated by

$$x_1$$
$$-2x_1x_3 + x_2^2$$
$$3x_1^2x_4 - 3x_1x_2x_3 + x_2^3$$
$$6x_1x_2x_3x_4 - 2x_2^3x_4 - 3x_1^2x_4^2 - \frac{8}{3}x_1x_3^3 + x_2^2x_3^2.$$

Let $K[X_4]$ be the polynomial algebras generated by $X_4 = \{x_1, x_2, x_3, x_4\}$. Then $K[X_4]^{\delta_4}$ is the algebras of constants of polynomial algebras where the basic Weitzenböck derivations is of the form of $\delta_4 : x_4 \longrightarrow x_3 \longrightarrow x_2 \longrightarrow x_1 \longrightarrow 0$. From the Maple computation, the Hilbert Series of polynomial algebra is

$$\begin{aligned}
H_{GL_2}(K[X_4], t_1, t_2, z) &= \frac{1 - t_1^2t_2z + t_1^4t_2^2z^2}{(1 - t_1^2t_2z)(1 - t_1^3t_2^3z^2)(1 + t_1^3t_2^3z^2)(1 - t_1^3z)} \\
&= \frac{(1 - t_1^2t_2z + t_1^4t_2^2z^2)(1 + t_1^2t_2z)}{(1 - t_1^2t_2z)(1 + t_1^2t_2z)(1 - t_1^6t_2^6z^4)(1 - t_1^3z)} \\
&= \frac{1 - t_1^2t_2z + t_1^4t_2^2z^2 + t_1^2t_2z - t_1^4t_2^2z^2 + t_1^6t_2^3z^3}{(1 - t_1^3z)(1 - t_1^4t_2^2z^2)(1 - t_1^6t_2^6z^4)}
\end{aligned}$$

$$= \frac{(1 + t_1^6 t_2^3 z^3)(1 - t_1^6 t_2^3 z^3)}{(1 - t_1^3 z)(1 - t_1^4 t_2^2 z^2)(1 - t_1^6 t_2^6 z^4)(1 - t_1^6 t_2^3 z^3)}$$

$$= \frac{(1 - t_1^{12} t_2^6 z^6)}{(1 - t_1^3 z)(1 - t_1^4 t_2^2 z^2)(1 - t_1^6 t_2^3 z^3)(1 - t_1^6 t_2^6 z^4)}$$

Here we have 4 generator of the form

$f_1 = (1 - t_1^3 z) = (3,0)1$ which means that f_1 has bidegree (3,0) and length 1,

$f_2 = (1 - t_1^4 t_2^2 z^2) = (4,2)2$ which means that f_2 has bidegree (4,2) and length 2,

$f_3 = (1 - t_1^6 t_2^3 z^3) = (6,3)3$ which means that f_3 has bidegree (6,3) and length 3,

$f_4 = (1 - t_1^6 t_2^6 z^4) = (6,6)4$ which means that f_4 has bidegree (6,6) and length 4.

The following table summarize the bidegree of x_i for $i = 1,2,3,4$.

x_i	Bidegree
x_1	(3,0)
x_2	(2,1)
x_3	(1,2)
x_4	(0,3)

- $f_1 = \alpha x_1, \alpha \in K$

  $\delta_4(f_1) = \delta_4(\alpha x_1) = \alpha \delta_4(x_1) = 0$

- $f_2 = \alpha x_1 x_3 + \beta x_2^2$

  $\delta_4(f_2) = 0$

  $0 = \delta_4(\alpha x_1 x_3 + \beta x_2^2)$

  $= \delta_4(\alpha x_1 x_3) + \delta_4(\beta x_2^2) = \alpha\big(\delta_4(x_1) x_3 + x_1 \delta_4(x_3)\big) + 2\beta x_2 \delta_4(x_2)$

  $= \alpha x_1 x_2 + 2\beta x_1 x_2 = (\alpha + 2\beta) x_1 x_2$

$(\alpha + 2\beta) = 0$

$\alpha = -2\beta$

Let $\beta = 1$ then $\alpha = -2$ therefore

$$f_2 = -2x_1 x_3 + x_2^2$$

- $f_3 = \alpha x_1^2 x_4 + \beta x_1 x_2 x_3 + \gamma x_2^3$

$0 = \delta_4(f_3) = \delta_4(\alpha x_1^2 x_4 + \beta x_1 x_2 x_3 + \gamma x_2^3)$

$= \alpha\delta_4(x_1^2 x_4) + \beta\delta_4(x_1 x_2 x_3) + \gamma\delta_4(x_2^3)$

$= \alpha\left(2x_1\delta_4(x_1)x_4 + x_1^2\delta_4(x_4)\right) + \beta\begin{pmatrix}\delta_4(x_1)x_2x_3 + x_1\delta_4(x_2)x_3\\ +(x_1x_2\delta_4(x_3)\end{pmatrix}$

$+3\gamma x_2^2\delta_4(x_2)$

$= \alpha x_1^2 x_3 + \beta x_1^2 x_3 + \beta x_1 x_2^2 + 3\gamma x_1 x_2^2 = (\alpha+\beta)x_1^2 x_3 + (\beta + 3\gamma)x_1 x_2^2$

$(\alpha+\beta) = 0 \Longrightarrow \alpha = -\beta$

$(\beta + 3\gamma)$=0 $\Longrightarrow \beta = -3\gamma$

Let $\gamma = 1$, then $\beta = -3$ and $\alpha = 3$ therefore

$$f_3 = 3x_1^2 x_4 - 3x_1 x_2 x_3 + x_2^3$$

- $f_4 = \alpha x_1 x_2 x_3 x_4 + \beta x_2^3 x_4 + \gamma x_1^2 x_4^2 + \theta x_1 x_3^3 + \varepsilon x_2^2 x_3^2$

$0 = \delta_4(f_4) = \delta_4(\alpha x_1 x_2 x_3 x_4 + \beta x_2^3 x_4 + \gamma x_1^2 x_4^2 + \theta x_1 x_3^3 + \varepsilon x_2^2 x_3^2)$

$= \alpha\delta_4(x_1x_2x_3x_4) + \beta\delta_4(x_2^3x_4) + \gamma\delta_4(x_1^2x_4^2) + \theta\delta_4(x_1x_3^3) + \varepsilon\delta_4(x_2^2x_3^2)$

$= \alpha\big(\delta_4(x_1)x_2x_3x_4 + x_1\delta_4(x_2)x_3x_4 + x_1x_2\delta_4(x_3)x_4 + x_1x_2x_3\delta_4(x_4)\big)$

$+\beta\left(\delta_4(x_2^3)x_4 + x_2^3\delta_4(x_4)\right) + \gamma\left(\delta_4(x_1^2)x_4^2 + x_1^2\delta_4(x_4^2)\right)$

$+\theta\left(\delta_4(x_1)x_3^3 + x_1\delta_4(x_3^3)\right) + \varepsilon\left(\delta_4(x_2^2)x_3^2 + x_2^2\delta_4(x_3^2)\right)$

$= \alpha(x_1^2x_3x_4 + x_1x_2^2x_4 + x_1x_2x_3^2) + \beta(3x_1x_2^2x_4 + x_2^3x_3) + 2\gamma x_1^2x_3x_4$

$+3\theta x_1x_2x_3^2 + 2\varepsilon(x_1x_2x_3^2 + 2x_2^3x_3)$

$= (\alpha+2\gamma)x_1^2x_3x_4 + (\alpha+3\beta)x_1x_2^2x_4 + (\alpha + 3\theta + 2\varepsilon)x_1x_2x_3^2$

$+(\beta + 2\varepsilon)x_2^3x_3 = 0$

$\alpha + 2\gamma = 0 \Longrightarrow \alpha = -2\gamma$

$\alpha + 3\beta = 0 \Longrightarrow \alpha = -3\beta$

$\beta + 2\varepsilon = 0 \Longrightarrow \beta = -2\varepsilon$

$\alpha + 3\theta + 2\varepsilon = 0$

Let $\varepsilon = 1$, then $\beta = -2$, $\alpha = 6$, $\gamma = -3$ and $3\theta = -\alpha - 2\varepsilon \Longrightarrow 3\theta = -8 \Longrightarrow \theta = -\frac{8}{3}$,

Therefore,

$$f_4 = 6x_1x_2x_3x_4 - 2x_2^3x_4 - 3x_1^2x_4^2 - \frac{8}{3}x_1x_3^3 + x_2^2x_3^2$$

Since the denominator of $H_{G_{L_2}}(K[X_4], t_1, t_2, z)$ is not constant its means that we the relation has bidegree (12,6) with length 6 and the following table show bidegree and length of f_i for $i = 1,2,3,4$.

f_i	Bidegree	Length
f_1	(3,0)	1
f_2	(4,2)	2
f_3	(6,3)	3
f_4	(6,6)	4

$$r := \alpha f_1^2 f_4 + \beta f_3^2 + \gamma f_2^3 = 0$$

$$0 = \alpha\left(x_1^2\left(6x_1x_2x_3x_4 - 2x_2^3x_4 - 3x_1^2x_4^2 - \frac{8}{3}x_1x_3^3 + x_2^2x_3^2\right)\right)$$
$$+\beta((3x_1^2x_4 - x_1x_2x_3 + x_2^3)^2) + \gamma((-x_1x_3 + x_2^2)^3)$$
$$= \alpha\left(6x_1^3x_2x_3x_4 - 2x_1^2x_2^3x_4 - 3x_1^4x_4^2 - \frac{8}{3}x_1^3x_3^3 + x_1^2x_2^2x_3^2\right)$$
$$+\beta\begin{pmatrix} 9x_1^4x_4^2 - 9x_1^3x_2x_3x_4 + 3x_1^2x_2^3x_4 - 9x_1^3x_2x_3x_4 + 9x_1^2x_2^2x_3^2 \\ -3x_1x_2^4x_3 + 3x_1^2x_2^3x_4 - 3x_1x_2^4x_3 + x_2^6 \end{pmatrix}$$
$$+\gamma(x_2^6 - 6x_1x_2^4x_3 + 12x_1^2x_2^2x_3^2 - 8x_1^3x_3^3)$$
$$= (6\alpha - 18\beta)x_1^3x_2x_3x_4 + (-2\alpha + 6\beta)x_1^2x_2^3x_4 + (-3\alpha + 9\beta)x_1^4x_4^2$$
$$+\left(-\tfrac{8}{3}\alpha - 8\gamma\right)x_1^3x_3^3 + (\alpha + 9\beta + 12\gamma)x_1^2x_2^2x_3^2 + (-6\beta - 6\gamma)x_1x_2^4x_3$$
$$+(\beta + \gamma)x_2^6$$

$(6\alpha - 18\beta) = 0 \implies \alpha = 3\beta$

$(-2\alpha + 6\beta) = 0 \implies \alpha = 3\beta$

$(-3\alpha + 9\beta) = 0 \implies \alpha = 3\beta$

$\left(-\frac{8}{3}\alpha - 8\gamma\right) = 0 \implies \alpha = -3\gamma$

$(\alpha + 9\beta + 12\gamma) = 0 \implies$

$(-6\beta - 6\gamma) = 0 \implies \beta = -\gamma$

$(\beta + \gamma) = 0 \implies \beta = -\gamma$

Let $\beta = 1$, then $\alpha = 3$ and $\gamma = -1$ therefore the relation is

$$r := 3f_1^2 f_4 + f_3^2 - f_2^3 = 0.$$

Example 3.2.4: The algebra of constants $K[X_5]^{\delta_5}$ is generated by

$$x_1$$
$$-2x_1x_3 + x_2^2$$
$$2x_1x_5 - 2x_2x_4 + x_3^2$$
$$-12x_1x_3x_5 - 6x_2x_3x_4 + 9x_1x_4^2 + 6x_2^2x_5 + 2x_3^3$$
$$3x_1^2x_4 - 3x_1x_2x_3 + x_2^3.$$

Let $K[X_5]$ be the polynomial algebras generated by $X_5 = \{x_1, x_2, x_3, x_4, x_5\}$. Then $K[X_5]^{\delta_5}$ is the algebras of constants of polynomial algebras where the basic Weitzenböck derivations is of the form of $\delta_5 : x_5 \longrightarrow x_4 \longrightarrow x_3 \longrightarrow x_2 \longrightarrow x_1 \longrightarrow 0$. From the Maple computation, the Hilbert Series of polynomial algebra is

$$H_{GL_2}(K[X_5], t_1, t_2, z) = \frac{1 - t_1^3t_2z + t_1^6t_2^2z^2}{(1 - t_1^4z)(1 - t_1^3t_2z)(1 - t_1^2t_2^2z)^2(1 + t_1^2t_2^2z)\begin{pmatrix}1 + t_1^2t_2^2z \\ +t_1^4t_2^4z^2\end{pmatrix}}$$

$$= \frac{(1 - t_1^3t_2z + t_1^6t_2^2z^2)(1 + t_1^3t_2z)}{(1 - t_1^4z)(1 - t_1^6t_2^2z^2)(1 - t_1^2t_2^2z)^2(1 + t_1^2t_2^2z)\begin{pmatrix}1 + t_1^2t_2^2z \\ +t_1^4t_2^4z^2\end{pmatrix}}$$

$$= \frac{1 - t_1^3t_2z + t_1^6t_2^2z^2 + t_1^3t_2z - t_1^6t_2^2z^2 + t_1^9t_2^3z^3}{(1 - t_1^4z)(1 - t_1^6t_2^2z^2)(1 - t_1^2t_2^2z)(1 + t_1^2t_2^2z)(1 - t_1^2t_2^2z)\begin{pmatrix}1 + t_1^2t_2^2z \\ +t_1^4t_2^4z^2\end{pmatrix}}$$

$$= \frac{(1 + t_1^9t_2^3z^3)}{(1 - t_1^4z)(1 - t_1^6t_2^2z^2)(1 - t_1^4t_2^4z^2)(1 - t_1^6t_2^6z^3)}$$

$$= \frac{(1+t_1^9t_2^3z^3)(1-t_1^9t_2^3z^3)}{(1-t_1^4z)(1-t_1^6t_2^2z^2)(1-t_1^4t_2^4z^2)(1-t_1^6t_2^6z^3)(1-t_1^9t_2^3z^3)}$$

$$= \frac{1-t_1^{18}t_2^6z^6}{(1-t_1^4z)(1-t_1^6t_2^2z^2)(1-t_1^4t_2^4z^2)(1-t_1^6t_2^6z^3)(1-t_1^9t_2^3z^3)}$$

Here we have 5 generator of the form

$f_1 = (1-t_1^4z) = (4,0)1$ which means that f_1 has bidegree (4,0) and length 1,

$f_2 = (1-t_1^6t_2^2z^2) = (6,2)2$ which means that f_2 has bidegree (6,2) and length 2,

$f_3 = (1-t_1^4t_2^4z^2) = (4,4)2$ which means that f_3 has bidegree (4,4) and length 2,

$f_4 = (1-t_1^6t_2^6z^3) = (6,6)3$ which means that f_4 has bidegree (6,6) and length 3,

$f_5 = (1-t_1^9t_2^3z^3) = (9,3)3$ which means that f_5 has bidegree (9,3) and length 3,

The following table summarize the bidegree of x_i for $i = 1,2,3,4,5$.

x_i	Bidegree
x_1	(4,0)
x_2	(3,1)
x_3	(2,2)
x_4	(1,3)
x_5	(0,4)

- $f_1 = \alpha x_1, \alpha \in K$

  $\delta_5(f_1) = \delta_5(\alpha x_1) = \alpha\delta_5(x_1) = 0$

- $f_2 = \alpha x_1x_3 + \beta x_2^2$

  $\delta_5(f_2) = 0$

  $0 = \delta_5(\alpha x_1x_3 + \beta x_2^2)$

  $= \delta_5(\alpha x_1x_3) + \delta_5(\beta x_2^2) = \alpha\big(\delta(x_1)x_3 + x_1\delta(x_3)\big) + 2\beta x_2\delta(x_2)$

  $= \alpha x_1x_2 + 2\beta x_1x_2 = (\alpha+2\beta)x_1x_2$

  $(\alpha+2\beta) = 0$

  $\alpha = -2\beta$

Let $\beta = 1$ then $\alpha = -2$ therefore

$$f_2 = -2x_1x_3 + x_2^2$$

- $f_3 = \alpha x_1x_5 + \beta x_2x_4 + \gamma x_3^2$

$0 = \delta_5(f_3) = \delta_5(\alpha x_1x_5 + \beta x_2x_4 + \gamma x_3^2)$

$= \alpha\big(\delta_5(x_1x_5)\big) + \beta\big(\delta_5(x_2x_4)\big) + \gamma\left(\delta_5(x_3^2)\right) = 0$

$= \alpha\big(\delta(x_1)x_5 + x_1\delta(x_5)\big) + \beta\big(\delta(x_2)x_4 + x_2\delta(x_4)\big) + 2\gamma x_3\delta(x_3)$

$= \alpha x_1x_4 + \beta x_1x_4 + \beta x_2x_3 + 2\gamma x_2x_3$

$= (\alpha+\beta)x_1x_4 + (\beta+2\gamma)x_2x_3$

$(\alpha+\beta) = 0 \implies \alpha = -\beta$

$(\beta+2\gamma) = 0 \implies \beta = -2\gamma$

Let $\gamma = 1$, then $\beta = -2$ and $\alpha = 2$ therefore

$$f_3 = 2x_1x_5 - 2x_2x_4 + x_3^2$$

- $f_4 = \alpha x_1x_3x_5 + \beta x_2x_3x_4 + \gamma x_1x_4^2 + \theta x_2^2x_5 + \varepsilon x_3^3$

$0 = \delta_5(f_4) = \delta_5(\alpha x_1x_3x_5 + \beta x_2x_3x_4 + \gamma x_1x_4^2 + \theta x_2^2x_5 + \varepsilon x_3^3)$

$= \alpha\big(\delta_5(x_1x_3x_5)\big) + \beta\big(\delta_5(x_2x_3x_4)\big) + \gamma\left(\delta_5(x_1x_4^2)\right) + \theta\left(\delta_5(x_2^2x_5)\right)$
$+\varepsilon\left(\delta_5(x_3^3)\right)$

$= \alpha\big(\delta_5(x_1)x_3x_5 + x_1\delta_5(x_3)x_5 + x_1x_3\delta_5(x_5)\big)$
$+\beta\big(\delta_5(x_2)x_3x_4 + x_2\delta_5(x_3)x_4 + x_2x_3\delta_5(x_4)\big)$
$+\gamma(\delta_5(x_1)x_4^2 + 2x_1\delta_5(x_4)x_4) + \theta\left(2\delta(x_2)x_2x_5 + x_2^2\delta(x_5)\right)$
$+3\varepsilon x_3^2\delta(x_3)$

$= \alpha x_1x_2x_5 + \alpha x_1x_3x_4 + \beta x_1x_3x_4 + \beta x_2^2x_4 + \beta x_2x_3^2 + 2\gamma x_1x_3x_4$
$+2\theta x_1x_2x_5 + \theta x_2^2x_4 + 3\varepsilon x_2x_3^2$

$= (\alpha+2\theta)x_1x_2x_5 + (\alpha+\beta+2\gamma)x_1x_3x_4 + (\beta+\theta)x_2x_3^2$
$+(\beta+3\varepsilon)x_2x_3^2$

$(\alpha+2\theta) = 0 \implies \alpha = -2\theta$

$(\beta+\theta) = 0 \implies \beta = -\theta$

$(\beta + 3\varepsilon) = 0 \implies \beta = -3\varepsilon$

$(\alpha + \beta + 2\gamma) = 0 \implies 2\gamma = -\alpha - \beta$

Let $\varepsilon = 2$, then $\beta = -6$, $\theta = 6$, $\alpha = -12$ and $\gamma = 9$ therefore

$$f_4 = -12x_1x_3x_5 - 6x_2x_3x_4 + 9x_1x_4^2 + 6x_2^2x_5 + 2x_3^3.$$

- $f_5 = \alpha x_1^2x_4 + \beta x_1x_2x_3 + \gamma x_2^3$

$$\begin{aligned}
0 = \delta_5(f_5) &= \delta_5(\alpha x_1^2x_4 + \beta x_1x_2x_3 + \gamma x_2^3) \\
&= \alpha\left(\delta_5(x_1^2x_4)\right) + \beta\big(\delta_5(x_1x_2x_3)\big) + \gamma\left(\delta_5(x_2^3)\right) \\
&= \alpha\left(2\delta_5(x_1)x_1x_4 + x_1^2\delta_5(x_4)\right) \\
&\quad +\beta\big(\delta_5(x_1)x_2x_3 + x_1\delta_5(x_2)x_3 + x_1x_2\delta_5(x_3)\big) + 3\gamma x_2^2\delta(x_2) \\
&= \alpha x_1^2x_3 + \beta x_1^2x_3 + \beta x_1x_2^2 + 3\gamma x_1x_2^2 \\
&= (\alpha + \beta)x_1^2x_3 + (\beta + 3\gamma)x_1x_2^2
\end{aligned}$$

$(\alpha + \beta) = 0 \implies \alpha = -\beta$

$(\beta + 3\gamma) = 0 \implies \beta = -3\gamma$

Let $\gamma = 1$, then $\beta = -3$ and $\alpha = 3$ therefore

$$f_5 = 3x_1^2x_4 - 3x_1x_2x_3 + x_2^3$$

Since the denominator of $H_{G_{L_2}}(K[X_5], t_1, t_2, z)$ is not constant its means that we the relation has bidegree (18,9)with length 6 and the following table show bidegree and length of f_i for $i = 1,2,3,4,5$.

f_i	Bidegree	Length
f_1	(4,0)	1
f_2	(6,2)	2
f_3	(4,4)	2
f_4	(6,6)	3
f_5	(9,3)	3

$$r := \alpha f_5^2 + \beta f_1^3 f_4 + \gamma f_1^2 f_2 f_3 + \theta f_2^3 = 0$$

$$0 = \alpha(3x_1^2x_4 - 3x_1x_2x_3 + x_2^3)^2$$
$$+\beta\left(x_1^3(-12x_1x_3x_5 - 6x_2x_3x_4 + 9x_1x_4^2 + 6x_2^2x_5 + 2x_3^3)\right)$$
$$+\gamma\left(x_1^2(-x_1x_3 + x_2^2)(2x_1x_5 - 2x_2x_4 + x_3^2)\right) + \theta(-x_1x_3 + x_2^2)^3$$
$$= \alpha\begin{pmatrix} 9x_1^4x_4^2 - 9x_1^3x_2x_3x_4 + 3x_1^2x_2^3x_4 - 9x_1^3x_2x_3x_4 + 9x_1^2x_2^2x_3^2 \\ -3x_1x_2^4x_3 + 3x_1^2x_2^3x_4 - 3x_1x_2^4x_3 + x_2^6 \end{pmatrix}$$
$$+\beta(-12x_1^4x_3x_5 - 6x_1^3x_2x_3x_4 + 9x_1^4x_4^2 + 6x_1^3x_2^2x_5 + 2x_1^3x_3^3)$$
$$+\gamma(-4x_1^4x_3x_5 + 4x_1^3x_2x_3x_4 - 2x_1^3x_3^3 + 2x_1^3x_2^2x_5 - 2x_1^2x_2^3x_4 + x_1^2x_2^2x_3^2)$$
$$+\theta(x_2^6 - 6x_1x_2^2x_3 + 12x_1^2x_2^2x_3^2 - 8x_1^3x_3^3)$$
$$= (9\alpha + 9\beta)x_1^4x_4^2 + (-18\alpha - 6\beta + 4\gamma)x_1^3x_2x_3x_4 + (6\alpha - 2\gamma)x_1^2x_2^3x_4$$
$$+(9\alpha + \gamma + 12\theta)x_1^2x_2^2x_3^2 + (-6\alpha - 6\theta)x_1x_2^4x_3 + (\alpha + \theta)x_2^6$$
$$+(-12\beta - 4\gamma)x_1^4x_3x_5 + (6\beta + 2\gamma)x_1^3x_2^2x_5 + (2\beta - 2\gamma - 8\theta)x_1^3x_3^3$$

$(9\alpha + 9\beta) = 0 \Longrightarrow \alpha = -\beta$

$(-18\alpha - 6\beta + 4\gamma) = 0 \Longrightarrow 18\beta - 6\beta + 4\gamma = 12\beta + 4\gamma = 0 \Longrightarrow \gamma = -3\beta$

$(6\alpha - 2\gamma) = 0 \Longrightarrow \gamma = 3\alpha$

$(9\alpha + \gamma + 12\theta) = 0 \Longrightarrow 12\alpha + 12\theta = 0 \Longrightarrow \alpha = -\theta$

$(-6\alpha - 6\theta) = 0 \Longrightarrow \alpha = -\theta$

$(\alpha + \theta) = 0 \Longrightarrow \alpha = -\theta$

$(-12\beta - 4\gamma) = 0 \Longrightarrow \gamma = -3\beta$

Let $\beta = 1$, then $\alpha = -1$, $\gamma = -3$ and $\theta = 1$ therefore the relation become

$$r := -f_5^2 + f_1^3f_4 - 3f_1^2f_2f_3 + f_2^3 = 0$$

Example 3.2.5: The algebra of constants $K[X_5]^{\delta_{3\times 2}}$ is generated by

$$x_1$$
$$-2x_1x_3 + x_2^2$$
$$x_1x_5^2 - 2x_2x_4x_5 + 2x_3x_4^2$$
$$x_4$$
$$x_1x_5 - x_2x_4.$$

Let $K[X_5]$ be the polynomial algebras generated by $X_5 = \{x_1, x_2, x_3, x_4, x_5\}$. Then $K[X_5]^{\delta_{3\times 2}}$ is the algebras of constants of polynomial algebras where the basic Weitzenböck derivations is of the form of

$$\delta_{3\times 2} : x_3 \longrightarrow x_2 \longrightarrow x_1 \longrightarrow 0,\ x_5 \longrightarrow x_4 \longrightarrow 0$$

From the Maple computation, the Hilbert Series of polynomial algebra is

$$H_{GL_2}(K[X_5], t_1, t_2, z) = \frac{1 + t_1^2 t_2 z^2}{(1 - t_1^2 z)(1 - t_1 t_2 z)(1 + t_1 t_2 z)(1 - t_1^2 t_2^2 z^3)(1 - t_1 z)}$$

$$= \frac{(1 + t_1^2 t_2 z^2)(1 - t_1^2 t_2 z^2)}{(1 - t_1^2 z)(1 - t_1^2 t_2^2 z^2)(1 - t_1^2 t_2^2 z^3)(1 - t_1 z)(1 - t_1^2 t_2 z^2)}$$

$$= \frac{(1 - t_1^4 t_2^2 z^4)}{(1 - t_1^2 z)(1 - t_1^2 t_2^2 z^2)(1 - t_1^2 t_2^2 z^3)(1 - t_1 z)(1 - t_1^2 t_2 z^2)}$$

Here we have 5 generator of the form

$f_1 = (1 - t_1^2 z) = (2{,}0)1$ which means that f_1 has bidegree (2,0) and length 1,

$f_2 = (1 - t_1^2 t_2^2 z^2) = (2{,}2)2$ which means that f_2 has bidegree (2,2) and length 2,

$f_3 = (1 - t_1^2 t_2^2 z^3) = (2{,}2)3$ which means that f_3 has bidegree (2,2) and length 3,

$f_4 = (1 - t_1 z) = (1{,}0)1$ which means that f_4 has bidegree (1,0) and length 1,

$f_5 = (1 - t_1^2 t_2 z^2) = (2{,}1)1$ which means that f_5 has bidegree (2,1) and length 1,

The following table summarize the bidegree of x_i for $i = 1,2,3,4,5$.

x_i	Bidegree
x_1	(2,0)
x_2	(1,1)
x_3	(0,2)
x_4	(1,0)
x_5	(0,1)

- $f_1 = \alpha x_1, \alpha \in K$

  $\delta_{3\times 2}(f_1) = \delta_{3\times 2}(\alpha x_1) = \alpha\delta_{3\times 2}(x_1) = 0$
- $f_2 = \alpha x_1 x_3 + \beta x_2^2$

  $0 = \delta_{3\times 2}(f_2) = \delta_{3\times 2}(\alpha x_1 x_3 + \beta x_2^2) = \delta_{3\times 2}(\alpha x_1 x_3) + \delta_{3\times 2}(\beta x_2^2)$

  $= \alpha\big(\delta_{3\times 2}(x_1)x_3 + x_1\delta_{3\times 2}(x_3)\big) + 2\beta x_2\delta_{3\times 2}(x_2)$

  $= \alpha x_1 x_2 + 2\beta x_1 x_2 = (\alpha + 2\beta)x_1 x_2$

  $(\alpha + 2\beta) = 0$

  $\alpha = -2\beta$

  Let $\beta = 1$ then $\alpha = -2$ therefore

  $$f_2 = -2x_1 x_3 + x_2^2$$
- $f_3 = \alpha x_1 x_5^2 + \beta x_2 x_4 x_5 + \gamma x_3 x_4^2$

  $0 = \delta_{3\times 2}(f_3) = \delta_{3\times 2}\big(\alpha x_1 x_5^2 + \beta x_2 x_4 x_5 + \gamma x_3 x_4^2\big)$

  $= \alpha\delta_{3\times 2}\big(x_1 x_5^2\big) + \beta\delta_{3\times 2}(x_2 x_4 x_5) + \gamma\delta_{3\times 2}(x_3 x_4^2)$

  $= 2\alpha x_1 x_4 x_5 + \beta(x_1 x_4 x_5 + x_2 x_4^2) + \gamma x_2 x_4^2$

  $= (2\alpha + \beta)x_1 x_4 x_5 + (\beta + \gamma)x_2 x_4^2$

  $(2\alpha + \beta) = 0 \implies \beta = -2\alpha$

  $(\beta + \gamma) = 0 \implies \beta = -\gamma$

  Let $\alpha = 1$, then $\beta = -2$ and $\gamma = 2$ therefore

  $$f_3 = x_1 x_5^2 - 2x_2 x_4 x_5 + 2x_3 x_4^2$$
- $f_4 = \alpha x_4, \alpha \in K$

  $\delta_{3\times 2}(f_4) = \delta_{3\times 2}(\alpha x_4) = \alpha\delta_{3\times 2}(x_4) = 0$
- $f_5 = \alpha x_1 x_5 + \beta x_2 x_4$

  $0 = \delta_{3\times 2}(f_5) = \delta_{3\times 2}(\alpha x_1 x_5 + \beta x_2 x_4) = \alpha\delta_{3\times 2}(x_1 x_5) + \beta\delta_{3\times 2}(x_2 x_4)$

  $= \alpha x_1 x_4 + \beta x_1 x_4 = (\alpha + \beta)x_1 x_4 = 0$

  $(\alpha + \beta) = 0 \implies \alpha = -\beta$

  Let $\alpha = 1$, then $\beta = -1$ therefore

  $$f_5 = x_1 x_5 - x_2 x_4$$

Since the denominator of $H_{G_{L_2}}(K[X_5], t_1, t_2, z)$ is not constant its means that we the relation has bidegree (4,2)with length 4 and the following table show bidegree and length of f_i for $i = 1,2,3,4,5$.

f_i	Bidegree	Length
f_1	(2,0)	1
f_2	(2,2)	2
f_3	(2,2)	3
f_4	(1,0)	1
f_5	(2,1)	2

$$r := \alpha f_5^2 + \beta f_2 f_4^2 + \gamma f_1 f_3 = 0$$

$$\begin{aligned}
0 &= \alpha(x_1x_5 - x_2x_4)^2 + \beta(-2x_1x_3 + x_2^2)x_4^2 + \gamma x_1\left(x_1x_5^2 - 2x_2x_4x_5 + 2x_3x_4^2\right) \\
&= \alpha x_1^2x_5^2 - 2\alpha x_1x_2x_4x_5 + \alpha x_2^2x_4^2 - 2\beta x_1x_3x_4^2 + \beta x_2^2x_4^2 + \gamma x_1^2x_5^2 \\
&\quad -2\gamma x_1x_2x_4x_5 + 2\gamma x_1x_3x_4^2 \\
&= (\alpha+\gamma)x_1^2x_5^2 + (-2\alpha - 2\gamma)x_1x_2x_4x_5 + (\alpha+\beta)x_2^2x_4^2 \\
&\quad +(-2\beta + 2\gamma)x_1x_3x_4^2
\end{aligned}$$

$(\alpha + \gamma) = 0 \Rightarrow \alpha = -\gamma$

$(-2\alpha - 2\gamma) = 0 \Rightarrow \alpha = -\gamma$

$(\alpha + \beta) = 0 \Rightarrow \alpha = -\beta$

$(-2\beta + 2\gamma) = 0 \Rightarrow \beta = \gamma$

Let $\alpha = 1$, then $\gamma = -1$ and $\beta = -1$ therefore

$$r := f_5^2 - f_2 f_4^2 - f_1 f_3 = 0$$

3.3. Algebras of Constants in Free Metabelian Lie Algebras

The examples in this section are based on elementary computation of generators of the algebra of constants as a $K[X_n]^{\delta_n}$-module $\left({L'_n}/{L''_n}\right)^{\delta_n}$ for $n \leq 4$.

Let consider a basic Weitzenböck derivation of free metabelian Lie algebra which has a single Jordan cell such that $\delta_n : x_n \longrightarrow x_{n-1} \longrightarrow \cdots \longrightarrow x_3 \longrightarrow x_2 \longrightarrow x_1 \longrightarrow 0$. Then by using simple maple computation we found the Hilbert series of the algebras of constant of free metabelian Lie algebra $\left(L'_n/L''_n\right)^{\delta_n}$ and from there we get the generating sets of the algebras of constant $K[X_n]^{\delta_n}$-module $\left(L'_n/L''_n\right)^{\delta_n}$ for $n \leq 4$. Again, we refer Dangovski et al (2013) for theoretical details.

Example 3.3.1: Let $n = 3$, then the $K[X_3]^{\delta_3}$-module $\left(L'_3/L''_3\right)^{\delta_3}$ is generated by

$$[x_2, x_1],$$
$$[x_3, x_1, x_1] - [x_2, x_1, x_2].$$

Let $K[X_3]$ be the polynomial algebras generated by $X_3 = \{x_1, x_2, x_3\}$. Then $K[X_3]^{\delta_3}$ is the algebras of constants of polynomial algebras for basic Weitzenböck derivations such that $\delta_3 : x_3 \longrightarrow x_2 \longrightarrow x_1 \longrightarrow 0$ and from Example 3.2.2, the algebras $K[X_3]^{\delta_3}$ is generated by the algebraically independent polynomials
x_1 and $x_2^2 - 2x_1x_3$.
The Hilbert Series of this polynomial algebra is

$$H_{GL_2}(K[X_3], t_1, t_2, z) = \frac{1}{(1 - t_1^2 z)(1 - t_1^2 t_2^2 z^2)}$$

Without loss of generality, the Hilbert series of polynomial algebra can be written in the form of power series

$$H_{G_{L_2}}(K[X_3], z_1, z_2) = \frac{1}{(1-t_1^2 z)(1-t_1^2 t_2^2 z^2)} = \frac{1}{(1-z_1)(1-z_2)}$$

From the basis of this polynomial algebra

$$\begin{aligned} H_{G_{L_2}}(K[X_3], z_1, z_2) = \frac{1}{(1-z_1)(1-z_2)} &= 1 + z_1 + z_2 + z_1 z_2 + z_1^2 + \cdots \\ &= 1 + t_1^2 z + t_1^2 t_2^2 z^2 + t_1^4 t_2^2 z^3 + t_1^4 z^2 + \cdots \\ &= 1 + t_1^2 z + (t_1^2 t_2^2 + t_1^4) z^2 + \cdots \end{aligned}$$

By the Maple computations we have the Hilbert series of free metabelian Lie algebra as follow

$$H_{G_{L_2}}\left(\left({L'_3}/{L''_3}\right)^{\delta_3}, t_1, t_2, z\right) = \frac{t_1^3 t_2 z^2}{(1-t_1^2 z)(1-t_1 t_2 z)}$$

this must have same denominator as the Hilbert series of polynomial algebras so, we have

$$H_{G_{L_2}}\left(\left({L'_3}/{L''_3}\right)^{\delta_3}, t_1, t_2, z\right) = \frac{t_1^3 t_2 z^2 (1 + t_1 t_2 z)}{(1-t_1^2 z)(1-t_1^2 t_2^2 z^2)} = \frac{t_1^3 t_2 z^2 + t_1^4 t_2^2 z^3}{(1-t_1^2 z)(1-t_1^2 t_2^2 z^2)}$$

We need to write this in the form of power series

So, we have

$$H_{G_{L_2}}\left(\left({L'_3}/{L''_3}\right)^{\delta_3}, t_1, t_2, z\right) = (t_1^3 t_2 z^2 + t_1^4 t_2^2 z^3) \cdot \begin{pmatrix} 1 + (t_1^2 + t_1) z \\ +(t_1^2 t_2^2 + t_1^3 + t_1^4) z^2 + \cdots \end{pmatrix}$$

Polynomial algebra of z_1, z_2 should be ignored when you are working on generators of $\left({L'_3}/{L''_3}\right)^{\delta_3}$ as a module of $K[z_1, z_2]$

$$H_{G_{L_2}}\left(\left({L'_3}/{L''_3}\right)^{\delta_3}, t_1, t_2, z\right) = t_1^3 t_2 z^2 + t_1^4 t_2^2 z^3$$

The following table summarized the x_i's with the corresponding bidegree for $n = 3$.

x_i	Bidegree
x_1	(2,0)
x_2	(1,1)
x_3	(0,2)

As we see in Hilbert series of algebras of constants of free metabelian Lie algebra, we have the bidegree (3,1) of length 2 and the bidegree (4,2) with 3 and this means that there are possibilities of two generator of length 2 and 3.

For a bidegree (3,1)2; the only possibility x_1x_2

This means we have only one generator $c_1$such that

$$c_1 = [x_2, x_1]$$

For a bidegree (4,2)3 ; we have two possibilities $x_1^2x_3$; $x_1x_2^2$ and it's can be written in this form

$$c_2 = \alpha_1[x_3, x_1, x_1] + \alpha_2[x_2, x_1, x_2]$$

such that $\alpha_1, \alpha_2 \in K$ and since c_2 is generator the derivation under δ_3 should be zero so,

$0 = \delta_3(c_2) = \delta_3(\alpha_1[x_3, x_1, x_1] + \alpha_2[x_2, x_1, x_2])$

$= \alpha_1[x_2, x_1, x_1] + \alpha_2[x_2, x_1, x_1] = (\alpha_1 + \alpha_2)[x_2, x_1, x_1]$

$(\alpha_1 + \alpha_2) = 0$

Here, $\alpha_2 = -\alpha_1$ this gives generator c_2 such that

$$c_2 = [x_3, x_1, x_1] - [x_2, x_1, x_2]$$

This gives that the $K[X_3]^{\delta_3}$ $-$module $\left({L'_3}/{L''_3} \right)^{\delta_3}$ has two generators c_1 and c_2 such that

$$c_1 = [x_2, x_1]$$
$$c_2 = [x_3, x_1, x_1] - [x_2, x_1, x_2]$$

Example 3.3.2: Let $n = 4$, then the $K[X_4]^{\delta_4}$-module $\left({L'_4}/{L''_4}\right)^{\delta_4}$ is generated by

$$[x_2, x_1],$$
$$[x_4, x_1] - [x_3, x_2],$$
$$[x_3, x_1, x_1] - [x_2, x_1, x_2],$$
$$3[x_2, x_1, x_4] - 2[x_3, x_1, x_3] + [x_3, x_2, x_2],$$
$$\begin{pmatrix} [x_3, x_2, x_2, x_2] - 3[x_3, x_1, x_1, x_4] + 4[x_4, x_1, x_1, x_3] + 3[x_2, x_1, x_2, x_4] \\ -2[x_4, x_1, x_2, x_2] - [x_3, x_1, x_2, x_3] \end{pmatrix}$$
$$\begin{pmatrix} 3[x_4, x_2, x_2, x_2, x_2] - 3[x_3, x_2, x_2, x_2, x_3] + 6[x_3, x_1, x_2, x_3, x_3] \\ -4[x_2, x_1, x_3, x_3, x_3] + 9[x_2, x_1, x_1, x_4, x_4] + 12[x_4, x_1, x_1, x_3, x_3] \\ -18[x_3, x_1, x_1, x_3, x_4] - 12[x_4, x_1, x_2, x_2, x_3] + 9[x_3, x_1, x_2, x_2, x_4] \end{pmatrix}$$
$$\begin{pmatrix} 3[x_4, x_2, x_2, x_2, x_2, x_2] - 3[x_3, x_2, x_2, x_2, x_2, x_3] - 8[x_3, x_1, x_1, x_3, x_3, x_3] \\ -18[x_4, x_1, x_1, x_1, x_3, x_4] + 18[x_3, x_1, x_1, x_1, x_4, x_4] - 15[x_4, x_1, x_2, x_2, x_2, x_3] \\ +3[x_3, x_1, x_2, x_2, x_2, x_4] + 9[x_2, x_1, x_2, x_2, x_3, x_4] + 12[x_3, x_1, x_2, x_2, x_3, x_3] \\ -10[x_2, x_1, x_2, x_3, x_3, x_3] + 9[x_4, x_1, x_1, x_2, x_2, x_4] - 18[x_2, x_1, x_1, x_2, x_4, x_4] \\ +18[x_4, x_1, x_1, x_2, x_3, x_3] - 18[x_3, x_1, x_1, x_2, x_3, x_4] + 18[x_2, x_1, x_1, x_3, x_3, x_4] \end{pmatrix}.$$

Let $K[X_4]$ be the polynomial algebras generated by $X_4 = \{x_1, x_2, x_3, x_4\}$. Then $K[X_4]^{\delta_4}$ is the algebras of constants of polynomial algebras for basic Weitzenböck derivations such that $\delta_4 : x_4 \longrightarrow x_3 \longrightarrow x_2 \longrightarrow x_1 \longrightarrow 0$ and from Example 3.2.3, the algebras $K[X_4]^{\delta_4}$ is generated by the algebraically independent polynomials x_1, $x_2^2 - 2x_1x_3$, $x_2^3 - 3x_1x_2x_3 + 3x_1^2x_4$ and $x_2^2x_3^2 - 2x_2^3x_4 - 3x_1^2x_4^2 - \frac{8}{3}x_1x_3^3 + 6x_1x_2x_3x_4$.

The Hilbert Series of this polynomial algebra is

$$\begin{aligned} H_{GL_2}(K[X_4], t_1, t_2, z) &= \frac{(1 - t_1^{12}t_2^6z^6)}{(1 - t_1^3z)(1 - t_1^4t_2^2z^2)(1 - t_1^6t_2^3z^3)(1 - t_1^6t_2^6z^4)} \\ &= \frac{(1 + t_1^6t_2^3z^3)}{(1 - t_1^3z)(1 - t_1^4t_2^2z^2)(1 - t_1^6t_2^6z^4)} \end{aligned}$$

Without loss of generality, the Hilbert series of polynomial algebra can be written in the form of power series

$$H_{G_{L_2}}(K[X_4], z_1, z_2, z_3) = \frac{1}{(1 - t_1^3 z)(1 - t_1^4 t_2^2 z^2)(1 - t_1^6 t_2^6 z^4)}$$
$$= \frac{1}{(1 - z_1)(1 - z_2)(1 - z_3)}$$

From the basis of this polynomial algebra

$$H_{G_{L_2}}(K[X_4], z_1, z_2, z_3) = \frac{1}{(1 - z_1)(1 - z_2)(1 - z_3)}$$
$$= 1 + z_1 + z_2 + z_3 + z_1 z_2 + z_1 z_3 + z_1^2 + \cdots$$
$$= 1 + t_1^3 z + t_1^4 t_2^2 z^2 + t_1^6 t_2^6 z^4 + t_1^7 t_2^2 z^3 + t_1^6 t_2^6 z^4$$
$$+ t_1^9 t_2^6 z^5 + t_1^6 z^2 + \cdots$$
$$= 1 + t_1^3 z + (t_1^4 t_2^2 + t_1^6) z^2 + \cdots$$

By the Maple computations we have the Hilbert series of free metabelian Lie algebra as follow

$$H_{G_{L_2}}\left(\left({L'_4}/{L''_4}\right)^{\delta_4}, t_1, t_2, z\right) = \frac{t_1^3 t_2 z^2 (t_1^2 + t_2^2 + t_1^4 t_2^4 z^2 + t_1^5 t_2^6 z^3 - t_1^8 t_2^6 z^4)}{(1 - t_1^3 z)(1 - t_1^2 t_2 z)(1 - t_1^6 t_2^6 z^4)}$$

this must have same denominator as the Hilbert series of polynomial algebras so, we have

$$H_{G_{L_2}}\left(\left({L'_4}/{L''_4}\right)^{\delta_4}, t_1, t_2, z\right) = \frac{t_1^3 t_2 z^2 \begin{pmatrix} t_1^2 + t_2^2 + t_1^4 t_2^4 z^2 \\ + t_1^5 t_2^6 z^3 - t_1^8 t_2^6 z^4 \end{pmatrix} (1 + t_1^2 t_2 z)}{(1 - t_1^3 z)(1 - t_1^4 t_2^2 z^2)(1 - t_1^6 t_2^6 z^4)}$$

We need to write this in the form of power series by denoting N as the numerator of Hilbert series of free metabelian Lie algebra such that

$$N = t_1^3 t_2 z^2 (t_1^2 + t_2^2 + t_1^4 t_2^4 z^2 + t_1^5 t_2^6 z^3 - t_1^8 t_2^6 z^4)(1 + t_1^2 t_2 z)$$

So, we have

$$H_{G_{L_2}}\left(\left(L_4'/L_4''\right)^{\delta_4}, t_1, t_2, z\right) = N \cdot 1 + t_1^3 z + (t_1^4 t_2^2 + t_1^6) z^2 + \cdots$$

Polynomial algebra of z_1, z_2, z_3 should be ignored when you are working on generators of $\left(L_4'/L_4''\right)^{\delta_4}$ as a module of $K[z_1, z_2, z_3]$

$$H_{G_{L_2}}\left(\left(L_4'/L_4''\right)^{\delta_4}, t_1, t_2, z\right) = N = t_1^3 t_2 z^2 \begin{pmatrix} t_1^2 + t_2^2 + t_1^4 t_2^4 z^2 \\ +t_1^5 t_2^6 z^3 - t_1^8 t_2^6 z^4 \end{pmatrix} (1 + t_1^2 t_2 z)$$

$$= (t_1^5 t_2 + t_1^3 t_2^3) z^2 + (t_1^7 t_2^2 + t_1^5 t_2^4) z^3 + t_1^7 t_2^5 z^4 + (t_1^8 t_2^7 + t_1^9 t_2^6) z^5 + (t_1^{10} t_2^8 - t_1^{11} t_2^7) z^6 - t_1^{13} t_2^7 z^7$$

The power of positive sign of t's in Hilbert series of algebras of constants of free metabelian Lie algebra represents the bidegree of generators and the power negative sign represents the relations while the power of z's represents the length.

The following table summarized the x_i's with the corresponding bidegree for $n = 4$.

x_i	Bidegree
x_1	(3,0)
x_2	(2,1)
x_3	(1,2)
x_4	(0,3)

As we see in Hilbert series of algebras of constants of free metabelian Lie algebra, we have the bidegree (5,1), (3,3) of length 2, the bidegree (7,2), (5,4)of 3, (7,5) of length 4, (8,7), (9,6) of length 5 and (10,8) of 6.

The Let us start with the bidegrees of length 2.

- For a bidegree $(5,1)2$; the only possibility x_1x_2

  This means we have only one generator $c_1$such that

$$c_1 = [x_2, x_1]$$

- For a bidegree $(3,3)2$; we have two possibilities x_1x_4 ; x_2x_3 and it's can be written in this form

$$c_2 = \alpha_1[x_4, x_1] + \alpha_2[x_3, x_2]$$

such that $\alpha_1, \alpha_2 \in K$ and since c_2 is generator the derivation under δ_4 should be zero so,

$0 = \delta_4(c_2) = \delta_4(\alpha_1[x_4, x_1] + \alpha_2[x_3, x_2])$

$= \alpha_1[x_3, x_1] + \alpha_2[x_3, x_1] = (\alpha_1 + \alpha_2)[x_3, x_1]$

$(\alpha_1 + \alpha_2) = 0$

Here, $\alpha_2 = -\alpha_1$ this gives generator c_2 such that

$$c_2 = [x_4, x_1] - [x_3, x_2]$$

This gives that the $K[X_4]^{\delta_4}$ −module $\left(L'_4/L''_4\right)^{\delta_4}$ has two generators c_1 and c_2 of length 2.

The next step is to found generators of length 3 with the following bidegrees $(7,2)$ and $(5,4)$. Therefore, we expect two generators of length three but before that we need to verify if this bidegree has new generators by comparing to the previous generators of algebra of constant of free metabelian Lie algebra we already have and the algebra of constants of polynomial algebra.

So, by comparing the bidegree of 2 previous generators of length 2 with f_1 : $(3,0)1$, we found if the new bidegree of length 3 has possible new generator. Let start with

- For a bidegree $(7,2)$ 3;

  $(7,2)\ 3 - (3,0)1 = (4,2)2$

We don't have any match from the previous bidegree of length 2 this means that (7,2)3 has new generator and let be denoted by c_3.

The possibilities of this bidegree are $x_1^2x_3$; $x_1x_2^2$ and c_3 can be written as linear combinations of all commutators in bidegree (7,2)3, so,

$$c_3 = \alpha_1[x_3, x_1, x_1] + \alpha_2[x_2, x_1, x_2]$$

such that $\alpha_1, \alpha_2 \in K$ and since c_3 is generator the derivation under δ_4 should be zero so,

$0 = \delta_4(c_3) = \delta_4(\alpha_1[x_3, x_1, x_1] + \alpha_2[x_2, x_1, x_2])$

$= \alpha_1[x_2, x_1, x_1] + \alpha_2[x_2, x_1, x_1] = (\alpha_1 + \alpha_2)[x_2, x_1, x_1]$

$(\alpha_1 + \alpha_2) = 0$

Here, $\alpha_2 = -\alpha_1$ this gives generator c_3 such that

$$c_3 = [x_3, x_1, x_1] - [x_2, x_1, x_2].$$

- For a bidegree (5,4) 3;

(5,4) 3 − (3,0)1 = (2,4)2

We don't have any match from the previous bidegree of length 2 this means that (5,4)3 has new generator and let be denoted by c_4.

The possibilities of this bidegree are $x_1x_2x_4$; $x_1x_3^2$; $x_2^2x_3$ and c_4 can be written as linear combinations of all commutators in bidegree (5,4)3, so,

$$c_4 = \alpha_1[x_2, x_1, x_4] + \alpha_2[x_4, x_1, x_2] + \alpha_3[x_3, x_1, x_3] + \alpha_4[x_3, x_2, x_2]$$

such that $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in K$ and since c_4 is generator the derivation under δ_4 should be zero so,

$0 = \delta_4(c_4) = \delta_4(\alpha_1[x_2, x_1, x_4] + \alpha_2[x_4, x_1, x_2] + \alpha_3[x_3, x_1, x_3] + \alpha_4[x_3, x_2, x_2])$

$= \alpha_1\delta_4[x_2, x_1, x_4] + \alpha_2\delta_4[x_4, x_1, x_2] + \alpha_3\delta_4[x_3, x_1, x_3] + \alpha_4\delta_4[x_3, x_2, x_2]$

$= \alpha_1[x_2, x_1, x_3] + \alpha_2([x_3, x_1, x_2] + [x_4, x_1, x_1]) + \alpha_3([x_2, x_1, x_3] + [x_3, x_1, x_2])$

$+\alpha_4(2[x_3, x_1, x_2] - [x_2, x_1, x_3])$

$=(\alpha_1 + \alpha_3 - \alpha_4)[x_2, x_1, x_3] + (\alpha_2 + \alpha_3 + 2\alpha_4)[x_3, x_1, x_2] + \alpha_2[x_4, x_1, x_1]$

$$(\alpha_1 + \alpha_3 - \alpha_4) = 0$$

$$(\alpha_2 + \alpha_3 + 2\alpha_4) = 0$$
$$\alpha_2 = 0$$

So, $\alpha_2 = 0$ and let $\alpha_4 = 1$ then, $\alpha_1 = 3$ and $\alpha_3 = -2$, this gives generator c_4 such that

$$c_4 = 3[x_2, x_1, x_4] - 2[x_3, x_1, x_3] + [x_3, x_2, x_2].$$

This gives that the $K[X_4]^{\delta_4}$ −module $\left({L'_4}/{L''_4} \right)^{\delta_4}$ has two generators c_3 and c_4 of length 3.

The next, we need to found generators of length 4. We have only one bidegree (7,5) of length 4 so, we expect one generator of length four but before that we need to verify if this bidegree has new generators by comparing to the previous generators of algebra of constant of free metabelian Lie algebra we already have and the algebra of constants of polynomial algebra.

So, by comparing the bidegree of 4 previous the generator of length 3 with f_1 : (3,0)1 and generators of length 2 with f_1^2: (6,0)2 , f_2 : (4,2)2, we found if the new bidegree of length 4 has possible new generator.

- For a bidegree (7,5)4;

$(7,5)4 - (3,0)1 = (4,5)3$

$(7,5)4 - (6,0)2 = (1,5)2$

$(7,5)4 - (4,2)2 = (3,3)2$

We don't have any match from the previous bidegree of length 2 and length 3 this means that (7,5)4 has new generator and let be denoted by c_5.

The possibilities of this bidegree are $x_2^3 x_3$; $x_1^2 x_3 x_4$; $x_2^2 x_1 x_4$; $x_3^2 x_1 x_2$ and c_5 can be written as linear combinations of all commutators in bidegree (7,5)4, so,

$$c_5 = \alpha_1[x_3, x_2, x_2, x_2] + \alpha_2[x_3, x_1, x_1, x_4] + \alpha_3[x_4, x_1, x_1, x_3] + \alpha_4[x_2, x_1, x_2, x_4] + \alpha_5[x_4, x_1, x_2, x_2] + \alpha_6[x_3, x_1, x_2, x_3] + \alpha_7[x_2, x_1, x_3, x_3]$$

such that $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \in K$ and since c_5 is generator the derivation under δ_4 should be zero so,

$$0 = \delta_4(c_5) = \delta_4\begin{pmatrix} \alpha_1[x_3, x_2, x_2, x_2] + \alpha_2[x_3, x_1, x_1, x_4] + \alpha_3[x_4, x_1, x_1, x_3] \\ +\alpha_4[x_2, x_1, x_2, x_4] + \alpha_5[x_4, x_1, x_2, x_2] + \alpha_6[x_3, x_1, x_2, x_3] \\ +\alpha_7[x_2, x_1, x_3, x_3] \end{pmatrix}$$

$$\begin{aligned}
&= \alpha_1\delta_4[x_3, x_2, x_2, x_2] + \alpha_2\delta_4[x_3, x_1, x_1, x_4] + \alpha_3\delta_4[x_4, x_1, x_1, x_3] \\
&\quad +\alpha_4\delta_4[x_2, x_1, x_2, x_4] + \alpha_5\delta_4[x_4, x_1, x_2, x_2] + \alpha_6\delta_4[x_3, x_1, x_2, x_3] \\
&\quad +\alpha_7\delta_4[x_2, x_1, x_3, x_3] \\
&= \alpha_1(3[x_3, x_1, x_2, x_2] - 2[x_2, x_1, x_2, x_3]) + \alpha_2([x_2, x_1, x_1, x_4] + [x_3, x_1, x_1, x_3]) \\
&\quad +\alpha_3([x_3, x_1, x_1, x_3] + [x_4, x_1, x_1, x_2]) + \alpha_4([x_2, x_1, x_1, x_4] + [x_2, x_1, x_2, x_3]) \\
&\quad +\alpha_5([x_3, x_1, x_2, x_2] + 2[x_4, x_1, x_1, x_2]) + \alpha_6([x_2, x_1, x_2, x_3] + [x_3, x_1, x_1, x_3] \\
&\quad +[x_3, x_1, x_2, x_2]) + 2\alpha_7[x_2, x_1, x_2, x_3] \\
&= (3\alpha_1 + \alpha_5 + \alpha_6)[x_3, x_1, x_2, x_2] + (-2\alpha_1 + \alpha_4 + \alpha_6 + 2\alpha_7)[x_2, x_1, x_2, x_3] \\
&\quad +(\alpha_2 + \alpha_4)[x_2, x_1, x_1, x_4] + (\alpha_2 + \alpha_3 + \alpha_6)[x_3, x_1, x_1, x_3] \\
&\quad +(\alpha_3 + 2\alpha_5)[x_4, x_1, x_1, x_2]
\end{aligned}$$

$$(3\alpha_1 + \alpha_5 + \alpha_6) = 0$$
$$(-2\alpha_1 + \alpha_4 + \alpha_6 + 2\alpha_7) = 0$$
$$(\alpha_2 + \alpha_4) = 0$$
$$(\alpha_2 + \alpha_3 + \alpha_6) = 0$$
$$(\alpha_3 + 2\alpha_5) = 0$$

So,

Let $\alpha_7 = 0$ and $\alpha_6 = -1$ then, $\alpha_1 = 1$; $\alpha_2 = -3$; $\alpha_3 = 4$; $\alpha_4 = 3$; $\alpha_5 = -2$

Therefore,

$$\begin{aligned}
c_5 = {} & [x_3, x_2, x_2, x_2] - 3[x_3, x_1, x_1, x_4] + 4[x_4, x_1, x_1, x_3] + 3[x_2, x_1, x_2, x_4] \\
& -2[x_4, x_1, x_2, x_2] - [x_3, x_1, x_2, x_3]
\end{aligned}$$

This gives that the $K[X_4]^{\delta_4}$ −module $\left({L'_4}/{L''_4}\right)^{\delta_4}$ has a generator c_5 of length 4.

The next, we need to found generators of length 5. We have only one bidegree (8,7)of length 5 so, we expect one generator of length five but before that we need to verify if this bidegree has new generators by comparing to the previous generators of algebra of constant of free metabelian Lie algebra we already have and the algebra of constants of polynomial algebra.

So, by comparing the bidegree of 5 previous the generator of length 4 with $f_1 : (3,0)1$, generators of length 3 with $f_1^2 : (6,0)2$, $f_2 : (4,2)2$ and generator of length 2 with $f_1^3 : (9,0)3$, $f_1 f_2 : (7,2)3$ and $f_3 : (6,3)3$. Then we found if the new bidegree of length 5 has possible new generator.

- For a bidegree (8,7)5;

$(8,7)5 - (3,0)1 = (5,7)4$

$(8,7)5 - (6,0)2 = (2,7)3$

$(8,7)5 - (4,2)2 = (4,5)3$

$(8,7)5 - (9,0)3 = (-1,7)2$

$(8,7)5 - (7,2)3 = (1,5)2$

$(8,7)5 - (6,3)3 = (2,4)2$

We don't have any match from the previous bidegree of length 2,3 and length 4 this means that (8,7)5 has new generator and let be denoted by c_6.

The possibilities of this bidegree are $x_2^4 x_4$; $x_2^3 x_3^2$; $x_3^3 x_1 x_2$; $x_1^2 x_2 x_4^2$; $x_1^2 x_3^2 x_4$; $x_2^2 x_1 x_3 x_4$ and c_6 can be written as linear combinations of all commutators in bidegree (8,7)5, so,

$$\begin{aligned} c_6 = {} & \alpha_1[x_4, x_2, x_2, x_2, x_2] + \alpha_2[x_3, x_2, x_2, x_2, x_3] + \alpha_3[x_3, x_1, x_2, x_3, x_3] \\ & + \alpha_4[x_2, x_1, x_3, x_3, x_3] + \alpha_5[x_4, x_1, x_1, x_2, x_4] + \alpha_6[x_2, x_1, x_1, x_4, x_4] \\ & + \alpha_7[x_4, x_1, x_1, x_3, x_3] + \alpha_8[x_3, x_1, x_1, x_3, x_4] + \alpha_9[x_4, x_1, x_2, x_2, x_3] \\ & + \alpha_{10}[x_3, x_1, x_2, x_2, x_4] + \alpha_{11}[x_2, x_1, x_2, x_3, x_4] \end{aligned}$$

such that $\alpha_1, \ldots, \alpha_{11} \in K$ and since c_6 is generator the derivation under δ_4 should be zero so,

$$0 = \delta_4(c_6)$$

$$= \delta_4 \begin{pmatrix} \alpha_1[x_4, x_2, x_2, x_2, x_2] + \alpha_2[x_3, x_2, x_2, x_2, x_3] + \alpha_3[x_3, x_1, x_2, x_3, x_3] \\ +\alpha_4[x_2, x_1, x_3, x_3, x_3] + \alpha_5[x_4, x_1, x_1, x_2, x_4] + \alpha_6[x_2, x_1, x_1, x_4, x_4] \\ +\alpha_7[x_4, x_1, x_1, x_3, x_3] + \alpha_8[x_3, x_1, x_1, x_3, x_4] + \alpha_9[x_4, x_1, x_2, x_2, x_3] \\ +\alpha_{10}[x_3, x_1, x_2, x_2, x_4] + \alpha_{11}[x_2, x_1, x_2, x_3, x_4] \end{pmatrix}$$

$$= \alpha_1\delta_4[x_4, x_2, x_2, x_2, x_2] + \alpha_2\delta_4[x_3, x_2, x_2, x_2, x_3] + \alpha_3\delta_4[x_3, x_1, x_2, x_3, x_3]$$
$$+\alpha_4\delta_4[x_2, x_1, x_3, x_3, x_3] + \alpha_5\delta_4[x_4, x_1, x_1, x_2, x_4] + \alpha_6\delta_4[x_2, x_1, x_1, x_4, x_4]$$
$$+\alpha_7\delta_4[x_4, x_1, x_1, x_3, x_3] + \alpha_8\delta_4[x_3, x_1, x_1, x_3, x_4] + \alpha_9\delta_4[x_4, x_1, x_2, x_2, x_3]$$
$$+\alpha_{10}\delta_4[x_3, x_1, x_2, x_2, x_4] + \alpha_{11}\delta_4[x_2, x_1, x_2, x_3, x_4]$$

$$= \alpha_1([x_3, x_2, x_2, x_2, x_2] + 4[x_4, x_1, x_2, x_2, x_2] - 3[x_2, x_1, x_2, x_2, x_4])$$
$$+\alpha_2(3[x_3, x_1, x_2, x_2, x_3] - 2[x_2, x_1, x_2, x_3, x_3] + [x_3, x_2, x_2, x_2, x_2])$$
$$+\alpha_3([x_2, x_1, x_2, x_3, x_3] + [x_3, x_1, x_1, x_3, x_3] + 2[x_3, x_1, x_2, x_2, x_3])$$
$$+3\alpha_4[x_2, x_1, x_2, x_3, x_3] + \alpha_5 \begin{pmatrix} [x_3, x_1, x_1, x_2, x_4] + [x_4, x_1, x_1, x_1, x_4] \\ +[x_4, x_1, x_1, x_2, x_3] \end{pmatrix}$$
$$+2\alpha_6[x_2, x_1, x_1, x_3, x_4] + \alpha_7([x_3, x_1, x_1, x_3, x_3] + 2[x_4, x_1, x_1, x_2, x_3])$$
$$+\alpha_8([x_2, x_1, x_1, x_3, x_4] + [x_3, x_1, x_1, x_2, x_4] + [x_3, x_1, x_1, x_3, x_3])$$
$$+\alpha_9([x_3, x_1, x_2, x_2, x_3] + 2[x_4, x_1, x_1, x_2, x_3] + [x_4, x_1, x_2, x_2, x_2])$$
$$+\alpha_{10}([x_3, x_1, x_2, x_2, x_4] + 2[x_3, x_1, x_1, x_2, x_4] + [x_3, x_1, x_2, x_2, x_3])$$
$$+\alpha_{11}([x_2, x_1, x_1, x_3, x_4] + [x_2, x_1, x_2, x_2, x_4] + [x_2, x_1, x_2, x_3, x_3])$$

$$= (\alpha_1 + \alpha_2)[x_3, x_2, x_2, x_2, x_2] + (4\alpha_1 + \alpha_9)[x_4, x_1, x_2, x_2, x_2]$$
$$+(-3\alpha_1 + \alpha_{10} + \alpha_{11})[x_2, x_1, x_2, x_2, x_4]$$
$$+(3\alpha_2 + 2\alpha_3 + \alpha_9 + \alpha_{10})[x_3, x_1, x_2, x_2, x_3]$$
$$+(-2\alpha_2 + \alpha_3 + 3\alpha_4 + \alpha_{11})[x_2, x_1, x_2, x_3, x_3]$$
$$+(\alpha_3 + \alpha_7 + \alpha_8)[x_3, x_1, x_1, x_3, x_3]$$
$$+(\alpha_5 + \alpha_8 + 2\alpha_{10})[x_3, x_1, x_1, x_2, x_4]$$
$$+\alpha_5[x_4, x_1, x_1, x_1, x_4]$$
$$+(\alpha_5 + 2\alpha_7 + 2\alpha_9)[x_4, x_1, x_1, x_2, x_3]$$
$$+(2\alpha_6 + \alpha_8 + \alpha_{11})[x_2, x_1, x_1, x_3, x_4]$$

$\alpha_1 + \alpha_2 = 0$, $4\alpha_1 + \alpha_9 = 0$, $-3\alpha_1 + \alpha_{10} + \alpha_{11} = 0$, $3\alpha_2 + 2\alpha_3 + \alpha_9 + \alpha_{10} = 0$

$-2\alpha_2 + \alpha_3 + 3\alpha_4 + \alpha_{11} = 0$, $\alpha_3 + \alpha_7 + \alpha_8 = 0$, $\alpha_5 + \alpha_8 + 2\alpha_{10} = 0$, $\alpha_5 = 0$

$\alpha_5 + 2\alpha_7 + 2\alpha_9 = 0$, $2\alpha_6 + \alpha_8 + \alpha_{11} = 0$, $(\alpha_3 + 2\alpha_5) = 0$

So,

Let $\alpha_5 = 0$, $\alpha_{11} = 0$ and $\alpha_{10} = 9$ then, $\alpha_1 = 3$; $\alpha_2 = -3$; $\alpha_3 = 6$; $\alpha_4 = -4$; $\alpha_6 = 9$; $\alpha_7 = 12$;

$\alpha_8 = -18$; $\alpha_9 = -12$

Therefore

$$\begin{aligned} c_6 = {} & 3[x_4,x_2,x_2,x_2,x_2] - 3[x_3,x_2,x_2,x_2,x_3] + 6[x_3,x_1,x_2,x_3,x_3] \\ & -4[x_2,x_1,x_3,x_3,x_3] + 9[x_2,x_1,x_1,x_4,x_4] + 12[x_4,x_1,x_1,x_3,x_3] \\ & -18[x_3,x_1,x_1,x_3,x_4] - 12[x_4,x_1,x_2,x_2,x_3] + 9[x_3,x_1,x_2,x_2,x_4] \end{aligned}$$

This gives that the $K[X_4]^{\delta_4}$ −module $\left({L'_4}/{L''_4} \right)^{\delta_4}$ has a generator c_6 of length 5.

Lastly, we need to found generators of length 6. We have only one bidegree (10,8)of length 6 so, we expect one generator of length six but before that we need to verify if this bidegree has new generators by comparing to the previous generators of algebra of constant of free metabelian Lie algebra we already have and the algebra of constants of polynomial algebra.

So, by comparing the bidegree of 6 previous the generator of length 5 with f_1 : (3,0)1, generators of length 3 with f_1^2: (6,0)2 , f_2 : (4,2)2, generator of length 3 with f_1^3: (9,0)3, $f_1 f_2$: (7,2)3 and f_3 : (6,3)3 and generator of length 2 with f_1^4: (12,0)4, $f_1^2 f_2$: (10,2)4, f_2^2 : (8,4)4, $f_1 f_3$: (9,3)4 and f_4 : (6,6)4. Then we found if the new bidegree of length 5 has possible new generator.

- For a bidegree(10,8)6;

(10,8)6 − (3,0)1 = (7,8)5

(10,8)6 − (6,0)2 = (4,8)4

$(10,8)6 - (4,2)2 = (6,6)4$

$(10,8)6 - (9,0)3 = (1,8)3$

$(10,8)6 - (7,2)3 \ \ = (3,6)3$

$(10,8)6 - (6,3)3 = (4,5)3$

$(10,8)6 - (12,0)4 = (-2,8)2$

$(10,8)6 - (10,2)4 = (0,6)2$

$(10,8)6 - (8,4)4 = (2,4)2$

$(10,8)6 - (9,3)4 \ \ = (1,5)2$

$(10,8)6 - (6,6)4 = (4,2)2$

We don't have any match from the previous bidegree of length 2,3,4 and length 5 this means that (10,8)6 has new generator and let be denoted by c_7.

The possibilities of this bidegree are $x_2^5x_4$; $x_2^4x_3^2$; $x_1^2x_3^4$; $x_1^3x_3x_4^2$; $x_1x_2^3x_3x_4$; $x_1x_2^2x_3^3$; $x_1^2x_2^2x_4^2$; $x_1^2x_2x_3^2x_4$ and c_7 can be written as linear combinations of all commutators in bidegree (10,8)6, so,

$$
\begin{aligned}
c_7 = {} & \alpha_1[x_4,x_2,x_2,x_2,x_2,x_2] + \alpha_2[x_3,x_2,x_2,x_2,x_2,x_3] \\
& +\alpha_3[x_3,x_1,x_1,x_3,x_3,x_3] + \alpha_4[x_4,x_1,x_1,x_1,x_3,x_4] \\
& +\alpha_5[x_3,x_1,x_1,x_1,x_4,x_4] + \alpha_6[x_4,x_1,x_2,x_2,x_2,x_3] \\
& +\alpha_7[x_3,x_1,x_2,x_2,x_2,x_4] + \alpha_8[x_2,x_1,x_2,x_2,x_3,x_4] \\
& +\alpha_9[x_3,x_1,x_2,x_2,x_3,x_3] + \alpha_{10}[x_2,x_1,x_2,x_3,x_3,x_3] \\
& +\alpha_{11}[x_4,x_1,x_1,x_2,x_2,x_4] + \alpha_{12}[x_2,x_1,x_1,x_2,x_4,x_4] \\
& +\alpha_{13}[x_4,x_1,x_1,x_2,x_3,x_3] + \alpha_{14}[x_3,x_1,x_1,x_2,x_3,x_4] \\
& +\alpha_{15}[x_2,x_1,x_1,x_3,x_3,x_4]
\end{aligned}
$$

such that $\alpha_1,\dots,\alpha_{15} \in K$ and since c_7 is generator the derivation under δ_4 should be zero so,

$$0=\delta_4(c_7)=\delta_4\begin{pmatrix}\alpha_1[x_4,x_2,x_2,x_2,x_2,x_2]+\alpha_2[x_3,x_2,x_2,x_2,x_2,x_3]\\+\alpha_3[x_3,x_1,x_1,x_3,x_3,x_3]+\alpha_4[x_4,x_1,x_1,x_1,x_3,x_4]\\+\alpha_5[x_3,x_1,x_1,x_1,x_4,x_4]+\alpha_6[x_4,x_1,x_2,x_2,x_2,x_3]\\+\alpha_7[x_3,x_1,x_2,x_2,x_2,x_4]+\alpha_8[x_2,x_1,x_2,x_2,x_3,x_4]\\+\alpha_9[x_3,x_1,x_2,x_2,x_3,x_3]+\alpha_{10}[x_2,x_1,x_2,x_3,x_3,x_3]\\+\alpha_{11}[x_4,x_1,x_1,x_2,x_2,x_4]+\alpha_{12}[x_2,x_1,x_1,x_2,x_4,x_4]\\+\alpha_{13}[x_4,x_1,x_1,x_2,x_3,x_3]+\alpha_{14}[x_3,x_1,x_1,x_2,x_3,x_4]\\+\alpha_{15}[x_2,x_1,x_1,x_3,x_3,x_4]\end{pmatrix}$$

$$\begin{aligned}&=\alpha_1\delta_4[x_4,x_2,x_2,x_2,x_2,x_2]+\alpha_2\delta_4[x_3,x_2,x_2,x_2,x_2,x_3]\\&+\alpha_3\delta_4[x_3,x_1,x_1,x_3,x_3,x_3]+\alpha_4\delta_4[x_4,x_1,x_1,x_1,x_3,x_4]\\&+\alpha_5\delta_4[x_3,x_1,x_1,x_1,x_4,x_4]+\alpha_6\delta_4[x_4,x_1,x_2,x_2,x_2,x_3]\\&+\alpha_7\delta_4[x_3,x_1,x_2,x_2,x_2,x_4]+\alpha_8\delta_4[x_2,x_1,x_2,x_2,x_3,x_4]\\&+\alpha_9\delta_4[x_3,x_1,x_2,x_2,x_3,x_3]+\alpha_{10}\delta_4[x_2,x_1,x_2,x_3,x_3,x_3]\\&+\alpha_{11}\delta_4[x_4,x_1,x_1,x_2,x_2,x_4]+\alpha_{12}\delta_4[x_2,x_1,x_1,x_2,x_4,x_4]\\&+\alpha_{13}\delta_4[x_4,x_1,x_1,x_2,x_3,x_3]+\alpha_{14}\delta_4[x_3,x_1,x_1,x_2,x_3,x_4]\\&+\alpha_{15}\delta_4[x_2,x_1,x_1,x_3,x_3,x_4]\end{aligned}$$

$$\begin{aligned}&=\alpha_1([x_3,x_2,x_2,x_2,x_2,x_2]+5[x_4,x_1,x_2,x_2,x_2,x_2]-4[x_2,x_1,x_2,x_2,x_2,x_4])\\&+\alpha_2(4[x_3,x_1,x_2,x_2,x_2,x_3]-3[x_2,x_1,x_2,x_2,x_3,x_3]+[x_3,x_2,x_2,x_2,x_2,x_2])\\&+\alpha_3([x_2,x_1,x_1,x_3,x_3,x_3]+3[x_3,x_1,x_1,x_2,x_3,x_3])\\&+\alpha_4([x_3,x_1,x_1,x_1,x_3,x_4]+[x_4,x_1,x_1,x_1,x_2,x_4]+[x_4,x_1,x_1,x_1,x_3,x_3])\\&+\alpha_5([x_3,x_1,x_1,x_1,x_4,x_4]+2[x_3,x_1,x_1,x_1,x_3,x_4])\\&+\alpha_6([x_3,x_1,x_2,x_2,x_2,x_3]+3[x_4,x_1,x_1,x_2,x_2,x_3]+[x_4,x_1,x_2,x_2,x_2,x_2])\\&+\alpha_7([x_2,x_1,x_2,x_2,x_2,x_4]+3[x_3,x_1,x_1,x_2,x_2,x_4]+[x_3,x_1,x_2,x_2,x_2,x_3])\\&+\alpha_8(2[x_2,x_1,x_1,x_2,x_3,x_4]+[x_2,x_1,x_2,x_2,x_2,x_4]+[x_2,x_1,x_2,x_2,x_3,x_3])\\&+\alpha_9([x_2,x_1,x_2,x_2,x_3,x_3]+2[x_3,x_1,x_1,x_2,x_3,x_3]+2[x_3,x_1,x_2,x_2,x_2,x_3])\\&+\alpha_{10}([x_2,x_1,x_1,x_3,x_3,x_3]+3[x_2,x_1,x_2,x_2,x_3,x_3])\\&+\alpha_{11}([x_3,x_1,x_1,x_2,x_2,x_4]+2[x_4,x_1,x_1,x_1,x_2,x_4]+[x_4,x_1,x_1,x_2,x_2,x_3])\\&+\alpha_{12}([x_2,x_1,x_1,x_1,x_4,x_4]+2[x_2,x_1,x_1,x_2,x_3,x_4])\\&+\alpha_{13}([x_3,x_1,x_1,x_2,x_3,x_3]+[x_4,x_1,x_1,x_1,x_3,x_3]+2[x_4,x_1,x_1,x_2,x_2,x_3])\end{aligned}$$

$$+\alpha_{14}\begin{pmatrix}[x_2,x_1,x_1,x_2,x_3,x_4]+[x_3,x_1,x_1,x_1,x_3,x_4]+[x_3,x_1,x_1,x_2,x_2,x_4]\\ +[x_3,x_1,x_1,x_2,x_3,x_3]\end{pmatrix}$$

$$+\alpha_{15}(2[x_2,x_1,x_1,x_2,x_3,x_4]+[x_2,x_1,x_1,x_3,x_3,x_3])$$

$$\begin{aligned}
&=(\alpha_1+\alpha_2)[x_3,x_2,x_2,x_2,x_2,x_2]\\
&\quad+(5\alpha_1+\alpha_6)[x_4,x_1,x_2,x_2,x_2,x_2]\\
&\quad+(-4\alpha_1+\alpha_7+\alpha_8)[x_2,x_1,x_2,x_2,x_2,x_4]\\
&\quad+(4\alpha_2+\alpha_6+\alpha_7+2\alpha_9)[x_3,x_1,x_2,x_2,x_2,x_3]\\
&\quad+(-3\alpha_2+\alpha_8+\alpha_9+3\alpha_{10})[x_2,x_1,x_2,x_2,x_3,x_3]\\
&\quad+(\alpha_3+\alpha_{10}+\alpha_{15})[x_2,x_1,x_1,x_3,x_3,x_3]\\
&\quad+(3\alpha_3+2\alpha_9+\alpha_{13}+\alpha_{14})[x_3,x_1,x_1,x_2,x_3,x_3]\\
&\quad+(\alpha_4+2\alpha_5+\alpha_{14})[x_3,x_1,x_1,x_1,x_3,x_4]\\
&\quad+(\alpha_4+2\alpha_{11})[x_4,x_1,x_1,x_1,x_2,x_4]\\
&\quad+(\alpha_4+\alpha_{13})[x_4,x_1,x_1,x_1,x_3,x_3]\\
&\quad+(\alpha_5+\alpha_{12})[x_2,x_1,x_1,x_1,x_4,x_4]\\
&\quad+(3\alpha_6+\alpha_{11}+2\alpha_{13})[x_4,x_1,x_1,x_2,x_2,x_3]\\
&\quad+(3\alpha_7+\alpha_{11}+\alpha_{14})[x_3,x_1,x_1,x_2,x_2,x_4]\\
&\quad+(2\alpha_8+2\alpha_{12}+\alpha_{14}+2\alpha_{15})[x_2,x_1,x_1,x_2,x_3,x_4]
\end{aligned}$$

$\alpha_1+\alpha_2=0, 5\alpha_1+\alpha_6=0, -4\alpha_1+\alpha_7+\alpha_8=0, 4\alpha_2+\alpha_6+\alpha_7+2\alpha_9=0$

$-3\alpha_2+\alpha_8+\alpha_9+3\alpha_{10}=0, \alpha_3+\alpha_{10}+\alpha_{15}=0, 3\alpha_3+2\alpha_9+\alpha_{13}+\alpha_{14}=0$

$\alpha_4+2\alpha_5+\alpha_{14}=0, \alpha_4+2\alpha_{11}=0, \alpha_4+\alpha_{13}=0, \alpha_5+\alpha_{12}=0$

$3\alpha_6+\alpha_{11}+2\alpha_{13}=0, 3\alpha_7+\alpha_{11}+\alpha_{14}=0, 2\alpha_8+2\alpha_{12}+\alpha_{14}+2\alpha_{15}=0$

$(\alpha_3+2\alpha_5)=0$

So, let $\alpha_{15}=18$ then, $\alpha_1=3$; $\alpha_2=-3$; $\alpha_3=-8$; $\alpha_4=-18$; $\alpha_5=18$; $\alpha_6=-15$; $\alpha_7=3$;

$\alpha_8=9$; $\alpha_9=12$; $\alpha_{10}=-10$; $\alpha_{11}=9$; $\alpha_{12}=-18$; $\alpha_{13}=18$; $\alpha_{14}=-18$

Therefore

$$\begin{aligned}
c_7&=3[x_4,x_2,x_2,x_2,x_2,x_2]-3[x_3,x_2,x_2,x_2,x_2,x_3]\\
&\quad-8[x_3,x_1,x_1,x_3,x_3,x_3]-18[x_4,x_1,x_1,x_1,x_3,x_4]
\end{aligned}$$

$+18[x_3, x_1, x_1, x_1, x_4, x_4] - 15[x_4, x_1, x_2, x_2, x_2, x_3]$
$+3[x_3, x_1, x_2, x_2, x_2, x_4] + 9[x_2, x_1, x_2, x_2, x_3, x_4]$
$+12[x_3, x_1, x_2, x_2, x_3, x_3] - 10[x_2, x_1, x_2, x_3, x_3, x_3]$
$+9[x_4, x_1, x_1, x_2, x_2, x_4] - 18[x_2, x_1, x_1, x_2, x_4, x_4]$
$+18[x_4, x_1, x_1, x_2, x_3, x_3] - 18[x_3, x_1, x_1, x_2, x_3, x_4]$
$+18[x_2, x_1, x_1, x_3, x_3, x_4]$

3.4. Symmetric Polynomials in Free Metabelian Lie Algebras

Let F_n be the free metabelian Lie algebra of rank n. An element $p \in F_n$ is called a symmetric polynomial if

$$p(x_1, \dots, x_n) = \pi * p(x_1, \dots, x_n) = p\big(x_{\pi(1)}, \dots, x_{\pi(n)}\big)$$

under the action $" * "$ of each $\pi \in S_n$.

Symmetric polynomials expressed as linear combinations of longer commutators are in the commutator ideal F_n' of F_n, and they form the Lie algebra $(F_n')^{S_n}$ which has a right $K[X_n]^{S_n}$-module structure as follows. Let $p(x_1, \dots, x_n) \in (F_n')^{S_n}$ and $q(x_1, \dots, x_n) \in K[X_n]^{S_n}$. Then for module multiplication $" * "$, we have

$$p(x_1, \dots, x_n) * q(x_1, \dots, x_n) = p(x_1, \dots, x_n) q(y_1, \dots, y_n)$$

where $y_i : F_n' \longrightarrow F_n'$, $1 \leq i \leq n$, is the adjoint operator; i.e., $py_i = [p, x_i]$ for all $p \in F_n'$.

due to Shmel'kin (1973) the free metabelian Lie algebra F_n can be embedded by ξ to an abelian wreath product $W(U_n)$. The image of each element in the commutator ideal F_n' via this embedding is as follows:

$$\xi : \sum_{i>k} [x_i, x_j]\, p_{ij}(Y_n) \longrightarrow \sum_{i>k} (u_i x_j - u_j x_i)\, p_{ij}(X_n) \in W(U_n).$$

See e.g. the paper by Drensky et al. (2020) for details. Recall that the algebra $K[X_n]^{S_n}$ is generated by the algebraically independent elementary symmetric polynomials

$$e_j^{(n)} = \sum_{i_1 < \cdots < i_j} x_{i_1} \cdots x_{i_j}\,,\ \ j = 1, \dots, n.$$

We compute the inverse image of $h_{ij}^{(n)}(U_n, X_n)$, $1 \le i < j \le n$, in order to obtain the generating set $\mathcal{G}_n$ of $(F_n')^{S_n}$ as a $K[X_n]^{S_n} = K\left[e_1^{(n)}, \dots, e_n^{(n)}\right]$-module.

The proofs of the following theorem and corollary were given in Drensky et al. (2020).

Theorem 3.4.1: The $K[X_n]^{S_n}$-module $(F_n')^{S_n}$ is generated by the set

$$\mathcal{G}_n = \left\{ \begin{matrix} \xi^{-1}\left(h_{ij}^{(n)}(U_n, X_n)\right) \mid h_{ij}^{(n)}(U_n, X_n) = j\varepsilon_i^{(n)}(U_n, X_n) e_j^{(n)}(X_n) \\ -i\varepsilon_j^{(n)}(U_n, X_n) e_i^{(n)}(X_n), \qquad 1 \le i < j \le n \end{matrix} \right\}$$

of polynomials, such that

$$\varepsilon_{q+1}^{(n)}(U_n, X_n) = \sum_{i=1}^{n} u_i \sum x_{j_1} \cdots x_{j_q}\,,\ \ 1 \le j_1 < \cdots < j_q \le n,\ \ i \ne j_l.$$

and

$e_1^{(n)} = x_1 + \cdots + x_n$

$e_2^{(n)} = x_1 x_2 + \cdots + x_1 x_n + x_2 x_3 + \cdots + x_{n-1} x_n$

$\vdots$

$e_n^{(n)} = x_1 \cdots x_n$

Corollary 3.4.2: The Lie algebra $F_n^{S_n}$ is generated by:

$$g_1(X_n) = \varepsilon_1(U_n, X_n) = x_1 + \cdots + x_n$$

(expressed as an element of F_n) and

$$h_{ij}^{(n)}(U_n, X_n) e_2^{a_2}(X_n) \cdots e_n^{a_n}(X_n), \qquad a_2, \dots, a_n \geq 0$$

(expressed as an element in the image of F_n in $(KU_n)wr(KT_n)$).

Example 3.4.3: We determine the expression of $h_{ij}^{(n)}(U_n, X_n)$ for $n \leq 5$ and We give the computations for the generator $\xi^{-1}\left(h_{ij}^{(n)}(U_n, X_n)\right) \in (F_n')^{S_n}$ as an illustration. Here all computations are straightforward according to Theorem 3.6.1.

Case $n = 2$. An element of F_2 with image in wreath product $(KU_2)wr(KT_2)$ is generated by $h_{12}^{(2)}(U_2, X_2)$ where

$$h_{12}^{(2)}(U_2, X_2) = 2\varepsilon_1^{(2)}(U_2, X_2) e_2^{(2)}(X_2) - \varepsilon_2^{(2)}(U_2, X_2) e_1^{(2)}(X_2)$$

Here $\varepsilon_1^{(2)}(U_2, X_2) = u_1 + u_2$, $\varepsilon_2^{(2)}(U_2, X_2) = u_2 x_1 + u_1 x_2$, $e_1^{(2)}(X_2) = x_1 + x_2$ and $e_2^{(2)}(X_2) = x_1 x_2$

We obtain that

$$\begin{aligned}
h_{12}^{(2)}(U_2, X_2) &= 2(u_1 + u_2)x_1 x_2 - (u_2 x_1 + u_1 x_2)(x_1 + x_2) \\
&= 2u_1 x_1 x_2 + 2u_2 x_1 x_2 - u_2 x_1 x_1 - u_2 x_1 x_2 - u_1 x_1 x_2 - u_1 x_2 x_2 \\
&= u_1 x_1 x_2 + u_2 x_1 x_2 - u_2 x_1 x_1 - u_1 x_2 x_2 \\
&= u_2 x_1 (x_2 - x_1) + u_1 x_2 (x_1 - x_2) \\
&= u_2 x_1 (x_2 - x_1) - u_1 x_2 (x_2 - x_1) \\
&= (u_2 x_1 - u_1 x_2)(x_2 - x_1)
\end{aligned}$$

In the wreath product $W(U_2)$. Therefore, the inverse of $h_{12}^{(2)}(U_2, X_2)$ as a $K[X_2]^{S_2}$-module $(F_2')^{S_2}$ in the algebra $(F_2')^{S_2}$ is equal to polynomial $g_{12}^{(2)}(X_2) = \xi^{-1}\left(h_{12}^{(2)}(U_2, X_2)\right) = [x_2, x_1, x_2 - x_1] \in (F_2')^{S_2}$.

Case $n = 3$. We have $1 \leq i < j \leq 3$. If $i = 1 \longrightarrow j = 2$ or 3 and if $i = 2 \longrightarrow j = 3$. Then, the elements of F_3 with their images in wreath product $(KU_3)wr(KT_3)$ is generated by $h_{12}^{(3)}(U_3, X_3)$, $h_{13}^{(3)}(U_3, X_3)$ and $h_{23}^{(3)}(U_3, X_3)$ where

$$h_{12}^{(3)}(U_3, X_3) = 2\varepsilon_1^{(3)}(U_3, X_3)e_2^{(3)}(X_3) - \varepsilon_2^{(3)}(U_3, X_3)e_1^{(3)}(X_3)$$

$$h_{13}^{(3)}(U_3, X_3) = 3\varepsilon_1^{(3)}(U_3, X_3)e_3^{(3)}(X_3) - \varepsilon_3^{(3)}(U_3, X_3)e_1^{(3)}(X_3)$$

$$h_{23}^{(3)}(U_3, X_3) = 3\varepsilon_2^{(3)}(U_3, X_3)e_3^{(3)}(X_3) - 2\varepsilon_3^{(3)}(U_3, X_3)e_2^{(3)}(X_3)$$

Here

$\varepsilon_1^{(3)}(U_3, X_3) = u_1 + u_2 + u_3$,

$\varepsilon_2^{(3)}(U_3, X_3) = u_2x_1 + u_1x_2 + u_3x_1 + u_1x_3 + u_3x_2 + u_2x_3$,

$\varepsilon_3^{(3)}(U_3, X_3) = u_1x_2x_3 + u_2x_1x_3 + u_3x_1x_2$,

$e_1^{(3)}(X_3) = x_1 + x_2 + x_3$, $e_2^{(3)}(X_3) = x_1x_2 + x_1x_3 + x_2x_3$ and

$e_3^{(3)}(X_3) = x_1x_2x_3$

In the wreath product $W(U_3)$, we obtain that

$$\begin{aligned}
h_{12}^{(3)}(U_3, X_3) &= 2(u_1 + u_2 + u_3)(x_1x_2 + x_1x_3 + x_2x_3) \\
&\quad -(u_2x_1 + u_1x_2 + u_3x_1 + u_1x_3 + u_3x_2 + u_2x_3)(x_1 + x_2 + x_3) \\
&= u_1x_1x_2 + u_1x_1x_3 + u_2x_1x_2 + u_2x_2x_3 + u_3x_1x_3 + u_3x_2x_3 \\
&\quad -u_2x_1x_1 - u_1x_2x_2 - u_3x_1x_1 - u_1x_3x_3 - u_3x_2x_2 - u_2x_3x_3 \\
&= u_2x_1(x_2 - x_1) + u_1x_2(x_1 - x_2) + u_3x_1(x_3 - x_1) \\
&\quad +u_1x_3(x_1 - x_3) + u_3x_2(x_3 - x_2) + u_2x_3(x_2 - x_3) \\
&= u_2x_1(x_2 - x_1) - u_1x_2(x_2 - x_1) + u_3x_1(x_3 - x_1) \\
&\quad -u_1x_3(x_3 - x_1) + u_3x_2(x_3 - x_2) - u_2x_3(x_3 - x_2) \\
&= (u_2x_1 - u_1x_2)(x_2 - x_1) + (u_3x_1 - u_1x_3)(x_3 - x_1) \\
&\quad +(u_3x_2 - u_2x_3)(x_3 - x_2)
\end{aligned}$$

By similarly computations we have

$$\begin{aligned}
h_{13}^{(3)}(U_3, X_3) &= (u_2x_1 - u_1x_2)(x_2 - x_1)x_3 + (u_3x_1 - u_1x_3)(x_3 - x_1)x_2 \\
&\quad +(u_3x_2 - u_2x_3)(x_3 - x_2)x_1
\end{aligned}$$

$h_{23}^{(3)}(U_3, X_3) = (u_2x_1 - u_1x_2)(x_2 - x_1)x_3x_3 + (u_3x_1 - u_1x_3)(x_3 - x_1)x_2x_2$
$\qquad +(u_3x_2 - u_2x_3)(x_3 - x_2)x_1x_1$

Therefore, the inverse of $h_{12}^{(3)}(U_3, X_3)$, $h_{13}^{(3)}(U_3, X_3)$ and $h_{23}^{(3)}(U_3, X_3)$ as a $K[X_3]^{S_3}$-module $(F_3')^{S_3}$ in the algebra $(F_3')^{S_3}$ are equal to polynomials $g_{12}^{(3)}(X_3)$, $g_{13}^{(3)}(X_3)$ and $g_{23}^{(3)}(X_3)$ respectively such that

$g_{12}^{(3)}(X_3) = [x_2, x_1, x_2 - x_1] + [x_3, x_1, x_3 - x_1] + [x_3, x_2, x_3 - x_2]$,

$g_{13}^{(3)}(X_3) = [x_2, x_1, x_2 - x_1, x_3] + [x_3, x_1, x_3 - x_1, x_2] + [x_3, x_2, x_3 - x_2, x_1]$,

$g_{23}^{(3)}(X_3) = [x_2, x_1, x_2 - x_1, x_3, x_3] + [x_3, x_1, x_3 - x_1, x_2, x_2]$
$\qquad +[x_3, x_2, x_3 - x_2, x_1, x_1]$.

Case $n = 4$. We have $1 \leq i < j \leq 4$. This means that

If $= 1 \longrightarrow j = 2$, 3 or 4;

$i = 2 \longrightarrow j = 3$ or 4; and

$i = 3 \longrightarrow j = 4$.

Then, the elements of F_4 with their images in wreath product $(KU_4)wr(KT_4)$ is generated by $h_{12}^{(4)}(U_4, X_4)$, $h_{13}^{(4)}(U_4, X_4)$, $h_{14}^{(4)}(U_4, X_4)$, $h_{23}^{(4)}(U_4, X_4)$, $h_{24}^{(4)}(U_4, X_4)$ and $h_{34}^{(4)}(U_4, X_4)$where

$h_{12}^{(4)}(U_4, X_4) = 2\varepsilon_1^{(4)}(U_4, X_4)e_2^{(4)}(X_4) - \varepsilon_2^{(4)}(U_4, X_4)e_1^{(4)}(X_4)$

$h_{13}^{(4)}(U_4, X_4) = 3\varepsilon_1^{(4)}(U_4, X_4)e_3^{(4)}(X_4) - \varepsilon_3^{(4)}(U_4, X_4)e_1^{(4)}(X_4)$

$h_{14}^{(4)}(U_4, X_4) = 4\varepsilon_1^{(4)}(U_4, X_4)e_4^{(4)}(X_4) - \varepsilon_4^{(4)}(U_4, X_4)e_1^{(4)}(X_4)$

$h_{23}^{(4)}(U_4, X_4) = 3\varepsilon_2^{(4)}(U_4, X_4)e_3^{(4)}(X_4) - 2\varepsilon_3^{(4)}(U_4, X_4)e_2^{(4)}(X_4)$

$h_{24}^{(4)}(U_4, X_4) = 4\varepsilon_2^{(4)}(U_4, X_4)e_4^{(4)}(X_4) - 2\varepsilon_4^{(4)}(U_4, X_4)e_2^{(4)}(X_4)$

$h_{34}^{(4)}(U_4, X_4) = 4\varepsilon_3^{(4)}(U_4, X_4)e_4^{(4)}(X_4) - 3\varepsilon_4^{(4)}(U_4, X_4)e_3^{(4)}(X_4)$

Here,

$\varepsilon_1^{(4)}(U_4, X_4) = u_1 + u_2 + u_3 + u_4$,

$\varepsilon_2^{(4)}(U_4, X_4) = u_2x_1 + u_1x_2 + u_3x_1 + u_1x_3 + u_4x_1 + u_1x_4 + u_3x_2 + u_2x_3$

$$+u_4x_2 + u_2x_4 + u_4x_3 + u_3x_4,$$

$$\varepsilon_3^{(4)}(U_4, X_4) = u_1x_2x_3 + u_1x_2x_4 + u_1x_3x_4 + u_2x_1x_3 + u_2x_1x_4 + u_2x_3x_4$$
$$+u_3x_1x_2 + u_3x_1x_4 + u_3x_2x_4 + u_4x_1x_2 + u_4x_1x_3 + u_4x_2x_3,$$

$$\varepsilon_4^{(4)}(U_4, X_4) = u_1x_2x_3x_4 + u_2x_1x_3x_4 + u_3x_1x_2x_4 + u_4x_1x_2x_3,$$

$$e_1^{(4)}(X_4) = x_1 + x_2 + x_3 + x_4,$$

$$e_2^{(4)}(X_4) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4,$$

$e_3^{(4)}(X_4) = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4$ and

$$e_4^{(4)}(X_4) = x_1x_2x_3x_4$$

In the wreath product $W(U_4)$, we obtain that

$$h_{24}^{(4)}(U_4, X_4) = 4\begin{pmatrix} u_2x_1 + u_1x_2 + u_3x_1 + u_1x_3 + u_4x_1 + u_1x_4 \\ +u_3x_2 + u_2x_3 + u_4x_2 + u_2x_4 + u_4x_3 + u_3x_4 \end{pmatrix} x_1x_2x_3x_4$$
$$-2\begin{pmatrix} u_1x_2x_3x_4 + u_2x_1x_3x_4 \\ +u_3x_1x_2x_4 + u_4x_1x_2x_3 \end{pmatrix}\begin{pmatrix} x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 \\ +x_2x_4 + x_3x_4 \end{pmatrix}$$
$$= 4u_2x_1x_1x_2x_3x_4 + 4u_1x_1x_2x_2x_3x_4 + 4u_3x_1x_1x_2x_3x_4$$
$$+4u_1x_1x_2x_3x_3x_4 + 4u_4x_1x_1x_2x_3x_4 + 4u_1x_1x_2x_3x_4x_4$$
$$+4u_3x_1x_2x_2x_3x_4 + 4u_2x_1x_2x_3x_3x_4 + 4u_4x_1x_2x_2x_3x_4$$
$$+4u_2x_1x_2x_3x_4x_4 + 4u_4x_1x_2x_3x_3x_4 + 4u_3x_1x_2x_3x_4x_4$$
$$-2u_1x_1x_2x_2x_3x_4 - 2u_2x_1x_1x_2x_3x_4 - 2u_3x_1x_1x_2x_2x_4$$
$$-2u_4x_1x_1x_2x_2x_3 - 2u_1x_1x_2x_3x_3x_4 - 2u_2x_1x_1x_3x_3x_4$$
$$-2u_3x_1x_1x_2x_3x_4 - 2u_4x_1x_1x_2x_3x_3 - 2u_1x_1x_2x_3x_4x_4$$
$$-2u_2x_1x_1x_3x_4x_4 - 2u_3x_1x_1x_2x_4x_4 - 2u_4x_1x_1x_2x_3x_4$$
$$-2u_1x_2x_2x_3x_3x_4 - 2u_2x_1x_2x_3x_3x_4 - 2u_3x_1x_2x_2x_3x_4$$
$$-2u_4x_1x_2x_2x_3x_3 - 2u_1x_2x_2x_3x_4x_4 - 2u_2x_1x_2x_3x_4x_4$$
$$-2u_3x_1x_2x_2x_4x_4 - 2u_4x_1x_2x_2x_3x_4 - 2u_1x_2x_3x_3x_4x_4$$
$$-2u_2x_1x_3x_3x_4x_4 - 2u_3x_1x_2x_3x_4x_4 - 2u_4x_1x_2x_3x_3x_4$$
$$= 2u_2x_1x_1x_2x_3x_4 + 2u_1x_1x_2x_2x_3x_4 + 2u_3x_1x_1x_2x_3x_4$$
$$+2u_1x_1x_2x_3x_3x_4 + 2u_4x_1x_1x_2x_3x_4 + 2u_1x_1x_2x_3x_4x_4$$
$$+2u_3x_1x_2x_2x_3x_4 + 2u_2x_1x_2x_3x_3x_4 + 2u_4x_1x_2x_2x_3x_4$$

$$+2u_2x_1x_2x_3x_4x_4 + 2u_4x_1x_2x_3x_3x_4 + 2u_3x_1x_2x_3x_4x_4$$
$$-2u_3x_1x_1x_2x_2x_4 - 2u_4x_1x_1x_2x_2x_3 - 2u_2x_1x_1x_3x_3x_4$$
$$-2u_4x_1x_1x_2x_3x_3 - 2u_2x_1x_1x_3x_4x_4 - 2u_3x_1x_1x_2x_4x_4$$
$$-2u_1x_2x_2x_3x_3x_4 - 2u_4x_1x_2x_2x_3x_3 - 2u_1x_2x_2x_3x_4x_4$$
$$-2u_3x_1x_2x_2x_4x_4 - 2u_1x_2x_3x_3x_4x_4 - 2u_2x_1x_3x_3x_4x_4$$
$$= u_2x_1x_3x_3x_4(x_2 - x_1) + u_1x_2x_3x_3x_4(x_1 - x_2)$$
$$+u_2x_1x_3x_4x_4(x_2 - x_1) + u_1x_2x_3x_4x_4(x_1 - x_2)$$
$$+u_3x_1x_2x_2x_4(x_3 - x_1) + u_1x_3x_2x_2x_4(x_1 - x_3)$$
$$+u_3x_1x_2x_4x_4(x_3 - x_1) + u_1x_3x_2x_4x_4(x_1 - x_3)$$
$$+u_4x_1x_2x_2x_3(x_4 - x_1) + u_1x_4x_2x_2x_3(x_1 - x_4)$$
$$+u_4x_1x_2x_3x_3(x_4 - x_1) + u_1x_4x_2x_3x_3(x_1 - x_4)$$
$$+u_3x_2x_1x_1x_4(x_3 - x_2) + u_2x_3x_1x_1x_4(x_2 - x_3)$$
$$+u_3x_2x_1x_4x_4(x_3 - x_2) + u_2x_3x_1x_4x_4(x_2 - x_3)$$
$$+u_4x_2x_1x_1x_3(x_4 - x_2) + u_2x_4x_1x_1x_3(x_2 - x_4)$$
$$+u_4x_2x_1x_3x_3(x_4 - x_2) + u_2x_4x_1x_3x_3(x_2 - x_4)$$
$$+u_4x_3x_1x_1x_2(x_4 - x_3) + u_3x_4x_1x_1x_2(x_3 - x_4)$$
$$+u_4x_3x_1x_2x_2(x_4 - x_3) + u_3x_4x_1x_2x_2(x_3 - x_4)$$
$$= (u_2x_1 - u_1x_2)(x_2 - x_1)x_3x_3x_4$$
$$+(u_2x_1 - u_1x_2)(x_2 - x_1)x_3x_4x_4$$
$$+(u_3x_1 - u_1x_3)(x_3 - x_1)x_2x_2x_4$$
$$+(u_3x_1 - u_1x_3)(x_3 - x_1)x_2x_4x_4$$
$$+(u_4x_1 - u_1x_4)(x_4 - x_1)x_2x_2x_3$$
$$+(u_4x_1 - u_1x_4)(x_4 - x_1)x_2x_3x_3$$
$$+(u_3x_2 - u_2x_3)(x_3 - x_2)x_1x_1x_4$$
$$+(u_3x_2 - u_2x_3)(x_3 - x_2)x_1x_4x_4$$
$$+(u_4x_2 - u_2x_4)(x_4 - x_2)x_1x_1x_3$$
$$+(u_4x_2 - u_2x_4)(x_4 - x_2)x_1x_3x_3$$
$$+(u_4x_3 - u_3x_4)(x_4 - x_3)x_1x_1x_2$$

$$
\begin{aligned}
&+(u_4x_3-u_3x_4)(x_4-x_3)x_1x_2x_2\\
&=(u_2x_1-u_1x_2)(x_2-x_1)(x_3+x_4)x_3x_4\\
&\quad+(u_3x_1-u_1x_3)(x_3-x_1)(x_2+x_4)x_2x_4\\
&\quad+(u_4x_1-u_1x_4)(x_4-x_1)(x_2+x_3)x_2x_3\\
&\quad+(u_3x_2-u_2x_3)(x_3-x_2)(x_1+x_4)x_1x_4\\
&\quad+(u_4x_2-u_2x_4)(x_4-x_2)(x_1+x_3)x_1x_3\\
&\quad+(u_4x_3-u_3x_4)(x_4-x_3)(x_1+x_2)x_1x_2
\end{aligned}
$$

with similarly computations we have

$$
\begin{aligned}
h_{12}^{(4)}(U_4,X_4)&=(u_2x_1-u_1x_2)(x_2-x_1)+(u_3x_1-u_1x_3)(x_3-x_1)\\
&+(u_4x_1-u_1x_4)(x_4-x_1)+(u_3x_2-u_2x_3)(x_3-x_2)\\
&+(u_4x_2-u_2x_4)(x_4-x_2)+(u_4x_3-u_3x_4)(x_4-x_3)
\end{aligned}
$$

$$
\begin{aligned}
h_{13}^{(4)}(U_4,X_4)&=(u_2x_1-u_1x_2)(x_2-x_1)(x_3+x_4)\\
&+(u_3x_1-u_1x_3)(x_3-x_1)(x_2+x_4)\\
&+(u_4x_1-u_1x_4)(x_4-x_1)(x_2+x_3)\\
&+(u_3x_2-u_2x_3)(x_3-x_2)(x_1+x_4)\\
&+(u_4x_2-u_2x_4)(x_4-x_2)(x_1+x_3)\\
&+(u_4x_3-u_3x_4)(x_4-x_3)(x_1+x_2)
\end{aligned}
$$

$$
\begin{aligned}
h_{14}^{(4)}(U_4,X_4)&=(u_2x_1-u_1x_2)(x_2-x_1)x_3x_4+(u_3x_1-u_1x_3)(x_3-x_1)x_2x_4\\
&+(u_4x_1-u_1x_4)(x_4-x_1)x_2x_3+(u_3x_2-u_2x_3)(x_3-x_2)x_1x_4\\
&+(u_4x_2-u_2x_4)(x_4-x_2)x_1x_3+(u_4x_3-u_3x_4)(x_4-x_3)x_1x_2
\end{aligned}
$$

$$
\begin{aligned}
h_{23}^{(4)}(U_4,X_4)&=(u_2x_1-u_1x_2)(x_2-x_1)(x_3+x_4)(x_3+x_4)\\
&-(u_2x_1-u_1x_2)(x_2-x_1)x_3x_4\\
&+(u_3x_1-u_1x_3)(x_3-x_1)(x_2+x_4)(x_2+x_4)\\
&-(u_3x_1-u_1x_3)(x_3-x_1)x_2x_4\\
&+(u_4x_1-u_1x_4)(x_4-x_1)(x_2+x_3)(x_2+x_3)\\
&-(u_4x_1-u_1x_4)(x_4-x_1)x_2x_3\\
&+(u_3x_2-u_2x_3)(x_3-x_2)(x_1+x_4)(x_1+x_4)
\end{aligned}
$$

$$-(u_3x_2-u_2x_3)(x_3-x_2)x_1x_4$$
$$+(u_4x_2-u_2x_4)(x_4-x_2)(x_1+x_3)(x_1+x_3)$$
$$-(u_4x_2-u_2x_4)(x_4-x_2)x_1x_3$$
$$+(u_4x_3-u_3x_4)(x_4-x_3)(x_1+x_2)(x_1+x_2)$$
$$-(u_4x_3-u_3x_4)(x_4-x_3)x_1x_2$$

$$h_{34}^{(4)}(U_4,X_4)=(u_2x_1-u_1x_2)(x_2-x_1)x_3x_3x_4x_4$$
$$+(u_3x_1-u_1x_3)(x_3-x_1)x_2x_2x_4x_4$$
$$+(u_4x_1-u_1x_4)(x_4-x_1)x_2x_2x_3x_3$$
$$+(u_3x_2-u_2x_3)(x_3-x_2)x_1x_1x_4x_4$$
$$+(u_4x_2-u_2x_4)(x_4-x_2)x_1x_1x_3x_3$$
$$+(u_4x_3-u_3x_4)(x_4-x_3)x_1x_1x_2x_2.$$

Therefore, the inverse of $h_{12}^{(4)}(U_4,X_4)$, $h_{13}^{(4)}(U_4,X_4)$, $h_{14}^{(4)}(U_4,X_4)$, $h_{23}^{(4)}(U_4,X_4)$, $h_{24}^{(4)}(U_4,X_4)$ and $h_{34}^{(4)}(U_4,X_4)$ as a $K[X_4]^{S_4}$-module $(F_4')^{S_4}$ in the algebra $(F_4')^{S_4}$ are equal to polynomials $g_{12}^{(4)}(X_4)$, $g_{13}^{(4)}(X_4)$, $g_{14}^{(4)}(X_4)$, $g_{23}^{(4)}(X_4)$, $g_{24}^{(4)}(X_4)$ and $g_{34}^{(4)}(X_4)$ respectively such that

$$g_{12}^{(4)}(X_4)=[x_2,x_1,x_2-x_1]+[x_3,x_1,x_3-x_1]+[x_4,x_1,x_4-x_1]$$
$$+[x_3,x_2,x_3-x_2]+[x_4,x_2,x_4-x_2]+[x_4,x_3,x_4-x_3],$$

$$g_{13}^{(4)}(X_4)=[x_2,x_1,x_2-x_1,x_3+x_4]+[x_3,x_1,x_3-x_1,x_2+x_4]$$
$$+[x_4,x_1,x_4-x_1,x_2+x_3]+[x_3,x_2,x_3-x_2,x_1+x_4]$$
$$+[x_4,x_2,x_4-x_2,x_1+x_3]+[x_4,x_3,x_4-x_3,x_1+x_2],$$

$$g_{14}^{(4)}(X_4)=[x_2,x_1,x_2-x_1,x_3,x_4]+[x_3,x_1,x_3-x_1,x_2,x_4]$$
$$+[x_4,x_1,x_4-x_1,x_2,x_3]+[x_3,x_2,x_3-x_2,x_1,x_4]$$
$$+[x_4,x_2,x_4-x_2,x_1,x_3]+[x_4,x_3,x_4-x_3,x_1,x_2],$$

$$g_{23}^{(4)}(X_4)=[x_2,x_1,x_2-x_1,x_3+x_4,x_3+x_4]-[x_2,x_1,x_2-x_1,x_3,x_4]$$
$$+[x_3,x_1,x_3-x_1,x_2+x_4,x_2+x_4]-[x_3,x_1,x_3-x_1,x_2,x_4]$$
$$+[x_4,x_1,x_4-x_1,x_2+x_3,x_2+x_3]-[x_4,x_1,x_4-x_1,x_2,x_3]$$

$$+[x_3,x_2,x_3-x_2,x_1+x_4,x_1+x_4]-[x_3,x_2,x_3-x_2,x_1,x_4]$$
$$+[x_4,x_2,x_4-x_2,x_1+x_3,x_1+x_3]-[x_4,x_2,x_4-x_2,x_1,x_3]$$
$$+[x_4,x_3,x_4-x_3,x_1+x_2,x_1+x_2]-[x_4,x_3,x_4-x_3,x_1,x_2],$$

$$g_{24}^{(4)}(X_4)=[x_2,x_1,x_2-x_1,x_3+x_4,x_3,x_4]+[x_3,x_1,x_3-x_1,x_2+x_4,x_2,x_4]$$
$$+[x_4,x_1,x_4-x_1,x_2+x_3,x_2,x_3]+[x_3,x_2,x_3-x_2,x_1+x_4,x_1,x_4]$$
$$+[x_4,x_2,x_4-x_2,x_1+x_3,x_1,x_3]+[x_4,x_3,x_4-x_3,x_1+x_2,x_1,x_2],$$

$$g_{34}^{(4)}(X_4)=[x_2,x_1,x_2-x_1,x_3,x_3,x_4,x_4]+[x_3,x_1,x_3-x_1,x_2,x_2,x_4,x_4]$$
$$+[x_4,x_1,x_4-x_1,x_2,x_2,x_3,x_3]+[x_3,x_2,x_3-x_2,x_1,x_1,x_4,x_4]$$
$$+[x_4,x_2,x_4-x_2,x_1,x_1,x_3,x_3]+[x_4,x_3,x_4-x_3,x_1,x_1,x_2,x_2].$$

Case $n=5$. We have $1\leq i<j\leq 5$. This means that

If $=1\longrightarrow j=2$, 3, 4 or 5;

$i=2\longrightarrow j=3$, 4 or 5;

$i=3\longrightarrow j=4$ or 5 and

$i=4\longrightarrow j=5$

Then, the elements of F_5 with their images in wreath product $(KU_5)wr(KT_5)$ is generated by $h_{12}^{(5)}(U_5,X_5)$, $h_{13}^{(5)}(U_5,X_5)$, $h_{14}^{(5)}(U_5,X_5)$, $h_{15}^{(5)}(U_5,X_5)$ $h_{23}^{(5)}(U_5,X_5)$, $h_{24}^{(5)}(U_5,X_5)$, $h_{25}^{(5)}(U_5,X_5)$, $h_{34}^{(5)}(U_5,X_5)$, $h_{35}^{(5)}(U_5,X_5)$ and $h_{45}^{(5)}(U_5,X_5)$ where

$$h_{12}^{(5)}(U_5,X_5)=2\varepsilon_1^{(5)}(U_5,X_5)e_2^{(5)}(X_5)-\varepsilon_2^{(5)}(U_5,X_5)e_1^{(5)}(X_5),$$
$$h_{13}^{(5)}(U_5,X_5)=3\varepsilon_1^{(5)}(U_5,X_5)e_3^{(5)}(X_5)-\varepsilon_3^{(5)}(U_5,X_5)e_1^{(5)}(X_5),$$
$$h_{14}^{(5)}(U_5,X_5)=4\varepsilon_1^{(5)}(U_5,X_5)e_4^{(5)}(X_5)-\varepsilon_4^{(5)}(U_5,X_5)e_1^{(5)}(X_5),$$
$$h_{15}^{(5)}(U_5,X_5)=5\varepsilon_1^{(5)}(U_5,X_5)e_5^{(5)}(X_5)-\varepsilon_5^{(5)}(U_5,X_5)e_1^{(5)}(X_5),$$
$$h_{23}^{(5)}(U_5,X_5)=3\varepsilon_2^{(5)}(U_5,X_5)e_3^{(5)}(X_5)-2\varepsilon_3^{(5)}(U_5,X_5)e_2^{(5)}(X_5),$$
$$h_{24}^{(5)}(U_5,X_5)=4\varepsilon_2^{(5)}(U_5,X_5)e_4^{(5)}(X_5)-2\varepsilon_4^{(5)}(U_5,X_5)e_2^{(5)}(X_5),$$
$$h_{25}^{(5)}(U_5,X_5)=5\varepsilon_2^{(5)}(U_5,X_5)e_5^{(5)}(X_5)-2\varepsilon_5^{(5)}(U_5,X_5)e_2^{(5)}(X_5),$$
$$h_{34}^{(5)}(U_5,X_5)=4\varepsilon_3^{(5)}(U_5,X_5)e_4^{(5)}(X_5)-3\varepsilon_4^{(5)}(U_5,X_5)e_3^{(5)}(X_5),$$
$$h_{35}^{(5)}(U_5,X_5)=5\varepsilon_3^{(5)}(U_5,X_5)e_5^{(5)}(X_5)-3\varepsilon_5^{(5)}(U_5,X_5)e_3^{(5)}(X_5),$$

$h_{45}^{(5)}(U_5,X_5) = 5\varepsilon_4^{(5)}(U_5,X_5)e_5^{(5)}(X_5) - 4\varepsilon_5^{(5)}(U_5,X_5)e_4^{(5)}(X_5)$.

With similarly computations as we did in the case $n = 2,3$ or 4 we obtain

$$\begin{aligned}
h_{12}^{(5)}(U_5,X_5) = {} & (u_2x_1 - u_1x_2)(x_2 - x_1) + (u_3x_1 - u_1x_3)(x_3 - x_1) \\
& +(u_4x_1 - u_1x_4)(x_4 - x_1) + (u_5x_1 - u_1x_5)(x_5 - x_1) \\
& +(u_3x_2 - u_2x_3)(x_3 - x_2) + (u_4x_2 - u_2x_4)(x_4 - x_2) \\
& +(u_5x_2 - u_2x_5)(x_5 - x_2) + (u_4x_3 - u_3x_4)(x_4 - x_3) \\
& +(u_5x_3 - u_3x_5)(x_5 - x_3) + (u_5x_4 - u_4x_5)(x_5 - x_4).
\end{aligned}$$

$$\begin{aligned}
h_{13}^{(5)}(U_5,X_5) = {} & (u_2x_1 - u_1x_2)(x_2 - x_1)(x_3 + x_4 + x_5) \\
& +(u_3x_1 - u_1x_3)(x_3 - x_1)(x_2 + x_4 + x_5) \\
& +(u_4x_1 - u_1x_4)(x_4 - x_1)(x_2 + x_3 + x_5) \\
& +(u_5x_1 - u_1x_5)(x_5 - x_1)(x_2 + x_3 + x_4) \\
& +(u_3x_2 - u_2x_3)(x_3 - x_2)(x_1 + x_4 + x_5) \\
& +(u_4x_2 - u_2x_4)(x_4 - x_2)(x_1 + x_3 + x_5) \\
& +(u_5x_2 - u_2x_5)(x_5 - x_2)(x_1 + x_3 + x_4) \\
& +(u_4x_3 - u_3x_4)(x_4 - x_3)(x_1 + x_2 + x_5) \\
& +(u_5x_3 - u_3x_5)(x_5 - x_3)(x_1 + x_2 + x_4) \\
& +(u_5x_4 - u_4x_5)(x_5 - x_4)(x_1 + x_2 + x_3)
\end{aligned}$$

$$\begin{aligned}
h_{14}^{(5)}(U_5,X_5) = {} & (u_2x_1 - u_1x_2)(x_2 - x_1)(x_3x_4 + x_3x_5 + x_4x_5) \\
& +(u_3x_1 - u_1x_3)(x_3 - x_1)(x_2x_4 + x_2x_5 + x_4x_5) \\
& +(u_4x_1 - u_1x_4)(x_4 - x_1)(x_2x_3 + x_2x_5 + x_3x_5) \\
& +(u_5x_1 - u_1x_5)(x_5 - x_1)(x_2x_3 + x_2x_4 + x_3x_4) \\
& +(u_3x_2 - u_2x_3)(x_3 - x_2)(x_1x_4 + x_1x_5 + x_4x_5) \\
& +(u_4x_2 - u_2x_4)(x_4 - x_2)(x_1x_3 + x_1x_5 + x_3x_5) \\
& +(u_5x_2 - u_2x_5)(x_5 - x_2)(x_1x_3 + x_1x_4 + x_3x_4) \\
& +(u_4x_3 - u_3x_4)(x_4 - x_3)(x_1x_2 + x_1x_5 + x_2x_5) \\
& +(u_5x_3 - u_3x_5)(x_5 - x_3)(x_1x_2 + x_1x_4 + x_2x_4) \\
& +(u_5x_4 - u_4x_5)(x_5 - x_4)(x_1x_2 + x_1x_3 + x_2x_3)
\end{aligned}$$

$$
\begin{aligned}
h_{15}^{(5)}(U_5,X_5) = {} & (u_2x_1-u_1x_2)(x_2-x_1)x_3x_4x_5 \\
& +(u_3x_1-u_1x_3)(x_3-x_1)x_2x_4x_5 \\
& +(u_4x_1-u_1x_4)(x_4-x_1)x_2x_3x_5 \\
& +(u_5x_1-u_1x_5)(x_5-x_1)x_2x_3x_4 \\
& +(u_3x_2-u_2x_3)(x_3-x_2)x_1x_4x_5 \\
& +(u_4x_2-u_2x_4)(x_4-x_2)x_1x_3x_5 \\
& +(u_5x_2-u_2x_5)(x_5-x_2)x_1x_3x_4 \\
& +(u_4x_3-u_3x_4)(x_4-x_3)x_1x_2x_5 \\
& +(u_5x_3-u_3x_5)(x_5-x_3)x_1x_2x_4 \\
& +(u_5x_4-u_4x_5)(x_5-x_4)x_1x_2x_3
\end{aligned}
$$

$$
\begin{aligned}
h_{23}^{(5)}(U_5,X_5) = {} & (u_2x_1-u_1x_2)(x_2-x_1)(x_3+x_4+x_5)(x_3+x_4+x_5) \\
& -(u_2x_1-u_1x_2)(x_2-x_1)(x_3x_4+x_3x_5+x_4x_5) \\
& +(u_3x_1-u_1x_3)(x_3-x_1)(x_2+x_4+x_5)(x_2+x_4+x_5) \\
& -(u_3x_1-u_1x_3)(x_3-x_1)(x_2x_4+x_2x_5+x_4x_5) \\
& +(u_4x_1-u_1x_4)(x_4-x_1)(x_2+x_3+x_5)(x_2+x_3+x_5) \\
& -(u_4x_1-u_1x_4)(x_4-x_1)(x_2x_3+x_2x_5+x_3x_5) \\
& +(u_5x_1-u_1x_5)(x_5-x_1)(x_2+x_3+x_4)(x_2+x_3+x_4) \\
& -(u_5x_1-u_1x_5)(x_5-x_1)(x_2x_3+x_2x_4+x_3x_4) \\
& +(u_3x_2-u_2x_3)(x_3-x_2)(x_1+x_4+x_5)(x_1+x_4+x_5) \\
& -(u_3x_2-u_2x_3)(x_3-x_2)(x_1x_4+x_1x_5+x_4x_5) \\
& +(u_4x_2-u_2x_4)(x_4-x_2)(x_1+x_3+x_5)(x_1+x_3+x_5) \\
& -(u_4x_2-u_2x_4)(x_4-x_2)(x_1x_3+x_1x_5+x_3x_5) \\
& +(u_5x_2-u_2x_5)(x_5-x_2)(x_1+x_3+x_4)(x_1+x_3+x_4) \\
& -(u_5x_2-u_2x_5)(x_5-x_2)(x_1x_3+x_1x_4+x_3x_4) \\
& +(u_4x_3-u_3x_4)(x_4-x_3)(x_1+x_2+x_5)(x_1+x_2+x_5) \\
& -(u_4x_3-u_3x_4)(x_4-x_3)(x_1x_2+x_1x_5+x_2x_5) \\
& +(u_5x_3-u_3x_5)(x_5-x_3)(x_1+x_2+x_4)(x_1+x_2+x_4) \\
& -(u_5x_3-u_3x_5)(x_5-x_3)(x_1x_2+x_1x_4+x_2x_4)
\end{aligned}
$$

$$+(u_5x_4 - u_4x_5)(x_5 - x_4)(x_1 + x_2 + x_3)(x_1 + x_2 + x_3)$$
$$-(u_5x_4 - u_4x_5)(x_5 - x_4)(x_1x_2 + x_1x_3 + x_2x_3).$$

$$h_{24}^{(5)}(U_5, X_5) = (u_2x_1 - u_1x_2)(x_2 - x_1)(x_3 + x_4 + x_5)(x_3x_4 + x_4x_5)$$
$$+(u_2x_1 - u_1x_2)(x_2 - x_1)(x_3 + x_5)x_3x_5$$
$$+(u_3x_1 - u_1x_3)(x_3 - x_1)(x_2 + x_4 + x_5)(x_2x_4 + x_4x_5)$$
$$+(u_3x_1 - u_1x_3)(x_3 - x_1)(x_2 + x_5)x_2x_5$$
$$+(u_4x_1 - u_1x_4)(x_4 - x_1)(x_2 + x_3 + x_5)(x_2x_3 + x_3x_5)$$
$$+(u_4x_1 - u_1x_4)(x_4 - x_1)(x_2 + x_5)x_2x_5$$
$$+(u_5x_1 - u_1x_5)(x_5 - x_1)(x_2 + x_3 + x_4)(x_2x_3 + x_3x_4)$$
$$+(u_5x_1 - u_1x_5)(x_5 - x_1)(x_2 + x_4)x_2x_4$$
$$+(u_3x_2 - u_2x_3)(x_3 - x_2)(x_1 + x_4 + x_5)(x_1x_4 + x_4x_5)$$
$$+(u_3x_2 - u_2x_3)(x_3 - x_2)(x_1 + x_5)x_1x_5$$
$$+(u_4x_2 - u_2x_4)(x_4 - x_2)(x_1 + x_3 + x_5)(x_1x_3 + x_3x_5)$$
$$+(u_4x_2 - u_2x_4)(x_4 - x_2)(x_1 + x_5)x_1x_5$$
$$+(u_5x_2 - u_2x_5)(x_5 - x_2)(x_1 + x_3 + x_4)(x_1x_3 + x_3x_4)$$
$$+(u_5x_2 - u_2x_5)(x_5 - x_2)(x_1 + x_4)x_1x_4$$
$$+(u_4x_3 - u_3x_4)(x_4 - x_3)(x_1 + x_2 + x_5)(x_1x_2 + x_2x_5)$$
$$+(u_4x_3 - u_3x_4)(x_4 - x_3)(x_1 + x_5)x_1x_5$$
$$+(u_5x_3 - u_3x_5)(x_5 - x_3)(x_1 + x_2 + x_4)(x_1x_2 + x_2x_4)$$
$$+(u_5x_3 - u_3x_5)(x_5 - x_3)(x_1 + x_4)x_1x_4$$
$$+(u_5x_4 - u_4x_5)(x_5 - x_4)(x_1 + x_2 + x_3)(x_1x_2 + x_2x_3)$$
$$+(u_5x_4 - u_4x_5)(x_5 - x_4)(x_1 + x_3)x_1x_3.$$

$$h_{25}^{(5)}(U_5, X_5) = (u_2x_1 - u_1x_2)(x_2 - x_1)(x_3 + x_4 + x_5)x_3x_4x_5$$
$$+(u_3x_1 - u_1x_3)(x_3 - x_1)(x_2 + x_4 + x_5)x_2x_4x_5$$
$$+(u_4x_1 - u_1x_4)(x_4 - x_1)(x_2 + x_3 + x_5)x_2x_3x_5$$
$$+(u_5x_1 - u_1x_5)(x_5 - x_1)(x_2 + x_3 + x_4)x_2x_3x_4$$
$$+(u_3x_2 - u_2x_3)(x_3 - x_2)(x_1 + x_4 + x_5)x_1x_4x_5$$
$$+(u_4x_2 - u_2x_4)(x_4 - x_2)(x_1 + x_3 + x_5)x_1x_3x_5$$

$$+(u_5x_2-u_2x_5)(x_5-x_2)(x_1+x_3+x_4)x_1x_3x_4$$
$$+(u_4x_3-u_3x_4)(x_4-x_3)(x_1+x_2+x_5)x_1x_2x_5$$
$$+(u_5x_3-u_3x_5)(x_5-x_3)(x_1+x_2+x_4)x_1x_2x_4$$
$$+(u_5x_4-u_4x_5)(x_5-x_4)(x_1+x_2+x_3)x_1x_2x_3$$

$$h_{34}^{(5)}(U_5,X_5)=(u_2x_1-u_1x_2)(x_2-x_1)(x_3+x_4+x_5)x_3x_4x_5$$
$$+(u_2x_1-u_1x_2)(x_2-x_1)(x_3x_3x_4x_4+x_3x_3x_5x_5+x_4x_4x_5x_5)$$
$$+(u_3x_1-u_1x_3)(x_3-x_1)(x_2+x_4+x_5)x_2x_4x_5$$
$$+(u_3x_1-u_1x_3)(x_3-x_1)(x_2x_2x_4x_4+x_2x_2x_5x_5+x_4x_4x_5x_5)$$
$$+(u_4x_1-u_1x_4)(x_4-x_1)(x_2+x_3+x_5)x_2x_3x_5$$
$$+(u_4x_1-u_1x_4)(x_4-x_1)(x_2x_2x_3x_3+x_2x_2x_5x_5+x_3x_3x_5x_5)$$
$$+(u_5x_1-u_1x_5)(x_5-x_1)(x_2+x_3+x_4)x_2x_3x_4$$
$$+(u_5x_1-u_1x_5)(x_5-x_1)(x_2x_2x_3x_3+x_2x_2x_4x_4+x_3x_3x_4x_4)$$
$$+(u_3x_2-u_2x_3)(x_3-x_2)(x_1+x_4+x_5)x_1x_4x_5$$
$$+(u_3x_2-u_2x_3)(x_3-x_2)(x_1x_1x_4x_4+x_1x_1x_5x_5+x_4x_4x_5x_5)$$
$$+(u_4x_2-u_2x_4)(x_4-x_2)(x_1+x_3+x_5)x_1x_3x_5$$
$$+(u_4x_2-u_2x_4)(x_4-x_2)(x_1x_1x_3x_3+x_1x_1x_5x_5+x_3x_3x_5x_5)$$
$$+(u_5x_2-u_2x_5)(x_5-x_2)(x_1+x_3+x_4)x_1x_3x_4$$
$$+(u_5x_2-u_2x_5)(x_5-x_2)(x_1x_1x_3x_3+x_1x_1x_4x_4+x_3x_3x_4x_4)$$
$$+(u_4x_3-u_3x_4)(x_4-x_3)(x_1+x_2+x_5)x_1x_2x_5$$
$$+(u_4x_3-u_3x_4)(x_4-x_3)(x_1x_1x_2x_2+x_1x_1x_5x_5+x_2x_2x_5x_5)$$
$$+(u_5x_3-u_3x_5)(x_5-x_3)(x_1+x_2+x_4)x_1x_2x_4$$
$$+(u_5x_3-u_3x_5)(x_5-x_3)(x_1x_1x_2x_2+x_1x_1x_4x_4+x_2x_2x_4x_4)$$
$$+(u_5x_4-u_4x_5)(x_5-x_4)(x_1+x_2+x_3)x_1x_2x_3$$
$$+(u_5x_4-u_4x_5)(x_5-x_4)(x_1x_1x_2x_2+x_1x_1x_3x_3+x_2x_2x_3x_3).$$

$$h_{35}^{(5)}(U_5,X_5)=(u_2x_1-u_1x_2)(x_2-x_1)(x_3x_4+x_3x_5+x_4x_5)x_3x_4x_5$$
$$+(u_3x_1-u_1x_3)(x_3-x_1)(x_2x_4+x_2x_5+x_4x_5)x_2x_4x_5$$
$$+(u_4x_1-u_1x_4)(x_4-x_1)(x_2x_3+x_2x_5+x_3x_5)x_2x_3x_5$$
$$+(u_5x_1-u_1x_5)(x_5-x_1)(x_2x_3+x_2x_4+x_3x_4)x_2x_3x_4$$

$$+(u_3x_2-u_2x_3)(x_3-x_2)(x_1x_4+x_1x_5+x_4x_5)x_1x_4x_5$$
$$+(u_4x_2-u_2x_4)(x_4-x_2)(x_1x_3+x_1x_5+x_3x_5)x_1x_3x_5$$
$$+(u_5x_2-u_2x_5)(x_5-x_2)(x_1x_3+x_1x_4+x_3x_4)x_1x_3x_4$$
$$+(u_4x_3-u_3x_4)(x_4-x_3)(x_1x_2+x_1x_5+x_2x_5)x_1x_2x_5$$
$$+(u_5x_3-u_3x_5)(x_5-x_3)(x_1x_2+x_1x_4+x_2x_4)x_1x_2x_4$$
$$+(u_5x_4-u_4x_5)(x_5-x_4)(x_1x_2+x_1x_3+x_2x_3)x_1x_2x_3.$$

$$\begin{aligned}h_{45}^{(5)}(U_5,X_5) &= (u_2x_1-u_1x_2)(x_2-x_1)x_3x_3x_4x_4x_5x_5\\
&+(u_3x_1-u_1x_3)(x_3-x_1)x_2x_2x_4x_4x_5x_5\\
&+(u_4x_1-u_1x_4)(x_4-x_1)x_2x_2x_3x_3x_5x_5\\
&+(u_5x_1-u_1x_5)(x_5-x_1)x_2x_2x_3x_3x_4x_4\\
&+(u_3x_2-u_2x_3)(x_3-x_2)x_1x_1x_4x_4x_5x_5\\
&+(u_4x_2-u_2x_4)(x_4-x_2)x_1x_1x_3x_3x_5x_5\\
&+(u_5x_2-u_2x_5)(x_5-x_2)x_1x_1x_3x_3x_4x_4\\
&+(u_4x_3-u_3x_4)(x_4-x_3)x_1x_1x_2x_2x_5x_5\\
&+(u_5x_3-u_3x_5)(x_5-x_3)x_1x_1x_2x_2x_4x_4\\
&+(u_5x_4-u_4x_5)(x_5-x_4)x_1x_1x_2x_2x_3x_3.\end{aligned}$$

Therefore, the inverse of $h_{12}^{(5)}(U_5,X_5)$, $h_{13}^{(5)}(U_5,X_5)$, $h_{14}^{(5)}(U_5,X_5)$, $h_{15}^{(5)}(U_5,X_5)$ $h_{23}^{(5)}(U_5,X_5)$, $h_{24}^{(5)}(U_5,X_5)$, $h_{25}^{(5)}(U_5,X_5)$, $h_{34}^{(5)}(U_5,X_5)$, $h_{35}^{(5)}(U_5,X_5)$ and $h_{45}^{(5)}(U_5,X_5)$ as a $K[X_5]^{S_5}$-module $(F_5')^{S_5}$ in the algebra $(F_5')^{S_5}$ are equal to polynomials $g_{12}^{(5)}(X_5)$, $g_{13}^{(5)}(X_5)$, $g_{14}^{(5)}(X_5)$, $g_{15}^{(5)}(X_5)$, $g_{23}^{(5)}(X_5)$, $g_{24}^{(5)}(X_5)$, $g_{25}^{(5)}(X_5)$, $g_{34}^{(5)}(X_5)$, $g_{35}^{(5)}(X_5)$ and $g_{45}^{(5)}(X_5)$ respectively such that

$$\begin{aligned}g_{12}^{(5)}(X_5) &= [x_2,x_1,x_2-x_1]+[x_3,x_1,x_3-x_1]+[x_4,x_1,x_4-x_1]\\
&+[x_5,x_1,x_5-x_1]+[x_3,x_2,x_3-x_2]+[x_4,x_2,x_4-x_2]\\
&+[x_5,x_2,x_5-x_2]+[x_4,x_3,x_4-x_3]+[x_5,x_3,x_5-x_3]\\
&+[x_5,x_4,x_5-x_4],\end{aligned}$$

$$\begin{aligned}g_{13}^{(5)}(X_5) &= [x_2,x_1,x_2-x_1,x_3+x_4+x_5]+[x_3,x_1,x_3-x_1,x_2+x_4+x_5]\\
&+[x_4,x_1,x_4-x_1,x_2+x_3+x_5]+[x_5,x_1,x_5-x_1,x_2+x_3+x_4]\end{aligned}$$

$$+[x_3,x_2,x_3-x_2,x_1+x_4+x_5]+[x_4,x_2,x_4-x_2,x_1+x_3+x_5]$$
$$+[x_5,x_2,x_5-x_2,x_1+x_3+x_4]+[x_4,x_3,x_4-x_3,x_1+x_2+x_5]$$
$$+[x_5,x_3,x_5-x_3,x_1+x_2+x_4]+[x_5,x_4,x_5-x_4,x_1+x_2+x_3],$$

$$g_{14}^{(5)}(X_5)=[x_2,x_1,x_2-x_1,x_3x_4+x_3x_5+x_4x_5]$$
$$+[x_3,x_1,x_3-x_1,x_2x_4+x_2x_5+x_4x_5]$$
$$+[x_4,x_1,x_4-x_1,x_2x_3+x_2x_5+x_3x_5]$$
$$+[x_5,x_1,x_5-x_1,x_2x_3+x_2x_4+x_3x_4]$$
$$+[x_3,x_2,x_3-x_2,x_1x_4+x_1x_5+x_4x_5]$$
$$+[x_4,x_2,x_4-x_2,x_1x_3+x_1x_5+x_3x_5]$$
$$+[x_5,x_2,x_5-x_2,x_1x_3+x_1x_4+x_3x_4]$$
$$+[x_4,x_3,x_4-x_3,x_1x_2+x_1x_5+x_2x_5]$$
$$+[x_5,x_3,x_5-x_3,x_1x_2+x_1x_4+x_2x_4]$$
$$+[x_5,x_4,x_5-x_4,x_1x_2+x_1x_3+x_2x_3],$$

$$g_{15}^{(5)}(X_5)=[x_2,x_1,x_2-x_1,x_3,x_4,x_5]+[x_3,x_1,x_3-x_1,x_2,x_4,x_5]$$
$$+[x_4,x_1,x_4-x_1,x_2,x_3,x_5]+[x_5,x_1,x_5-x_1,x_2,x_3,x_4]$$
$$+[x_3,x_2,x_3-x_2,x_1,x_4,x_5]+[x_4,x_2,x_4-x_2,x_1,x_3,x_5]$$
$$+[x_5,x_2,x_5-x_2,x_1,x_3,x_4]+[x_4,x_3,x_4-x_3,x_1,x_2,x_5]$$
$$+[x_5,x_3,x_5-x_3,x_1,x_2,x_4]+[x_5,x_4,x_5-x_4,x_1,x_2,x_3],$$

$$g_{23}^{(5)}(X_5)=[x_2,x_1,x_2-x_1,x_3+x_4+x_5,x_3+x_4+x_5]$$
$$-[x_2,x_1,x_2-x_1,x_3x_4+x_3x_5+x_4x_5]$$
$$+[x_3,x_1,x_3-x_1,x_2+x_4+x_5,x_2+x_4+x_5]$$
$$-[x_3,x_1,x_3-x_1,x_2x_4+x_2x_5+x_4x_5]$$
$$+[x_4,x_1,x_4-x_1,x_2+x_3+x_5,x_2+x_3+x_5]$$
$$-[x_4,x_1,x_4-x_1,x_2x_3+x_2x_5+x_3x_5]$$
$$+[x_5,x_1,x_5-x_1,x_2+x_3+x_4,x_2+x_3+x_4]$$
$$-[x_5,x_1,x_5-x_1,x_2x_3+x_2x_4+x_3x_4]$$
$$+[x_3,x_2,x_3-x_2,x_1+x_4+x_5,x_1+x_4+x_5]$$
$$-[x_3,x_2,x_3-x_2,x_1x_4+x_1x_5+x_4x_5]$$

$$
\begin{aligned}
&+[x_4, x_2, x_4 - x_2, x_1 + x_3 + x_5, x_1 + x_3 + x_5]\\
&-[x_4, x_2, x_4 - x_2, x_1x_3 + x_1x_5 + x_3x_5]\\
&+[x_5, x_2, x_5 - x_2, x_1 + x_3 + x_4, x_1 + x_3 + x_4]\\
&-[x_5, x_2, x_5 - x_2, x_1x_3 + x_1x_4 + x_3x_4]\\
&+[x_4, x_3, x_4 - x_3, x_1 + x_2 + x_5, x_1 + x_2 + x_5]\\
&-[x_4, x_3, x_4 - x_3, x_1x_2 + x_1x_5 + x_2x_5]\\
&+[x_5, x_3, x_5 - x_3, x_1 + x_2 + x_4, x_1 + x_2 + x_4]\\
&-[x_5, x_3, x_5 - x_3, x_1x_2 + x_1x_4 + x_2x_4]\\
&+[x_5, x_4, x_5 - x_4, x_1 + x_2 + x_3, x_1 + x_2 + x_3]\\
&-[x_5, x_4, x_5 - x_4, x_1x_2 + x_1x_3 + x_2x_3],
\end{aligned}
$$

$$
\begin{aligned}
g_{24}^{(5)}(X_5) = {}&[x_2, x_1, x_2 - x_1, x_3 + x_4 + x_5, x_3x_4 + x_4x_5]\\
&+[x_2, x_1, x_2 - x_1, x_3 + x_5, x_3, x_5]\\
&+[x_3, x_1, x_3 - x_1, x_2 + x_4 + x_5, x_2x_4 + x_4x_5]\\
&+[x_3, x_1, x_3 - x_1, x_2 + x_5, x_2, x_5]\\
&+[x_4, x_1, x_4 - x_1, x_2 + x_3 + x_5, x_2x_3 + x_3x_5]\\
&+[x_4, x_1, x_4 - x_1, x_2 + x_5, x_2, x_5]\\
&+[x_5, x_1, x_5 - x_1, x_2 + x_3 + x_4, x_2x_3 + x_3x_4]\\
&+[x_5, x_1, x_5 - x_1, x_2 + x_4, x_2, x_4]\\
&+[x_3, x_2, x_3 - x_2, x_1 + x_4 + x_5, x_1x_4 + x_4x_5]\\
&+[x_3, x_2, x_3 - x_2, x_1 + x_5, x_1, x_5]\\
&+[x_4, x_2, x_4 - x_2, x_1 + x_3 + x_5, x_1x_3 + x_3x_5]\\
&+[x_4, x_2, x_4 - x_2, x_1 + x_5, x_1, x_5]\\
&+[x_5, x_2, x_5 - x_2, x_1 + x_3 + x_4, x_1x_3 + x_3x_4]\\
&+[x_5, x_2, x_5 - x_2, x_1 + x_4, x_1, x_4]\\
&+[x_4, x_3, x_4 - x_3, x_1 + x_2 + x_5, x_1x_2 + x_2x_5]\\
&+[x_4, x_3, x_4 - x_3, x_1 + x_5, x_1, x_5]\\
&+[x_5, x_3, x_5 - x_3, x_1 + x_2 + x_4, x_1x_2 + x_2x_4]\\
&+[x_5, x_3, x_5 - x_3, x_1 + x_4, x_1, x_4]
\end{aligned}
$$

$$+[x_5, x_4, x_5 - x_4, x_1 + x_2 + x_3, x_1x_2 + x_2x_3]$$
$$+[x_5, x_4, x_5 - x_4, x_1 + x_3, x_1, x_3],$$

$$g_{25}^{(5)}(X_5) = [x_2, x_1, x_2 - x_1, x_3 + x_4 + x_5, x_3, x_4, x_5]$$
$$+[x_3, x_1, x_3 - x_1, x_2 + x_4 + x_5, x_2, x_4, x_5]$$
$$+[x_4, x_1, x_4 - x_1, x_2 + x_3 + x_5, x_2, x_3, x_5]$$
$$+[x_5, x_1, x_5 - x_1, x_2 + x_3 + x_4, x_2, x_3, x_4]$$
$$+[x_3, x_2, x_3 - x_2, x_1 + x_4 + x_5, x_1, x_4, x_5]$$
$$+[x_4, x_2, x_4 - x_2, x_1 + x_3 + x_5, x_1, x_3, x_5]$$
$$+[x_5, x_2, x_5 - x_2, x_1 + x_3 + x_4, x_1, x_3, x_4]$$
$$+[x_4, x_3, x_4 - x_3, x_1 + x_2 + x_5, x_1, x_2, x_5]$$
$$+[x_5, x_3, x_5 - x_3, x_1 + x_2 + x_4, x_1, x_2, x_4]$$
$$+[x_5, x_4, x_5 - x_4, x_1 + x_2 + x_3, x_1, x_2, x_3],$$

$$g_{34}^{(5)}(X_5) = \begin{bmatrix} x_2, x_1, x_2 - x_1, x_3 + x_4 + x_5, x_3x_3x_4x_4 + x_3x_3x_5x_5 + x_4x_4x_5x_5, \\ x_3, x_4, x_5 \end{bmatrix}$$
$$+\begin{bmatrix} x_3, x_1, x_3 - x_1, x_2 + x_4 + x_5, x_2x_2x_4x_4 + x_2x_2x_5x_5 + x_4x_4x_5x_5, \\ x_2, x_4, x_5 \end{bmatrix}$$
$$+\begin{bmatrix} x_4, x_1, x_4 - x_1, x_2 + x_3 + x_5, x_2x_2x_3x_3 + x_2x_2x_5x_5 + x_3x_3x_5x_5, \\ x_2, x_3, x_5 \end{bmatrix}$$
$$+\begin{bmatrix} x_5, x_1, x_5 - x_1, x_2 + x_3 + x_4, x_2x_2x_3x_3 + x_2x_2x_4x_4 + x_3x_3x_4x_4, \\ x_2, x_3, x_4 \end{bmatrix}$$
$$+\begin{bmatrix} x_3, x_2, x_3 - x_2, x_1 + x_4 + x_5, x_1x_1x_4x_4 + x_1x_1x_5x_5 + x_4x_4x_5x_5, \\ x_1, x_4, x_5 \end{bmatrix}$$
$$+\begin{bmatrix} x_4, x_2, x_4 - x_2, x_1 + x_3 + x_5, x_1x_1x_3x_3 + x_1x_1x_5x_5 + x_3x_3x_5x_5, \\ x_1, x_3, x_5 \end{bmatrix}$$
$$+\begin{bmatrix} x_5, x_2, x_5 - x_2, x_1 + x_3 + x_4, x_1x_1x_3x_3 + x_1x_1x_4x_4 + x_3x_3x_4x_4, \\ x_1, x_3, x_4 \end{bmatrix}$$
$$+\begin{bmatrix} x_4, x_3, x_4 - x_3, x_1 + x_2 + x_5, x_1x_1x_2x_2 + x_1x_1x_5x_5 + x_2x_2x_5x_5, \\ x_1, x_2, x_5 \end{bmatrix}$$
$$+\begin{bmatrix} x_5, x_3, x_5 - x_3, x_1 + x_2 + x_4, x_1x_1x_2x_2 + x_1x_1x_4x_4 + x_2x_2x_4x_4, \\ x_1, x_2, x_4 \end{bmatrix}$$
$$+\begin{bmatrix} x_5, x_4, x_5 - x_4, x_1 + x_2 + x_3, x_1x_1x_2x_2 + x_1x_1x_3x_3 + x_2x_2x_3x_3, \\ x_1, x_2, x_3 \end{bmatrix},$$

$$g_{35}^{(5)}(X_5) = [x_2, x_1, x_2 - x_1, x_3x_4 + x_3x_5 + x_4x_5, x_3, x_4, x_5]$$
$$+[x_3, x_1, x_3 - x_1, x_2x_4 + x_2x_5 + x_4x_5, x_2, x_4, x_5]$$
$$+[x_4, x_1, x_4 - x_1, x_2x_3 + x_2x_5 + x_3x_5, x_2, x_3, x_5]$$

$$+[x_5, x_1, x_5 - x_1, x_2x_3 + x_2x_4 + x_3x_4, x_2, x_3, x_4]$$
$$+[x_3, x_2, x_3 - x_2, x_1x_4 + x_1x_5 + x_4x_5, x_1, x_4, x_5]$$
$$+[x_4, x_2, x_4 - x_2, x_1x_3 + x_1x_5 + x_3x_5, x_1, x_3, x_5]$$
$$+[x_5, x_2, x_5 - x_2, x_1x_3 + x_1x_4 + x_3x_4, x_1, x_3, x_4]$$
$$+[x_4, x_3, x_4 - x_3, x_1x_2 + x_1x_5 + x_2x_5, x_1, x_2, x_5]$$
$$+[x_5, x_3, x_5 - x_3, x_1x_2 + x_1x_4 + x_2x_4, x_1, x_2, x_4]$$
$$+[x_5, x_4, x_5 - x_4, x_1x_2 + x_1x_3 + x_2x_3, x_1, x_2, x_3],$$

$$g_{45}^{(5)}(X_5) = [x_2, x_1, x_2 - x_1, x_3, x_3, x_4, x_4, x_5, x_5]$$
$$+[x_3, x_1, x_3 - x_1, x_2, x_2, x_4, x_4, x_5, x_5]$$
$$+[x_4, x_1, x_4 - x_1, x_2, x_2, x_3, x_3, x_5, x_5]$$
$$+[x_5, x_1, x_5 - x_1, x_2, x_2, x_3, x_3, x_4, x_4]$$
$$+[x_3, x_2, x_3 - x_2, x_1, x_1, x_4, x_4, x_5, x_5]$$
$$+[x_4, x_2, x_4 - x_2, x_1, x_1, x_3, x_3, x_5, x_5]$$
$$+[x_5, x_2, x_5 - x_2, x_1, x_1, x_3, x_3, x_4, x_4]$$
$$+[x_4, x_3, x_4 - x_3, x_1, x_1, x_2, x_2, x_5, x_5]$$
$$+[x_5, x_3, x_5 - x_3, x_1, x_1, x_2, x_2, x_4, x_4]$$
$$+[x_5, x_4, x_5 - x_4, x_1, x_1, x_2, x_2, x_3, x_3].$$

4. MAIN RESULTS

4.1 Generators of Constants in the Free Metabelian Lie Algebras of rank 5

Theorem 4.1.1: Let $\delta_{3\times 2}$ be the Weitzenböck derivation which has two cells of size 3×3 and 2×2 respectively, i.e, $\delta_{3\times 2} : x_3 \longrightarrow x_2 \longrightarrow x_1 \longrightarrow 0$ and $\delta_{3\times 2} : x_5 \longrightarrow x_4 \longrightarrow 0$. Then $(F'_{n-1})^{\delta_{3\times 2}}$ is a $K[X_{n-1}]^{\delta_{3\times 2}}$-module generated by $\{c_1, \ldots, c_{11}\}$, where

$$c_1 = [x_2, x_1],$$
$$c_2 = [x_4, x_1],$$
$$c_3 = [x_5, x_1] - [x_4, x_2],$$
$$c_4 = [x_5, x_4],$$
$$c_5 = [x_2, x_1, x_5] + [x_4, x_2, x_2] - 2[x_4, x_1, x_3]$$
$$c_6 = [x_2, x_1, x_5] - [x_3, x_1, x_4],$$
$$c_7 = [x_3, x_1, x_1] - [x_2, x_1, x_2],$$
$$c_8 = [x_5, x_1, x_5] - [x_5, x_2, x_4] - [x_4, x_2, x_5] + 2[x_4, x_3, x_4],$$
$$c_9 = [x_2, x_1, x_5, x_5] - 2[x_3, x_1, x_4, x_5] + 2[x_3, x_2, x_4, x_4],$$
$$c_{10} = [x_3, x_1, x_1, x_5] - [x_2, x_1, x_2, x_5] - [x_3, x_1, x_2, x_4] + 2[x_2, x_1, x_3, x_4],$$
$$\begin{aligned} c_{11} = {} & [x_5, x_2, x_2, x_2, x_4] + [x_4, x_2, x_2, x_2, x_5] + 2[x_5, x_1, x_1, x_3, x_5] \\ & -[x_3, x_1, x_1, x_5, x_5] - [x_5, x_1, x_2, x_2, x_5] + [x_2, x_1, x_2, x_5, x_5] \\ & -2[x_4, x_2, x_2, x_3, x_4] + 4[x_4, x_1, x_3, x_3, x_4] - 2[x_3, x_1, x_3, x_4, x_4] \\ & -2[x_5, x_1, x_2, x_3, x_4] - 2[x_4, x_1, x_2, x_3, x_5] + 2[x_3, x_1, x_2, x_4, x_5]. \end{aligned}$$

Proof: Let $K[X_5]$ be the polynomial algebras generated by $X_5 = \{x_1, x_2, x_3, x_4, x_5\}$. Then $K[X_5]^{\delta_{3\times 2}}$ is the algebras of constants of polynomial algebras for nonbasic Weitzenböck derivations such that $\delta_{3\times 2} : x_3 \longrightarrow x_2 \longrightarrow x_1 \longrightarrow 0$ and $x_5 \longrightarrow x_4 \longrightarrow 0$ and from Example 3.2.5, the algebras $K[X_5]^{\delta_{3\times 2}}$ is generated by the algebraically independent polynomials

$f_1 = x_1$, $f_2 = x_2^2 - 2x_1x_3$, $f_3 = x_1x_5^2 - 2x_2x_4x_5 + 2x_3x_4^2$, $f_4 = x_4$ and $f_5 = x_1x_5 - x_2x_4$.

With single relation

$$f_5^2 = f_2f_4^2 + f_1f_3$$

The Hilbert Series of this polynomial algebra is

$$H_{G_{L_2}}(K[X_5], t_1, t_2, z) = \frac{(1 - t_1^4t_2^2z^4)}{(1 - t_1^2z)(1 - t_1^2t_2^2z^2)(1 - t_1^2t_2^2z^3)(1 - t_1z)(1 - t_1^2t_2z^2)}$$

$$= \frac{1 + t_1^2t_2z^2}{(1 - t_1^2z)(1 - t_1^2t_2^2z^2)(1 - t_1^2t_2^2z^3)(1 - t_1z)}$$

by write this Hilbert series in the form of power series, we know that

$$H_{G_{L_2}}(K[X_5], z_1, z_2, z_3, z_4) = \sum \dim K[X_5]^{\alpha_1,\alpha_2,\alpha_3,\alpha_4} \cdot z_1^{\alpha_1} \cdot z_2^{\alpha_2} \cdot z_3^{\alpha_3} \cdot z_4^{\alpha_4}$$

So, let take on part of this Hilbert Series to facilitate to be written as power series

$$H_{G_{L_2}}(K[X_5], z_1, z_2, z_3, z_4) = \frac{1}{(1 - t_1^2z)(1 - t_1^2t_2^2z^2)(1 - t_1^2t_2^2z^3)(1 - t_1z)}$$

$$= \frac{1}{(1-z_1)(1-z_2)(1-z_3)(1-z_4)}$$

From the basis of this polynomial algebra

$$H_{G_{L_2}}(K[X_5], z_1, z_2, z_3, z_4) = \frac{1}{(1 - z_1)(1 - z_2)(1 - z_3)(1 - z_4)}$$

$$= 1 + z_1 + z_2 + z_3 + z_4 + z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3$$
$$+ z_2z_4 + z_3z_4 + z_1^2 + \cdots$$
$$= 1 + t_1^2z + t_1^2t_2^2z^2 + t_1^2t_2^2z^3 + t_1z + t_1^4t_2^2z^3$$
$$+ t_1^4t_2^2z^4 + t_1^6z^2 + \cdots$$
$$= 1 + (t_1^2 + t_1)z + (t_1^2t_2^2 + t_1^3 + t_1^4)z^2 + \cdots$$

By the Maple computations we have the Hilbert series of free metabelian Lie algebra as follow

$$H_{G_{L_2}}\left(\left({L'_5}/{L''_5}\right)^{\delta_{3\times 2}}, t_1, t_2, z\right) = \frac{\begin{matrix}(t_1^3t_2 + t_1^3 + t_1^2t_2 + t_1t_2)z^2 + (t_1^3t_2^2 - t_1^4t_2)z^3 \\ +(t_1^3t_2^2 + t_1^3t_2^3)z^4 + \left(-t_1^4t_2^3 - t_1^5t_2^2 - t_1^5t_2^3\right)z^5 \\ +t_1^6t_2^3z^6\end{matrix}}{(1-t_1^2z)(1-t_1t_2z)(1-t_1^2t_2^2z^3)(1-t_1z)}$$

this must have same denominator as the Hilbert series of polynomial algebras so, we have

$$H_{G_{L_2}}\left(\left({L'_5}/{L''_5}\right)^{\delta_{3\times 2}}, t_1, t_2, z\right) = \frac{\begin{pmatrix}(t_1^3t_2+t_1^3+t_1^2t_2+t_1t_2)z^2+(t_1^3t_2^2-t_1^4t_2)z^3 \\ +(t_1^3t_2^2+t_1^3t_2^3)z^4+\left(-t_1^4t_2^3-t_1^5t_2^2-t_1^5t_2^3\right)z^5 \\ +t_1^6t_2^3z^6\end{pmatrix}(1+t_1t_2z)}{(1-t_1^2z)(1-t_1^2t_2^2z^2)(1-t_1^2t_2^2z^3)(1-t_1z)}$$

We need to write this in the form of power series by denoting N as the numerator of Hilbert series of free metabelian Lie algebra such that

$$N = \begin{pmatrix}(t_1^3t_2 + t_1^3 + t_1^2t_2 + t_1t_2)z^2 + (t_1^3t_2^2 - t_1^4t_2)z^3 \\ +(t_1^3t_2^2 + t_1^3t_2^3)z^4 + \left(-t_1^4t_2^3 - t_1^5t_2^2 - t_1^5t_2^3\right)z^5 + t_1^6t_2^3z^6\end{pmatrix}(1+t_1t_2z)$$

So, we have

$$H_{G_{L_2}}\left(\left({L'_5}/{L''_5}\right)^{\delta_{3\times 2}}, t_1, t_2, z\right) = N \cdot 1 + (t_1^2 + t_1)z + (t_1^2t_2^2 + t_1^3 + t_1^4)z^2 + \cdots$$

Remark 4.1.2: $c_i \in (L')^{\delta_{3\times 2}} \subseteq L'$

$< c_i > \subset (L')^{\delta_{3\times 2}}$ where c_i is a generator.

Polynomial algebra of z_1, z_2, z_3, z_4 should be ignored when you are working on generators of $\left({L'_5}/{L''_5}\right)^{\delta_{3\times 2}}$ as a module of $K[z_1, z_2, z_3, z_4]$.

$$H_{G_{L_2}}\left(\left({L'_5}/{L''_5}\right)^{\delta_{3\times 2}}, t_1, t_2, z\right) = N$$

$$= \begin{pmatrix} (t_1^3t_2 + t_1^3 + t_1^2t_2 + t_1t_2)z^2 + (t_1^3t_2^2 - t_1^4t_2)z^3 \\ +(t_1^3t_2^2 + t_1^3t_2^3)z^4 + \left(-t_1^4t_2^3 - t_1^5t_2^2 - t_1^5t_2^3\right)z^5 \\ +t_1^6t_2^3z^6 \end{pmatrix}(1 + t_1t_2z)$$

$$= (t_1^3t_2 + t_1^3 + t_1^2t_2 + t_1t_2)z^2 + \begin{pmatrix} t_1^3t_2^2 - t_1^4t_2 + t_1^4t_2^2 + t_1^4t_2 + t_1^3t_2^2 \\ +t_1^2t_2^2 \end{pmatrix} z^3$$

$$+\left(t_1^3t_2^2 + t_1^3t_2^3 + t_1^4t_2^3 - t_1^5t_2^2\right)z^4 + \begin{pmatrix} -t_1^4t_2^3 - t_1^5t_2^2 - t_1^5t_2^3 + t_1^4t_2^3 \\ +t_1^4t_2^4 \end{pmatrix} z^5$$

$$+\left(t_1^6t_2^3 - t_1^5t_2^4 - t_1^6t_2^3 - t_1^6t_2^4\right)z^6 + t_1^7t_2^4z^7.$$

The power of positive sign of t's in Hilbert series of algebras of constants of free metabelian Lie algebra represents the bidegree of generators and the power negative sign represents the relations while the power of z's represents the length.

The following table summarized the x_i's with the corresponding bidegree for $n = 5$.

x_i	Bidegree
x_1	(2,0)
x_2	(1,1)
x_3	(0,2)
x_4	(1,0)
x_5	(0,1)

The Let us start with the bidegrees of length 2.

As we see in Hilbert series of algebras of constants of free metabelian Lie algebra the length 2 has four bidegrees which are (3,1)2; (3,0)2; (2,1)2 and (2,2)2 and each one corresponds to the generators c_1, c_2, c_3 and c_4 respectively.

- For a bidegree $(3,1)2$; the only possibility x_1x_2

This means we have only one generator $c_1$such that

$$c_1 = [x_2, x_1]$$

- For a bidegree $(3,0)2$; the only possibility x_1x_4

we have generator $c_2$such that

$$c_2 = [x_4, x_1]$$

- For a bidegree $(2,1)2$; we have two possibilities x_1x_5, x_2x_4 and it's can be written in this form

$$c_3 = \alpha_1[x_5, x_1] + \alpha_2[x_4, x_2].$$

such that $\alpha_1, \alpha_2 \in K$ and since c_3 is generator the derivation under $\delta_{3\times 2}$ should be zero so,

$$\begin{aligned}\delta_{3\times2}(c_3) &= \delta_{3\times2}(\alpha_1[x_5, x_1] + \alpha_2[x_4, x_2]) \\ &= \alpha_1[x_4, x_1] + \alpha_2[x_4, x_1] = (\alpha_1 + \alpha_2)[x_4, x_1] = 0\end{aligned}$$

Here, $\alpha_2 = -\alpha_1$ this gives generator c_3 such that

$$c_3 = [x_5, x_1] - [x_4, x_2].$$

- For a bidegree $(1,1)2$; the only possibility x_4x_5, then we have a generator c_4 such that

$$c_4 = [x_5, x_4].$$

$K[X_5]^{\delta_{3\times2}}$ $-$module $\left({L'_5}/{L''_5}\right)^{\delta_{3\times2}}$ has four generators c_1, c_2, c_3 and c_4 of length 2.

The next step is to found generators of length 3 with the following bidegrees $(3,2)$; $(3,2)$; $(4,2)$; $(4,1)$and $(2,2)$. Here this double bidegree means we have two possible bidegrees. Therefore, we expect four generators of length three but before that we need to verify if this bidegree has new generators by comparing to the previous generators of algebra of constant of free metabelian Lie algebra we already have and the algebra of constants of polynomial algebra.

So, by comparing the bidegree of 4 previous generators of length 2 with $f_1 : (2,0)1$ and $f_4 : (1,0)1$ we found if the new bidegree of length 3 has possible new generator.

- Let start with bidegree $(3,2)3$;

$(3,2)3 - (2,0)1 = (1,2)2$

$(3,2)3 - (1,0)1 = (2,2)2$

We don't have any match from the previous bidegree of length 2 this means that $(3,2)3$ has new generator and since the coefficient of this bidegree in the Hilbert series is 2 this bidegree generates two different generators and let be denoted by c_5 and c_6 respectively.

The possibilities of this bidegree are $x_1x_2x_5$; $x_2^2x_4$ and $x_1x_3x_4$ and c_5 can be written as linear combinations of all commutators in bidegree $(3,2)3$, so,

$$c_5 = \alpha_1[x_5, x_1, x_2] + \alpha_2[x_2, x_1, x_5] + \alpha_3[x_4, x_2, x_2] + \alpha_4[x_4, x_1, x_3] + \alpha_5[x_3, x_1, x_4].$$

For all $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in K$. Since c_5 is generator the derivation under $\delta_{3\times2}$ should be zero so,

$$\begin{aligned}\delta_{3\times2}(c_5) &= \delta_{3\times2}\begin{pmatrix}\alpha_1[x_5, x_1, x_2] + \alpha_2[x_2, x_1, x_5] + \alpha_3[x_4, x_2, x_2] \\ +\alpha_4[x_4, x_1, x_3] + \alpha_5[x_3, x_1, x_4]\end{pmatrix} \\ &= \alpha_1([x_4, x_1, x_2] + [x_5, x_1, x_1]) + \alpha_2[x_2, x_1, x_4] \\ &\quad +\alpha_3(2[x_4, x_1, x_2] - [x_2, x_1, x_4]) + \alpha_4[x_4, x_1, x_2] + \alpha_5[x_2, x_1, x_4] \\ &= (\alpha_1 + 2\alpha_3 + \alpha_4)[x_4, x_1, x_2] + \alpha_1[x_5, x_1, x_1] \\ &\quad +(\alpha_2 - \alpha_3 + \alpha_5)[x_2, x_1, x_4] \\ &= 0\end{aligned}$$

$\alpha_1 + 2\alpha_3 + \alpha_4 = 0$, $\alpha_1 = 0$ and $\alpha_2 - \alpha_3 + \alpha_5 = 0$.

By Gaussian elimination we have

$\alpha_1 = 0$, $\alpha_2 = \frac{-1}{2}\alpha_4 - \alpha_5$, $\alpha_3 = \frac{-1}{2}\alpha_4$, $\alpha_4 = \alpha_4$ and $\alpha_5 = \alpha_5$,

We know that $\alpha_1 = 0$, so, let $\alpha_4 = -2$ and $\alpha_5 = 0$, then $\alpha_2 = 1$ and $\alpha_3 = 1$

$$c_5 = [x_2, x_1, x_5] + [x_4, x_2, x_2] - 2[x_4, x_1, x_3]$$

As we have two new generators; the next generator come out for $\alpha_1 = 0$, $\alpha_4 = 0$ and $\alpha_5 = -1$, then $\alpha_2 = 1$ and $\alpha_3 = 0$

$$c_6 = [x_2, x_1, x_5] - [x_3, x_1, x_4].$$

- For a bidegree $(4,2)3$;

$(4,2)3 - (2,0)1 = (2,2)2$

$(4,2)3 - (1,0)1 = (3,2)2$

We don't have any match from the previous generators of length 2 this means that $(4,2)3$ has new generator and let be denoted by c_7.

The possibilities of this bidegree are $x_1^2x_3$ and $x_1x_2^2$ and c_7 can be written as linear combination of all possible commutator in bidegree $(4,2)3$. Hence,

$$c_7 = \alpha_1[x_3, x_1, x_1] + \alpha_2[x_2, x_1, x_2]$$

For $\alpha_1, \alpha_2 \in K$. Since c_7 is generator, the derivations under $\delta_{3\times 2}$ should be zero so,

$\delta_{3\times 2}(c_7) = \delta_{3\times 2}(\alpha_1[x_3, x_1, x_1] + \alpha_2[x_2, x_1, x_2])$

$= \alpha_1[x_2, x_1, x_1] + \alpha_2[x_2, x_1, x_1] = (\alpha_1 + \alpha_2)[x_2, x_1, x_1]$

$= 0$

$\alpha_2 = -\alpha_1$

$$c_7 = [x_3, x_1, x_1] - [x_2, x_1, x_2]$$

- For a bidegree $(4,1)3$;

$(4,1)3 - (2,0)1 = (2,1)2$ match with c_3f_1

$(4,1)3 - (1,0)1 = (3,1)2$ match with c_1f_4

Since we have match from the previous generators of length 2 this means that probably $(4,1)3$ has no new generator but we need to verify by denoting this by c_i.

The possibilities of this bidegree are $x_1^2x_5$ and $x_1x_2x_4$ and c_i can be written in this form such that $\alpha_1, \alpha_2, \alpha_3 \in K$

$$c_i = \alpha_1[x_5, x_1, x_1] + \alpha_2[x_4, x_1, x_2] + \alpha_3[x_2, x_1, x_4]$$

$\delta_{3\times 2}(c_i) = \delta_{3\times 2}(\alpha_1[x_5, x_1, x_1] + \alpha_2[x_4, x_1, x_2] + \alpha_3[x_2, x_1, x_4])$

$= \alpha_1[x_4, x_1, x_1] + \alpha_2[x_4, x_1, x_1] + \alpha_3 0$

$= 0$

$\alpha_2 = -\alpha_1$

$c_i = [x_5, x_1, x_1] - [x_4, x_1, x_2] = [x_5, x_1, x_1] - [x_4, x_2, x_1] - [x_2, x_1, x_4]$

$= c_3f_1 - c_1f_4$

Therefore, the bidegree $(4,1)3$ has no new generators.

- For a bidegree $(2,2)3$;

$(2,2)3-(2,0)1=(0,2)2$

$(2,2)3-(1,0)1=(1,2)2$

We don't have any match from the previous generators of length 2 this means that $(2,2)3$ has new generator and let be denoted by c_8.

The possibilities of this bidegree are $x_1x_5^2$; $x_2x_4x_5$ and $x_3x_4^2$. Let c_8 be the generator and is linear combination of all commutators of bidegree $(2,2)3$ such that

$$c_8=\alpha_1[x_5,x_1,x_5]+\alpha_2[x_5,x_2,x_4]+\alpha_3[x_4,x_2,x_5]+\alpha_4[x_4,x_3,x_4]$$

For all α_1, $\alpha_1,\alpha_2,\alpha_3,\alpha_4\in K$. Since c_8 is generator the derivation under $\delta_{3\times 2}$ should be zero so,

$$\begin{aligned}\delta_{3\times2}(c_8)&=\delta_{3\times2}(\alpha_1[x_5,x_1,x_5]+\alpha_2[x_5,x_2,x_4]+\alpha_3[x_4,x_2,x_5]+\alpha_4[x_4,x_3,x_4])\\&=\alpha_1([x_4,x_1,x_5]+[x_5,x_1,x_4])+\alpha_2([x_4,x_2,x_4]+[x_5,x_1,x_4])\\&\quad+\alpha_3([x_4,x_1,x_5]+[x_4,x_2,x_4])+\alpha_4[x_4,x_2,x_4]\\&=(\alpha_1+\alpha_3)[x_4,x_1,x_5]+(\alpha_1+\alpha_2)[x_5,x_1,x_4]\\&\quad+(\alpha_2+\alpha_3+\alpha_4)[x_4,x_2,x_4]\\&=0\end{aligned}$$

$\alpha_1=-\alpha_3$, $\alpha_1=-\alpha_2$, $\alpha_3=\alpha_2$ and $\alpha_4=-2\alpha_2$

Let $\alpha_2=-1$, then $\alpha_1=1$; $\alpha_3=-1$ and $\alpha_4=2$

$$c_8=[x_5,x_1,x_5]-[x_5,x_2,x_4]-[x_4,x_2,x_5]+2[x_4,x_3,x_4].$$

$K[X_5]^{\delta_{3\times2}}$ −module $\left({L'_5}/{L''_5}\right)^{\delta_{3\times2}}$ has four generators c_5,c_6,c_7 and c_8 of length 3.

The following step is to found generators of length 4 with the bidegree (3,2); (3,3) and (4,3). Here we need to verify if each bidegrees has new generators by comparing to the previous generators of algebra of constant of free metabelian Lie algebra we already have and the algebra of constants of polynomial algebra. So, we have to compare the bidegree of generators of length 3 with $f_1:(2,0)1$ and

f_4 : $(1,0)1$ while we compare the bidegree of generators of length 2 with f_1^2 : $(4,0)2$; f_4^2 : $(2,0)2$; $f_1 f_4$: $(3,0)2$; f_2 : $(2,2)2$ and f_5 : $(2,1)2$. Let's start with bidegree

- $(3,2)4$;

$(3,2)4 - (2,0)1 = (1,2)3$

$(3,2)4 - (1,0)1 = (2,2)3$ yes, we have match of $c_8 f_4$

$(3,2)4 - (4,0)2 = (-1,2)2$

$(3,2)4 - (2,0)2 = (1,2)2$

$(3,2)4 - (3,0)2 = (0,2)2$

$(3,2)4 - (2,2)2 = (1,0)2$

$(3,2)4 - (2,1)2 = (1,1)2$ yes, by $c_4 f_5$

Since we have match from the previous generators of length 2 and length 3 this means that the bidegree $(3,2)4$ has no new generator but we need to verify by denoting this by c_j.

The possibilities are $x_4^3 x_3$; $x_4^2 x_2 x_5$ and $x_5^2 x_1 x_4$. c_j can be written as linear combination of all commutator in bidegree such that $\alpha_1, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in K$

$$c_j = \alpha_1[x_4, x_3, x_4, x_4] + \alpha_2[x_5, x_2, x_4, x_4] + \alpha_3[x_4, x_2, x_4, x_5] + \alpha_4[x_5, x_1, x_4, x_5] \\ + \alpha_5[x_4, x_1, x_5, x_5]$$

The condition $\delta_{3\times 2}(c_j) = 0$ gives that

$$\begin{aligned}
\delta_{3\times 2}(c_j) &= \delta_{3\times 2}\begin{pmatrix}\alpha_1[x_4, x_3, x_4, x_4] + \alpha_2[x_5, x_2, x_4, x_4] + \alpha_3[x_4, x_2, x_4, x_5] \\ +\alpha_4[x_5, x_1, x_4, x_5] + \alpha_5[x_4, x_1, x_5, x_5]\end{pmatrix} \\
&= \alpha_1[x_4, x_2, x_4, x_4] + \alpha_2([x_4, x_2, x_4, x_4] + [x_5, x_1, x_4, x_4]) \\
&\quad +\alpha_3([x_4, x_2, x_4, x_5] + [x_4, x_2, x_4, x_5]) \\
&\quad +\alpha_4([x_4, x_1, x_4, x_5] + [x_5, x_1, x_4, x_4]) + 2\alpha_5[x_4, x_1, x_4, x_5] \\
&= (\alpha_1 + \alpha_2 + \alpha_3)[x_4, x_2, x_4, x_4] + (\alpha_2 + \alpha_4)[x_5, x_1, x_4, x_4] \\
&\quad +(\alpha_3 + \alpha_4 + 2\alpha_5)[x_4, x_1, x_4, x_5] \\
&= 0
\end{aligned}$$

$\alpha_2 = -\alpha_4$, $\alpha_3 = -\alpha_4 - 2\alpha_5$, $\alpha_1 = 2\alpha_4 + 2\alpha_5$, $\alpha_4 = \alpha_4$ and $\alpha_5 = \alpha_5$.

Let $\alpha_4 = 1$ and $\alpha_5 = 0$, then $\alpha_1 = 2$; $\alpha_2 = -1$ and $\alpha_3 = 2$

$c_j = 2[x_4, x_3, x_4, x_4] - [x_5, x_2, x_4, x_4] + 2[x_4, x_2, x_4, x_5] + [x_5, x_1, x_4, x_5] = c_8 f_4$

And,

Let $\alpha_4 = 1$ and $\alpha_5 = -1$, then $\alpha_1 = 0$; $\alpha_2 = -1$ and $\alpha_3 = 1$

$c_j = -[x_5, x_2, x_4, x_4] + [x_4, x_2, x_4, x_5] + [x_5, x_1, x_4, x_5] - [x_4, x_1, x_5, x_5] = c_4 f_5$

Therefore, the bidegree $(3,2)4$ has no new generators.

- For a bidegree $(3,3)4$;

$(3,3)4 - (2,0)1 = (1,3)3$

$(3,3)4 - (1,0)1 = (2,3)3$

$(3,3)4 - (4,0)2 = (-1,3)2$

$(3,3)4 - (2,0)2 = (1,3)2$

$(3,3)4 - (3,0)2 = (0,3)2$

$(3,3)4 - (2,2)2 = (1,1)2$ yes, by $c_4 f_2$

$(3,3)4 - (2,1)2 = (1,2)2$

Since we have only one match from the previous generators of length 2 this means that we have new generator from the bidegree $(3,3)4$ and will be denoted by c_9. The possibilities are $x_1 x_2 x_5^2$; $x_1 x_3 x_4 x_5$; $x_2 x_3 x_4^2$ and $x_2^2 x_4 x_5$. Here c_9 can be written as linear combination of all possible commutators in bidegree $(3,3)4$. So,

$$\begin{aligned} c_9 = {} & \alpha_1[x_5, x_1, x_2, x_5] + \alpha_2[x_2, x_1, x_5, x_5] + \alpha_3[x_5, x_1, x_3, x_4] + \alpha_4[x_4, x_1, x_3, x_5] \\ & +\alpha_5[x_3, x_1, x_4, x_5] + \alpha_6[x_4, x_2, x_3, x_4] + \alpha_7[x_3, x_2, x_4, x_4] \\ & +\alpha_8[x_5, x_2, x_2, x_4] + \alpha_9[x_4, x_2, x_2, x_5] \end{aligned}$$

For all $\alpha_1, \alpha_i \in K, i = 1, \ldots, 9$ and the condition $\delta_{3\times 2}(c_9) = 0$ gives that

$$\begin{aligned} \delta_{3\times 2}(c_9) &= \delta_{3\times 2}\begin{pmatrix} \alpha_1[x_5, x_1, x_2, x_5] + \alpha_2[x_2, x_1, x_5, x_5] + \alpha_3[x_5, x_1, x_3, x_4] \\ +\alpha_4[x_4, x_1, x_3, x_5] + \alpha_5[x_3, x_1, x_4, x_5] + \alpha_6[x_4, x_2, x_3, x_4] \\ +\alpha_7[x_3, x_2, x_4, x_4] + \alpha_8[x_5, x_2, x_2, x_4] + \alpha_9[x_4, x_2, x_2, x_5] \end{pmatrix} \\ &= \alpha_1([x_4, x_1, x_2, x_5] + [x_5, x_1, x_1, x_5] + [x_5, x_1, x_2, x_4]) \\ &\quad +2\alpha_2[x_2, x_1, x_4, x_5] + \alpha_3([x_4, x_1, x_3, x_4] + [x_5, x_1, x_2, x_4]) \\ &\quad +\alpha_4([x_4, x_1, x_2, x_5] + [x_4, x_1, x_3, x_4]) \end{aligned}$$

$$
\begin{aligned}
&+\alpha_5([x_2,x_1,x_4,x_5]+[x_3,x_1,x_4,x_4])\\
&+\alpha_6([x_4,x_1,x_3,x_4]+[x_4,x_2,x_2,x_4])+\alpha_7[x_3,x_1,x_4,x_4]\\
&+\alpha_8([x_4,x_2,x_2,x_4]+2[x_5,x_1,x_2,x_4]-[x_2,x_1,x_4,x_5])\\
&+\alpha_9(2[x_4,x_1,x_2,x_5]-[x_2,x_1,x_4,x_5]+[x_4,x_2,x_2,x_4])\\
&=(\alpha_1+\alpha_4+2\alpha_9)[x_4,x_1,x_2,x_5]+\alpha_1[x_5,x_1,x_1,x_5]\\
&+(\alpha_1+\alpha_3+2\alpha_8)[x_5,x_1,x_2,x_4]\\
&+(2\alpha_2+\alpha_5-\alpha_8-\alpha_9)[x_2,x_1,x_4,x_5]\\
&+(\alpha_3+\alpha_4+\alpha_6)[x_4,x_1,x_3,x_4]+(\alpha_5+\alpha_7)[x_3,x_1,x_4,x_4]\\
&+(\alpha_6+\alpha_8+\alpha_9)[x_4,x_2,x_2,x_4]\\
&=0
\end{aligned}
$$

By the Gaussian elimination we have;

$\alpha_1=0$, $\alpha_2={}^1\!/_2\,\alpha_7$, $\alpha_3=2\alpha_9$, $\alpha_4=-2\alpha_9$, $\alpha_5=-\alpha_7$, $\alpha_6=0$, $\alpha_7=\alpha_7$, $\alpha_8=-\alpha_9$, and $\alpha_9=\alpha_9$.

Let's suppose that $\alpha_1=0$; $\alpha_6=0$; $\alpha_7=0$ and $\alpha_9=-1$ then, $\alpha_2=0$; $\alpha_3=-2$; $\alpha_4=2$; $\alpha_5=0$ and $\alpha_8=1$ so,

$$
\begin{aligned}
c_9&=-2[x_5,x_1,x_3,x_4]+2[x_4,x_1,x_3,x_5]+[x_5,x_2,x_2,x_4]-[x_4,x_2,x_2,x_5]\\
&=c_4f_2
\end{aligned}
$$

Hence, this does not give new generator. Let's assume that $\alpha_1=0$; $\alpha_6=0$; $\alpha_7=2$ and $\alpha_9=0$ then, $\alpha_2=1$; $\alpha_3=0$; $\alpha_4=0$; $\alpha_5=-2$ and $\alpha_8=0$ hence,

$$c_9=[x_2,x_1,x_5,x_5]-2[x_3,x_1,x_4,x_5]+2[x_3,x_2,x_4,x_4]$$

Therefore, the bidegree (3,3)4 has new generator which is c_9.

- (4,3)4;

(4,3)4 − (2,0)1 = (2,3)3

(4,3)4 − (1,0)1 = (3,3)3

(4,3)4 − (4,0)2 = (0,3)2

(4,3)4 − (2,0)2 = (2,3)2

(4,3)4 − (3,0)2 = (1,3)2

$(4,3)4 - (2,2)2 = (2,1)2$ yes, by $c_3 f_2$

$(4,3)4 - (2,1)2 = (2,2)2$

Since we have only one match from the previous generators of length 2 this means that we have new generator from the bidegree $(4,3)4$ and will be denoted by c_{10}. The possibilities are $x_1^2x_3x_5$; $x_1x_2^2x_5$; $x_1x_2x_3x_4$ and $x_2^3x_4$. Here c_{10} can be written as linear combination of all possible commutators in bidegree $(4,3)4$. So,

$$
\begin{aligned}
c_{10} = {} & \alpha_1[x_5,x_1,x_1,x_3] + \alpha_2[x_3,x_1,x_1,x_5] + \alpha_3[x_5,x_1,x_2,x_2] \\
& +\alpha_4[x_2,x_1,x_2,x_5] + \alpha_5[x_4,x_1,x_2,x_3] + \alpha_6[x_3,x_1,x_2,x_4] \\
& +\alpha_7[x_2,x_1,x_3,x_4] + \alpha_8[x_4,x_2,x_2,x_2]
\end{aligned}
$$

For all α_1, $\alpha_i \in K, i = 1, \dots, 8$ and the condition $\delta_{3\times 2}(c_{10}) = 0$ gives that

$$
\begin{aligned}
\delta_{3\times 2}(c_{10}) = {} & \delta_{3\times 2}\begin{pmatrix} \alpha_1[x_5,x_1,x_1,x_3] + \alpha_2[x_3,x_1,x_1,x_5] + \alpha_3[x_5,x_1,x_2,x_2] \\ +\alpha_4[x_2,x_1,x_2,x_5] + \alpha_5[x_4,x_1,x_2,x_3] + \alpha_6[x_3,x_1,x_2,x_4] \\ +\alpha_7[x_2,x_1,x_3,x_4] + \alpha_8[x_4,x_2,x_2,x_2] \end{pmatrix} \\
= {} & \alpha_1([x_4,x_1,x_1,x_3] + [x_5,x_1,x_1,x_2]) \\
& +\alpha_2([x_2,x_1,x_1,x_5] + [x_3,x_1,x_1,x_4]) \\
& +\alpha_3([x_4,x_1,x_2,x_2] + 2[x_5,x_1,x_1,x_2]) \\
& +\alpha_4([x_2,x_1,x_1,x_5] + [x_2,x_1,x_2,x_4]) \\
& +\alpha_5([x_4,x_1,x_1,x_3] + [x_4,x_1,x_2,x_2]) \\
& +\alpha_6([x_2,x_1,x_2,x_4] + [x_3,x_1,x_1,x_4]) + +\alpha_7[x_2,x_1,x_2,x_4] \\
& +\alpha_8(3[x_4,x_1,x_2,x_2] - 2[x_2,x_1,x_2,x_4]) \\
= {} & (\alpha_1+\alpha_5)[x_4,x_1,x_1,x_3] + (\alpha_1+2\alpha_3)[x_5,x_1,x_1,x_2] \\
& +(\alpha_2+\alpha_4)[x_2,x_1,x_1,x_5] + (\alpha_2+\alpha_6)[x_3,x_1,x_1,x_4] \\
& +(\alpha_3+\alpha_5+3\alpha_8)[x_4,x_1,x_2,x_2] \\
& +(\alpha_4+\alpha_6+\alpha_7-2\alpha_8)[x_2,x_1,x_2,x_4] \\
= {} & 0
\end{aligned}
$$

By the Gaussian elimination we have;

$\alpha_1 = 2\alpha_8$, $\alpha_2 = {}^1\!/_2\,\alpha_7 - \alpha_8$, $\alpha_3 = -\alpha_8$, $\alpha_4 = -{}^1\!/_2\,\alpha_7 + \alpha_8$, $\alpha_5 = -2\alpha_8$, $\alpha_6 = -{}^1\!/_2\,\alpha_7 + \alpha_8$, $\alpha_7 = \alpha_7$ and $\alpha_8 = \alpha_8$.

Since we suppose that $\alpha_7 = -2$ and $\alpha_8 = -1$ then, $\alpha_1 = -2$; $\alpha_2 = 0$; $\alpha_3 = 1$; $\alpha_4 = 0$; $\alpha_5 = 2$ and $\alpha_6 = 0$

$$c_{10} = -2[x_5, x_1, x_1, x_3] + [x_5, x_1, x_2, x_2] + 2[x_4, x_1, x_2, x_3] - 2[x_2, x_1, x_3, x_4] - [x_4, x_2, x_2, x_2] = c_4 f_2$$

Hence, this does not give new generator and let $\alpha_7 = 2$ and $\alpha_8 = 0$ then, $\alpha_1 = 0$; $\alpha_2 = 1$; $\alpha_3 = 0$; $\alpha_4 = -1$; $\alpha_5 = 0$ and $\alpha_6 = -1$

$$c_{10} = [x_3, x_1, x_1, x_5] - [x_2, x_1, x_2, x_5] - [x_3, x_1, x_2, x_4] + 2[x_2, x_1, x_3, x_4]$$

Therefore, the bidegree (4,3)4 has new generator which is c_{10}.

$K[X_5]^{\delta_{3\times 2}}$ −module $\left({L_5'}/{L_5''}\right)^{\delta_{3\times 2}}$ has two generators c_9 and c_{10} of length 4.

Continuing in the same way we found another the bidegree (4,4)5 has new generator and is denoted as c_{11} so,

$K[X_5]^{\delta_{3\times 2}}$ −module $\left({L_5'}/{L_5''}\right)^{\delta_{3\times 2}}$ has two generators c_{11} of length 5 such that

$$\begin{aligned} c_{11} = {} & [x_5, x_2, x_2, x_2, x_4] + [x_4, x_2, x_2, x_2, x_5] + 2[x_5, x_1, x_1, x_3, x_5] \\ & -[x_3, x_1, x_1, x_5, x_5] - [x_5, x_1, x_2, x_2, x_5] + [x_2, x_1, x_2, x_5, x_5] \\ & -2[x_4, x_2, x_2, x_3, x_4] + 4[x_4, x_1, x_3, x_3, x_4] - 2[x_3, x_1, x_3, x_4, x_4] \\ & -2[x_5, x_1, x_2, x_3, x_4] - 2[x_4, x_1, x_2, x_3, x_5] + 2[x_3, x_1, x_2, x_4, x_5]. \end{aligned}$$

Since the bidegree (4,3) of length 5, bidegree (6,3) of length 6 and bidegree (7,4) of length 7 have the correspondent with the previous generators, the we conclude that this bidegrees does not generate the new generators.

4.2. The Nowicki Conjecture for Free Metabelian Poisson Algebras

In this section we give spanning elements of the algebra of constants of a special Weitzenböck derivation, called the Nowicki derivation, of the free metabelian Poisson algebra of rank $2m$. Let P_{2m} be the free metabelian Poisson algebra generated by $x_1 < x_2 < \cdots < x_m < y_1 < y_2 < \cdots < y_m$, and let

$$Z_{2m} = X_m \cup Y_m = \{x_1, \ldots, x_m, y_1, \ldots, y_m\} = \{z_1, \ldots, z_{2m}\}.$$

The Nowicki derivation of the free metabelian Poisson algebra is defined as δ_{2m} such that

$$\delta_{2m}: y_i \longrightarrow x_i \longrightarrow 0.$$

Let

$\mathcal{B}_{1m} = Z_m \cup \left\{[z_{i_1}, \ldots, z_{i_k}] \mid i_1 > i_2 \leq i_3 \leq \cdots \leq i_k, \qquad k \geq 2\right\}$

$\mathcal{B}_{2m} = \left\{z_i \cdot z_j, z_i \cdot z_j \cdot z_k \mid i \leq j \leq k\right\}$

$\mathcal{B}_{3m} = \left\{[z_i, z_j] \cdot z_k \mid i > j\right\}$

$\mathcal{B}_{4m} = \left\{[z_i, z_j, z_k] \cdot z_l \mid j < i \leq l, j \leq k < l\right\}$.

Then the union $K\mathcal{B}_{1m} \cup K\mathcal{B}_{2m} \cup K\mathcal{B}_{3m} \cup K\mathcal{B}_{4m}$ forms a linear basis for the free metabelian Poisson algebra P_{2m} by (Zhang, Z., et al 2019) that is $P_{2m} = K\mathcal{B}_{1m} \oplus K\mathcal{B}_{2m} \oplus K\mathcal{B}_{3m} \oplus K\mathcal{B}_{4m}$.

Lemma 4.2.1: The subspaces$(K\mathcal{B}_{im})^{\delta_{2m}}$, $i = 1,2,3,4$, are invariant spaces.

Proof: We need to show that $(K\mathcal{B}_{1m})^{\delta_{2m}} \subseteq K\mathcal{B}_{1m}$, $(K\mathcal{B}_{2m})^{\delta_{2m}} \subseteq K\mathcal{B}_{2m}$, $(K\mathcal{B}_{3m})^{\delta_{2m}} \subseteq K\mathcal{B}_{3m}$ and $(K\mathcal{B}_{4m})^{\delta_{2m}} \subseteq K\mathcal{B}_{4m}$.

Let start with $(K\mathcal{B}_{1m})^{\delta_{2m}} \subseteq K\mathcal{B}_{1m}$. Let $p \in K\mathcal{B}_{1m}$, then p is of the form of

$$p = \sum_{i>j\leq j_1\leq\cdots\leq j_{2m}} \alpha_a \left[z_i, z_j\right] \cdot adZ_m^a \quad | \; z_i \in X_m \cup Y_m = Z_m, adZ_m^a$$

$$= adz_{j_1}^{a_1} \cdots adz_{j_{2m}}^{a_{2m}}, \alpha_a = \alpha_{a_1,\ldots,a_{2m}}$$

$$\delta_{2m}(p) = \sum_{i>j\leq j_1\leq\cdots\leq j_{2m}} \alpha_a \left[\delta_{2m}(z_i), z_j\right] \cdot adZ_m^a + \sum_{i>j\leq j_1\leq\cdots\leq j_{2m}} \alpha_a \left[z_i, \delta_{2m}(z_j)\right] \cdot adZ_m^a$$

$$+\sum_{i>j\le j_1\le\cdots\le j_{2m}}\alpha_a\left[z_i,z_j\right]\cdot ad\delta_{2m}\left(z_{j_1}^{a_1}\right)\cdot adz_{j_2}^{a_2}\cdots adz_{j_{2m}}^{a_{2m}}+\cdots$$

$$+\sum_{i>j\le j_1\le\cdots\le j_{2m}}\alpha_a\left[z_i,z_j\right]\cdot adz_{j_1}^{a_1}\cdots adz_{j_{2m-1}}^{a_{2m-1}}ad\delta_{2m}\left(z_{j_{2m}}^{a_{2m}}\right)$$

This implies that $\delta_{2m}(p)\in K\mathcal{B}_{1m}$, hence $(K\mathcal{B}_{1m})^{\delta_{2m}}\subseteq K\mathcal{B}_{1m}$.

For $(K\mathcal{B}_{2m})^{\delta_{2m}}\subseteq K\mathcal{B}_{2m}$, let $p\in K\mathcal{B}_{2m}$,then p is of the form of

$$p=\sum_{i\ge j}\alpha_{ij}\,z_i\cdot z_j+\sum_{i\ge j\ge k}\beta_{ijk}\,z_i\cdot z_j\cdot z_k$$

$$\delta_{2m}(p)=\sum_{i\ge j}\alpha_{ij}\left(\delta_{2m}(z_i)\cdot z_j+z_i\cdot\delta_{2m}(z_j)\right)$$

$$+\sum_{i\ge j\ge k}\beta_{ijk}\left(\delta_{2m}(z_i)\cdot z_j\cdot z_k+z_i\cdot\delta_{2m}(z_j)\cdot z_k+z_i\cdot z_j\cdot\delta_{2m}(z_k)\right)$$

This implies that $\delta_{2m}(p)\in K\mathcal{B}_{2m}$, hence $(K\mathcal{B}_{2m})^{\delta_{2n}}\subseteq K\mathcal{B}_{2m}$.

For $(K\mathcal{B}_{3m})^{\delta_{2m}}\subseteq K\mathcal{B}_{3m}$, let $p\in K\mathcal{B}_{3m}$,then p is of the form of

$$p=\sum_{j<i,1\le k\le m}\alpha_{ijk}\left[z_i,z_j\right]\cdot z_k$$

$$\delta_{2m}(p)=\sum_{j<i,1\le k\le m}\alpha_{ijk}\begin{pmatrix}\left[\delta_{2m}(z_i),z_j\right]\cdot z_k+\left[z_i,\delta_{2m}(z_j)\right]\cdot z_k\\ +\left[z_i,z_j\right]\cdot\delta_{2m}(z_k)\end{pmatrix}$$

This implies that $\delta_{2m}(p)\in K\mathcal{B}_{3m}$, hence $(K\mathcal{B}_{3m})^{\delta_{2n}}\subseteq K\mathcal{B}_{3m}$.

Lastly, $(K\mathcal{B}_{4m})^{\delta_{2m}}\subseteq K\mathcal{B}_{4m}$, let $p\in K\mathcal{B}_{4m}$, then p is of the form of

$$p=\sum_{i>j\le k<l\ge i}\alpha_{ijkl}\left[z_i,z_j,z_k\right]\cdot z_l$$

$$\delta_{2m}(p)=\sum_{i>j\le k<l\ge i}\alpha_{ijkl}\begin{pmatrix}\left[\delta_{2m}(z_i),z_j,z_k\right]\cdot z_l+\left[z_i,\delta_{2m}(z_j),z_k\right]\cdot z_l\\ +\left[z_i,z_j,\delta_{2m}(z_k)\right]\cdot z_l+\left[z_i,z_j,z_k\right]\cdot\delta_{2m}(z_l)\end{pmatrix}$$

Here the element $\delta_{2m}(p)$ must be rearranged as linear sum of basis element in $K\mathcal{B}_{4m}$, hence $\delta_{2m}(p)\in K\mathcal{B}_{4m}$ and $(K\mathcal{B}_{4m})^{\delta_{2m}}\subseteq K\mathcal{B}_{4m}$.

Therefore, from above Lemma 4.2.1 we have

$$P_{2m}^{\delta_{2m}} = (K\mathcal{B}_{1m})^{\delta_{2m}} \oplus (K\mathcal{B}_{2m})^{\delta_{2m}} \oplus (K\mathcal{B}_{3m})^{\delta_{2m}} \oplus (K\mathcal{B}_{4m})^{\delta_{2m}}$$

Thus, the action of Weitzenböck derivation δ_{2m} in the subspaces $K\mathcal{B}_{im}$, $i = 1, 2, 3, 4$ is sufficient to give the full description of the algebra $P_{2m}^{\delta_{2m}}$. Let F_{2m} be the free metabelian Lie algebra and let $\mathcal{B}_{1m}$ be the basis for the free metabelian Lie algebra, then $F_{2m}^{\delta_{2m}} = (K\mathcal{B}_{1m})^{\delta_{2m}}$ has generators as $K[X_{2m}]^{\delta_{2m}}$-module see e.g., in (Dangovski, R., et al 2013) and (Drensky and Fındık, 2020).

Let us consider that

$$K[Z_m] = K[X_m \cup Y_m] = W_0 \oplus W_1 \oplus W_2 \oplus W_3 \oplus \cdots$$

where the subvector spaces of homogenous degrees are as follows:

$W_0 = K\{c \cdot 1 \mid c \in K\} = Sp\{1\}$

$W_1 = \{c_1x_1 + \cdots + c_mx_m + d_1y_1 + \cdots + d_my_m \mid c_i, d_j \in K\} = Sp\{X_m \cup Y_m\}$

$W_2 = \{\alpha_{ij}z_iz_j \mid z_i, z_j \in K\} = Sp\{ z_i, z_j \in X_n \cup Y_n\}$

$W_3 = \{\alpha_{ijk}z_iz_jz_k \mid z_i, z_j, z_k \in K\} = Sp\{ z_i, z_j, z_k \in X_n \cup Y_n\}$

By the Lemma 4.2.1, it is known that $W_i^{\delta_{2m}} \subseteq W_i$ this means that

$$K[Z_m]^{\delta_{2m}} = \oplus W_i^{\delta_{2m}}$$

Therefore, we have

$W_0^{\delta_{2m}} = 0,$

$W_1^{\delta_{2m}} = Sp\{X_m\} = \{c_1x_1 + \cdots + c_mx_m \mid c_i \in K\},$

$W_2^{\delta_{2m}} = Sp\{u_{pq} \mid 1 \le p < q \le m\} = \left\{\sum_{\alpha_{pq}\in K} u_{pq} \mid 1 \le p < q \le m\right\},$

$W_3^{\delta_{2m}} = Sp\{x_iu_{pq} \mid x_iu_{jk} - x_ju_{ik} + x_ku_{ij}\},$

$\vdots$

$K[Z_m]^{\delta_{2m}} = K\left[X_m, u_{pq} \mid u_{pq} = \begin{vmatrix} x_p & y_p \\ x_q & y_q \end{vmatrix} = x_py_q - x_qy_p, R\right]$ where R is relation such that

$$R:\begin{cases} x_i u_{jk} - x_j u_{ik} + x_k u_{ij} = 0 & 1 \le i < j < k \le m \\ u_{ij} u_{kl} - u_{ik} u_{jl} + u_{il} u_{jk} = 0 & 1 \le i < j < k < l \le m \end{cases}$$

We conclude from here that $(K\mathcal{B}_{2m})^{\delta_{2m}} = W_2^{\delta_{2m}} \oplus W_3^{\delta_{2m}}$.

Let $K\mathcal{B}_{3m} = KA_1 \oplus KA_2 \oplus KA_3 \oplus KA_4 \oplus KA_5 \oplus KA_6$ such that

$A_1 = \{[x_i, x_j] \cdot x_k \ : \ 1 \le \ j < i \le m, \ 1 \le k \le m\}$,

$A_2 = \{[y_i, y_j] \cdot y_k \ : \ 1 \le \ j < i \le m, \ 1 \le k \le m\}$,

$A_3 = \{[x_i, x_j] \cdot y_k \ : \ 1 \le \ j < i \le m, \ 1 \le k \le m\}$,

$A_4 = \{[y_i, y_j] \cdot x_k \ : \ 1 \le \ j < i \le m, \ 1 \le k \le m\}$,

$A_5 = \{[y_i, x_j] \cdot y_k \ : 1 \le \ i, j, k \le m\}$,

$A_6 = \{[y_i, x_j] \cdot x_k \ : \ 1 \le \ i, j, k \le m\}$.

Theorem 4.2.2: The vector space $(K\mathcal{B}_{3m})^{\delta_{2m}}$ is spanned on the elements of the form

$$[x_i, x_j] \cdot x_k \ : \ 1 \le \ j < i \le m, \qquad 1 \le k \le m$$
$$[y_i, x_i] \cdot x_k \ : \ 1 \le i, k \le m$$
$$[x_i, x_j] \cdot y_k + [y_j, x_i] \cdot x_k \ : 1 \le \ j < i \le m, \qquad 1 \le k \le m$$
$$[y_i, x_j] \cdot x_k + [y_j, x_i] \cdot x_k \ : 1 \le \ j < i \le m, \qquad 1 \le k \le m.$$

Proof: Let $p \in K\mathcal{B}_{3m}$, then there exist $P_i \in KA_i \ i = 1, \dots, 6$ such that $p = p_1 + p_2 + p_3 + p_4 + p_5 + p_6$. If $p \in (K\mathcal{B}_{3m})^{\delta_{2m}}$ and δ_{2m} is linear on element, then

$0 = \delta_{2m}(p) = \delta_{2m}(p_1 + \dots + p_6) = \delta_{2m}(p_1) + \dots + \delta_{2m}(p_6)$

Observation:

Clearly, $\delta_{2m}(p_1) = 0$, where $p_1 \in KA_1$.

Since $p_2 \in KA_2$, then p_2 is of the form

$p_2 = \sum_{\substack{1 \le j < i \le m \\ 1 \le k \le m}} \alpha_{ijk} [y_i, y_j] \cdot y_k$ for some $\alpha_{ijk} \in K$. Then

$$\delta_{2m}(p_2) = \sum_{\substack{1\le j<i\le m\\ 1\le k\le m}} \alpha_{ijk}\begin{pmatrix}[\delta_{2m}(y_i), y_j]\cdot y_k + [y_i, \delta_{2m}(y_j)]\cdot y_k\\ +[y_i, y_j]\cdot \delta_{2m}(y_k)\end{pmatrix}$$

$$= \sum_{\substack{1\le j<i\le m\\ 1\le k\le m}} \alpha_{ijk}\left(-[y_j, x_i]\cdot y_k + [y_i, x_j]\cdot y_k + [y_i, y_j]\cdot x_k\right)$$

$\Longrightarrow \delta_{2m}(p_2) \in KA_4 \oplus KA_5$.

For $p_3 \in KA_3$, then p_3 is of the form

$$p_3 = \sum_{\substack{1\le j<i\le m\\ 1\le k\le m}} \alpha_{ijk}\,[x_i, x_j]\cdot y_k \text{ for some } \alpha_{ijk} \in K.\text{ Then}$$

$$\delta_{2m}(p_3) = \sum_{\substack{1\le j<i\le m\\ 1\le k\le m}} \alpha_{ijk}\,[x_i, x_j]\cdot \delta_{2m}(y_k) = \sum_{\substack{1\le j<i\le m\\ 1\le k\le m}} \alpha_{ijk}\,[x_i, x_j]\cdot x_k$$

$\Longrightarrow \delta_{2m}(p_3) \in KA_1$.

For $p_4 \in KA_4$, then p_4 is of the form

$$p_4 = \sum_{\substack{1\le j<i\le m\\ 1\le k\le m}} \alpha_{ijk}\,[y_i, y_j]\cdot x_k \text{ for some } \alpha_{ijk} \in K.\text{ Then}$$

$$\delta_{2m}(p_4) = \sum_{\substack{1\le j<i\le m\\ 1\le k\le m}} \alpha_{ijk}\left([\delta_{2m}(y_i), y_j]\cdot x_k + [y_i, \delta_{2m}(y_j)]\cdot x_k\right)$$

$$= \sum_{\substack{1\le j<i\le m\\ 1\le k\le m}} \alpha_{ijk}\left(-[y_j, x_i]\cdot x_k + [y_i, x_j]\cdot x_k\right)$$

$\Longrightarrow \delta_{2m}(p_4) \in KA_6$.

For $p_5 \in KA_5$, then p_5 is of the form

$$p_5 = \sum_{1\le i,j,k\le m} \alpha_{ijk}\,[y_i, x_j]\cdot y_k \text{ for some } \alpha_{ijk} \in K.\text{ Then}$$

$$\delta_{2m}(p_5) = \sum_{1\le i,j,k\le m} \alpha_{ijk}\left([\delta_{2m}(y_i), x_j]\cdot y_k + [y_i, x_j]\cdot \delta_{2m}(y_k)\right)$$

$$= \sum_{1\le i,j,k\le m} \alpha_{ijk}\left([x_i, x_j]\cdot y_k + [y_i, x_j]\cdot x_k\right)$$

$\Longrightarrow \delta_{2m}(p_5) \in KA_3 \oplus KA_6$.

For $p_6 \in KA_6$, then p_6 is of the form

$$p_6 = \sum_{1\le i,j,k\le m} \alpha_{ijk}\,[y_i, x_j]\cdot x_k \ \text{ for some } \alpha_{ijk} \in K. \text{Then}$$

$$\delta_{2m}(p_6) = \sum_{1\le i,j,k\le m} \alpha_{ijk}\,[\delta_{2m}(y_i), x_j]\cdot x_k = \sum_{1\le i,j,k\le m} \alpha_{ijk}\,[x_i, x_j]\cdot x_k$$

$\Longrightarrow \delta_{2m}(p_6) \in KA_1$.

Since $\delta_{2m}(p_1) = 0$, $\delta_{2m}(p_2) \in KA_4 \oplus KA_5$, $\delta_{2m}(p_3) \in KA_1$, $\delta_{2m}(p_4) \in KA_6$, $\delta_{2m}(p_5) \in KA_3 \oplus KA_6$ and $\delta_{2m}(p_6) \in KA_1$, then $\delta_{2m}(p) = \delta_{2m}(p_1 + +p_2 + p_3 + p_4 + p_5 + p_6) = \delta_{2m}(p_1) + \delta_{2n}(p_2) + \delta_{2m}(p_3 + p_6) + \delta_{2m}(p_4 + p_5)$. This means that $\delta_{2m}(p) = 0 = \delta_{2m}(p_1) = \delta_{2m}(p_2) = \delta_{2m}(p_3 + p_6) = \delta_{2m}(p_4 + p_5)$. From the observation we have the following fact $(KA_1)^{\delta_{2m}} = \{0\}$, $(KA_2)^{\delta_{2m}} \subseteq KA_4 \oplus KA_5$, $(KA_3 \oplus KA_6)^{\delta_{2m}} \subseteq KA_1$ and $(KA_4 \oplus KA_5)^{\delta_{2m}} \subseteq KA_3 \oplus KA_6$.

Therefore, since $\delta_{2m}(p_1) = 0$, then $KA_1 \subseteq (K\mathcal{B}_{3m})^{\delta_{2m}}$, i.e., elements of the form

$$\sum_{\substack{1\le j<i\le m\\ 1\le k\le m}} \alpha_{ijk}\,[x_i, x_j]\cdot x_k \ \text{ are constants.}$$

For

$$\begin{aligned}
\delta_{2m}(p_2) = 0 = \delta_{2m}&\left(\sum_{\substack{1\le j<i\le m\\ 1\le k\le m}} \alpha_{ijk}\,[y_i, y_j]\cdot y_k\right)\\
&= \sum_{\substack{1\le j<i\le m\\ 1\le k\le m}} \alpha_{ijk}\left(-[y_j, x_i]\cdot y_k + [y_i, x_j]\cdot y_k + [y_i, y_j]\cdot x_k\right)\\
&= \sum_{\substack{1\le j<i\le m\\ 1\le k\le m}} \alpha_{ijk}\left(-[y_j, x_i]\cdot y_k + [y_i, x_j]\cdot y_k\right) + \sum_{\substack{1\le j<i\le m\\ 1\le k\le m}} \alpha_{ijk}\,[y_i, y_j]\cdot x_k\\
&= 0
\end{aligned}$$

By Remark 2.10.2 if $a + b = 0$ such that $a \in V$, $b \in W$ and $V \cap W = \{0\}$, then $a = b = 0$. Hence,

$$\sum_{\substack{1\leq j<i\leq m \\ 1\leq k\leq m}} \alpha_{ijk}\,[y_i, y_j]\cdot x_k = 0$$

Since $[y_i, y_j]\cdot x_k$ is a basis element in A_4 it cannot be zero hence $\alpha_{ijk} = 0$.

Therefore,

$$p_2 = \sum_{\substack{1\leq j<i\leq m \\ 1\leq k\leq m}} \alpha_{ijk}\,[y_i, y_j]\cdot y_k = 0$$

and this means that $(KA_2)^{\delta_{2m}} = \{0\}$.

Let

$$\begin{aligned}
p_3 + p_6 &= \sum_{\substack{1\leq j<i\leq m \\ 1\leq k\leq m}} \alpha_{ijk}\,[x_i, x_j]\cdot y_k + \sum_{1\leq i,j,k\leq m} \beta_{ijk}\,[y_i, x_j]\cdot x_k \\
&= \sum_{\substack{1\leq j<i\leq m \\ 1\leq k\leq m}} \alpha_{ijk}\,[x_i, x_j]\cdot y_k + \sum_{\substack{1\leq j=i\leq m \\ 1\leq k\leq m}} \beta_{iik}\,[y_i, x_i]\cdot x_k \\
&\quad + \sum_{\substack{1\leq j<i\leq m \\ 1\leq k\leq m}} \beta_{ijk}\,[y_i, x_j]\cdot x_k \\
&\quad + \sum_{\substack{1\leq j<i\leq m \\ 1\leq k\leq m}} \beta_{jik}\,[y_j, x_i]\cdot x_k
\end{aligned}$$

$$\begin{aligned}
\delta_{2m}(p_3 + p_6) &= 0 \\
&= \delta_{2m}\left(\begin{array}{c} \displaystyle\sum_{\substack{1\leq j<i\leq m \\ 1\leq k\leq m}} \alpha_{ijk}\,[x_i, x_j]\cdot y_k + \sum_{\substack{1\leq j=i\leq m \\ 1\leq k\leq m}} \beta_{iik}\,[y_i, x_i]\cdot x_k \\ \displaystyle + \sum_{\substack{1\leq j<i\leq m \\ 1\leq k\leq m}} \beta_{ijk}\,[y_i, x_j]\cdot x_k + \sum_{\substack{1\leq j<i\leq m \\ 1\leq k\leq m}} \beta_{jik}\,[y_j, x_i]\cdot x_k \end{array}\right) \\
&= \sum_{\substack{1\leq j<i\leq m \\ 1\leq k\leq m}} \alpha_{ijk}\,[x_i, x_j]\cdot x_k + \sum_{\substack{1\leq j=i\leq m \\ 1\leq k\leq m}} \beta_{iik}\,[x_i, x_i]\cdot x_k
\end{aligned}$$

$$+\sum_{\substack{1\le j<i\le m\\1\le k\le m}}\beta_{ijk}\left[x_i,x_j\right]\cdot x_k-\sum_{\substack{1\le j<i\le m\\1\le k\le m}}\beta_{jik}\left[x_i,x_j\right]\cdot x_k$$

$$=\sum_{\substack{1\le j<i\le m\\1\le k\le m}}\left(\alpha_{ijk}+\beta_{ijk}-\beta_{jik}\right)\left[x_i,x_j\right]\cdot x_k+\sum_{\substack{1\le j=i\le m\\1\le k\le m}}\beta_{iik}\left[x_i,x_i\right]$$

$$\cdot x_k$$

$$=0$$

$[x_i,x_i]\cdot x_k=0$ this mean β_{iik} can be any. Since $\left[x_i,x_j\right]\cdot x_k$ is a basis element in A_1, then it cannot be zero hence, $\alpha_{ijk}+\beta_{ijk}-\beta_{jik}=0$. So, $\beta_{jik}=\alpha_{ijk}+\beta_{ijk}$. Therefore,

$$p_3+p_6=\sum_{\substack{1\le j<i\le m\\1\le k\le m}}\alpha_{ijk}\left[x_i,x_j\right]\cdot y_k\sum_{\substack{1\le j<i\le m\\1\le k\le m}}\beta_{ijk}\left[y_i,x_j\right]\cdot x_k$$

$$+\sum_{\substack{1\le j<i\le m\\1\le k\le m}}\left(\alpha_{ijk}+\beta_{ijk}\right)\left[y_j,x_i\right]\cdot x_k+\sum_{\substack{1\le j=i\le m\\1\le k\le m}}\beta_{iik}\left[y_i,x_i\right]\cdot x_k$$

$$=\sum_{\substack{1\le j<i\le m\\1\le k\le m}}\alpha_{ijk}\left(\left[x_i,x_j\right]\cdot y_k+\left[y_j,x_i\right]\cdot x_k\right)$$

$$+\sum_{\substack{1\le j<i\le m\\1\le k\le m}}\beta_{ijk}\left(\left[y_i,x_j\right]\cdot x_k+\left[y_j,x_i\right]\cdot x_k\right)+\sum_{\substack{1\le j=i\le m\\1\le k\le m}}\beta_{iik}\left[y_i,x_i\right]\cdot x_k$$

This proved that all element here are constant in $KA_3\oplus KA_6$. Therefore, $(KA_3\oplus KA_6)^{\delta_{2m}}$ is spanned on the elements of the form

$$[y_i,x_i]\cdot x_k\ :\ 1\le i,k\le m$$

$$\left[x_i,x_j\right]\cdot y_k+\left[y_j,x_i\right]\cdot x_k\ :\ 1\le\ j<i\le m,\qquad 1\le k\le m$$

$$\left[y_i,x_j\right]\cdot x_k+\left[y_j,x_i\right]\cdot x_k\ :\ 1\le\ j<i\le m,\qquad 1\le k\le m.$$

Finally, we have to show that $(KA_4\oplus KA_5)^{\delta_{2m}}=\{0\}$. Let

$$p_4+p_5=\sum_{\substack{1\le j<i\le m\\1\le k\le m}}\alpha_{ijk}\left[y_i,y_j\right]\cdot x_k+\sum_{1\le i,j,k\le m}\beta_{ijk}\left[y_i,x_j\right]\cdot y_k$$

$$= \sum_{\substack{1\le j<i\le m \\ 1\le k\le m}} \alpha_{ijk}\left[y_i, y_j\right]\cdot x_k + \sum_{\substack{1\le j=i\le m \\ 1\le k\le m}} \beta_{iik}\left[y_i, x_i\right]\cdot y_k$$

$$+ \sum_{\substack{1\le j<i\le m \\ 1\le k\le m}} \beta_{ijk}\left[y_i, x_j\right]\cdot y_k$$

$$+ \sum_{\substack{1\le j<i\le m \\ 1\le k\le n}} \beta_{jik}\left[y_j, x_i\right]\cdot y_k$$

$$\delta_{2m}(p_4 + p_5) = 0$$

$$= \delta_{2m}\begin{pmatrix} \displaystyle\sum_{\substack{1\le j<i\le m \\ 1\le k\le m}} \alpha_{ijk}\left[y_i, y_j\right]\cdot x_k + \sum_{\substack{1\le j=i\le m \\ 1\le k\le m}} \beta_{iik}\left[y_i, x_i\right]\cdot y_k \\ \displaystyle + \sum_{\substack{1\le j<i\le m \\ 1\le k\le m}} \beta_{ijk}\left[y_i, x_j\right]\cdot y_k + \sum_{\substack{1\le j<i\le m \\ 1\le k\le m}} \beta_{jik}\left[y_j, x_i\right]\cdot y_k \end{pmatrix}$$

$$= \sum_{\substack{1\le j<i\le m \\ 1\le k\le m}} \alpha_{ijk}\left(-\left[y_j, x_i\right]\cdot x_k + \left[y_i, x_j\right]\cdot x_k\right)$$

$$+ \sum_{\substack{1\le j=i\le m \\ 1\le k\le m}} \beta_{iik}\left[y_i, x_i\right]\cdot x_k$$

$$+ \sum_{\substack{1\le j<i\le m \\ 1\le k\le m}} \beta_{ijk}\left(\left[x_i, x_j\right]\cdot y_k + \left[y_i, x_j\right]\cdot x_k\right)$$

$$+ \sum_{\substack{1\le j<i\le m \\ 1\le k\le m}} \beta_{jik}\left(-\left[x_i, x_j\right]\cdot y_k + \left[y_j, x_i\right]\cdot x_k\right)$$

$$= \sum_{\substack{1\le j<i\le m \\ 1\le k\le m}} \left(-\alpha_{ijk} + \beta_{jik}\right)\left[y_j, x_i\right]\cdot x_k$$

$$+ \sum_{\substack{1\le j<i\le m \\ 1\le k\le m}} \left(\alpha_{ijk} + \beta_{ijk}\right)\left[y_i, x_j\right]\cdot x_k$$

$$+\sum_{\substack{1\le j<i\le m\\ 1\le k\le m}} (\beta_{ijk}-\beta_{jik})[x_i,x_j]\cdot y_k + \sum_{\substack{1\le j=i\le m\\ 1\le k\le m}} \beta_{iik}\,[y_i,x_i]\cdot x_k$$

$$=0$$

Since $[y_j,x_i]\cdot x_k$, $[y_i,x_j]\cdot x_k$, and $[y_i,x_i]\cdot x_k$are the basis element in A_6 and $[x_i,x_j]\cdot y_k$ is a basis element in basis A_3, then they cannot be zero hence, $(-\alpha_{ijk}+\beta_{jik})=(\alpha_{ijk}+\beta_{ijk})=(\beta_{ijk}-\beta_{jik})=\beta_{iik}=0$. This means that for

$$\alpha_{ijk}=\beta_{ijk}=\beta_{jik}=\beta_{iik}=0$$

Therefore, $p_4+p_5=0$ and this means that $(KA_4\oplus KA_5)^{\delta_{2m}}=\{0\}$.

Theorem 4.2.3: The vector space $(K\mathcal{B}_{4m})^{\delta_{2m}}$ is spanned on the elements of the form

$$[x_i,x_j,x_k]\cdot x_l, \text{ where } 1\le j<i\le l\le m,\ 1\le j\le k<l\le m,$$

$$[x_i,x_j,x_j]\cdot y_j,$$

$$[x_i,x_j,x_i]\cdot y_i,$$

$$[x_i,x_j,x_i]\cdot y_j+[x_i,x_j,x_j]\cdot y_i,$$

where $1\le j<i\le m$,

$$[x_i,x_j,x_k]\cdot y_k,$$

$$[x_k,x_j,x_i]\cdot y_i,$$

$$[x_i,x_j,x_k]\cdot y_i+[x_i,x_j,x_i]\cdot y_k,$$

$$[x_k,x_i,x_i]\cdot y_j+[x_i,x_j,x_i]\cdot y_k,$$

$$-[x_k,x_j,x_i]\cdot y_j+[x_i,x_j,x_k]\cdot y_j,$$

$$[x_i,x_j,x_j]\cdot y_k+[x_i,x_j,x_k]\cdot y_j,$$

$$[x_k,x_j,x_j]\cdot y_i+[x_i,x_j,x_k]\cdot y_j,$$

$$-[x_k,x_i,x_k]\cdot y_j+[x_k,x_j,x_k]\cdot y_i,$$

$$[x_k,x_j,x_i]\cdot y_k+[x_k,x_j,x_k]\cdot y_i,$$

where $1\le j<i<k\le m$,

$$[x_i,x_j,x_k]\cdot y_l+[x_i,x_j,x_l]\cdot y_k,$$

$$[x_k, x_j, x_i] \cdot y_l + [x_k, x_j, x_l] \cdot y_i,$$

$$[x_k, x_j, x_i] \cdot y_l + [x_l, x_i, x_k] \cdot y_j,$$

$$[x_i, x_j, x_k] \cdot y_l + [x_l, x_j, x_i] \cdot y_k - [x_k, x_j, x_i] \cdot y_l,$$

$$-[x_i, x_j, x_k] \cdot y_l + [x_k, x_i, x_l] \cdot y_j + [x_k, x_j, x_i] \cdot y_l,$$

$$-[x_i, x_j, x_k] \cdot y_l + [x_k, x_j, x_i] \cdot y_l + [x_l, x_j, x_k] \cdot y_i,$$

where $1 \leq j < i < k < l \leq m$,

and

$$[y_i, x_i, x_i] \cdot y_i, \text{ where } 1 \leq i \leq m,$$

$$[y_j, x_j, x_i] \cdot y_i,$$

$$[y_i, x_j, x_i] \cdot y_i + [y_j, x_i, x_i] \cdot y_i,$$

$$[y_j, x_j, x_j] \cdot y_i + [y_j, x_j, x_i] \cdot y_j,$$

$$-2[x_i, x_j, y_j] \cdot y_i + [y_i, x_j, x_j] \cdot y_i + [y_j, x_i, x_i] \cdot y_j,$$

where $1 \leq j < i \leq m$,

$$[y_i, x_j, x_k] \cdot y_k + [y_j, x_i, x_k] \cdot y_k,$$

$$[y_j, x_j, x_i] \cdot y_k + [y_j, x_j, x_k] \cdot y_i,$$

$$[x_k, x_j, y_i] \cdot y_k + [x_k, x_i, y_j] \cdot y_k - [y_k, x_j, x_i] \cdot y_k - [y_j, x_k, x_k] \cdot y_i,$$

$$[x_i, x_j, y_j] \cdot y_k + [x_k, x_j, y_j] \cdot y_i - [y_i, x_j, x_j] \cdot y_k - [y_j, x_i, x_k] \cdot y_j,$$

$$[x_i, x_j, y_i] \cdot y_k + [y_i, x_j, x_k] \cdot y_i + [x_k, x_i, y_j] \cdot y_i + [y_j, x_i, x_i] \cdot y_k,$$

$$-[x_i, x_j, y_i] \cdot y_k + [y_j, x_i, x_k] \cdot y_i - [x_k, x_i, y_j] \cdot y_i + [y_i, x_j, x_i] \cdot y_k,$$

where $1 \leq j < i < k \leq m$,

$$[x_i, x_j, y_k] \cdot y_l + [x_l, x_k, y_j] \cdot y_i + [y_j, x_i, x_k] \cdot y_l + [y_i, x_j, x_l] \cdot y_k,$$

$$\begin{pmatrix} [x_k, x_i, y_j] \cdot y_l + [x_l, x_j, y_i] \cdot y_k - [y_j, x_i, x_k] \cdot y_l - [y_i, x_j, x_l] \cdot y_k \\ -[x_l, x_i, y_j] \cdot y_k - [x_k, x_j, y_i] \cdot y_l \end{pmatrix},$$

$$[y_i, x_j, x_k] \cdot y_l + [y_j, x_i, x_l] \cdot y_k + [y_j, x_i, x_k] \cdot y_l + [y_i, x_j, x_l] \cdot y_k,$$

$$-[x_l, x_i, y_j] \cdot y_k + [y_j, x_k, x_l] \cdot y_i - [x_k, x_j, y_i] \cdot y_l + [y_k, x_j, x_i] \cdot y_l,$$

where $1 \leq j < i < k < l \leq m$.

Proof: Let us define

$A_1 = \{[x_i, x_j, x_k] \cdot x_l \ : \ 1 \le j < i \le l \le m,\ 1 \le j \le k < l \le m\}$,

$A_2 = \{[y_i, y_j, y_k] \cdot y_l \ : \ 1 \le j < i \le l \le m,\ 1 \le j \le k < l \le m\}$,

$A_3 = \{[x_i, x_j, x_k] \cdot y_l \ : \ 1 \le i > j \le k \le m\}$,

$A_4 = \{[x_i, x_j, y_k] \cdot y_l \ : \ 1 \le j < i \le m,\ 1 \le k < l \le m\}$,

$A_5 = \{[y_i, x_j, x_k] \cdot y_l \ : \ 1 \le j \le k \le m,\ 1 \le i \le l \le m\}$,

$A_6 = \{[y_i, x_j, y_k] \cdot y_l \ : \ 1 \le k < l \le m,\ 1 \le i \le l \le m\}$.

Then $K\mathcal{B}_{4m} = KA_1 \oplus KA_2 \oplus KA_3 \oplus KA_4 \oplus KA_5 \oplus KA_6$.

Let $p \in K\mathcal{B}_{4m}$, then it is of the form $p = p_1 + p_2 + p_3 + p_4 + p_5 + p_6$, for some $P_i \in KA_i$, $i = 1, \dots, 6$. If $p \in (K\mathcal{B}_{4m})^{\delta_{2m}}$ and δ_{2m} is linear on element, then

$0 = \delta_{2m}(p) = \delta_{2m}(p_1 + \cdots + p_6) = \delta_{2m}(p_1) + \cdots + \delta_{2m}(p_6)$

Observation:

Clearly, $\delta_{2m}(p_1) = 0$, (which implies that $KA_1 \subseteq (K\mathcal{B}_{4m})^{\delta_{2m}}$).

Since $p_2 \in KA_2$, then p_2 is of the form

$$p_2 = \sum_{\substack{1 \le j < i \le l \le m \\ 1 \le j \le k < l \le m}} \varepsilon_{ijkl} [y_i, y_j, y_k] \cdot y_l \ \text{ for some } \varepsilon_{ijkl} \in K. \text{Then}$$

$$\delta_{2m}(p_2) = \sum_{\substack{1 \le j < i \le l \le m \\ 1 \le j \le k < l \le m}} \varepsilon_{ijkl} \begin{pmatrix} [\delta_{2m}(y_i), y_j, y_k] \cdot y_l + [y_i, \delta_{2m}(y_j), y_k] \cdot y_l \\ +[y_i, y_j, \delta_{2m}(y_k)] \cdot y_l + [y_i, y_j, y_k] \cdot \delta_{2m}(y_l) \end{pmatrix}$$

$$= \sum_{\substack{1 \le j < i \le l \le m \\ 1 \le j \le k < l \le m}} \varepsilon_{ijkl} \begin{pmatrix} [x_i, y_j, y_k] \cdot y_l + [y_i, x_j, y_k] \cdot y_l \\ +[y_i, y_j, x_k] \cdot y_l + [y_i, y_j, y_k] \cdot x_l \end{pmatrix}$$

$$= \sum_{\substack{1 \le j < i \le l \le m \\ 1 \le j \le k < l \le m}} \varepsilon_{ijkl} \begin{pmatrix} -[y_j, x_i, y_k] \cdot y_l + [y_i, x_j, y_k] \cdot y_l - [y_j, x_k, y_i] \cdot y_l \\ +[y_i, x_k, y_j] \cdot y_l - [y_i, x_l, y_j] \cdot y_k + [y_j, x_l, y_i] \cdot y_k \end{pmatrix}$$

Clearly one can see $\delta_{2m}(p_2) \in KA_6$.

For $p_3 \in KA_3$, then p_3 is of the form

$$p_3 = \sum_{1 \le i > j \le k \le m} \varepsilon_{ijkl} \left[x_i, x_j, x_k\right] \cdot y_l \text{ for some } \varepsilon_{ijkl} \in K. \text{Then}$$

$$\begin{aligned}
\delta_{2m}(p_3) &= \sum_{1 \le i > j \le k \le m} \varepsilon_{ijkl} \begin{pmatrix} \left[\delta_{2m}(x_i), x_j, x_k\right] \cdot y_l + \left[x_i, \delta_{2m}(x_j), x_k\right] \cdot y_l \\ + \left[x_i, x_j, \delta_{2m}(x_k)\right] \cdot y_l + \left[x_i, x_j, x_k\right] \cdot \delta_{2m}(y_l) \end{pmatrix} \\
&= \sum_{1 \le i > j \le k \le m} \varepsilon_{ijkl} \left[x_i, x_j, x_k\right] \cdot x_l
\end{aligned}$$

$\Longrightarrow \delta_{2m}(p_3) \in KA_1$.

For $p_4 \in KA_4$, then p_4 is of the form

$$p_4 = \sum_{\substack{1 \le j < i \le m \\ 1 \le k < l \le m}} \varepsilon_{ijkl} \left[x_i, x_j, y_k\right] \cdot y_l \text{ for some } \varepsilon_{ijkl} \in K. \text{Then}$$

$$\begin{aligned}
\delta_{2m}(p_4) &= \sum_{\substack{1 \le j < i \le m \\ 1 \le k < l \le m}} \varepsilon_{ijkl} \begin{pmatrix} \left[\delta_{2m}(x_i), x_j, y_k\right] \cdot y_l + \left[x_i, \delta_{2m}(x_j), y_k\right] \cdot y_l \\ + \left[x_i, x_j, \delta_{2m}(y_k)\right] \cdot y_l + \left[x_i, x_j, y_k\right] \cdot \delta_{2m}(y_l) \end{pmatrix} \\
&= \sum_{\substack{1 \le j < i \le m \\ 1 \le k < l \le m}} \varepsilon_{ijkl} \left(\left[x_i, x_j, x_k\right] \cdot y_l + \left[x_i, x_j, y_k\right] \cdot x_l\right) \\
&= \sum_{\substack{1 \le j < i \le m \\ 1 \le k < l \le m}} \varepsilon_{ijkl} \left(\left[x_i, x_j, x_k\right] \cdot y_l - \left[x_i, x_j, x_l\right] \cdot y_k\right)
\end{aligned}$$

$\Longrightarrow \delta_{2m}(p_4) \in KA_3$.

For $p_5 \in KA_5$, then p_5 is of the form

$$p_5 = \sum_{\substack{1 \le j \le k \le m \\ 1 \le i \le l \le m}} \varepsilon_{ijkl} \left[y_i, x_j, x_k\right] \cdot y_l \text{ for some } \varepsilon_{ijkl} \in K. \text{Then}$$

$$\begin{aligned}
\delta_{2m}(p_5) &= \sum_{\substack{1 \le j \le k \le m \\ 1 \le i \le l \le m}} \varepsilon_{ijkl} \begin{pmatrix} \left[\delta_{2m}(y_i), x_j, x_k\right] \cdot y_l + \left[y_i, \delta_{2m}(x_j), x_k\right] \cdot y_l \\ + \left[y_i, x_j, \delta_{2m}(x_k)\right] \cdot y_l + \left[y_i, x_j, x_k\right] \cdot \delta_{2m}(y_l) \end{pmatrix} \\
&= \sum_{\substack{1 \le j \le k \le m \\ 1 \le i \le l \le m}} \varepsilon_{ijkl} \left(\left[x_i, x_j, x_k\right] \cdot y_l + \left[y_i, x_j, x_k\right] \cdot x_l\right)
\end{aligned}$$

$$= \sum_{\substack{1\le j\le k\le m\\ 1\le i\le l\le m}} \varepsilon_{ijkl} \begin{pmatrix} [x_i, x_j, x_k] \cdot y_l + [x_l, x_j, x_k] \cdot y_i \\ -[x_k, x_j, x_l] \cdot y_i \end{pmatrix}$$

$\Longrightarrow \delta_{2m}(p_5) \in KA_3$.

For $p_6 \in KA_6$, then p_6 is of the form

$$p_6 = \sum_{\substack{1\le k<l\le m\\ 1\le i\le l\le m}} \varepsilon_{ijkl}\, [y_i, x_j, y_k] \cdot y_l \text{ for some } \varepsilon_{ijkl} \in K. \text{Then}$$

$$\delta_{2m}(p_6) = \sum_{\substack{1\le k<l\le m\\ 1\le i\le l\le m}} \varepsilon_{ijkl} \begin{pmatrix} [\delta_{2m}(y_i), x_j, y_k] \cdot y_l + [y_i, \delta_{2m}(x_j), y_k] \cdot y_l \\ +[y_i, x_j, \delta_{2m}(y_k)] \cdot y_l + [y_i, x_j, y_k] \cdot \delta_{2m}(y_l) \end{pmatrix}$$

$$= \sum_{\substack{1\le k<l\le m\\ 1\le i\le l\le m}} \varepsilon_{ijkl} \left([x_i, x_j, y_k] \cdot y_l + [y_i, x_j, x_k] \cdot y_l + [y_i, x_j, y_k] \cdot x_l\right)$$

$$= \sum_{\substack{1\le k<l\le m\\ 1\le i\le l\le m}} \varepsilon_{ijkl} \left([x_i, x_j, y_k] \cdot y_l + [y_i, x_j, x_k] \cdot y_l - [y_i, x_j, x_l] \cdot y_k\right)$$

$\Longrightarrow \delta_{2m}(p_6) \in KA_4 \oplus KA_5$.

Therefore, $0 = \delta_{2m}(p_1) + \delta_{2m}(p_2) + \delta_{2m}(p_3) + \delta_{2m}(p_4 + p_5) + \delta_{2m}(p_6)$. This means that $\delta_{2m}(p) = 0 = \delta_{2m}(p_1) = \delta_{2m}(p_2) = \delta_{2m}(p_3) = \delta_{2m}(p_4 + p_5) = \delta_{2m}(p_6)$.

If $\delta_{2m}(p_1) = 0$, then $KA_1 \subseteq (K\mathcal{B}_{4m})^{\delta_{2m}}$, i.e., elements of the form

$$[x_i, x_j, x_k] \cdot x_l,$$

is a constant where $1 \le j < i \le l \le m$, $1 \le j \le k < l \le m$.

For p_2. Let

$$p_2 = \sum_{\substack{1\le j<i\le l\le m\\ 1\le j\le k<l\le m}} \varepsilon_{ijkl}\, [y_i, y_j, y_k] \cdot y_l$$

$$= \sum_{1\le j<i\le k<l\le m} \varepsilon_{ijkl}\, [y_i, y_j, y_k] \cdot y_l + \sum_{1\le j<k<i<l\le m} \varepsilon_{ijkl}\, [y_i, y_j, y_k] \cdot y_l$$

$$+ \sum_{1\le j<k<i=l\le m} \varepsilon_{ijki}\, [y_i, y_j, y_k] \cdot y_i + \sum_{1\le j<i=k<l\le m} \varepsilon_{ijjl}\, [y_i, y_j, y_j] \cdot y_l$$

$$+\sum_{1\le j=k<i=l\le m} \varepsilon_{ijji}\left[y_i, y_j, y_j\right]\cdot y_i$$

$$=\sum_{1\le j<i\le k<l\le m} \varepsilon_{ijkl}\left[y_i, y_j, y_k\right]\cdot y_l + \sum_{1\le j<i<k<l\le m} \varepsilon_{kjil}\left[y_k, y_j, y_i\right]\cdot y_l$$

$$+\sum_{1\le j<i<k\le m} \varepsilon_{kjik}\left[y_k, y_j, y_i\right]\cdot y_k + \sum_{1\le j<i<l\le m} \varepsilon_{ijjl}\left[y_i, y_j, y_j\right]\cdot y_l$$

$$+\sum_{1\le j<i\le m} \varepsilon_{ijji}\left[y_i, y_j, y_j\right]\cdot y_i$$

$$=\sum_{1\le j<i<k<l\le m} \varepsilon_{ijkl}\left[y_i, y_j, y_k\right]\cdot y_l + \sum_{1\le j<i<k<l\le m} \varepsilon_{kjil}\left[y_k, y_j, y_i\right]\cdot y_l$$

$$+\sum_{1\le j<i<l\le m} \varepsilon_{ijil}\left[y_i, y_j, y_i\right]\cdot y_l + \sum_{1\le j<i<k\le m} \varepsilon_{kjik}\left[y_k, y_j, y_i\right]\cdot y_k$$

$$+\sum_{1\le j<i<l\le m} \varepsilon_{ijjl}\left[y_i, y_j, y_j\right]\cdot y_l + \sum_{1\le j<i\le m} \varepsilon_{ijji}\left[y_i, y_j, y_j\right]\cdot y_i$$

So, $p_2 = p_{21} + p_{22} + p_{23}$ such that

$$p_{21} = \sum_{1\le j<i<k<l\le m} \varepsilon_{ijkl}\left[y_i, y_j, y_k\right]\cdot y_l + \sum_{1\le j<i<k<l\le m} \varepsilon_{kjil}\left[y_k, y_j, y_i\right]\cdot y_l,$$

$$p_{22} = \sum_{1\le j<i<l\le m} \varepsilon_{ijil}\left[y_i, y_j, y_i\right]\cdot y_l + \sum_{1\le j<i<l\le m} \varepsilon_{ljil}\left[y_l, y_j, y_i\right]\cdot y_l$$

$$+\sum_{1\le j<i<l\le m} \varepsilon_{ijjl}\left[y_i, y_j, y_j\right]\cdot y_l,$$

$$p_{23} = \sum_{1\le j<i\le m} \varepsilon_{ijji}\left[y_i, y_j, y_j\right]\cdot y_i.$$

Then

$\delta_{2m}(p_2) = 0 = \delta_{2m}(p_{21} + p_{22} + p_{23}) = \delta_{2m}(p_{21}) + \delta_{2m}(p_{22}) + \delta_{2m}(p_{23})$.

Let V_{21} be the subspace of KA_2 containing monomials with exactly 4 indices, V_{22} be the subspace of KA_2 containing monomials with exactly 3 indices and V_{21} be the subspace of KA_2 containing monomials with exactly 2 indices then $V_{21} \cap$

$V_{22} = \{0\} = V_{21} \cap V_{23} = V_{22} \cap V_{23}$. Then $\delta_{2m}(p_{21}) \in V_{21}$, $\delta_{2m}(p_{22}) \in V_{22}$, $\delta_{2m}(p_{23}) \in V_{23}$

Therefore, $\delta_{2m}(p_{21}) = 0 = \delta_{2m}(p_{22}) = \delta_{2m}(p_{23})$.

$$
\begin{aligned}
0 &= \delta_{2m}(p_{21}) \\
&= \delta_{2m}\left(\sum_{1\le j<i<k<l\le m} \varepsilon_{ijkl}\left[y_i, y_j, y_k\right]\cdot y_l + \sum_{1\le j<i<k<l\le m} \varepsilon_{kjil}\left[y_k, y_j, y_i\right]\cdot y_l\right) \\
&= \delta_{2m}\left(\sum_{1\le j<i<k<l\le m} \varepsilon_{ijkl}\left[y_i, y_j, y_k\right]\cdot y_l\right) \\
&\quad +\delta_{2m}\left(\sum_{1\le j<i<k<l\le m} \varepsilon_{kjil}\left[y_k, y_j, y_i\right]\cdot y_l\right) \\
&= \sum_{1\le j<i<k<l\le m} \varepsilon_{ijkl}\begin{pmatrix} \left[\delta_{2m}(y_i), y_j, y_k\right]\cdot y_l + \left[y_i, \delta_{2m}(y_j), y_k\right]\cdot y_l \\ +\left[y_i, y_j, \delta_{2m}(y_k)\right]\cdot y_l + \left[y_i, y_j, y_k\right]\cdot \delta_{2m}(y_l)\end{pmatrix} \\
&\quad + \sum_{1\le j<i<k<l\le m} \varepsilon_{kjil}\begin{pmatrix} \left[\delta_{2m}(y_k), y_j, y_i\right]\cdot y_l + \left[y_k, \delta_{2m}(y_j), y_i\right]\cdot y_l \\ +\left[y_k, y_j, \delta_{2m}(y_i)\right]\cdot y_l + \left[y_k, y_j, y_i\right]\cdot \delta_{2m}(y_l)\end{pmatrix} \\
&= \sum_{1\le j<i<k<l\le m} \varepsilon_{ijkl}\begin{pmatrix} \left[x_i, y_j, y_k\right]\cdot y_l + \left[y_i, x_j, y_k\right]\cdot y_l + \left[y_i, y_j, x_k\right]\cdot y_l \\ +\left[y_i, y_j, y_k\right]\cdot x_l\end{pmatrix} \\
&\quad + \sum_{1\le j<i<k<l\le m} \varepsilon_{kjil}\begin{pmatrix} \left[x_k, y_j, y_i\right]\cdot y_l + \left[y_k, x_j, y_i\right]\cdot y_l + \left[y_k, y_j, x_i\right]\cdot y_l \\ +\left[y_k, y_j, y_i\right]\cdot x_l\end{pmatrix} \\
&= \sum_{1\le j<i<k<l\le m} \varepsilon_{ijkl}\begin{pmatrix} -\left[y_j, x_i, y_k\right]\cdot y_l + \left[y_i, x_j, y_k\right]\cdot y_l - \left[y_j, x_k, y_i\right]\cdot y_l \\ +\left[y_i, x_k, y_j\right]\cdot y_l - \left[y_i, x_l, y_j\right]\cdot y_k + \left[y_j, x_l, y_i\right]\cdot y_k\end{pmatrix} \\
&\quad + \sum_{1\le j<i<k<l\le m} \varepsilon_{kjil}\begin{pmatrix} -\left[y_j, x_k, y_i\right]\cdot y_l + \left[y_k, x_j, y_i\right]\cdot y_l - \left[y_j, x_i, y_k\right]\cdot y_l \\ +\left[y_k, x_i, y_j\right]\cdot y_l - \left[y_i, x_l, y_j\right]\cdot y_k\end{pmatrix} \\
&= \sum_{1\le j<i<k<l\le m} \left(-\varepsilon_{ijkl} - \varepsilon_{kjil}\right)\left[y_j, x_i, y_k\right]\cdot y_l \\
&\quad + \sum_{1\le j<i<k<l\le m} \varepsilon_{ijkl}\left[y_i, x_j, y_k\right]\cdot y_l
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{1\le j<i<k<l\le m}(-\varepsilon_{ijkl}-\varepsilon_{kjil})\left[y_j,x_k,y_i\right]\cdot y_l\\
&+\sum_{1\le j<i<k<l\le m}\varepsilon_{ijkl}\left[y_i,x_k,y_j\right]\cdot y_l\\
&+\sum_{1\le j<i<k<l\le m}(-\varepsilon_{ijkl}-\varepsilon_{kjil})\left[y_i,x_l,y_j\right]\cdot y_k\\
&+\sum_{1\le j<i<k<l\le m}\varepsilon_{ijkl}\left[y_j,x_l,y_i\right]\cdot y_k\\
&+\sum_{1\le j<i<k<l\le m}\varepsilon_{kjil}\left[y_k,x_j,y_i\right]\cdot y_l+\sum_{1\le j<i<k<l\le m}\varepsilon_{kjil}\left[y_k,x_i,y_j\right]\cdot y_l\\
&=0.
\end{aligned}
$$

Clearly, since there is no coincidence, we have $\varepsilon_{ijkl}=\varepsilon_{kjil}=0,\ 1\le j<i<k<l\le m$. Hence,

$$
p_{21}=\sum_{1\le j<i<k<l\le m}\varepsilon_{ijkl}\left[y_i,y_j,y_k\right]\cdot y_l+\sum_{1\le j<i<k<l\le m}\varepsilon_{kjil}\left[y_k,y_j,y_i\right]\cdot y_l=0
$$

For p_{22}, let

$$
\begin{aligned}
0&=\delta_{2m}(p_{22})\\
&=\delta_{2m}\left(\begin{array}{c}\displaystyle\sum_{1\le j<i<l\le m}\varepsilon_{ijil}\left[y_i,y_j,y_i\right]\cdot y_l+\sum_{1\le j<i<l\le m}\varepsilon_{ljil}\left[y_l,y_j,y_i\right]\cdot y_l\\ \displaystyle+\sum_{1\le j<i<l\le m}\varepsilon_{ijjl}\left[y_i,y_j,y_j\right]\cdot y_l\end{array}\right)\\
&=\delta_{2m}\left(\sum_{1\le j<i<l\le m}\varepsilon_{ijil}\left[y_i,y_j,y_i\right]\cdot y_l\right)+\delta_{2m}\left(\sum_{1\le j<i<l\le m}\varepsilon_{ljil}\left[y_l,y_j,y_i\right]\cdot y_l\right)\\
&\quad+\delta_{2m}\left(\sum_{1\le j<i<l\le m}\varepsilon_{ijjl}\left[y_i,y_j,y_j\right]\cdot y_l\right)\\
&=\sum_{1\le j<i<l\le m}\varepsilon_{ijil}\begin{pmatrix}\left[\delta_{2m}(y_i),y_j,y_i\right]\cdot y_l+\left[y_i,\delta_{2m}(y_j),y_i\right]\cdot y_l\\+\left[y_i,y_j,\delta_{2m}(y_i)\right]\cdot y_l+\left[y_i,y_j,y_i\right]\cdot\delta_{2m}(y_l)\end{pmatrix}
\end{aligned}
$$

$$+\sum_{1\le j<i<l\le m}\varepsilon_{ljil}\begin{pmatrix}[\delta_{2m}(y_l),y_j,y_i]\cdot y_l+[y_l,\delta_{2m}(y_j),y_i]\cdot y_l\\+[y_l,y_j,\delta_{2m}(y_i)]\cdot y_l+[y_l,y_j,y_i]\cdot\delta_{2m}(y_l)\end{pmatrix}$$

$$+\sum_{1\le j<i<l\le m}\varepsilon_{ijjl}\begin{pmatrix}[\delta_{2m}(y_i),y_j,y_j]\cdot y_l+[y_i,\delta_{2m}(y_j),y_j]\cdot y_l\\+[y_i,y_j,\delta_{2m}(y_j)]\cdot y_l+[y_i,y_j,y_j]\cdot\delta_{2m}(y_l)\end{pmatrix}$$

$$=\sum_{1\le j<i<l\le m}\varepsilon_{ijil}\begin{pmatrix}[x_i,y_j,y_i]\cdot y_l+[y_i,x_j,y_i]\cdot y_l+[y_i,y_j,x_i]\cdot y_l\\+[y_i,y_j,y_i]\cdot x_l\end{pmatrix}$$

$$+\sum_{1\le j<i<l\le m}\varepsilon_{ljil}\begin{pmatrix}[x_l,y_j,y_i]\cdot y_l+[y_l,x_j,y_i]\cdot y_l+[y_l,y_j,x_i]\cdot y_l\\+[y_l,y_j,y_i]\cdot x_l\end{pmatrix}$$

$$+\sum_{1\le j<i<l\le m}\varepsilon_{ijjl}\begin{pmatrix}[x_i,y_j,y_j]\cdot y_l+[y_i,x_j,y_j]\cdot y_l+[y_i,y_j,x_j]\cdot y_l\\+[y_i,y_j,y_j]\cdot x_l\end{pmatrix}$$

$$=\sum_{1\le j<i<l\le m}\varepsilon_{ijil}\begin{pmatrix}-2[y_j,x_i,y_i]\cdot y_l+[y_i,x_j,y_i]\cdot y_l+[y_i,x_i,y_j]\cdot y_l\\-[y_i,x_l,y_j]\cdot y_i\end{pmatrix}$$

$$+\sum_{1\le j<i<l\le m}\varepsilon_{ljil}\begin{pmatrix}-[y_j,x_l,y_i]\cdot y_l+[y_l,x_j,y_i]\cdot y_l+[y_l,x_i,y_j]\cdot y_l\\-[y_i,x_l,y_j]\cdot y_l\end{pmatrix}$$

$$+\sum_{1\le j<i<l\le m}\varepsilon_{ijjl}\begin{pmatrix}-[y_j,x_i,y_j]\cdot y_l+2[y_i,x_j,y_j]\cdot y_l-[y_j,x_j,y_i]\cdot y_l\\-[y_j,x_l,y_j]\cdot y_i\end{pmatrix}$$

This gives that

$[y_i,x_j,y_i]\cdot y_l$ with $1\le j<i<l\le m$ does not appear in another sum; i.e., $\varepsilon_{ijil}=0$. $[y_l,x_j,y_i]\cdot y_l$ with $1\le j<i<l\le m$ does not appear in another sum; i.e., $\varepsilon_{ljil}=0$. $[y_j,x_i,y_j]\cdot y_l$ with $1\le j<i<l\le m$ does not appear in another sum; i.e., $\varepsilon_{ijjl}=0$. Hence if $\varepsilon_{ijil}=\varepsilon_{ljil}=\varepsilon_{ijil}=0$ for $1\le j<i<l\le m$, then

$$p_{22}=\sum_{1\le j<i<l\le m}\varepsilon_{ijil}\,[y_i,y_j,y_i]\cdot y_l+\sum_{1\le j<i<l\le m}\varepsilon_{ljil}\,[y_l,y_j,y_i]\cdot y_l$$
$$+\sum_{1\le j<i<l\le m}\varepsilon_{ijil}\,[y_i,y_j,y_i]\cdot y_l$$
$$=0$$

For p_{23}, let

$$
\begin{aligned}
0 = \delta_{2m}(p_{23}) &= \delta_{2m}\left(\sum_{1\le j<i\le m} \varepsilon_{ijji}\,[y_i, y_j, y_j]\cdot y_i\right)\\
&= \sum_{1\le j<i\le m} \varepsilon_{ijji}\begin{pmatrix} [\delta_{2m}(y_i), y_j, y_j]\cdot y_i + [y_i, \delta_{2m}(y_j), y_j]\cdot y_i \\ +[y_i, y_j, \delta_{2m}(y_j)]\cdot y_i + [y_i, y_j, y_j]\cdot \delta_{2m}(y_i)\end{pmatrix}\\
&= \sum_{1\le j<i\le m} \varepsilon_{ijji}\begin{pmatrix} [x_i, y_j, y_j]\cdot y_i + [y_i, x_j, y_j]\cdot y_i + [y_i, y_j, x_j]\cdot y_i \\ +[y_i, y_j, y_j]\cdot x_i\end{pmatrix}\\
&= \sum_{1\le j<i\le m} \varepsilon_{ijji}\left(-2[y_j, x_i, y_j]\cdot y_i + 2[y_i, x_j, y_j]\cdot y_i\right)\\
&= \sum_{1\le j<i\le m} 2\varepsilon_{ijji}\left([y_i, x_j, y_j]\cdot y_i - [y_j, x_i, y_j]\cdot y_i\right)\\
&= \sum_{1\le j<i\le m} 2\varepsilon_{ijji}\,[y_i, x_j, y_j]\cdot y_i - \sum_{1\le j<i\le m} 2\varepsilon_{ijji}\,[y_j, x_i, y_j]\cdot y_i
\end{aligned}
$$

Since $[y_i, x_j, y_j]\cdot y_i$ and $[y_j, x_i, y_j]\cdot y_i$ does not coincide then

$$\sum_{1\le j<i\le m} 2\varepsilon_{ijji}\,[y_i, x_j, y_j]\cdot y_i = \sum_{1\le j<i\le m} 2\varepsilon_{ijji}\,[y_j, x_i, y_j]\cdot y_i = 0$$

So $\varepsilon_{ijji} = 0$, $1 \le j < i \le m$. This implies that

$$p_{23} = \sum_{1\le j<i\le m} \varepsilon_{ijji}\,[y_i, y_j, y_j]\cdot y_i = 0.$$

Therefore, since $p_2 = p_{21} + p_{22} + p_{23}$ and $p_{21} = p_{22} = p_{23} = 0$ we can conclude that

$$p_2 = \sum_{\substack{1\le j<i\le l\le m \\ 1\le j\le k<l\le m}} \varepsilon_{ijkl}\,[y_i, y_j, y_k]\cdot y_l = 0$$

This means that $(KA_2)^{\delta_{2m}} = \{0\}$.

For

$$p_3 = \sum_{1\le i>j\le k\le m} \varepsilon_{ijkl}\,[x_i, x_j, x_k]\cdot y_l$$

We have the condition that $1 \leq i > j \leq k \leq m$ here $\boldsymbol{l}$ can be anywhere in this position

$$\boldsymbol{j} \qquad \boldsymbol{i=k}$$

This gives 5 conditions which are $j < i = k < l, j < i = k = l, j < l < i = k$, $j = l < i = k$ and $l < j < i = k$.

For the position $\boldsymbol{j} \quad \boldsymbol{i} \quad \boldsymbol{k}$ we have 7 different positions which are

$j < i < k < l, j < i < k = l, j < i < l < k, j < i = l < k, j < l < i < k$, $j = l < i < k$ and $l < j < i < k$

For the position $\boldsymbol{j} \quad \boldsymbol{k} \quad \boldsymbol{i}$ we obtain 7 different positions which are

$j < k < i < l, j < k < i = l, j < k < l < i, j < k = l < i, j < l < k < i$, $j = l < k < i$ and $l < j < k < i$.

For last the position we have $\boldsymbol{j=k} \quad \boldsymbol{i}$ with 5 different position which are

$j = k < i < l, j = k < i = l, j = k < l < i$, $j = k = l < i$ and $l < j = k < i$.

Now p_3 become

$$\begin{aligned}
p_3 = &\sum_{1 \leq j < i = k < l \leq m} \varepsilon_{ijil} \left[x_i, x_j, x_i\right] \cdot y_l + \sum_{1 \leq j < i = k = l \leq m} \varepsilon_{ijii} \left[x_i, x_j, x_i\right] \cdot y_i \\
&+ \sum_{1 \leq j < l < i = k \leq m} \varepsilon_{ijik} \left[x_i, x_j, x_i\right] \cdot y_l + \sum_{1 \leq j = l < i = k \leq m} \varepsilon_{ijij} \left[x_i, x_j, x_i\right] \cdot y_j \\
&+ \sum_{1 \leq l < j < i = k \leq m} \varepsilon_{ijil} \left[x_i, x_j, x_i\right] \cdot y_l + \sum_{1 \leq j < i < k < l \leq m} \varepsilon_{ijkl} \left[x_i, x_j, x_k\right] \cdot y_l \\
&+ \sum_{1 \leq j < i < k = l \leq m} \varepsilon_{ijkk} \left[x_i, x_j, x_k\right] \cdot y_k + \sum_{1 \leq j < i < l < k \leq m} \varepsilon_{ijkl} \left[x_i, x_j, x_k\right] \cdot y_l \\
&+ \sum_{1 \leq j < i = l < k \leq m} \varepsilon_{ijki} \left[x_i, x_j, x_k\right] \cdot y_i + \sum_{1 \leq j < l < i < k \leq m} \varepsilon_{ijkl} \left[x_i, x_j, x_k\right] \cdot y_l \\
&+ \sum_{1 \leq j = l < i < k \leq m} \varepsilon_{ijkj} \left[x_i, x_j, x_k\right] \cdot y_j + \sum_{1 \leq j < i < k < l \leq m} \varepsilon_{ijkl} \left[x_i, x_j, x_k\right] \cdot y_l
\end{aligned}$$

$$+\sum_{1\le j<k<i<l\le m}\varepsilon_{ijkl}\left[x_i,x_j,x_k\right]\cdot y_l+\sum_{1\le j<k<i=l\le m}\varepsilon_{ijki}\left[x_i,x_j,x_k\right]\cdot y_i$$

$$+\sum_{1\le j<k<l<i\le m}\varepsilon_{ijkl}\left[x_i,x_j,x_k\right]\cdot y_l+\sum_{1\le j<k=l<i\le m}\varepsilon_{ijkk}\left[x_i,x_j,x_k\right]\cdot y_k$$

$$+\sum_{1\le j<l<k<i\le m}\varepsilon_{ijkl}\left[x_i,x_j,x_k\right]\cdot y_l+\sum_{1\le j=l<k<i\le m}\varepsilon_{ijkj}\left[x_i,x_j,x_k\right]\cdot y_j$$

$$+\sum_{1\le l<j<k<i\le m}\varepsilon_{ijkl}\left[x_i,x_j,x_k\right]\cdot y_l+\sum_{1\le j=k<i<l\le m}\varepsilon_{ijkl}\left[x_i,x_j,x_j\right]\cdot y_l$$

$$+\sum_{1\le j=k<i=l\le m}\varepsilon_{ijkl}\left[x_i,x_j,x_j\right]\cdot y_i+\sum_{1\le j=k<l<i\le m}\varepsilon_{ijkl}\left[x_i,x_j,x_j\right]\cdot y_l$$

$$+\sum_{1\le j=k=l<i\le m}\varepsilon_{ijkl}\left[x_i,x_j,x_j\right]\cdot y_j+\sum_{1\le l<j=k<i\le m}\varepsilon_{ijkl}\left[x_i,x_j,x_j\right]\cdot y_l$$

$$=\sum_{1\le j<i<l\le m}\varepsilon_{ijil}\left[x_i,x_j,x_i\right]\cdot y_l+\sum_{1\le j<i\le m}\varepsilon_{ijii}\left[x_i,x_j,x_i\right]\cdot y_i$$

$$+\sum_{1\le j<i<l\le m}\varepsilon_{ljli}\left[x_l,x_j,x_l\right]\cdot y_i+\sum_{1\le j<i\le m}\varepsilon_{ijij}\left[x_i,x_j,x_i\right]\cdot y_j$$

$$+\sum_{1\le j<i<l\le m}\varepsilon_{lilj}\left[x_l,x_i,x_l\right]\cdot y_j+\sum_{1\le j<i<k<l\le m}\varepsilon_{ijkl}\left[x_i,x_j,x_k\right]\cdot y_l$$

$$+\sum_{1\le j<i<k\le m}\varepsilon_{ijkk}\left[x_i,x_j,x_k\right]\cdot y_k+\sum_{1\le j<i<k<l\le m}\varepsilon_{ijlk}\left[x_i,x_j,x_l\right]\cdot y_k$$

$$+\sum_{1\le j<i<k\le m}\varepsilon_{ijki}\left[x_i,x_j,x_k\right]\cdot y_i+\sum_{1\le j<i<k<l\le m}\varepsilon_{kjli}\left[x_k,x_j,x_l\right]\cdot y_i$$

$$+\sum_{1\le j<i<k\le m}\varepsilon_{ijkj}\left[x_i,x_j,x_k\right]\cdot y_j+\sum_{1\le j<i<k<l\le m}\varepsilon_{kilj}\left[x_k,x_i,x_l\right]\cdot y_j$$

$$+\sum_{1\le j<i<k<l\le m}\varepsilon_{kjil}\left[x_k,x_j,x_i\right]\cdot y_l+\sum_{1\le j<i<k\le m}\varepsilon_{kjik}\left[x_k,x_j,x_i\right]\cdot y_k$$

$$+\sum_{1\le j<i<k<l\le m}\varepsilon_{ljik}\left[x_l,x_j,x_i\right]\cdot y_k+\sum_{1\le j<i<k\le m}\varepsilon_{kjii}\left[x_k,x_j,x_i\right]\cdot y_i$$

$$+\sum_{1\le j<i<k<l\le m}\varepsilon_{ljki}\left[x_l,x_j,x_k\right]\cdot y_i+\sum_{1\le j<i<k\le m}\varepsilon_{kjij}\left[x_k,x_j,x_i\right]\cdot y_j$$

$$+\sum_{1\le j<i<k<l\le m}\varepsilon_{likj}\left[x_l,x_i,x_k\right]\cdot y_j+\sum_{1\le j<i<l\le m}\varepsilon_{ijjl}\left[x_i,x_j,x_j\right]\cdot y_l$$

$$+\sum_{1\le j<i\le m}\varepsilon_{ijji}\left[x_i,x_j,x_j\right]\cdot y_i+\sum_{1\le j<i<l\le m}\varepsilon_{ljji}\left[x_l,x_j,x_j\right]\cdot y_i$$

$$+\sum_{1\le j<i\le m}\varepsilon_{ijjj}\left[x_i,x_j,x_j\right]\cdot y_j+\sum_{1\le j<i<l\le m}\varepsilon_{liij}\left[x_l,x_i,x_i\right]\cdot y_j$$

$$=\sum_{1\le j<i\le m}\begin{pmatrix}\varepsilon_{ijii}\left[x_i,x_j,x_i\right]\cdot y_i+\varepsilon_{ijij}\left[x_i,x_j,x_i\right]\cdot y_j+\varepsilon_{ijji}\left[x_i,x_j,x_j\right]\cdot y_i\\ +\varepsilon_{ijjj}\left[x_i,x_j,x_j\right]\cdot y_j\end{pmatrix}$$

$$+\sum_{1\le j<i<k\le m}\begin{pmatrix}\varepsilon_{ijik}\left[x_i,x_j,x_i\right]\cdot y_k+\varepsilon_{kjki}\left[x_k,x_j,x_k\right]\cdot y_i\\ +\varepsilon_{kikj}\left[x_k,x_i,x_k\right]\cdot y_j+\varepsilon_{ijkk}\left[x_i,x_j,x_k\right]\cdot y_k\\ +\varepsilon_{ijki}\left[x_i,x_j,x_k\right]\cdot y_i+\varepsilon_{ijkj}\left[x_i,x_j,x_k\right]\cdot y_j\\ +\varepsilon_{kjik}\left[x_k,x_j,x_i\right]\cdot y_k+\varepsilon_{kjii}\left[x_k,x_j,x_i\right]\cdot y_i\\ +\varepsilon_{kjij}\left[x_k,x_j,x_i\right]\cdot y_j+\varepsilon_{ijjk}\left[x_i,x_j,x_j\right]\cdot y_k\\ +\varepsilon_{kjji}\left[x_k,x_j,x_j\right]\cdot y_i+\varepsilon_{kiij}\left[x_k,x_i,x_i\right]\cdot y_j\end{pmatrix}$$

$$+\sum_{1\le j<i<k<l\le m}\begin{pmatrix}\varepsilon_{ijkl}\left[x_i,x_j,x_k\right]\cdot y_l+\varepsilon_{ijlk}\left[x_i,x_j,x_l\right]\cdot y_k\\ +\varepsilon_{kjli}\left[x_k,x_j,x_l\right]\cdot y_i+\varepsilon_{kilj}\left[x_k,x_i,x_l\right]\cdot y_j\\ +\varepsilon_{kjil}\left[x_k,x_j,x_i\right]\cdot y_l+\varepsilon_{ljik}\left[x_l,x_j,x_i\right]\cdot y_k\\ +\varepsilon_{ljki}\left[x_l,x_j,x_k\right]\cdot y_i+\varepsilon_{likj}\left[x_l,x_i,x_k\right]\cdot y_j\end{pmatrix}$$

So, $p_3=(p_3)_2+(p_3)_3+(p_3)_4$ such that

$$(p_3)_2=\sum_{1\le j<i\le m}\begin{pmatrix}\varepsilon_{ijii}\left[x_i,x_j,x_i\right]\cdot y_i+\varepsilon_{ijij}\left[x_i,x_j,x_i\right]\cdot y_j+\varepsilon_{ijji}\left[x_i,x_j,x_j\right]\cdot y_i\\ +\varepsilon_{ijjj}\left[x_i,x_j,x_j\right]\cdot y_j\end{pmatrix},$$

$$(p_3)_3=\sum_{1\le j<i<k\le m}\begin{pmatrix}\varepsilon_{ijik}\left[x_i,x_j,x_i\right]\cdot y_k+\varepsilon_{kjki}\left[x_k,x_j,x_k\right]\cdot y_i\\ +\varepsilon_{kikj}\left[x_k,x_i,x_k\right]\cdot y_j+\varepsilon_{ijkk}\left[x_i,x_j,x_k\right]\cdot y_k\\ +\varepsilon_{ijki}\left[x_i,x_j,x_k\right]\cdot y_i+\varepsilon_{ijkj}\left[x_i,x_j,x_k\right]\cdot y_j\\ +\varepsilon_{kjik}\left[x_k,x_j,x_i\right]\cdot y_k+\varepsilon_{kjii}\left[x_k,x_j,x_i\right]\cdot y_i\\ +\varepsilon_{kjij}\left[x_k,x_j,x_i\right]\cdot y_j+\varepsilon_{ijjk}\left[x_i,x_j,x_j\right]\cdot y_k\\ +\varepsilon_{kjji}\left[x_k,x_j,x_j\right]\cdot y_i+\varepsilon_{kiij}\left[x_k,x_i,x_i\right]\cdot y_j\end{pmatrix},$$

$$(p_3)_4 = \sum_{1\le j<i<k<l\le m} \begin{pmatrix} \varepsilon_{ijkl}[x_i, x_j, x_k]\cdot y_l + \varepsilon_{ijlk}[x_i, x_j, x_l]\cdot y_k \\ +\varepsilon_{kjli}[x_k, x_j, x_l]\cdot y_i + \varepsilon_{kilj}[x_k, x_i, x_l]\cdot y_j \\ +\varepsilon_{kjil}[x_k, x_j, x_i]\cdot y_l + \varepsilon_{ljik}[x_l, x_j, x_i]\cdot y_k \\ +\varepsilon_{ljki}[x_l, x_j, x_k]\cdot y_i + \varepsilon_{likj}[x_l, x_i, x_k]\cdot y_j \end{pmatrix}.$$

Then

$\delta_{2m}(p_3) = 0 = \delta_{2m}((p_3)_2 + (p_3)_3 + (p_3)_4) = \delta_{2m}((p_3)_2) + \delta_{2m}((p_3)_3) + \delta_{2m}((p_3)_4)$. Let V_{32} be the subspace of KA_3 containing monomials with exactly 2 indices, V_{33} be the subspace of KA_3 containing monomials with exactly 3 indices and V_{34} be the subspace of KA_3 containing monomials with exactly 4 indices then $V_{32}\cap V_{33} = \{0\} = V_{32}\cap V_{34} = V_{33}\cap V_{34}$. Then $\delta_{2m}((p_3)_2)\in V_{32}$, $\delta_{2m}((p_3)_3)\in V_{33}$, $\delta_{2m}((p_3)_4)\in V_{34}$

Therefore, $\delta_{2m}((p_3)_2) = 0 = \delta_{2m}((p_3)_3) = \delta_{2m}((p_3)_4)$.

for

$$\begin{aligned} 0 &= \delta_{2m}((p_3)_2) \\ &= \delta_{2m}\left(\sum_{1\le j<i\le m}\begin{pmatrix} \varepsilon_{ijii}[x_i, x_j, x_i]\cdot y_i + \varepsilon_{ijij}[x_i, x_j, x_i]\cdot y_j \\ +\varepsilon_{ijji}[x_i, x_j, x_j]\cdot y_i + \varepsilon_{ijjj}[x_i, x_j, x_j]\cdot y_j \end{pmatrix}\right) \\ &= \sum_{1\le j<i\le m}\begin{pmatrix} \varepsilon_{ijii}[x_i, x_j, x_i]\cdot \delta_{2m}(y_i) + \varepsilon_{ijij}[x_i, x_j, x_i]\cdot \delta_{2m}(y_j) \\ +\varepsilon_{ijji}[x_i, x_j, x_j]\cdot \delta_{2m}(y_i) + \varepsilon_{ijjj}[x_i, x_j, x_j]\cdot \delta_{2m}(y_j) \end{pmatrix} \\ &= \sum_{1\le j<i\le m}\begin{pmatrix} \varepsilon_{ijii}[x_i, x_j, x_i]\cdot x_i + \varepsilon_{ijij}[x_i, x_j, x_i]\cdot x_j + \varepsilon_{ijji}[x_i, x_j, x_j]\cdot x_i \\ +\varepsilon_{ijjj}[x_i, x_j, x_j]\cdot x_j \end{pmatrix} \end{aligned}$$

We rearrange the basis by using the condition in Remark 2.10.3, we get that

$$\begin{aligned} 0 &= \sum_{1\le j<i\le m}\begin{pmatrix} \varepsilon_{ijii}[x_i, x_j, x_i]\cdot x_i - \varepsilon_{ijij}[x_i, x_j, x_j]\cdot x_i + \varepsilon_{ijji}[x_i, x_j, x_j]\cdot x_i \\ +\varepsilon_{ijjj}[x_i, x_j, x_j]\cdot x_j \end{pmatrix} \\ &= \sum_{1\le j<i\le m}\begin{pmatrix} \varepsilon_{ijii}[x_i, x_j, x_i]\cdot x_i + (\varepsilon_{ijji} - \varepsilon_{ijij})[x_i, x_j, x_j]\cdot x_i \\ +\varepsilon_{ijjj}[x_i, x_j, x_j]\cdot x_j \end{pmatrix}. \end{aligned}$$

Since $[x_i, x_j, x_i] \cdot x_i = 0$ and $[x_i, x_j, x_j] \cdot x_j = 0$, by the condition in Remark 2.10.3, then this mean ε_{ijii} and ε_{ijjj} can be any. Since $[x_i, x_j, x_j] \cdot x_i$ is a basis element in KA_3, then it cannot be zero hence, $\varepsilon_{ijji} - \varepsilon_{ijij} = 0$. So, $\varepsilon_{ijji} = \varepsilon_{ijij}$.

Therefore,

$$(p_3)_2 = \sum_{1 \le j < i \le m} \begin{pmatrix} \varepsilon_{ijii}[x_i, x_j, x_i] \cdot y_i + \varepsilon_{ijij}[x_i, x_j, x_i] \cdot y_j + \varepsilon_{ijij}[x_i, x_j, x_j] \cdot y_i \\ +\varepsilon_{ijjj}[x_i, x_j, x_j] \cdot y_j \end{pmatrix}$$

$$= \sum_{1 \le j < i \le m} \begin{pmatrix} \varepsilon_{ijii}[x_i, x_j, x_i] \cdot y_i + \varepsilon_{ijij}\left([x_i, x_j, x_i] \cdot y_j + [x_i, x_j, x_j] \cdot y_i\right) \\ +\varepsilon_{ijjj}[x_i, x_j, x_j] \cdot y_j \end{pmatrix}.$$

Therefore,

$$[x_i, x_j, x_j] \cdot y_j,$$

$$[x_i, x_j, x_i] \cdot y_i,$$

$$[x_i, x_j, x_i] \cdot y_j + [x_i, x_j, x_j] \cdot y_i,$$

are constant elements in KA_3 where $1 \le j < i \le m$.

For

$0 = \delta_{2m}((p_3)_3)$

$$= \delta_{2m}\left(\sum_{1 \le j < i < k \le m} \begin{pmatrix} \varepsilon_{ijik}[x_i, x_j, x_i] \cdot y_k + \varepsilon_{kjki}[x_k, x_j, x_k] \cdot y_i \\ +\varepsilon_{kikj}[x_k, x_i, x_k] \cdot y_j + \varepsilon_{ijkk}[x_i, x_j, x_k] \cdot y_k \\ +\varepsilon_{ijki}[x_i, x_j, x_k] \cdot y_i + \varepsilon_{ijkj}[x_i, x_j, x_k] \cdot y_j \\ +\varepsilon_{kjik}[x_k, x_j, x_i] \cdot y_k + \varepsilon_{kjii}[x_k, x_j, x_i] \cdot y_i \\ +\varepsilon_{kjij}[x_k, x_j, x_i] \cdot y_j + \varepsilon_{ijjk}[x_i, x_j, x_j] \cdot y_k \\ +\varepsilon_{kjji}[x_k, x_j, x_j] \cdot y_i + \varepsilon_{kiij}[x_k, x_i, x_i] \cdot y_j \end{pmatrix} \right)$$

$$= \sum_{1 \le j < i < k \le m} \begin{pmatrix} \varepsilon_{ijik}[x_i, x_j, x_i] \cdot \delta_{2m}(y_k) + \varepsilon_{kjki}[x_k, x_j, x_k] \cdot \delta_{2m}(y_i) \\ +\varepsilon_{kikj}[x_k, x_i, x_k] \cdot \delta_{2m}(y_j) + \varepsilon_{ijkk}[x_i, x_j, x_k] \cdot \delta_{2m}(y_k) \\ +\varepsilon_{ijki}[x_i, x_j, x_k] \cdot \delta_{2m}(y_i) + \varepsilon_{ijkj}[x_i, x_j, x_k] \cdot \delta_{2m}(y_j) \\ +\varepsilon_{kjik}[x_k, x_j, x_i] \cdot \delta_{2m}(y_k) + \varepsilon_{kjii}[x_k, x_j, x_i] \cdot \delta_{2m}(y_i) \\ +\varepsilon_{kjij}[x_k, x_j, x_i] \cdot \delta_{2m}(y_j) + \varepsilon_{ijjk}[x_i, x_j, x_j] \cdot \delta_{2m}(y_k) \\ +\varepsilon_{kjji}[x_k, x_j, x_j] \cdot \delta_{2m}(y_i) + \varepsilon_{kiij}[x_k, x_i, x_i] \cdot \delta_{2m}(y_j) \end{pmatrix}$$

$$= \sum_{1\le j<i<k\le m} \begin{pmatrix} \varepsilon_{ijik}[x_i,x_j,x_i]\cdot x_k + \varepsilon_{kjki}[x_k,x_j,x_k]\cdot x_i \\ +\varepsilon_{kikj}[x_k,x_i,x_k]\cdot x_j + \varepsilon_{ijkk}[x_i,x_j,x_k]\cdot x_k \\ +\varepsilon_{ijki}[x_i,x_j,x_k]\cdot x_i + \varepsilon_{ijkj}[x_i,x_j,x_k]\cdot x_j \\ +\varepsilon_{kjik}[x_k,x_j,x_i]\cdot x_k + \varepsilon_{kjii}[x_k,x_j,x_i]\cdot x_i \\ +\varepsilon_{kjij}[x_k,x_j,x_i]\cdot x_j + \varepsilon_{ijjk}[x_i,x_j,x_j]\cdot x_k \\ +\varepsilon_{kjji}[x_k,x_j,x_j]\cdot x_i + \varepsilon_{kiij}[x_k,x_i,x_i]\cdot x_j \end{pmatrix}$$

By using the condition in Remark 2.10.3, we rearrange the basis to obtain that

$$0 = \sum_{1\le j<i<k\le m} \begin{pmatrix} \varepsilon_{ijik}[x_i,x_j,x_i]\cdot x_k - \varepsilon_{kjki}[x_k,x_j,x_i]\cdot x_k \\ -\varepsilon_{kikj}[x_k,x_j,x_i]\cdot x_k + \varepsilon_{ijkk}[x_i,x_j,x_k]\cdot x_k \\ -\varepsilon_{ijki}[x_i,x_j,x_i]\cdot x_k - \varepsilon_{ijkj}[x_i,x_j,x_j]\cdot x_k \\ +\varepsilon_{kjik}[x_k,x_j,x_i]\cdot x_k + \varepsilon_{kjii}[x_k,x_j,x_i]\cdot x_i \\ -\varepsilon_{kjij}[x_i,x_j,x_j]\cdot x_k + \varepsilon_{ijjk}[x_i,x_j,x_j]\cdot x_k \\ +\varepsilon_{kjji}[x_i,x_j,x_j]\cdot x_k - \varepsilon_{kiij}[x_i,x_j,x_i]\cdot x_k \end{pmatrix}$$

$$= \sum_{1\le j<i<k\le m} \begin{pmatrix} (\varepsilon_{ijik} - \varepsilon_{ijki} - \varepsilon_{kiij})[x_i,x_j,x_i]\cdot x_k \\ +(-\varepsilon_{kjki} - \varepsilon_{kikj} + \varepsilon_{kjik})[x_k,x_j,x_i]\cdot x_k \\ +\varepsilon_{ijkk}[x_i,x_j,x_k]\cdot x_k \\ +(-\varepsilon_{ijkj} - \varepsilon_{kjij} + \varepsilon_{ijjk} + \varepsilon_{kjji})[x_i,x_j,x_j]\cdot x_k \\ +\varepsilon_{kjii}[x_k,x_j,x_i]\cdot x_i \end{pmatrix}$$

Since $[x_i,x_j,x_k]\cdot x_k = 0$ and $[x_k,x_j,x_i]\cdot x_i = 0$, from the condition in Remark 2.10.3, then this mean ε_{ijkk} and ε_{kjii} can be any. Since $[x_i,x_j,x_i]\cdot x_k$, $[x_k,x_j,x_i]\cdot x_k$ and $[x_i,x_j,x_j]\cdot x_k$ are the basis element in KA_3, then it cannot be zero hence, $\varepsilon_{ijik} - \varepsilon_{ijki} - \varepsilon_{kiij} = 0$, $-\varepsilon_{kjki} - \varepsilon_{kikj} + \varepsilon_{kjik} = 0$ and $-\varepsilon_{ijkj} - \varepsilon_{kjij} + \varepsilon_{ijjk} + \varepsilon_{kjji} = 0$. So,

$$\varepsilon_{ijik} = \varepsilon_{ijki} + \varepsilon_{kiij}$$

$$\varepsilon_{kjki} = -\varepsilon_{kikj} + \varepsilon_{kjik}$$

$$\varepsilon_{ijkj} = -\varepsilon_{kjij} + \varepsilon_{ijjk} + \varepsilon_{kjji}$$

Therefore,

$$(p_3)_3 = \sum_{1\le j<i<k\le m} \left(\begin{array}{c} (\varepsilon_{ijki} + \varepsilon_{kiij})[x_i, x_j, x_i] \cdot y_k \\ +(-\varepsilon_{kikj} + \varepsilon_{kjik})[x_k, x_j, x_k] \cdot y_i \\ +\varepsilon_{kikj}[x_k, x_i, x_k] \cdot y_j + \varepsilon_{ijkk}[x_i, x_j, x_k] \cdot y_k \\ +\varepsilon_{ijki}[x_i, x_j, x_k] \cdot y_i \\ +(-\varepsilon_{kjij} + \varepsilon_{ijjk} + \varepsilon_{kjji})[x_i, x_j, x_k] \cdot y_j \\ +\varepsilon_{kjik}[x_k, x_j, x_i] \cdot y_k + \varepsilon_{kjii}[x_k, x_j, x_i] \cdot y_i \\ +\varepsilon_{kjij}[x_k, x_j, x_i] \cdot y_j + \varepsilon_{ijjk}[x_i, x_j, x_j] \cdot y_k \\ +\varepsilon_{kjji}[x_k, x_j, x_j] \cdot y_i + \varepsilon_{kiij}[x_k, x_i, x_i] \cdot y_j \end{array} \right)$$

$$= \sum_{1\le j<i<k\le m} \left(\begin{array}{c} \varepsilon_{ijki}([x_i, x_j, x_i] \cdot y_k + [x_i, x_j, x_k] \cdot y_i) \\ +\varepsilon_{kiij}([x_i, x_j, x_i] \cdot y_k + [x_k, x_i, x_i] \cdot y_j) \\ +\varepsilon_{kikj}(-[x_k, x_j, x_k] \cdot y_i + [x_k, x_i, x_k] \cdot y_j) \\ +\varepsilon_{kjik}([x_k, x_j, x_k] \cdot y_i + [x_k, x_j, x_i] \cdot y_k) \\ +\varepsilon_{ijkk}[x_i, x_j, x_k] \cdot y_k \\ +\varepsilon_{kjij}(-[x_i, x_j, x_k] \cdot y_j + [x_k, x_j, x_i] \cdot y_j) \\ +\varepsilon_{ijjk}([x_i, x_j, x_k] \cdot y_j + [x_i, x_j, x_j] \cdot y_k) \\ +\varepsilon_{kjji}([x_i, x_j, x_k] \cdot y_j + [x_k, x_j, x_j] \cdot y_i) \\ +\varepsilon_{kjii}[x_k, x_j, x_i] \cdot y_i \end{array} \right)$$

Hence,

$$[x_i, x_j, x_k] \cdot y_k,$$

$$[x_k, x_j, x_i] \cdot y_i,$$

$$[x_i, x_j, x_i] \cdot y_k + [x_i, x_j, x_k] \cdot y_i,$$

$$[x_i, x_j, x_i] \cdot y_k + [x_k, x_i, x_i] \cdot y_j,$$

$$-[x_k, x_j, x_k] \cdot y_i + [x_k, x_i, x_k] \cdot y_j,$$

$$[x_k, x_j, x_k] \cdot y_i + [x_k, x_j, x_i] \cdot y_k,$$

$$-[x_i, x_j, x_k] \cdot y_j + [x_k, x_j, x_i] \cdot y_j,$$

$$[x_i, x_j, x_k] \cdot y_j + [x_i, x_j, x_j] \cdot y_k,$$

$$[x_i, x_j, x_k] \cdot y_j + [x_k, x_j, x_j] \cdot y_i,$$

are constant elements in KA_3 where $1 \le j < i < k \le m$.

For

$$0 = \delta_{2m}((p_3)_4)$$

$$= \delta_{2m}\left(\sum_{1\le j<i<k<l\le m}\begin{pmatrix} \varepsilon_{ijkl}[x_i,x_j,x_k]\cdot y_l + \varepsilon_{ijlk}[x_i,x_j,x_l]\cdot y_k \\ +\varepsilon_{kjli}[x_k,x_j,x_l]\cdot y_i + \varepsilon_{kilj}[x_k,x_i,x_l]\cdot y_j \\ +\varepsilon_{kjil}[x_k,x_j,x_i]\cdot y_l + \varepsilon_{ljik}[x_l,x_j,x_i]\cdot y_k \\ +\varepsilon_{ljki}[x_l,x_j,x_k]\cdot y_i + \varepsilon_{likj}[x_l,x_i,x_k]\cdot y_j \end{pmatrix}\right)$$

$$= \sum_{1\le j<i<k<l\le m}\begin{pmatrix} \varepsilon_{ijkl}[x_i,x_j,x_k]\cdot \delta_{2m}(y_l) + \varepsilon_{ijlk}[x_i,x_j,x_l]\cdot \delta_{2m}(y_k) \\ +\varepsilon_{kjli}[x_k,x_j,x_l]\cdot \delta_{2m}(y_i) + \varepsilon_{kilj}[x_k,x_i,x_l]\cdot \delta_{2m}(y_j) \\ +\varepsilon_{kjil}[x_k,x_j,x_i]\cdot \delta_{2m}(y_l) + \varepsilon_{ljik}[x_l,x_j,x_i]\cdot \delta_{2m}(y_k) \\ +\varepsilon_{ljki}[x_l,x_j,x_k]\cdot (y_i) + \varepsilon_{likj}[x_l,x_i,x_k]\cdot \delta_{2m}(y_j) \end{pmatrix}$$

$$= \sum_{1\le j<i<k<l\le m}\begin{pmatrix} \varepsilon_{ijkl}[x_i,x_j,x_k]\cdot x_l + \varepsilon_{ijlk}[x_i,x_j,x_l]\cdot x_k \\ +\varepsilon_{kjli}[x_k,x_j,x_l]\cdot x_i + \varepsilon_{kilj}[x_k,x_i,x_l]\cdot x_j \\ +\varepsilon_{kjil}[x_k,x_j,x_i]\cdot x_l + \varepsilon_{ljik}[x_l,x_j,x_i]\cdot x_k \\ +\varepsilon_{ljki}[x_l,x_j,x_k]\cdot x_i + \varepsilon_{likj}[x_l,x_i,x_k]\cdot x_j \end{pmatrix}$$

By using the condition in Remark 2.10.3, we rearrange the basis to obtain that

$$0 = \sum_{1\le j<i<k<l\le m}\begin{pmatrix} \varepsilon_{ijkl}[x_i,x_j,x_k]\cdot x_l - \varepsilon_{ijlk}[x_i,x_j,x_k]\cdot x_l \\ -\varepsilon_{kjli}([x_k,x_j,x_i]\cdot x_l) \\ +\varepsilon_{kilj}([x_i,x_j,x_k]\cdot x_l - [x_k,x_j,x_i]\cdot x_l) \\ +\varepsilon_{kjil}[x_k,x_j,x_i]\cdot x_l \\ +\varepsilon_{ljik}([x_k,x_j,x_i]\cdot x_l - [x_i,x_j,x_k]\cdot x_l) \\ +\varepsilon_{ljki}(-[x_k,x_j,x_i]\cdot x_l + [x_i,x_j,x_k]\cdot x_l) \\ -\varepsilon_{likj}[x_k,x_j,x_i]\cdot x_l \end{pmatrix}$$

$$= \sum_{1\le j<i<k<l\le m}\begin{pmatrix} (\varepsilon_{ijkl} - \varepsilon_{ijlk} + \varepsilon_{kilj} - \varepsilon_{ljik} + \varepsilon_{ljki})[x_i,x_j,x_k]\cdot x_l \\ +\begin{pmatrix} -\varepsilon_{kjli} - \varepsilon_{kilj} + \varepsilon_{kjil} + \varepsilon_{ljik} \\ -\varepsilon_{ljki} - \varepsilon_{likj} \end{pmatrix}[x_k,x_j,x_i]\cdot x_l \end{pmatrix}$$

Since $[x_i,x_j,x_k]\cdot x_l$ and $[x_k,x_j,x_i]\cdot x_l$ are the basis element in KA_3, then it cannot be zero hence, $\varepsilon_{ijkl} - \varepsilon_{ijlk} + \varepsilon_{kilj} - \varepsilon_{ljik} + \varepsilon_{ljki} = 0$ and $-\varepsilon_{kjli} - \varepsilon_{kilj} + \varepsilon_{kjil} + \varepsilon_{ljik} - \varepsilon_{ljki} - \varepsilon_{likj} = 0$ So,

$$\varepsilon_{ijkl} = \varepsilon_{ijlk} - \varepsilon_{kilj} + \varepsilon_{ljik} - \varepsilon_{ljki}$$

$$\varepsilon_{kjil} = \varepsilon_{kjli} + \varepsilon_{kilj} - \varepsilon_{ljik} + \varepsilon_{ljki} + \varepsilon_{likj}$$

Therefore,

$$(p_3)_4 = \sum_{1\le j<i<k<l\le m} \begin{pmatrix} (\varepsilon_{ijlk} - \varepsilon_{kilj} + \varepsilon_{ljik} - \varepsilon_{ljki})[x_i, x_j, x_k]\cdot y_l \\ +\varepsilon_{ijlk}[x_i, x_j, x_l]\cdot y_k + \varepsilon_{kjli}[x_k, x_j, x_l]\cdot y_i \\ +\varepsilon_{kilj}[x_k, x_i, x_l]\cdot y_j \\ +(\varepsilon_{kjli} + \varepsilon_{kilj} - \varepsilon_{ljik} + \varepsilon_{ljki} + \varepsilon_{likj})[x_k, x_j, x_i]\cdot y_l \\ +\varepsilon_{ljik}[x_l, x_j, x_i]\cdot y_k + \varepsilon_{ljki}[x_l, x_j, x_k]\cdot y_i \\ +\varepsilon_{likj}[x_l, x_i, x_k]\cdot y_j \end{pmatrix}$$

$$= \sum_{1\le j<i<k<l\le m} \begin{pmatrix} \varepsilon_{ijlk}([x_i, x_j, x_k]\cdot y_l + [x_i, x_j, x_l]\cdot y_k) \\ +\varepsilon_{kilj}\begin{pmatrix} -[x_i, x_j, x_k]\cdot y_l + [x_k, x_i, x_l]\cdot y_j \\ +[x_k, x_j, x_i]\cdot y_l \end{pmatrix} \\ +\varepsilon_{ljik}\begin{pmatrix} [x_i, x_j, x_k]\cdot y_l - [x_k, x_j, x_i]\cdot y_l \\ +[x_l, x_j, x_i]\cdot y_k \end{pmatrix} \\ +\varepsilon_{ljki}\begin{pmatrix} -[x_i, x_j, x_k]\cdot y_l + [x_k, x_j, x_i]\cdot y_l \\ +[x_l, x_j, x_k]\cdot y_i \end{pmatrix} \\ +\varepsilon_{kjli}([x_k, x_j, x_l]\cdot y_i + [x_k, x_j, x_i]\cdot y_l) \\ +\varepsilon_{likj}([x_k, x_j, x_i]\cdot y_l + [x_l, x_i, x_k]\cdot y_j) \end{pmatrix}$$

Hence,

$$[x_i, x_j, x_k]\cdot y_l + [x_i, x_j, x_l]\cdot y_k,$$
$$[x_k, x_j, x_l]\cdot y_i + [x_k, x_j, x_i]\cdot y_l,$$
$$[x_k, x_j, x_i]\cdot y_l + [x_l, x_i, x_k]\cdot y_j,$$
$$-[x_i, x_j, x_k]\cdot y_l + [x_k, x_i, x_l]\cdot y_j + [x_k, x_j, x_i]\cdot y_l,$$
$$[x_i, x_j, x_k]\cdot y_l - [x_k, x_j, x_i]\cdot y_l + [x_l, x_j, x_i]\cdot y_k,$$
$$-[x_i, x_j, x_k]\cdot y_l + [x_k, x_j, x_i]\cdot y_l + [x_l, x_j, x_k]\cdot y_i,$$

are constant elements in KA_3 where $1 \le j < i < k < l \le m$. Therefore, this prove that the vector space $(KA_3)^{\delta_{2m}}$ is spanned on the elements of the form

$$[x_i, x_j, x_j]\cdot y_j,$$

$$[x_i, x_j, x_i] \cdot y_i,$$
$$[x_i, x_j, x_i] \cdot y_j + [x_i, x_j, x_j] \cdot y_i,$$

where $1 \leq j < i \leq m$,

$$[x_i, x_j, x_k] \cdot y_k,$$
$$[x_k, x_j, x_i] \cdot y_i,$$
$$[x_i, x_j, x_i] \cdot y_k + [x_i, x_j, x_k] \cdot y_i,$$
$$[x_i, x_j, x_i] \cdot y_k + [x_k, x_i, x_i] \cdot y_j,$$
$$-[x_k, x_j, x_k] \cdot y_i + [x_k, x_i, x_k] \cdot y_j,$$
$$[x_k, x_j, x_k] \cdot y_i + [x_k, x_j, x_i] \cdot y_k,$$
$$-[x_i, x_j, x_k] \cdot y_j + [x_k, x_j, x_i] \cdot y_j,$$
$$[x_i, x_j, x_k] \cdot y_j + [x_i, x_j, x_j] \cdot y_k,$$
$$[x_i, x_j, x_k] \cdot y_j + [x_k, x_j, x_j] \cdot y_i,$$

where $1 \leq j < i < k \leq m$,

and

$$[x_i, x_j, x_k] \cdot y_l + [x_i, x_j, x_l] \cdot y_k,$$
$$[x_k, x_j, x_l] \cdot y_i + [x_k, x_j, x_i] \cdot y_l,$$
$$[x_k, x_j, x_i] \cdot y_l + [x_l, x_i, x_k] \cdot y_j,$$
$$-[x_i, x_j, x_k] \cdot y_l + [x_k, x_i, x_l] \cdot y_j + [x_k, x_j, x_i] \cdot y_l,$$
$$[x_i, x_j, x_k] \cdot y_l - [x_k, x_j, x_i] \cdot y_l + [x_l, x_j, x_i] \cdot y_k,$$
$$-[x_i, x_j, x_k] \cdot y_l + [x_k, x_j, x_i] \cdot y_l + [x_l, x_j, x_k] \cdot y_i,$$

where $1 \leq j < i < k < l \leq m$.

For $p_4 + p_5$. Let

$$p_4 = \sum_{\substack{1 \leq j < i \leq m \\ 1 \leq k < l \leq m}} \varepsilon_{ijkl} [x_i, x_j, y_k] \cdot y_l$$

We have the condition that $1 \leq j < i \leq m$ and $1 \leq k < l \leq m$ according to this condition we have $\boldsymbol{l}$ in different position.

Let start with $\boldsymbol{j} \quad \boldsymbol{i} \quad \boldsymbol{k} \quad l$, this gives 1 condition which is $j < i < k < l$. For the position $\boldsymbol{j} \quad \boldsymbol{i} = \boldsymbol{k} \quad l$ and this gives 1 condition which is $j < i = k < l$. For the position $\boldsymbol{j} \quad \boldsymbol{k} \quad l \quad \boldsymbol{i} = l \quad l$ we have $j < k < i < l$, $j < k < i = l$ and $j < k < l < i$. For the position $\boldsymbol{j} = \boldsymbol{k} \quad l \quad \boldsymbol{i} = l \quad l$ we obtain $j = k < i < l$, $j = k < i = l$ and $j = k < l < i$. For the position $\boldsymbol{k} \quad l \quad \boldsymbol{j} = l \quad l \quad \boldsymbol{i} = l \quad l$ we 5 different conditions $k < j < i < l$, $k < j < i = l$, $k < j < l < i$, $k < j = l < i$ and $k < l < j < i$.

Now p_4 become

$$
\begin{aligned}
p_4 = & \sum_{1 \le j < i < k < l \le m} \varepsilon_{ijkl} \left[x_i, x_j, y_k\right] \cdot y_l + \sum_{1 \le j < i = k < l \le m} \varepsilon_{ijil} \left[x_i, x_j, y_i\right] \cdot y_l \\
& + \sum_{1 \le j < k < i < l \le m} \varepsilon_{ijkl} \left[x_i, x_j, y_k\right] \cdot y_l + \sum_{1 \le j < k < i = l \le m} \varepsilon_{ijki} \left[x_i, x_j, y_k\right] \cdot y_i \\
& + \sum_{1 \le j < k < l < i \le m} \varepsilon_{ijkl} \left[x_i, x_j, y_k\right] \cdot y_l + \sum_{1 \le j = k < i < l \le m} \varepsilon_{ijjl} \left[x_i, x_j, y_j\right] \cdot y_l \\
& + \sum_{1 \le j = k < i = l \le m} \varepsilon_{ijji} \left[x_i, x_j, y_j\right] \cdot y_i + \sum_{1 \le j = k < l < i \le m} \varepsilon_{ijjl} \left[x_i, x_j, y_j\right] \cdot y_l \\
& + \sum_{1 \le k < j < i < l \le m} \varepsilon_{ijkl} \left[x_i, x_j, y_k\right] \cdot y_l + \sum_{1 \le k < j < i = l \le m} \varepsilon_{ijki} \left[x_i, x_j, y_k\right] \cdot y_i \\
& + \sum_{1 \le k < j < l < i \le m} \varepsilon_{ijkl} \left[x_i, x_j, y_k\right] \cdot y_l + \sum_{1 \le k < j = l < i \le m} \varepsilon_{ijkj} \left[x_i, x_j, y_k\right] \cdot y_j \\
& + \sum_{1 \le k < l < j < i \le m} \varepsilon_{ijkl} \left[x_i, x_j, y_k\right] \cdot y_l \\
= & \sum_{1 \le j < i < k < l \le m} \varepsilon_{ijkl} \left[x_i, x_j, y_k\right] \cdot y_l + \sum_{1 \le j < i < l \le m} \varepsilon_{ijil} \left[x_i, x_j, y_i\right] \cdot y_l \\
& + \sum_{1 \le j < i < k < l \le m} \varepsilon_{kjil} \left[x_k, x_j, y_i\right] \cdot y_l + \sum_{1 \le j < i < k \le m} \varepsilon_{kjik} \left[x_k, x_j, y_i\right] \cdot y_k \\
& + \sum_{1 \le j < i < k < l \le m} \varepsilon_{ljik} \left[x_l, x_j, y_i\right] \cdot y_k + \sum_{1 \le j < i < l \le m} \varepsilon_{ijjl} \left[x_i, x_j, y_j\right] \cdot y_l
\end{aligned}
$$

$$+\sum_{1\le j<i\le m}\varepsilon_{ijji}\left[x_i,x_j,y_j\right]\cdot y_i+\sum_{1\le j<i<l\le m}\varepsilon_{ljji}\left[x_l,x_j,y_j\right]\cdot y_i$$
$$+\sum_{1\le j<i<k<l\le m}\varepsilon_{kijl}\left[x_k,x_i,y_j\right]\cdot y_l+\sum_{1\le j<i<k\le m}\varepsilon_{kijk}\left[x_k,x_i,y_j\right]\cdot y_k$$
$$+\sum_{1\le j<i<k<l\le m}\varepsilon_{lijk}\left[x_l,x_i,y_j\right]\cdot y_k+\sum_{1\le j<i<k\le m}\varepsilon_{kiji}\left[x_k,x_i,y_j\right]\cdot y_i$$
$$+\sum_{1\le j<i<k<l\le m}\varepsilon_{lkji}\left[x_l,x_k,y_j\right]\cdot y_i$$

$$=\sum_{1\le j<i\le m}\varepsilon_{ijji}\left[x_i,x_j,y_j\right]\cdot y_i$$
$$+\sum_{1\le j<i<k\le m}\begin{pmatrix}\varepsilon_{ijik}\left[x_i,x_j,y_i\right]\cdot y_k+\varepsilon_{kjik}\left[x_k,x_j,y_i\right]\cdot y_k\\+\varepsilon_{ijjk}\left[x_i,x_j,y_j\right]\cdot y_k+\varepsilon_{kjji}\left[x_k,x_j,y_j\right]\cdot y_i\\+\varepsilon_{kijk}\left[x_k,x_i,y_j\right]\cdot y_k+\varepsilon_{kiji}\left[x_k,x_i,y_j\right]\cdot y_i\end{pmatrix}$$
$$+\sum_{1\le j<i<k<l\le m}\begin{pmatrix}\varepsilon_{ijkl}\left[x_i,x_j,y_k\right]\cdot y_l+\varepsilon_{kjil}\left[x_k,x_j,y_i\right]\cdot y_l\\+\varepsilon_{ljik}\left[x_l,x_j,y_i\right]\cdot y_k+\varepsilon_{kijl}\left[x_k,x_i,y_j\right]\cdot y_l\\+\varepsilon_{lijk}\left[x_l,x_i,y_j\right]\cdot y_k+\varepsilon_{lkji}\left[x_l,x_k,y_j\right]\cdot y_i\end{pmatrix}.$$

and

$$p_5=\sum_{\substack{1\le j\le k\le m\\1\le i\le l\le m}}\gamma_{ijkl}\left[y_i,x_j,x_k\right]\cdot y_l$$

We have the condition that $1\le j\le k\le m$ and $1\le i\le l\le m$ and this can be split in 4 different part which are $1\le j<k\le m$, $\le j=k\le m$, $1\le i<l\le m$ and $\le i=l\le m$. From this condition we have $\boldsymbol{l}$ in the following different positions.

If we have $1\le j<k\le m$ and $1\le i<l\le m$ then the first position is $\boldsymbol{j}\quad\boldsymbol{k}\quad\boldsymbol{i}\quad l$, this gives 1 condition which is $j<k<i<l$. For the position $\boldsymbol{j}\quad\boldsymbol{i}=\boldsymbol{k}\quad l$ and this gives 1 condition which is $j<i=k<l$. For the position $\boldsymbol{j}\quad\boldsymbol{i}\quad l\quad\boldsymbol{k}=l\quad l$ we have $j<i<k<l$, $j<i<k=l$ and $j<i<l<k$. For

the position $\boldsymbol{i}=\boldsymbol{j}$ l $\boldsymbol{k}=l$ l we obtain $i=j<k<l$, $i=j<k=l$ and $i=j<l<k$. For the position $\boldsymbol{i}$ l $\boldsymbol{j}=l$ l $\boldsymbol{k}=l$ l we 5 different conditions $i<j<k<l$, $i<j<k=l$, $i<j<l<k$, $i<j=l<k$ and $i<l<j<k$.

If $1\leq j<k\leq m$ and $1\leq i=l\leq m$ then we have $j<k<i=l$, $j<k=i=l$, $j<i=l<k$, $j=i=l<k$ and $i=l<j<k$.

If we have $1\leq j=k\leq m$ and $1\leq i<l\leq m$ then $j=k<i<l$, $j=k=i<l$, $i<j=k<l$, $i<j=k=l$ and $i<l<j=k$.

Lastly, for $1\leq j=k\leq m$ and $1\leq i=l\leq m$ we have $j=k<i=l$, $j=k=i=l$ and $i=l<j=k$.

Now p_5 become

$$\begin{aligned}
p_5 = &\sum_{1\leq j<k<i<l\leq m}\gamma_{ijkl}\left[y_i,x_j,x_k\right]\cdot y_l+\sum_{1\leq j<i=k<l\leq m}\gamma_{ijil}\left[y_i,x_j,x_i\right]\cdot y_l\\
&+\sum_{1\leq j<i<k<l\leq m}\gamma_{ijkl}\left[y_i,x_j,x_k\right]\cdot y_l+\sum_{1\leq j<i<k=l\leq m}\gamma_{ijkk}\left[y_i,x_j,x_k\right]\cdot y_k\\
&+\sum_{1\leq j<i<l<k\leq m}\gamma_{ijkl}\left[y_i,x_j,x_k\right]\cdot y_l+\sum_{1\leq i=j<k<l\leq m}\gamma_{iikl}\left[y_i,x_i,x_k\right]\cdot y_l\\
&+\sum_{1\leq i=j<k=l\leq m}\gamma_{iikk}\left[y_i,x_i,x_k\right]\cdot y_k+\sum_{1\leq i=j<l<k\leq m}\gamma_{iikl}\left[y_i,x_i,x_k\right]\cdot y_l\\
&+\sum_{1\leq i<j<k<l\leq m}\gamma_{ijkl}\left[y_i,x_j,x_k\right]\cdot y_l+\sum_{1\leq i<j<k=l\leq m}\gamma_{ijkk}\left[y_i,x_j,x_k\right]\cdot y_k\\
&+\sum_{1\leq i<j<l<k\leq m}\gamma_{ijkl}\left[y_i,x_j,x_k\right]\cdot y_l+\sum_{1\leq i<j=l<k\leq m}\gamma_{ijkj}\left[y_i,x_j,x_k\right]\cdot y_j\\
&+\sum_{1\leq i<l<j<k\leq m}\gamma_{ijkl}\left[y_i,x_j,x_k\right]\cdot y_l+\sum_{1\leq j<k<i=l\leq m}\gamma_{ijki}\left[y_i,x_j,x_k\right]\cdot y_i\\
&+\sum_{1\leq j<k=i=l\leq m}\gamma_{ijii}\left[y_i,x_j,x_i\right]\cdot y_i+\sum_{1\leq j<i=l<k\leq m}\gamma_{ijki}\left[y_i,x_j,x_k\right]\cdot y_i\\
&+\sum_{1\leq j=i=l<k\leq m}\gamma_{iiki}\left[y_i,x_i,x_k\right]\cdot y_i+\sum_{1\leq i=l<j<k\leq m}\gamma_{ijki}\left[y_i,x_j,x_k\right]\cdot y_i
\end{aligned}$$

$$+\sum_{1\le j=k<i<l\le m}\gamma_{ijjl}\left[y_i,x_j,x_j\right]\cdot y_l+\sum_{1\le j=k=i<l\le m}\gamma_{iiil}\left[y_i,x_i,x_i\right]\cdot y_l$$

$$+\sum_{1\le i<j=k<l\le m}\gamma_{ijjl}\left[y_i,x_j,x_j\right]\cdot y_l+\sum_{1\le i<j=k=l\le m}\gamma_{ijjj}\left[y_i,x_j,x_j\right]\cdot y_j$$

$$+\sum_{1\le i<l<j=k\le m}\gamma_{ijjl}\left[y_i,x_j,x_j\right]\cdot y_l+\sum_{1\le j=k<i=l\le m}\gamma_{ijji}\left[y_i,x_j,x_j\right]\cdot y_i$$

$$+\sum_{1\le j=k=i=l\le m}\gamma_{iiii}\left[y_i,x_i,x_i\right]\cdot y_i+\sum_{1\le i=l<j=k\le m}\gamma_{ijji}\left[y_i,x_j,x_j\right]\cdot y_i$$

$$=\sum_{1\le j<i<k<l\le m}\gamma_{kjil}\left[y_k,x_j,x_i\right]\cdot y_l+\sum_{1\le j<i<l\le m}\gamma_{ijil}\left[y_i,x_j,x_i\right]\cdot y_l$$

$$+\sum_{1\le j<i<k<l\le m}\gamma_{ijkl}\left[y_i,x_j,x_k\right]\cdot y_l+\sum_{1\le j<i<k\le m}\gamma_{ijkk}\left[y_i,x_j,x_k\right]\cdot y_k$$

$$+\sum_{1\le j<i<k<l\le m}\gamma_{ijlk}\left[y_i,x_j,x_l\right]\cdot y_k+\sum_{1\le i<k<l\le m}\gamma_{iikl}\left[y_i,x_i,x_k\right]\cdot y_l$$

$$+\sum_{1\le i<k\le m}\gamma_{iikk}\left[y_i,x_i,x_k\right]\cdot y_k+\sum_{1\le i<k<l\le m}\gamma_{iilk}\left[y_i,x_i,x_l\right]\cdot y_k$$

$$+\sum_{1\le j<i<k<l\le m}\gamma_{jikl}\left[y_j,x_i,x_k\right]\cdot y_l+\sum_{1\le j<i<k\le m}\gamma_{jikk}\left[y_j,x_i,x_k\right]\cdot y_k$$

$$+\sum_{1\le j<i<k<l\le m}\gamma_{jilk}\left[y_j,x_i,x_l\right]\cdot y_k+\sum_{1\le j<i<k\le m}\gamma_{jiki}\left[y_j,x_i,x_k\right]\cdot y_i$$

$$+\sum_{1\le j<i<k<l\le m}\gamma_{jkli}\left[y_j,x_k,x_l\right]\cdot y_i+\sum_{1\le j<i<k\le m}\gamma_{kjik}\left[y_k,x_j,x_i\right]\cdot y_k$$

$$+\sum_{1\le j<i\le m}\gamma_{ijii}\left[y_i,x_j,x_i\right]\cdot y_i+\sum_{1\le j<i<k\le m}\gamma_{ijki}\left[y_i,x_j,x_k\right]\cdot y_i$$

$$+\sum_{1\le i<k\le m}\gamma_{iiki}\left[y_i,x_i,x_k\right]\cdot y_i+\sum_{1\le j<i<k\le m}\gamma_{jikj}\left[y_j,x_i,x_k\right]\cdot y_j$$

$$+\sum_{1\le j<i<l\le m}\gamma_{ijjl}\left[y_i,x_j,x_j\right]\cdot y_l+\sum_{1\le i<l\le m}\gamma_{iiil}\left[y_i,x_i,x_i\right]\cdot y_l$$

$$+\sum_{1\le j<i<l\le m}\gamma_{jiil}\left[y_j,x_i,x_i\right]\cdot y_l+\sum_{1\le i<j\le m}\gamma_{ijjj}\left[y_i,x_j,x_j\right]\cdot y_j$$

$$+\sum_{1\le j<i<l\le m}\gamma_{jlli}\left[y_j,x_l,x_l\right]\cdot y_i+\sum_{1\le j<i\le m}\gamma_{ijji}\left[y_i,x_j,x_j\right]\cdot y_i$$

$$+\sum_{1\le i\le m}\gamma_{iiii}\left[y_i,x_i,x_i\right]\cdot y_i+\sum_{1\le i<j\le m}\gamma_{ijji}\left[y_i,x_j,x_j\right]\cdot y_i$$

$$=\sum_{1\le i\le m}\gamma_{iiii}\left[y_i,x_i,x_i\right]\cdot y_i$$

$$+\sum_{1\le j<i\le m}\begin{pmatrix}\gamma_{jjii}\left[y_j,x_j,x_i\right]\cdot y_i+\gamma_{ijii}\left[y_i,x_j,x_i\right]\cdot y_i+\gamma_{jjij}\left[y_j,x_j,x_i\right]\cdot y_j\\+\gamma_{jjji}\left[y_j,x_j,x_j\right]\cdot y_i+\gamma_{jiii}\left[y_j,x_i,x_i\right]\cdot y_i+\gamma_{ijji}\left[y_i,x_j,x_j\right]\cdot y_i\\+\gamma_{jiij}\left[y_j,x_i,x_i\right]\cdot y_j\end{pmatrix}$$

$$+\sum_{1\le j<i<k\le m}\begin{pmatrix}\gamma_{ijik}\left[y_i,x_j,x_i\right]\cdot y_k+\gamma_{ijkk}\left[y_i,x_j,x_k\right]\cdot y_k\\+\gamma_{jjik}\left[y_j,x_j,x_i\right]\cdot y_k+\gamma_{jjki}\left[y_j,x_j,x_k\right]\cdot y_i\\+\gamma_{jikk}\left[y_j,x_i,x_k\right]\cdot y_k+\gamma_{jiki}\left[y_j,x_i,x_k\right]\cdot y_i\\+\gamma_{kjik}\left[y_k,x_j,x_i\right]\cdot y_k+\gamma_{ijki}\left[y_i,x_j,x_k\right]\cdot y_i\\+\gamma_{jikj}\left[y_j,x_i,x_k\right]\cdot y_j+\gamma_{ijjk}\left[y_i,x_j,x_j\right]\cdot y_k\\+\gamma_{jiik}\left[y_j,x_i,x_i\right]\cdot y_k+\gamma_{jkki}\left[y_j,x_k,x_k\right]\cdot y_i\end{pmatrix}$$

$$+\sum_{1\le j<i<k<l\le m}\begin{pmatrix}\gamma_{kjil}\left[y_k,x_j,x_i\right]\cdot y_l+\gamma_{ijkl}\left[y_i,x_j,x_k\right]\cdot y_l\\+\gamma_{ijlk}\left[y_i,x_j,x_l\right]\cdot y_k+\gamma_{jikl}\left[y_j,x_i,x_k\right]\cdot y_l\\+\gamma_{jilk}\left[y_j,x_i,x_l\right]\cdot y_k+\gamma_{jkli}\left[y_j,x_k,x_l\right]\cdot y_i\end{pmatrix}.$$

Now,

$$p_4+p_5=\sum_{1\le i\le m}\gamma_{iiii}\left[y_i,x_i,x_i\right]\cdot y_i$$

$$+\sum_{1\le j<i\le m}\begin{pmatrix}\varepsilon_{ijji}\left[x_i,x_j,y_j\right]\cdot y_i+\gamma_{jjii}\left[y_j,x_j,x_i\right]\cdot y_i\\+\gamma_{ijii}\left[y_i,x_j,x_i\right]\cdot y_i+\gamma_{jjij}\left[y_j,x_j,x_i\right]\cdot y_j\\+\gamma_{jjji}\left[y_j,x_j,x_j\right]\cdot y_i+\gamma_{jiii}\left[y_j,x_i,x_i\right]\cdot y_i\\+\gamma_{ijji}\left[y_i,x_j,x_j\right]\cdot y_i+\gamma_{jiij}\left[y_j,x_i,x_i\right]\cdot y_j\end{pmatrix}$$

$$+\sum_{1\le j<i<k\le m}\begin{pmatrix} \varepsilon_{ijik}[x_i,x_j,y_i]\cdot y_k+\varepsilon_{kjik}[x_k,x_j,y_i]\cdot y_k \\ +\varepsilon_{ijjk}[x_i,x_j,y_j]\cdot y_k+\varepsilon_{kjji}[x_k,x_j,y_j]\cdot y_i \\ +\varepsilon_{kijk}[x_k,x_i,y_j]\cdot y_k+\varepsilon_{kiji}[x_k,x_i,y_j]\cdot y_i \\ \gamma_{ijik}[y_i,x_j,x_i]\cdot y_k+\gamma_{ijkk}[y_i,x_j,x_k]\cdot y_k \\ +\gamma_{jjik}[y_j,x_j,x_i]\cdot y_k+\gamma_{jjki}[y_j,x_j,x_k]\cdot y_i \\ +\gamma_{jikk}[y_j,x_i,x_k]\cdot y_k+\gamma_{jiki}[y_j,x_i,x_k]\cdot y_i \\ +\gamma_{kjik}[y_k,x_j,x_i]\cdot y_k+\gamma_{ijki}[y_i,x_j,x_k]\cdot y_i \\ +\gamma_{jikj}[y_j,x_i,x_k]\cdot y_j+\gamma_{ijjk}[y_i,x_j,x_j]\cdot y_k \\ +\gamma_{jiik}[y_j,x_i,x_i]\cdot y_k+\gamma_{jkki}[y_j,x_k,x_k]\cdot y_i \end{pmatrix}$$

$$+\sum_{1\le j<i<k<l\le m}\begin{pmatrix} \varepsilon_{ijkl}[x_i,x_j,y_k]\cdot y_l+\varepsilon_{kjil}[x_k,x_j,y_i]\cdot y_l \\ +\varepsilon_{ljik}[x_l,x_j,y_i]\cdot y_k+\varepsilon_{kijl}[x_k,x_i,y_j]\cdot y_l \\ +\varepsilon_{lijk}[x_l,x_i,y_j]\cdot y_k+\varepsilon_{lkji}[x_l,x_k,y_j]\cdot y_i \\ \gamma_{kjil}[y_k,x_j,x_i]\cdot y_l+\gamma_{ijkl}[y_i,x_j,x_k]\cdot y_l \\ +\gamma_{ijlk}[y_i,x_j,x_l]\cdot y_k+\gamma_{jikl}[y_j,x_i,x_k]\cdot y_l \\ +\gamma_{jilk}[y_j,x_i,x_l]\cdot y_k+\gamma_{jkli}[y_j,x_k,x_l]\cdot y_i \end{pmatrix}$$

So, $p_4+p_5=(p_4+p_5)_1+(p_4+p_5)_2+(p_4+p_5)_3+(p_4+p_5)_4$ such that

$$(p_5)_1=\sum_{1\le i\le m}\gamma_{iiii}\,[y_i,x_i,x_i]\cdot y_i$$

$$(p_4+p_5)_2=\sum_{1\le j<i\le m}\begin{pmatrix} \varepsilon_{ijji}[x_i,x_j,y_j]\cdot y_i+\gamma_{jjii}[y_j,x_j,x_i]\cdot y_i \\ +\gamma_{ijii}[y_i,x_j,x_i]\cdot y_i+\gamma_{jjij}[y_j,x_j,x_i]\cdot y_j \\ +\gamma_{jjji}[y_j,x_j,x_j]\cdot y_i+\gamma_{jiii}[y_j,x_i,x_i]\cdot y_i \\ +\gamma_{ijji}[y_i,x_j,x_j]\cdot y_i+\gamma_{jiij}[y_j,x_i,x_i]\cdot y_j \end{pmatrix},$$

$$(p_4+p_5)_3=\sum_{1\le j<i<k\le m}\begin{pmatrix} \varepsilon_{ijik}[x_i,x_j,y_i]\cdot y_k+\varepsilon_{kjik}[x_k,x_j,y_i]\cdot y_k \\ +\varepsilon_{ijjk}[x_i,x_j,y_j]\cdot y_k+\varepsilon_{kjji}[x_k,x_j,y_j]\cdot y_i \\ +\varepsilon_{kijk}[x_k,x_i,y_j]\cdot y_k+\varepsilon_{kiji}[x_k,x_i,y_j]\cdot y_i \\ +\gamma_{ijik}[y_i,x_j,x_i]\cdot y_k+\gamma_{ijkk}[y_i,x_j,x_k]\cdot y_k \\ +\gamma_{jjik}[y_j,x_j,x_i]\cdot y_k+\gamma_{jjki}[y_j,x_j,x_k]\cdot y_i \\ +\gamma_{jikk}[y_j,x_i,x_k]\cdot y_k+\gamma_{jiki}[y_j,x_i,x_k]\cdot y_i \\ +\gamma_{kjik}[y_k,x_j,x_i]\cdot y_k+\gamma_{ijki}[y_i,x_j,x_k]\cdot y_i \\ +\gamma_{jikj}[y_j,x_i,x_k]\cdot y_j+\gamma_{ijjk}[y_i,x_j,x_j]\cdot y_k \\ +\gamma_{jiik}[y_j,x_i,x_i]\cdot y_k+\gamma_{jkki}[y_j,x_k,x_k]\cdot y_i \end{pmatrix},$$

$$(p_4+p_5)_4=\sum_{1\le j<i<k<l\le m}\begin{pmatrix}\varepsilon_{ijkl}[x_i,x_j,y_k]\cdot y_l+\varepsilon_{kjil}[x_k,x_j,y_i]\cdot y_l\\+\varepsilon_{ljik}[x_l,x_j,y_i]\cdot y_k+\varepsilon_{kijl}[x_k,x_i,y_j]\cdot y_l\\+\varepsilon_{lijk}[x_l,x_i,y_j]\cdot y_k+\varepsilon_{lkji}[x_l,x_k,y_j]\cdot y_i\\+\gamma_{kjil}[y_k,x_j,x_i]\cdot y_l+\gamma_{ijkl}[y_i,x_j,x_k]\cdot y_l\\+\gamma_{ijlk}[y_i,x_j,x_l]\cdot y_k+\gamma_{jikl}[y_j,x_i,x_k]\cdot y_l\\+\gamma_{jilk}[y_j,x_i,x_l]\cdot y_k+\gamma_{jkli}[y_j,x_k,x_l]\cdot y_i\end{pmatrix}.$$

Then

$\delta_{2m}(p_4+p_5)=0=\delta_{2m}((p_5)_1+(p_4+p_5)_2+(p_4+p_5)_3+(p_4+p_5)_4)=$ $\delta_{2m}((p_5)_1)+\delta_{2m}((p_4+p_5)_2)+\delta_{2m}((p_4+p_5)_3)+\delta_{2m}((p_4+p_5)_4)$. Let $V_{(5)_1}$ be the subspace of KA_5 containing monomials with exactly 1 indice, $V_{(4+5)_2}$ be the subspace of $KA_4\oplus KA_5$ containing monomials withexactly 2 indices, $V_{(4+5)_3}$ be the subspace of $KA_4\oplus KA_5$ containing monomials with exactly 3 indices and $V_{(4+5)_4}$ be the subspace of $KA_4\oplus KA_5$ containing monomials with exactly 4 indices then $V_{(5)_1}\cap V_{(4+5)_2}=\{0\}=V_{(5)_1}\cap V_{(4+5)_3}=V_{(5)_1}\cap V_{(4+5)_4}=V_{(4+5)_2}\cap V_{(4+5)_3}=V_{(4+5)_2}\cap V_{(4+5)_4}=V_{(4+5)_3}\cap V_{(4+5)_4}$.

Then $\delta_{2m}((p_5)_1)\in V_{(5)_1}$, $\delta_{2m}((p_4+p_5)_2)\in V_{(4+5)_2}$, $\delta_{2m}((p_4+p_5)_3)\in V_{(4+5)_3}$, $\delta_{2m}((p_4+p_5)_4)\in V_{(4+5)_4}$.

Therefore, $\delta_{2m}((p_5)_1)=0=\delta_{2m}((p_4+p_5)_2)=\delta_{2m}((p_4+p_5)_3)=$ $\delta_{2m}((p_4+p_5)_4)$.

$$\begin{aligned}0&=\delta_{2m}((p_5)_1)\\&=\delta_{2m}\left(\sum_{1\le i\le m}\gamma_{iiii}\,[y_i,x_i,x_i]\cdot y_i\right)\\&=\sum_{1\le i\le m}\gamma_{iiii}\left([\delta_{2m}(y_i),x_i,x_i]\cdot y_i+[y_i,x_i,x_i]\cdot\delta_{2m}(y_i)\right)\\&=\sum_{1\le i\le m}\gamma_{iiii}\left([x_i,x_i,x_i]\cdot y_i+[y_i,x_i,x_i]\cdot x_i\right)\end{aligned}$$

By using the conditions in Remark 2.10.3, $[x_i, x_i, x_i] \cdot y_i = 0$ and $[y_i, x_i, x_i] \cdot x_i = 0$, so, this mean γ_{iiii} can be any. Hence,

$$(p_5)_1 = \sum_{1 \le i \le m} \gamma_{iiii} \, [y_i, x_i, x_i] \cdot y_i$$

and

$$[y_i, x_i, x_i] \cdot y_i$$

is a constant element in KA_5 where $1 \le i \le m$.

For

$0 = \delta_{2m}((p_4 + p_5)_2)$

$$0 = \delta_{2m}\left(\sum_{1 \le j < i \le m} \begin{pmatrix} \varepsilon_{ijji}[x_i, x_j, y_j] \cdot y_i + \gamma_{jjii}[y_j, x_j, x_i] \cdot y_i \\ +\gamma_{ijii}[y_i, x_j, x_i] \cdot y_i + \gamma_{jjij}[y_j, x_j, x_i] \cdot y_j \\ +\gamma_{jjji}[y_j, x_j, x_j] \cdot y_i + \gamma_{jiii}[y_j, x_i, x_i] \cdot y_i \\ +\gamma_{ijji}[y_i, x_j, x_j] \cdot y_i + \gamma_{jiij}[y_j, x_i, x_i] \cdot y_j \end{pmatrix} \right)$$

$$= \sum_{1 \le j < i \le m} \begin{pmatrix} \varepsilon_{ijji}\left([x_i, x_j, \delta_{2m}(y_j)] \cdot y_i + [x_i, x_j, y_j] \cdot \delta_{2m}(y_i) \right) \\ +\gamma_{jjii}\left([\delta_{2m}(y_j), x_j, x_i] \cdot y_i + [y_j, x_j, x_i] \cdot \delta_{2m}(y_i) \right) \\ +\gamma_{ijii}\left([\delta_{2m}(y_i), x_j, x_i] \cdot y_i + [y_i, x_j, x_i] \cdot \delta_{2m}(y_i) \right) \\ +\gamma_{jjij}\left([\delta_{2m}(y_j), x_j, x_i] \cdot y_j + [y_j, x_j, x_i] \cdot \delta_{2m}(y_j) \right) \\ +\gamma_{jjji}\left([\delta_{2m}(y_j), x_j, x_j] \cdot y_i + [y_j, x_j, x_j] \cdot \delta_{2m}(y_i) \right) \\ +\gamma_{jiii}\left([\delta_{2m}(y_j), x_i, x_i] \cdot y_i + [y_j, x_i, x_i] \cdot \delta_{2m}(y_i) \right) \\ +\gamma_{ijji}\left([\delta_{2m}(y_i), x_j, x_j] \cdot y_i + [y_i, x_j, x_j] \cdot \delta_{2m}(y_i) \right) \\ +\gamma_{jiij}\left([\delta_{2m}(y_j), x_i, x_i] \cdot y_j + [y_j, x_i, x_i] \cdot \delta_{2m}(y_j) \right) \end{pmatrix}$$

$$= \sum_{1\le j<i\le m}\begin{pmatrix} \varepsilon_{ijji}([x_i,x_j,x_j]\cdot y_i + [x_i,x_j,y_j]\cdot x_i) \\ +\gamma_{jjii}([x_j,x_j,x_i]\cdot y_i + [y_j,x_j,x_i]\cdot x_i) \\ +\gamma_{ijii}([x_i,x_j,x_i]\cdot y_i + [y_i,x_j,x_i]\cdot x_i) \\ +\gamma_{jjij}([x_j,x_j,x_i]\cdot y_j + [y_j,x_j,x_i]\cdot x_j) \\ +\gamma_{jjji}([x_j,x_j,x_j]\cdot y_i + [y_j,x_j,x_j]\cdot x_i) \\ +\gamma_{jiii}([x_j,x_i,x_i]\cdot y_i + [y_j,x_i,x_i]\cdot x_i) \\ +\gamma_{ijji}([x_i,x_j,x_j]\cdot y_i + [y_i,x_j,x_j]\cdot x_i) \\ +\gamma_{jiij}([x_j,x_i,x_i]\cdot y_j + [y_j,x_i,x_i]\cdot x_j) \end{pmatrix}$$

We rearrange by using the condition in Remark 2.10.3 in other to get the basis, we obtain that

$$0 = \sum_{1\le j<i\le m}\begin{pmatrix} \varepsilon_{ijji}([x_i,x_j,x_j]\cdot y_i - [x_i,x_j,x_i]\cdot y_j) + \gamma_{jjii}(0) \\ +\gamma_{ijii}([x_i,x_j,x_i]\cdot y_i) \\ +\gamma_{jjij}(-[x_i,x_j,x_j]\cdot y_j) + \gamma_{jjji}([x_i,x_j,x_j]\cdot y_j) \\ +\gamma_{jiii}(-[x_i,x_j,x_i]\cdot y_i) \\ +\gamma_{ijji}(2[x_i,x_j,x_j]\cdot y_i) + \gamma_{jiij}(-2[x_i,x_j,x_i]\cdot y_j) \end{pmatrix}$$

$$= \sum_{1\le j<i\le m}\begin{pmatrix} (\varepsilon_{ijji} + 2\gamma_{ijji})[x_i,x_j,x_j]\cdot y_i + (-\varepsilon_{ijji} - 2\gamma_{jiij})[x_i,x_j,x_i]\cdot y_j \\ +(\gamma_{ijii} - \gamma_{jiii})[x_i,x_j,x_i]\cdot y_i + (-\gamma_{jjij} + \gamma_{jjji})[x_i,x_j,x_i]\cdot y_i \end{pmatrix}$$

Since $[x_j,x_j,x_i]\cdot y_i = 0$ and $[y_j,x_j,x_i]\cdot x_i = 0$, by the condition in Remark 2.10.3, then this mean γ_{jjii} can be any. Since $[x_i,x_j,x_j]\cdot y_i$, $[x_i,x_j,x_i]\cdot y_j$, $[x_i,x_j,x_i]\cdot y_i$ and $[x_i,x_j,x_i]\cdot y_i$ are the basis elements in $KA_4 \oplus KA_5$, then it cannot be zero hence, $\varepsilon_{ijji} + 2\gamma_{ijji} = 0$, $-\varepsilon_{ijji} - 2\gamma_{jiij} = 0$, $\gamma_{ijii} - \gamma_{jiii} = 0$ and $-\gamma_{jjij} + \gamma_{jjji} = 0$.

So,

$$\varepsilon_{ijji} = -2\gamma_{ijji} = -2\gamma_{jiij}.$$

$$\gamma_{ijii} = \gamma_{jiii}$$

$$\gamma_{jjij} = \gamma_{jjji}$$

Therefore,

$$(p_4+p_5)_2=\sum_{1\le j<i\le m}\begin{pmatrix}-2\gamma_{ijji}[x_i,x_j,y_j]\cdot y_i+\gamma_{jjii}[y_j,x_j,x_i]\cdot y_i\\ +\gamma_{ijii}[y_i,x_j,x_i]\cdot y_i+\gamma_{jjji}[y_j,x_j,x_i]\cdot y_j\\ +\gamma_{jjji}[y_j,x_j,x_j]\cdot y_i+\gamma_{ijii}[y_j,x_i,x_i]\cdot y_i\\ +\gamma_{ijji}[y_i,x_j,x_j]\cdot y_i+\gamma_{ijji}[y_j,x_i,x_i]\cdot y_j\end{pmatrix}$$

$$=\sum_{1\le j<i\le m}\begin{pmatrix}\gamma_{ijji}\begin{pmatrix}-2[x_i,x_j,y_j]\cdot y_i+[y_i,x_j,x_j]\cdot y_i\\ +[y_j,x_i,x_i]\cdot y_j\end{pmatrix}\\ +\gamma_{jjii}[y_j,x_j,x_i]\cdot y_i\\ +\gamma_{ijii}([y_i,x_j,x_i]\cdot y_i+[y_j,x_i,x_i]\cdot y_i)\\ +\gamma_{jjji}([y_j,x_j,x_i]\cdot y_j+[y_j,x_j,x_j]\cdot y_i)\end{pmatrix}$$

Therefore,

$$[y_j,x_j,x_i]\cdot y_i,$$
$$[y_i,x_j,x_i]\cdot y_i+[y_j,x_i,x_i]\cdot y_i,$$
$$[y_j,x_j,x_i]\cdot y_j+[y_j,x_j,x_j]\cdot y_i,$$
$$-2[x_i,x_j,y_j]\cdot y_i+[y_i,x_j,x_j]\cdot y_i+[y_j,x_i,x_i]\cdot y_j$$

are constant elements in $KA_4\oplus KA_5$ where $1\le j<i\le m$.

For

$0=\delta_{2m}((p_4+p_5)_3)$

$$0=\delta_{2m}\left(\sum_{1\le j<i<k\le m}\begin{pmatrix}\varepsilon_{ijik}[x_i,x_j,y_i]\cdot y_k+\varepsilon_{kjik}[x_k,x_j,y_i]\cdot y_k\\ +\varepsilon_{ijjk}[x_i,x_j,y_j]\cdot y_k+\varepsilon_{kjji}[x_k,x_j,y_j]\cdot y_i\\ +\varepsilon_{kijk}[x_k,x_i,y_j]\cdot y_k+\varepsilon_{kiji}[x_k,x_i,y_j]\cdot y_i\\ +\gamma_{ijik}[y_i,x_j,x_i]\cdot y_k+\gamma_{ijkk}[y_i,x_j,x_k]\cdot y_k\\ +\gamma_{jjik}[y_j,x_j,x_i]\cdot y_k+\gamma_{jjki}[y_j,x_j,x_k]\cdot y_i\\ +\gamma_{jikk}[y_j,x_i,x_k]\cdot y_k+\gamma_{jiki}[y_j,x_i,x_k]\cdot y_i\\ +\gamma_{kjik}[y_k,x_j,x_i]\cdot y_k+\gamma_{ijki}[y_i,x_j,x_k]\cdot y_i\\ +\gamma_{jikj}[y_j,x_i,x_k]\cdot y_j+\gamma_{ijjk}[y_i,x_j,x_j]\cdot y_k\\ +\gamma_{jiik}[y_j,x_i,x_i]\cdot y_k+\gamma_{jkki}[y_j,x_k,x_k]\cdot y_i\end{pmatrix}\right)$$

$$
= \sum_{1\le j<i<k\le m} \begin{pmatrix}
\varepsilon_{ijik}\left(\left[x_i, x_j, \delta_{2m}(y_i)\right]\cdot y_k + \left[x_i, x_j, y_i\right]\cdot \delta_{2m}(y_k)\right) \\
+\varepsilon_{kjik}\left(\left[x_k, x_j, \delta_{2m}(y_i)\right]\cdot y_k + \left[x_k, x_j, y_i\right]\cdot \delta_{2m}(y_k)\right) \\
+\varepsilon_{ijjk}\left(\left[x_i, x_j, \delta_{2m}(y_j)\right]\cdot y_k + \left[x_i, x_j, y_j\right]\cdot \delta_{2m}(y_k)\right) \\
+\varepsilon_{kjji}\left(\left[x_k, x_j, \delta_{2m}(y_j)\right]\cdot y_i + \left[x_k, x_j, y_j\right]\cdot \delta_{2m}(y_i)\right) \\
+\varepsilon_{kijk}\left(\left[x_k, x_i, \delta_{2m}(y_j)\right]\cdot y_k + \left[x_k, x_i, y_j\right]\cdot \delta_{2m}(y_k)\right) \\
+\varepsilon_{kiji}\left(\left[x_k, x_i, \delta_{2m}(y_j)\right]\cdot y_i + \left[x_k, x_i, y_j\right]\cdot \delta_{2m}(y_i)\right) \\
+\gamma_{ijik}\left(\left[\delta_{2m}(y_i), x_j, x_i\right]\cdot y_k + \left[y_i, x_j, x_i\right]\cdot \delta_{2m}(y_k)\right) \\
+\gamma_{ijkk}\left(\left[\delta_{2m}(y_i), x_j, x_k\right]\cdot y_k + \left[y_i, x_j, x_k\right]\cdot \delta_{2m}(y_k)\right) \\
+\gamma_{jjik}\left(\left[\delta_{2m}(y_j), x_j, x_i\right]\cdot y_k + \left[y_j, x_j, x_i\right]\cdot \delta_{2m}(y_k)\right) \\
+\gamma_{jjki}\left(\left[\delta_{2m}(y_j), x_j, x_k\right]\cdot y_i + \left[y_j, x_j, x_k\right]\cdot \delta_{2m}(y_i)\right) \\
+\gamma_{jikk}\left(\left[\delta_{2m}(y_j), x_i, x_k\right]\cdot y_k + \left[y_j, x_i, x_k\right]\cdot \delta_{2m}(y_k)\right) \\
+\gamma_{jiki}\left(\left[\delta_{2m}(y_j), x_i, x_k\right]\cdot y_i + \left[y_j, x_i, x_k\right]\cdot \delta_{2m}(y_i)\right) \\
+\gamma_{kjik}\left(\left[\delta_{2m}(y_k), x_j, x_i\right]\cdot y_k + \left[y_k, x_j, x_i\right]\cdot \delta_{2m}(y_k)\right) \\
+\gamma_{ijki}\left(\left[\delta_{2m}(y_i), x_j, x_k\right]\cdot y_i + \left[y_i, x_j, x_k\right]\cdot \delta_{2m}(y_i)\right) \\
+\gamma_{jikj}\left(\left[\delta_{2m}(y_j), x_i, x_k\right]\cdot y_j + \left[y_j, x_i, x_k\right]\cdot \delta_{2m}(y_j)\right) \\
+\gamma_{ijjk}\left(\left[\delta_{2m}(y_i), x_j, x_j\right]\cdot y_k + \left[y_i, x_j, x_j\right]\cdot \delta_{2m}(y_k)\right) \\
+\gamma_{jiik}\left(\left[\delta_{2m}(y_j), x_i, x_i\right]\cdot y_k + \left[y_j, x_i, x_i\right]\cdot \delta_{2m}(y_k)\right) \\
+\gamma_{jkki}\left(\left[\delta_{2m}(y_j), x_k, x_k\right]\cdot y_i + \left[y_j, x_k, x_k\right]\cdot \delta_{2m}(y_i)\right)
\end{pmatrix}
$$

$$= \sum_{1\le j<i<k\le m} \begin{pmatrix} \varepsilon_{ijik}\left([x_i, x_j, x_i]\cdot y_k + [x_i, x_j, y_i]\cdot x_k\right) \\ +\varepsilon_{kjik}\left([x_k, x_j, x_i]\cdot y_k + [x_k, x_j, y_i]\cdot x_k\right) \\ +\varepsilon_{ijjk}\left([x_i, x_j, x_j]\cdot y_k + [x_i, x_j, y_j]\cdot x_k\right) \\ +\varepsilon_{kjji}\left([x_k, x_j, x_j]\cdot y_i + [x_k, x_j, y_j]\cdot x_i\right) \\ +\varepsilon_{kijk}\left([x_k, x_i, x_j]\cdot y_k + [x_k, x_i, y_j]\cdot x_k\right) \\ +\varepsilon_{kiji}\left([x_k, x_i, x_j]\cdot y_i + [x_k, x_i, y_j]\cdot x_i\right) \\ +\gamma_{ijik}\left([x_i, x_j, x_i]\cdot y_k + [y_i, x_j, x_i]\cdot x_k\right) \\ +\gamma_{ijkk}\left([x_i, x_j, x_k]\cdot y_k + [y_i, x_j, x_k]\cdot x_k\right) \\ +\gamma_{jjik}\left([x_j, x_j, x_i]\cdot y_k + [y_j, x_j, x_i]\cdot x_k\right) \\ +\gamma_{jjki}\left([x_j, x_j, x_k]\cdot y_i + [y_j, x_j, x_k]\cdot x_i\right) \\ +\gamma_{jikk}\left([x_j, x_i, x_k]\cdot y_k + [y_j, x_i, x_k]\cdot x_k\right) \\ +\gamma_{jiki}\left([x_j, x_i, x_k]\cdot y_i + [y_j, x_i, x_k]\cdot x_i\right) \\ +\gamma_{kjik}\left([x_k, x_j, x_i]\cdot y_k + [y_k, x_j, x_i]\cdot x_k\right) \\ +\gamma_{ijki}\left([x_i, x_j, x_k]\cdot y_i + [y_i, x_j, x_k]\cdot x_i\right) \\ +\gamma_{jikj}\left([x_j, x_i, x_k]\cdot y_j + [y_j, x_i, x_k]\cdot x_j\right) \\ +\gamma_{ijjk}\left([x_i, x_j, x_j]\cdot y_k + [y_i, x_j, x_j]\cdot x_k\right) \\ +\gamma_{jiik}\left([x_j, x_i, x_i]\cdot y_k + [y_j, x_i, x_i]\cdot x_k\right) \\ +\gamma_{jkki}\left([x_j, x_k, x_k]\cdot y_i + [y_j, x_k, x_k]\cdot x_i\right) \end{pmatrix}$$

We rearrange the basis by using the conditions in Remark 2.10.3, we get that

$$0=\sum_{1\le j<i<k\le m}\begin{pmatrix}
\varepsilon_{ijik}([x_i,x_j,x_i]\cdot y_k-[x_i,x_j,x_k]\cdot y_i)\\
+\varepsilon_{kjik}([x_k,x_j,x_i]\cdot y_k-[x_k,x_j,x_k]\cdot y_i)\\
+\varepsilon_{ijjk}([x_i,x_j,x_j]\cdot y_k-[x_i,x_j,x_k]\cdot y_j)\\
+\varepsilon_{kjji}([x_k,x_j,x_j]\cdot y_i-[x_k,x_j,x_i]\cdot y_j)\\
+\varepsilon_{kijk}\begin{pmatrix}-[x_i,x_j,x_k]\cdot y_k+[x_k,x_j,x_i]\cdot y_k\\ -[x_k,x_i,x_k]\cdot y_j\end{pmatrix}\\
+\varepsilon_{kiji}\begin{pmatrix}-[x_i,x_j,x_k]\cdot y_i+[x_k,x_j,x_i]\cdot y_i\\ -[x_k,x_i,x_i]\cdot y_j\end{pmatrix}\\
+\gamma_{ijik}\begin{pmatrix}[x_i,x_j,x_i]\cdot y_k+[x_k,x_j,x_i]\cdot y_i\\ -[x_i,x_j,x_k]\cdot y_i\end{pmatrix}\\
+\gamma_{ijkk}[x_i,x_j,x_k]\cdot y_k\\
+\gamma_{jjik}([x_k,x_j,x_i]\cdot y_k-[x_i,x_j,x_k]\cdot y_j)\\
+\gamma_{jjki}([x_i,x_j,x_k]\cdot y_j-[x_k,x_j,x_i]\cdot y_j)\\
-\gamma_{jikk}[x_i,x_j,x_k]\cdot y_k\\
+\gamma_{jiki}(-[x_i,x_j,x_k]\cdot y_i-[x_k,x_i,x_i]\cdot y_j)\\
+\gamma_{kjik}(2[x_k,x_j,x_i]\cdot y_k-[x_i,x_j,x_k]\cdot y_k)\\
+\gamma_{ijki}(2[x_i,x_j,x_k]\cdot y_i-[x_k,x_j,x_i]\cdot y_i)\\
+\gamma_{jikj}(-[x_i,x_j,x_k]\cdot y_j-[x_k,x_j,x_i]\cdot y_j)\\
+\gamma_{ijjk}([x_i,x_j,x_j]\cdot y_k+[x_k,x_j,x_j]\cdot y_i)\\
+\gamma_{jiik}(-[x_i,x_j,x_i]\cdot y_k+[x_k,x_i,x_i]\cdot y_j)\\
+\gamma_{jkki}(-[x_k,x_j,x_k]\cdot y_i-[x_k,x_i,x_k]\cdot y_j)
\end{pmatrix}$$

$$= \sum_{1\le j<i<k\le m} \left(\begin{array}{c} (\varepsilon_{ijik}+\gamma_{ijik}-\gamma_{jiik})[x_i,x_j,x_i]\cdot y_k \\ +(-\varepsilon_{ijik}-\varepsilon_{kiji}-\gamma_{ijik}-\gamma_{jiki}+2\gamma_{ijki})[x_i,x_j,x_k]\cdot y_i \\ +(\varepsilon_{kjik}+\varepsilon_{kijk}+\gamma_{jjik}+2\gamma_{kjik})[x_k,x_j,x_i]\cdot y_k \\ +(-\varepsilon_{kjik}-\gamma_{jkki})[x_k,x_j,x_k]\cdot y_i \\ +(\varepsilon_{ijjk}+\gamma_{ijjk})[x_i,x_j,x_j]\cdot y_k \\ +(-\varepsilon_{ijjk}-\gamma_{jjik}+\gamma_{jjki}-\gamma_{jikj})[x_i,x_j,x_k]\cdot y_j \\ +(\varepsilon_{kjji}+\gamma_{ijjk})[x_k,x_j,x_j]\cdot y_i \\ +(-\varepsilon_{kjji}-\gamma_{jjki}-\gamma_{jikj})[x_k,x_j,x_i]\cdot y_j \\ +(-\varepsilon_{kijk}+\gamma_{ijkk}-\gamma_{jikk}-\gamma_{kjik})[x_i,x_j,x_k]\cdot y_k \\ +(-\varepsilon_{kijk}-\gamma_{jkki})[x_k,x_i,x_k]\cdot y_j \\ +(\varepsilon_{kiji}+\gamma_{ijik}-\gamma_{ijki})[x_k,x_j,x_i]\cdot y_i \\ +(-\varepsilon_{kiji}-\gamma_{jiki}+\gamma_{jiik})[x_k,x_i,x_i]\cdot y_j \end{array} \right)$$

So, we have

$$\varepsilon_{ijik}+\gamma_{ijik}-\gamma_{jiik}=0$$

$$-\varepsilon_{ijik}-\varepsilon_{kiji}-\gamma_{ijik}-\gamma_{jiki}+2\gamma_{ijki}=0$$

$$\varepsilon_{kiji}+\gamma_{ijik}-\gamma_{ijki}=0$$

$$-\varepsilon_{kiji}-\gamma_{jiki}+\gamma_{jiik}=0$$

$$\varepsilon_{kjik}+\varepsilon_{kijk}+\gamma_{jjik}+2\gamma_{kjik}=0$$

$$-\varepsilon_{kjik}-\gamma_{jkki}=0$$

$$-\varepsilon_{kijk}+\gamma_{ijkk}-\gamma_{jikk}-\gamma_{kjik}$$

$$-\varepsilon_{kijk}-\gamma_{jkki}=0$$

$$\varepsilon_{ijjk}+\gamma_{ijjk}=0$$

$$\varepsilon_{ijjk}-\gamma_{jjik}+\gamma_{jjki}-\gamma_{jikj}=0$$

$$\varepsilon_{kjji}+\gamma_{ijjk}=0$$

$$-\varepsilon_{kjji}-\gamma_{jjki}-\gamma_{jikj}=0$$

After computation we have

$\varepsilon_{ijik}=\gamma_{ijki}-\gamma_{jiki}=\varepsilon_{kiji}, \gamma_{ijik}=\gamma_{jiki}, \gamma_{jiik}=\gamma_{ijki}$

$\varepsilon_{kjik} = \varepsilon_{kijk} = -\gamma_{jkki} = -\gamma_{kjik}, \gamma_{ijkk} = \gamma_{jikk}$

$\varepsilon_{ijjk} = \varepsilon_{kjji} = -\gamma_{ijjk} = -\gamma_{jikj}, \gamma_{jjik} = \gamma_{jjki}$

Therefore,

$$(p_4+p_5)_3 = \sum_{1\le j<i<k\le m} \begin{pmatrix} (\gamma_{ijki}-\gamma_{jiki})[x_i,x_j,y_i]\cdot y_k + \varepsilon_{kjik}[x_k,x_j,y_i]\cdot y_k \\ +\varepsilon_{ijjk}[x_i,x_j,y_j]\cdot y_k + \varepsilon_{ijjk}[x_k,x_j,y_j]\cdot y_i \\ +\varepsilon_{kjik}[x_k,x_i,y_j]\cdot y_k + (\gamma_{ijki}-\gamma_{jiki})[x_k,x_i,y_j]\cdot y_i \\ +\gamma_{jiki}[y_i,x_j,x_i]\cdot y_k + \gamma_{ijkk}[y_i,x_j,x_k]\cdot y_k \\ +\gamma_{jjik}[y_j,x_j,x_i]\cdot y_k + \gamma_{jjki}[y_j,x_j,x_k]\cdot y_i \\ +\gamma_{ijkk}[y_j,x_i,x_k]\cdot y_k + \gamma_{jiki}[y_j,x_i,x_k]\cdot y_i \\ -\varepsilon_{kjik}[y_k,x_j,x_i]\cdot y_k + \gamma_{ijki}[y_i,x_j,x_k]\cdot y_i \\ -\varepsilon_{ijjk}[y_j,x_i,x_k]\cdot y_j - \varepsilon_{ijjk}[y_i,x_j,x_j]\cdot y_k \\ +\gamma_{ijki}[y_j,x_i,x_i]\cdot y_k - \varepsilon_{kjik}[y_j,x_k,x_k]\cdot y_i \end{pmatrix}$$

$$= \sum_{1\le j<i<k\le m} \begin{pmatrix} \gamma_{ijki}\begin{pmatrix} [x_i,x_j,y_i]\cdot y_k + [x_k,x_i,y_j]\cdot y_i + [y_i,x_j,x_k]\cdot y_i \\ +\gamma_{ijki}[y_j,x_i,x_i]\cdot y_k \end{pmatrix} \\ +\gamma_{jiki}\begin{pmatrix} -[x_i,x_j,y_i]\cdot y_k - [x_k,x_i,y_j]\cdot y_i + [y_i,x_j,x_i]\cdot y_k \\ +[y_j,x_i,x_k]\cdot y_i \end{pmatrix} \\ +\varepsilon_{kjik}\begin{pmatrix} [x_k,x_j,y_i]\cdot y_k + [x_k,x_i,y_j]\cdot y_k - [y_k,x_j,x_i]\cdot y_k \\ -[y_j,x_k,x_k]\cdot y_i \end{pmatrix} \\ +\gamma_{ijkk}([y_i,x_j,x_k]\cdot y_k + [y_j,x_i,x_k]\cdot y_k) \\ +\varepsilon_{ijjk}\begin{pmatrix} [x_i,x_j,y_j]\cdot y_k + [x_k,x_j,y_j]\cdot y_i - [y_j,x_i,x_k]\cdot y_j \\ -[y_i,x_j,x_j]\cdot y_k \end{pmatrix} \\ +\gamma_{jjik}([y_j,x_j,x_i]\cdot y_k + [y_j,x_j,x_k]\cdot y_i) \end{pmatrix}$$

Therefore,

$$[y_i,x_j,x_k]\cdot y_k + [y_j,x_i,x_k]\cdot y_k,$$

$$[y_j,x_j,x_i]\cdot y_k + [y_j,x_j,x_k]\cdot y_i,$$

$$[x_i,x_j,y_j]\cdot y_k + [x_k,x_j,y_j]\cdot y_i - [y_j,x_i,x_k]\cdot y_j - [y_i,x_j,x_j]\cdot y_k,$$

$$[x_k,x_j,y_i]\cdot y_k + [x_k,x_i,y_j]\cdot y_k - [y_k,x_j,x_i]\cdot y_k - [y_j,x_k,x_k]\cdot y_i,$$

$$[x_i,x_j,y_i]\cdot y_k + [x_k,x_i,y_j]\cdot y_i + [y_i,x_j,x_k]\cdot y_i + \gamma_{ijki}[y_j,x_i,x_i]\cdot y_k,$$

$$[x_i,x_j,y_i]\cdot y_k-[x_k,x_i,y_j]\cdot y_i+[y_i,x_j,x_i]\cdot y_k+[y_j,x_i,x_k]\cdot y_i,$$

are constant elements in $KA_4\oplus KA_5$ where $1\le j<i<k\le m$.

For

$$0=\delta_{2m}((p_4+p_5)_4)$$

$$=\delta_{2m}\left(\sum_{1\le j<i<k<l\le m}\begin{pmatrix}\varepsilon_{ijkl}[x_i,x_j,y_k]\cdot y_l+\varepsilon_{kjil}[x_k,x_j,y_i]\cdot y_l\\ +\varepsilon_{ljik}[x_l,x_j,y_i]\cdot y_k+\varepsilon_{kijl}[x_k,x_i,y_j]\cdot y_l\\ +\varepsilon_{lijk}[x_l,x_i,y_j]\cdot y_k+\varepsilon_{lkji}[x_l,x_k,y_j]\cdot y_i\\ +\gamma_{kjil}[y_k,x_j,x_i]\cdot y_l+\gamma_{ijkl}[y_i,x_j,x_k]\cdot y_l\\ +\gamma_{ijlk}[y_i,x_j,x_l]\cdot y_k+\gamma_{jikl}[y_j,x_i,x_k]\cdot y_l\\ +\gamma_{jilk}[y_j,x_i,x_l]\cdot y_k+\gamma_{jkli}[y_j,x_k,x_l]\cdot y_i\end{pmatrix}\right)$$

$$=\sum_{1\le j<i<k<l\le m}\begin{pmatrix}\varepsilon_{ijkl}\left([x_i,x_j,\delta_{2m}(y_k)]\cdot y_l+[x_i,x_j,y_k]\cdot\delta_{2m}(y_l)\right)\\ +\varepsilon_{kjil}\left([x_k,x_j,\delta_{2m}(y_i)]\cdot y_l+[x_k,x_j,y_i]\cdot\delta_{2m}(y_l)\right)\\ +\varepsilon_{ljik}\left([x_l,x_j,\delta_{2m}(y_i)]\cdot y_k+[x_l,x_j,y_i]\cdot\delta_{2m}(y_k)\right)\\ +\varepsilon_{kijl}\left([x_k,x_i,\delta_{2m}(y_j)]\cdot y_l+[x_k,x_i,y_j]\cdot\delta_{2m}(y_l)\right)\\ +\varepsilon_{lijk}\left([x_l,x_i,\delta_{2m}(y_j)]\cdot y_k+[x_l,x_i,y_j]\cdot\delta_{2m}(y_k)\right)\\ +\varepsilon_{lkji}\left([x_l,x_k,\delta_{2m}(y_j)]\cdot y_i+[x_l,x_k,y_j]\cdot\delta_{2m}(y_i)\right)\\ +\gamma_{kjil}\left([\delta_{2m}(y_k),x_j,x_i]\cdot y_l+[y_k,x_j,x_i]\cdot\delta_{2m}(y_l)\right)\\ +\gamma_{ijkl}\left([\delta_{2m}(y_i),x_j,x_k]\cdot y_l+[y_i,x_j,x_k]\cdot\delta_{2m}(y_l)\right)\\ +\gamma_{ijlk}\left([\delta_{2m}(y_i),x_j,x_l]\cdot y_k+[y_i,x_j,x_l]\cdot\delta_{2m}(y_k)\right)\\ +\gamma_{jikl}\left([\delta_{2m}(y_j),x_i,x_k]\cdot y_l+[y_j,x_i,x_k]\cdot\delta_{2m}(y_l)\right)\\ +\gamma_{jilk}\left([\delta_{2m}(y_j),x_i,x_l]\cdot y_k+[y_j,x_i,x_l]\cdot\delta_{2m}(y_k)\right)\\ +\gamma_{jkli}\left([\delta_{2m}(y_j),x_k,x_l]\cdot y_i+[y_j,x_k,x_l]\cdot\delta_{2m}(y_i)\right)\end{pmatrix}$$

$$= \sum_{1\le j<i<k<l\le m} \begin{pmatrix} \varepsilon_{ijkl}\left(\left[x_i,x_j,x_k\right]\cdot y_l + \left[x_i,x_j,y_k\right]\cdot x_l\right) \\ +\varepsilon_{kjil}\left(\left[x_k,x_j,x_i\right]\cdot y_l + \left[x_k,x_j,y_i\right]\cdot x_l\right) \\ +\varepsilon_{ljik}\left(\left[x_l,x_j,x_i\right]\cdot y_k + \left[x_l,x_j,y_i\right]\cdot x_k\right) \\ +\varepsilon_{kijl}\left(\left[x_k,x_i,x_j\right]\cdot y_l + \left[x_k,x_i,y_j\right]\cdot x_l\right) \\ +\varepsilon_{lijk}\left(\left[x_l,x_i,x_j\right]\cdot y_k + \left[x_l,x_i,y_j\right]\cdot x_k\right) \\ +\varepsilon_{lkji}\left(\left[x_l,x_k,x_j\right]\cdot y_i + \left[x_l,x_k,y_j\right]\cdot x_i\right) \\ +\gamma_{kjil}\left(\left[x_k,x_j,x_i\right]\cdot y_l + \left[y_k,x_j,x_i\right]\cdot x_l\right) \\ +\gamma_{ijkl}\left(\left[x_i,x_j,x_k\right]\cdot y_l + \left[y_i,x_j,x_k\right]\cdot x_l\right) \\ +\gamma_{ijlk}\left(\left[x_i,x_j,x_l\right]\cdot y_k + \left[y_i,x_j,x_l\right]\cdot x_k\right) \\ +\gamma_{jikl}\left(\left[x_j,x_i,x_k\right]\cdot y_l + \left[y_j,x_i,x_k\right]\cdot x_l\right) \\ +\gamma_{jilk}\left(\left[x_j,x_i,x_l\right]\cdot y_k + \left[y_j,x_i,x_l\right]\cdot x_k\right) \\ +\gamma_{jkli}\left(\left[x_j,x_k,x_l\right]\cdot y_i + \left[y_j,x_k,x_l\right]\cdot x_i\right) \end{pmatrix}$$

We rearrange the basis by using the condition in Remark 2.10.3, we obtain that

$$0 = \sum_{1\le j<i<k<l\le m} \begin{pmatrix} \varepsilon_{ijkl}\left(\left[x_i,x_j,x_k\right]\cdot y_l - \left[x_i,x_j,x_l\right]\cdot y_k\right) \\ +\varepsilon_{kjil}\left(\left[x_k,x_j,x_i\right]\cdot y_l - \left[x_k,x_j,x_l\right]\cdot y_i\right) \\ +\varepsilon_{ljik}\left(\left[x_l,x_j,x_i\right]\cdot y_k - \left[x_l,x_j,x_k\right]\cdot y_i\right) \\ +\varepsilon_{kijl}\left(-\left[x_i,x_j,x_k\right]\cdot y_l + \left[x_k,x_j,x_i\right]\cdot y_l - \left[x_k,x_i,x_l\right]\cdot y_j\right) \\ +\varepsilon_{lijk}\left(-\left[x_i,x_j,x_l\right]\cdot y_k + \left[x_l,x_j,x_i\right]\cdot y_k - \left[x_l,x_i,x_k\right]\cdot y_j\right) \\ +\varepsilon_{lkji}\begin{pmatrix} -\left[x_k,x_j,x_l\right]\cdot y_i + \left[x_l,x_j,x_k\right]\cdot y_i + \left[x_k,x_i,x_l\right]\cdot y_j \\ -\left[x_l,x_i,x_k\right]\cdot y_j \end{pmatrix} \\ +\gamma_{kjil}\left(\left[x_k,x_j,x_i\right]\cdot y_l + \left[x_l,x_j,x_i\right]\cdot y_k - \left[x_i,x_j,x_l\right]\cdot y_k\right) \\ +\gamma_{ijkl}\left(\left[x_i,x_j,x_k\right]\cdot y_l + \left[x_l,x_j,x_k\right]\cdot y_i - \left[x_k,x_j,x_l\right]\cdot y_i\right) \\ +\gamma_{ijlk}\left(\left[x_i,x_j,x_l\right]\cdot y_k - \left[x_l,x_j,x_k\right]\cdot y_i + \left[x_k,x_j,x_l\right]\cdot y_i\right) \\ +\gamma_{jikl}\left(-\left[x_i,x_j,x_k\right]\cdot y_l + \left[x_l,x_i,x_k\right]\cdot y_j - \left[x_k,x_i,x_l\right]\cdot y_j\right) \\ +\gamma_{jilk}\left(-\left[x_i,x_j,x_l\right]\cdot y_k - \left[x_l,x_i,x_k\right]\cdot y_j + \left[x_k,x_i,x_l\right]\cdot y_j\right) \\ +\gamma_{jkli}\left(-\left[x_k,x_j,x_l\right]\cdot y_i - \left[x_l,x_i,x_k\right]\cdot y_j\right) \end{pmatrix}$$

$$= \sum_{1\le j<i<k<l\le m} \begin{pmatrix} (\varepsilon_{ijkl} - \varepsilon_{kijl} + \gamma_{ijkl} - \gamma_{jikl})[x_i, x_j, x_k] \cdot y_l \\ +(-\varepsilon_{ijkl} - \varepsilon_{lijk} - \gamma_{kjil} + \gamma_{ijlk} - \gamma_{jilk})[x_i, x_j, x_l] \cdot y_k \\ +(\varepsilon_{kjil} + \varepsilon_{kijl} + \gamma_{kjil})[x_k, x_j, x_i] \cdot y_l \\ +(-\varepsilon_{kjil} - \varepsilon_{lkji} - \gamma_{ijkl} + \gamma_{ijlk} - \gamma_{jkli})[x_k, x_j, x_l] \cdot y_i \\ +(\varepsilon_{ljik} + \varepsilon_{lijk} + \gamma_{kjil})[x_l, x_j, x_i] \cdot y_k \\ +(-\varepsilon_{ljik} + \varepsilon_{lkji} + \gamma_{ijkl} - \gamma_{ijlk})[x_l, x_j, x_k] \cdot y_i \\ +(-\varepsilon_{kijl} + \varepsilon_{lkji} - \gamma_{jikl} + \gamma_{jilk})[x_k, x_i, x_l] \cdot y_j \\ +(-\varepsilon_{lijk} - \varepsilon_{lkji} + \gamma_{jikl} - \gamma_{jkli})[x_l, x_i, x_k] \cdot y_j \end{pmatrix}$$

Now we have

$$\varepsilon_{ijkl} - \varepsilon_{kijl} + \gamma_{ijkl} - \gamma_{jikl} = 0$$

$$-\varepsilon_{ijkl} - \varepsilon_{lijk} - \gamma_{kjil} + \gamma_{ijlk} - \gamma_{jilk} = 0$$

$$\varepsilon_{kjil} + \varepsilon_{kijl} + \gamma_{kjil} = 0$$

$$\varepsilon_{kjil} - \varepsilon_{lkji} - \gamma_{ijkl} + \gamma_{ijlk} - \gamma_{jkli} = 0$$

$$\varepsilon_{ljik} + \varepsilon_{lijk} + \gamma_{kjil} = 0$$

$$-\varepsilon_{ljik} + \varepsilon_{lkji} + \gamma_{ijkl} - \gamma_{ijlk} = 0$$

$$-\varepsilon_{kijl} + \varepsilon_{lkji} - \gamma_{jikl} + \gamma_{jilk} = 0$$

$$-\varepsilon_{lijk} - \varepsilon_{lkji} + \gamma_{jikl} - \gamma_{jkli} = 0$$

After computation we have

$\varepsilon_{ijkl} = \varepsilon_{lkji}, \varepsilon_{kijl} = \varepsilon_{ljik}, \gamma_{ijkl} = \gamma_{jilk}, \gamma_{jikl} = \gamma_{jilk} + \varepsilon_{lkji} - \varepsilon_{ljik} = \gamma_{ijlk},$

$\varepsilon_{lijk} = -\gamma_{jkli} - \varepsilon_{ljik} = \varepsilon_{kjil}, \gamma_{kjil} = \gamma_{jkli}$

Hence,

$$(p_4 + p_5)_4 = \sum_{1\le j<i<k<l\le m} \begin{pmatrix} \varepsilon_{lkji}[x_i, x_j, y_k] \cdot y_l + (-\gamma_{jkli} - \varepsilon_{ljik})[x_k, x_j, y_i] \cdot y_l \\ +\varepsilon_{ljik}[x_l, x_j, y_i] \cdot y_k + \varepsilon_{ljik}[x_k, x_i, y_j] \cdot y_l \\ +(-\gamma_{jkli} - \varepsilon_{ljik})[x_l, x_i, y_j] \cdot y_k \\ +\varepsilon_{lkji}[x_l, x_k, y_j] \cdot y_i + \gamma_{jkli}[y_k, x_j, x_i] \cdot y_l \\ +\gamma_{jilk}[y_i, x_j, x_k] \cdot y_l \\ +(\gamma_{jilk} + \varepsilon_{lkji} - \varepsilon_{ljik})[y_i, x_j, x_l] \cdot y_k \\ +(\gamma_{jilk} + \varepsilon_{lkji} - \varepsilon_{ljik})[y_j, x_i, x_k] \cdot y_l \\ +\gamma_{jilk}[y_j, x_i, x_l] \cdot y_k + \gamma_{jkli}[y_j, x_k, x_l] \cdot y_i \end{pmatrix}$$

$$= \sum_{1\le j<i<k<l\le m} \left(\begin{array}{l} \varepsilon_{lkji} \begin{pmatrix} [x_i, x_j, y_k] \cdot y_l + [x_l, x_k, y_j] \cdot y_i \\ +[y_i, x_j, x_l] \cdot y_k + [y_j, x_i, x_k] \cdot y_l \end{pmatrix} \\ +\gamma_{jkli} \begin{pmatrix} -[x_k, x_j, y_i] \cdot y_l - [x_l, x_i, y_j] \cdot y_k \\ +[y_k, x_j, x_i] \cdot y_l + [y_j, x_k, x_l] \cdot y_i \end{pmatrix} \\ +\varepsilon_{ljik} \begin{pmatrix} -[x_k, x_j, y_i] \cdot y_l + [x_l, x_j, y_i] \cdot y_k \\ +[x_k, x_i, y_j] \cdot y_l - [x_l, x_i, y_j] \cdot y_k \\ -[y_i, x_j, x_l] \cdot y_k - [y_j, x_i, x_k] \cdot y_l \end{pmatrix} \\ +\gamma_{jilk} \begin{pmatrix} [y_i, x_j, x_k] \cdot y_l + [y_i, x_j, x_l] \cdot y_k \\ +[y_j, x_i, x_k] \cdot y_l + [y_j, x_i, x_l] \cdot y_k \end{pmatrix} \end{array} \right)$$

Therefore,

$$[x_i, x_j, y_k] \cdot y_l + [x_l, x_k, y_j] \cdot y_i + [y_i, x_j, x_l] \cdot y_k + [y_j, x_i, x_k] \cdot y_l,$$

$$-[x_k, x_j, y_i] \cdot y_l - [x_l, x_i, y_j] \cdot y_k + [y_k, x_j, x_i] \cdot y_l + [y_j, x_k, x_l] \cdot y_i,$$

$$[y_i, x_j, x_k] \cdot y_l + [y_i, x_j, x_l] \cdot y_k + [y_j, x_i, x_k] \cdot y_l + [y_j, x_i, x_l] \cdot y_k,$$

$$\begin{pmatrix} -[x_k, x_j, y_i] \cdot y_l + [x_l, x_j, y_i] \cdot y_k + [x_k, x_i, y_j] \cdot y_l - [x_l, x_i, y_j] \cdot y_k \\ -[y_i, x_j, x_l] \cdot y_k - [y_j, x_i, x_k] \cdot y_l \end{pmatrix}$$

are constant elements in $KA_4 \oplus KA_5$ where $1 \le j < i < k < l \le m$.

Therefore, this shows that the vector space $(KA_4 \oplus KA_5)^{\delta_{2m}}$ is spanned on the elements of the form

$$[y_i, x_i, x_i] \cdot y_i$$

where $1 \le i \le m$,

$$[y_j, x_j, x_i] \cdot y_i,$$

$$[y_i, x_j, x_i] \cdot y_i + [y_j, x_i, x_i] \cdot y_i,$$

$$[y_j, x_j, x_i] \cdot y_j + [y_j, x_j, x_j] \cdot y_i,$$

$$-2[x_i, x_j, y_j] \cdot y_i + [y_i, x_j, x_j] \cdot y_i + [y_j, x_i, x_i] \cdot y_j$$

where $1 \le j < i \le m$,

$$[y_i, x_j, x_k] \cdot y_k + [y_j, x_i, x_k] \cdot y_k,$$

$$[y_j, x_j, x_i] \cdot y_k + [y_j, x_j, x_k] \cdot y_i,$$

$$[x_i,x_j,y_j]\cdot y_k+[x_k,x_j,y_j]\cdot y_i-[y_j,x_i,x_k]\cdot y_j-[y_i,x_j,x_j]\cdot y_k,$$
$$[x_k,x_j,y_i]\cdot y_k+[x_k,x_i,y_j]\cdot y_k-[y_k,x_j,x_i]\cdot y_k-[y_j,x_k,x_k]\cdot y_i,$$
$$[x_i,x_j,y_i]\cdot y_k+[x_k,x_i,y_j]\cdot y_i+[y_i,x_j,x_k]\cdot y_i+\gamma_{ijki}[y_j,x_i,x_i]\cdot y_k,$$
$$[x_i,x_j,y_i]\cdot y_k-[x_k,x_i,y_j]\cdot y_i+[y_i,x_j,x_i]\cdot y_k+[y_j,x_i,x_k]\cdot y_i,$$

where $1\le j<i<k\le m$, and

$$[x_i,x_j,y_k]\cdot y_l+[x_l,x_k,y_j]\cdot y_i+[y_i,x_j,x_l]\cdot y_k+[y_j,x_i,x_k]\cdot y_l,$$
$$-[x_k,x_j,y_i]\cdot y_l-[x_l,x_i,y_j]\cdot y_k+[y_k,x_j,x_i]\cdot y_l+[y_j,x_k,x_l]\cdot y_i,$$
$$[y_i,x_j,x_k]\cdot y_l+[y_i,x_j,x_l]\cdot y_k+[y_j,x_i,x_k]\cdot y_l+[y_j,x_i,x_l]\cdot y_k,$$
$$\begin{pmatrix}-[x_k,x_j,y_i]\cdot y_l+[x_l,x_j,y_i]\cdot y_k+[x_k,x_i,y_j]\cdot y_l\\ -[x_l,x_i,y_j]\cdot y_k-[y_i,x_j,x_l]\cdot y_k-[y_j,x_i,x_k]\cdot y_l\end{pmatrix}$$

where $1\le j<i<k<l\le m$.

For p_6. Let

$$p_6=\sum_{\substack{1\le k<l\le m\\ 1\le i\le l\le m}}\varepsilon_{ijkl}\,[y_i,x_j,y_k]\cdot y_l$$

We have two conditions $1\le k<l\le m$ and $1\le i\le l\le m$ so, we can split the last one into $1\le i<l\le m$ and $\le i=l\le m$.

If we have $1\le k<l\le m$ and $1\le i<l\le m$ then the following are 19 different conditions $k<i<l<j$, $k<i<l=j$, $k<i<j<l$, $k<i=j<l$, $k<j<i<l$, $k=j<i<l$, $j<k<i<l$, $i=k<l<j$, $i=k<l=j$, $i=k<j<l$, $i=k=j<l$, $j<i=k<j$, $i<k<l<j$, $i<k<l=j$, $i<k<j<l$, $i<k=j<l$, $i<j<k<l$, $i=j<k<l$ and $j<i<k<l$.

If $1\le k<l\le m$ and $1\le i=l\le m$ then we have 5 different conditions which are $k<i=l<j$, $k<i=l=j$, $k<j<i=l$, $k=j<i=l$ and $j<k<i=l$.

So, p_6 become

$$p_6=\sum_{1\le k<i<l<j\le m}\varepsilon_{ijkl}\,[y_i,x_j,y_k]\cdot y_l+\sum_{1\le k<i<l=j\le m}\varepsilon_{ijkj}\,[y_i,x_j,y_k]\cdot y_j$$

$$+\sum_{1\le k<i<j<l\le m}\varepsilon_{ijkl}\left[y_i,x_j,y_k\right]\cdot y_l+\sum_{1\le k<i=j<l\le m}\varepsilon_{iikl}\left[y_i,x_i,y_k\right]\cdot y_l$$

$$+\sum_{1\le k<j<i<l\le m}\varepsilon_{ijkl}\left[y_i,x_j,y_k\right]\cdot y_l+\sum_{1\le k=j<i<l\le m}\varepsilon_{ijkl}\left[y_i,x_j,y_j\right]\cdot y_l$$

$$+\sum_{1\le j<k<i<l\le m}\varepsilon_{ijkl}\left[y_i,x_j,y_k\right]\cdot y_l+\sum_{1\le i=k<l<j\le m}\varepsilon_{ijil}\left[y_i,x_j,y_i\right]\cdot y_l$$

$$+\sum_{1\le i=k<l=j\le m}\varepsilon_{ijij}\left[y_i,x_j,y_i\right]\cdot y_j+\sum_{1\le i=k<j<l\le m}\varepsilon_{ijil}\left[y_i,x_j,y_i\right]\cdot y_l$$

$$+\sum_{1\le i=k=j<l\le m}\varepsilon_{iiil}\left[y_i,x_i,y_i\right]\cdot y_l+\sum_{1\le j<i=k<j\le m}\varepsilon_{ijil}\left[y_i,x_j,y_i\right]\cdot y_l$$

$$+\sum_{1\le i<k<l<j\le m}\varepsilon_{ijkl}\left[y_i,x_j,y_k\right]\cdot y_l+\sum_{1\le i<k<l=j\le m}\varepsilon_{ijkj}\left[y_i,x_j,y_k\right]\cdot y_j$$

$$+\sum_{1\le i<k<j<l\le m}\varepsilon_{ijkl}\left[y_i,x_j,y_k\right]\cdot y_l+\sum_{1\le i<k=j<l\le m}\varepsilon_{ijjl}\left[y_i,x_j,y_j\right]\cdot y_l$$

$$+\sum_{1\le i<j<k<l\le m}\varepsilon_{ijkl}\left[y_i,x_j,y_k\right]\cdot y_l+\sum_{1\le i=j<k<l\le m}\varepsilon_{iikl}\left[y_i,x_i,y_k\right]\cdot y_l$$

$$+\sum_{1\le j<i<k<l\le m}\varepsilon_{ijkl}\left[y_i,x_j,y_k\right]\cdot y_l+\sum_{1\le k<i=l<j\le m}\varepsilon_{ijki}\left[y_i,x_j,y_k\right]\cdot y_i$$

$$+\sum_{1\le k<i=l=j\le m}\varepsilon_{iiki}\left[y_i,x_i,y_k\right]\cdot y_i+\sum_{1\le k<j<i=l\le m}\varepsilon_{ijki}\left[y_i,x_j,y_k\right]\cdot y_i$$

$$+\sum_{1\le k=j<i=l\le m}\varepsilon_{ijji}\left[y_i,x_j,y_j\right]\cdot y_i+\sum_{1\le j<k<i=l\le m}\varepsilon_{ijki}\left[y_i,x_j,y_k\right]\cdot y_i$$

$$=\sum_{1\le j<i<k<l\le m}\varepsilon_{iljk}\left[y_i,x_l,y_j\right]\cdot y_k+\sum_{1\le j<i<k\le m}\varepsilon_{ikjk}\left[y_i,x_k,y_j\right]\cdot y_k$$

$$+\sum_{1\le j<i<k<l\le m}\varepsilon_{ikjl}\left[y_i,x_k,y_j\right]\cdot y_l+\sum_{1\le i<k<l\le m}\varepsilon_{kkil}\left[y_k,x_k,y_i\right]\cdot y_l$$

$$+\sum_{1\le j<i<k<l\le m}\varepsilon_{kijl}\left[y_i,x_j,y_k\right]\cdot y_l+\sum_{1\le j<i<l\le m}\varepsilon_{ijjl}\left[y_i,x_j,y_j\right]\cdot y_l$$

$$+\sum_{1\le j<i<k<l\le m}\varepsilon_{kjil}\left[y_k,x_j,y_i\right]\cdot y_l+\sum_{1\le j<i<l\le m}\varepsilon_{jlji}\left[y_j,x_l,y_j\right]\cdot y_i$$

$$+\sum_{1\le i<j\le m}\varepsilon_{ijij}\left[y_i,x_j,y_i\right]\cdot y_j+\sum_{1\le j<i<l\le m}\varepsilon_{jijl}\left[y_j,x_i,y_j\right]\cdot y_l$$

$$+\sum_{1\le i<l\le m}\varepsilon_{iiil}\left[y_i,x_i,y_i\right]\cdot y_l+\sum_{1\le j<i<l\le m}\varepsilon_{ijil}\left[y_i,x_j,y_i\right]\cdot y_l$$

$$+\sum_{1\le j<i<k<l\le m}\varepsilon_{jlik}\left[y_j,x_l,y_i\right]\cdot y_k+\sum_{1\le i<k<j\le m}\varepsilon_{ijkj}\left[y_i,x_j,y_k\right]\cdot y_j$$

$$+\sum_{1\le j<i<k<l\le m}\varepsilon_{jkil}\left[y_j,x_k,y_i\right]\cdot y_l+\sum_{1\le j<i<l\le m}\varepsilon_{jiil}\left[y_j,x_i,y_i\right]\cdot y_l$$

$$+\sum_{1\le j<i<k<l\le m}\varepsilon_{jikl}\left[y_j,x_i,y_k\right]\cdot y_l+\sum_{1\le i<k<l\le m}\varepsilon_{iikl}\left[y_i,x_i,y_k\right]\cdot y_l$$

$$+\sum_{1\le j<i<k<l\le m}\varepsilon_{ijkl}\left[y_i,x_j,y_k\right]\cdot y_l+\sum_{1\le j<i<k\le m}\varepsilon_{ikji}\left[y_i,x_k,y_j\right]\cdot y_i$$

$$+\sum_{1\le k<i\le m}\varepsilon_{iiki}\left[y_i,x_i,y_k\right]\cdot y_i+\sum_{1\le j<i<k\le m}\varepsilon_{kijk}\left[y_k,x_i,y_j\right]\cdot y_k$$

$$+\sum_{1\le j<i\le m}\varepsilon_{ijji}\left[y_i,x_j,y_j\right]\cdot y_i+\sum_{1\le j<i<k\le m}\varepsilon_{kjik}\left[y_k,x_j,y_i\right]\cdot y_k$$

$$=\sum_{1\le j<i\le m}\begin{pmatrix}\varepsilon_{jiji}\left[y_j,x_i,y_j\right]\cdot y_i+\varepsilon_{jjji}\left[y_j,x_j,y_j\right]\cdot y_i+\varepsilon_{iiji}\left[y_i,x_i,y_j\right]\cdot y_i\\ +\varepsilon_{ijji}\left[y_i,x_j,y_j\right]\cdot y_i\end{pmatrix}$$

$$+\sum_{1\le j<i<k\le m}\begin{pmatrix}\varepsilon_{ikjk}\left[y_i,x_k,y_j\right]\cdot y_k+\varepsilon_{iijk}\left[y_i,x_i,y_j\right]\cdot y_k\\ +\varepsilon_{ijjk}\left[y_i,x_j,y_j\right]\cdot y_k+\varepsilon_{jkji}\left[y_j,x_k,y_j\right]\cdot y_i\\ +\varepsilon_{jijk}\left[y_j,x_i,y_j\right]\cdot y_k+\varepsilon_{ijik}\left[y_i,x_j,y_i\right]\cdot y_k\\ +\varepsilon_{jkik}\left[y_j,x_k,y_i\right]\cdot y_k+\varepsilon_{jiik}\left[y_j,x_i,y_i\right]\cdot y_k\\ +\varepsilon_{jjik}\left[y_j,x_j,y_i\right]\cdot y_k+\varepsilon_{ikji}\left[y_i,x_k,y_j\right]\cdot y_i\\ +\varepsilon_{kijk}\left[y_k,x_i,y_j\right]\cdot y_k+\varepsilon_{kjik}\left[y_k,x_j,y_i\right]\cdot y_k\end{pmatrix}$$

$$+\sum_{1\le j<i<k<l\le m}\begin{pmatrix}\varepsilon_{iljk}[y_i,x_l,y_j]\cdot y_k+\varepsilon_{ikjl}[y_i,x_k,y_j]\cdot y_l\\+\varepsilon_{kijl}[y_i,x_j,y_k]\cdot y_l+\varepsilon_{kjil}[y_k,x_j,y_i]\cdot y_l\\+\varepsilon_{jlik}[y_j,x_l,y_i]\cdot y_k+\varepsilon_{jkil}[y_j,x_k,y_i]\cdot y_l\\+\varepsilon_{jikl}[y_j,x_i,y_k]\cdot y_l+\varepsilon_{ijkl}[y_i,x_j,y_k]\cdot y_l\end{pmatrix}$$

So, $p_6=(p_6)_2+(p_6)_3+(p_6)_4$ such that

$$(p_6)_2=\sum_{1\le j<i\le m}\begin{pmatrix}\varepsilon_{jiji}[y_j,x_i,y_j]\cdot y_i+\varepsilon_{jjji}[y_j,x_j,y_j]\cdot y_i+\varepsilon_{iiji}[y_i,x_i,y_j]\cdot y_i\\+\varepsilon_{ijji}[y_i,x_j,y_j]\cdot y_i\end{pmatrix},$$

$$(p_6)_3=\sum_{1\le j<i<k\le m}\begin{pmatrix}\varepsilon_{ikjk}[y_i,x_k,y_j]\cdot y_k+\varepsilon_{iijk}[y_i,x_i,y_j]\cdot y_k\\+\varepsilon_{ijjk}[y_i,x_j,y_j]\cdot y_k+\varepsilon_{jkji}[y_j,x_k,y_j]\cdot y_i\\+\varepsilon_{jijk}[y_j,x_i,y_j]\cdot y_k+\varepsilon_{ijik}[y_i,x_j,y_i]\cdot y_k\\+\varepsilon_{jkik}[y_j,x_k,y_i]\cdot y_k+\varepsilon_{jiik}[y_j,x_i,y_i]\cdot y_k\\+\varepsilon_{jjik}[y_j,x_j,y_i]\cdot y_k+\varepsilon_{ikji}[y_i,x_k,y_j]\cdot y_i\\+\varepsilon_{kijk}[y_k,x_i,y_j]\cdot y_k+\varepsilon_{kjik}[y_k,x_j,y_i]\cdot y_k\end{pmatrix},$$

$$(p_6)_4=\sum_{1\le j<i<k<l\le m}\begin{pmatrix}\varepsilon_{iljk}[y_i,x_l,y_j]\cdot y_k+\varepsilon_{ikjl}[y_i,x_k,y_j]\cdot y_l\\+\varepsilon_{kijl}[y_i,x_j,y_k]\cdot y_l+\varepsilon_{kjil}[y_k,x_j,y_i]\cdot y_l\\+\varepsilon_{jlik}[y_j,x_l,y_i]\cdot y_k+\varepsilon_{jkil}[y_j,x_k,y_i]\cdot y_l\\+\varepsilon_{jikl}[y_j,x_i,y_k]\cdot y_l+\varepsilon_{ijkl}[y_i,x_j,y_k]\cdot y_l\end{pmatrix}.$$

Then,

$$\begin{aligned}\delta_{2m}(p_6)=0&=\delta_{2m}((p_6)_2+(p_6)_3+(p_6)_4)\\&=\delta_{2m}((p_6)_2)+\delta_{2m}((p_6)_3)+\delta_{2m}((p_6)_4).\end{aligned}$$

Let V_{62} be the subspace of KA_6 containing monomials with exactly 2 indices, V_{63} be the subspace of KA_6 containing monomials with exactly 3 indices and V_{64} be the subspace of KA_6 containing monomials with exactly 4 indices then $V_{62}\cap V_{63}=\{0\}=V_{62}\cap V_{64}=V_{63}\cap V_{64}$. Then,

$$\delta_{2m}((p_6)_2)\in V_{62},\ \delta_{2m}((p_6)_3)\in V_{63},\ \delta_{2m}((p_6)_4)\in V_{64}$$

Therefore,

$$\delta_{2m}((p_6)_2)=0=\delta_{2m}((p_6)_3)=\delta_{2m}((p_6)_4).$$

For,

$$0=\delta_{2m}((p_6)_2)$$

$$=\delta_{2m}\left(\sum_{1\le j<i\le m}\begin{pmatrix}\varepsilon_{jiji}[y_j,x_i,y_j]\cdot y_i+\varepsilon_{jjji}[y_j,x_j,y_j]\cdot y_i\\ +\varepsilon_{iiji}[y_i,x_i,y_j]\cdot y_i+\varepsilon_{ijji}[y_i,x_j,y_j]\cdot y_i\end{pmatrix}\right)$$

$$=\sum_{1\le j<i\le m}\begin{pmatrix}\varepsilon_{jiji}\begin{pmatrix}[\delta_{2m}(y_j),x_i,y_j]\cdot y_i+[y_j,x_i,\delta_{2m}(y_j)]\cdot y_i\\ +[y_j,x_i,y_j]\cdot\delta_{2m}(y_i)\end{pmatrix}\\ +\varepsilon_{jjji}\begin{pmatrix}[\delta_{2m}(y_j),x_j,y_j]\cdot y_i+[y_j,x_j,\delta_{2m}(y_j)]\cdot y_i\\ +[y_j,x_j,y_j]\cdot\delta_{2m}(y_i)\end{pmatrix}\\ +\varepsilon_{iiji}\begin{pmatrix}[\delta_{2m}(y_i),x_i,y_j]\cdot y_i+[y_i,x_i,\delta_{2m}(y_j)]\cdot y_i\\ +[y_i,x_i,y_j]\cdot\delta_{2m}(y_i)\end{pmatrix}\\ +\varepsilon_{ijji}\begin{pmatrix}[\delta_{2m}(y_i),x_j,y_j]\cdot y_i+[y_i,x_j,\delta_{2m}(y_j)]\cdot y_i\\ +[y_i,x_j,y_j]\cdot\delta_{2m}(y_i)\end{pmatrix}\end{pmatrix}$$

$$=\sum_{1\le j<i\le m}\begin{pmatrix}\varepsilon_{jiji}([x_j,x_i,y_j]\cdot y_i+[y_j,x_i,x_j]\cdot y_i+[y_j,x_i,y_j]\cdot x_i)\\ +\varepsilon_{jjji}([x_j,x_j,y_j]\cdot y_i+[y_j,x_j,x_j]\cdot y_i+[y_j,x_j,y_j]\cdot x_i)\\ +\varepsilon_{iiji}([x_i,x_i,y_j]\cdot y_i+[y_i,x_i,x_j]\cdot y_i+[y_i,x_i,y_j]\cdot x_i)\\ +\varepsilon_{ijji}([x_i,x_j,y_j]\cdot y_i+[y_i,x_j,x_j]\cdot y_i+[y_i,x_j,y_j]\cdot x_i)\end{pmatrix}$$

By using the condition in Remark 2.10.3, we rearrange the basis and we obtain that

$$0=\sum_{1\le j<i\le m}\begin{pmatrix}\varepsilon_{jiji}(-2[x_i,x_j,y_j]\cdot y_i+[y_j,x_j,x_i]\cdot y_i-[y_j,x_i,x_i]\cdot y_j)\\ +\varepsilon_{jjji}([y_j,x_j,x_j]\cdot y_i-[y_j,x_j,x_i]\cdot y_j)\\ +\varepsilon_{iiji}([y_i,x_j,x_i]\cdot y_i-[y_j,x_i,x_i]\cdot y_i)\\ +\varepsilon_{ijji}(2[x_i,x_j,y_j]\cdot y_i+[y_i,x_j,x_j]\cdot y_i-[y_j,x_j,x_i]\cdot y_i)\end{pmatrix}$$

$$=\sum_{1\le j<i\le m}\begin{pmatrix}(-2\varepsilon_{jiji}+2\varepsilon_{ijji})[x_i,x_j,y_j]\cdot y_i\\ +(\varepsilon_{jiji}-\varepsilon_{jjji}+\varepsilon_{iiji}-\varepsilon_{ijji})[y_j,x_j,x_i]\cdot y_i\\ +(-\varepsilon_{jiji}-\varepsilon_{iiji})[y_j,x_i,x_i]\cdot y_j+\varepsilon_{jjji}[y_j,x_j,x_j]\cdot y_i\\ -\varepsilon_{iiji}[y_j,x_i,x_i]\cdot y_i\end{pmatrix}$$

Hence,

$-2\varepsilon_{jiji}+2\varepsilon_{ijji}=0$, $\varepsilon_{jiji}-\varepsilon_{jjji}+\varepsilon_{iiji}-\varepsilon_{ijji}=0$, $-\varepsilon_{jiji}-\varepsilon_{iiji}=0$, $\varepsilon_{jjji}=\varepsilon_{iiji}=0$.

So, we have

$$0=\varepsilon_{jjji}=\varepsilon_{iiji}=\varepsilon_{jiji}=\varepsilon_{ijji}$$

This implies that

$$(p_6)_2 = 0$$

For

$0 = \delta_{2m}((p_6)_3)$

$$= \delta_{2m}\left(\sum_{1\le j<i<k\le m}\begin{pmatrix} \varepsilon_{ikjk}[y_i, x_j, y_k]\cdot y_j + \varepsilon_{iijk}[y_i, x_i, y_j]\cdot y_k \\ +\varepsilon_{ijjk}[y_i, x_j, y_j]\cdot y_k + \varepsilon_{jkji}[y_j, x_k, y_j]\cdot y_i \\ +\varepsilon_{jijk}[y_j, x_i, y_j]\cdot y_k + \varepsilon_{ijik}[y_i, x_j, y_i]\cdot y_k \\ +\varepsilon_{jkik}[y_j, x_k, y_i]\cdot y_k + \varepsilon_{jiik}[y_j, x_i, y_i]\cdot y_k \\ +\varepsilon_{jjik}[y_j, x_j, y_i]\cdot y_k + \varepsilon_{ikji}[y_i, x_k, y_j]\cdot y_i \\ +\varepsilon_{kijk}[y_k, x_i, y_j]\cdot y_k + \varepsilon_{kjik}[y_k, x_j, y_i]\cdot y_k \end{pmatrix}\right)$$

$$= \sum_{1\le j<i<k\le m} \left(\begin{array}{l} \varepsilon_{ikjk}\begin{pmatrix} [\delta_{2m}(y_i), x_k, y_j]\cdot y_k + [y_i, x_k, \delta_{2m}(y_j)]\cdot y_k \\ +[y_i, x_k, y_j]\cdot \delta_{2m}(y_k) \end{pmatrix} \\ +\varepsilon_{iijk}\begin{pmatrix} [\delta_{2m}(y_i), x_i, y_j]\cdot y_k + [y_i, x_i, \delta_{2m}(y_j)]\cdot y_k \\ +[y_i, x_i, y_j]\cdot \delta_{2m}(y_k) \end{pmatrix} \\ +\varepsilon_{ijjk}\begin{pmatrix} [\delta_{2m}(y_i), x_j, y_j]\cdot y_k + [y_i, x_j, \delta_{2m}(y_j)]\cdot y_k \\ +[y_i, x_j, y_j]\cdot \delta_{2m}(y_k) \end{pmatrix} \\ +\varepsilon_{jkji}\begin{pmatrix} [\delta_{2m}(y_j), x_k, y_j]\cdot y_i + [y_j, x_k, \delta_{2m}(y_j)]\cdot y_i \\ +[y_j, x_k, y_j]\cdot \delta_{2m}(y_i) \end{pmatrix} \\ +\varepsilon_{jijk}\begin{pmatrix} [\delta_{2m}(y_j), x_i, y_j]\cdot y_k + [y_j, x_i, \delta_{2m}(y_j)]\cdot y_k \\ +[y_j, x_i, y_j]\cdot \delta_{2m}(y_k) \end{pmatrix} \\ +\varepsilon_{ijik}\begin{pmatrix} [\delta_{2m}(y_i), x_j, y_i]\cdot y_k + [y_i, x_j, \delta_{2m}(y_i)]\cdot y_k \\ +[y_i, x_j, y_i]\cdot \delta_{2m}(y_k) \end{pmatrix} \\ +\varepsilon_{jkik}\begin{pmatrix} [\delta_{2m}(y_j), x_k, y_i]\cdot y_k + [y_j, x_k, \delta_{2m}(y_i)]\cdot y_k \\ +[y_j, x_k, y_i]\cdot \delta_{2m}(y_k) \end{pmatrix} \\ +\varepsilon_{jiik}\begin{pmatrix} [\delta_{2m}(y_j), x_i, y_i]\cdot y_k + [y_j, x_i, \delta_{2m}(y_i)]\cdot y_k \\ +[y_j, x_i, y_i]\cdot \delta_{2m}(y_k) \end{pmatrix} \\ +\varepsilon_{jjik}\begin{pmatrix} [\delta_{2m}(y_j), x_j, y_i]\cdot y_k + [y_j, x_j, \delta_{2m}(y_i)]\cdot y_k \\ +[y_j, x_j, y_i]\cdot \delta_{2m}(y_k) \end{pmatrix} \\ +\varepsilon_{ikji}\begin{pmatrix} [\delta_{2m}(y_i), x_k, y_j]\cdot y_i + [y_i, x_k, \delta_{2m}(y_j)]\cdot y_i \\ +[y_i, x_k, y_j]\cdot \delta_{2m}(y_i) \end{pmatrix} \\ +\varepsilon_{kijk}\begin{pmatrix} [\delta_{2m}(y_k), x_i, y_j]\cdot y_k + [y_k, x_i, \delta_{2m}(y_j)]\cdot y_k \\ +[y_k, x_i, y_j]\cdot \delta_{2m}(y_k) \end{pmatrix} \\ +\varepsilon_{kjik}\begin{pmatrix} [\delta_{2m}(y_k), x_j, y_i]\cdot y_k + [y_k, x_j, \delta_{2m}(y_i)]\cdot y_k \\ +[y_k, x_j, y_i]\cdot \delta_{2m}(y_k) \end{pmatrix} \end{array} \right)$$

$$= \sum_{1\le j<i<k\le m}\begin{pmatrix} \varepsilon_{ikjk}\left([x_i,x_k,y_j]\cdot y_k+[y_i,x_k,x_j]\cdot y_k+[y_i,x_k,y_j]\cdot x_k\right) \\ +\varepsilon_{iijk}\left([x_i,x_i,y_j]\cdot y_k+[y_i,x_i,x_j]\cdot y_k+[y_i,x_i,y_j]\cdot x_k\right) \\ +\varepsilon_{ijjk}\left([x_i,x_j,y_j]\cdot y_k+[y_i,x_j,x_j]\cdot y_k+[y_i,x_j,y_j]\cdot x_k\right) \\ +\varepsilon_{jkji}\left([x_j,x_k,y_j]\cdot y_i+[y_j,x_k,x_j]\cdot y_i+[y_j,x_k,y_j]\cdot x_i\right) \\ +\varepsilon_{jijk}\left([x_j,x_i,y_j]\cdot y_k+[y_j,x_i,x_j]\cdot y_k+[y_j,x_i,y_j]\cdot x_k\right) \\ +\varepsilon_{ijik}\left([x_i,x_j,y_i]\cdot y_k+[y_i,x_j,x_i]\cdot y_k+[y_i,x_j,y_i]\cdot x_k\right) \\ +\varepsilon_{jkik}\left([x_j,x_k,y_i]\cdot y_k+[y_j,x_k,x_i]\cdot y_k+[y_j,x_k,y_i]\cdot x_k\right) \\ +\varepsilon_{jiik}\left([x_j,x_i,y_i]\cdot y_k+[y_j,x_i,x_i]\cdot y_k+[y_j,x_i,y_i]\cdot x_k\right) \\ +\varepsilon_{jjik}\left([x_j,x_j,y_i]\cdot y_k+[y_j,x_j,x_i]\cdot y_k+[y_j,x_j,y_i]\cdot x_k\right) \\ +\varepsilon_{ikji}\left([x_i,x_k,y_j]\cdot y_i+[y_i,x_k,x_j]\cdot y_i+[y_i,x_k,y_j]\cdot x_i\right) \\ +\varepsilon_{kijk}\left([x_k,x_i,y_j]\cdot y_k+[y_k,x_i,x_j]\cdot y_k+[y_k,x_i,y_j]\cdot x_k\right) \\ +\varepsilon_{kjik}\left([x_k,x_j,y_i]\cdot y_k+[y_k,x_j,x_i]\cdot y_k+[y_k,x_j,y_i]\cdot x_k\right) \end{pmatrix}$$

We rearrange by using the condition in Remark 2.10.3 in other to get the basis

$$
0=\sum_{1\leq j<i<k\leq m}\left(\begin{matrix}
\varepsilon_{ikjk}\begin{pmatrix}-[x_k,x_i,y_j]\cdot y_k-[x_k,x_j,y_i]\cdot y_k\\ +[y_i,x_j,x_k]\cdot y_k-[y_j,x_k,x_k]\cdot y_i\end{pmatrix}\\
+\varepsilon_{iijk}\begin{pmatrix}-[x_i,x_j,y_i]\cdot y_k+[y_i,x_j,x_i]\cdot y_k\\ -[y_j,x_j,x_k]\cdot y_i+[x_k,x_j,y_j]\cdot y_i\end{pmatrix}\\
+\varepsilon_{ijjk}\begin{pmatrix}[x_i,x_j,y_j]\cdot y_k+[y_i,x_j,x_j]\cdot y_k\\ -[y_j,x_j,x_k]\cdot y_i+[x_k,x_j,y_j]\cdot y_i\end{pmatrix}\\
+\varepsilon_{jkji}\begin{pmatrix}-2[x_k,x_j,y_j]\cdot y_i+[y_j,x_j,x_k]\cdot y_i\\ -[y_j,x_i,x_k]\cdot y_j\end{pmatrix}\\
+\varepsilon_{jijk}\begin{pmatrix}-2[x_i,x_j,y_j]\cdot y_k+[y_j,x_j,x_i]\cdot y_k\\ -[y_j,x_i,x_k]\cdot y_j\end{pmatrix}\\
+\varepsilon_{ijik}\begin{pmatrix}[x_i,x_j,y_i]\cdot y_k+[y_i,x_j,x_i]\cdot y_k\\ -[y_i,x_j,x_k]\cdot y_i\end{pmatrix}\\
+\varepsilon_{jkik}\begin{pmatrix}-[x_k,x_j,y_i]\cdot y_k-[x_k,x_i,y_j]\cdot y_k\\ +[y_j,x_i,x_k]\cdot y_k-[y_j,x_k,x_k]\cdot y_i\end{pmatrix}\\
+\varepsilon_{jiik}\begin{pmatrix}-[x_i,x_j,y_i]\cdot y_k+[y_j,x_i,x_i]\cdot y_k\\ -[y_j,x_i,x_k]\cdot y_i\end{pmatrix}\\
+\varepsilon_{jjik}\left([y_j,x_j,x_i]\cdot y_k-[y_j,x_j,x_k]\cdot y_i\right)\\
+\varepsilon_{ikji}\begin{pmatrix}-[x_k,x_i,y_j]\cdot y_i+[y_i,x_j,x_k]\cdot y_i\\ -[y_j,x_i,x_k]\cdot y_i\end{pmatrix}\\
+\varepsilon_{kijk}\begin{pmatrix}2[x_k,x_i,y_j]\cdot y_k+[y_k,x_j,x_i]\cdot y_k\\ -[y_j,x_i,x_k]\cdot y_k\end{pmatrix}\\
+\varepsilon_{kjik}\begin{pmatrix}2[x_k,x_j,y_i]\cdot y_k+[y_k,x_j,x_i]\cdot y_k\\ -[y_i,x_j,x_k]\cdot y_k\end{pmatrix}
\end{matrix}\right)
$$

$$= \sum_{1\le j<i<k\le m} \begin{pmatrix} (-\varepsilon_{ikjk} - \varepsilon_{jkik} + 2\varepsilon_{kijk})[x_k, x_i, y_j] \cdot y_k \\ +(-\varepsilon_{ikjk} - \varepsilon_{jkik} + 2\varepsilon_{kjik})[x_k, x_j, y_i] \cdot y_k \\ +(\varepsilon_{ikjk} - \varepsilon_{kjik})[y_i, x_j, x_k] \cdot y_k \\ +(-\varepsilon_{ikjk} - \varepsilon_{jkik})[y_j, x_k, x_k] \cdot y_i \\ +(-\varepsilon_{iijk} + \varepsilon_{ijik} - \varepsilon_{jiik})[x_i, x_j, y_i] \cdot y_k \\ +(\varepsilon_{iijk} + \varepsilon_{ijik})[y_i, x_j, x_i] \cdot y_k \\ +(-\varepsilon_{iijk} - \varepsilon_{ijjk} + \varepsilon_{jkji} - \varepsilon_{jjik})[y_j, x_j, x_k] \cdot y_i \\ +(\varepsilon_{iijk} + \varepsilon_{ijjk} - 2\varepsilon_{jkji})[x_k, x_j, y_j] \cdot y_i \\ +(\varepsilon_{ijjk} - 2\varepsilon_{jijk})[x_i, x_j, y_j] \cdot y_k + \varepsilon_{ijjk}[y_i, x_j, x_j] \cdot y_k \\ +(-\varepsilon_{jkji} - \varepsilon_{jijk})[y_j, x_i, x_k] \cdot y_j \\ +(\varepsilon_{jijk} + \varepsilon_{jjik})[y_j, x_j, x_i] \cdot y_k \\ +(-\varepsilon_{ijik} + \varepsilon_{ikji})[y_i, x_j, x_k] \cdot y_i \\ +(\varepsilon_{jkik} - \varepsilon_{kijk})[y_j, x_i, x_k] \cdot y_k + \varepsilon_{jiik}[y_j, x_i, x_i] \cdot y_k \\ +(-\varepsilon_{jiik} - \varepsilon_{ikji})[y_j, x_i, x_k] \cdot y_i \\ -\varepsilon_{ikji}[x_k, x_i, y_j] \cdot y_i + (\varepsilon_{kijk} + \varepsilon_{kjik})[y_k, x_j, x_i] \cdot y_k \end{pmatrix}$$

So,

$-\varepsilon_{iijk} + \varepsilon_{ijik} - \varepsilon_{jiik} = 0$, $\varepsilon_{iijk} + \varepsilon_{ijik} = 0$, $-\varepsilon_{ijik} + \varepsilon_{ikji} = 0$, $\varepsilon_{jiik} = 0$, $-\varepsilon_{jiik} - \varepsilon_{ikji} = 0$ and $\varepsilon_{ikji} = 0$ gives

$$0 = \varepsilon_{ikji} = \varepsilon_{jiik} = \varepsilon_{ijik} = \varepsilon_{iijk},$$

$-\varepsilon_{iijk} - \varepsilon_{ijjk} + \varepsilon_{jkji} - \varepsilon_{jjik} = 0$, $\varepsilon_{iijk} + \varepsilon_{ijjk} - 2\varepsilon_{jkji} = 0$, $\varepsilon_{ijjk} - 2\varepsilon_{jijk} = 0$, $\varepsilon_{ijjk} = 0$, $-\varepsilon_{jkji} - \varepsilon_{jijk} = 0$, $\varepsilon_{jijk} + \varepsilon_{jjik} = 0$ give that

$$0 = \varepsilon_{ijjk} = \varepsilon_{jijk} = \varepsilon_{jkji} = \varepsilon_{jjik},$$

$-\varepsilon_{ikjk} - \varepsilon_{jkik} + 2\varepsilon_{kijk} = 0$, $-\varepsilon_{ikjk} - \varepsilon_{jkik} + 2\varepsilon_{kjik} = 0$, $\varepsilon_{ikjk} - \varepsilon_{kjik} = 0$, $-\varepsilon_{ikjk} - \varepsilon_{jkik} = 0$, $\varepsilon_{jkik} - \varepsilon_{kijk} = 0$, $\varepsilon_{kijk} + \varepsilon_{kjik} = 0$ gives that

$$0 = \varepsilon_{jkik} = \varepsilon_{kijk} = \varepsilon_{ikjk} = \varepsilon_{kjik}.$$

This implies that

$$(p_6)_3 = 0$$

For

$0 = \delta_{2m}((p_6)_4)$

$$= \delta_{2m}\left(\sum_{1\le j<i<k<l\le m}\begin{pmatrix}\varepsilon_{iljk}[y_i,x_l,y_j]\cdot y_k + \varepsilon_{ikjl}[y_i,x_k,y_j]\cdot y_l\\ +\varepsilon_{kijl}[y_i,x_j,y_k]\cdot y_l + \varepsilon_{kjil}[y_k,x_j,y_i]\cdot y_l\\ +\varepsilon_{jlik}[y_j,x_l,y_i]\cdot y_k + \varepsilon_{jkil}[y_j,x_k,y_i]\cdot y_l\\ +\varepsilon_{jikl}[y_j,x_i,y_k]\cdot y_l + \varepsilon_{ijkl}[y_i,x_j,y_k]\cdot y_l\end{pmatrix}\right)$$

$$= \sum_{1\le j<i<k<l\le m}\begin{pmatrix}\varepsilon_{iljk}\begin{pmatrix}[\delta_{2m}(y_i),x_l,y_j]\cdot y_k + [y_i,x_l,\delta_{2m}(y_j)]\cdot y_k\\ +[y_i,x_l,y_j]\cdot\delta_{2m}(y_k)\end{pmatrix}\\ +\varepsilon_{ikjl}\begin{pmatrix}[\delta_{2m}(y_i),x_k,y_j]\cdot y_l + [y_i,x_k,\delta_{2m}(y_j)]\cdot y_l\\ +[y_i,x_k,y_j]\cdot\delta_{2m}(y_l)\end{pmatrix}\\ +\varepsilon_{kijl}\begin{pmatrix}[\delta_{2m}(y_i),x_j,y_k]\cdot y_l + [y_i,x_j,\delta_{2m}(y_k)]\cdot y_l\\ +[y_i,x_j,y_k]\cdot\delta_{2m}(y_l)\end{pmatrix}\\ +\varepsilon_{kjil}\begin{pmatrix}[\delta_{2m}(y_k),x_j,y_i]\cdot y_l + [y_k,x_j,\delta_{2m}(y_i)]\cdot y_l\\ +[y_k,x_j,y_i]\cdot\delta_{2m}(y_l)\end{pmatrix}\\ +\varepsilon_{jlik}\begin{pmatrix}[\delta_{2m}(y_j),x_l,y_i]\cdot y_k + [y_j,x_l,\delta_{2m}(y_i)]\cdot y_k\\ +[y_j,x_l,y_i]\cdot\delta_{2m}(y_k)\end{pmatrix}\\ +\varepsilon_{jkil}\begin{pmatrix}[\delta_{2m}(y_j),x_k,y_i]\cdot y_l + [y_j,x_k,\delta_{2m}(y_i)]\cdot y_l\\ +[y_j,x_k,y_i]\cdot\delta_{2m}(y_l)\end{pmatrix}\\ +\varepsilon_{jikl}\begin{pmatrix}[\delta_{2m}(y_j),x_i,y_k]\cdot y_l + [y_j,x_i,\delta_{2m}(y_k)]\cdot y_l\\ +[y_j,x_i,y_k]\cdot\delta_{2m}(y_l)\end{pmatrix}\\ +\varepsilon_{ijkl}\begin{pmatrix}[\delta_{2m}(y_i),x_j,y_k]\cdot y_l + [y_i,x_j,\delta_{2m}(y_k)]\cdot y_l\\ +[y_i,x_j,y_k]\cdot\delta_{2m}(y_l)\end{pmatrix}\end{pmatrix}$$

$$= \sum_{1\le j<i<k<l\le m}\begin{pmatrix}\varepsilon_{iljk}([x_i,x_l,y_j]\cdot y_k + [y_i,x_l,x_j]\cdot y_k + [y_i,x_l,y_j]\cdot x_k)\\ +\varepsilon_{ikjl}([x_i,x_k,y_j]\cdot y_l + [y_i,x_k,x_j]\cdot y_l + [y_i,x_k,y_j]\cdot x_l)\\ +\varepsilon_{kijl}([x_i,x_j,y_k]\cdot y_l + [y_i,x_j,x_k]\cdot y_l + [y_i,x_j,y_k]\cdot x_l)\\ +\varepsilon_{kjil}([x_k,x_j,y_i]\cdot y_l + [y_k,x_j,x_i]\cdot y_l + [y_k,x_j,y_i]\cdot x_l)\\ +\varepsilon_{jlik}([x_j,x_l,y_i]\cdot y_k + [y_j,x_l,x_i]\cdot y_k + [y_j,x_l,y_i]\cdot x_k)\\ +\varepsilon_{jkil}([x_j,x_k,y_i]\cdot y_l + [y_j,x_k,x_i]\cdot y_l + [y_j,x_k,y_i]\cdot x_l)\\ +\varepsilon_{jikl}([x_j,x_i,y_k]\cdot y_l + [y_j,x_i,x_k]\cdot y_l + [y_j,x_i,y_k]\cdot x_l)\\ +\varepsilon_{ijkl}([x_i,x_j,y_k]\cdot y_l + [y_i,x_j,x_k]\cdot y_l + [y_i,x_j,y_k]\cdot x_l)\end{pmatrix}$$

Now we use the conditions in Remark 2.10.3, and get that

$$0 = \sum_{1\le j<i<k<l\le m} \begin{pmatrix} \varepsilon_{iljk}\begin{pmatrix} -[x_l,x_i,y_j]\cdot y_k - [x_l,x_j,y_i]\cdot y_k + [y_i,x_j,x_l]\cdot y_k \\ -[y_j,x_k,x_l]\cdot y_i \end{pmatrix} \\ +\varepsilon_{ikjl}\begin{pmatrix} -[x_k,x_i,y_j]\cdot y_l - [x_k,x_j,y_i]\cdot y_l + [y_i,x_j,x_k]\cdot y_l \\ -[y_j,x_k,x_l]\cdot y_i + [x_l,x_k,y_j]\cdot y_i \end{pmatrix} \\ +\varepsilon_{kijl}([x_i,x_j,y_k]\cdot y_l + [y_i,x_j,x_k]\cdot y_l - [y_i,x_j,x_l]\cdot y_k) \\ +\varepsilon_{kjil}\begin{pmatrix} [x_k,x_j,y_i]\cdot y_l + [y_k,x_j,x_i]\cdot y_l - [y_i,x_j,x_l]\cdot y_k \\ +[x_l,x_j,y_i]\cdot y_k \end{pmatrix} \\ +\varepsilon_{jlik}\begin{pmatrix} -[x_l,x_j,y_i]\cdot y_k - [x_l,x_i,y_j]\cdot y_k + [y_j,x_i,x_l]\cdot y_k \\ +[x_l,x_k,y_j]\cdot y_i - [y_j,x_k,x_l]\cdot y_i \end{pmatrix} \\ +\varepsilon_{jkil}\begin{pmatrix} -[x_k,x_j,y_i]\cdot y_l - [x_k,x_i,y_j]\cdot y_l + [y_j,x_i,x_k]\cdot y_l \\ -[y_j,x_k,x_l]\cdot y_i \end{pmatrix} \\ +\varepsilon_{jikl}(-[x_i,x_j,y_k]\cdot y_l + [y_j,x_i,x_k]\cdot y_l - [y_j,x_i,x_l]\cdot y_k) \\ +\varepsilon_{ijkl}([x_i,x_j,y_k]\cdot y_l + [y_i,x_j,x_k]\cdot y_l - [y_i,x_j,x_l]\cdot y_k) \end{pmatrix}$$

$$= \sum_{1\le j<i<k<l\le m} \begin{pmatrix} (-\varepsilon_{iljk} - \varepsilon_{jlik})[x_l,x_i,y_j]\cdot y_k \\ +(-\varepsilon_{iljk} + \varepsilon_{kjil} - \varepsilon_{jlik})[x_l,x_j,y_i]\cdot y_k \\ +(\varepsilon_{iljk} - \varepsilon_{kijl} - \varepsilon_{kjil} - \varepsilon_{ijkl})[y_i,x_j,x_l]\cdot y_k \\ +(-\varepsilon_{iljk} - \varepsilon_{ikjl} - \varepsilon_{jlik} - \varepsilon_{jkil})[y_j,x_k,x_l]\cdot y_i \\ +(-\varepsilon_{ikjl} - \varepsilon_{jkil})[x_k,x_i,y_j]\cdot y_l \\ +(-\varepsilon_{ikjl} + \varepsilon_{kjil} - \varepsilon_{jkil})[x_k,x_j,y_i]\cdot y_l \\ +(\varepsilon_{ikjl} + \varepsilon_{kijl} + \varepsilon_{ijkl})[y_i,x_j,x_k]\cdot y_l \\ +(\varepsilon_{ikjl} + \varepsilon_{jlik})[x_l,x_k,y_j]\cdot y_i \\ +(\varepsilon_{kijl} - \varepsilon_{jikl} + \varepsilon_{ijkl})[x_i,x_j,y_k]\cdot y_l \\ +\varepsilon_{kjil}[y_k,x_j,x_i]\cdot y_l \\ +(\varepsilon_{jlik} - \varepsilon_{jikl})[y_j,x_i,x_l]\cdot y_k \\ +(\varepsilon_{jkil} + \varepsilon_{jikl})[y_j,x_i,x_k]\cdot y_l \end{pmatrix}$$

So, $-\varepsilon_{iljk} - \varepsilon_{jlik} = 0$, $-\varepsilon_{iljk} + \varepsilon_{kjil} - \varepsilon_{jlik} = 0$, $\varepsilon_{iljk} - \varepsilon_{kijl} - \varepsilon_{kjil} - \varepsilon_{ijkl} = 0$, $-\varepsilon_{iljk} - \varepsilon_{ikjl} - \varepsilon_{jlik} - \varepsilon_{jkil} = 0$, $-\varepsilon_{ikjl} - \varepsilon_{jkil} = 0$, $-\varepsilon_{ikjl} + \varepsilon_{kjil} - \varepsilon_{jkil} = 0$, $\varepsilon_{ikjl} + \varepsilon_{kijl} + \varepsilon_{ijkl} = 0$, $\varepsilon_{ikjl} + \varepsilon_{jlik} = 0$, $\varepsilon_{kijl} - \varepsilon_{jikl} + \varepsilon_{ijkl} = 0$, $\varepsilon_{kjil} = 0$, $\varepsilon_{jlik} - \varepsilon_{jikl} = 0$ and $\varepsilon_{jkil} + \varepsilon_{jikl} = 0$

$$0 = \varepsilon_{kjil} = \varepsilon_{jkil} = \varepsilon_{ikjl} = \varepsilon_{jikl} = \varepsilon_{jlik} = \varepsilon_{iljk} = \varepsilon_{kijl} = \varepsilon_{ijkl}$$

This implies that

$$(p_6)_4 = 0$$

Therefore, that the vector space

$$(KA_6)^{\delta_{2m}} = \{0\}.$$

4.3. Symmetric Polynomials in Free Metabelian Poisson Algebras

This section contains the spanning elements of the algebra of symmetric polynomials in the free metabelian Poisson algebra $P_n^{S_n}$. Recall that the algebra $P_n^{S_n}$ is a direct sum of $(K\mathcal{B}_{1n})^{S_n}$, $(K\mathcal{B}_{2n})^{S_n}$, $(K\mathcal{B}_{3n})^{S_n}$ and $(K\mathcal{B}_{4n})^{S_n}$ as a vector space. A set of generators $\mathcal{G}_n$ for the symmetric polynomials $(K\mathcal{B}_{1n})^{S_n} = F_n^{S_n}$ as $K[X_n]^{S_n}$-module has been given in Theorem 3.4.1.

Now consider the K-basis elements of length ≤ 3 of the algebra $K[X_n]^{S_n}$ as

$$1_K,\, g^{(n)} = x_1 + \cdots + x_n,\, x_1^2 + \cdots + x_n^2,\, x_1x_2 + \cdots + x_1x_n + \cdots + x_{n-1}x_n,$$
$$x_1^3 + \cdots + x_n^3,\, x_1x_1x_2 + \cdots + x_1x_1x_n + \cdots + x_nx_nx_{n-1},$$
$$x_1x_2x_3 + \cdots + x_{n-2}x_{n-1}x_n.$$

Therefore, $(K\mathcal{B}_{2n})^{S_n} = K\left\{p_1^{(n)},\ p_2^{(n)},\ p_3^{(n)},\ p_4^{(n)},\ p_5^{(n)}\right\}$, where

$$p_1^{(n)} = \sum_{1\leq i\leq n} x_i \cdot x_i,\ \ p_2^{(n)} = \sum_{1\leq i\neq j\leq n} x_i \cdot x_j,\ \ p_3^{(n)} = \sum_{1\leq i\leq n} x_i \cdot x_i \cdot x_i,$$
$$p_4^{(n)} = \sum_{1\leq i\neq j\leq n} x_i \cdot x_i \cdot x_j,\quad p_5^{(n)} = \sum_{1\leq i<j<k\leq n} x_i \cdot x_j \cdot x_k.$$

Note that $p_k^{(n)} \cdot p_l^{(n)} = \left[p_k^{(n)}, p_l^{(n)}\right] = 0$, $\forall\ 1 \leq k, l \leq 5$ and $n \geq 2$.

The following theorem gives that $(K\mathcal{B}_{3n})^{S_n}$ is a one-dimensional vector space.

Theorem 4.3.1: The K-vector space $(K\mathcal{B}_{3n})^{S_n}$ is spanned on the symmetric polynomial

$$p_6^{(n)} = \sum_{1\leq j<i\leq n} [x_i, x_j] \cdot (x_i - x_j)$$

that is $(K\mathcal{B}_{3n})^{S_n} = K\left\{p_6^{(n)}\right\}$.

Proof: Let us start by defining the multi-homogeneous subspaces

$$(K\mathcal{B}_{3n})_{(1,1,1)} = K\{[x_i, x_j] \cdot x_k \mid 1 \le j < i \le n, \qquad k \ne i, j\}$$

$$(K\mathcal{B}_{3n})_{(2,1)} = K\{[x_i, x_j] \cdot x_j, [x_i, x_j] \cdot x_i \mid 1 \le j < i \le n\}$$

of $K\mathcal{B}_{3n} = (K\mathcal{B}_{3n})_{(1,1,1)} \oplus (K\mathcal{B}_{3n})_{(2,1)}$. Then clearly $(K\mathcal{B}_{3n})_{(1,1,1)}^{S_n} \subset (K\mathcal{B}_{3n})_{(1,1,1)}$ and $(K\mathcal{B}_{3n})_{(2,1)}^{S_n} \subset (K\mathcal{B}_{3n})_{(2,1)}$. As a results, it is enough to determine symmetric polynomials in these subspaces in order to obtain the description of $(K\mathcal{B}_{3n})^{S_n}$. Let $p(X_n) \in (K\mathcal{B}_{3n})_{(1,1,1)}^{S_n}$ of the form

$$p(X_n) = \sum_{j<i, k\ne i,j} \alpha_{ijk} \, [x_i, x_j] \cdot x_k$$

for some $\alpha_{ijk} \in K$. Because the polynomial $p(X_n)$ is symmetric, then $\tau_{ij} * p(X_n) = p(X_n)$, for any transposition $\tau_{ij} = (ij) \in S_n$, $1 \le j < i \le n$. We may fix (i_0, j_0) and consider $\tau_{i_0 j_0} \in S_n$.

Then $p(X_n) = \tau_{i_0 j_0} * p(X_n)$ gives

$$p(X_n) = \tau_{i_0 j_0} * \left(\sum_{k \ne i_0, j_0} \alpha_{i_0 j_0 k} \, [x_{i_0}, x_{j_0}] \cdot x_k + \sum_{(i,j) \ne (i_0, j_0)} \alpha_{ijk} \, [x_i, x_j] \cdot x_k \right)$$

$$= - \sum_{k \ne i_0, j_0} \alpha_{i_0 j_0 k} \, [x_{i_0}, x_{j_0}] \cdot x_k + \sum_{(i,j) \ne (i_0, j_0)} \alpha_{ijk} \, [x_{\tau_{i_0 j_0}(i)}, x_{\tau_{i_0 j_0}(j)}] \cdot x_{\tau_{i_0 j_0}(k)}$$

It is obvious that basis elements $[x_{i_0}, x_{j_0}] \cdot x_k \in \mathcal{B}_{3n}$ does not appear in the second sum for any k. Hence, as consequence from the equation we have

$$\sum_{k \ne i_0, j_0} 2\alpha_{i_0 j_0 k} \, [x_{i_0}, x_{j_0}] \cdot x_k = 0,$$

and that $\alpha_{i_0 j_0 k} = 0$, for every $k \ne i_0, j_0$. By continuing the same procedure for different pairs (i, j) proves that $\alpha_{ijk} = 0$ for all (i, j, k). As a result, if $p(X_n) = 0$, then the vector space $(K\mathcal{B}_{3n})_{(1,1,1)}$ does not have nonzero symmetric polynomials.

Next, let $q(X_n) \in (K\mathcal{B}_{3n})_{(2,1)}^{S_n}$ has element of the form

$$q(X_n) = \sum_{i>j} \alpha_{ijj}\,[x_i, x_j] \cdot x_j + \sum_{i>j} \alpha_{iji}\,[x_i, x_j] \cdot x_i$$

for some $\alpha_{ijj}, \alpha_{iji} \in K$. Considering $\tau_{i_0 j_0} * q(X_n) = q(X_n)$ for a fixed pair (i_0, j_0), same calculations give that

$$\begin{aligned}\tau_{i_0 j_0} * q(X_n) &= \alpha_{i_0 j_0 j_0}[x_{j_0}, x_{i_0}] \cdot x_{i_0} + \alpha_{i_0 j_0 i_0}[x_{j_0}, x_{i_0}] \cdot x_{j_0} \\ &+ \sum_{(i,j)\neq(i_0,j_0)} \alpha_{ijj}\,[x_{\tau_{i_0 j_0}(i)}, x_{\tau_{i_0 j_0}(j)}] \cdot x_{\tau_{i_0 j_0}(j)} \\ &+ \sum_{(i,j)\neq(i_0,j_0)} \alpha_{iji}\,[x_{\tau_{i_0 j_0}(i)}, x_{\tau_{i_0 j_0}(j)}] \cdot x_{\tau_{i_0 j_0}(i)}\end{aligned}$$

and hence

$$\begin{aligned}&\alpha_{i_0 j_0 j_0}[x_{i_0}, x_{j_0}] \cdot x_{j_0} + \alpha_{i_0 j_0 i_0}[x_{i_0}, x_{j_0}] \cdot x_{i_0} \\ &\qquad = -\alpha_{i_0 j_0 j_0}[x_{i_0}, x_{j_0}] \cdot x_{i_0} - \alpha_{i_0 j_0 i_0}[x_{i_0}, x_{j_0}] \cdot x_{j_0}\end{aligned}$$

which implies that $\alpha_{i_0 j_0 j_0} + \alpha_{i_0 j_0 i_0} = 0$. We get $\alpha_{ijj} = -\alpha_{iji}$ for any pair $(i,j), j < i$, by applying the similar method. Consequently, the polynomial $q(X_n)$ can be expressed as follows.

$$q(X_n) = \sum_{i>j} -\alpha_{iji}\,[x_i, x_j] \cdot x_j + \sum_{i>j} \alpha_{iji}\,[x_i, x_j] \cdot x_i = \sum_{i>j} \alpha_{iji}\,[x_i, x_j] \cdot (x_i - x_j)$$

The remaining part of the proof will be focused to prove that $\alpha_{i_0 j_0 i_0} = \alpha_{i_1 j_1 i_1}$ for arbitrary fixed $(i_0, j_0) \neq (i_1, j_1)$. Since $\pi \in S_n$ is a permutation such that $\pi(i_0) = i_1, \pi(j_0) = j_1$. Then

$$\begin{aligned}\pi * q(X_n) &= \alpha_{i_0 j_0 i_0}[x_{i_1}, x_{j_1}] \cdot (x_{i_1} - x_{i_1}) + \alpha_{i_1 j_1 i_1}[x_{\pi(i_1)}, x_{\pi(j_1)}] \cdot (x_{\pi(i_1)} - x_{\pi(j_1)}) \\ &+ \sum_{(i,j)\neq(i_0,j_0),(i_1,j_1)} \alpha_{iji}\,[x_{\pi(i)}, x_{\pi(j)}] \cdot \left(x_{\pi(i)} - x_{\pi(j)}\right) = q(X_n)\end{aligned}$$

Keep in mind that $\left(\pi(i), \pi(j)\right) \neq (i_1, j_1), (j_1, i_1)$ when $(i,j) \neq (i_0, j_0)$. Hence, $[x_{\pi(i)}, x_{\pi(j)}] \cdot x_{\pi(i)} \neq \pm[x_{i_1}, x_{j_1}] \cdot x_{i_1}$ and $[x_{\pi(i)}, x_{\pi(j)}] \cdot x_{\pi(i)} \neq \pm[x_{i_1}, x_{j_1}] \cdot x_{j_1}$ for $(i,j) \neq (i_0, j_0)$. Now, one may consider the coefficient of the basis element $[x_{i_1}, x_{j_1}] \cdot x_{i_1}$ in the equation $\pi * q(X_n) - q(X_n) = 0$, it is simple to conclude that $\alpha_{i_1 j_1 i_1} - \alpha_{i_0 j_0 i_0} = 0$. Therefore, $q(X_n) = \alpha q(X_n)$ for some $\alpha \in K$, such that

$$q(X_n) = \sum_{i>j} [x_i, x_j] \cdot (x_i - x_j) \in (K\mathcal{B}_{3n})^{S_n}.$$

In the following theorem, we determine the vector space $(K\mathcal{B}_{4n})^{S_n}$ of symmetric polynomials in $K\mathcal{B}_{4n}$.

Theorem 4.3.2: If $s(X_n)$ is a symmetric polynomial in the vector space $(K\mathcal{B}_{4n})^{S_n}$. Then $s(X_n) = \alpha_1 p_7^{(n)} + \alpha_2 p_8^{(n)}$ for some $\alpha_1, \alpha_2 \in K$, where

$$s(X_2) = \alpha_1 [x_2, x_1, x_1] \cdot x_2, \;\; p_7^{(n)} = \sum_{1 \le j < i \le n} [x_i, x_j, x_j] \cdot x_i,$$

$$p_8^{(n)} = \sum_{1 \le j < i < k \le n} (-[x_i, x_j, x_i] \cdot x_k + [x_i, x_j, x_j] \cdot x_k + [x_k, x_j, x_i] \cdot x_k), n \ge 3.$$

Proof: For $n = 2$, the only basis element in $\mathcal{B}_{4n}$ is $[x_2, x_1, x_1] \cdot x_2$ which is symmetric:

$$\tau_{12} * [x_2, x_1, x_1] \cdot x_2 = [x_1, x_2, x_2] \cdot x_1 = -[x_1, x_2, x_1] \cdot x_2 = [x_2, x_1, x_1] \cdot x_2$$

by the identities satisfied in P_2. Therefore, $(K\mathcal{B}_{42})^{S_2} = K\mathcal{B}_{42}$.

For $n \ge 3$, let us define the multi-homogeneous subspaces of $K\mathcal{B}_{4n}$ as follows:

$$(K\mathcal{B}_{4n})_{(2,2)} = K\{[x_i, x_j, x_j] \cdot x_i \mid j < i\},$$

$$(K\mathcal{B}_{4n})_{(1,1,1,1)} = K\{[x_i, x_j, x_k] \cdot x_l, [x_k, x_j, x_i] \cdot x_l \mid j < i < k < l\},$$

$$(K\mathcal{B}_{4n})_{(2,1,1)} = K\{[x_i, x_j, x_i] \cdot x_k, [x_i, x_j, x_j] \cdot x_k, [x_k, x_j, x_i] \cdot x_k \mid j < i < k\}.$$

The above subspaces are S_n-invariant, and it is obvious that

$$K\mathcal{B}_{4n} = (K\mathcal{B}_{4n})_{(2,2)} \oplus (K\mathcal{B}_{4n})_{(1,1,1,1)} \oplus (K\mathcal{B}_{4n})_{(2,1,1)}$$

this implies that the descriptions of the subspaces $(K\mathcal{B}_{4n})^{S_n}_{(1,1,1,1)}$, $(K\mathcal{B}_{4n})^{S_n}_{(2,2)}$ and $(K\mathcal{B}_{4n})^{S_n}_{(2,1,1)}$ are enough.

Let us start by proving that the subspace $(K\mathcal{B}_{4n})_{(1,1,1,1)}$ contains no nonzero symmetric polynomials. Let

$$p(X_n) = \sum_{j<i<k<l} (\alpha_{ijkl} [x_i, x_j, x_k] \cdot x_l + \beta_{kjil} [x_k, x_j, x_i] \cdot x_l)$$

for some $\alpha_{ijkl}, \beta_{kjil} \in K$. We may fix (i_0, j_0, k_0, l_0), and consider $\tau_{i_0 j_0} * p(X_n) = p(X_n)$, $\tau_{k_0 j_0} * p(X_n) = p(X_n)$, for transpositions$\tau_{i_0 j_0}$ and $\tau_{k_0 j_0}$. Because

$$\begin{aligned}
&\tau_{i_0 j_0} * \left(\alpha_{i_0 j_0 k_0 l_0}\left[x_{i_0}, x_{j_0}, x_{k_0}\right] \cdot x_{l_0} + \beta_{k_0 j_0 i_0 l_0}\left[x_{k_0}, x_{j_0}, x_{i_0}\right] \cdot x_{l_0}\right)\\
&\quad = \alpha_{i_0 j_0 k_0 l_0}\left[x_{j_0}, x_{i_0}, x_{k_0}\right] \cdot x_{l_0} + \beta_{k_0 j_0 i_0 l_0}\left[x_{k_0}, x_{i_0}, x_{j_0}\right] \cdot x_{l_0}\\
&\quad = -\alpha_{i_0 j_0 k_0 l_0}\left[x_{i_0}, x_{j_0}, x_{k_0}\right] \cdot x_{l_0} - \beta_{k_0 j_0 i_0 l_0}\left(\left[x_{i_0}, x_{j_0}, x_{k_0}\right] + \left[x_{j_0}, x_{k_0}, x_{i_0}\right]\right) \cdot x_{l_0}\\
&\quad = \left(-\alpha_{i_0 j_0 k_0 l_0} - \beta_{k_0 j_0 i_0 l_0}\right)\left[x_{i_0}, x_{j_0}, x_{k_0}\right] \cdot x_{l_0} + \beta_{k_0 j_0 i_0 l_0}\left[x_{k_0}, x_{j_0}, x_{i_0}\right] \cdot x_{l_0}
\end{aligned}$$

the coefficients of $\left[x_{i_0}, x_{j_0}, x_{k_0}\right] \cdot x_{l_0}$ and $\left[x_{k_0}, x_{j_0}, x_{i_0}\right] \cdot x_{l_0}$ in the equation

$$\begin{aligned}
\tau_{i_0 j_0} * p(X_n) = {}& \tau_{i_0 j_0} * \left(\alpha_{i_0 j_0 k_0 l_0}\left[x_{i_0}, x_{j_0}, x_{k_0}\right] \cdot x_{l_0} + \beta_{k_0 j_0 i_0 l_0}\left[x_{k_0}, x_{j_0}, x_{i_0}\right] \cdot x_{l_0}\right)\\
&+ \sum_{(i,j,k,l) \neq (i_0, j_0 k_0, l_0)} \tau_{i_0 j_0} * \left(\alpha_{ijkl}\left[x_i, x_j, x_k\right] \cdot x_l + \beta_{kjil}\left[x_k, x_j, x_i\right] \cdot x_l\right)
\end{aligned}$$

implies that

$$2\alpha_{i_0 j_0 k_0 l_0} + \beta_{k_0 j_0 i_0 l_0} = 0. \qquad (1)$$

On the other hand,

$$\begin{aligned}
&\tau_{k_0 j_0} * \left(\alpha_{i_0 j_0 k_0 l_0}\left[x_{i_0}, x_{j_0}, x_{k_0}\right] \cdot x_{l_0} + \beta_{k_0 j_0 i_0 l_0}\left[x_{k_0}, x_{j_0}, x_{i_0}\right] \cdot x_{l_0}\right)\\
&\quad = \alpha_{i_0 j_0 k_0 l_0}\left[x_{i_0}, x_{k_0}, x_{j_0}\right] \cdot x_{l_0} + \beta_{k_0 j_0 i_0 l_0}\left[x_{j_0}, x_{k_0}, x_{i_0}\right] \cdot x_{l_0}\\
&\quad = -\alpha_{i_0 j_0 k_0 l_0}\left(\left[x_{k_0}, x_{j_0}, x_{i_0}\right] + \left[x_{j_0}, x_{i_0}, x_{k_0}\right]\right) \cdot x_{l_0} - \beta_{k_0 j_0 i_0 l_0}\left[x_{k_0}, x_{j_0}, x_{i_0}\right] \cdot x_{l_0}\\
&\quad = \alpha_{i_0 j_0 k_0 l_0}\left[x_{i_0}, x_{j_0}, x_{k_0}\right] \cdot x_{l_0} + \left(-\alpha_{i_0 j_0 k_0 l_0} - \beta_{k_0 j_0 i_0 l_0}\right)\left[x_{k_0}, x_{j_0}, x_{i_0}\right] \cdot x_{l_0}
\end{aligned}$$

and hence,

$$\alpha_{i_0 j_0 k_0 l_0} + 2\beta_{k_0 j_0 i_0 l_0} = 0. \qquad (2)$$

by $\tau_{i_0 j_0} * p(X_n) = p(X_n)$. The equations (1) and (2) show that $\alpha_{i_0 j_0 k_0 l_0} = \beta_{k_0 j_0 i_0 l_0} = 0$. Similarly, $\alpha_{ijkl} = \beta_{kjil} = 0$ for all (i, j, k, l) implying that $p(X_n) = 0$. Therefore, $(K\mathcal{B}_{4n})^{S_n}_{(1,1,1,1)} = \{0\}$.

The following step of the proof is to prove that $(K\mathcal{B}_{4n})^{S_n}_{(2,2)}$ is spanned on the single element

$$\sum_{1 \leq j < i \leq n} \left[x_i, x_j, x_j\right] \cdot x_i.$$

Let $p(X_n) \in (K\mathcal{B}_{4n})_{(2,2)}{}^{S_n}$ be any symmetric polynomial of the form

$$p(X_n) = \sum_{1\leq j<i\leq n} \alpha_{ijji} \left[x_i, x_j, x_j\right] \cdot x_i$$

for some $\alpha_{ijji} \in K$, and fix $(i_0, j_0), (i_1, j_1)$. Now consider a permutation $\pi \in S_n$ in which $\pi(i_0) = i_1, \pi(j_0) = j_1$. Then

$$\pi * p(X_n) = \alpha_{i_0 j_0 j_0 i_0}\left[x_{i_1}, x_{j_1}, x_{j_1}\right] \cdot x_{i_1} + \alpha_{i_1 j_1 j_1 i_1}\left[x_{\pi(i_1)}, x_{\pi(j_1)}, x_{\pi(j_1)}\right] \cdot x_{\pi(i_1)}$$
$$+ \sum_{(i,j)\neq(i_0,j_0),(i_1,j_1)} \alpha_{ijji} \left[x_{\pi(i)}, x_{\pi(j)}, x_{\pi(j)}\right] \cdot x_{\pi(i)}$$

and hence $\alpha_{i_0 j_0 j_0 i_0} = \alpha_{i_1 j_1 j_1 i_1}$. Correspondingly, all coefficients α_{ijji} are equal. Therefore,

$$p(X_n) = \alpha \sum_{j<i} \left[x_i, x_j, x_j\right] \cdot x_i$$

for some $\alpha \in K$. Let $p(X_n) \in (K\mathcal{B}_{4n})^{S_n}_{(2,1,1)}$ be an arbitrary symmetric polynomial of the form

$$p(X_n) = \sum_{1\leq j<i<k\leq n} \big(\alpha_{ijik}[x_i, x_j, x_i] \cdot x_k + \alpha_{ijjk}[x_i, x_j, x_j] \cdot x_k + \alpha_{kjik}[x_k, x_j, x_i] \cdot x_k\big)$$

for some $\alpha_{ijik}, \alpha_{ijjk}, \alpha_{kjik} \in K$. One may fix (i_0, j_0, k_0) and consider the transposition $\tau_{i_0 j_0}$. Then, we obtain

$$\tau_{i_0 j_0} * \alpha_{i_0 j_0 i_0 k_0}[x_{i_0}, x_{j_0}, x_{i_0}] \cdot x_{k_0} + \alpha_{i_0 j_0 j_0 k_0}[x_{i_0}, x_{j_0}, x_{j_0}] \cdot x_{k_0}$$
$$+ \alpha_{k_0 j_0 i_0 k_0}[x_{k_0}, x_{j_0}, x_{i_0}] \cdot x_{k_0}$$
$$= \alpha_{i_0 j_0 i_0 k_0}[x_{j_0}, x_{i_0}, x_{j_0}] \cdot x_{k_0} + \alpha_{i_0 j_0 j_0 k_0}[x_{j_0}, x_{i_0}, x_{i_0}] \cdot x_{k_0}$$
$$+\alpha_{k_0 j_0 i_0 k_0}[x_{k_0}, x_{i_0}, x_{j_0}] \cdot x_{k_0}$$
$$= -\alpha_{i_0 j_0 i_0 k_0}[x_{i_0}, x_{j_0}, x_{j_0}] \cdot x_{k_0} - \alpha_{i_0 j_0 j_0 k_0}[x_{i_0}, x_{j_0}, x_{i_0}] \cdot x_{k_0}$$
$$-\alpha_{k_0 j_0 i_0 k_0}[x_{i_0}, x_{j_0}, x_{k_0}] \cdot x_{k_0} - \alpha_{k_0 j_0 i_0 k_0}[x_{j_0}, x_{k_0}, x_{i_0}] \cdot x_{k_0}$$
$$= -\alpha_{i_0 j_0 i_0 k_0}[x_{i_0}, x_{j_0}, x_{j_0}] \cdot x_{k_0} - \alpha_{i_0 j_0 j_0 k_0}[x_{i_0}, x_{j_0}, x_{i_0}] \cdot x_{k_0}$$
$$+\alpha_{k_0 j_0 i_0 k_0}[x_{k_0}, x_{j_0}, x_{i_0}] \cdot x_{k_0}$$

and hence $\tau_{i_0j_0} * p(X_n) = p(X_n)$ shows that $\alpha_{i_0j_0i_0k_0} = -\alpha_{i_0j_0j_0k_0}$.

Accordingly, we get from $\tau_{i_0k_0} * p(X_n) = p(X_n)$ that

$$\tau_{i_0k_0} * \alpha_{i_0j_0i_0k_0}[x_{i_0}, x_{j_0}, x_{i_0}] \cdot x_{k_0} + \alpha_{i_0j_0j_0k_0}[x_{i_0}, x_{j_0}, x_{j_0}] \cdot x_{k_0}$$
$$+ \alpha_{k_0j_0i_0k_0}[x_{k_0}, x_{j_0}, x_{i_0}] \cdot x_{k_0}$$
$$= \alpha_{i_0j_0i_0k_0}[x_{k_0}, x_{j_0}, x_{k_0}] \cdot x_{i_0} + \alpha_{i_0j_0j_0k_0}[x_{k_0}, x_{j_0}, x_{j_0}] \cdot x_{i_0}$$
$$+\alpha_{k_0j_0i_0k_0}[x_{i_0}, x_{j_0}, x_{k_0}] \cdot x_{i_0}$$
$$= -\alpha_{i_0j_0i_0k_0}[x_{k_0}, x_{j_0}, x_{i_0}] \cdot x_{k_0}$$
$$+\alpha_{i_0j_0j_0k_0}([x_{i_0}, x_{j_0}, x_{j_0}] \cdot x_{k_0} - [x_{j_0}, x_{j_0}, x_{i_0}] \cdot x_{k_0})$$
$$-\alpha_{k_0j_0i_0k_0}[x_{i_0}, x_{j_0}, x_{i_0}] \cdot x_{k_0}$$
$$= -\alpha_{i_0j_0i_0k_0}[x_{k_0}, x_{j_0}, x_{i_0}] \cdot x_{k_0} + \alpha_{i_0j_0j_0k_0}[x_{i_0}, x_{j_0}, x_{j_0}] \cdot x_{k_0}$$
$$-\alpha_{k_0j_0i_0k_0}[x_{i_0}, x_{j_0}, x_{i_0}] \cdot x_{k_0}$$

and that $\alpha_{i_0j_0i_0k_0} = -\alpha_{k_0j_0i_0k_0}$. Thus, similar computations show that

$$\alpha_{ijik} = -\alpha_{ijjk} = -\alpha_{kjik}$$

for every $j < i < k$, and $p(X_n)$ is of the form

$$p(X_n) = \sum_{j<i<k} \alpha_{ijik}\, r_{ijk}$$

for some $\alpha_{ijik} \in K$, where

$$r_{ijk} = r_{ijk}(X_n) = -[x_i, x_j, x_i] \cdot x_k + [x_i, x_j, x_j] \cdot x_k + [x_k, x_j, x_i] \cdot x_k$$

Since (i_1, j_1, k_1) is another triple and $\pi \in S_n$ is a permutation such that $\pi(i_0) = i_1$, $\pi(j_0) = j_1$, $\pi(k_0) = k_1$. Then

$$\pi * p(X_n) = \pi * \left(\alpha_{i_0j_0i_0k_0} r_{i_0j_0k_0} + \alpha_{i_1j_1i_1k_1} r_{i_1j_1k_1}\right)$$
$$+ \sum_{(i,j,k)\neq(i_0,j_0,k_0),(i_1,j_1,k_1)} \alpha_{ijik}\left(\pi * r_{ijk}\right)$$
$$= \alpha_{i_0j_0i_0k_0} r_{i_1j_1k_1} + \alpha_{i_1j_1i_1k_1} r_{\pi(i_1)\pi(j_1)\pi(k_1)}$$
$$+ \sum_{(i,j,k)\neq(i_0,j_0,k_0),(i_1,j_1,k_1)} \alpha_{ijik}\, r_{\pi(i)\pi(j)\pi(k)}$$

So, $\pi * p(X_n) = p(X_n)$ gives that $\alpha_{i_0 j_0 i_0 k_0} = \alpha_{i_1 j_1 i_1 k_1}$. This implies that all coefficients α_{ijik} are equivalent. Therefore, $p(X_n) = \alpha r(X_n)$ where

$$r(X_n) = \alpha \sum_{j<i<k} r_{ijk}(X_n)$$

for some $\alpha \in K$. Eventually, it is trivial to verify that $p_7^{(2)}$, $p_7^{(n)}$, $p_8^{(n)}$, $n \geq 3$, are symmetric, which completes the proof.

Corollary 4.3.3: The subspace $(K\mathcal{B}_{2n})^{S_n} \oplus (K\mathcal{B}_{3n})^{S_n} \oplus (K\mathcal{B}_{4n})^{S_n}$ is of a basis

$$\mathcal{B}_n = \left\{p_1^{(n)}, p_2^{(n)}, p_3^{(n)}, p_4^{(n)}, p_5^{(n)}, p_6^{(n)}, p_7^{(n)}, p_8^{(n)}\right\},$$

where $p_5^{(2)} = p_8^{(2)} = 0$, and the algebra $P_n^{S_n}$ of symmetric polynomials in the free metabelian Poisson algebra P_n is spanned on the set

$$P_n = \left\{g^{(n)}\right\} \cup \mathcal{G}_n * K[X_n]^{S_n} \cup \mathcal{B}_n$$

as a K-vector space.

In the following example, the explicit generators $\mathcal{G}_n = g_{ij}^{(n)} = \xi^{-1}\left(h_{ij}^{(n)}(U_n, X_n)\right)$ are given in Example 3.4.3 of the set P_n, which spans the vector space $P_n^{S_n}$ for $n = 2,3,4,5$ is determined.

Example 4.3.4: Here we give the explicit expression which spans the vector basis

$$\mathcal{B}_n = \left\{p_1^{(n)}, p_2^{(n)}, p_3^{(n)}, p_4^{(n)}, p_5^{(n)}, p_6^{(n)}, p_7^{(n)}, p_8^{(n)}\right\},$$

where $p_5^{(2)} = p_8^{(2)} = 0$.

For $n = 2$. The vector space $\mathcal{B}_2$ is spanned on

$$\left\{p_1^{(2)}, p_2^{(2)}, p_3^{(2)}, p_4^{(2)}, p_6^{(2)}, p_7^{(2)}\right\}$$

Such that

$p_1^{(2)} = x_1 \cdot x_1 + x_2 \cdot x_2$, $p_2^{(2)} = x_1 \cdot x_2$, $p_3^{(2)} = x_1 \cdot x_1 \cdot x_1 + x_2 \cdot x_2 \cdot x_2$

$p_4^{(2)} = x_1 \cdot x_1 \cdot x_2 + x_1 \cdot x_2 \cdot x_2$, $p_6^{(2)} = [x_2, x_1] \cdot (x_2 - x_1)$ and

$p_7^{(2)} = [x_2, x_1, x_1] \cdot x_2$.

For $n = 3$. The vector space $\mathcal{B}_3$ is spanned on

$$\left\{p_1^{(3)}, p_2^{(3)}, p_3^{(3)}, p_4^{(3)}, p_5^{(3)}, p_6^{(3)}, p_7^{(3)}, p_8^{(3)}\right\}$$

Such that

$p_1^{(3)} = x_1 \cdot x_1 + x_2 \cdot x_2 + x_3 \cdot x_3$, $p_2^{(3)} = x_1 \cdot x_2 + x_1 \cdot x_3 + x_2 \cdot x_3$,

$p_3^{(3)} = x_1 \cdot x_1 \cdot x_1 + x_2 \cdot x_2 \cdot x_2 + x_3 \cdot x_3 \cdot x_3$,

$p_4^{(3)} = x_1 \cdot x_1 \cdot x_2 + x_1 \cdot x_1 \cdot x_3 + x_1 \cdot x_2 \cdot x_2 + x_1 \cdot x_3 \cdot x_3 + x_2 \cdot x_2 \cdot x_3$
$+ x_2 \cdot x_3 \cdot x_3$,

$p_5^{(3)} = x_1 \cdot x_2 \cdot x_3$,

$p_6^{(3)} = [x_2, x_1] \cdot (x_2 - x_1) + [x_3, x_1] \cdot (x_3 - x_1) + [x_3, x_2] \cdot (x_3 - x_2)$,

$p_7^{(3)} = [x_2, x_1, x_1] \cdot x_2 + [x_3, x_1, x_1] \cdot x_3 + [x_3, x_2, x_2] \cdot x_3$ and

$p_8^{(3)} = [x_2, x_1, x_1] \cdot x_3 - [x_2, x_1, x_2] \cdot x_3 + [x_3, x_1, x_2] \cdot x_3$.

For $n = 4$. The vector space $\mathcal{B}_4$ is spanned on

$$\left\{p_1^{(4)}, p_2^{(4)}, p_3^{(4)}, p_4^{(4)}, p_5^{(4)}, p_6^{(4)}, p_7^{(4)}, p_8^{(4)}\right\}$$

Such that

$p_1^{(4)} = x_1 \cdot x_1 + x_2 \cdot x_2 + x_3 \cdot x_3 + x_4 \cdot x_4$,

$p_2^{(4)} = x_1 \cdot x_2 + x_1 \cdot x_3 + x_1 \cdot x_4 + x_2 \cdot x_3 + x_2 \cdot x_4 + x_3 \cdot x_4$,

$p_3^{(4)} = x_1 \cdot x_1 \cdot x_1 + x_2 \cdot x_2 \cdot x_2 + x_3 \cdot x_3 \cdot x_3 + x_4 \cdot x_4 \cdot x_4$,

$p_4^{(4)} = x_1 \cdot x_1 \cdot x_2 + x_1 \cdot x_1 \cdot x_3 + x_1 \cdot x_1 \cdot x_4 + x_1 \cdot x_2 \cdot x_2 + x_1 \cdot x_3 \cdot x_3$
$+ x_1 \cdot x_4 \cdot x_4 + x_2 \cdot x_2 \cdot x_3 + x_2 \cdot x_2 \cdot x_4 + x_2 \cdot x_3 \cdot x_3 + x_2 \cdot x_4 \cdot x_4$
$+ x_3 \cdot x_3 \cdot x_4 + x_3 \cdot x_4 \cdot x_4$,

$p_5^{(4)} = x_1 \cdot x_2 \cdot x_3 + x_1 \cdot x_2 \cdot x_4 + x_1 \cdot x_3 \cdot x_4 + x_2 \cdot x_3 \cdot x_4$,

$p_6^{(4)} = [x_2, x_1] \cdot (x_2 - x_1) + [x_3, x_1] \cdot (x_3 - x_1) + [x_4, x_1] \cdot (x_4 - x_1)$
$+ [x_3, x_2] \cdot (x_3 - x_2) + [x_4, x_2] \cdot (x_4 - x_2) + [x_4, x_3] \cdot (x_4 - x_3)$,

$p_7^{(4)} = [x_2, x_1, x_1] \cdot x_2 + [x_3, x_1, x_1] \cdot x_3 + [x_4, x_1, x_1] \cdot x_4 + [x_3, x_2, x_2] \cdot x_3$

$$+[x_4,x_2,x_2]\cdot x_4+[x_4,x_3,x_3]\cdot x_4\ ,$$

$$p_8^{(4)}=[x_2,x_1,x_1]\cdot x_3-[x_2,x_1,x_2]\cdot x_3+[x_3,x_1,x_2]\cdot x_3+[x_2,x_1,x_1]\cdot x_4$$
$$-[x_2,x_1,x_2]\cdot x_4+[x_4,x_1,x_2]\cdot x_4+[x_3,x_1,x_1]\cdot x_4-[x_3,x_1,x_3]\cdot x_4$$
$$+[x_4,x_1,x_3]\cdot x_4+[x_3,x_2,x_2]\cdot x_4-[x_3,x_2,x_3]\cdot x_4+[x_4,x_2,x_3]\cdot x_4$$

For $n=5$. The vector space $\mathcal{B}_5$ is spanned on

$$\left\{p_1^{(5)},p_2^{(5)},p_3^{(5)},p_4^{(5)},p_5^{(5)},p_6^{(5)},p_7^{(5)},p_8^{(5)}\right\}$$

Such that

$$p_1^{(5)}=x_1\cdot x_1+x_2\cdot x_2+x_3\cdot x_3+x_4\cdot x_4+x_5\cdot x_5,$$

$$p_2^{(5)}=x_1\cdot x_2+x_1\cdot x_3+x_1\cdot x_4+x_1\cdot x_5+x_2\cdot x_3+x_2\cdot x_4+x_2\cdot x_5$$
$$+x_3\cdot x_4+x_3\cdot x_5+x_4\cdot x_5,$$

$$p_3^{(5)}=x_1\cdot x_1\cdot x_1+x_2\cdot x_2\cdot x_2+x_3\cdot x_3\cdot x_3+x_4\cdot x_4\cdot x_4+x_5\cdot x_5\cdot x_5,$$

$$p_4^{(5)}=x_1\cdot x_1\cdot x_2+x_1\cdot x_1\cdot x_3+x_1\cdot x_1\cdot x_4+x_1\cdot x_1\cdot x_5+x_1\cdot x_2\cdot x_2$$
$$+x_1\cdot x_3\cdot x_3+x_1\cdot x_4\cdot x_4+x_1\cdot x_5\cdot x_5+x_2\cdot x_2\cdot x_3+x_2\cdot x_2\cdot x_4$$
$$+x_2\cdot x_2\cdot x_5+x_2\cdot x_3\cdot x_3+x_2\cdot x_4\cdot x_4+x_2\cdot x_5\cdot x_5+x_3\cdot x_3\cdot x_4$$
$$+x_3\cdot x_3\cdot x_5+x_3\cdot x_4\cdot x_4+x_3\cdot x_5\cdot x_5+x_4\cdot x_4\cdot x_5+x_4\cdot x_5\cdot x_5,$$

$$p_5^{(5)}=x_1\cdot x_2\cdot x_3+x_1\cdot x_2\cdot x_4+x_1\cdot x_2\cdot x_5+x_1\cdot x_3\cdot x_4+x_1\cdot x_3\cdot x_5$$
$$+x_1\cdot x_4\cdot x_5+x_2\cdot x_3\cdot x_4+x_2\cdot x_3\cdot x_5+x_2\cdot x_4\cdot x_5+x_3\cdot x_4\cdot x_5,$$

$$p_6^{(5)}=[x_2,x_1]\cdot(x_2-x_1)+[x_3,x_1]\cdot(x_3-x_1)+[x_4,x_1]\cdot(x_4-x_1)$$
$$+[x_5,x_1]\cdot(x_5-x_1)+[x_3,x_2]\cdot(x_3-x_2)+[x_4,x_2]\cdot(x_4-x_2)$$
$$+[x_5,x_2]\cdot(x_5-x_2)+[x_4,x_3]\cdot(x_4-x_3)+[x_5,x_3]\cdot(x_5-x_3)$$
$$+[x_5,x_4]\cdot(x_5-x_4),$$

$$p_7^{(5)}=[x_2,x_1,x_1]\cdot x_2+[x_3,x_1,x_1]\cdot x_3+[x_4,x_1,x_1]\cdot x_4+[x_5,x_1,x_1]\cdot x_5$$
$$+[x_3,x_2,x_2]\cdot x_3+[x_4,x_2,x_2]\cdot x_4+[x_5,x_2,x_2]\cdot x_5+[x_4,x_3,x_3]\cdot x_4$$
$$+[x_5,x_3,x_3]\cdot x_5+[x_5,x_4,x_4]\cdot x_5,$$

$$p_8^{(5)}=[x_2,x_1,x_1]\cdot x_3-[x_2,x_1,x_2]\cdot x_3+[x_3,x_1,x_2]\cdot x_3+[x_2,x_1,x_1]\cdot x_4$$
$$-[x_2,x_1,x_2]\cdot x_4+[x_4,x_1,x_2]\cdot x_4+[x_2,x_1,x_1]\cdot x_5-[x_2,x_1,x_2]\cdot x_5$$

$$+[x_5,x_1,x_2]\cdot x_5+[x_3,x_1,x_1]\cdot x_4-[x_3,x_1,x_3]\cdot x_4+[x_4,x_1,x_3]\cdot x_4$$
$$+[x_3,x_1,x_1]\cdot x_5-[x_3,x_1,x_3]\cdot x_5+[x_5,x_1,x_3]\cdot x_5+[x_4,x_1,x_1]\cdot x_5$$
$$-[x_4,x_1,x_4]\cdot x_5+[x_5,x_1,x_4]\cdot x_5+[x_3,x_2,x_2]\cdot x_4-[x_3,x_2,x_3]\cdot x_4$$
$$+[x_4,x_2,x_3]\cdot x_4+[x_3,x_2,x_2]\cdot x_5-[x_3,x_2,x_3]\cdot x_5+[x_5,x_2,x_3]\cdot x_5$$
$$+[x_4,x_2,x_2]\cdot x_5-[x_4,x_2,x_4]\cdot x_5+[x_5,x_2,x_4]\cdot x_5+[x_4,x_3,x_3]\cdot x_5$$
$$-[x_4,x_3,x_4]\cdot x_5+[x_5,x_3,x_4]\cdot x_5.$$

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