THE METHOD OF INVERSE SCATTERING AND BÄCKLUND TRANSFORMATIONS

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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Solving integrable nonlinear partial differential equations by the method of inverse scattering and Bäcklund transformations are very effective. Both of these methods depend on the existence of Lax pair. The Bäcklund transformations provide us some particular solutions of the integrable nonlinear partial differential equations but the method of inverse scattering solves the initial value problem of these equations. In this work, we use these methods to solve the Korteweg-de Vries equation, the Korteweg-de Vries hierarchy and AKNS system.

Keywords: Bäcklund transformations, Inverse scattering method, Korteweg de Vries equation, AKNS system.

ÖZET

TERS SAÇILMA YÖNTEMİ VE BÄCKLUND DÖNÜŞÜMLERİ

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Matematik, Yüksek Lisans
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İntegrallenebilir doğrusal olmayan kısmi diferansiyel denklemlerin çözülmesinde ters saçılma ve Bäcklund dönüşümü yöntemleri çok etkilidir. Bu yöntemlerin her ikisi de Lax çiftinin varlığına bağlıdır. Bäcklund dönüşümleri bize integrallenebilir doğrusal olmayan kısmi diferansiyel denklemin bazı özel çözümlerini sağlar, ancak ters saçılma yöntemi ise bu denklemlerin başlangıç değer problemlerini çözer. Bu çalışmada Korteweg-de Vries denklemini, Korteweg-de Vries hiyerarşisini ve AKNS sistemini çözmek için bu yöntemleri kullanıyoruz.

Anahtar sözcükler: Bäcklund dönüşümü, Ters saçılma yöntemi, Korteweg de Vries denklemi, AKNS sistemi.

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Chapter 1

Introduction

Integrable systems of nonlinear partial differential equations (pdes) have been the center of interest for mathematicians and mathematical physicists. Since nonlinear pdes appear in many different branches of physics e.g., fluid mechanics, plasma physics, etc. It is also known that the integrable systems of nonlinear partial differential equations have various mathematical structures, some of which are the existence of infinitely many local conservation laws, Lax pairs, the applicability of inverse scattering method, and Bäcklund transformations.

The solitary wave (a soliton) was first observed by J. Scott Russell [1] on the Edinburgh-Glasgow Canal in 1834. After a number of experiments, he observed the relation between the speed and the amplitude of the wave as

$$\nu^2 = g(h + \alpha), \tag{1.1}$$

where g is the acceleration of gravity, h is the undisturbed depth of water, ν is the speed of the wave, and α is the amplitude of the wave. Up to that time, solitons were unknown in the theory of partial differential equations. However, in 1895, Korteweg and de Vries [2] derived an equation that has a solitary wave solution, which is known as the Korteweg-de Vries (KdV) equation. The KdV equation can be written as

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1.2}$$

which describes the propagation of waves on shallow water surfaces. The solitary wave solution can be obtained by seeking a travelling wave solution

$$u(x,t) = f(x - \nu t), \tag{1.3}$$

where f is an arbitrary function and ν is the speed of the wave. By inserting (1.3) into (1.2), we get

$$f''' - (6f + \nu)f' = 0, \tag{1.4}$$

which can be integrated once, then multiply by f' and integrate once more yielding

$$\frac{1}{2}(f')^2 = f^3 + \frac{1}{2}\nu f^2 + Af + B, \tag{1.5}$$

where A and B are integration constants. We assume that the solution u and its x-derivatives vanish at infinity, then $A = B = 0$. One more integration, and we obtain

$$u(x,t) = -\frac{1}{2}\nu\,\mathrm{sech}^2\Big\{\frac{1}{2}\nu^{1/2}(x - \nu t - x_0)\Big\}, \tag{1.6}$$

which is a solitary wave solution where arbitrary x_0 denotes the location of the peak at $t = 0$. Note that the wave moves to the right with a speed ν which is proportional to the amplitude. After this observation, a question arises: what happens after the collision of two solitary waves? The answer to this question was given by Zabusky and Kruskal [3] in their study of the Fermi-Pasta-Ulam problem. They observed that after the collision, waves retained their identities, such as speed, width, and shape. Since these waves resemble the particles, they were named as solitons. The numerical discovery of solitons by Kruskal and Zabusky played a key role in the revival of the modern theory of integrable systems.

Definition 1.0.1. A soliton solution of a nonlinear partial differential equation or a system is a solution that

i) represents a wave of permanent form

ii) is localized so that it decays or approaches a constant at infinity

iii) can collide with other solitons and retain its identity, i.e., speed and shape.

Besides having a soliton solution, the KdV equation possesses another property, which is the existence of infinitely many local conservation laws. The equation in the form

$$T_t + X_x = 0 \tag{1.7}$$

is called a conservation law where T is the conserved density and X is the flux. In particular, T and X depend on x, t, u, and its x-derivatives, but not t-derivatives of u. If $T = F_x$, then the conservation law is trivial. The first three conservation laws for the KdV equation are given as

$$\begin{gathered}
(u)_t + (u_{xx} - 3u^2)_x = 0, \\
\big(\frac{1}{2}u^2\big)_t + \big(uu_{xx} - \frac{1}{2}u_x^2 - 2u^3\big)_x = 0, \\
\big(u^3 + \frac{1}{2}u_x^2\big)_t + \big(-\frac{9}{2}u^4 + 3u^2u_{xx} - 6uu_x^2 + u_xu_{xxx} - \frac{1}{2}u_{xx}^2\big)_x = 0,
\end{gathered} \tag{1.8}$$

where $u \to 0$ as $|x| \to +\infty$. Hence, from (1.8), we obtain constants of the motion

$$\begin{gathered}
\int_{-\infty}^{\infty} u\,dx \;= \text{constant}, \qquad \int_{-\infty}^{\infty} u^2\,dx \;= \text{constant}, \\
\int_{-\infty}^{\infty} \big(u^3 + \frac{1}{2}u_x^2\big)\,dx \;= \text{constant},
\end{gathered} \tag{1.9}$$

which are known as the conservation of mass, momentum, and energy, respectively. In addition to these, Miura, Gardner, and Kruskal [4] found six more conservation laws for the KdV equation and they conjectured that there were infinitely many. They attempted to find conservation laws for the equation

$$u_t - 6u^p u_x + u_{xxx} = 0, \quad p = 1, 2, 3... \tag{1.10}$$

and found that only for $p = 1, 2$ has infinitely many conservation laws. For $p = 1$, it is the KdV equation, and for $p = 2$, it is the modified KdV (mKdV) equation. Miura [5] discovered a transformation from the conserved densities of the mKdV equation to the conserved densities of the KdV equation which is known as the Miura transformation. It also transforms the solutions of the mKdV equation to the solutions of the KdV equation by

$$u = v^2 + v_x \tag{1.11}$$

where u satisfies the KdV equation and v satisfies the mKdV equation. By using the Wentzel–Kramers–Brillouin (WKB) approximation and the Miura transformation (1.11), Miura, Gardner, and Kruskal proved that the KdV equation and the mKdV equation have infinitely many conservation laws.

The Fourier transform can be used to solve the initial value problem for linear partial differential equations. A generalization of the Fourier transform to the nonlinear partial differential equation is known as the method of inverse scattering transform (IST). In 1967, Gardner, Green, Kruskal, and Miura [6] introduced the method of inverse scattering transform for the KdV equation and solved the initial value problem of the KdV equation. In 1968, Lax [7] introduced a new method for solving nonlinear pdes which is known as the Lax pair or Lax representation. This method can be used to write a nonlinear pde as the integrability condition of a system of linear equations involving two operators L and M, where L is the operator of the eigenvalue problem and M is the operator of the time evolution equation such that

$$L\psi = \lambda\psi, \tag{1.12}$$

$$\psi_t = M\psi. \tag{1.13}$$

By taking the t-derivative of (1.12) and, using (1.13) and (1.12), we obtain the evolution equation as

$$L_t = [M, L] = ML - LM, \tag{1.14}$$

where $\lambda_t = 0$. The details are provided in Section 2.1, which also includes a method for finding the KdV hierarchy and the recursion operator for the KdV equation. The versatility of the Lax pair method in dealing with a broad range of nonlinear pdes has enabled the extension of the IST to general evolution equations by IST. In 1972, Zakharov and Shabat [8] solved the nonlinear Schrödinger equation. In 1973, Wadati [9] applied the same method to the modified KdV equation. In 1974, Ablowitz, Kaup, Newell, and Segur solved the sine-Gordon equation [10] and they developed a new method called the AKNS scheme [11], which was a generalization of the work of Zakharov and Shabat.

The method of Fourier transform consists of three steps. Let us consider the linear partial differential equation

$$u_t(x, t) = -if(-i\partial_x)u(x, t) \tag{1.15}$$

with the initial value $u(x, 0) = u_0(x)$ which vanishes asymptotically. Here, $f(z)$ is an analytic function i.e., a polynomial or a rational function. For simplicity, we assume that $f(z)$ is a real and odd function, then $u(z,t)$ is real for $t > 0$ whenever the initial value is real. The Fourier transform is given by

$$\hat{u}(k,t) = \int_{-\infty}^{\infty} u(x,t)e^{-ikx}\,dx \ . \tag{1.16}$$

The Steps are as follows:

I) Determine the Fourier coefficients $\hat{u}(k,0)$ from the initial value $u_0(x)$ (This step is analogous to the process of determining the scattering data at $t = 0$ in the IST).

$$\hat{u}(k,0) = \int_{-\infty}^{\infty} u_0(x)e^{-ikx}\,dx \ . \tag{1.17}$$

II) Find the time evolution of the Fourier coefficient $\hat{u}(k,t)$ (This step is analogous to the process of finding the time evolution of the scattering data in the IST). By taking the Fourier transform of (1.15) and integrating with respect to t once, we obtain

$$\hat{u}(k,t) = \hat{u}(k,0)e^{-if(k)t}. \tag{1.18}$$

III) The last step is to obtain the solution $u(x,t)$ of (1.15) by using the inverse Fourier transform of (1.18) (This step is analogous to solving the inverse problem in the IST), then

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(x)e^{ik(x-y)-if(k)t}\,dy\,dk \ . \tag{1.19}$$

Note that the spectrum k which denotes the number of wave is independent of time [12].

$$\begin{tikzcd}
u(x,0) \arrow[r, "\text{I.Fourier Transform}"] \arrow[d, dashed, "\text{PDE}"'] & \hat{u}(k,0) \arrow[d, "\text{II.Time Evolution}"] \\
u(x,t) & \hat{u}(k,t) \arrow[l, "\substack{\text{III.Inverse} \\ \text{Fourier Transform}}"]
\end{tikzcd}$$

Figure 1.1: Steps of the Method of Fourier Transform

The number of steps in the IST is one greater than the number of steps in the method of Fourier transform. The IST consists of four steps: the direct problem (determining the scattering data at an initial time), the time evolution of the scattering data (determining the scattering data at any time), solving the Gelfand-Levitan-Marchenko (GLM) equation, and the inverse problem. In Section 2.2, the IST is used to solve the initial value problem of the KdV equation, and details are provided. In Section 2.3, we solved the initial value problem of the KdV hierarchy by using the IST. In Chapter 3, we first provided the necessary information about the AKNS scheme and use the IST for this system.

The Bäcklund transformations were first introduced by the Swedish mathematician B. Bäcklund [13] in the late 19th century. These transformations provide particular solutions to integrable nonlinear partial differential equations. There are two types of Bäcklund transformations (BT): one of which relate solutions of different integrable systems, and the other one relates two solutions of the same integrable system which is known as the auto-Bäcklund transformation. To find the auto-BT for the KdV equation, Clairin's method, developed by the French mathematician Georges Clairin in 1974, is used by Lamb [14]. Wahlquist and Estabrook used the differential form to derive the same auto-BT [15], [16]. Furthermore, Chen derived the Bäcklund transformations for the AKNS scheme [17], [18]. Wadati, Sanuki, and Konno [19] also derived the BT for the same system independently of Chen's work. In the same year, Konno and Wadati [20] extended the previous work by considering the auto-BT of the nonlinear Schrödinger equation. In Chapter 4, we present several examples of Bäcklund transformations for various integrable systems. We then proceed to derive the auto-BT for the KdV equation by using the Miura transformation. Finally, we derive the BT for the AKNS scheme, which also includes the KdV hierarchy.

Chapter 2

Inverse Scattering Transform for the KdV Equation

2.1 Lax Formulation of the KdV Equation

Our aim is to solve the initial value problem for u, where $u(x,t)$ satisfies a nonlinear evolution equation of the form

$$u_t = N(u) \tag{2.1}$$

with the initial value, $u(x,0) = f(x)$. Let $\mathscr{B}$ be an appropriate function space and we assume that $u \in \mathscr{B}$ for each t and $N : \mathscr{B} \mapsto \mathscr{B}$ is a nonlinear operator which depends on x and derivatives with respect x, but is independent of t.

We can associate a self-adjoint operator L over a Hilbert space for each function u in $\mathscr{B}$ with the property: if u changes with respect to t according to the equation (2.1) then the operators $L(t)$ also change with respect to t which are unitarily equivalent. Then, there is a one parameter family of unitary operators $U(t)$ where

$$U(t)^{-1}L(t)U(t) = L(t_0) \tag{2.2}$$

where t_0 is a fixed value of t. Take t-derivative, to yield

$$-U^{-1}U_tLU + U^{-1}L_tU + U^{-1}LU_t = 0. \tag{2.3}$$

We suppose that $U(t)$ satisfies a differential equation

$$U_t = MU \tag{2.4}$$

where $M(t)$ is a skew-adjoint operator which is acting on $\mathscr{B}$, i.e., $M^\dagger = -M$. By substituting (2.4) into (2.3), we obtain

$$-U^{-1}MLU + U^{-1}L_tU + U^{-1}LMU = 0 \tag{2.5}$$

and then, multiplying by U from the left and by U^{-1} from the right, we get

$$-ML + L_t + LM = 0 \tag{2.6}$$

or

$$L_t = ML - LM = [M, L]. \tag{2.7}$$

Thus, the evolution equation (2.1) is identical to (2.7) where L and M are some linear operators in x acting on $\mathscr{B}$ [7]. The shortcoming of this process is that one must take a correct guess for the operator L to obtain (2.1).

Gardner, Kruskal, and Miura [6] discovered that for the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{2.8}$$

the operator L is the Schrödinger operator, i.e.,

$$L = -\frac{\partial^2}{\partial x^2} + u(x,t) \quad \text{and} \quad L_t = u_t. \tag{2.9}$$

Now, let $\mathscr{B}$ be a Hilbert space with the inner product $(\phi, \psi) = \displaystyle\int_{-\infty}^{\infty} \phi\psi \, dx$. We assume L is self-adjoint i.e., $L = L^\dagger$ so that

$$(L\phi, \psi) = (\phi, L\psi) \tag{2.10}$$

for all $\phi, \psi \in \mathscr{B}$. We introduce the eigenvalue equation or the Sturm-Liouville equation

$$L\psi = \lambda\psi \tag{2.11}$$

for $t \geq 0$ and $x \in \mathbb{R}$ where, in general, $\lambda = \lambda(t)$. Take t-derivative of (2.11), we have

$$L_t\psi + L\psi_t = \lambda_t\psi + \lambda\psi_t. \tag{2.12}$$

Use the equation (2.7) yielding

$$[M, L]\psi + (L - \lambda)\psi_t = \lambda_t \psi \tag{2.13}$$

or

$$ML\psi - LM\psi + (L - \lambda)\psi_t = \lambda_t \psi. \tag{2.14}$$

We insert the equation (2.11) and obtain

$$\lambda M\psi - LM\psi + (L - \lambda)\psi_t = \lambda_t \psi \tag{2.15}$$

or equivalently

$$(L - \lambda)(\psi_t - M\psi) = \lambda_t \psi. \tag{2.16}$$

Now, take inner product with ψ, to obtain

$$\big(\psi, (L - \lambda)(\psi_t - M\psi)\big) = \lambda_t(\psi, \psi). \tag{2.17}$$

Since $L - \lambda$ is a self-adjoint operator, we can write

$$\big((L - \lambda)\psi, (\psi_t - M\psi)\big) = \lambda_t(\psi, \psi) \tag{2.18}$$

and, since $(L - \lambda)\psi = 0$ then we find that λ does not depend on t ($\lambda_t = 0$) since $(\psi, \psi) \neq 0$.

From (2.16) and (2.11), we obtain

$$\psi_t - M\psi = \alpha\psi \tag{2.19}$$

where α is a constant. Since (2.7) is invariant under the transformation $M \mapsto M + \alpha I$, we can let $\alpha = 0$, then the time-evolution for ψ, for $t > 0$

$$\psi_t = M\psi. \tag{2.20}$$

For the (third order) KdV equation, Lax pair is determined as

$$L = -\frac{\partial^2}{\partial x^2} + u(x, t) \tag{2.21}$$

and

$$M = -u_x + 2(u + 2\lambda)\frac{\partial}{\partial x} \tag{2.22}$$

where L is a self-adjoint operator and M is a skew-adjoint operator. Then the Lax equation $L_t = [M, L]$ is satisfied if and only if the KdV equation holds.

2.1.1 KdV hierarchy

With the help of Lax formulation, we can obtain the hierarchy of KdV equation. Let L be the Schrödinger operator, which is the same operator in the case of KdV, but let us choose a different operator M as

$$M = A + B\partial_x \tag{2.23}$$

where A and B are functions of u and its x-derivatives. Hence, the eigenvalue equation and the time-evolution equation are given respectively

$$\psi_{xx} = (u - \lambda)\psi \tag{2.24}$$

and

$$\psi_t = A\psi + B\psi_x. \tag{2.25}$$

The operator M is required to have enough freedom to make the operator $L_t - [M, L]$ of degree zero, i.e. multiplicative operator.

After an elementary calculation, we find that

$$\begin{aligned}[M, L] &= (A + B\partial_x)(-\partial_x^2 + u) - (-\partial_x^2 + u)(A + B\partial_x) \\ &= A_{xx} + Bu_x + (2A_x + B_{xx})\partial_x + 2B_x\partial_x^2.\end{aligned} \tag{2.26}$$

Since $L\psi = \lambda\psi$ where $L = -\partial_x^2 + u$, then

$$[M, L] = A_{xx} + Bu_x + 2(u - \lambda)B_x + (2A_x + B_{xx})\partial_x \tag{2.27}$$

which is a multiplicative operator if

$$2A_x + B_{xx} = 0 \tag{2.28}$$

or

$$A = -\frac{1}{2}B_x + h(t), \tag{2.29}$$

where h is an arbitrary function of t, then $A_{xx} = -\frac{1}{2}B_{xxx}$, and it yields

$$u_t = -\frac{1}{2}B_{xxx} + Bu_x + 2(u - \lambda)B_x. \tag{2.30}$$

Now, assuming B is analytic on λ then we can expand it as a power series of λ

$$B = \sum_{n=0}^{N} b_{N-n}\lambda^n = b_0\lambda^N + b_1\lambda^{N-1} + ... + b_N \tag{2.31}$$

where b_i's depend on u and its derivatives with respect to x. We insert (2.31) into (2.30), and obtain

$$\begin{aligned} u_t =& u_x \sum_{n=0}^{N} b_{N-n}\lambda^n - \frac{1}{2}\sum_{n=0}^{N} b_{N-n,xxx}\lambda^n \\ &+ 2u\sum_{n=0}^{N} b_{N-n,x}\lambda^n - 2\sum_{n=0}^{N} b_{N-n,x}\lambda^{n+1} = 0. \end{aligned} \tag{2.32}$$

From the coefficient of λ^0, we have

$$u_t = b_N u_x - \frac{1}{2} b_{N,xxx} + 2u b_{N,x} \tag{2.33}$$

and from the coefficients of λ^n, $n > 0$, we have

$$u_x b_{N-n} - \frac{1}{2} b_{N-n,xxx} + 2u b_{N-n,x} - 2b_{N-n+1,x} = 0. \tag{2.34}$$

Let $N - n = k$, then the equation (2.34) becomes

$$\begin{aligned} b_{k+1,x} &= -\frac{1}{4}\big(\partial_x^3 - 4u\partial_x - 2u_x\big) b_k \\ &= -\frac{1}{4}\big(\partial_x^2 - 4u - 2u_x\partial_x^{-1}\big) b_{k,x} \end{aligned} \tag{2.35}$$

We define $\mathcal{R} = \partial_x^2 - 4u - 2u_x\partial_x^{-1}$ and then

$$b_{k+1,x} = -\frac{1}{4}\mathcal{R} b_{k,x} \quad \text{for} \quad k = 0, 1, 2, ..., N-1 \tag{2.36}$$

where $\mathcal{R}$ is called the recursion operator. The equation (2.33) becomes

$$u_t = -\frac{1}{2}\mathcal{R} b_{N,x} \tag{2.37}$$

where $b_{k+1,x} = -\frac{1}{4}\mathcal{R} b_{k,x}$ for $k = 0, 1, ...$

Let b_0 be constant, the equation (2.36) provides

$$b_1 = \frac{b_0}{2} u \tag{2.38}$$

and

$$b_{k+1,x} = -\frac{1}{4}\mathcal{R}b_{k,x} = \left(-\frac{1}{4}\right)^2\mathcal{R}^2 b_{k-1,x} = \left(-\frac{1}{4}\right)^k\mathcal{R}^k b_{1,x} = \frac{b_0}{2}\left(-\frac{1}{4}\right)^k\mathcal{R}^k u_x \tag{2.39}$$

then,

$$u_t = -\frac{1}{2}\mathcal{R}b_{N,x} = -\frac{b_0}{4}\left(-\frac{1}{4}\right)^{N-1}\mathcal{R}^N u_x = b_0\left(-\frac{1}{4}\right)^N\mathcal{R}^N u_x. \tag{2.40}$$

We choose $b_0\left(-\frac{1}{4}\right)^N = -1$. Hence (2.40) becomes

$$u_t + \mathcal{R}^N u_x = 0, \quad N = 1, 2, \ldots \quad . \tag{2.41}$$

The above equation defines the hierarchy of KdV equation where $\mathcal{R} = \partial_x^2 - 4u - 2u_x\partial_x^{-1}$ is the recursion operator of the KdV equation.

All the members of the hierarchy have the same eigenvalue equation or the operator L, but the time-evolution equation or operator M changes. The first few members of this hierarchy are obtained as follows:

Take $N = 1$, we obtain the third-order KdV equation

$$u_t + \mathcal{R}u_x = u_t - 6uu_x + u_{xxx} = 0, \tag{2.42}$$

for $N = 2$, the fifth-order KdV equation is yielded

$$u_t + \mathcal{R}^2 u_x = u_t + \mathcal{R}(u_{xxx} - 6uu_x) = 0 \tag{2.43}$$

or

$$u_t + u_5 - 10uu_{xxx} - 20u_xu_{xx} + 30u^2u_x = 0, \tag{2.44}$$

for $N = 3$, the seventh-order KdV equation is

$$\begin{aligned} &u_t - 140u^3u_x + 70u_x^3 + 280uu_xu_{xx} + 70u^2u_{xxx} \\ &- 70u_{xx}u_{xxx} - 42u_xu_4 - 14uu_5 + u_7 = 0 \end{aligned} \tag{2.45}$$

and, lastly, for $N = 4$, the ninth-order KdV equation is

$$\begin{aligned} &u_t - 1260uu_x^3 - 2520u^2u_xu_{xx} - 420u^3u_{xxx} + 1302u_xu_{xx}^2 \\ &+ 966u_x^2u_{xxx} + 1260uu_{xx}u_{xxx} + 756uu_xu_4 + 126u^2u_5 \\ &- 252u_{xxx}u_4 - 168u_{xx}u_5 - 72u_xu_6 - 28uu_7 + 630u_xu^4 \\ &+ u_9 = 0. \end{aligned} \tag{2.46}$$

where u_m for $m \in \mathbb{N}$ denotes the mth order derivative of u with respect to x.

2.2 The inverse scattering transform for the KdV equation

There is a very nice connection between the KdV equation and the Schrödinger equation in quantum mechanics which is a special case of the Sturm-Liouville equation in mathematics. By using this equation, we are going to solve the initial value problem of the KdV equation.

Now, we shall see how the KdV equation is connected to the Schrödinger equation. The KdV equation is given in the standard form by

$$u_t - 6uu_x + u_{xxx} = 0. \tag{2.47}$$

To show this connection, let us define a function

$$u = v^2 + v_x \tag{2.48}$$

which is called the Miura transformation where u satisfies the KdV equation. Insert (2.48) into (2.47) to yield

$$\Big(\frac{\partial}{\partial x} + 2v\Big)v_t - 6(v_x + v^2)(v_{xx} + 2vv_x) + \Big(\frac{\partial}{\partial x} + 2v\Big)v_{xxx} + 6vv_x = 0 \tag{2.49}$$

and rearrange to get

$$\Big(\frac{\partial}{\partial x} + 2v\Big)(v_t + v_{xxx} - 6v^2v_x) = 0. \tag{2.50}$$

Here, v satisfies the modified KdV (mKdV) equation which is given by

$$v_t - 6v^2v_x + v_{xxx} = 0. \tag{2.51}$$

We note that the Miura transformation is a restricted transformation. If v is a solution of the mKdV equation, then u is a solution of the KdV equation, not vice versa. The reason is that the equation (2.50) contains an additional operator. There are solutions of (2.50) which are not solutions of the mKdV equation.

$$\text{mKdV} \xrightarrow[]{\substack{\text{Miura} \\ \text{transformation}}} \text{KdV}$$

The Miura transformation (2.48) is a Riccati equation for v. Let us substitute

$$v = \psi_x/\psi \tag{2.52}$$

for some differentiable function $\psi(x,t) \neq 0$ to yield

$$\psi_{xx} - u(x,t)\psi = 0. \tag{2.53}$$

We know that the KdV equation is invariant under the Galilean transformation

$$\begin{aligned} u &\longrightarrow u - \lambda \\ x &\longrightarrow x + 6\lambda t \\ t &\longrightarrow t \end{aligned} \tag{2.54}$$

for arbitrary real λ. Under such a transformation, we can replace the equation (2.53) by

$$\psi_{xx} + (\lambda - u(x,t))\psi = 0 \tag{2.55}$$

which is the time-independent Schrödinger equation from quantum mechanics where u is a potential or a solution of the KdV equation, λ is the energy level and ψ is the wave function. We note that, here in (2.55), t plays the role of a parameter.

If it is possible to find the wave function ψ then the potential $u(x,t)$ can be determined by (2.55). This method is called the Inverse Scattering Transform (IST). The aim is to solve the initial value problem

$$\begin{aligned} &u_t - 6uu_x + u_{xxx} = 0 \\ &u(x,0) = u_0(x) \end{aligned} \tag{2.56}$$

of the KdV equation. It consists of

1) Direct problem: to find the scattering data,

2) Time evolution of the scattering data,

3) Solve the Gelfand-Levitan-Marchenko (GLM) equation,

4) Determine $u(x,t)$.

$$
\begin{tikzcd}
u(x,0) \arrow[rr, "\text{1. direct problem}"] \arrow[d, "\text{4. IST}"'] & & S(0) \arrow[d, "\text{2. time evolution of the scattering data}"] \\
u(x,t) & & S(t) \arrow[ll, "\text{inverse problem}"', "\text{3. Gelfand-Levitan-Marchenko equation}"]
\end{tikzcd}
$$

Figure 2.1: Steps of the Inverse Scattering Method

2.2.1 Direct problem

Direct problem is the first step in the method of inverse scattering transform. In this step we solve the time-independent Schrödinger equation with $u(x,0) = u_0(x)$ as the initial potential. In the scattering problem of the Schrödinger equation, the initial value $u_0(x)$ is required to be integrable, i.e.

$$\int_{-\infty}^{\infty} |u_0(x)|\, dx \; < \infty \tag{2.57}$$

and $u_0(x)$ must decay sufficiently rapidly at infinity so that the Faddeev condition must also be satisfied [21]

$$\int_{-\infty}^{\infty} \big(1 + |x|\big)|u_0(x)|\, dx \; < \infty. \tag{2.58}$$

Now, we consider the Sturm-Liouville equation

$$\psi_{xx} + (\lambda - u)\psi = 0. \tag{2.59}$$

There are two types of spectrum for λ, one of which is for $\lambda > 0$ and the other is for $\lambda < 0$. Note that if $u_0(x) \neq 0$ for some x, $\lambda = 0$ cannot occur. Since $u(x,t) \to 0$ as $|x| \to \infty$ then the Sturm-Liouville equation asymptotically becomes

$$\psi_{xx} + \lambda\psi = 0 \quad \text{as} \quad |x| \to \infty \tag{2.60}$$

Likewise, there are two types of eigenfunctions, one of which is a linear combination of $e^{\pm i\sqrt{\lambda}x}$ for all $\lambda > 0$ (continuous spectrum), and another one is of $e^{\pm\sqrt{\lambda}x}$ for all $\lambda < 0$ (discrete spectrum). However, for discrete spectrum, we consider a solution that decays at infinity.

Now, let us integrate the Sturm-Liouville equation to observe the nature of eigenfunctions i.e., continuity, boundedness, and differentiability. Integrate (2.59) on the interval $[a, b]$ to yield

$$[\psi_x]_b^a = \int_b^a (u - \lambda)\psi \, dx \tag{2.61}$$

When $b \to a$, if $[\psi_x]_b^a$ goes to zero, then ψ_x is continuous. However a problem arises if u is a delta function i.e, $u = \delta(x - y)$ where $b < y < a$. So we need to be more careful. This time, first integrate (2.59) from c to x, then from b to a to obtain

$$[\psi]_b^a = (a - b)\psi_x(c) + \int_b^a \left\{ \int_c^x (u - \lambda)\psi \, dx \right\} dx \tag{2.62}$$

and for all functions $u(x,t)$, ψ is continuous everywhere. Hence, we shall consider eigenfunctions ψ which are bounded, continuous, and at least once differentiable. Since the discrete eigenfunctions are bounded and exponentially decay as $|x| \to \infty$, then they are absolutely and square integrable i.e.,

$$\int_{-\infty}^{\infty} |\psi| \, dx \ < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} |\psi|^2 \, dx \ = \int_{-\infty}^{\infty} \psi^2 \, dx \ < \infty \tag{2.63}$$

respectively. However, the continuous eigenfunction is not square integrable.

We shall study the discrete spectrum first. Let us define $\lambda = -\kappa_n^2$ for $n = 1, 2, .., N$ and $\kappa_1 < \kappa_2 < ... < \kappa_N$. The discrete eigenfunction corresponding to the eigenvalue κ_n is given by

$$\psi_n \sim \begin{cases} \alpha_n e^{-\kappa_n x}, & x > 0 \\ \beta_n e^{\kappa_n x}, & x < 0 \end{cases} \tag{2.64}$$

Since ψ_n is continuous, then $\alpha_n = \beta_n$. The eigenfunction is normalized via the square-integrability

$$\int_{-\infty}^{\infty} \psi_n^2 \, dx \ = 1. \tag{2.65}$$

Substitute (2.64) into (2.65) and we obtain

$$\alpha_n^2 \int_{-\infty}^{0} e^{2\kappa_n x}\, dx \ + \alpha_n^2 \int_{0}^{\infty} e^{-2\kappa_n x}\, dx \ = 1 \tag{2.66}$$

yielding $\alpha_n = \sqrt{\kappa_n}$.

Now, consider two different eigenfunctions corresponding to eigenvalues κ_n and κ_m with the same potential u

$$\psi_n^{''} = (\kappa_n^2 + u)\psi_n \quad \text{and} \quad \psi_m^{''} = (\kappa_m^2 + u)\psi_m \tag{2.67}$$

and then

$$(\kappa_n^2 - \kappa_m^2)\psi_n\psi_m = \psi_m\psi_n^{''} - \psi_n\psi_m^{''} = \frac{d}{dx}(\psi_m\psi_n^{'} - \psi_n\psi_m^{'}). \tag{2.68}$$

We use the Wronskian of ψ_m and ψ_n

$$(\kappa_n^2 - \kappa_m^2)\psi_n\psi_m = \frac{d}{dx}W(\psi_m, \psi_n) \tag{2.69}$$

and integrate over $\mathbb{R}$

$$[W(\psi_m, \psi_n)]_{-\infty}^{\infty} = (\kappa_n^2 - \kappa_m^2)\int_{-\infty}^{\infty} \psi_n\psi_m\, dx\ . \tag{2.70}$$

Since ψ_n and ψ_m are decaying solutions, the left hand side is zero. If $\kappa_n \neq \kappa_m$, then ψ_n and ψ_m are orthogonal, but if $\kappa_n = \kappa_m$ then $\psi_n = \alpha\psi_m$, where $\alpha = \pm 1$. In other words, discrete eigenfunctions which correspond to different eigenvalues are orthogonal.

We shall now study the continuous spectrum. Let $\lambda = k^2$, and we can define the continuous eigenfunction in two ways. First, the incident wave comes from the right and the continuous eigenfunction is given by

$$\hat{\psi}_R \sim \begin{cases} e^{-ikx} + b(k)e^{ikx}, & \text{as} \quad x \to \infty \\ a(k)e^{-ikx}, & \text{as} \quad x \to -\infty \end{cases} \tag{2.71}$$

or, second, the incident wave comes from the left, and the continuous eigenfunction is given by

$$\hat{\psi}_L \sim \begin{cases} \tilde{a}(k)e^{ikx}, & \text{as} \quad x \to \infty \\ e^{ikx} + \tilde{b}(k)e^{-ikx}, & \text{as} \quad x \to -\infty \end{cases} \tag{2.72}$$

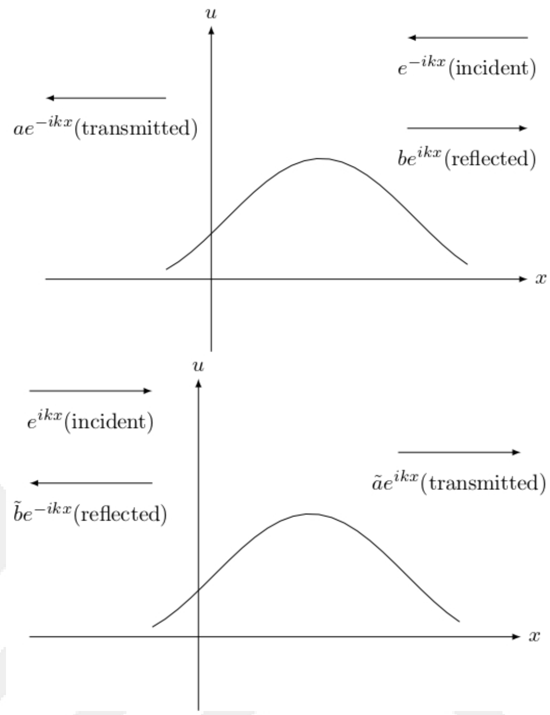


Figure 2.2: Representation of scattering region: (a) incident wave of unit amplitude from right, (b) incident wave of unit amplitude from left

where the complex constants a, $\tilde{a}$ are called transmission coefficients and the complex constants b, $\tilde{b}$ are called reflection coefficients.

Since the potential u is the same, both continuous eigenfunctions are valid for direct problem, and, for inverse problem, they give us the same potential $u(x,t)$ [22]. Therefore, in our calculations, we shall consider the case in which the incident wave comes from $+\infty$ and let $\hat{\psi}_R = \hat{\psi}$ [23].

Let us give the interpretation of the eigenfunction $\hat{\psi}$: The incident wave of unit amplitude is sent from $+\infty$ to interact with the potential u and, as a result of interaction, a function of wave is sent back to $+\infty$ (reflected wave), and the remainder is transmitted to $-\infty$ (transmitted wave). Here $\hat{\psi}$ and its complex conjugate $\hat{\psi}^*$ satisfy

$$\begin{aligned} &\hat{\psi}'' + (k^2 - u)\hat{\psi} = 0, \\ &\hat{\psi}^{*\prime\prime} + (k^2 - u)\hat{\psi}^* = 0, \end{aligned} \tag{2.73}$$

and the Wronskian, $W(\hat{\psi}, \hat{\psi}^*)$ at $+\infty$ and $-\infty$ becomes

$$W(\hat{\psi}, \hat{\psi}^*) = 2ikaa^* = 2ik(1 - bb^*) \tag{2.74}$$

or

$$|a|^2 + |b|^2 = 1 \tag{2.75}$$

which is called the conservation of energy in the scattering.

We discussed the continuous spectrum ($\lambda = k^2$) and the discrete spectrum ($\lambda = -\kappa_n^2$). Now, let us have some examples.

Example 2.2.1. We choose the initial value as

$$u(x, 0) = -U_0 \delta(x) \tag{2.76}$$

where U_0 is a constant and $\delta(x)$ is Dirac's delta function

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0. \end{cases} \tag{2.77}$$

We should determine the behaviours of the solution at $x = 0$. The Sturm-Liouville equation

$$\psi^{''} + \lambda\psi = -U_0 \delta(x)\psi \tag{2.78}$$

is integrated once on the interval $[-\varepsilon, \varepsilon]$ to yield

$$\int_{-\varepsilon}^{\varepsilon} \psi^{''}\, dx \ + \lambda \int_{-\varepsilon}^{\varepsilon} \psi\, dx \ = -U_0 \psi(0). \tag{2.79}$$

Let $\varepsilon \to 0$, then

$$[\psi^{'}] = -U_0 \psi(0) \tag{2.80}$$

where $[\quad]$ denotes the change in $\psi^{'}$. We notice that ψ is continuous everywhere, but its derivative has a jump at $x = 0$.

For the discrete spectrum ($\lambda = -\kappa_n^2$), the discrete eigenfunction is given by

$$\psi_n = \begin{cases} \alpha_n e^{-\kappa_n x}, & x > 0 \\ \beta_n e^{\kappa_n x}, & x < 0. \end{cases} \tag{2.81}$$

Since ψ_n is continuous, $\alpha_n = \beta_n$. We have already found that $\alpha_n = \sqrt{\kappa_n}$ since the solution is normalized. From the discontinuity of $\psi_n^{'}$,

$$[\psi_n^{'}] = -\kappa_n(\alpha_n) - \kappa_n(\alpha_n) = -U_0\alpha_n. \tag{2.82}$$

We find that $\kappa_n = U_0/2$. There is only one discrete eigenvalue if $U_0 > 0$. (α_n must be real, $\alpha_n = \sqrt{\kappa_n} = \sqrt{U_0/2}$)

For the continuous spectrum ($\lambda = k^2$), the continuous eigenfunction is

$$\hat{\psi} = \begin{cases} e^{-ikx} + b(k)e^{ikx}, & x > 0 \\ a(k)e^{-ikx}, & x < 0. \end{cases} \tag{2.83}$$

Since $\hat{\psi}$ is continuous at $x = 0$, then $a = 1 + b$. The jump condition for $\hat{\psi}$ at zero provides

$$[\psi^{'}] = -ik + bik - (-ika) = -U_0(1 + b) \tag{2.84}$$

or

$$b(k) = -\frac{U_0}{U_0 + 2ik} \quad \text{and} \quad a(k) = \frac{2ik}{U_0 + 2ik}. \tag{2.85}$$

The transmission and reflection coefficients have a pole at $k = i\frac{U_0}{2}$. Since $k = i\kappa_n$, then $\kappa_n = \frac{U_0}{2}$. We note that the poles of $b(k)$ or $a(k)$ in the upper-half plane provide us the discrete eigenvalues if k is extended into the complex plane. This conclusion is quite general.

Example 2.2.2. In this example, let

$$u(x, 0) = -U_0 \operatorname{sech}^2 x, \quad x \in \mathbb{R} \tag{2.86}$$

where U_0 is constant, then

$$\psi_{xx} + (\lambda + U_0 \operatorname{sech}^2 x)\psi = 0 \tag{2.87}$$

will be solved for eigenfunctions ψ. Let $y = \tanh x$, then (2.87) becomes

$$(1 - y^2)\frac{d}{dy}\Big\{(1 - y^2)\frac{d\psi}{dy}\Big\} + \big\{\lambda + U_0(1 - y^2)\big\}\psi = 0 \tag{2.88}$$

or

$$\frac{d}{dy}\Big((1 - y^2)\frac{d\psi}{dy}\Big) + \Big[U_0 + \frac{\lambda}{1 - y^2}\Big]\psi = 0 \tag{2.89}$$

which is the well-known associated Legendre equation. Let us suppose $U_0 = N(N+1)$ where N is a positive integer, and consider the discrete spectrum $\lambda = -\kappa_n^2$ where $\kappa_n = n$, $n = 1, 2, ..., N$, then

$$\frac{d}{dy}\Big((1-y^2)\frac{d\psi_n}{dy}\Big)+\Big[N(N+1) - \frac{n^2}{1-y^2}\Big]\psi_n = 0. \tag{2.90}$$

This equation has N bounded solutions $\psi_n(y) = \alpha_n P_N^n(y)$ which are proportional to the associated Legendre polynomials, $P_N^n(y)$, where

$$P_N^n(y) = (-1)^n(1-y^2)^{n/2}\frac{d^n}{dy^n}P_N(y) \tag{2.91}$$

and

$$P_N(y) = \frac{1}{2^N N!}\frac{d^N}{dy^N}(y^2-1)^N, \tag{2.92}$$

and $P_N(y)$ is the Legendre polynomial of order N. For example, if $N = 1$,

$$P_1(y) = y \quad \text{then} \quad P_1^1(y) = -(1-y^2)^{1/2} = -\operatorname{sech} x \tag{2.93}$$

and, if $N = 2$, $P_2(y) = \frac{1}{2}(3y^2 - 1)$, then

$$P_2^1(y) = -3\operatorname{sech} x \tanh x \quad \text{and} \quad P_2^2(y) = 3\operatorname{sech}^2 x \tag{2.94}$$

Since $\psi_n(y) = \alpha_n P_2^n(y)$, then $\psi_1(y) = -3\alpha_1 \operatorname{sech} x \tanh x$ corresponding to $\kappa_1 = 1$ ($\lambda_1 = -1$) and $\psi_2(y) = 3\alpha_2 \operatorname{sech}^2 x$ corresponding to $\kappa_2 = 2$ ($\lambda_2 = -4$). We know that $\psi_1(y)$ and $\psi_2(y)$ are normalized then the discrete eigenfunctions for $N = 2$ are

$$\psi_1 = \Big(\frac{3}{2}\Big)^{1/2}\tanh x \operatorname{sech} x \quad \text{and} \quad \psi_2 = \frac{3^{1/2}}{2}\operatorname{sech}^2 x. \tag{2.95}$$

Now, let us consider the continuous spectrum ($\lambda = k^2$). The equation (2.89) becomes

$$\frac{d}{dy}\Big((1-y^2)\frac{d\psi}{dy}\Big)+\Big(U_0 + \frac{k^2}{1-y^2}\Big)\psi = 0 \tag{2.96}$$

or

$$(1-y^2)\psi'' - 2y\psi' + \Big(U_0 + \frac{k^2}{1-y^2}\Big)\psi = 0. \tag{2.97}$$

Let $z = \frac{1}{2}(1+y)$, the equation (2.97) yields

$$z(1-z)\psi_{zz} - (2z-1)\psi_z + \Big(U_0 + \frac{k^2}{4z(1-z)}\Big)\psi = 0. \tag{2.98}$$

Let $\hat{\psi}(z;k) = a(k)2^{ik}z^{-ik/2}(1-z)^{-ik/2}g(z)$ and substitute into (2.98), then we clearly see that $g(z)$ satisfies

$$z(1-z)g^{''} + [\tilde{c} - (\tilde{a} + \tilde{b} - 1)z]g^{'} - \tilde{a}\tilde{b}g = 0 \tag{2.99}$$

where

$$\begin{aligned} \tilde{a} &= \frac{1}{2} - ik + \big(U_0 + \frac{1}{4}\big)^{1/2}, \\ \tilde{b} &= \frac{1}{2} - ik - \big(U_0 + \frac{1}{4}\big)^{1/2}, \\ \tilde{c} &= 1 - ik. \end{aligned} \tag{2.100}$$

The function $g(z) = F(\tilde{a}, \tilde{b}, \tilde{c}; z)$ is the hypergeometric function where $z = \frac{1}{2}(1 + y)$. Hence the continuous eigenfunction is

$$\hat{\psi}(x;k) = a(k).2^{ik}(\operatorname{sech} x)^{-ik}F\Big(\tilde{a}, \tilde{b}, \tilde{c}; \frac{1}{2}(1+y)\Big). \tag{2.101}$$

As $z \to 0^+$ corresponds to $x \to -\infty$ and as $z \to 1^-$ corresponds to $x \to +\infty$ where $z = \frac{1}{2}(1+y)$. It is easy to show that

$$\hat{\psi}(x;k) \sim a(k)e^{-ikx} \quad \text{as} \quad x \to -\infty. \tag{2.102}$$

To find the asymptotic behaviour of $\hat{\psi}$ at $+\infty$, use the linear combination [24]

$$\begin{aligned} F(\tilde{a}, \tilde{b}, \tilde{c}; z) =& \frac{\Gamma(\tilde{c})\Gamma(\tilde{c} - \tilde{a} - \tilde{b})}{\Gamma(\tilde{c} - \tilde{a})\Gamma(\tilde{c} - \tilde{b})} F(\tilde{a}, \tilde{b}, \tilde{a} + \tilde{b} + 1 - \tilde{c}; 1 - z) \\ &+ \frac{\Gamma(\tilde{c})\Gamma(\tilde{a} + \tilde{b} - \tilde{c})}{\Gamma(\tilde{a})\Gamma(\tilde{b})}(1-z)^{\tilde{c} - \tilde{a} - \tilde{b}} F(\tilde{c} - \tilde{a}, \tilde{c} - \tilde{b}, 1 + \tilde{c} - \tilde{a} - \tilde{b}; 1 - z) \end{aligned} \tag{2.103}$$

then, as $x \to \infty$ $(z \to 1^-)$

$$\hat{\psi}(x;k) \sim \frac{a(k)\Gamma(\tilde{c})\Gamma(\tilde{a} + \tilde{b} - \tilde{c})}{\Gamma(\tilde{a})\Gamma(\tilde{b})}e^{-ikx} + \frac{a(k)\Gamma(\tilde{c})\Gamma(\tilde{c} - \tilde{a} - \tilde{b})}{\Gamma(\tilde{c} - \tilde{a})\Gamma(\tilde{c} - \tilde{b})}e^{ikx}. \tag{2.104}$$

Comparing these results by (2.71), we obtain that

$$a(k) = \frac{\Gamma(\tilde{a})\Gamma(\tilde{b})}{\Gamma(\tilde{c})\Gamma(\tilde{a} + \tilde{b} - \tilde{c})} \quad \text{and} \quad b(k) = \frac{a(k)\Gamma(\tilde{c})\Gamma(\tilde{c} - \tilde{a} - \tilde{b})}{\Gamma(\tilde{c} - \tilde{a})\Gamma(\tilde{c} - \tilde{b})} \tag{2.105}$$

and, $a(k) \to 1$ and $b(k) \to 0$ as $k \to \infty$. The scattering coefficients are determined.

We have two important remarks which are derived from (2.105). The first one is when $b(k) = 0$. A property of the Gamma function is

$$\Gamma\Big(\frac{1}{2} - z\Big)\Gamma\Big(\frac{1}{2} + z\Big) = \frac{\pi}{\cos(\pi z)} \tag{2.106}$$

then

$$\begin{aligned}\Gamma(\tilde{c} - \tilde{a})\Gamma(\tilde{c} - \tilde{b}) &= \Gamma\Big(\frac{1}{2} - \Big(U_0 + \frac{1}{4}\Big)^{1/2}\Big)\Gamma\Big(\frac{1}{2} + \Big(U_0 + \frac{1}{4}\Big)^{1/2}\Big) \\ &= \frac{\pi}{\cos\pi\Big(U_0 + \frac{1}{4}\Big)^{1/2}}\end{aligned} \tag{2.107}$$

and so,

$$b(k) = \frac{a(k)}{\pi}\Gamma(\tilde{c})\Gamma(\tilde{c} - \tilde{a} - \tilde{b})\cos\pi\Big(U_0 + \frac{1}{4}\Big)^{1/2}. \tag{2.108}$$

Consequently, $b(k) = 0$ if $\Big(U_0 + \frac{1}{4}\Big)^{1/2} = N + \frac{1}{2}$ i.e., $U_0 = N(N+1)$, where N is a positive integer. This means that reflectionless potentials ($b = 0$) have only a discrete spectrum.

For the second remark, we should check the poles of $b(k)$ (or $a(k)$). These two coefficients have poles if $\Gamma(\tilde{a})$ or $\Gamma(\tilde{b})$ are undefined. For instance, if $\tilde{b} = -m$ where m is a positive integer, then

$$\frac{1}{2} - ik - \Big(U_0 + \frac{1}{4}\Big)^{1/2} = -m \tag{2.109}$$

giving

$$k = i\Big\{\Big(U_0 + \frac{1}{4}\Big)^{1/2} - \Big(m + \frac{1}{2}\Big)\Big\}, \tag{2.110}$$

so there exist finite numbers of eigenvalues if $\Big(U_0 + \frac{1}{4}\Big)^{1/2} > 0$ with $U_0 > 0$. The total number of eigenvalues is given by

$$\Big[\Big(U_0 + \frac{1}{4}\Big)^{1/2} - \frac{1}{2}\Big] + 1 \tag{2.111}$$

where [ ] denotes the integral part.

2.2.2 Time evolution of the scattering data

Our purpose now is to solve the initial value problem of the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0, \quad x \in \mathbb{R}, \quad t > 0 \tag{2.112}$$

with $u(x,0) = u_0(x)$. By using the Lax pair of the KdV equation, we write the eigenvalue problem (or Sturm-Liouville equation)

$$\psi_{xx} + (\lambda - u(x,t))\psi = 0 \quad \text{or} \quad L\psi = \lambda\psi \tag{2.113}$$

where t is a parameter. In the first step, we have found the scattering data at $t = 0$

$$S(0) = \{\kappa_n, c_n, k, a(k), b(k)\} \tag{2.114}$$

by using (2.113), where c_n is a normalization constant and, $a(k)$ and $b(k)$ are transmission and reflection coefficients, respectively.

The second step is to find the scattering data at any time t

$$S(t) = \{\kappa_n(t), c_n(t), k(t), a(k,t), b(k,t)\}. \tag{2.115}$$

For this purpose, we need the time evolution equation. Again by using Lax pair, we write the time evolution of ψ as

$$\psi_t + u_x\psi - 2(u + 2\lambda)\psi_x = R(x,t) \tag{2.116}$$

or

$$\psi_t - M\psi = R(x,t) \tag{2.117}$$

where, in general, $\lambda = \lambda(t)$. To find $R(x,t)$, take x-derivative of (2.116),

$$\psi_{tx} + u_{xx}\psi + u_x\psi_x - 2u_x\psi_x - 2(u + 2\lambda)\psi_{xx} = R_x. \tag{2.118}$$

Insert (2.113) into (2.118)

$$\psi_{tx} + u_{xx}\psi - u_x\psi_x + 2(u + 2\lambda)(\lambda - u)\psi = R_x. \tag{2.119}$$

Re-arrange the equation to obtain

$$\psi_{tx} + [u_{xx} + 4\lambda^2 - 2u^2 - 2\lambda u]\psi - u_x\psi_x = R_x \tag{2.120}$$

then, take one more x-derivative and use (2.113) to yield

$$\psi_{txx} + (4\lambda^2 - 2u^2 - 2\lambda u)\psi_x + (u_{xxx} - 5uu_x - \lambda u_x)\psi = R_{xx}. \tag{2.121}$$

Now, take t-derivative of (2.113)

$$\psi_{xxt} + (\lambda_t - u_t)\psi + (\lambda - u)\psi_t = 0 \tag{2.122}$$

and use (2.116) to yield

$$\psi_{xxt} = (u_t - \lambda_t)\psi + (\lambda - u)[R - u_x\psi + 2(u + 2\lambda)\psi]. \tag{2.123}$$

Lastly, insert (2.123) into (2.121) and use the KdV equation to obtain

$$-\lambda_t\psi = (\lambda - u)R + R_{xx} = -\frac{\psi_{xx}}{\psi}R + R_{xx} \tag{2.124}$$

or

$$\frac{\partial}{\partial x}(R_x\psi - \psi_x R) = -\lambda_t\psi^2. \tag{2.125}$$

Integrate (2.125) over all x, we get

$$[R_x\psi - \psi_x R]_{-\infty}^{\infty} = -\lambda_t \int_{-\infty}^{\infty} \psi^2\, dx\ . \tag{2.126}$$

Left-hand side is zero for both spectrum, and the integral on the right-hand side does not vanish, then λ is independent of t ($\lambda_t = 0$). We find that λ is constant and

$$R_x\psi - \psi_x R = g(t) \tag{2.127}$$

for an arbitrary function $g(t)$. Again, the left-hand side vanishes as $|x| \to \infty$, then $g(t)$ is zero function

$$R_x\psi - \psi_x R = 0. \tag{2.128}$$

Integrate once more to obtain

$$R = h(t)\psi \tag{2.129}$$

where $h(t)$ is an arbitrary function. The equation (2.116) becomes

$$\psi_t + u_x\psi - 2(u + 2\lambda)\psi_x = h(t)\psi. \tag{2.130}$$

Multiply by ψ and we get

$$\frac{1}{2}(\psi^2)_t + u_x\psi^2 - 2(u + 2\lambda)\psi\psi_x = h(t)\psi^2 \tag{2.131}$$

or

$$\frac{1}{2}(\psi^2)_t + [u\psi^2 - 2\psi_x^2 - 4\lambda\psi^2]_x = h(t)\psi^2. \tag{2.132}$$

Integrate (2.132) over all x, we get

$$\frac{1}{2}\frac{d}{dt}\Big(\int_{-\infty}^{\infty} \psi^2\, dx\ \Big)+[u\psi^2 - 2\psi_x^2 - 4\lambda\psi^2]_{-\infty}^{\infty} = h(t)\int_{-\infty}^{\infty} \psi^2\, dx\ . \tag{2.133}$$

For discrete spectrum, the eigenfunction decays at infinity then, the second term on the left-hand side vanishes. Hence, we obtain

$$\frac{1}{2}\frac{d}{dt}\Big(\int_{-\infty}^{\infty} \psi^2\, dx\ \Big)= h(t)\int_{-\infty}^{\infty} \psi^2\, dx\ . \tag{2.134}$$

Since the discrete eigenfunctions are normalized and $\int_{-\infty}^{\infty} \psi^2\, dx$ is constant, then $h(t) = 0$ for $\lambda = -\kappa_n^2 (< 0)$. For continuous spectrum, $h(t)$ must be evaluated.

Now, we are going to use the system of equations

$$\begin{aligned} &\psi_{xx} + (\lambda - u(x,t))\psi = 0 \\ &\psi_t + u_x\psi - 2(u + 2\lambda)\psi_x = h(t)\psi \end{aligned} \tag{2.135}$$

to find the time evolution of the scattering data. Here $h(t) = 0$ for the discrete spectrum.

For the discrete spectrum, we can find the time evolution of the normalization constant. Since u and u_x vanish at infinity, we obtain

$$\psi_n(x,t) \sim c_n(t)e^{-\kappa_n x} \quad x \to \infty \tag{2.136}$$

from the eigenvalue equation, and use the time evolution of ψ when $h(t) = 0$. We have

$$\psi_t + u_x\psi - 2(u - 2\kappa_n^2)\psi_x = 0 \tag{2.137}$$

then

$$\psi_t + 4\kappa_n^2\psi_x = 0 \quad x \to \infty \tag{2.138}$$

since u and u_x vanish at infinity. Insert (2.136), we get

$$\frac{d}{dt}c_n(t)e^{-\kappa_n x} - 4\kappa_n^3 e^{-\kappa_n x}c_n(t) = 0 \tag{2.139}$$

or

$$\frac{d}{dt}c_n(t) - 4\kappa_n^3 c_n(t) = 0. \tag{2.140}$$

We find

$$c_n(t) = c_n(0)e^{4\kappa_n^3 t} \quad \text{and} \quad \kappa_n(t) = \kappa_n \tag{2.141}$$

since $\lambda_t = 0$ and $c_n(0)$ is the normalization constant at $t = 0$.

For continuous spectrum ($\lambda = k^2$), recall that the continuous eigenfunction is

$$\hat{\psi}(x,t) = \begin{cases} e^{-ikx} + b(k,t)e^{ikx} & x \to \infty \\ a(k,t)e^{-ikx} & x \to -\infty \end{cases} \tag{2.142}$$

and the time evolution for $\hat{\psi}$ becomes

$$\hat{\psi}_t - 4k^2\hat{\psi}_x = h(t)\hat{\psi} \quad \text{as} \quad |x| \to \infty. \tag{2.143}$$

Insert (2.142) and calculate the equation at $+\infty$ and $-\infty$, separately. As $x \to \infty$, (2.143) becomes

$$\frac{d}{dt}be^{ikx} - 4k^2(-ike^{-ikx} + ikbe^{ikx}) = h(e^{-ikx} + be^{ikx}) \tag{2.144}$$

and, then comparing the coefficients of e^{ikx} and e^{-ikx}, we obtain

$$\frac{d}{dt}b - 4ik^3b = hb \quad \text{and} \quad h = 4ik^3. \tag{2.145}$$

We evaluate the time evolution of the reflection coefficient

$$b(k,t) = b(k,0)e^{8ik^3t} \quad t > 0 \tag{2.146}$$

where $b(k,0)$ is the reflection coefficient at $t = 0$. As $x \to -\infty$, (2.143) becomes

$$\frac{d}{dt}ae^{-ikx} - 4k^2(-ikae^{-ikx}) = hae^{-ikx}. \tag{2.147}$$

Since $h = 4ik^3$, then $\frac{d}{dt}a = 0$ or

$$a(k,t) = a(k,0), \quad t > 0 \tag{2.148}$$

where $a(k,0)$ is the transmission coefficient at $t = 0$.

Therefore, the scattering data at any time t is given by

$$S(t) = \{\kappa_n, c_n(t) = c_n(0)e^{4\kappa_n^3 t}, k, a(k,t) = a(k,0), b(k,t) = b(k,0)e^{8ik^3t}\} \tag{2.149}$$

where

$$S(0) = \{\kappa_n, c_n(0), k, a(k,0), b(k,0)\} \tag{2.150}$$

is evaluated from the eigenvalue equation $\psi_{xx} + (\lambda - u(x,0))\psi = 0$ where $u(x,0)$ is the initial value of the KdV equation [6].

2.2.3 Gelfand-Levitan-Marchenko (GLM) equation

The third step in the method of inverse scattering transform is to solve the Gelfand-Levitan-Marchenko (GLM) equation. First, we shall derive the GLM equation. We start with the eigenvalue equation

$$\psi_{xx} + (k^2 - u)\psi = 0 \tag{2.151}$$

where $\lambda = k^2$. In this case, let us ignore the time (variable t) and let $u = u(x)$ and $\psi = \psi(x, k)$. For the equation (2.151), we seek a solution ψ such that $\psi \sim e^{ikx}$ as $x \to +\infty$. We can assume the solution as

$$\psi_+(x, k) = e^{ikx} + \int_x^\infty K(x, z)e^{ikz}\, dz \quad \text{as} \quad x \to +\infty \tag{2.152}$$

where $K(x, z)$ is the kernel such that $K(x, z) \neq 0$ for $z > x$. Now, we shall find the conditions which the kernel $K(x, z)$ must satisfy. First, take x-derivative of (2.152) twice. We get

$$\psi_{+xx} = e^{ikx}\Big(-k^2 - \frac{d}{dx}K(x, x) - ikK(x, x) - K_x(x, x)\Big) + \int_x^\infty K_{xx}(x, z)e^{ikz}\, dz \tag{2.153}$$

where $\frac{d}{dx}$ is a total derivative i.e., $\frac{d}{dx}K(x, x) = K_x(x, x) + K_z(x, x)$. Using integration by parts twice in (2.152), we get

$$\psi_+(x, k) = e^{ikx}\Big(1 + \frac{i}{k}K(x, x) - \frac{K_z(x, x)}{k^2}\Big) - \frac{1}{k^2}\int_x^\infty K_{zz}e^{ikz}\, dz \tag{2.154}$$

provided that $K(x, z)$ and $K_z(x, z)$ vanish as $z \to \infty$, for fixed x. Inserting (2.153) and (2.154) into the Sturm-Liouville equation, we get

$$\begin{aligned} e^{ikx}\Big(-u - \frac{d}{dx}K(x, x) - K_x(x, x) - K_z(x, x) - \frac{i}{k}K(x, x)u + \frac{1}{k^2}K_z(x, x)u\Big) \\ + \int_x^\infty (K_{xx} - K_{zz} + \frac{u}{k^2}K_{zz})e^{ikz}\, dz \ = 0. \end{aligned} \tag{2.155}$$

For the term $\frac{u(x)}{k^2}\int_x^\infty K_{zz}e^{ikz}\, dz$, using integration by parts twice, we obtain

$$-\frac{u}{k^2}e^{ikx}K_z(x, x) + \frac{iu}{k}e^{ikx}K(x, x) - \int_x^\infty ue^{ikz}K\, dz \tag{2.156}$$

again, provided that $K_z(x,z)$ and $K(x,z)$ vanish as $z \to \infty$. Consequently, the Sturm-Liouville equation reduces to

$$-e^{ikx}\Big(u + 2\frac{d}{dx}K(x,x)\Big) + \int_x^\infty \big(K_{xx} - K_{zz} - u(x)K\big)e^{ikz}\,dz \ = 0 \tag{2.157}$$

which is satisfied if

$$K_{xx} - K_{zz} - u(x)K = 0 \tag{2.158}$$

and

$$u = -\frac{d}{dx}K(x,x) \tag{2.159}$$

with the conditions $K(x,z) \to 0$ and $K_z(x,z) \to 0$ as $z \to \infty$. We found that if the kernel $K(x,z)$ is known, then the potential $u(x)$ (or $u(x,t)$) can be determined.

We are one step closer to deriving the GLM equation. The continuous eigenfunction can be written as

$$\hat{\psi} = \psi_+^* + b(k)\psi_+ \tag{2.160}$$

which gives the correct behaviour as $x \to +\infty$ since $\psi_+ \sim e^{ikx}$ as $x \to +\infty$. Using (2.152), we get

$$\hat{\psi}(x,k) = e^{-ikx} + b(k)e^{ikx} + \int_{-\infty}^\infty K(x,z)e^{-ikz}\,dz \ + b(k)\int_{-\infty}^\infty K(x,z)e^{ikz}\,dz \tag{2.161}$$

where $K(x,z) \neq 0$ for $z > x$. Rearranging this equation, we obtain

$$\int_{-\infty}^\infty K(x,z)e^{-ikz}\,dz \ = \hat{\psi} - e^{-ikx} - b(k)e^{ikx} - b(k)\int_{-\infty}^\infty K(x,z)e^{ikz}\,dz\ . \tag{2.162}$$

Take the inverse Fourier transform of both sides. On the left hand side, we obtain

$$\frac{1}{2\pi}\int_{-\infty}^\infty \Big\{\int_{-\infty}^\infty K(x,y)e^{-iky}\,dy\ \Big\}e^{ikz}\,dk \tag{2.163}$$

and, then interchange the order of integration to yield

$$\begin{aligned}\int_{-\infty}^\infty K(x,y)\Big\{\frac{1}{2\pi}\int_{-\infty}^\infty e^{ik(z-y)}\,dk\ \Big\}\,dy \ &= \int_{-\infty}^\infty K(x,y)\delta(z-y)\,dz \\ &= K(x,z).\end{aligned} \tag{2.164}$$

Hence, the equation (2.162) becomes

$$K(x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big\{\hat{\psi} - e^{-ikx} - b(k)e^{ikx} - b(k)\int_{-\infty}^{\infty} K(x,y)e^{iky}\,dy\;\Big\}e^{ikz}\,dk\;. \tag{2.165}$$

At this point, let us define a function

$$F(X) = \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k)e^{ikX}\,dk \tag{2.166}$$

and insert this function into (2.165) and interchange the order of integration to yield

$$K(x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(\hat{\psi}-e^{-ikx})e^{ikz}\,dk - F(x+z) - \int_{-\infty}^{\infty} K(x,y)F(y+z)\,dy\;. \tag{2.167}$$

We need to evaluate the first integral on the right-hand side. There are two cases; one of which is that $b(k)$ (or $a(k)$) does not have poles or any discrete eigenvalues, other case is that there are some discrete eigenvalues. Note that all discrete eigenvalues are placed in the upper-half plane of the complex k-plane.

Let us consider the first case. We introduce contour C, in the upper-half of the complex k-plane, which consists of the arc C_R and the real line $[-R, R]$, see Figure 2.3.

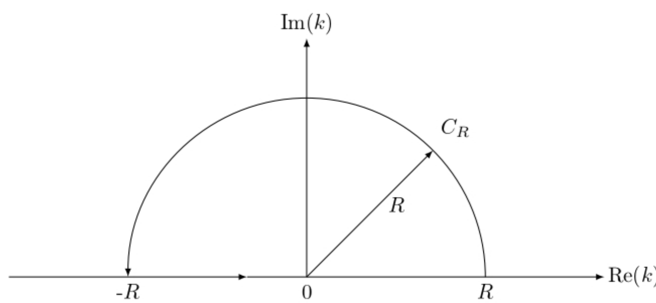


Figure 2.3: The contour C in the complex k-plane

Then, by using the contour integral, we have

$$\oint_C P(x,k)\,dk \;= \int_{-R}^{R} P(x,k)\,dk \;+ \int_{C_R} P(x,k)\,dk \tag{2.168}$$

where $P(x,k) = (\hat{\psi} - e^{-ikx})e^{ikz}$. Here, $\hat{\psi}e^{ikx}$ has no pole since at the beginning we assume there is no discrete spectrum. Then

$$\oint_C P(x,k)\,dk \;= 0 \tag{2.169}$$

and, also

$$\int_{C_R} P(x,k)\,dk \;=\int_{C_R}\{\hat{\psi}e^{ikx}-1\}e^{ik(z-x)}\,dk \;\to 0 \quad \text{as} \quad R\to\infty \tag{2.170}$$

for $z>x$. The reason is that $b(k)\to 0$ as $|k|\to\infty$, then $|\hat{\psi}e^{ikx}|\to 1$ as $|k|\to\infty$. Consequently, we show that

$$\int_{-\infty}^{\infty}(\hat{\psi}-e^{-ikx})e^{ikz}\,dk \;=0 \tag{2.171}$$

then, the equation (2.167) becomes

$$K(x,z)+F(x+z)+\int_x^{\infty}K(x,y)F(y+z)\,dy \;=0, \quad z>x>-\infty \tag{2.172}$$

which is the famous Gelfand-Levitan-Marchenko equation for the kernel $K(x,z)$ where the function is given by

$$F(X)=\frac{1}{2\pi}\int_{-\infty}^{\infty}b(k)e^{ikX}\,dk\ . \tag{2.173}$$

For the second case, where there is a discrete spectrum, the result of the first integral will change. We suppose that the contour C covers all the discrete eigenvalues and, since there are poles in the upper half of the complex k-plane, then

$$\oint_C P(x,k)\,dk \;=2\pi i\sum_{c\in C}\mathrm{res}_c P(x,k) \tag{2.174}$$

where $P(x,k)=(\hat{\psi}-e^{-ikx})e^{ikz}$. Since $e^{ik(z-x)}$ for $z>x$ has no poles then as $R\to\infty$

$$\int_{-\infty}^{\infty}(\hat{\psi}-e^{-ikx})e^{ikz}\,dk \;=2\pi i\sum_{n=1}^{N}R_n \tag{2.175}$$

where R_n is the residue of $\hat{\psi}e^{ikz}$ at $k=i\kappa_n$. To evaluate R_n, we shall define a new solution with an appropriate kernel $L(x,z)$ as

$$\psi_-(x,k)=e^{-ikx}+\int_{-\infty}^{x}L(x,z)e^{-ikz}\,dz \tag{2.176}$$

where $L(x,z)\neq 0$ for $z<x$. If we insert this solution into the Sturm-Liouville equation, we get

$$e^{-ikx}\Big\{-u+2\frac{d}{dx}L(x,x)\Big\}+\int_{-\infty}^{x}\{L_{xx}-L_{zz}+u(x)L\}e^{-ikz}\,dz \;=0 \tag{2.177}$$

which is satisfied if

$$L_{xx}(x,z) - L_{zz}(x,z) + u(x)L(x,z) = 0, \quad z < x \tag{2.178}$$

and

$$u(x) = 2\frac{d}{dx}L(x,x) \tag{2.179}$$

with the conditions $L(x,z) \to 0$ and $L_z(x,z) \to 0$ as $z \to -\infty$ then $L(x,x) = -K(x,x)$. We note that if the incident wave comes from $-\infty$, we write the GLM equation for the kernel $L(x,z)$.

Hence, $\psi_-(x,k) \sim e^{-ikx}$ as $x \to -\infty$. The continuous eigenfunction can be written as

$$\hat{\psi} = a(k)\psi_- \tag{2.180}$$

which gives the correct behaviour as $x \to -\infty$. By comparing it with (2.160), we get

$$a(k)\psi_- = \psi_+^* + b(k)\psi_+. \tag{2.181}$$

Likewise, the discrete eigenfunction can be written as ($k = i\kappa_n$)

$$\psi_n(x) = c_n\psi_+(x, i\kappa_n) = d_n\psi_-(x, i\kappa_n) \tag{2.182}$$

since $\psi_+(x; i\kappa_n) \sim e^{-\kappa_n x}$ as $x \to +\infty$, and $\psi_-(x; i\kappa_n) \sim e^{\kappa_n x}$ as $x \to -\infty$. Here c_n is a normalization constant and d_n is a real constant. The constants $a(k)$ and $b(k)$ have only simple poles in the upper half plane of the complex k-plane and, $|a(k)| \leq 1$ and $|b(k)| \leq 1$ for all k [25]. Now, evaluate the Wronskian $W(\psi_-, \psi_+)$

$$a(k)W(\psi_-, \psi_+) = W(a(k)\psi_-, \psi_+) = W(\psi_+^* + b(k)\psi_+, \psi_+). \tag{2.183}$$

As $x \to +\infty$

$$W(\psi_-, \psi_+) = 2ika^{-1} \tag{2.184}$$

and if $k = i\kappa_n$, $W(\psi_-, \psi_+) = 0$ since a^{-1} has zeros at $k = i\kappa_n$. Take derivative of (2.184) with respect to k to yield

$$W(\psi_{-k}, \psi_+) + W(\psi_-, \psi_{+k}) = 2ia^{-1} - 2ik\Big(\frac{a'}{a^2}\Big) \tag{2.185}$$

where $a' = \frac{da}{dk}$. We, again, write the Sturm-Liouville equation

$$\psi_{xx} + (k^2 - u)\psi = 0 \tag{2.186}$$

where $\lambda = k^2$, take k-derivative of (2.186)

$$\psi_{xxk} + 2k\psi + (k^2 - u)\psi_k = 0 \tag{2.187}$$

and multiply (2.186) by ψ_k,

$$\psi_k \psi_{xx} + (k^2 - u)\psi\psi_k = 0. \tag{2.188}$$

Combine the above equations to yield

$$\psi_k \psi_{xx} - \psi\psi_{xxk} = 2k\psi^2 \tag{2.189}$$

and integrate it

$$\int^x (\psi_k \psi_{xx} - \psi\psi_{xxk})\, dx \ = 2k \int^x \psi^2\, dx \tag{2.190}$$

or

$$W(\psi_k, \psi) = 2k \int^x \psi^2\, dx\ . \tag{2.191}$$

If we let $\psi = \psi_+$ and set $k = i\kappa_n$, (2.191) becomes

$$W^+(\psi_{+k}, \psi_+) - W^-(\psi_{+k}, \psi_+) = 2i\kappa_n \int_{-\infty}^{\infty} \psi_+^2\, dx \tag{2.192}$$

on the interval from $-\infty$ to $+\infty$. Use $\psi_n(x) = c_n\psi_+(x; i\kappa_n)$ since ψ_n is normalized, we get

$$W^+(\psi_{+k}, \psi_+) - W^-(\psi_{+k}, \psi_+) = 2i\kappa_n \frac{1}{c_n^2}. \tag{2.193}$$

Now, set $k = i\kappa_n$, and (2.185) becomes

$$W(\psi_{-k}, \psi_+) + W(\psi_-, \psi_{+k}) = 2\kappa_n \Big(\frac{a^{'}}{a^2}\Big)_n \tag{2.194}$$

where $(\quad)_n$ denotes evaluation at $k = i\kappa_n$, since $a(k)$ has poles. Use $c_n\psi_+ = d_n\psi_-$ at $k = i\kappa_n$, then we get

$$\frac{d_n}{c_n} W^-(\psi_{-k}, \psi_-) + \frac{c_n}{d_n} W^-(\psi_+, \psi_{+k}) = 2\kappa_n \Big(\frac{a^{'}}{a^2}\Big)_n \tag{2.195}$$

as $x \to -\infty$. Using (2.193) and (2.195), eliminating the term $W^-(\psi_+, \psi_{+k})$, we get

$$W^+(\psi_{+k}, \psi_+) - \frac{d_n^2}{c_n^2} W^-(\psi_{-k}, \psi_-) = \frac{2i\kappa_n}{c_n^2} - \frac{2d_n}{c_n}\kappa_n \Big(\frac{a^{'}}{a^2}\Big)_n. \tag{2.196}$$

Since $W(\psi_{+k}, \psi_+) \to 0$ as $x \to +\infty$, and $W(\psi_{-k}, \psi_-) \to 0$ as $x \to -\infty$, right-hand side of the equation is zero. Then

$$\frac{2i\kappa_n}{c_n^2} = \frac{2d_n}{c_n}\kappa_n\Big(\frac{a'}{a^2}\Big)_n \tag{2.197}$$

or

$$\Big(\frac{a'}{a^2}\Big)_n = \frac{i}{c_n d_n}. \tag{2.198}$$

Now, we can evaluate the residue of $\hat{\psi}e^{ikz}$ at $k = i\kappa_n$. We write

$$\hat{\psi}e^{ikz} = a(k)\psi_-(x,k)e^{ikz} = \frac{c_n}{d_n}a\psi_+e^{ikz} \tag{2.199}$$

then, the residue at $k = i\kappa_n$ is

$$R_n = \frac{c_n}{d_n}\psi_+(x, i\kappa_n)e^{-\kappa_n z} \times \mathrm{res}_{i\kappa_n} a(k) \tag{2.200}$$

since only $a(k)$ has poles and they are simple poles, the residue of $a(k)$ at $k = i\kappa_n$ is

$$\mathrm{res}_{i\kappa_n} a(k) = -\Big(\frac{a^2}{a'}\Big)_n. \tag{2.201}$$

Therefore, we get

$$R_n = ic_n^2\psi_+(x, i\kappa_n)e^{-\kappa_n z}. \tag{2.202}$$

We can determine the integral

$$\int_{-\infty}^{\infty}(\hat{\psi} - e^{-ikx})e^{ikz}\,dz \;= -2\pi\sum_{n=1}^{N} c_n^2\psi_+(x; i\kappa_n)e^{-\kappa_n z} \tag{2.203}$$

that is

$$\int_{-\infty}^{\infty}(\hat{\psi} - e^{-ikx})e^{ikz}\,dz \;= -2\pi\sum_{n=1}^{N} c_n^2 e^{-\kappa_n z}\Big\{e^{-\kappa_n x} + \int_x^{\infty} K(x,y)e^{-\kappa_n y}\,dy\;\Big\}. \tag{2.204}$$

Now, let us define a new function

$$F(X) = \sum_{n=1}^{N} c_n^2 e^{-\kappa_n X} + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k)e^{ikX}\,dk\;. \tag{2.205}$$

Then, (2.167) becomes

$$K(x,z) + F(x+z) + \int_x^{\infty} K(x,y)F(y+z)\,dy \;= 0 \tag{2.206}$$

which is the GLM equation for $K(x,y)$. Lastly, we shall write this equation for the kernel $K(x,z;t)$ at any time t by using the time evolution of the scattering data

$$S(t) = \{\kappa_n, c_n(t) = c_n(0)e^{4\kappa_n^3 t}, k, a(k,t) = a(k,0), b(k,t) = b(k,0)e^{8ik^3 t}\}. \quad (2.207)$$

The GLM equation for $K(x,y;t)$ is

$$K(x,z;t) + F(x+z;t) + \int_x^\infty K(x,y;t)F(y+z;t)\,dy \ = 0 \quad (2.208)$$

where

$$F(X;t) = \sum_{n=1}^{N} c_n^2(0)e^{8\kappa_n^3 t - \kappa_n X} + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k,0)e^{8ik^3 t + ikX}\,dk \ , \quad (2.209)$$

and the potential is

$$u(x,t) = -2\frac{\partial}{\partial x}K(x,x;t). \quad (2.210)$$

The existence and uniqueness of the solution of (2.208) have been proved in [26].

If $b(k,0) = 0$ (reflectionless potential), then $F(x+z;t)$ is a separable function

$$F(x+z;t) = \sum_{n=1}^{N} X_n(x,t)Z_n(z) \quad (2.211)$$

where N is a natural number, then $K(x,y;t)$ is also separable

$$K(x,y;t) = \sum_{n=1}^{N} L_n(x,t)Z_n(z). \quad (2.212)$$

Hence the Marchenko equation becomes

$$L_n(x,t) + X_n(x,t) + \sum_{m=1}^{N} L_m(x,t)\int_x^\infty Z_m(y)X_n(y,t)\,dy \ = 0 \quad (2.213)$$

which are N algebraic equations for N unknowns, $L_n(x,t)$.

We can replace this integral system with an algebraic system. Beforehand, let

$$F(x+z;t) = \sum_{n=1}^{N} c_n^2(0)e^{8\kappa_n^3 t - \kappa_n(x+z)}, \quad (b(k,0) = 0) \quad (2.214)$$

and

$$K(x,z;t) = \sum_{n=1}^{N} L_n(x,t)e^{-\kappa_n z} \tag{2.215}$$

where $L_n(x,t)$ are unknowns. The Marchenko equation for $K(x,z;t)$ becomes

$$L_n(x,t)+c_n^2(0)e^{8\kappa_n^3 t-\kappa_n x}+c_n^2(0)e^{8\kappa_n^3 t}\sum_{m=1}^{N} L_m(x,t)\int_x^{\infty} e^{-(\kappa_n+\kappa_m)y}\,dy \ = 0. \tag{2.216}$$

Evaluating the integral, we get

$$L_n(x,t) + c_n^2(0)e^{8\kappa_n^3 t-\kappa_n x} + c_n^2(0)e^{8\kappa_n^3 t}\sum_{m=1}^{N} \frac{L_m(x,t)e^{-(\kappa_n+\kappa_m)x}}{\kappa_n+\kappa_m} = 0. \tag{2.217}$$

Now, we can replace (2.217) with the algebraic system

$$A\mathbf{L} + \mathbf{B} = 0 \tag{2.218}$$

where $\mathbf{L}$ and $\mathbf{B}$ are column vectors with elements L_n and $B_n = c_n^2(0)e^{8\kappa_n^3 t-\kappa_n x}$, respectively. A is a $N \times N$ matrix with elements

$$A_{mn} = \delta_{mn} + \frac{c_m^2(0)}{\kappa_m+\kappa_n}e^{8\kappa_m^3 t-(\kappa_m+\kappa_n)x} \tag{2.219}$$

where δ_{mn} is the Kronecker delta. The solution for $\mathbf{L}$ is

$$\mathbf{L} = -A^{-1}\mathbf{B} \tag{2.220}$$

and, let

$$\tilde{K}(x,x;t) = \mathbf{E}^T\mathbf{L} \tag{2.221}$$

where $\mathbf{E}$ is the column vector with the elements $E_n = e^{-\kappa_n x}$. Note that

$$\frac{d}{dx}A_{mn} = -c_m^2(0)e^{8\kappa_m^3 t-(\kappa_m+\kappa_n)x} = -B_m E_n \tag{2.222}$$

where B_m is mth element of $\mathbf{B}$ vector and E_n is nth element of $\mathbf{E}$ vector.

$$\tilde{K}_m = E_m L_m = -E_m(A_{mn}^{-1}B_n) = A_{mn}^{-1}\Big(\frac{d}{dx}A_{mn}\Big) \tag{2.223}$$

and, the kernel $K(x,x;t)$ is the trace of $\tilde{K}$ matrix

$$K(x,x;t) = tr(\tilde{K}) = tr\Big(A^{-1}\frac{dA}{dx}\Big) = \frac{1}{|A|}\frac{d|A|}{dx} = \frac{\partial}{\partial x}\log|A|, \tag{2.224}$$

then the potential is [27]

$$u(x) = -2\frac{\partial}{\partial x}K(x,x;t) = -2\frac{\partial^2}{\partial x^2}\log|A|. \tag{2.225}$$

However if $b(k) \neq 0$, then the GLM equation may not be solved for the kernel $K(x,z;t)$ due to the integral part of the function $F(X;t)$. We need to use numerical analysis. The case in which the reflection coefficient is a rational function has studied by Kay [28].

2.2.4 Inverse problem

Now, we are ready to solve the initial value problem of the KdV equation and determine the potential $u(x,t)$ which satisfies the KdV equation.

If the scattering data $S(0)$ is known for the KdV equation, we solve the GLM equation [29], [30]

$$K(x,z;t) + F(x+z;t) + \int_x^\infty K(x,y;t)F(y+z;t)\,dy \;= 0 \tag{2.226}$$

for the kernel $K(x,z;t)$, where the function $F(X,t)$ is given by

$$F(X;t) = \sum_{n=1}^{N} c_n^2(0)e^{8\kappa_n^3 t - \kappa_n X} + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k,0)e^{8ik^3t+ikX}\,dk\ , \tag{2.227}$$

then the solution of the KdV equation is determined by

$$u(x,t) = -2\frac{\partial}{\partial x}K(x,x;t). \tag{2.228}$$

Now, let us look at some examples.

Example 2.2.3. Choose the initial value as

$$u(x,0) = -2\,\mathrm{sech}^2\,x \tag{2.229}$$

where $U_0 = N(N+1) = 2$ corresponding to $N = 1$. Then, for the discrete spectrum, there is only one eigenvalue $\lambda_1 = -1$ ($\lambda_1 = -\kappa_1^2$ with $\kappa_1 = 1$). By using associated Legendre polynomials, we found that

$$\psi_1(x) = \frac{1}{\sqrt{2}}\,\mathrm{sech}\,x \tag{2.230}$$

so that $\int_{-\infty}^{\infty} \psi_1^2 \, dx \; = 1$. The asymptotic behaviour of the solution provides

$$\psi_1(x) \to \sqrt{2}e^{-x} \quad \text{as} \quad x \to \infty \tag{2.231}$$

so that $c_1(0) = \sqrt{2}$, and then $c_1(t) = \sqrt{2}e^{4t}$. If $U_0 = 2$, then for this initial value, the reflection coefficient is zero ($b(k) = 0$). Hence the function

$$F(X,t) = 2e^{8t-X} \tag{2.232}$$

is separable. Inserting this into the GLM equation, we obtain

$$K(x,z;t) + 2e^{8t-(x+z)} + 2e^{8t-z} \int_x^{\infty} K(x,y;t)e^{-y} \, dy \; = 0 \tag{2.233}$$

and this implies that $K(x,z;t)$ is also separable. Let

$$K(x,z;t) = L(x,t)e^{-z}. \tag{2.234}$$

The Marchenko equation (2.233) becomes

$$e^{-z}L(x,t) + 2e^{8t-(x+z)} + 2e^{8t-z} \int_x^{\infty} L(x,t)e^{-2y} \, dy \; = 0 \tag{2.235}$$

and we find

$$L(x,t) = -\frac{2e^{8t-z}}{1+e^{8t-2x}}. \tag{2.236}$$

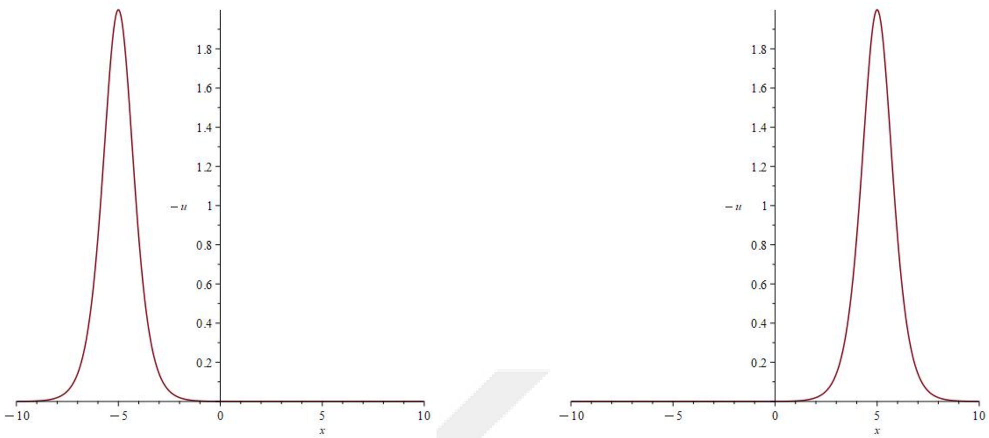
Hence, the kernel $K(x,z;t)$

$$K(x,z;t) = -\frac{2e^{8t-x-z}}{1+e^{8t-2x}}. \tag{2.237}$$

Since $u(x,t) = -2\frac{\partial}{\partial x}K(x,x;t)$, then

$$\begin{aligned} u(x,t) &= 2\frac{\partial}{\partial x}\Big(\frac{2e^{8t-2x}}{1+e^{8t-2x}}\Big) \\ &= -\frac{8e^{2x-8t}}{(1+e^{2x-8t})^2} \\ &= -2\,\mathrm{sech}^2(x-4t) \end{aligned} \tag{2.238}$$

which is the solitary wave solution (one-soliton solution) of the KdV equation that propagates to the right with amplitude 2 and speed 4. As shown in Figure 2.4, the wave is observed to maintain its identity while traveling.



(a) $t = -1.25$ (b) $t = 1.25$

Figure 2.4: The one-soliton solution with $u(x,0) = -2\,\mathrm{sech}^2\, x$

Example 2.2.4. Let the initial value be

$$u(x,0) = -6\,\mathrm{sech}^2\, x \tag{2.239}$$

which corresponds to $N = 2$, then there are two discrete eigenvalues and the reflection coefficient $b(k) = 0$. We found that

$$\psi_1(x) = \left(\frac{3}{2}\right)^{1/2} \tanh x\, \mathrm{sech}\, x \quad \text{and} \quad \psi_2(x) = \frac{\sqrt{3}}{2}\,\mathrm{sech}^2\, x \tag{2.240}$$

which are normalized. The asymptotic behaviour of these solutions yield

$$\psi_1(x) \sim \sqrt{6}e^{-x} \quad \text{and} \quad \psi_2(x) \sim 2\sqrt{3}e^{-2x} \quad \text{as} \quad x \to +\infty \tag{2.241}$$

so that $c_1(0) = \sqrt{6}$ and $c_2(0) = 2\sqrt{3}$. We write the function

$$F(X;t) = 6e^{8t-X} + 12e^{64t-2X}, \tag{2.242}$$

then the GLM equation becomes

$$\begin{aligned} K(x,z;t) &+ 6e^{8t-(x+z)} + 12e^{64t-2(x+z)} \\ &+ \int_x^\infty K(x,y;t)\Big[6e^{8t-(y+z)} + 12e^{64t-2(y+z)}\Big]\, dy \; = 0 \end{aligned} \tag{2.243}$$

and this implies that K is separable. Let

$$K(x,z;t) = L_1(x,t)e^{-z} + L_2(x,t)e^{-2z} \tag{2.244}$$

where L_1, L_2 are unknowns. Insert this into the Marchenko equation to get

$$\begin{aligned} L_1(x,t) + 6e^{8t-x} + 6e^{8t}\Big[L_1(x,t)\int_x^\infty e^{-2y}dy + L_2(x,t)\int_x^\infty e^{-3y}dy\Big] = 0, \\ L_2(x,t) + 12e^{64t-2x} + 12e^{64t}\Big[L_1(x,t)\int_x^\infty e^{-3y}dy + L_2(x,t)\int_x^\infty e^{-4y}dy\Big] = 0. \end{aligned} \tag{2.245}$$

After the calculation of integrals, we find unknowns as

$$\begin{aligned} L_1(x,t) &= \frac{6\big(e^{72t-5x} - e^{8t-x}\big)}{D} \\ L_2(x,t) &= -\frac{12\big(e^{64t-2x} + e^{72t-4x}\big)}{D} \end{aligned} \tag{2.246}$$

where $D = 1 + 3e^{8t-2x} + 3e^{64t-4x} + e^{72t-6x}$. Hence the solution

$$u(x,t) = -12\frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{\big[3\cosh(x - 28t) + \cosh(3x - 36t)\big]^2} \tag{2.247}$$

which is the two-soliton solution of the KdV equation and both solitons travel to the right.

Let us look at the asymptotic behaviour of the solution. Let $\xi = x - 16t$ ($\xi = x - 4\kappa_2^2 t$ with $\kappa_2 = 2$), then (2.247) becomes

$$u(x,t) = -12\frac{3 + 4\cosh(2\xi + 24t) + \cosh(4\xi)}{\big\{3\cosh(\xi - 12t) + \cosh(3\xi + 12t)\big\}^2} \tag{2.248}$$

for fixed ξ. We find that

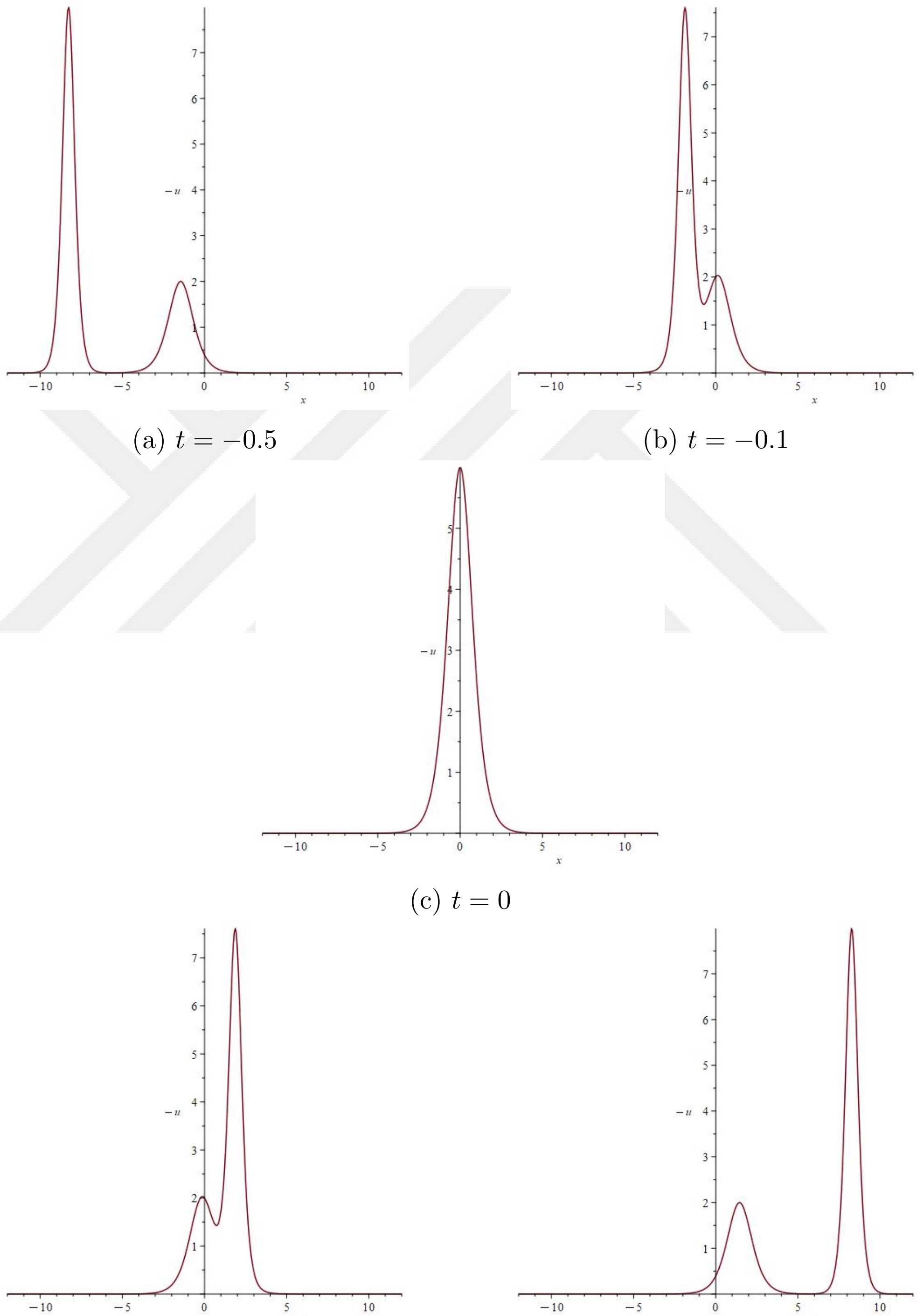
$$u(x,t) = -8\,\mathrm{sech}^2\big(2\xi - 1/2\log(3)\big) \quad \text{as} \quad t \to +\infty \tag{2.249}$$

travels with speed 16 ($= 4\kappa_2^2$) and amplitude 8 ($= 2\kappa_2^2$). Let $\eta = x - 4t$ ($\eta = x - 4\kappa_1^2 t$ with $\kappa_1 = 1$), then (2.247) becomes

$$u(x,t) = -12\frac{3 + 4\cosh(2\eta) + \cosh(4\eta - 48t)}{\big\{3\cosh(\eta - 24t) + \cosh(3\eta - 24t)\big\}^2} \tag{2.250}$$

for fixed η and we find that

$$u(x,t) = -2\,\mathrm{sech}^2\big(\eta + 1/2\log(3)\big) \quad \text{as} \quad t \to \infty \tag{2.251}$$



(a) $t = -0.5$ (b) $t = -0.1$

(c) $t = 0$

(d) $t = 0.1$ (e) $t = 0.5$

Figure 2.5: The two-soliton solution with $u(x, 0) = -6\,\mathrm{sech}^2\, x$.

travels with speed 4 ($= 4\kappa_1^2$) and amplitude 2 ($= 2\kappa_1^2$). We find that the taller wave moves forward by the amount of $1/2\log(3)$ and, the shorter one moves backward by the amount of $\log(3)$. Hence the asymptotic form of the solution is [31]

$$u(x,t) = -8\,\mathrm{sech}^2\big(2\xi - 1/2\log(3)\big) - 2\,\mathrm{sech}^2\big(\eta + 1/2\log(3)\big) \quad \text{as} \quad t \to \infty. \tag{2.252}$$

Figure 2.5 shows that initially, the taller wave is positioned to the left of the shorter wave. Since the taller wave travels faster, the taller one catches the shorter one and the collision occurs at $t = 0$, then the taller one moves away from the shorter one. After the interaction, both waves retain their identities i.e., speed and shape. However, we observe phase shift for both waves. The taller wave moves forward and the shorter one moves backward after interaction.

Example 2.2.5. Let us find N-soliton solution of the KdV equation, so choose the initial value as

$$u(x,0) = -N(N+1)\,\mathrm{sech}^2 x, \quad N = 1, 2, \ldots \tag{2.253}$$

so that there are N discrete eigenvalues $\lambda = -\kappa_n^2$ ($\kappa_n = n$ and $n = 1, 2, ..., N$) and the reflection $b(k) = 0$. The eigenfunctions are proportional to the associated Legendre polynomials

$$\psi_n(x) \propto P_N^n(\tanh x) \tag{2.254}$$

and, find the normalization constant $c_n(0)$

$$\psi_n(x) \sim c_n(0)e^{-\kappa_n x} \quad \text{as} \quad x \to +\infty. \tag{2.255}$$

Then the function F is

$$F(X;t) = \sum_{n=1}^{N} c_n^2(0)e^{8n^3t - nX}. \tag{2.256}$$

We have already studied this case, see calculations after the equation (2.214). If we set $\kappa_n = n$, we can write the solution as

$$u(x,t) = -2\frac{\partial^2}{\partial x^2}\log|A| \tag{2.257}$$

where A is $N \times N$ matrix with elements

$$A_{mn} = \delta_{mn} + \frac{c_m^2(0)}{m+n} e^{8m^3 t - (m+n)x}. \tag{2.258}$$

If we choose $\xi_n = x - 4\kappa_n^2 t = x - 4n^2 t$, and for fixed ξ_n, the asymptotic behaviour of $u(x,t)$ is

$$u(x,t) \sim -2n^2 \operatorname{sech}^2\big\{n(x - 4n^2 t) - x_n\big\} \quad \text{as} \quad t \to +\infty. \tag{2.259}$$

It travels to the right with the speed $4n^2$ and amplitude $2n^2$, where x_n is the phase. Then the solution $u(x,t)$ is

$$u(x,t) = -2\sum_{n=1}^{N} n^2 \operatorname{sech}^2\big\{n(x - 4n^2 t) - x_n\big\} \quad \text{as} \quad t \to +\infty \tag{2.260}$$

which is the N-soliton solution of the KdV equation. All solitons interact at $t = 0$ and move away from each other without changing their identities. We note that phase shift happens for all solitons after the interaction.

Now, we shall look at some examples and observe what happens if the reflection coefficient $b(k) \neq 0$.

Example 2.2.6. Let the initial value be

$$u(x,0) = -4 \operatorname{sech}^2 x \tag{2.261}$$

and $U_0 = 4$. There is no positive integer N such that $4 = N(N+1)$. To find the total number of discrete eigenvalues, we use

$$\Big[\big(U_0 + \frac{1}{4}\big)^{1/2} - \frac{1}{2}\Big] + 1 \tag{2.262}$$

where [ ] denotes the integral part, then when $U_0 = 4$, there are two discrete eigenvalues which provide two different solitons. However, the solution is not pure-soliton, but includes a dispersive wave (or tail) which travels to the left. In general, the upper bound for the number of solitons, N, is [32]

$$N \leq 1 + \frac{1}{2}\int_{-\infty}^{\infty} |x|\big\{1 + \operatorname{sgn} u(x,0)\big\} u(x,0)\, dx\ . \tag{2.263}$$

Hence in this case, we can write the solution $u(x,t)$ as

$$u(x,t) = u_n + u_s = u_n - 2\sum_{n=1}^{N} n^2 \operatorname{sech}^2\big\{n(x - 4n^2 t) - x_n\big\} \quad \text{as} \quad t \to +\infty \tag{2.264}$$

where u_n denotes non-soliton solution and u_s denotes soliton solutions.

Example 2.2.7. Choose the initial value as

$$u(x,0) = \operatorname{sech}^2 x \tag{2.265}$$

since $U_0 = -1$ (< 0), the initial pulse will collapse and it will give rise to only a dispersive wave. The solution of the KdV equation with the initial value $\operatorname{sech}^2 x$ is related to the Airy function [24].

2.3 The inverse scattering transform for the KdV hierarchy

Our aim is to solve the initial value problem of the KdV hierarchy which is given by

$$u_t + \mathcal{R}^N u_x = 0, \quad N = 1, 2, ... \tag{2.266}$$

with $u(x,0) = f(x)$, where $\mathcal{R} = \partial_x^2 - 4u - 2u_x\partial_x^{-1}$ is the recursion operator of the KdV equation. When $N = 1$, the initial value problem is for the third-order KdV equation and we have already studied this case. The Lax pair for the KdV hierarchy is given by

$$L = -\partial_x^2 + u(x,t) \tag{2.267}$$

and

$$M = A + B\partial_x. \tag{2.268}$$

Here $A = -\frac{1}{2}B_x + h(t)$ and

$$B = \sum_{n=0}^{N} b_{N-n}\lambda^n = b_0\lambda^N + b_1\lambda^{N-1} + ... \tag{2.269}$$

with

$$b_{k+1,x} = \frac{b_0}{2}\Big(-\frac{1}{4}\Big)^k \mathcal{R}^k u_x, \quad b_1 = \frac{b_0}{2}u(x,t) \tag{2.270}$$

where $b_0(-1/4)^N = -1$. The eigenvalue equation and the time-evolution equation for ψ, for $t > 0$, are given by

$$L\psi = \lambda\psi \tag{2.271}$$

and

$$\psi_t = M\psi \tag{2.272}$$

respectively. The scattering problem for the KdV hierarchy is given by the Sturm-Liouville equation [33], [34]

$$\psi_{xx} + \big(\lambda - u(x,t)\big)\psi = 0 \tag{2.273}$$

since the operator L remains the same and we have already evaluated the scattering data at $t = 0$ as

$$S(0) = \Big\{\kappa_n, c_n(0), k, a(k,0), b(k,0)\Big\}. \tag{2.274}$$

However, the scattering data at any time t must be evaluated, since the time-evolution equation for ψ (or the operator M) changes. Now, we shall find the scattering data $S(t)$ at any time t. The evolution equation for ψ is

$$\psi_t = A\psi + B\psi_x \tag{2.275}$$

where

$$\begin{aligned} A &= -\frac{1}{2}B_x + h(t) \\ B &= b_0\lambda^N + b_1\lambda^{N-1} + b_2\lambda^{N-2} + ... \end{aligned} \tag{2.276}$$

We know that $u(x,t)$ and its x-derivatives vanish as $|x| \to \infty$, then $b_i \to 0$ as $|x| \to \infty$ for $i = 1, 2, ..., N$. Hence, we get

$$\psi_t = h(t)\psi + b_0\lambda^N\psi_x \quad \text{as} \quad |x| \to \infty \tag{2.277}$$

and substitute $b_0 = -(-4)^N$ to yield

$$\psi_t = h(t)\psi - (-4)^N\lambda^N\psi_x \quad \text{as} \quad |x| \to \infty. \tag{2.278}$$

For discrete spectrum ($\lambda = -\kappa_n^2 < 0$), we calculated that $h(t) = 0$ and the discrete eigenfunction corresponding to the eigenvalue $\lambda = -\kappa_n^2$ is given by

$$\psi \sim c_n e^{-\kappa_n x} \quad \text{as} \quad x \to +\infty. \tag{2.279}$$

Inserting this into (2.278), we get

$$\frac{d}{dt}c_n(t)e^{-\kappa_n x} = (-4)^N\lambda^N\kappa_n c_n(t)e^{-\kappa_n x} \tag{2.280}$$

that is

$$\frac{d}{dt}c_n(t) = 4^N \kappa_n^{2N+1} c_n(t) \tag{2.281}$$

since $\lambda = -\kappa_n^2$. Then, we obtain the normalization constant at any time t as

$$c_n(t) = c_n(0)e^{4^N \kappa_n^{2N+1} t} \tag{2.282}$$

where $c_n(0)$ is the normalization constant at $t = 0$.

For continuous spectrum ($\lambda = k^2 > 0$), $h(t)$ must be calculated and the continuous eigenfunction is given by

$$\hat{\psi} \sim \begin{cases} e^{-ikx} + b(k,t)e^{ikx}, & \text{as} \quad x \to +\infty \\ a(k,t)e^{-ikx}, & \text{as} \quad x \to -\infty \end{cases} \tag{2.283}$$

We calculate the time evolution equation for $\hat{\psi}$ as $x \to +\infty$, then we obtain

$$\frac{d}{dt}b(k,t)e^{ikx} = h(t)\Big\{e^{-ikx} + b(k,t)e^{ikx}\Big\} - (-4)^N k^{2N}\Big\{-ike^{-ikx} + ikb(k,t)e^{ikx}\Big\} \tag{2.284}$$

and comparing the coefficients of e^{ikx} and e^{-ikx}, we find that $h(t) = -ik^{2N+1}(-4)^N$. Then the reflection coefficient at any time t is

$$b(k,t) = b(k,0)e^{-2ik^{2N+1}(-4)^N t} \tag{2.285}$$

where $b(k,0)$ is the reflection coefficient at $t = 0$. Now, we calculate the time evolution equation for $\hat{\psi}$ as $x \to -\infty$, then we obtain

$$\frac{d}{dt}a(k,t)e^{-ikx} = -ik^{2N+1}(-4)^N\{a(k,t)e^{-ikx}\} - (-4)^N k^{2N}\{-ika(k,t)e^{-ikx}\} \tag{2.286}$$

or

$$\frac{d}{dt}a(k,t) = 0 \tag{2.287}$$

and we find that the transmission coefficient does not depend on t.

We have the scattering data $S(t)$ at any time

$$S(t) = \Big\{\kappa_n, c_n(0)e^{4^N \kappa_n^{2N+1} t}, k, a(k,0), b(k,0)e^{-2ik^{2N+1}(-4)^N t}\Big\} \tag{2.288}$$

where $S(0)$ has been found from the scattering problem with initial value $u(x,0)$.

If $S(0)$ is given and we solve the GLM equation for the kernel $K(x,z;t)$

$$K(x,z;t)+F(x+z;t)+\int_x^{\infty}K(x,y;t)F(y+z;t)\,dy \ =0 \tag{2.289}$$

where F is given by

$$F(X;t)=\sum_{n=1}^{N}c_n^2(0)e^{2\kappa_n^{2N+1}4^N t-\kappa_n X}+\frac{1}{2\pi}\int_{-\infty}^{\infty}b(k;0)e^{ikX-2ik^{2N+1}(-4)^N t}\,dk \tag{2.290}$$

then the potential is

$$u(x,t)=-2\frac{\partial}{\partial x}K(x,x;t) \tag{2.291}$$

which is the solution of the KdV hierarchy.

Example 2.3.1. Let us choose the initial value as

$$u(x,0)=-2\,\mathrm{sech}^2\,x. \tag{2.292}$$

We have already studied that this initial value corresponds to only one discrete eigenvalue, $\lambda=-1(\kappa_1=1)$, and the potential is reflectionless, $b(k,t)=0$. The discrete eigenvalue is

$$\psi_1(x;0)=\frac{1}{\sqrt{2}}\,\mathrm{sech}\,x\to\sqrt{2}e^{-x}\quad \text{as}\quad x\to\infty \tag{2.293}$$

where the normalization constant at $t=0$ is $c_1(0)=\sqrt{2}$. Then we write the function

$$F(X;t)=2e^{2(4)^N t-X}. \tag{2.294}$$

Then the GLM equation becomes

$$K(x,z;t)+2e^{2(4)^N t-(x+z)}+2\int_x^{\infty}K(x,y;t)e^{2(4)^N t-(y+z)}\,dy \ =0 \tag{2.295}$$

then, this implies that the kernel $K(x,y;t)$ is

$$K(x,z;t)=L(x,t)e^{-z} \tag{2.296}$$

which is a separable function. Substitute it to yield

$$L(x,t)+2e^{2(4)^N t-x}+2L(x,t)e^{2(4)^N t}\int_x^{\infty}e^{-2y}\,dy \ =0 \tag{2.297}$$

then, we get

$$L(x,t) = -\frac{2e^{2(4)^N t - x}}{1 + e^{2(4)^N t - 2x}}. \tag{2.298}$$

Hence the kernel K is

$$K(x,x;t) = -\frac{2e^{2(4)^N t - 2x}}{1 + e^{2(4)^N t - 2x}} \tag{2.299}$$

then the potential is

$$\begin{aligned} u(x,t) &= -2\frac{4e^{2(4)^N t - 2x}}{\left\{1 + e^{2(4)^N t - 2x}\right\}^2} \\ &= -2\,\mathrm{sech}^2\left\{(4)^N t - x\right\} \end{aligned} \tag{2.300}$$

which is a solitary (one-soliton) solution of the KdV hierarchy with the speed $(4)^N$ and the amplitude 2. Hence, we can make this conclusion: Let us say that the solution of Mth order KdV equation and Nth order KdV equation correspond to the same discrete eigenvalue. If $M > N > 0$, then the soliton solution of the Mth order KdV equation travels faster that the one of the Nth order KdV equation, but their amplitude are the same.

Chapter 3

The AKNS Scheme

The inverse scattering transform was, first, developed for the Korteweg-de Vries equation by Gardner, Greene, Kruskal, and Miura [6] [31]. It was not clear whether this method can be applied to any nonlinear evolution equations. Zakharov and Shabat [8] showed that the inverse scattering transform can be applied to the nonlinear Schrödinger equation

$$i\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+\kappa|u|^2u=0 \quad \text{for} \quad \kappa>0 \tag{3.1}$$

where $u(x,t)$ decreases sufficiently rapidly as $|x|\to\infty$. At about the same time, Wadati [9] also proved that this method is applicable to the evolution equation

$$q_t+6q^2q_x+q_{xxx}=0 \tag{3.2}$$

which is known as the modified KdV (mKdV) equation. Later, Ablowitz, Kaup, Newell, and Segur [10] found the inverse scattering transform for the sine-Gordon equation

$$u_{xt}=\sin u. \tag{3.3}$$

After that, Ablowitz, Kaup, Newell, and Segur developed the AKNS scheme from which the nonlinear evolution equation can be derived, and the initial value problem of these equations can be solved by IST [11].

3.1 Zero curvature formalism for AKNS

Let us introduce a linear 2×2 eigenvalue problem

$$\begin{aligned} v_{1x} &= -i\zeta v_1 + qv_2 \\ v_{2x} &= i\zeta v_2 + rv_1 \end{aligned} \tag{3.4}$$

and the time evolution equation of v_1 and v_2

$$\begin{aligned} v_{1t} &= Av_1 + Bv_2 \\ v_{2t} &= Cv_1 + Dv_2 \end{aligned} \tag{3.5}$$

where $q(x,t)$ and $r(x,t)$ are potentials and ζ is the eigenvalue and, A, B, C, D are scalar functions which depend on the potentials q and r and their x-derivatives, and x, t, ζ, but are independent of v_1 and v_2. (3.4) and (3.5) are known as the AKNS scheme.

For the choice $r = -1$, (3.4) can be reduced to the Sturm-Liouville equation for v_2,

$$v_{2x} = i\zeta v_2 - v_1 \tag{3.6}$$

or

$$v_1 = i\zeta v_2 - v_2 x. \tag{3.7}$$

Substitute it to the first equation, we obtain

$$v_{2xx} + (\zeta^2 + q)v_2 = 0 \tag{3.8}$$

where $\lambda = \zeta^2$ and $u = -q$. Note that ζ is independent of t.

Now, we shall find the conditions which A, B, C, D must satisfy. We write (3.4) and (3.5) as

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_x = \begin{pmatrix} -i\zeta & q \\ r & i\zeta \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_t = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{3.9}$$

It is required that $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_{xt} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_{tx}$, then take t-derivative of (3.4) and x-derivative of (3.5) to yield

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_{xt} = \begin{pmatrix} 0 & q_t \\ r_t & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} -i\zeta & q \\ r & i\zeta \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_t \tag{3.10}$$

and

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_{tx} = \begin{pmatrix} A_x & B_x \\ C_x & D_x \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_x . \tag{3.11}$$

We find that

$$\begin{pmatrix} -i\zeta & q \\ r & i\zeta \end{pmatrix}_t - \begin{pmatrix} A & B \\ C & D \end{pmatrix}_x + \left[\begin{pmatrix} -i\zeta & q \\ r & i\zeta \end{pmatrix}, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right] = 0 \tag{3.12}$$

must be satisfied which is known as zero-curvature conditions. Here, [ ] denotes the commutator of two matrices. Hence we have the following conditions

$$\begin{aligned} A_x &= qC - rB \\ B_x + 2i\zeta B &= q_t - (A - D)q \\ C_x - 2i\zeta C &= r_t + (A - D)r \\ -D_x &= qC - rB \end{aligned} \tag{3.13}$$

We found that $-A_x = D_x$, then $D = -A + h(t)$ for arbitrary $h(t)$. Without loss of generality, we choose $h(t) = 0$, $(D = -A)$, then (3.13) reduces to

$$\begin{aligned} A_x &= qC - rB \\ B_x + 2i\zeta B &= q_t - 2Aq \\ C_x - 2i\zeta C &= r_t + 2Ar \end{aligned} \tag{3.14}$$

for $A(x, t, \zeta)$, $B(x, t, \zeta)$, and $C(x, t, \zeta)$. By using the equations (3.14), we obtain the nonlinear evolution equation. Since the eigenvalue ζ is a free parameter, we can set

$$A = \sum_{n=0}^{N} a_n \zeta^n, \quad B = \sum_{n=0}^{N} b_n \zeta^n, \quad C = \sum_{n=0}^{N} c_n \zeta^n. \tag{3.15}$$

Inserting them into (3.14), we get the following conditions for a_i, b_i, and c_i's

$$b_{n,x} + 2ib_{n-1} + 2qa_n = 0 \quad \text{for} \quad 1 \le n \le N \tag{3.16a}$$

$$b_{0,x} = q_t - 2qa_0 \quad \text{and} \quad b_N = 0 \tag{3.16b}$$

and

$$c_{n,x} - 2ic_{n-1} - 2ra_n = 0 \quad \text{for} \quad 1 \le n \le N \tag{3.17a}$$

$$c_{0,x} = r_t + 2ra_0 \quad \text{and} \quad c_N = 0 \tag{3.17b}$$

and

$$a_{n,x} = qc_n - rb_n \quad \text{for} \quad 0 \le n \le N-1 \tag{3.18a}$$

$$a_N = \text{constant}. \tag{3.18b}$$

We found the evolution equations from (3.16b)

$$q_t = b_{0,x} + 2qa_0 \tag{3.19}$$

and from (3.17b)

$$r_t = c_{0,x} - 2ra_0. \tag{3.20}$$

For $N = 3$, we have

$$\begin{aligned} A &= a_2\zeta^2 + a_1\zeta + a_0 \\ B &= b_2\zeta^2 + b_1\zeta + b_0 \\ C &= c_2\zeta^2 + c_1\zeta + c_0. \end{aligned} \tag{3.21}$$

We already found that $b_3 = 0$, $c_3 = 0$. Let $a_3 = \varepsilon_3$ where ε_3 is a constant. From (3.16a) and (3.17a), we have

$$\begin{aligned} 2ib_2 + 2qa_2 &= 0 \\ 2ic_2 + 2ra_2 &= 0. \end{aligned} \tag{3.22}$$

We find that $b_2 = i\varepsilon_3 q$, $c_2 = i\varepsilon_3 r$. From (3.18a)

$$a_{2,x} = qc_2 - rb_2 = q(i\varepsilon_3 r) - r(i\varepsilon_3 q) = 0, \tag{3.23}$$

we find that $a_2 = \varepsilon_2$, where ε_2 is a constant. Again, (3.16a), (3.17a) provide $b_1 = i\varepsilon_2 q - \frac{1}{2}\varepsilon_3 q_x$ and $c_1 = i\varepsilon_2 r + \frac{1}{2}\varepsilon_3 r_x$ and, (3.18a) provides $a_1 = \frac{1}{2}\varepsilon_3 qr + \varepsilon_1$. Finally, (3.16a), (3.17a) give b_0, c_0 and, (3.18a) gives a_0.

$$\begin{aligned} A &= \varepsilon_3\zeta^3 + \varepsilon_2\zeta^2 + \left(\frac{1}{2}\varepsilon_3 qr + \varepsilon_1\right)\zeta + \left(\frac{1}{2}\varepsilon_2 qr - \frac{i}{4}\varepsilon_3\big(qr_x - q_x r\big) + \varepsilon_0\right) \\ B &= i\varepsilon_3 q\zeta^2 + \left(i\varepsilon_2 q - \frac{1}{2}\varepsilon_3 q_x\right)\zeta + \left(i\varepsilon_1 q + \frac{i}{2}\varepsilon_3 q^2 r - \frac{1}{2}\varepsilon_2 q_x - \frac{i}{4}\varepsilon_3 q_{xx}\right) \\ C &= i\varepsilon_3 r\zeta^2 + \left(i\varepsilon_2 r + \frac{1}{2}\varepsilon_3 r_x\right)\zeta + \left(i\varepsilon_1 r + \frac{i}{2}\varepsilon_3 qr^2 + \frac{1}{2}\varepsilon_2 r_x - \frac{i}{4}\varepsilon_3 r_{xx}\right) \end{aligned} \tag{3.24}$$

and the evolution equations (3.19) and (3.20) become

$$\begin{aligned} q_t &= \frac{i}{4}\varepsilon_3\big(6qrq_x - q_{xxx}\big) + \frac{1}{2}\varepsilon_2\big(2q^2r - q_{xx}\big) + i\varepsilon_1 q_x + 2\varepsilon_0 q \\ r_t &= \frac{i}{4}\varepsilon_3\big(6qrr_x - r_{xxx}\big) - \frac{1}{2}\varepsilon_2\big(2qr^2 - r_{xx}\big) + i\varepsilon_1 r_x - 2\varepsilon_0 r \end{aligned} \tag{3.25}$$

where ε_i's are constants. We list some special cases below:

(i) $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 0$, $\varepsilon_3 = -4i$ (for $N = 3$), the evolution equations are

$$\begin{aligned} q_t &= 6qrq_x - q_{xxx} \\ r_t &= 6qrr_x - r_{xxx} \end{aligned} \tag{3.26}$$

(a) $r = \pm 1$, we obtain the KdV equation

$$q_t \mp 6qq_x + q_{xxx} = 0 \tag{3.27}$$

(b) $r = \pm q$, we have the modified KdV equation

$$q_t \mp 6q^2q_x + q_{xxx} = 0 \tag{3.28}$$

(ii) $\varepsilon_0 = \varepsilon_1 = \varepsilon_3 = 0$, $\varepsilon_2 = -2i$ (for $N = 2$), the evolution equations are

$$\begin{aligned} q_t &= -i2q^2r + iq_{xx} \\ r_t &= i2qr^2 - ir_{xx} \end{aligned} \tag{3.29}$$

(a) $r = \pm q^*$, we obtain the nonlinear Schrödinger equation

$$q_t \pm 2iq^2q^* - iq_{xx} = 0 \tag{3.30}$$

where * denotes the complex conjugate. We found the evolution equations corresponding to the expansions of A, B, C in positive powers of ζ. We can also find different evolution equations for the expansion in inverse power of ζ. We set

$$A(x,t,\zeta) = \frac{a(x,t)}{\zeta}, \quad B(x,t,\zeta) = \frac{b(x,t)}{\zeta}, \quad C(x,t,\zeta) = \frac{c(x,t)}{\zeta}. \tag{3.31}$$

The equation (3.14) reduces to

$$\begin{aligned} &b_x = -2qa, \quad 2ib = q_t \\ &c_x = 2ra, \quad -2ic = r_t \\ &a = qc - rb = \frac{i}{2}(qr)_t \end{aligned} \tag{3.32}$$

and we find

$$
\begin{aligned}
q_{xt} &= 2ib_x = 2i(-2qa) = -4qa \\
r_{xt} &= -2ic_x = -2i(2ra) = -4ira.
\end{aligned}
\tag{3.33}
$$

We list some special cases below:

(i) $a = \frac{i}{4}\cos u$, $b = c = \frac{i}{4}\sin u$, then $r = -q = \frac{u_x}{2}$. Since $r_t = -2ic$ we get the sine-Gordon equation

$$u_{xt} = \sin u. \tag{3.34}$$

(ii) $a = \frac{i}{4}\cosh u$, $-b = c = \frac{i}{4}\sinh u$, then $r = q = \frac{u_x}{2}$. Since $r_t = -2ic$ we get the sinh-Gordon equation

$$u_{xt} = \sinh u. \tag{3.35}$$

From the AKNS scheme, we obtain the evolution equations which can be solved by the inverse scattering method [35].

3.2 Inverse scattering transform for AKNS

3.2.1 Direct problem

Now, let us introduce 2×2 eigenvalue problem

$$\psi_x = \begin{pmatrix} -i\zeta & q \\ r & i\zeta \end{pmatrix} \psi \tag{3.36}$$

or

$$\begin{pmatrix} \partial/\partial x & -q \\ r & -\partial/\partial x \end{pmatrix} \psi = -i\zeta\psi \tag{3.37}$$

where $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, the bounded functions $q(x)$, $r(x)$ are potentials and ζ is the eigenvalue ($\zeta_t = 0$). Here, it is required that q, r decay sufficiently rapidly so that q, r are absolutely integrable (i.e, $q, r \in L_1$)

$$\int_{-\infty}^{\infty} |q(x)|\, dx \ < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} |r(x)|\, dx \ < \infty \tag{3.38}$$

and, for convenience, we assume

$$\int_{-\infty}^{\infty} |x|^n \left\{ \begin{vmatrix} q(x) \\ r(x) \end{vmatrix} \right\} dx \ < \infty \tag{3.39}$$

for all n. We will see later why we make such an assumption. We also note that t plays the role of a parameter and, here, we shall solve the direct problem for fixed t i.e., $t = 0$.

We define four eigenfunctions

$$\psi_+(x;\zeta) \sim e^{-i\zeta x} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \hat{\psi}_+(x;\zeta) \sim e^{i\zeta x} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{as} \quad x \to +\infty \tag{3.40}$$

and

$$\psi_-(x;\zeta) \sim e^{-i\zeta x} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \hat{\psi}_-(x;\zeta) \sim e^{i\zeta x} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad \text{as} \quad x \to -\infty \tag{3.41}$$

which are the solution of (3.36) at a fixed time. The Wronskian of ψ_+, $\hat{\psi}_+$ and the Wronskian of ψ_-, $\hat{\psi}_-$ are

$$W(\psi_+, \hat{\psi}_+) = 1 \quad \text{and} \quad W(\psi_-, \hat{\psi}_-) = -1 \tag{3.42}$$

in which the Wronskian is defined as $W(u, v) = u_1 v_2 - u_2 v_1$. Since the solution ψ_+ and $\hat{\psi}_+$ are linearly independent, then we can write

$$\begin{aligned} \psi_- &= a(\zeta)\psi_+ + b(\zeta)\hat{\psi}_+ \\ \hat{\psi}_- &= \hat{b}(\zeta)\psi_+ - \hat{a}(\zeta)\hat{\psi}_+. \end{aligned} \tag{3.43}$$

Then,

$$\begin{pmatrix} \psi_- & \hat{\psi}_- \end{pmatrix} = \begin{pmatrix} \psi_+ & \hat{\psi}_+ \end{pmatrix} \begin{pmatrix} a(\zeta) & \hat{b}(\zeta) \\ b(\zeta) & -\hat{a}(\zeta) \end{pmatrix} \tag{3.44}$$

and by using $W(\psi_-, \hat{\psi}_-) = -1$ and (3.44), we find

$$a(\zeta)\hat{a}(\zeta) + b(\zeta)\hat{b}(\zeta) = 1. \tag{3.45}$$

As long as (3.38) is satisfied, $e^{i\zeta x}\psi_-$ and $e^{-i\zeta x}\hat{\psi}_+$ are analytic in the upper half plane and, $e^{-i\zeta x}\hat{\psi}_-$ and $e^{i\zeta x}\psi_+$ are analytic in the lower half plane.

Hence, we have

$$W(\psi_-, \hat{\psi}_+) = W(a\psi_+ + b\hat{\psi}_+, \hat{\psi}_+) = a(\zeta) \tag{3.46}$$

where $a(\zeta)$ is analytic in the upper half plane and

$$W(\hat{\psi}_-, \psi_+) = W(\hat{b}\psi_+ - \hat{a}\hat{\psi}_+, \psi_+) = -\hat{a}(\zeta) \tag{3.47}$$

$\hat{a}(\zeta)$ is analytic in the lower half plane. In general,

$$W(\hat{\psi}_-, \hat{\psi}_+) = \hat{b}(\zeta) \quad \text{and} \quad W(\psi_-, \psi_+) = b(\zeta) \tag{3.48}$$

are not required to be analytic. We find that q, r decay sufficiently rapidly then $a(\zeta)$ and $\hat{a}(\zeta)$ are analytic functions. We also assume that, for convenience, (3.39) is satisfied. If (3.39) holds, this assures us that $a(\zeta)$ and $\hat{a}(\zeta)$ are also analytic functions on the real axis. Then $a(\zeta)$ and $\hat{a}(\zeta)$ have finitely many zeros in the upper half plane and in the lower half plane, respectively i.e., $\hat{a}(\zeta)$ is analytic and $\hat{a}(\zeta) \to 1$ as $|\zeta| \to +\infty$, then all the zeros of $\hat{a}(\zeta)$ are isolated and lie in the lower half plane. The same is valid for $a(\zeta)$, since $a(\zeta) \to 1$ as $|\zeta| \to +\infty$, except, this time, all zeros lie in the upper half plane.

Now, we can rewrite (3.44) as

$$\Psi_- = \Psi_+ S \tag{3.49}$$

where $\Psi_- = \begin{pmatrix} \psi_- & \hat{\psi}_- \end{pmatrix}$, $\Psi_+ = \begin{pmatrix} \psi_+ & \hat{\psi}_+ \end{pmatrix}$ and $S(\zeta)$ is a scattering matrix

$$S(\zeta) = \begin{pmatrix} a(\zeta) & \hat{b}(\zeta) \\ b(\zeta) & -\hat{a}(\zeta) \end{pmatrix} \tag{3.50}$$

for fixed t (say $t = 0$). The solution Ψ_+ can be expressed by an appropriate kernel, then we can write

$$\Psi_+(x;\zeta) = \Psi_0(x;\zeta) + \int_x^\infty K(x,y)\Psi_0(y;\zeta)\, dy \tag{3.51}$$

where $K(x, y)$ is a 2×2 kernel matrix and

$$\Psi_0(x;\zeta) = \begin{pmatrix} e^{-i\zeta x} & 0 \\ 0 & e^{i\zeta x} \end{pmatrix} \tag{3.52}$$

which describes the asymptotic behaviour of $\Psi_+(x;\zeta)$ since $\Psi_+(x;\zeta) \to \Psi_0(x;\zeta)$ as $x \to +\infty$. We note that the kernel matrix does not depend on the eigenvalue ζ. Insert (3.51) into (3.37), on the right hand side, we obtain

$$-i\zeta\Psi_+ = -i\zeta\Big(\Psi_0(x;\zeta) + \int_x^\infty K(x,y)\Psi_0(y;\zeta)\,dy\Big). \tag{3.53}$$

On left hand side, from the first term $\Psi_0(x;\zeta)$, we obtain

$$\begin{pmatrix} \partial_x & -q \\ r & -\partial_x \end{pmatrix}\begin{pmatrix} e^{-i\zeta x} & 0 \\ 0 & e^{i\zeta x} \end{pmatrix} = -i\zeta\begin{pmatrix} e^{-i\zeta x} & 0 \\ 0 & e^{i\zeta x} \end{pmatrix} + \begin{pmatrix} 0 & -q \\ r & 0 \end{pmatrix}\begin{pmatrix} e^{-i\zeta x} & 0 \\ 0 & e^{i\zeta x} \end{pmatrix}. \tag{3.54}$$

Then

$$\begin{pmatrix} \partial_x & -q \\ r & -\partial_x \end{pmatrix}\Psi_0 = -i\zeta\Psi_0 + Q\Psi_0 \tag{3.55}$$

where $Q(x) = \begin{pmatrix} 0 & -q(x) \\ r(x) & 0 \end{pmatrix}$.

On the left hand side, from the second term $\int_x^\infty K\Psi_0\,dy$, we obtain

$$\begin{aligned}
\begin{pmatrix} \partial_x & -q \\ r & -\partial_x \end{pmatrix}\int_x^\infty \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} e^{-i\zeta y} & 0 \\ 0 & e^{i\zeta y} \end{pmatrix} dy \;=\; & \\
\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} K_{11}(x,x) & K_{12}(x,x) \\ K_{21}(x,x) & K_{22}(x,x) \end{pmatrix}\begin{pmatrix} e^{-i\zeta x} & 0 \\ 0 & e^{i\zeta x} \end{pmatrix} & \\
-\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\int_x^\infty \begin{pmatrix} K_{11x} & K_{12x} \\ K_{21x} & K_{22x} \end{pmatrix}\begin{pmatrix} e^{-i\zeta y} & 0 \\ 0 & e^{i\zeta y} \end{pmatrix} dy & \\
+\begin{pmatrix} 0 & -q \\ r & 0 \end{pmatrix}\int_x^\infty \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} e^{-i\zeta y} & 0 \\ 0 & e^{i\zeta y} \end{pmatrix} dy &
\end{aligned} \tag{3.56}$$

then

$$\begin{pmatrix} \partial_x & -q \\ r & -\partial_x \end{pmatrix}\int_x^\infty K\Psi_0\,dy \;= \hat{I}K(x,x)\Psi_0 - \hat{I}\int_x^\infty K_x\Psi_0\,dy \;+ Q\int_x^\infty K\Psi_0\,dy \tag{3.57}$$

where $\hat{I} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & -q \\ r & 0 \end{pmatrix}$. We note that $\hat{I}^2 = I$ (identity matrix).

After combining all the terms, we obtain

$$
\begin{aligned}
-i\zeta\Big(\Psi_0+\int_x^\infty K\Psi_0\,dy\Big) =& -i\zeta\Psi_0+Q\Psi_0+\hat{I}K(x,x)\Psi_0 \\
&-\hat{I}\int_x^\infty K_x\Psi_0\,dy+Q\int_x^\infty K\Psi_0\,dy
\end{aligned} \tag{3.58}
$$

that is

$$
\begin{aligned}
-i\zeta\Big(\Psi_0+\int_x^\infty K\Psi_0\,dy\Big) =& -\Big(i\zeta I-\hat{I}K(x,x)\Big)\Psi_0-\hat{I}\int_x^\infty K_x\Psi_0\,dy \\
&+Q\Big(\Psi_0+\int_x^\infty K\Psi_0\,dy\Big).
\end{aligned} \tag{3.59}
$$

Here, $K(x,y)=0$ for $y<x$.

Now, we shall find the condition which $K(x,y)$ must satisfy. If $K(x,y)\to 0$ as $y\to\infty$, then use integration by parts once to find

$$
\begin{aligned}
-i\zeta\int_x^\infty K\Psi_0\,dy &= -\Big[K(x,y)\Psi_0(y;\zeta)\hat{I}\Big|_{y=x}^{\infty}-\int_x^\infty K_y\Psi_0\hat{I}\,dy\Big] \\
&= K(x,x)\Psi_0(x;\zeta)\hat{I}+\int_x^\infty K_y\Psi_0\hat{I}\,dy\ .
\end{aligned} \tag{3.60}
$$

Substitute it into (3.59), we obtain

$$
\begin{aligned}
&\hat{I}K(x,x)\Psi_0(x;\zeta)-K(x,x)\Psi_0(x;\zeta)\hat{I}+Q\Psi_0(x;\zeta) \\
&-\int_x^\infty\Big\{\hat{I}K_x(x,y)\Psi_0(y;\zeta)+K_y(x,y)\hat{I}\Psi_0(y;\zeta)-Q(x)K(x,y)\Psi(y;\zeta)\Big\}\,dy\ =0
\end{aligned} \tag{3.61}
$$

where $\Psi_0\hat{I}=\hat{I}\Psi_0$. The equation (3.61) is satisfied if the kernel $K(x,y)$ satisfies the followings:

$$
\hat{I}K_x(x,y)+K_y(x,y)\hat{I}-Q(x)K(x,y)=0,\quad y>x \tag{3.62}
$$

and

$$
\hat{I}K(x,x)-K(x,x)\hat{I}+Q(x)=0 \tag{3.63}
$$

and provided that $K(x,y)\to 0$ as $y\to\infty$. From (3.62), we obtain

$$
\begin{aligned}
&\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}K_{11x} & K_{12x}\\ K_{21x} & K_{22x}\end{pmatrix}+\begin{pmatrix}K_{11y} & K_{12y}\\ K_{21y} & K_{22y}\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} \\
&-\begin{pmatrix}0 & q(x)\\ r(x) & 0\end{pmatrix}\begin{pmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{pmatrix}=0
\end{aligned} \tag{3.64}
$$

i.e.,

$$\begin{pmatrix} -K_{11x} - K_{11y} + qK_{21} & -K_{12x} + K_{12y} + qK_{22} \\ K_{21x} - K_{21y} - rK_{11} & K_{22x} + K_{22y} - rK_{12} \end{pmatrix} = 0. \tag{3.65}$$

From (3.63), we obtain

$$\begin{aligned} &\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} K_{11}(x,x) & K_{12}(x,x) \\ K_{21}(x,x) & K_{22}(x,x) \end{pmatrix} - \begin{pmatrix} K_{11}(x,x) & K_{12}(x,x) \\ K_{21}(x,x) & K_{22}(x,x) \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ &\qquad + \begin{pmatrix} 0 & -q(x) \\ r(x) & 0 \end{pmatrix} = 0 \end{aligned} \tag{3.66}$$

that is

$$\begin{pmatrix} 0 & -2K_{12}(x,x) - q(x) \\ 2K_{21}(x,x) + r(x) & 0 \end{pmatrix} = 0. \tag{3.67}$$

We can find the potentials from the kernel $K(x,y)$ which are given by

$$\begin{aligned} q(x) &= -2K_{12}(x,x) \\ r(x) &= -2K_{21}(x,x) \end{aligned} \tag{3.68}$$

for fixed t.

3.2.2 Matrix Gelfand-Levitan-Marchenko equation

We have seen that if the potentials $q(x) = q(x,0)$ and $r(x) = r(x,0)$ are given, we can solve the scattering problem (direct problem). This time, we shall solve the inverse scattering problem which is to find the potentials for given scattering data. In direct problem, we found that $a, b, \hat{a}, \hat{b}$ are entire functions if r, q decay sufficiently rapidly. Hence, here, we also assume that scattering data satisfy (3.38) and (3.39). Let us rewrite (3.44) as

$$\begin{pmatrix} \psi_-/a(\zeta) & \hat{\psi}_-/\hat{a}(\zeta) \end{pmatrix} = \begin{pmatrix} \psi_+ & \hat{\psi}_+ \end{pmatrix} \begin{pmatrix} 1 & \hat{b}(\zeta)/\hat{a}(\zeta) \\ b(\zeta)/a(\zeta) & -1 \end{pmatrix}. \tag{3.69}$$

Let $\hat{\Psi}_- = \begin{pmatrix} \psi_-/a & \hat{\psi}_-/\hat{a} \end{pmatrix}$ then (3.69) becomes

$$\hat{\Psi}_- = \Psi_+ T \tag{3.70}$$

where $T = \begin{pmatrix} 1 & \hat{b}/\hat{a} \\ b/a & -1 \end{pmatrix}$ which is known as a transmission matrix. Insert (3.51) into (3.70) to yield

$$\hat{\Psi}_{-}(x;\zeta) = \Big(\Psi_0(x;\zeta) + \int_x^\infty K(x,y)\Psi_0(y;\zeta)\,dy\,\Big)T(\zeta). \tag{3.71}$$

In order to derive $K(x,y)$, we multiply both sides with $\frac{1}{2\pi}\Psi_0(z;-\zeta)$ for $z > x$ and then integrate along an appropriate contour C in the complex ζ-plane from $-\infty + i0$ to $+\infty + i0$.

$$\begin{aligned} \frac{1}{2\pi}\int_C \hat{\Psi}_{-}(x;\zeta)\Psi_0(z;-\zeta)\,d\zeta \;&= \frac{1}{2\pi}\int_C \Psi_0(x;\zeta)T(\zeta)\Psi_0(z;-\zeta)\,d\zeta \\ &+ \frac{1}{2\pi}\int_C \Big\{\int_x^\infty K(x,y)\Psi_0(y,\zeta)\,dy\,\Big\}T(\zeta)\Psi_0(z;-\zeta)\,d\zeta\;. \end{aligned} \tag{3.72}$$

We note that contour C represents two contours C_+ and C_-, on the upper half plane and on the lower half plane, respectively. We will use a suitable one for integral. On the right hand side, change the order of integration of the second term, then we obtain

$$\frac{1}{2\pi}\int_x^\infty K(x,y)\Big\{\int_C \Psi_0(y;\zeta)T(\zeta)\Psi_0(z;-\zeta)\,d\zeta\,\Big\}\,dy\;. \tag{3.73}$$

Insert this into (3.72) to get

$$\begin{aligned} \frac{1}{2\pi}\int_C \hat{\Psi}_{-}(x;\zeta)\Psi_0(z;-\zeta)\,d\zeta \;&= \frac{1}{2\pi}\int_C \Psi_0(x;\zeta)T(\zeta)\Psi_0(z;-\zeta)\,d\zeta \\ &+ \frac{1}{2\pi}\int_x^\infty K(x,y)\Big\{\int_C \Psi_0(y;\zeta)T(\zeta)\Psi_0(z;-\zeta)\,d\zeta\,\Big\}\,dy\;. \end{aligned} \tag{3.74}$$

Now, we find that

$$\begin{aligned} \Psi_0(x;\zeta)T(\zeta)\Psi_0(z;-\zeta) &= \begin{pmatrix} e^{-i\zeta x} & 0 \\ 0 & e^{i\zeta x} \end{pmatrix} \begin{pmatrix} 1 & \hat{b}/\hat{a} \\ b/a & -1 \end{pmatrix} \begin{pmatrix} e^{i\zeta z} & 0 \\ 0 & e^{-i\zeta z} \end{pmatrix} \\ &= \begin{pmatrix} e^{i\zeta(z-x)} & \frac{\hat{b}}{\hat{a}}e^{-i\zeta(z+x)} \\ \frac{b}{a}e^{i\zeta(z+x)} & -e^{-i\zeta(z-x)} \end{pmatrix}. \end{aligned} \tag{3.75}$$

We define the following functions

$$f(X) = \frac{1}{2\pi}\int_{C_+} \frac{b(\zeta)}{a(\zeta)}e^{i\zeta X}\,d\zeta \quad \text{and} \quad \hat{f}(X) = \frac{1}{2\pi}\int_{C_-} \frac{\hat{b}(\zeta)}{\hat{a}(\zeta)}e^{-i\zeta X}\,d\zeta \tag{3.76}$$

where $a(\zeta)$ has zeros on the upper half plane and $\hat{a}(\zeta)$ has zeros on the lower half plane. We also know that

$$\frac{1}{2\pi}\int_C e^{i\zeta(z-x)}\,d\zeta \;= \delta(z-x). \tag{3.77}$$

If we integrate (3.75) along the contour C, we obtain

$$\frac{1}{2\pi}\int_C \Psi_0(x;\zeta)T(\zeta)\Psi_0(z;-\zeta)\,d\zeta \;= \begin{pmatrix} \delta(z-x) & \hat{f}(x+z) \\ f(x+z) & -\delta(x-z) \end{pmatrix}, \tag{3.78}$$

therefore, (3.74) becomes

$$\begin{aligned}\frac{1}{2\pi}\int_C \hat{\Psi}_-(x;\zeta)\Psi_0(z;-\zeta)\,d\zeta \;&= \begin{pmatrix} \delta(z-x) & \hat{f}(x+z) \\ f(x+z) & -\delta(x-z) \end{pmatrix} \\ &+\int_x^\infty K(x,y)\begin{pmatrix} \delta(z-y) & \hat{f}(y+z) \\ f(y+z) & -\delta(y-z) \end{pmatrix} dy\;.\end{aligned} \tag{3.79}$$

For $z > x$, $\delta(z-x) = 0$, due to the definition of Dirac's delta function and letting $\hat{F}(X) = \begin{pmatrix} 0 & \hat{f}(X) \\ f(X) & 0 \end{pmatrix}$, (3.79) reduces to

$$\frac{1}{2\pi}\int_C \hat{\Psi}_-(x;\zeta)\Psi_0(z;-\zeta)\,d\zeta \;= \hat{F}(x+z) - K(x,z)\hat{I} + \int_x^\infty K(x,y)\hat{F}(y+z)\,dy\;. \tag{3.80}$$

Now, we shall show that left hand side is zero

$$\hat{\Psi}_-(x;\zeta)\Psi_0(z;-\zeta) = \begin{pmatrix} \psi_-/a & \hat{\psi}_-/\hat{a} \end{pmatrix}\begin{pmatrix} e^{i\zeta z} & 0 \\ 0 & e^{-i\zeta z} \end{pmatrix} = \begin{pmatrix} \frac{\psi_-}{a}e^{i\zeta z} & \frac{\hat{\psi_-}}{\hat{a}}e^{-i\zeta z} \end{pmatrix} \tag{3.81}$$

then

$$\begin{aligned}\frac{1}{2\pi}\int_C \hat{\Psi}_-(x;\zeta)\Psi_0(z;-\zeta)\,d\zeta \;&= \begin{pmatrix} \frac{1}{2\pi}\int_{C_+}\frac{\psi_-}{a}e^{i\zeta z}\,d\zeta & \frac{1}{2\pi}\int_{C_-}\frac{\hat{\psi}_-}{\hat{a}}e^{-i\zeta z}\,d\zeta \end{pmatrix} \\ &= \begin{pmatrix} I_1 & I_2 \end{pmatrix}.\end{aligned} \tag{3.82}$$

Since $e^{i\zeta z}\psi_-$ is an analytic function on the upper half plane for $z > x$, then $I_1 = 0$ and, since $e^{-i\zeta z}\hat{\psi}_-$ is an analytic function on the lower half plane for $z > x$, then $I_2 = 0$. Hence, (3.80) reduces to

$$\hat{F}(x+z) - K(x,z)\hat{I} + \int_x^\infty K(x,y)\hat{F}(y+z)\,dy \;= 0 \tag{3.83}$$

which is known as the Marchenko equation for the kernel $K(x,z)$. For convenience, we can write (3.83) again. We multiply (3.83) by $-\hat{I}$ (note that $\hat{I}^2 = I$)

$$K(x,z) - \hat{F}(x+z)\hat{I} - \int_x^{\infty} K(x,y)\hat{F}(y+z)\hat{I}\,dy \ = 0 \tag{3.84}$$

and let $-\hat{F}(X)\hat{I} = F$.

Hence, the matrix GLM equation for $K(x,z)$ is

$$K(x,z) + F(x+z) + \int_x^{\infty} K(x,y)F(y+z)\,dy \ = 0 \tag{3.85}$$

where

$$F(X) = \begin{pmatrix} 0 & -\hat{f}(X) \\ f(X) & 0 \end{pmatrix} \tag{3.86}$$

and the functions $f(X)$ and $\hat{f}(X)$ are given.

If $a(\zeta)$ has N simple zeros at $\zeta = \zeta_n$, $(n = 1, 2, ..., N)$, on the upper half plane then, the function f is

$$f(X) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{b(\zeta)}{a(\zeta)} e^{i\zeta X}\,d\zeta \ - i\sum_{n=1}^{N} c_n e^{i\zeta_n X} \tag{3.87}$$

where $c_n = \frac{b(\zeta)}{da(\zeta)/d\zeta}$ at $\zeta = \zeta_n$, and c_n's are the normalization constants and $\frac{b(\zeta)}{a(\zeta)}$ is the reflection coefficient.

If $\hat{a}(\zeta)$ has M simple zeros at $\zeta = \zeta_m$, $(m = 1, 2, ..., M)$, on the lower half-plane then the function $\hat{f}$ is

$$\hat{f}(X) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\hat{b}(\zeta)}{\hat{a}(\zeta)} e^{-i\zeta X}\,d\zeta \ + i\sum_{m=1}^{M} \hat{c}_m e^{-i\zeta_m X} \tag{3.88}$$

where $\hat{c}_m = \frac{\hat{b}(\zeta)}{d\hat{a}(\zeta)/d\zeta}$ at $\zeta = \zeta_m$, and $\hat{c}_m$'s are the normalization constants and $\frac{b(\zeta)}{a(\zeta)}$ is the reflection coefficient.

Note that we obtain $f(X)$ and $\hat{f}(X)$ by integrating (3.76) along the appropriate contour.

If we solve the equation (3.85) for the kernel $K(x,z)$, then we can find the potentials

$$q(x) = -2K_{12}(x,x) \quad \text{and} \quad r(x) = -2K_{21}(x,x) \tag{3.89}$$

at a fixed time [36].

3.2.3 Time evolution of the scattering data

Until now, the potentials q, r and the solution of the eigenvalue problem, ψ, were dependent of x for fixed t (say $t = 0$). We shall, now, find the potentials for all t. Beforehand, we need to find the scattering data for all t. Let $\psi = \psi(x,t)$ and it satisfies

$$\psi_x = \begin{pmatrix} -i\zeta & q(x,t) \\ r(x,t) & i\zeta \end{pmatrix} \psi \quad \text{and} \quad \psi_t = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \psi \tag{3.90}$$

where A, B, C satisfy the following

$$\begin{aligned} A_x &= qC - rB \\ B_x + 2i\zeta B &= q_t - 2Aq \\ C_x - 2i\zeta C &= r_t + 2Ar \end{aligned} \tag{3.91}$$

It is required that $A \to \alpha(\zeta)$, $B \to 0$ and $C \to 0$ as $|x| \to \infty$ since $q(x,t) \to 0$ and $r(x,t) \to 0$ as $|x| \to \infty$. The time evolution equation becomes

$$\psi_t \sim \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \psi \quad \text{as} \quad |x| \to \infty \tag{3.92}$$

and the asymptotic behaviour of ψ is

$$\psi \sim \begin{pmatrix} e^{\alpha t} & 0 \\ 0 & e^{-\alpha t} \end{pmatrix} f(x) \quad \text{as} \quad |x| \to \infty \tag{3.93}$$

where $f(x)$ is an arbitrary column vector with two elements. Let us write Φ_+ defined as Ψ_+, and Φ_- defined as Ψ_-

$$\Phi_+ \sim \begin{pmatrix} e^{-i\zeta x} & 0 \\ 0 & e^{i\zeta x} \end{pmatrix} \quad \text{as} \quad x \to +\infty \tag{3.94}$$

and

$$\Phi_- \sim \begin{pmatrix} e^{-i\zeta x} & 0 \\ 0 & -e^{i\zeta x} \end{pmatrix} \quad \text{as} \quad x \to -\infty \tag{3.95}$$

and, they are connected via the scattering matrix S

$$\Phi_- = \Phi_+ S \tag{3.96}$$

as $\Psi_- = \Psi_+ S$ where $S = \begin{pmatrix} a & \hat{b} \\ b & -\hat{a} \end{pmatrix}$. Hence we can write two time-dependent eigenfunctions as

$$\psi_1 = \Phi_+ E \quad \text{and} \quad \psi_2 = \Phi_- E \tag{3.97}$$

where $E = \begin{pmatrix} e^{\alpha t} & 0 \\ 0 & e^{-\alpha t} \end{pmatrix}$. Insert ψ_2 into the time evolution equation for ψ in (3.90), we obtain

$$\Phi_{-t} E + \Phi_- E_t = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \Phi_- E \tag{3.98}$$

We find that

$$\begin{aligned} E_t = \begin{pmatrix} \alpha e^{\alpha t} & 0 \\ 0 & -\alpha e^{-\alpha t} \end{pmatrix} &= -\alpha \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\alpha t} & 0 \\ 0 & e^{-\alpha t} \end{pmatrix} \\ &= -\alpha \hat{I} E \end{aligned} \tag{3.99}$$

where $\hat{I} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Insert this into (3.98) to yield

$$\Phi_{-t} E + \Phi_-(-\alpha \hat{I} E) = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \Phi_- E \tag{3.100}$$

or

$$\Phi_{-t} E = \left\{ \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \Phi_- + \alpha \Phi_- \hat{I} \right\} E \tag{3.101}$$

then we obtain

$$\Phi_{-t} = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \Phi_- + \alpha \Phi_- \hat{I}. \tag{3.102}$$

Now, take t-derivative of (3.96) to yield

$$\Phi_{-t} = \Phi_{+t} S + \Phi_+ S_t \tag{3.103}$$

and substitute (3.102), we get

$$\begin{pmatrix} A & B \\ C & -A \end{pmatrix} \Phi_- + \alpha \Phi_- \hat{I} = \Phi_{+t} S + \Phi_+ S_t. \tag{3.104}$$

Use (3.96) to change Φ_- to Φ_+

$$\begin{pmatrix} A & B \\ C & -A \end{pmatrix} \Phi_+ S + \alpha \Phi_+ S \hat{I} = \Phi_{+t} S + \Phi_+ S_t \tag{3.105}$$

and let $x \to +\infty$, then we obtain

$$\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} e^{-i\zeta x} & 0 \\ 0 & e^{i\zeta x} \end{pmatrix} \begin{pmatrix} a & \hat{b} \\ b & -\hat{a} \end{pmatrix} + \alpha \begin{pmatrix} e^{-i\zeta x} & 0 \\ 0 & e^{i\zeta x} \end{pmatrix} \begin{pmatrix} a & \hat{b} \\ b & -\hat{a} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} e^{-i\zeta x} & 0 \\ 0 & e^{i\zeta x} \end{pmatrix} \begin{pmatrix} a_t & \hat{b}_t \\ b_t & -\hat{a}_t \end{pmatrix} \tag{3.106}$$

where $\Phi_{+t} \to 0$ as $x \to +\infty$. Since $\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$ and $\begin{pmatrix} e^{-i\zeta x} & 0 \\ 0 & e^{i\zeta x} \end{pmatrix}$ commute, (3.106) reduces to

$$\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} a & \hat{b} \\ b & -\hat{a} \end{pmatrix} + \begin{pmatrix} a & \hat{b} \\ b & -\hat{a} \end{pmatrix} \begin{pmatrix} -\alpha & 0 \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} a_t & \hat{b}_t \\ b_t & -\hat{a}_t \end{pmatrix} \tag{3.107}$$

or

$$\begin{pmatrix} 0 & 2\alpha\hat{b} \\ -2\alpha b & 0 \end{pmatrix} = \begin{pmatrix} a_t & \hat{b}_t \\ b_t & -\hat{a}_t \end{pmatrix} \tag{3.108}$$

then the scattering data $a(\zeta, t)$, $b(\zeta, t)$, $\hat{a}(\zeta, t)$, $\hat{b}(\zeta, t)$ become

$$\begin{aligned} &a(\zeta, t) = a_0(\zeta), \quad \hat{a}(\zeta, t) = \hat{a}_0(\zeta) \\ &b(\zeta, t) = b_0(\zeta) e^{-2\alpha t}, \quad \hat{b}(\zeta, t) = \hat{b}_0(\zeta) e^{2\alpha t}, \quad \forall t > 0 \end{aligned} \tag{3.109}$$

where $a(\zeta, 0) = a_0(\zeta)$, $\hat{a}(\zeta, 0) = \hat{a}_0(\zeta)$, $b(\zeta, 0) = b_0(\zeta)$, and $\hat{b}(\zeta, 0) = \hat{b}_0(\zeta)$ are scattering data at $t = 0$.

Now, we have all the information to solve the initial value problem for

$$\psi_x = \begin{pmatrix} -i\zeta & q(x,t) \\ r(x,t) & i\zeta \end{pmatrix} \psi \tag{3.110}$$

on the interval $-\infty < x < \infty$ with the initial values $q(x,0)$ and $r(x,0)$. First, we solve the direct problem and obtain the scattering data at $t = 0$. Then we solve the Marchenko equation

$$K(x,z;t) + F(x+z;t) + \int_x^\infty K(x,y;t)F(y+z;t)\,dy \ = 0 \tag{3.111}$$

for $K(x,z;t)$ where $F(X) = \begin{pmatrix} 0 & -\hat{f}(X;t) \\ f(X;t) & 0 \end{pmatrix}$. The function f is given by

$$f(X;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{b_0(\zeta)}{a_0(\zeta)} e^{i\zeta X - 2\alpha t}\,d\zeta \ - i\sum_{n=1}^{N} c_n(0)e^{i\zeta_n X - 2\alpha t} \tag{3.112}$$

where $c_n(0) = \frac{b_0(\zeta)}{da_0(\zeta)/d\zeta}$ at $\zeta = \zeta_n$, and the function $\hat{f}$ is given by

$$\hat{f}(X;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\hat{b}_0(\zeta)}{\hat{a}_0(\zeta)} e^{-i\zeta X + 2\alpha t}\,d\zeta \ + i\sum_{m=1}^{M} \hat{c}_m(0)e^{-i\zeta_m X + 2\alpha t} \tag{3.113}$$

where $\hat{c}_m(0) = \frac{\hat{b}_0(\zeta)}{d\hat{a}_0(\zeta)/d\zeta}$ at $\zeta = \zeta_m$. Here, α is determined from $A \to \alpha(\zeta)$ as $|x| \to +\infty$.

Hence the potentials are

$$q(x,t) = -2K_{12}(x,x;t) \quad \text{and} \quad r(x,t) = -2K_{21}.(x,x;t). \tag{3.114}$$

The class of evolution equations q and r is described by a relation between $q(x,t)$ and $r(x,t)$ e.g., $r = -q$ or $r = q^*$ [36].

3.2.4 An example : $r(x,t) = -q(x,t)$, $q(x,0) = \lambda \operatorname{sech} \lambda x$

We choose $r(x,t) = -q(x,t)$ to which the modified KdV and the sine-Gordon equation correspond. We choose the initial value as

$$q(x,0) = \lambda \operatorname{sech} \lambda x \tag{3.115}$$

where λ is a positive constant. We, first, find the scattering data for fixed t ($t = 0$). Therefore we need to solve

$$\begin{pmatrix} \partial_x & -q \\ -q & -\partial_x \end{pmatrix} \psi = -i\zeta\psi. \tag{3.116}$$

For convenience, write the solution as either $\psi = e^{i\zeta x}\phi$ or $\psi = e^{-i\zeta x}\phi$ where ϕ is a column vector with two elements.

If $E = e^{i\zeta x}$, then insert $\psi = e^{i\zeta x}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$ into (3.116) and (3.116) reduces to

$$\begin{pmatrix}\partial_x & -q\\ -q & -\partial_x\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = -2i\zeta\begin{pmatrix}\phi_1\\ 0\end{pmatrix} \tag{3.117}$$

and, if $E = e^{-i\zeta x}$, then insert $\psi = e^{-i\zeta x}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$ into (3.116) and (3.116) reduces to

$$\begin{pmatrix}\partial_x & -q\\ -q & -\partial_x\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = -2i\zeta\begin{pmatrix}0\\ \phi_2\end{pmatrix}. \tag{3.118}$$

When $E = e^{i\zeta x}$, we suppose that $\phi_1 = \mu \operatorname{sech}\lambda x$ where μ is an arbitrary constant. From (3.117), we have

$$-q(x)\phi_1(x) - \phi_{2x}(x) = 0 \tag{3.119}$$

where $q(x)$ and $\phi_1(x)$ are given, then we get

$$\phi_2(x) = -\mu\tanh\lambda x + \eta \tag{3.120}$$

where η is an integration constant. Again, from (3.117), we have

$$\phi_{1x} - q\phi_2 = -2i\zeta\phi_1 \tag{3.121}$$

then $\eta = 2i\zeta\mu/\lambda$. Therefore, by using

$$\phi_1 = \mu\operatorname{sech}\lambda x \quad \text{and} \quad \phi_2 = -\mu\tanh\lambda x + 2i\zeta\mu/\lambda \tag{3.122}$$

we find the solution

$$\psi = e^{i\zeta x}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = e^{i\zeta x}\begin{pmatrix}\mu\operatorname{sech} x\\ \frac{2i\zeta}{\lambda}\mu - \mu\tanh\lambda x\end{pmatrix}. \tag{3.123}$$

We must choose an appropriate constant μ so that the asymptotic behaviour of ψ must be

$$\begin{aligned} &\text{either} \quad \psi \sim e^{i\zeta x}\begin{pmatrix}0\\ 1\end{pmatrix} \quad \text{as} \quad x \to +\infty \\ &\text{or} \quad \psi \sim e^{i\zeta x}\begin{pmatrix}0\\ -1\end{pmatrix} \quad \text{as} \quad x \to -\infty \end{aligned} \tag{3.124}$$

(see the eigenfunctions in (3.40) and (3.41)). Therefore, we obtain two solutions

$$\hat{\psi}_+ = \frac{e^{i\zeta x}}{2i\zeta - \lambda}\begin{pmatrix}\lambda \operatorname{sech}\lambda x \\ 2i\zeta - \lambda\tanh\lambda x\end{pmatrix} \sim e^{i\zeta x}\begin{pmatrix}0\\1\end{pmatrix} \quad \text{as} \quad x \to +\infty \tag{3.125}$$

and

$$\hat{\psi}_- = -\frac{e^{i\zeta x}}{2i\zeta + \lambda}\begin{pmatrix}\lambda \operatorname{sech}\lambda x \\ 2i\zeta - \lambda\tanh\lambda x\end{pmatrix} \sim e^{i\zeta x}\begin{pmatrix}0\\-1\end{pmatrix} \quad \text{as} \quad x \to -\infty. \tag{3.126}$$

Now, for the case $E = e^{-i\zeta x}$, we suppose that $\phi_2 = \mu \operatorname{sech}\lambda x$ where μ is an arbitrary constant. From (3.118), by following the previous steps, we find

$$\phi_1 = \mu\tanh\lambda x + 2i\zeta\mu/\lambda \quad \text{as} \quad \phi_2 = \mu\operatorname{sech}\lambda x \tag{3.127}$$

and we write the solution as

$$\psi = e^{-i\zeta x}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = e^{-i\zeta x}\begin{pmatrix}\frac{2i\zeta}{\lambda}\mu + \mu\tanh\lambda x \\ \mu\operatorname{sech}\lambda x\end{pmatrix}. \tag{3.128}$$

The asymptotic behaviour of ψ is

$$\begin{aligned} &\text{either} \quad \psi \sim e^{-i\zeta x}\begin{pmatrix}1\\0\end{pmatrix} \quad \text{as} \quad x \to +\infty \\ &\text{or} \quad \psi \sim e^{-i\zeta x}\begin{pmatrix}1\\0\end{pmatrix} \quad \text{as} \quad x \to -\infty. \end{aligned} \tag{3.129}$$

We choose an appropriate constant μ and find two solutions

$$\psi_+ = \frac{e^{-i\zeta x}}{2i\zeta + \lambda}\begin{pmatrix}2i\zeta + \lambda\tanh\lambda x \\ \lambda\operatorname{sech}\lambda x\end{pmatrix} \sim e^{-i\zeta x}\begin{pmatrix}1\\0\end{pmatrix} \quad \text{as} \quad x \to +\infty \tag{3.130}$$

and

$$\psi_- = \frac{e^{-i\zeta x}}{2i\zeta - \lambda}\begin{pmatrix}2i\zeta + \lambda\tanh\lambda x \\ \lambda\operatorname{sech}\lambda x\end{pmatrix} \sim e^{-i\zeta x}\begin{pmatrix}1\\0\end{pmatrix} \quad \text{as} \quad x \to -\infty. \tag{3.131}$$

We already know that these four solutions are related via the scattering matrix $S(\zeta)$. Use $\psi_- = a\psi_+ + b\hat{\psi}_+$, then we obtain

$$\begin{aligned} \frac{e^{-i\zeta x}}{2i\zeta - \lambda}\begin{pmatrix}2i\zeta + \lambda\tanh\lambda x \\ \lambda\operatorname{sech}\lambda x\end{pmatrix} &= a\frac{e^{-i\zeta x}}{2i\zeta + \lambda}\begin{pmatrix}2i\zeta + \lambda\tanh\lambda x \\ \lambda\operatorname{sech}\lambda x\end{pmatrix} \\ &\quad + b\frac{e^{i\zeta x}}{2i\zeta - \lambda}\begin{pmatrix}\lambda\operatorname{sech}\lambda x \\ 2i\zeta - \lambda\tanh\lambda x\end{pmatrix}. \end{aligned} \tag{3.132}$$

If $a \neq 0$ then $b = 0$ and we find

$$a(\zeta) = \frac{2i\zeta + \lambda}{2i\zeta - \lambda}. \tag{3.133}$$

If $a = 0$ (when $\zeta = \frac{1}{2}i\lambda$), let $b\left(\frac{1}{2}i\lambda\right) = b_1$, then it yields

$$e^{\frac{1}{2}x\lambda} \begin{pmatrix} -\lambda + \lambda \tanh \lambda x \\ \lambda \operatorname{sech} \lambda x \end{pmatrix} = b_1 e^{-\frac{1}{2}x\lambda} \begin{pmatrix} \lambda \operatorname{sech} \lambda x \\ -\lambda - \lambda \tanh \lambda x \end{pmatrix} \tag{3.134}$$

which is satisfied if $b_1 = -1$. Therefore, we have

$$b(\zeta) = \begin{cases} -1, & \zeta = \frac{1}{2}i\lambda \\ 0, & \text{otherwise.} \end{cases} \tag{3.135}$$

Now, use $\hat{\psi}_- = \hat{b}\psi_+ - \hat{a}\hat{\psi}_+$, then we obtain

$$\begin{aligned} -\frac{e^{i\zeta x}}{2i\zeta + \lambda} \begin{pmatrix} \lambda \operatorname{sech} \lambda x \\ 2i\zeta - \lambda \tanh \lambda x \end{pmatrix} &= \hat{b} \frac{e^{-i\zeta x}}{2i\zeta + \lambda} \begin{pmatrix} 2i\zeta + \lambda \tanh \lambda x \\ \lambda \operatorname{sech} \lambda x \end{pmatrix} \\ &\quad - \hat{a} \frac{e^{i\zeta x}}{2i\zeta - \lambda} \begin{pmatrix} \lambda \operatorname{sech} \lambda x \\ 2i\zeta - \lambda \tanh \lambda x \end{pmatrix}. \end{aligned} \tag{3.136}$$

If $\hat{a} \neq 0$ then $\hat{b} = 0$ and we find

$$\hat{a} = \frac{2i\zeta - \lambda}{2i\zeta + \lambda}. \tag{3.137}$$

We observed that $\hat{a}(\zeta) = a^*(\zeta)$ (conjugate of $a(\zeta)$).

If $\hat{a}(\zeta) = 0$ (when $\zeta = -\frac{1}{2}i\lambda$), let $\hat{b}\left(-\frac{1}{2}i\lambda\right) = b_2$, then it yields

$$-e^{\frac{1}{2}\lambda x} \begin{pmatrix} \lambda \operatorname{sech} \lambda x \\ \lambda - \lambda \tanh \lambda x \end{pmatrix} = b_2 e^{-\frac{1}{2}\lambda x} \begin{pmatrix} \lambda + \lambda \tanh \lambda x \\ \lambda \operatorname{sech} \lambda x \end{pmatrix} \tag{3.138}$$

which is satisfied if $b_2 = -1$. Therefore, we have

$$\hat{b}(\zeta) = \begin{cases} -1, & \zeta = -\frac{1}{2}i\lambda \\ 0, & \text{otherwise.} \end{cases} \tag{3.139}$$

We obtained all the scattering data [37]. Now we can find $q(x,t)$ at any time by solving the inverse problem. We shall use (3.111), (3.112), (3.113), (3.114).

Let us write the scattering data at $t=0$, again

$$a(\zeta)=\frac{2i\zeta+\lambda}{2i\zeta-\lambda}\quad\text{and}\quad \hat{a}(\zeta)=a^*(\zeta)=\frac{2i\zeta-\lambda}{2i\zeta+\lambda}$$
$$b(\zeta)=\begin{cases}0, & \zeta\neq\frac{1}{2}i\lambda\\ -1, & \zeta=\frac{1}{2}i\lambda\end{cases}\quad\text{and}\quad \hat{b}(\zeta)=\begin{cases}0, & \zeta\neq-\frac{1}{2}i\lambda\\ -1, & \zeta=-\frac{1}{2}i\lambda\end{cases} \tag{3.140}$$

for $\lambda>0$. By using $c_1(0)=\frac{b_0(\zeta)}{da_0(\zeta)/d\zeta}$ at $\zeta=\frac{1}{2}i\lambda$ and $\hat{c}_1(0)=\frac{\hat{b}_0(\zeta)}{d\hat{a}_0(\zeta)/d\zeta}$ at $\zeta=-\frac{1}{2}i\lambda$, we find

$$c_1(0)=-i\lambda\quad\text{and}\quad \hat{c}_1(0)=i\lambda. \tag{3.141}$$

Since $b(\zeta)$ and $\hat{b}(\zeta)$ are equal to zero except for the discrete eigenvalues, then we find

$$\begin{aligned}f(X;t)&=-\lambda e^{-\frac{1}{2}\lambda x}e^{-2\alpha(\frac{1}{2}i\lambda)t},\\ \hat{f}(X;t)&=-\lambda e^{-\frac{1}{2}\lambda x}e^{2\alpha(-\frac{1}{2}i\lambda)t}.\end{aligned} \tag{3.142}$$

Here, α is determined from

$$A\to\alpha(\zeta)\quad\text{as}\quad |x|\to+\infty. \tag{3.143}$$

Hence, we need to know which expansion procedure we used for A,B,C. Let us choose the expansion in inverse power of ζ as

$$A(x,t,\zeta)=\frac{a(x,t)}{\zeta},\quad B(x,t,\zeta)=\frac{b(x,t)}{\zeta},\quad C(x,t,\zeta)=\frac{c(x,t)}{\zeta}. \tag{3.144}$$

We have already studied this case. Since $r=-q$, it will lead to the sine-Gordon equation and

$$A(x,t,\zeta)=\frac{i}{4\zeta}\cos u(x,t). \tag{3.145}$$

Since $u(x,t)\to 0$ as $|x|\to+\infty$, then

$$A(x,t,\zeta)\to\alpha(\zeta)=\frac{i}{4\zeta} \tag{3.146}$$

yielding

$$f(X;t)=\hat{f}(X;t)=-\lambda e^{-\frac{1}{2}\lambda x}e^{-\frac{1}{\lambda}t}. \tag{3.147}$$

Then $F=\begin{pmatrix}0 & -\hat{f}\\ f & 0\end{pmatrix}=\begin{pmatrix}0 & -f\\ f & 0\end{pmatrix}$ is antisymmetric. We can write the Marchenko equation for $K(x,z;t)$ in (3.111) as

$$\begin{pmatrix}K_{11}+\int_x^\infty K_{12}f\,dy & K_{12}-f-\int_x^\infty K_{11}f\,dy\\ K_{21}+f+\int_x^\infty K_{22}f\,dy & K_{22}-\int_x^\infty K_{21}f\,dy\end{pmatrix}=0. \tag{3.148}$$

Here, $K_{11} = K_{22}$ and $K_{21} = -K_{12}$, the latter one leads to $r = -q$, then we have two equations

$$\begin{aligned} K_{11} + \int_x^\infty K_{12} f \, dy \ &= 0, \\ K_{12} - f - \int_x^\infty K_{11} f \, dy \ &= 0. \end{aligned} \tag{3.149}$$

Substitute the function to yield

$$K_{11}(x,z;t) - \lambda e^{-\frac{1}{\lambda}t} e^{-\frac{1}{2}\lambda z} \int_x^\infty K_{12}(x,y;t) e^{-\frac{1}{2}\lambda y} \, dy \ = 0, \tag{3.150}$$

$$K_{12}(x,z;t) + \lambda e^{-\frac{1}{2}\lambda(x+z)} e^{-\frac{1}{\lambda}t} + \lambda e^{-\frac{1}{\lambda}t} e^{-\frac{1}{2}\lambda z} \int_x^\infty K_{11}(x,y;t) e^{-\frac{1}{2}\lambda y} \, dy \ = 0. \tag{3.151}$$

The equation (3.150) implies that

$$K_{11}(x,z;t) = L(x,t) e^{-\frac{1}{2}\lambda z} \tag{3.152}$$

and insert it into (3.150), then we get

$$L(x,t) = \lambda e^{-\frac{1}{\lambda}t} \int_x^\infty K_{12}(x,y;t) e^{-\frac{1}{2}\lambda y} \, dy \ . \tag{3.153}$$

Now insert (3.153) into (3.151) to yield

$$\begin{aligned} K_{12}(x,z;t) &= -\lambda e^{-\frac{1}{2}\lambda(x+z)} e^{-\frac{1}{\lambda}t} - \lambda e^{-\frac{1}{\lambda}t} e^{-\frac{1}{2}\lambda z} L(x,t) \int_x^\infty e^{-\lambda y} \, dy \\ &= \Big\{ -\lambda e^{-\frac{1}{2}\lambda x} - L(x,t) e^{-\lambda x} \Big\} e^{-\frac{1}{2}\lambda z} e^{-\frac{1}{\lambda}t}. \end{aligned} \tag{3.154}$$

The equality (3.153) can be rewritten as

$$\begin{aligned} L(x,t) &= \lambda e^{-\frac{1}{\lambda}t} e^{-\frac{1}{\lambda}t} \Big\{ -\lambda e^{-\frac{1}{2}\lambda x} - L(x,t) e^{-\lambda x} \Big\} \int_x^\infty e^{-\lambda y} \, dy \\ &= e^{-\frac{2}{\lambda}t} e^{-\lambda x} \Big\{ -\lambda e^{-\frac{1}{2}\lambda x} - L(x,t) e^{-\lambda x} \Big\}. \end{aligned} \tag{3.155}$$

Then we find that

$$L(x,t) = \frac{-\lambda e^{-\frac{3}{2}\lambda x} e^{-\frac{2}{\lambda}t}}{1 + e^{-2\lambda x} e^{-\frac{2}{\lambda}t}}. \tag{3.156}$$

Since we know $L(x,t)$, we can find $K_{12}(x,z;t)$ from (3.154). Insert $L(x,t)$ into (3.154), then we get

$$K_{12}(x,z;t) = \frac{-\lambda e^{-\frac{1}{2}\lambda x} e^{-\frac{1}{2}\lambda z} e^{-\frac{1}{\lambda}t}}{1 + e^{-2\lambda x} e^{-\frac{2}{\lambda}t}} \tag{3.157}$$

and, then

$$\begin{aligned} K_{12}(x,x;t) &= \frac{-\lambda e^{-\lambda x} e^{-\frac{1}{\lambda}t}}{1+e^{-2\lambda x} e^{-\frac{2}{\lambda}t}} \\ &= \frac{-\lambda}{e^{\lambda x} e^{\frac{1}{\lambda}t} + e^{-\lambda x} e^{-\frac{1}{\lambda}t}} \\ &= -\frac{\lambda}{2} \operatorname{sech}\Big(\lambda x + \frac{1}{\lambda}t\Big) \end{aligned} \tag{3.158}$$

then the potential is

$$q(x,t) = -2K_{12}(x,x;t) = \lambda \operatorname{sech}\Big(\lambda x + \frac{1}{\lambda}t\Big). \tag{3.159}$$

For the sine-Gordon equation

$$q(x,t) = -\frac{-u_x(x,t)}{2} \tag{3.160}$$

then we have

$$u_x(x,t) = -2\lambda \operatorname{sech}\Big(\lambda x + \frac{1}{\lambda}t\Big). \tag{3.161}$$

We integrate it with respect to x, then we obtain

$$u(x,t) = -4\arctan\Bigg\{\exp\Big(\lambda x + \frac{1}{\lambda}t\Big)\Bigg\} \tag{3.162}$$

which is the solution of the sine-Gordon equation

$$u_{xt} = \sin u(x,t). \tag{3.163}$$

Lastly, Ablowitz and Segur proved that the matrix $\begin{pmatrix} A & B \\ C & -A \end{pmatrix}$ exists only if $q(x,t)$ and $r(x,t)$ satisfy the following via orthogonality condition

$$\Bigg\{-\hat{I}\frac{\partial}{\partial t} + 2\alpha(i\mathscr{L})\Bigg\}\begin{pmatrix} r \\ q \end{pmatrix} = 0 \tag{3.164}$$

where $A \to \alpha(\zeta)$ as $|x| \to \infty$, and the operator $\mathscr{L}$ is given by

$$\mathscr{L} = \frac{1}{2}\hat{I}\frac{\partial}{\partial x} - \begin{pmatrix} r & 0 \\ 0 & q \end{pmatrix} \int_x^\infty dx \begin{pmatrix} q & -r \\ q & -r \end{pmatrix} \tag{3.165}$$

and $\hat{I} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ [36].

Chapter 4

Bäcklund Trasformations

Bäcklund transformations were introduced by Bäcklund around 1875, and the first known evolution equation that owns these transformations was the sine-Gordon equation [13].

Definition 4.0.1. Suppose that we have two partial differential equations for the functions u and v depending on the variables x and t. These equations are given as

$$P[u] = P(u, u_x, u_t, ...) = 0 \quad \text{and} \quad Q[v] = Q(v, v_x, v_t, ...) = 0 \tag{4.1}$$

where P and Q are, in general, nonlinear operators. Let $R_i = 0$ be a set of relations,

$$R_i(u, v, u_x, v_x, u_t, v_t, ...; x, t) = 0, \quad i = 1, 2 \tag{4.2}$$

between the two functions u and v. Then $R_i = 0$ are called Bäcklund transformations (BT) between P and Q, if the followings hold:
(i) R_i's are integrable for u if and only if $Q[v] = 0$
(ii) R_i's are integrable for v if and only if $P[u] = 0$
(iii) When $Q[v] = 0$, then u is a solution of $P[u] = 0$
(iv) When $P[u] = 0$, then v is a solution of $Q[v] = 0$.
(Note that $u_x = f(x, t)$ and $u_t = g(x, t)$ are integrable for u, if and only if they are compatible i.e., $u_{xt} = u_{tx}$.)

If these two operators are equal i.e., $P = Q$, then the pair of the relations $R_i = 0$ are called auto-Bäcklund transformations and they are unique. Let the order of $P[u]$ is m, and the order of $Q[v]$ is n, then the order of R_i for $i = 1, 2$ must be smaller than the minimum of m and n; otherwise, there is no use for Bäcklund transformations.

4.1 Examples of Bäcklund transformations for some partial differential equations

Example 4.1.1. Let u and v satisfy Laplace's equation

$$P[u] = u_{xx} + u_{yy} = 0, \quad Q[v] = v_{xx} + v_{yy} = 0. \tag{4.3}$$

Consider the pair of first order equations which are known as the Cauchy-Riemann equations

$$R_1(u_x, v_y) = u_x - v_y = 0, \quad R_2(u_y, v_x) = u_y + v_x = 0 \tag{4.4}$$

and cross-differentiate them to verify the integrability of R_1 and R_2 then, we yield

$$u_{xy} - v_{yy} = 0 \quad \text{and} \quad u_{yx} + v_{xx} = 0. \tag{4.5}$$

By subtracting, we obtain $Q[v] = 0$. Again, cross-differentiate in another way to yield

$$u_{xx} - v_{yx} = 0 \quad \text{and} \quad u_{yy} + v_{xy} = 0. \tag{4.6}$$

By adding, we obtain $P[u] = 0$. The pair $R_1 = 0$ and $R_2 = 0$ are the auto-Bäcklund transformations for Laplace's equation.

By using the auto-Bäcklund transformations, we can produce a new solution from the know one of the same system. For illustration, if $v(x, y) = xy$ is a solution of Laplace's equation, the BT are reduced to

$$u_x = x, \quad u_y = -y \tag{4.7}$$

and we obtain another solution to Laplace's equation

$$u(x, y) = \frac{1}{2}(x^2 - y^2). \tag{4.8}$$

Example 4.1.2. Liouville's equation and the wave equation are given by

$$P[u] = u_{xt} - e^u = 0, \quad Q[v] = v_{xt} = 0 \tag{4.9}$$

respectively. Consider the pair of first-order equations

$$R_1 = u_x + v_x - \sqrt{2}e^{(u-v)/2} = 0 \quad \text{and} \quad R_2 = u_t - v_t - \sqrt{2}e^{(u+v)/2} = 0 \tag{4.10}$$

and cross-differentiate them to verify the integrability of R_1 and R_2 then, we find

$$u_{xt} + v_{xt} = \frac{1}{\sqrt{2}}e^{(u-v)/2}(u_t - v_t) = e^u \tag{4.11}$$

and

$$u_{tx} - v_{tx} = \frac{1}{\sqrt{2}}e^{(u+v)/2}(u_x + v_x) = e^u. \tag{4.12}$$

By adding, we get Liouville's equation and by subtracting, we get the wave equation. Thus, the given pair of relations $R_1 = 0$ and $R_2 = 0$ are Bäcklund transformations between Liouville's equation and the wave equation. These Bäcklund transformations can be derived from Clairin's method which was introduced by Clairin around 1900 [38].

By using BT, we can find the most general solution to Liouville's equation from the known general solution of the wave equation. For illustration, the most general solution of the wave equation is

$$v(x,t) = f(x) + g(t) \tag{4.13}$$

where f and g are arbitrary differentiable functions. By using BT, we obtain

$$u_x + f_x = \sqrt{2}e^{u/2}e^{-(f+g)/2} \tag{4.14}$$

and

$$u_x - g_t = \sqrt{2}e^{u/2}e^{(f+g)/2}. \tag{4.15}$$

Let us suppose that the function u is given as

$$u = -2\ln(\rho) - f(x) + g(t) \tag{4.16}$$

and so, BT are reduced to

$$\rho_x = -\frac{1}{\sqrt{2}}e^{-f(x)} \quad \text{and} \quad \rho_t = -\frac{1}{\sqrt{2}}e^{g(t)}. \tag{4.17}$$

Integrate these equations to yield

$$\rho(x,t) = -\frac{1}{\sqrt{2}} \int^{x} e^{-f(x')}\, dx' - \frac{1}{\sqrt{2}} \int^{t} e^{g(t')}\, dt'. \tag{4.18}$$

Therefore we obtain

$$u(x,t) = -2\ln\Big\{ -\frac{1}{\sqrt{2}} \int^{x} e^{-f(x')}\, dx' - \frac{1}{\sqrt{2}} \int^{t} e^{g(t')}\, dt' \Big\} - f(x) + g(t) \tag{4.19}$$

which is the general solution of the Liouville's equation.

Example 4.1.3. Now, let us consider the sine-Gordon equation

$$u_{xt} = \sin(u). \tag{4.20}$$

The auto-Bäcklund transformations are given as

$$R_1 = \frac{1}{2}(u+v)_x - a\sin\Big(\frac{u-v}{2}\Big) = 0 \tag{4.21}$$

and

$$R_2 = \frac{1}{2}(u-v)_t - \frac{1}{a}\sin\Big(\frac{u+v}{2}\Big) = 0 \tag{4.22}$$

where a $(\neq 0)$ is an arbitrary constant. The integrability of BT can be verified by cross-differentiating, then we get

$$\frac{1}{2}(u+v)_{xt} = \frac{a}{2}(u-v)_t \cos\Big(\frac{u-v}{2}\Big) = \sin\Big(\frac{u+v}{2}\Big)\cos\Big(\frac{u-v}{2}\Big) \tag{4.23}$$

and

$$\frac{1}{2}(u-v)_{tx} = \frac{1}{2a}(u+v)_x \cos\Big(\frac{u+v}{2}\Big) = \sin\Big(\frac{u-v}{2}\Big)\cos\Big(\frac{u+v}{2}\Big). \tag{4.24}$$

By adding and subtracting, we clearly see that u and v satisfy the sine-Gordon equation. Hence, $R_1 = 0$ and $R_2 = 0$ are the auto-Bäcklund transformations for the sine-Gordon equation.

Now, we shall generate a new solution from the trivial one $v(x,y) = 0$ which is a solution of the sine-Gordon equation. Let $v = 0$, then the BT become

$$u_x = 2a\sin\Big(\frac{u}{2}\Big) \quad \text{and} \quad u_t = \frac{2}{a}\sin\Big(\frac{u}{2}\Big), \quad a \neq 0 \tag{4.25}$$

We integrate both equations to yield

$$2ax = \int^u \frac{du}{\sin(u/2)} = 2\log\left|\tan\left(\frac{u}{4}\right)\right| + f(t) \tag{4.26}$$

and

$$\frac{2t}{a} = \int^u \frac{du}{\sin(u/2)} = 2\log\left|\tan\left(\frac{u}{4}\right)\right| + g(x) \tag{4.27}$$

where f and g are two arbitrary functions, respectively. By differentiating (4.26) with respect to t and using u_t in (4.25), we obtain

$$f(t) = -\frac{2}{a}t + \alpha \tag{4.28}$$

and, by differentiating (4.27) with respect to x and using u_x in (4.25), we get

$$g(x) = -2ax + \beta \tag{4.29}$$

where α and β are arbitrary constants. Then we get from (4.26) or (4.27)

$$\tan\left(\frac{u}{4}\right) = C\exp\left(ax + \frac{t}{a}\right) \tag{4.30}$$

or

$$u(x,t) = 4\arctan\left\{C\exp\left(ax + \frac{t}{a}\right)\right\} \tag{4.31}$$

where C is an arbitrary constant. Here, we obtained the solitary (one-soliton) wave solution of the sine-Gordon equation which is a wave that preserves its speed and shape as it propagates.

Each time using a different Bäcklund parameter, we can derive a different solution of the sine-Gordon equation by using one parameter family of Bäcklund transformations.

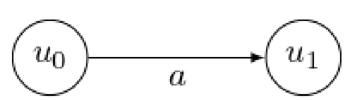


Figure 4.1: Diagram of BT for sine-Gordon with one parameter

In this process, we can avoid integration since we know that the BT admits a theorem of permutability. Bianchi [39] first recognized that this theorem holds for the sine-Gordon equation.

Theorem 4.1.4 (Theorem of permutability [40]). Let $S(u, v; a) = 0$ be a Bäcklund transformation where u and v are solutions of a nonlinear partial differential equation and a is a parameter. If the Bäcklund transformation is applied to the solution u_0 with two different parameters a_1, and a_2, i.e., $S(u_0, u_1; a_1) = 0$ and $S(u_0, u_2; a_2) = 0$, then fourth solution u_3 can be obtained by either $S(u_1, u_3; a_2) = 0$ or $S(u_2, u_3; a_1) = 0$.

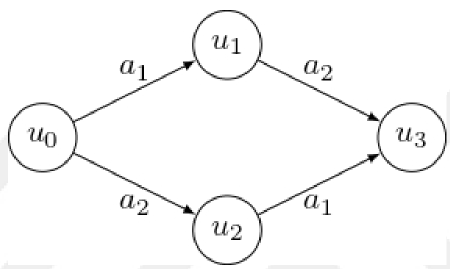


Figure 4.2: Diagram of the theorem of permutability

Now, we shall find the nonlinear superposition for the sine-Gordon equation. Let u_1 and u_2 be solutions of the sine-Gordon equation derived from a solution u_0 with parameters a_1 and a_2, respectively by using (4.21)

$$\frac{1}{2}(u_1 + u_0)_x = a_1 \sin\Big(\frac{u_1 - u_0}{2}\Big) \tag{4.32}$$

$$\frac{1}{2}(u_2 + u_0)_x = a_2 \sin\Big(\frac{u_2 - u_0}{2}\Big) \tag{4.33}$$

and by using (4.22)

$$\frac{1}{2}(u_1 - u_0)_t = \frac{1}{a_1} \sin\Big(\frac{u_1 + u_0}{2}\Big) \tag{4.34}$$

$$\frac{1}{2}(u_2 - u_0)_t = \frac{1}{a_2} \sin\Big(\frac{u_2 + u_0}{2}\Big). \tag{4.35}$$

Let u_3 be a solution obtained from u_1 with parameter a_2

$$\frac{1}{2}(u_3 + u_1)_x = a_2 \sin\Big(\frac{u_3 - u_1}{2}\Big) \tag{4.36}$$

$$\frac{1}{2}(u_3 - u_1)_t = \frac{1}{a_2} \sin\Big(\frac{u_3 + u_1}{2}\Big) \tag{4.37}$$

and obtained from u_2 with the parameter a_1

$$\frac{1}{2}(u_3 + u_2)_x = a_1 \sin\Big(\frac{u_3 - u_2}{2}\Big) \tag{4.38}$$

$$\frac{1}{2}(u_3 - u_2)_t = \frac{1}{a_1} \sin\Big(\frac{u_3 + u_2}{2}\Big). \tag{4.39}$$

Subtracting (4.36) from (4.38), and subtracting (4.32) from (4.33), we obtain

$$a_1 \sin\left(\frac{u_3 - u_2}{2}\right) - a_2 \sin\left(\frac{u_3 - u_1}{2}\right) = a_2 \sin\left(\frac{u_2 - u_0}{2}\right) - a_1 \sin\left(\frac{u_1 - u_0}{2}\right). \quad (4.40)$$

Subtracting (4.39) from (4.37), and subtracting (4.34) from (4.35), we obtain

$$\frac{1}{a_2} \sin\left(\frac{u_3 + u_1}{2}\right) - \frac{1}{a_1} \sin\left(\frac{u_3 + u_2}{2}\right) = \frac{1}{a_2} \sin\left(\frac{u_2 + u_0}{2}\right) - \frac{1}{a_1} \sin\left(\frac{u_1 + u_0}{2}\right) \quad (4.41)$$

or

$$a_1 \sin\left(\frac{u_3 + u_1}{2}\right) - a_2 \sin\left(\frac{u_3 + u_2}{2}\right) = a_1 \sin\left(\frac{u_2 + u_0}{2}\right) - a_2 \sin\left(\frac{u_1 + u_0}{2}\right). \quad (4.42)$$

We add (4.40) and (4.42) to yield

$$\begin{aligned} &a_1 \left\{ \sin\left(\frac{u_3 - u_2}{2}\right) + \sin\left(\frac{u_1 - u_0}{2}\right) - \sin\left(\frac{u_2 + u_0}{2}\right) + \sin\left(\frac{u_3 + u_1}{2}\right) \right\} \\ &\quad = a_2 \left\{ \sin\left(\frac{u_2 - u_0}{2}\right) + \sin\left(\frac{u_3 - u_1}{2}\right) + \sin\left(\frac{u_3 + u_2}{2}\right) - \sin\left(\frac{u_1 + u_0}{2}\right) \right\}. \end{aligned} \quad (4.43)$$

By using $\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x$, we get

$$\frac{\sin\left(\frac{u_3}{2}\right) - \sin\left(\frac{u_0}{2}\right)}{\cos\left(\frac{u_3}{2}\right) + \cos\left(\frac{u_0}{2}\right)} = \left(\frac{a_1 + a_2}{a_1 - a_2}\right) \frac{\sin\left(\frac{u_2}{2}\right) - \sin\left(\frac{u_1}{2}\right)}{\cos\left(\frac{u_2}{2}\right) + \cos\left(\frac{u_1}{2}\right)}. \quad (4.44)$$

Substitute $\frac{u_0}{2} = \frac{u_0+u_3}{4} + \frac{u_0-u_3}{4}$, $\frac{u_3}{2} = \frac{u_0+u_3}{4} - \frac{u_0-u_3}{4}$, and $\frac{u_1}{2} = \frac{u_1+u_2}{4} + \frac{u_1-u_2}{4}$, $\frac{u_2}{2} = \frac{u_1+u_2}{4} - \frac{u_1-u_2}{4}$, then we obtain

$$\tan\left(\frac{u_3 - u_0}{4}\right) = \left(\frac{a_1 + a_2}{a_1 - a_2}\right) \tan\left(\frac{u_2 - u_1}{4}\right) \quad (4.45)$$

or

$$u_3 = u_0 + 4 \arctan\left(\frac{a_1 + a_2}{a_1 - a_2} \tan\left(\frac{u_2 - u_1}{4}\right)\right) \quad (4.46)$$

which is the nonlinear superposition for the sine-Gordon equation and illustrates the relation among four solutions. Lamb [41] showed that (4.46) can be used to provide N-soliton solution of the sine-Gordon equation.

4.1.1 A list of BT for some evolution equations

1) A Bäcklund transformations for the KdV equation

$$P[u] = u_t + 6uu_x + u_{xxx} = 0 \tag{4.47}$$

and the (λ-dependent) modified Korteweg-de Vries equation

$$Q[v] = v_t + v_{xxx} - 6(\lambda + v^2)v_x = 0 \tag{4.48}$$

are given by

$$v_x = \lambda + u + v^2 \tag{4.49}$$

and

$$v_t = -u_{xx} - 2u^2 + 2\lambda u + 4\lambda^2 - 2u_x v + v^2(4\lambda - 2u). \tag{4.50}$$

Here, $R_1 = v_x - \lambda - u - v^2 = 0$ is a Bäcklund transformation from the mKdV equation to the KdV equation.

2)The auto-BT for the potential KdV (pKdV) equation

$$P = Q = w_t + w_{xxx} + 3(w_x)^2 = 0 \tag{4.51}$$

are given by

$$v_x + w_x = -2\lambda - \frac{1}{2}(w - v)^2 \tag{4.52}$$

and

$$v_t + w_t = -2(w_x)^2 - 4\lambda w_x - 8\lambda^2 + 2w_{xx}(w - v) + (w_x - 2\lambda)(w - v)^2. \tag{4.53}$$

Here, $R_1 = v_x + w_x + 2\lambda + \frac{1}{2}(w - v)^2 = 0$ is the auto-Bäcklund transformation for the pKdV equation.

3) The auto-BT for the potential mKdV (pmKdV) equation

$$P = Q = w_t + w_{xxx} - 2(w_x)^3 = 0 \tag{4.54}$$

are given by

$$v_x - w_x = \lambda \sinh(v + w) \tag{4.55}$$

and

$$v_t - w_t = -2\lambda^2 w_x - 2\lambda w_{xx} \cosh(v+w) + \lambda\Big(2(w_x)^2 - \lambda^2\Big)\sinh(v+w). \quad (4.56)$$

Here, $R_1 = v_x - w_x - \lambda \sinh(v+w) = 0$ is the auto-Bäcklund for the pmKdV equation.

4) The auto-BT for the sinh-Gordon equation

$$P = Q = w_{xt} - \sinh w = 0 \quad (4.57)$$

are given by

$$\frac{1}{2}(v_x + w_x) = \lambda \sinh\left(\frac{v-w}{2}\right) \quad (4.58)$$

and

$$\frac{1}{2}(v_t - w_t) = \frac{1}{\lambda}\sinh\left(\frac{v+w}{2}\right). \quad (4.59)$$

Here, $R_1 = \frac{1}{2}(v_x + w_x) - \lambda \sinh\left(\frac{v-w}{2}\right) = 0$ is the auto-Bäcklund for the sinh-Gordon equation. [42]

4.2 Bäcklund transformations for the KdV equation

The Korteweg-de Vries equation, that describes the propagation of waves on shallow water surfaces, is given in the standard form by

$$u_t - 6uu_x + u_{xxx} = 0. \quad (4.60)$$

To find the Bäcklund transformations for the KdV equation, one can use the method of Clairin as in the paper [14] and, one can as well use differential forms as in the paper [15]. However, in this paper, we consider the Miura transformation

$$u = v^2 + v_x \quad (4.61)$$

where u satisfies the KdV equation and v satisfies the modified KdV (mKdV) equation

$$v_t - 6v^2 v_x + v_{xxx} = 0 \quad (4.62)$$

and, (4.61) is a transformation from the mKdV equation to the KdV equation. We have already mentioned that the Miura transformation is a restricted transformation.

It is easy to show that the KdV equation is invariant under the Galilean transformation that is

$$t \to t, \quad x \to x + 6\lambda t, \quad u \to u - \lambda \tag{4.63}$$

thus, the Miura transformation takes the form

$$u = \lambda + v^2 + v_x \tag{4.64}$$

where λ is a real parameter, and the mKdV equation becomes

$$v_t - 6(v^2 + \lambda)v_x + v_{xxx} = 0 \tag{4.65}$$

which is known as the Gardner equation. The Gardner equation with (4.64) implies the KdV equation.

$$\begin{tikzcd} v_t - 6(v^2 + \lambda)v_x + v_{xxx} = 0 \arrow[d, "u = \lambda + v^2 + v_x"] \\ u_t - 6uu_x + u_{xxx} = 0 \end{tikzcd}$$

There is a discrete symmetry of (4.65), if the function v is a solution of the equation (4.65), then $-v$ is also a solution. Thus, when we write the Miura equation (4.64) for v and $-v$, we get u_1 and u_2 as the solutions of the KdV equation

$$u_1 = \lambda + v^2 + v_x, \quad u_2 = \lambda + v^2 - v_x \tag{4.66}$$

for given v and parameter λ and, we obtain from these equations

$$u_1 - u_2 = 2v_x \quad \text{and} \quad u_1 + u_2 = 2(\lambda + v^2). \tag{4.67}$$

Let us introduce

$$u_i = \frac{\partial w_i}{\partial x} \quad (i = 1, 2) \tag{4.68}$$

which corresponds to the potential KdV (pKdV) equation

$$w_t - 3w_x^2 + w_{xxx} = 0 \tag{4.69}$$

so that the equations (4.67) become

$$w_1 - w_2 = 2v \tag{4.70}$$

and

$$(w_1 + w_2)_x = 2\lambda + \frac{1}{2}(w_1 - w_2)^2 \tag{4.71}$$

where w_1 and w_2 are solutions to the pKdV equation. Then, we can write

$$(w_1 + w_2)_t - 3(w_{1x}^2 + w_{2x}^2) + (w_1 + w_2)_{xxx} = 0. \tag{4.72}$$

Inserting (4.71) to yield

$$(w_1 + w_2)_t - 2(w_{1x}^2 + w_{1x}w_{2x} + w_{2x}^2) + (w_1 - w_2)(w_1 - w_2)_{xx} = 0. \tag{4.73}$$

Hence, the auto-Bäcklund transformations for the potential KdV equation are given by

$$(w_1 + w_2)_x = 2\lambda + \frac{1}{2}(w_1 - w_2)^2 \tag{4.74}$$

$$(w_1 + w_2)_t = 2(w_{1x}^2 + w_{1x}w_{2x} + w_{2x}^2) - (w_1 - w_2)(w_1 - w_2)_{xx} \tag{4.75}$$

which are also the auto-BT for the KdV equation through the equation $u_i = \partial w_i/\partial x$, where (4.74) describes the x-part and (4.75) describes the t-part or time evolution part. The equations (4.74) and (4.75) are compatible and integrable.

Now, we shall verify the integrability of the BT. First, we differentiate the equation (4.74) with respect to x twice to yield

$$(w_1 + w_2)_{xxx} = (w_1 - w_2)_x^2 + (w_1 - w_2)(w_1 - w_2)_{xx} \tag{4.76}$$

and by adding the equation (4.75), we obtain

$$(w_{1t} - 3w_{1x}^2 + w_{1xxx}) + (w_{2t} - 3w_{2x}^2 + w_{2xxx}) = 0. \tag{4.77}$$

Secondly, we take t-derivative of (4.74)

$$(w_1 + w_2)_{xt} = (w_1 - w_2)(w_1 - w_2)_t \tag{4.78}$$

and take x-derivative of the equation (4.75)

$$\begin{aligned}(w_1+w_2)_{xt} =& 2(2w_{1x}w_{1xx}+w_{1xx}w_{2x}+w_{1x}w_{2xx}+2w_{2x}w_{2xx}) \\ &- (w_1-w_2)_x(w_1-w_2)_{xx}-(w_1-w_2)(w_1-w_2)_{xxx}\end{aligned} \tag{4.79}$$

and, subtract (4.78) from (4.79) to yield

$$3(w_1+w_2)_x(w_1+w_2)_{xx}-3(w_1-w_2)\big[(w_1-w_2)_{xxx}+(w_1-w_2)_t\big]=0, \tag{4.80}$$

and by using the x-derivative of (4.74)

$$(w_1+w_2)_{xx}=(w_1-w_2)(w_1-w_2)_x \tag{4.81}$$

we get

$$(w_{1t}-3w_{1x}^2+w_{1xxx})-(w_{2t}-3w_{2x}^2+w_{2xxx})=0. \tag{4.82}$$

By subtracting and adding the equations (4.77) and (4.82), we find that w_1 and w_2 satisfy the potential KdV equation and, also satisfy the KdV equation through the relation $u_i=\partial w_i/\partial x$.

Now, we shall show that (4.74) is suitable for all hierarchies of the KdV equation. Substitute $v=\frac{1}{2}(w_1-w_2)$ into (4.74), we get

$$v_x+w_{2x}=\lambda+v^2 \tag{4.83}$$

and let $u=w_{2x}$ since it is a solution of the KdV equation, then

$$v_x+u=\lambda+v^2 \tag{4.84}$$

and the substitution of $v=-\frac{\psi_x}{\psi}$ where $\psi\neq 0$ provides

$$\psi_{xx}+(\lambda-u)\psi=0 \tag{4.85}$$

which is the Schrödinger equation. Since this equation is the eigenvalue equation for all hierarchies of KdV, (4.74) is the BT for the KdV hierarchy. However, the t-part of the BT changes [36].

4.2.1 Soliton solutions

Now, we shall derive pure soliton solutions of the KdV equation by using the Bäcklund transformations. Note that not all choices of solution and parameters lead to a pure soliton solution. Let us consider the trivial solution $w_2 = 0$, then the equation (4.74) becomes

$$w_{1x} = 2\lambda + \frac{1}{2}w_1^2. \tag{4.86}$$

Let $\lambda = -\kappa^2(< 0)$, by integrating once when $|w_1|< 2\kappa$, we get a regular solution which is

$$w_1(x,t) = -2\kappa \tanh\{\kappa x + f(t)\} \tag{4.87}$$

where f is an arbitrary function. The t-part of the BT becomes

$$w_{1t} - 2w_{1x}^2 + w_1 w_{1xx} = 0. \tag{4.88}$$

By inserting (4.86) into (4.88), we get

$$w_{1t} + 4\kappa^2 w_{1x} = 0 \tag{4.89}$$

where it has a general solution which is

$$w_1(x,t) = g(x - 4\kappa^2 t) \tag{4.90}$$

for the arbitrary function g. For consistency, we choose

$$f(t) = -4\kappa^3 t - \kappa x_0 \tag{4.91}$$

where x_0 is an arbitrary constant. Thus, we get a new solution of the potential KdV equation

$$w_1(x,t) = -2\kappa \tanh\{\kappa(x - x_0 - 4\kappa^2 t)\} \tag{4.92}$$

and so, we obtain the regular (bounded) solitary-wave solution, or one-soliton solution of the KdV equation

$$u_1(x,t) = -2\kappa^2 \operatorname{sech}^2\{\kappa(x - x_0 - 4\kappa^2 t)\} \tag{4.93}$$

by using $u_i = \partial w_i/\partial x$.

If $|w_1| > 2\kappa$, from (4.86) we obtain a singular solution

$$w_1(x,t) = -2\kappa \coth\{\kappa(x - x_0 - 4\kappa^2 t)\} \tag{4.94}$$

where the singular (unbounded) solitary wave solution is

$$u_1(x,t) = 2\kappa^2 \operatorname{cosech}^2\{\kappa(x - x_0 - 4\kappa^2 t)\} \tag{4.95}$$

which satisfies the KdV equation. Even though this solution is not of physical interest, it will help us to obtain a pure-soliton solution.

Example 4.2.1. If we choose $\lambda = -1$ (or $\kappa = 1$) and $x_0 = 0$, (4.93) provides one-soliton solution with $u(x,0) = -2\operatorname{sech}^2 x$ in Figure 2.4.

We know that the Bäcklund transformation admits the Bianchi permutability theorem. By using this theorem, we shall find a nonlinear superposition formula for the KdV equation, and then, use it to derive a two-soliton solution.

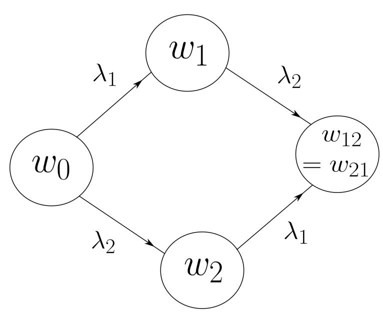


Figure 4.3: Bianchi diagram for constructing two-soliton solution

The auto-Bäcklund transformations for the KdV equation are given by

$$(w_1 + w_2)_x = 2\lambda + \frac{1}{2}(w_1 - w_2)^2 \tag{4.96}$$

and

$$(w_1 + w_2)_t = 2(w_{1x}^2 + w_{1x}w_{2x} + w_{2x}^2) - (w_1 - w_2)(w_1 - w_2)_{xx} \tag{4.97}$$

where λ is a parameter. Let w_0 be a solution of the KdV equation (seed solution). To find another solution w_1 of the KdV equation, we use the BT (4.96) and obtain

$$(w_1 + w_0)_x = 2\lambda_1 + \frac{1}{2}(w_1 - w_0)^2 \tag{4.98}$$

when $\lambda = \lambda_1$. Let w_2 be another solution when $\lambda = \lambda_2$ i.e.,

$$(w_2 + w_0)_x = 2\lambda_2 + \frac{1}{2}(w_2 - w_0)^2. \tag{4.99}$$

Now, we derive two more solutions, one is w_{12} from w_1 with λ_2, and another is w_{21} from w_2 with λ_1, by using (4.96), then

$$(w_{12} + w_1)_x = 2\lambda_2 + \frac{1}{2}(w_{12} - w_1)^2 \tag{4.100}$$

and

$$(w_{21} + w_2)_x = 2\lambda_1 + \frac{1}{2}(w_{21} - w_2)^2 \tag{4.101}$$

where we take $w_{12} = w_{21}$. We, first, subtract (4.99) from (4.98) to yield

$$w_{1x} - w_{2x} = 2(\lambda_1 - \lambda_2) + \frac{1}{2}(w_1 - w_0)^2 - \frac{1}{2}(w_2 - w_0)^2 \tag{4.102}$$

then, we take $w_{12} = w_{21}$ and subtract (4.101) from (4.100) to yield

$$w_{1x} - w_{2x} = 2(\lambda_2 - \lambda_1) + \frac{1}{2}(w_{12} - w_1)^2 - \frac{1}{2}(w_{12} - w_2)^2. \tag{4.103}$$

By subtracting (4.102) from (4.103), we obtain

$$0 = 4(\lambda_2 - \lambda_1) - (w_1 - w_2)(w_{12} - w_0) \tag{4.104}$$

or

$$w_{12} = w_0 - 4\frac{\lambda_2 - \lambda_1}{w_2 - w_1} \tag{4.105}$$

which is known as a nonlinear superposition for the solution of the KdV equation.

Now, we shall find a two-soliton solution. Only way to obtain a regular two-soliton solution is to choose a singular (unbounded) solution corresponding to λ_2 and a regular (bounded) solution corresponding to λ_1, where $\lambda_2 < \lambda_1$ and the initial solution is the vacuum solution $w_0 = 0$. The regular two-soliton solution can be obtained by the nonlinear superposition (4.105)

$$w_{12} = -4\frac{\lambda_2 - \lambda_1}{\overline{w}_2 - w_1} \tag{4.106}$$

where bar denotes the singular (unbounded) solution. By choosing singular solution $\overline{w}_2$, we avoid the denominator to vanish at some points [16].

Example 4.2.2. We take $w_0 = 0$ and a regular solution $w_1 = -2\tanh(x-4t)$ corresponding to $\lambda_1 = -1$ (or $\kappa = 1$), and a singular solution $w_2 = -4\coth(2x - 32t)$ corresponding to $\lambda_2 = -4$ (or $\kappa = 2$),thus we obtain

$$w_{12}(x,t) = \frac{-6}{2\coth(2x-32t) - \tanh(x-4t)} \tag{4.107}$$

which is a solution of the potential KdV equation. We use the equation $u_{12} = \partial w_{12}/\partial x$, to yield

$$u_{12}(x,t) = -12\frac{3 + 4\cosh(2x-8t) + \cosh(4x-64t)}{\{3\cosh(x-28t) + \cosh(3x-36t)\}^2} \tag{4.108}$$

which is known as the two-soliton solution of the KdV equation. This is the same solution in (2.247) derived from IST when $u(x,0) = -6\,\mathrm{sech}^2\, x$ and the figure is given by Figure 2.5.

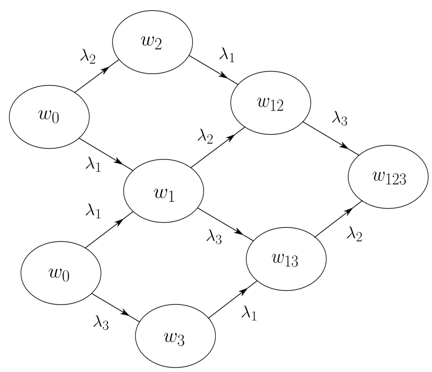


Figure 4.4: Bianchi diagram for constructing three-soliton solution

By using the equation (4.105) and Figure 4.4, we can write the third-order transformed solution

$$w_{123} = w_1 - 4\frac{\lambda_3 - \lambda_2}{w_{13} - w_{12}} \tag{4.109}$$

and, again by using (4.105) for the denominator, we get

$$w_{123} = \frac{\lambda_1 w_1(w_2 - w_3) + \lambda_2 w_2(w_3 - w_1) + \lambda_3 w_3(w_1 - w_2)}{\lambda_1(w_2 - w_3) + \lambda_2(w_3 - w_1) + \lambda_3(w_1 - w_2)} \tag{4.110}$$

which is a three parameter nonlinear superposition for the solution of the KdV equation.

Example 4.2.3. We choose $\lambda_1 = -1$, $\lambda_2 = -4$ and $\lambda_3 = -9$, then derive a regular solution w_1, a singular solution $\overline{w}_2$ and a regular solution w_3 from the equation (4.86) with a trivial solution $w_0 = 0$, respectively, to yield the regular three-soliton solution of the KdV equation

$$u(x,t) = -2\frac{F(x,t)}{G(x,t)} \tag{4.111}$$

where

$$\begin{aligned} F(x,t) =& 1512 + 600\cosh(2x-8t) + 12\cosh(10x-280t) + 360\cosh(6x-216t) \\ &+ 180\cosh(6x-72t) + 960\cosh(4x-64t) + 120\cosh(8x-224t) \\ &+ 1620\cosh(2x-56t) + 300\cosh(2x-152t) + 480\cosh(4x-208t) \end{aligned} \tag{4.112}$$

and

$$G(x,t) = \{\cosh(6x-144t) + 6\cosh(4x-136t) + 10\cosh(72t) + 15\cosh(2x-80t)\}^2. \tag{4.113}$$

The same result is found by IST when $u(x,0) = -12\,\mathrm{sech}^2\, x$. Since the derivation of three-soliton solution from IST is tedious, the calculations are not included here.

When we applied the transformation recursively, we obtain a new soliton solution from the previous one. After n-applications, we generate the n-soliton solution of the KdV equation. Hence the nth step is expressed as

$$w_{(n)} = w_{(n-2)} - 4\frac{\lambda_n - \lambda_{n-1}}{w_{(n-1)'} - w_{(n-1)}} \tag{4.114}$$

where $(n-1)$ denotes the set of $n-1$ parameters $\{1, 2, ..., n-1\}$ and $(n-1)'$ denotes the set of $n-1$ parameters $\{1, 2, ..., n-2, n\}$ [16].

Hence we try to find a more general representation of nth step solution of the KdV equation by matrices. Let us consider two cases for even and odd integer numbers.

Firstly, we can write the equations (4.105) starting from a trivial solution as division of determinant of two matrices

$$w_2 = \frac{U_2}{V_2} \tag{4.115}$$

where

$$U_2 = \begin{vmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{vmatrix} \quad \text{and} \quad V_2 = \begin{vmatrix} 1 & w_1 \\ 1 & w_2 \end{vmatrix}. \tag{4.116}$$

For $n = 4$, nonlinear superposition can also be found, and it is represented by

$$w_4 = \frac{U_4}{V_4} \tag{4.117}$$

where

$$U_4 = \begin{vmatrix} 1 & w_1 & \lambda_1 & \lambda_1^2 \\ 1 & w_2 & \lambda_2 & \lambda_2^2 \\ 1 & w_3 & \lambda_3 & \lambda_3^2 \\ 1 & w_4 & \lambda_4 & \lambda_4^2 \end{vmatrix} \quad \text{and} \quad V_4 = \begin{vmatrix} 1 & w_1 & \lambda_1 & \lambda_1 w_1 \\ 1 & w_2 & \lambda_2 & \lambda_2 w_2 \\ 1 & w_3 & \lambda_3 & \lambda_3 w_3 \\ 1 & w_4 & \lambda_4 & \lambda_4 w_4 \end{vmatrix}. \tag{4.118}$$

By looking previous result, we can write the $2n$th step as follows:

$$w_{2n} = \frac{U_{2n}}{V_{2n}} \quad \text{for } n = 1, 2, 3, ... \tag{4.119}$$

where

$$U_{2n} = \begin{vmatrix} 1 & w_1 & \lambda_1 & ... & \lambda_1^{n-2} w_1 & \lambda_1^{n-1} & \lambda_1^n \\ 1 & w_2 & \lambda_2 & ... & \lambda_2^{n-2} w_2 & \lambda_2^{n-1} & \lambda_2^n \\ ... & ... & ... & ... & ... & ... & ... \\ 1 & w_{2n} & \lambda_{2n} & ... & \lambda_{2n}^{n-2} w_{2n} & \lambda_{2n}^{n-1} & \lambda_{2n}^n \end{vmatrix} \tag{4.120}$$

and

$$V_{2n} = \begin{vmatrix} 1 & w_1 & \lambda_1 & ... & \lambda_1^{n-2} w_1 & \lambda_1^{n-1} & \lambda_1^{n-1} w_1 \\ 1 & w_2 & \lambda_2 & ... & \lambda_2^{n-2} w_2 & \lambda_2^{n-1} & \lambda_2^{n-1} w_2 \\ ... & ... & ... & ... & ... & ... & ... \\ 1 & w_{2n} & \lambda_{2n} & ... & \lambda_{2n}^{n-2} w_{2n} & \lambda_{2n}^{n-1} & \lambda_{2n}^{n-1} w_{2n} \end{vmatrix}. \tag{4.121}$$

Secondly, for $n = 3$, from the equation (4.110), we find that

$$w_3 = \frac{U_3}{V_3} \tag{4.122}$$

where

$$U_3 = \begin{vmatrix} 1 & w_1 & \lambda_1 w_1 \\ 1 & w_2 & \lambda_2 w_2 \\ 1 & w_3 & \lambda_3 w_3 \end{vmatrix} \quad \text{and} \quad V_3 = \begin{vmatrix} 1 & w_1 & \lambda_1 \\ 1 & w_2 & \lambda_2 \\ 1 & w_3 & \lambda_3 \end{vmatrix}. \tag{4.123}$$

We consider the equation (4.114) for $2n$th step

$$w_{(2n)} = w_{(2n-2)} - 4\frac{\lambda_{2n} - \lambda_{2n-1}}{w_{(2n-1)'} - w_{(2n-1)}} \tag{4.124}$$

and, we see $w_{(2n)}$ is derived from $w_{(2n-1)'}$ and $w_{(2n-1)}$. Therefore, U_{2n-1} resembles V_{2n} and V_{2n-1} resembles U_{2n}. Hence we conclude that

$$w_{2n-1} = \frac{U_{2n-1}}{V_{2n-1}} \quad \text{for } n = 1, 2, 3, ... \tag{4.125}$$

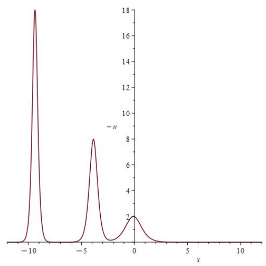
where

$$U_{2n-1} = \begin{vmatrix} 1 & w_1 & \lambda_1 & ... & \lambda_1^{n-2} & \lambda_1^{n-2} w_1 & \lambda_1^{n-1} w_1 \\ 1 & w_2 & \lambda_2 & ... & \lambda_2^{n-2} & \lambda_2^{n-2} w_2 & \lambda_2^{n-1} w_2 \\ ... & ... & ... & ... & ... & ... & ... \\ 1 & w_{2n-1} & \lambda_{2n-1} & ... & \lambda_{2n-1}^{n-2} & \lambda_{2n-1}^{n-2} w_{2n-1} & \lambda_{2n-1}^{n-1} w_{2n-1} \end{vmatrix} \tag{4.126}$$

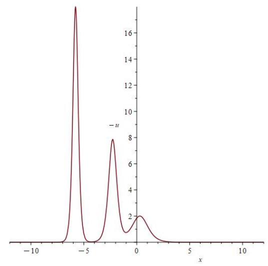
and

$$V_{2n-1} = \begin{vmatrix} 1 & w_1 & \lambda_1 & ... & \lambda_1^{n-2} & \lambda_1^{n-2} w_1 & \lambda_1^{n-1} \\ 1 & w_2 & \lambda_2 & ... & \lambda_2^{n-2} & \lambda_2^{n-2} w_2 & \lambda_2^{n-1} \\ ... & ... & ... & ... & ... & ... & ... \\ 1 & w_{2n-1} & \lambda_{2n-1} & ... & \lambda_{2n-1}^{n-2} & \lambda_{2n-1}^{n-2} w_{2n-1} & \lambda_{2n-1}^{n-1} \end{vmatrix}. \tag{4.127}$$

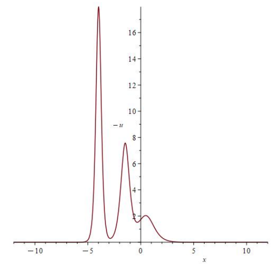
To obtain a pure multi-soliton, choose the starting solution as a vacuum solution $w_0 = 0$ and let eigenvalues be well-ordered such that $... < -\kappa_n^2 < -\kappa_{n-1}^2 < ... < -\kappa_2^2 < -\kappa_1^2$ i.e., $\kappa_1 < \kappa_2 < ... < \kappa_n < ...$ for $\kappa_i = 0, 1, 2, ...$ If we use the singular (unbounded) soliton solution corresponding to κ_{2n}, and the regular (bounded) soliton solution corresponding to κ_{2n-1} for all n, then we can derive a pure N-soliton solution by using either (4.119) or (4.125) [16].



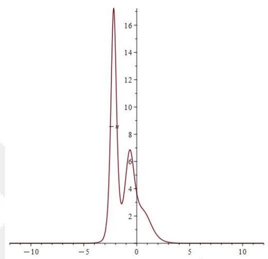
(a) $t = -0.25$



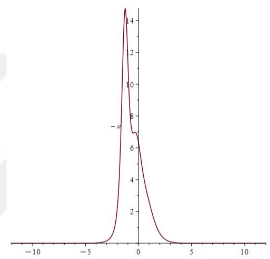
(b) $t = -0.15$



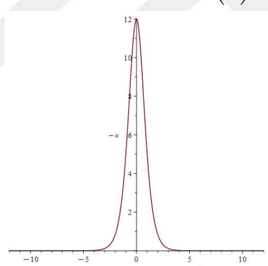
(c) $t = -0.1$



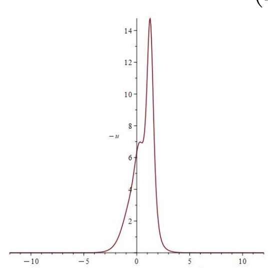
(d) $t = -0.05$



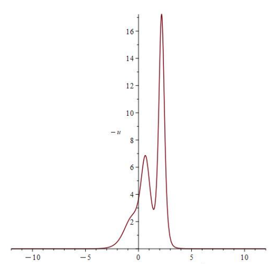
(e) $t = -0.025$



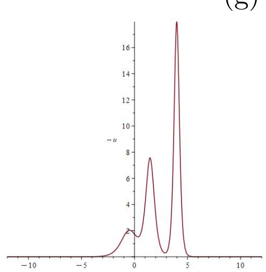
(f) $t = 0$



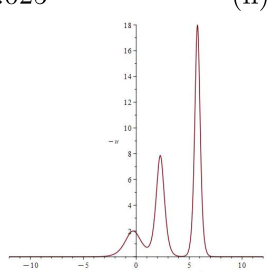
(g) $t = 0.025$



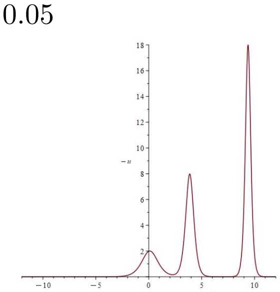
(h) $t = 0.05$



(i) $t = 0.1$



(j) $t = 0.15$



(k) $t = 0.25$

Figure 4.5: The three-soliton solution with $u(x,0) = -12\,\mathrm{sech}^2\, x$ for different t-values (motion of three-soliton)

4.3 Derivation of BT from the AKNS scheme

Our aim is now to derive Bäcklund transformations from the Riccati form of the equations in the inverse scattering method. We shall consider the eigenvalue problem

$$\begin{aligned} \psi_{1x} - \eta\psi_1 &= q(x,t)\psi_2 \\ \psi_{2x} + \eta\psi_2 &= r(x,t)\psi_1 \end{aligned} \tag{4.128}$$

and the time evolution of ψ_1 and ψ_2

$$\begin{aligned} \psi_{1t} &= A(x,t,\eta)\psi_1 + B(x,t,\eta)\psi_2 \\ \psi_{2t} &= C(x,t,\eta)\psi_1 - A(x,t,\eta)\psi_2. \end{aligned} \tag{4.129}$$

These equations (4.128) and (4.129) are the AKNS scheme in (3.4) and (3.5) when $\eta = -i\zeta$, and η is independent of t. (4.128) and (4.129) are compatible if A, B, C satisfy

$$\begin{aligned} A_x &= qC - rB \\ B_x - 2\eta B &= q_t - 2Aq \\ C_x + 2\eta C &= r_t + 2Ar. \end{aligned} \tag{4.130}$$

Different choices in A, B, and C lead the equations (4.130) to the various nonlinear evolution equations. We have already given the list of the evolution equations. This time by using (3.24), we shall evaluate A, B, and C.

(a) the Korteweg-de Vries equation

$$\begin{aligned} &q_t + 6qq_x + q_{xxx} = 0, \qquad r = -1 \\ &A = -4\eta^3 - 2\eta q - q_x \\ &B = -4\eta^2 q - 2\eta q_x - 2q^2 - q_{xx} \\ &C = 4\eta^2 + 2q \end{aligned} \tag{4.131}$$

(b) the sine-Gordon equation

$$\begin{aligned} & u_{xt} = \sin u, \qquad r = -q = \frac{u_x}{2} \\ & A = \frac{1}{4\eta}\cos u \\ & B = C = \frac{1}{4\eta}\sin u \end{aligned} \tag{4.132}$$

(c) the modified Korteweg-de Vries equation

$$\begin{aligned} & q_t + 6q^2 q_x + q_{xxx} = 0, \qquad r = -q \\ & A = -4\eta^3 - 2\eta q^2 \\ & B = -4\eta^2 q - 2\eta q_x - 2q^3 - q_{xx} \\ & C = 4\eta^2 q - 2\eta q_x + 2q^3 + q_{xx} \end{aligned} \tag{4.133}$$

(d) the nonlinear Schrödinger equation

$$\begin{aligned} & q_t - 2iq^2 q^* - iq_{xx} = 0, \qquad r = -q^* \\ & A = 2i\eta^2 + i|q|^2 \\ & B = 2i\eta q + iq_x \\ & C = -2i\eta q^* + iq_x^* \end{aligned} \tag{4.134}$$

Derivation of Bäcklund transformation from the AKNS scheme was studied by Chen [17], and in that paper, gauge-like transformation that leaves the evolution equation invariant was considered. Since the relation between this transformation and Bäcklund transformation is ambiguous physically and mathematically, we are going to use the method of Wadati, Sanuki and Konno [19]. They showed that the Bäcklund transformation is used to add one more soliton to the solution of the same evolution equation by using a property of associated Sturm-Liouville equations in the paper by Crum [43]. It states that

$$-\psi_{xx} + \eta^2\psi = q(x,t)\psi \tag{4.135}$$

is related to

$$-\psi'_{xx} + \eta^2\psi' = q'(x,t)\psi' \tag{4.136}$$

via the transformation

$$q'(x,t) = q(x,t) - 2\frac{d^2}{dx^2}\log\psi' \tag{4.137}$$

where $\psi' = 1/\psi$. Now, we shall introduce a function

$$\Gamma = \frac{\psi_1}{\psi_2}. \tag{4.138}$$

From (4.128), we get

$$\psi_2\big(\psi_{1x} - \eta\psi_1 - q\psi_2\big) = \psi_1\big(\psi_{2x} + \eta\psi_2 - r\psi_1\big) \tag{4.139}$$

and dividing by $(\psi_2)^2$ and using (4.138), we obtain that (4.128) reduces to

$$\Gamma_x = 2\eta\Gamma + q - r\Gamma^2. \tag{4.140}$$

From (4.129), we get

$$\psi_2\big(\psi_{1t} - A\psi_1 - B\psi_2\big) = \psi_1\big(\psi_{2t} + A\psi_2 - C\psi_1\big) \tag{4.141}$$

and by dividing by $(\psi_2)^2$ and using (4.138), (4.129) reduces to

$$\Gamma_t = B + 2A\Gamma - C\Gamma^2. \tag{4.142}$$

We use the Riccati forms (4.140) and (4.142) to derive BT. We shall consider three classes of AKNS.

Class I : $(r = -1)$
When $r = -1$, (4.128) reduces to the Sturm-Liouville equation

$$-\psi_{2xx} + \eta^2\psi_2 = q(x,t)\psi_2. \tag{4.143}$$

Let ψ_2' also satisfy the Sturm-Liouville equation with $q'(x,t)$, and by using (4.137) we get

$$q'(x,t) = q(x,t) - 2\frac{d^2}{dx^2}\log\Big(\frac{1}{\psi_2}\Big). \tag{4.144}$$

Let $w_x = q$ and $w_x' = q'$, then

$$w' - w = 2\frac{\psi_{2x}}{\psi}. \tag{4.145}$$

Since $\psi_{2x} + \eta\psi_2 = -\psi_1$, then $\Gamma = -\frac{\psi_{2x}}{\psi_2} - \eta$. We obtain that

$$w' - w = -2(\eta + \Gamma) \tag{4.146}$$

or

$$\Gamma = -\eta + \frac{1}{2}(w - w'). \tag{4.147}$$

Insert (4.147) into (4.140) with $r = -1$ to yield

$$w_x + w'_x = 2\eta^2 - \frac{1}{2}(w - w')^2 \tag{4.148}$$

and insert (4.147) into (4.142) to yield

$$w'_t - w_t = -2B + 4A\Big(\eta + \frac{1}{2}(w' - w)\Big) + 2C\Big(\eta + \frac{1}{2}(w' - w)\Big)^2. \tag{4.149}$$

The equations (4.148) and (4.149) are the Bäcklund transformations for this class. By using (4.131), we can obtain the BT for the KdV equation

$$q_t + 6qq_x + q_{xxx} = 0. \tag{4.150}$$

Insert (4.131) into (4.149), we obtain

$$\begin{aligned} w'_t - w_t =& (w' - w)(-2w_{xx}) + (w' - w)^2(2\eta^2 + w_x) \\ &- 8\eta^4 + 4w_x^2 + 4\eta^2 w_x + 2w_{xxx}. \end{aligned} \tag{4.151}$$

By using (4.148) and (4.150), we obtain

$$w'_t + w_t = -4\eta^2 w'_x - 2w_x^2 - 2w_{xx}(w' - w) + w_x(w' - w)^2 \tag{4.152}$$

where w and w' satisfy the potential KdV equation

$$w_t + 3w_x^2 + w_{xxx} = 0. \tag{4.153}$$

The equations (4.148) and (4.152) are the Bäcklund transformations for the KdV equation (4.150), and they are also found by Wahlquist and Estabrook [16].

Class II : $(r = -q)$

When $r = -q$, we insert $-q\psi_1$ into the first equation in (4.128) to yield

$$\psi_{2xx} - \eta^2\psi_2 = -q_x\psi_1 - q^2\psi_2 \tag{4.154}$$

and, insert $q\psi_2$ into the second equation in (4.128) to yield

$$\psi_{1xx} - \eta^2\psi_1 = q_x\psi_2 - q^2\psi_1. \tag{4.155}$$

By using (4.154) and (4.155), we can write

$$\begin{aligned} &-(\psi_1 + i\psi_2)_{xx} + \eta^2(\psi_1 + i\psi_2) = (iq_x + q^2)(\psi_1 + i\psi_2) \\ &-(\psi_1 - i\psi_2)_{xx} + \eta^2(\psi_1 - i\psi_2) = (-iq_x + q^2)(\psi_1 - i\psi_2). \end{aligned} \tag{4.156}$$

Let us define $\phi_1 = \psi_1 + i\psi_2$ and $\phi_2 = \psi_1 - i\psi_2$ and $U = iq_x + q^2$, and its conjugate $U^* = -iq_x + q^2$. Hence, the system (4.128) reduces to

$$\begin{aligned} &-(\phi_1)_{xx} + \eta^2(\phi_1) = U\phi_1 \\ &-(\phi_2)_{xx} + \eta^2(\phi_2) = U^*\phi_2. \end{aligned} \tag{4.157}$$

This can be mapped to the second system

$$\begin{aligned} &-(\phi_1')_{xx} + \eta^2(\phi_1') = U'\phi_1' \\ &-(\phi_2')_{xx} + \eta^2(\phi_2') = U'^*\phi_2' \end{aligned} \tag{4.158}$$

under the transformation, derived from (4.137)

$$U' = U - 2\frac{d^2}{dx^2}\log\phi_1' \quad \text{where} \quad \phi_1' = \frac{1}{\phi_1} \tag{4.159}$$

and

$$U'^* = U^* - 2\frac{d^2}{dx^2}\log\phi_2' \quad \text{where} \quad \phi_2' = \frac{1}{\phi_2}. \tag{4.160}$$

Subtracting (4.160) from (4.159), we get

$$q_x' = q_x - i\frac{d^2}{dx^2}\log\Big(\frac{\phi_1}{\phi_2}\Big). \tag{4.161}$$

Let $q' = w_x'$ and $q = w_x$, and integrate (4.161) twice to yield

$$w' = w - i\log\Big(\frac{\phi_1}{\phi_2}\Big). \tag{4.162}$$

We use the property

$$\log\Big(\frac{\phi_1}{\phi_2}\Big) = 2i\arctan\Big(\frac{\psi_2}{\psi_1}\Big) \tag{4.163}$$

where $\phi_1 = \psi_1 + i\psi_2$ and $\phi_2 = \psi_1 - i\psi_2$, $(\phi_2 = \phi_1^*)$, then (4.162) becomes

$$w' = w + 2\arctan\Big(\frac{\psi_2}{\psi_1}\Big) \tag{4.164}$$

or

$$\frac{1}{2}(w'-w)=\arctan\Big(\frac{\psi_2}{\psi_1}\Big). \tag{4.165}$$

Since we define the function Γ as ψ_1/ψ_2, then

$$\Gamma=\cot\Big\{\frac{1}{2}(w'-w)\Big\}. \tag{4.166}$$

Now, insert (4.166) into (4.140) with $r=-q$ to yield

$$w_x+w'_x=2\eta\sin(w-w') \tag{4.167}$$

and insert (4.166) into (4.142) to get

$$w_t-w'_t=2A\sin(w'-w)-(B+C)\cos(w'-w)+B-C. \tag{4.168}$$

Therefore, (4.167) and (4.168) are the BT for this class. Let us use (4.132) to find the BT for the sine-Gordon equation. By inserting (4.132) into (4.167) and (4.168), we obtain

$$\frac{1}{2}(u_x+u'_x)=2\eta\sin\Big(\frac{1}{2}(u-u')\Big) \tag{4.169}$$

and

$$\frac{1}{2}(u_t-u'_t)=\frac{1}{2\eta}\sin\Big(\frac{1}{2}(u+u')\Big) \tag{4.170}$$

respectively. These auto-BT for the sine-Gordon equation are equal to (4.21) and (4.22) when $a=2\eta$.

To find the BT for the modified Korteweg-de Vries equation, we use (4.133). We insert (4.133) into (4.168) and, obtain

$$w_t+w'_t=8\eta^2w_x+4\eta\big(2\eta^2+w_x^2\big)\sin(w-w')-4\eta w_{xx}\cos(w'-w) \tag{4.171}$$

where w and w' satisfy the potential mKdV equation

$$w_t+2w_x^3+w_{xxx}=0. \tag{4.172}$$

Hence, (4.167) and (4.171) are the auto-BT for the mKdV equation that is given in (4.133).

We can consider that $r=q$ belongs to this class. However, this time we do not have to find the transformations that keep the equations in (4.128) invariant. If we let $u\to iu$, then the sine-Gordon equation $u_{xt}=\sin u$ becomes

$$u_{xt}=\sinh u \tag{4.173}$$

which is the sinh-Gordon equation. The BT (4.169) and (4.170) become

$$\frac{1}{2}(u_x + u'_x) = 2\eta \sinh\Big(\frac{1}{2}(u - u')\Big) \tag{4.174}$$

and

$$\frac{1}{2}(u_t - u'_t) = \frac{1}{2\eta} \sinh\Big(\frac{1}{2}(u + u')\Big) \tag{4.175}$$

which are equal to the auto-BT of the sinh-Gordon equation in (4.58) and (4.59) when $\lambda = 2\eta$. If we let $q \to iq$, then the mKdV equation corresponding to $r = -q$ becomes

$$q_t - 6q^2 q_x + q_{xxx} = 0 \tag{4.176}$$

which corresponds to $r = q$. The BT (4.167) and (4.171) become

$$w'_x - w_x = -2\eta \sinh(w + w') \tag{4.177}$$

and

$$w_t - w'_t = 8\eta^2 w_x + 4\eta\big(w_x^2 - 2\eta^2\big)\sinh(w + w') - 4\eta w_{xx} \cosh(w + w') \tag{4.178}$$

where w and w' satisfy the potential mKdV equation

$$w_t - 2w_x^3 + w_{xxx} = 0. \tag{4.179}$$

(4.177) and (4.178) are equal to the auto-BT for the mKdV in (4.55) and (4.56) when $\lambda = -2\eta$ through $q = w_x$.

The nonlinear evolution equations that belong to the same class have the same nonlinear superposition formula. Since the sine-Gordon equation and the mKdV equation corresponding to $r = -q$ are in the same class, then the solutions of the mKdV equation

$$q_t + 6q^2 q_x + q_{xxx} = 0 \tag{4.180}$$

satisfy the following nonlinear superposition

$$\tan\Big\{\frac{1}{2}(w_3 - w_0)\Big\} = \frac{\eta_1 + \eta_2}{\eta_1 - \eta_2} \tan\Big\{\frac{1}{2}(w_2 - w_1)\Big\}. \tag{4.181}$$

If $a_i = 2\eta_i$ and $w_i = -u_i/2$, then (4.181) provides us the nonlinear superposition for the sine-Gordon equation in (4.46) [18].

Class III : $(r = -q^*)$

When $r = -q^*$, the Riccati form of 2x2 eigenvalue problem, (4.140) becomes

$$\Gamma_x = 2\eta\Gamma + q + q^*\Gamma^2. \tag{4.182}$$

We define a second system

$$\Gamma'_x = 2\eta\Gamma' + q' + q'^*(\Gamma')^2. \tag{4.183}$$

Konno and Wadati [20] found that (4.182) can be mapped to (4.183) under the transformations

$$q' = q + 2\frac{\Gamma^2\Gamma_x^* - \Gamma_x}{1 - |\Gamma|^4} \tag{4.184}$$

or

$$q' = -q - 4\eta\frac{\Gamma}{1 + |\Gamma|^2} \tag{4.185}$$

and

$$\Gamma' = \frac{1}{\Gamma^*}. \tag{4.186}$$

If we insert (4.185) and (4.186) into (4.183), we obtain (4.182). Therefore, these transformations keep the Riccati equation (4.182) invariant. We find the function Γ as

$$\Gamma = -\frac{2\eta \pm \sqrt{4\eta^2 - |q' + q|^2}}{(q'^* + q^*)}. \tag{4.187}$$

It is easy to show that (4.187) satisfies the equation (4.185). Since the function Γ in (4.187) with either plus or minus sign provides us the same BT, then we shall use

$$\Gamma = -\frac{2\eta + \sqrt{4\eta^2 - |q' + q|^2}}{(q'^* + q^*)}. \tag{4.188}$$

By inserting (4.188) into (4.182), we yield

$$\begin{aligned}(q'^* + q^*)_x\{4\eta\sqrt{4\eta^2 - |q' + q|^2} + 8\eta^2 - |q' + q|^2\} &+ (q'^* + q^*)^2(q' + q)_x \\ = 4\eta(q^* - q'^*)\{2\eta\sqrt{4\eta^2 - |q' + q|^2} &+ 4\eta^2 - |q' + q|^2\} \\ - |q' + q|^2(q^* - q'^*)&\sqrt{4\eta^2 - |q' + q|^2} \\ + (q'^* + q^*)^2(q - q')&\sqrt{4\eta^2 - |q' + q|^2}\end{aligned} \tag{4.189}$$

then, from (4.189), we find the x-part of the BT as

$$q_x + q'_x = (q - q')\sqrt{4\eta^2 - |q + q'|^2}. \tag{4.190}$$

By inserting (4.188) into (4.142), we yield

$$\begin{aligned}
&(q'^* + q^*)^2(q' + q)_t + \left\{4\eta\sqrt{4\eta^2 - |q' + q|^2} + 8\eta^2 - |q' + q|^2\right\}(q'^* + q^*)_t \\
=&2\sqrt{4\eta^2 - |q' + q|^2}\left\{2\eta + \sqrt{4\eta^2 - |q' + q|^2}\right\}\left\{-2A(q'^* + q^*) - 4\eta C\right\} \\
&+ 2\sqrt{4\eta^2 - |q' + q|^2}(q'^* + q^*)\left\{B(q'^* + q^*) + C(q' + q)\right\}
\end{aligned} \tag{4.191}$$

which is the t-part of the BT and, if A, B, C are given, it can be simplified further. Therefore, (4.190) and (4.191) are the auto-BT for this class ($r = -q^*$).

We consider that $r = q^*$ belongs to this class and the transformations for this case are found as

$$\Gamma' = \frac{1}{\Gamma^*} \tag{4.192}$$

and

$$q' = q + 4\eta\frac{\Gamma}{1 - |\Gamma|^2}. \tag{4.193}$$

By solving (4.193), we find the function Γ as

$$\Gamma = -\frac{2\eta \pm \sqrt{4\eta^2 + |q' - q|^2}}{(q'^* - q^*)} \tag{4.194}$$

and, by inserting (4.194) into (4.140) and (4.142) with $r = q^*$, we can obtain the auto-BT for this class ($r = q^*$).

Now, let us use (4.134) and (4.191) to derive the BT for the nonlinear Schrödinger equation (NLSE)

$$q_t - 2iq^2q^* - iq_{xx} = 0. \tag{4.195}$$

By inserting A, B, C in (4.134) into (4.191), we get

$$\begin{aligned}
q_t + q'_t =&i(q_x - q'_x)\sqrt{4\eta^2 - |q + q'|^2} \\
&+ \frac{i}{2}(q + q')\left(|q + q'|^2 + |q - q'|^2\right).
\end{aligned} \tag{4.196}$$

Therefore, (4.190) and (4.196) are the auto-BT for the NLSE in (4.195). By using the transformations

$$x \to x - 2kt, \quad t \to t, \quad q \to \exp(ikx - ik^2t)q \tag{4.197}$$

which keep the nonlinear Schrödinger equation invariant, we can reduce (4.190) and (4.196) to the Bäcklund transformation found by Lamb [14].

By following the same procedure in the derivation of the nonlinear superposition for the KdV equation, we can find the nonlinear superposition for the NLSE by using (4.190). Therefore, the solutions of (4.195) must satisfy

$$\begin{aligned}(q_0 - q_1)\sqrt{4\eta_1^2 - |q_0 + q_1|^2} - (q_0 - q_2)\sqrt{4\eta_2^2 - |q_0 + q_2|^2} \\ + (q_2 - q_3)\sqrt{4\eta_1^2 - |q_2 + q_3|^2} - (q_1 - q_3)\sqrt{4\eta_2^2 - |q_1 + q_3|^2} = 0\end{aligned} \tag{4.198}$$

where q_1 and q_2 are generated by (4.190) from q_0 with η_1 and η_2 respectively, and q_3 are derived either from q_1 with η_2 or from q_2 with η_1 [44] .

We shall remark that class II is a special case of class III since the potentials are real.

Chapter 5

Conclusion

In this thesis, we have studied the method of inverse scattering and Bäcklund transformations to solve integrable nonlinear partial differential equations. The method of inverse scattering is used to solve the initial value problem of integrable evolution equations. We have found the Lax pairs of the KdV equation and the KdV hierarchy. Then we have solved the initial value problem of the KdV equation by the method of inverse scattering. We have derived one- and two-soliton solutions of the KdV equation by using the inverse scattering transform. Moreover, we have solved the initial value problem for the KdV hierarchy and observed that the speed of a soliton also depends on the order of the KdV hierarchy. We have applied the same method to the AKNS scheme which includes the KdV equation, the mKdV equation, the nonlinear Schrödinger equation, the sine-Gordon equation, and the sinh-Gordon equation. We have derived a solitary solution for the sine-Gordon equation.

We have studied the Bäcklund transformations which provide a particular solution of an integrable system. We have listed Bäcklund transformations for some known evolution equations. Then, we have derived the auto-Bäcklund transformations for the KdV equation and constructed one-, two-, and three-soliton solutions of the KdV equation by using the nonlinear superposition of that equation. Additionally, we have examined the technique for determining the N-soliton (pure soliton) solution of the KdV equation. We have derived the auto-Bäcklund transformations for the AKNS scheme which also includes the KdV hierarchy. We have found the nonlinear superposition formula for the KdV equation, the nonlinear Schrödinger equation and the sine-Gordon equation, which is the same as that for the mKdV equation.

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