A Theoretical Overview of Persistent Homology

by

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A Theoretical Overview of Persistent Homology

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Graduate School of Sciences and Engineering

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Anneme;

Bana verdiği emek, gösterdiği sevgi ve destekle buraya gelmemde en büyük rolü oynayan insana.

ABSTRACT

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This thesis presents a theoretical overview of persistent homology, a key method in topological data analysis for capturing the evolution of homological features across a filtration. The theory of persistence modules is developed within a general framework, including the interleaving and bottleneck distances, along with their fundamental properties. The classical Nerve Theorem is proved with particular emphasis on its applicability to manifolds. The persistent homological version of this theorem provides a theoretical justification for replacing topological spaces with computationally tractable simplicial complexes over a filtration. Moreover, the stability properties of persistence modules are discussed, and several results from the literature are reviewed.

ÖZETÇE

Kalıcı Homolojiye Kuramsal Bir Bakış
Kaan ÇİM
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Bu tez, kalıcı homolojinin kuramsal bir incelemesini sunmaktadır. Kalıcı homoloji, bir filtrasyon boyunca homolojik özelliklerin evrimini yakalamaya yarayan, topolojik veri analizinde temel bir yöntemdir. Kalıcılık modüllerinin kuramı genel bir çerçevede geliştirilmiş; serpiştirme (interleaving) ve darboğaz (bottleneck) uzaklıkları ile bunların özelliklerine yer verilmiştir. Klasik Sinir (Nerve) Teoremi kanıtlanmış, özellikle manifoldlar üzerindeki uygulanabilirliğine vurgu yapılmıştır. Bu teoremin kalıcı homoloji bağlamındaki sürümü, bir filtrasyon üzerinde topolojik uzayların hesaplanabilir simpleksel komplekslerle değiştirilmesine kuramsal bir gerekçe sunmaktadır. Ayrıca, kalıcılık modüllerinin kararlılık özellikleri ele alınmış ve yazındaki bazı önermelere yer verilmiştir.

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ABBREVIATIONS

$\mathbb{N}$	The Set of Natural Numbers
$\mathbb{Z}$	The Set of Integers
$\mathbb{R}$	The Set of Real Numbers
F	Field
$\mathbf{C}$	Category
$\cong$	Isomorphism in The Respective Category
$H_k(X)$	Homology in Degree k
P; $(\mathbf{P})$	Poset; (as Category)
$\mathbb{M}$ (or $\mathbb{V}$)	Persistence Module (over a Field)
$\mathcal{F}$	Filtration
$\mathcal{O}(X)$	Offset Filtration
$\check{\mathcal{C}}(X)$	Čech Filtration
$\mathcal{VR}(X)$	Vietoris Rips Filtration
$\mathcal{MB}(X, \mu)$	Measure Bifiltration
$\mathcal{SR}(Y)$	Subdivision Rips Bifiltration
$\mathbb{H}_k(\mathcal{F})$	Persistence Homology Module in Degree k
$\mathbb{H}_{k-sub}(X, f)$	Sublevelset Persistence Homology Module
$\mathcal{B}(\mathbb{V})$	Barcode
d_i	Interleaving Distance
$\mathrm{dgm}(\mathbb{V})$	Persistence Diagram
d_b	Bottleneck Distance
d_H	Hausdorff Distance
d_{GH}	Gromov-Hausdorff Distance
d_{Pr}	Prohorov Distance
d_{GPr}	Gromov-Prohorov Distance

Chapter 1

INTRODUCTION

Persistent homology has emerged as a powerful tool for capturing topological features that persist across scales. Rather than focusing on individual snapshots of a topological space, it encodes how these features evolve. It provides a systematic way to track the appearance and disappearance of homological structures, such as path-connected components, holes, and voids, as one moves through a filtration of a topological space. This theoretical framework has also found significant applications, particularly in data analysis. In this thesis, however, we focus on the mathematical foundations of persistent homology, rather than its applications.

This thesis aims to present the basic theoretical building blocks of persistent homology and to demonstrate how the persistent version of the classical Nerve Theorem provides theoretical justification for topological data analysis (TDA) on manifold-like data. The thesis explains the central concept of persistence modules, which establishes a general model for persistent homology. Through these modules, important distance measures such as interleaving and bottleneck distances are defined. The stability properties based on these distances provide practical assurance for applications.

To make persistent homology practically applicable, we need computationally accessible models of topological spaces. Simplicial complexes serve this purpose by providing combinatorial representations that retain essential topological features and are suitable for algorithmic analysis. A key theoretical result that bridges the gap between these combinatorial models and the underlying topology is the Nerve Theorem, which asserts that under appropriate conditions, the homotopy type of a space can be recovered from the nerve simplicial complex associated with the cover.

Persistent homology is designed to operate on topological spaces, but real-world data is rarely presented in a topological form. Instead, data is typically given as finite point clouds sampled from some unknown space. When observed from a distance, these point clouds often appear to trace out a coherent geometric shape, leading to the common assumption that the underlying space possesses the structure of a manifold. In this thesis, we give a detailed proof of the classical Nerve Theorem and explain how its persistent homological version allows us to carry out TDA techniques on data that is assumed to be sampled from manifolds.

Roadmap. This thesis consists of eight chapters. In Chapter 2, we provide the necessary background, covering basic concepts from topology, analysis, algebra, and category theory to establish a self-contained foundation. In Chapter 3, we focus on simplicial complexes, which serve as combinatorial models of topological spaces. In Chapter 4, we present a proof of the Nerve Theorem, building connections between simplicial complexes and their underlying topological spaces. This chapter also includes a detailed discussion of paracompactness, which is a prerequisite for the theorem and a fundamental property of manifolds. In Chapter 5, we review the basics of singular homology and simplicial homology. In Chapter 6, we develop the framework of persistence modules, along with the interleaving and bottleneck distances between them. In Chapter 7, we build on this structure to study persistent homology and state the persistent nerve theorem. In Chapter 8, we discuss the stability theorems that ensure that small perturbations in the data do not lead to drastic changes.

Chapter 2

BACKGROUND

This chapter contains the essential background required to comprehend the subsequent material presented in this thesis.

2.1 Topology

Since persistent homology fundamentally relies on topological notions, we first outline the necessary background in topology. This section is based on [21].

Definition 2.1. *Let X be a set and $\mathcal{T} \subseteq \mathcal{P}(X)$. $\mathcal{T}$ is called a* ***topology*** *on X and the pair $(X, \mathcal{T})$ is called a* ***topological space*** *if $\mathcal{T}$ satisfies the following conditions:*

1. *The empty set $\emptyset$ and the entire set X are in $\mathcal{T}$.*
2. *If $\{U_i\}_{i \in I}$ is a collection of sets in $\mathcal{T}$, then $\bigcup_{i \in I} U_i \in \mathcal{T}$.*
3. *For all $n \in \mathbb{N}$ if $U_1, U_2, \ldots, U_n \in \mathcal{T}$, then $\bigcap_{k=1}^{n} U_k \in \mathcal{T}$.*

Elements of $\mathcal{T}$ are called ***open sets*** *and their complements in X are called* ***closed sets****.*

Definition 2.2. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $f : X \to Y$ be a function. f is called* ***continuous*** *if for every $V \in \mathcal{T}_Y$, the preimage $f^{-1}(V) \in \mathcal{T}_X$. For a bijective function f, if both f and f^{-1} are continuous, then f (also f^{-1}) is called a* ***homeomorphism****, and in this case, $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are called* ***homeomorphic****. Thus, a homeomorphism preserves the topological structure and properties.*

Definition 2.3. *Let X be a set. A collection $\mathcal{B} \subseteq \mathcal{P}(X)$ is called a* ***basis*** *for a topology on X (i.e., a basis that generates a topology) if (i) for each $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$ and (ii) for all $B_1, B_2 \in \mathcal{B}$ and all $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. The collection of all unions of elements of $\mathcal{B}$ defines a topology $\mathcal{T}_{\mathcal{B}}$ on X, called the* ***topology generated by*** *$\mathcal{B}$. The elements of $\mathcal{B}$ are called* ***basis elements****.*

Now let $(X, \mathcal{T})$ be a topological space. A subcollection $\mathcal{B} \subseteq \mathcal{T}$ is called a ***basis for the topology*** *$\mathcal{T}$ if one of the following equivalent conditions holds:*

- *For every $U \in \mathcal{T}$, there exists a subcollection $\{B_i\}_{i\in I} \subseteq \mathcal{B}$ such that $U = \bigcup_{i\in I} B_i$.*
- *For every $U \in \mathcal{T}$ and every $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.*

Definition 2.4. *Let $(X, \mathcal{T})$ be a topological space and $x \in X$. A* ***local basis*** *at x is a collection of open sets $\mathcal{B}_x$ containing x such that for every open set U containing x, there exists $B \in \mathcal{B}_x$ such that $x \in B \subseteq U$.*

Remark 2.1. Let $(X, \mathcal{T})$ be a topological space. If $\{\mathcal{B}_x\}_{x\in X}$ is a collection of local bases then the set $\mathcal{B} = \bigcup_{x\in X} \mathcal{B}_x$ is a basis for $(X, \mathcal{T})$.

Definition 2.5. *Let $(X, \mathcal{T})$ be a topological space.*

X is called ***first countable*** *if every point $x \in X$ has a countable local basis.*

X is called ***second countable*** *if there exists a countable basis for the topology τ.*

Definition 2.6. *Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$.*

The ***interior*** *of A, denoted by A°, is defined as follows:*

$$A^\circ := \{x \in A \mid \exists U \in \mathcal{T} \text{ such that } x \in U \subseteq A\} = \bigcup_{U\subseteq A,\, U\in\mathcal{T}} U$$

The ***closure*** *of A, denoted by $\overline{A}$, is defined as follows:*

$$\overline{A} := \{x \in X \mid \forall U \in \mathcal{T}\ (x \in U \implies U \cap A \neq \emptyset)\} = \bigcap_{C\supseteq A,\, C \text{ closed}} C$$

Definition 2.7. *A topological space $(X, \mathcal{T})$ is called a* ***Hausdorff space*** *if for any $x, y \in X$ with $x \neq y$, there exist $U, V \in \mathcal{T}$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.*

Definition 2.8. *Let $(X, \mathcal{T}_X)$ be a topological space, and let $Y \subseteq X$ be a subset. The* ***subspace topology*** *on Y is defined as $\mathcal{T}_Y := \{U \cap Y \mid U \in \mathcal{T}_X\}$.*

Definition 2.9. *Let $(X, \mathcal{T}_X)$ be a topological space and let $\sim$ be an equivalence relation on X. Define the set of equivalence classes as $Y = X/\sim$, and define $q : X \to Y$ by $q(x) = [x]$. The* ***quotient topology*** *on Y is defined as $\mathcal{T}_Y := \{U \subseteq Y \mid q^{-1}(U) \in \mathcal{T}_X\}$.*

Definition 2.10. *Let $(X, \mathcal{T})$ be a topological space. A collection of open sets $\{U_i\}_{i\in I}$ is called an* ***open cover*** *of K if $K \subseteq \bigcup_{i\in I} U_i$.*

Definition 2.11. *Let $(X, \mathcal{T})$ be a topological space. A subset $K \subseteq X$ is called* ***compact*** *if every open cover of K has a finite subcover, i.e., for any open cover $\{U_i\}_{i \in I}$ of K there exists a finite subset $J \subseteq I$ such that $K \subseteq \bigcup_{j \in J} U_j$.*

Remark 2.2. Some basic properties of compact spaces are worth recalling:

- The continuous image of a compact set is compact.
- Any closed subspace of a compact space is compact.
- In a Hausdorff space, every compact subset is closed.

Definition 2.12. *A topological space $(X, \mathcal{T})$ is called* ***connected*** *if it cannot be written as the union of two non-empty, disjoint open subsets. A subset $A \subseteq X$ is* ***connected*** *if $(A, \mathcal{T}_A)$ is a connected space, where $\mathcal{T}_A$ is the subspace topology on A.*

Remark 2.3. The **connected component** of a point $x \in X$ is the union of all connected subsets of X that contain x. This is a connected subset of X and is the maximal connected subset containing x. The connected components of X form a partition of X.

Definition 2.13. *A topological space $(X, \mathcal{T})$ is called* ***path-connected*** *if for any two points $x, y \in X$, there exists a continuous map $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. The* ***path connected component*** *of $x \in X$ is a maximal path-connected subset of X containing x, and it contains y if there exists a continuous path from x to y.*

Remark 2.4. The continuous image of a connected (respectively, path-connected) set is connected (respectively, path-connected). Also, every path-connected space is connected.

Definition 2.14. *A* ***homotopy*** *between two continuous functions $f, g : X \to Y$ is a continuous map $H : X \times [0, 1] \to Y$ such that for all $x \in X$, we have $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. In this case, f and g are called* ***homotopic*** *and are denoted by $f \simeq g$. A continuous map $f : X \to Y$ is called a* ***homotopy equivalence*** *if there exists a continuous map $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. In this case, X and Y are called* ***homotopy equivalent****. A topological space X is called* ***contractible*** *if it is homotopy equivalent to a one-point space. This is equivalent to saying that the identity map id_X is homotopic to a constant map.*

Remark 2.5. Homotopy is an equivalence relation on the set of continuous functions from X to Y. Homotopy equivalence is an equivalence relation on the set of topological spaces.

2.2 Analysis

Many topological constructions rely on analytical notions. In this section, we briefly review the analytical background needed to formulate these concepts within a topological framework. This section covers standard material; see, for example, [21].

Definition 2.15. *A **metric space** (X, d) is a set X together with a function $d : X \times X \to \mathbb{R}$ called a **metric** such that for all $x, y, z \in X$*

1. *$d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.*
2. *$d(x, y) = d(y, x)$.*
3. *$d(x, z) \leq d(x, y) + d(y, z)$.*

*The **open ball** with **center** $x \in X$ and **radius** $r > 0$ is the set $B(x,r) := \{y \in X \mid d(x, y) < r\}$. The **closed ball** with **center** $x \in X$ and **radius** $r > 0$ is the set $\overline{B(x,r)} := \{y \in X \mid d(x, y) \leq r\}$. A subset $U \subseteq X$ is called **open** if for every $x \in U$, there exists an $\epsilon > 0$ such that the open ball $B_d(x, \epsilon) \subset U$. The collection of such open sets defines the **metric topology** on (X, d).*

*Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is called an **isometry** if for all $x_1, x_2 \in X$, $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$.*

Remark 2.6. Metric spaces are Hausdorff and first countable.

Proposition 2.1. *Let $f : X \to Y$ be a function from a metric space (X, d_X) to a metric $Y, (d_Y)$. f is continuous at $x \in X$ if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $y \in X$, $d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon$.*

Definition 2.16. *Let $f : X \to Y$ be a function between two metric spaces (X, d_X) and (Y, d_Y). The function f is **uniformly continuous** if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x_1, x_2 \in X$, if $d_X(x_1, x_2) < \delta$, then $d_Y(f(x_1), f(x_2)) < \epsilon$.*

Proposition 2.2. *If $f : X \to Y$ is a continuous function from a compact metric space X to a metric space Y, then f is uniformly continuous.*

Definition 2.17. *A subset A of a metric space (X, d) is called **bounded** if there exists a real number $M > 0$ and a point $x \in X$ such that for all $y \in A$, $d(x, y) < M$. A metric space (X, d) is called **totally bounded** if for every $\epsilon > 0$, there exists a finite cover of X by open balls with radius ϵ.*

2.3 Algebra

This section reviews related algebraic notions (see, for example, [3]) to prepare for the algebraic machinery of persistent homology.

Definition 2.18. *A **ring** is a set R equipped with two binary operations $+$ (addition) and $\cdot$ (multiplication) such that the following properties hold:*

1. $(R, +)$ *is an abelian group, i.e.,*
   (a) *For all* $a, b, c \in R$, $(a + b) + c = a + (b + c)$.
   (b) *There exists an element* $0 \in R$ *such that for all* $a \in R$, $0 + a = a + 0 = a$.
   (c) *For each* $a \in R$, *there exists an element* $b \in R$ *such that* $a + b = b + a = 0$.
   (d) *For all* $a, b \in R$, $a + b = b + a$.
2. $(R, \cdot)$ *satisfies* $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ *for all* $a, b, c \in R$.
3. *For all* $a, b, c \in R$,
   (a) $a \cdot (b + c) = a \cdot b + a \cdot c$.
   (b) $(a + b) \cdot c = a \cdot c + b \cdot c$.

*A ring R is called a **commutative ring with unity** if also*

4. *There exists an element* $1 \in R$ *called **unity** such that for all* $a \in R$, $a \cdot 1 = 1 \cdot a = a$.
5. *For all* $a, b \in R$, $a \cdot b = b \cdot a$.

Definition 2.19. *A **field** is a commutative ring F with unity such that $1 \neq 0$ and every non-zero element of F has a multiplicative inverse, i.e., for every $a \in F$ with $a \neq 0$, there exists an element $a^{-1} \in F$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$.*

Definition 2.20. *Let R be a commutative ring with unity. An R-**module** is an abelian group $(M, +_M)$ equipped with a function $R \times M \to M$ (called **scalar multiplication**) sending $(r, m) \mapsto rm$ such that for all $r, s \in R$ and $m, n \in M$*

1. $r(m +_M n) = rm +_M rn$.
2. $(r +_R s)m = rm +_M sm$.
3. $r(sm) = (rs)m$.
4. $1_R m = m$, *where* 1_R *is the multiplicative identity in* R.

Remark 2.7. Let R be a field. An R-module is also known as a **vector space** over R. R-module is a natural generalization of the concept of vector space from fields to rings.

Definition 2.21. *Let R be a commutative ring with unity and M be an R-module. A nonempty subset $N \subseteq M$ is called a* ***submodule*** *of M if N is an R-module under the restriction of addition and scalar multiplication of M on N. Equivalently, for all $m, n \in N$, $m +_M n \in N$ and for all $r \in R$ and $n \in N$, $rn \in N$.*

Definition 2.22. *Let M be an R-module and N be a submodule of M. Since M is an abelian group, every subgroup is normal, and in particular N is a normal subgroup of M. Hence, the set of cosets M/N is equipped with the well-defined operations (i) addition $(m+N)+(n+N) := (m+n)+N$ and (ii) scalar multiplication $r(m+N) := (rm)+N$ for all $m, n \in M$ and $r \in R$. With these operations, M/N becomes an R-module and is called the* ***quotient module****.*

Definition 2.23. *A* ***R-module homomorphism*** *(or* ***R-linear map****) is a map $\varphi : M \to N$ between two R-modules M and N such that for all $m, m' \in M$ and $r \in R$ (i) $\varphi(m +_M m') = \varphi(m) +_N \varphi(m')$ and (ii) $\varphi(r \cdot_M m) = r \cdot_N \varphi(m)$.*

The ***kernel*** *of φ is $\ker(\varphi) := \{m \in M \mid \varphi(m) = 0_N\}$, where 0_N is the zero element in N. The* ***image*** *of φ is $\mathrm{im}(\varphi) := \{\varphi(m) \mid m \in M\}$.*

An ***R-module isomorphism*** *is an R-module homomorphism $\varphi : M \to N$ that has an inverse R-module homomorphism $\varphi^{-1} : N \to M$ such that $\varphi \circ \varphi^{-1} = \mathrm{id}_N$ and $\varphi^{-1} \circ \varphi = \mathrm{id}_M$. This is equivalent to being a bijective R-module homomorphism. If such a R-module homomorphism exists, M and N are called* ***isomorphic****.*

Proposition 2.3. *Let $\varphi : M \to N$ be an R-module homomorphism. Then the image of φ is an R-module and $M/\ker(\varphi) \cong \mathrm{im}(\varphi)$.*

Definition 2.24. *Let $\{M_i\}_{i\in I}$ consist of R-modules. The* ***direct sum*** *of the family $\{M_i\}_{i\in I}$, denoted by $\bigoplus_{i\in I} M_i$, is the R-module consisting of all tuples $(m_i)_{i\in I}$ where $m_i \in M_i$ and $m_i = 0$ for all but finitely many $i \in I$. The* ***direct product*** *of the family $\{M_i\}_{i\in I}$, denoted by $\prod_{i\in I} M_i$, is the R-module consisting of all tuples $(m_i)_{i\in I}$ where $m_i \in M_i$. In both constructions, addition and scalar multiplication are defined component-wise.*

Definition 2.25. *Let R be a commutative ring. An R-module M is called a* ***free module*** *if it is isomorphic to a direct sum of copies of R, i.e., $M \cong \bigoplus_{i\in I} R$ for some index set I.*

Definition 2.26. *Let M be an R-module. A set $\{e_i\}_{i \in I} \subseteq M$ is called a **basis** for M if every element $m \in M$ can be uniquely expressed as a finite linear combination $m = \sum_{i \in I} r_i e_i$, where $r_i \in R$ and all but finitely many r_i are zero.*

Proposition 2.4. *An R-module M is free if and only if it has a basis.*

Remark 2.8. Every vector space is a free module over its underlying field. Hence, every vector space has a basis.

Definition 2.27. *A **vector space** V over a field F is an F-module. In this case, an F-module homomorphism is also known as a **linear map**. Moreover, all bases of a vector space V have the same cardinality. The **dimension** of V, denoted by $\dim(V)$, is the cardinality of any basis. The space V is called **finite-dimensional** if it has a finite basis, and **infinite-dimensional** otherwise.*

Remark 2.9. If $T : V \to W$ is a linear map between vector spaces and V is a finite-dimensional vector space then $\dim(V) = \dim(\ker(T)) + \dim(\mathrm{im}(T))$. This result is known as the rank-nullity theorem.

2.4 Category Theory

A basic familiarity with category theory is useful for developing ideas related to persistent homology. See, for example, the lecture notes [22] for an introduction.

Definition 2.28. *A **category** $\mathcal{C}$ consists of (i) a collection of **objects**, denoted by Ob($\mathcal{C}$), (ii) a collection of **morphisms** (also called **arrows**), denoted by $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ for each pair of objects $X, Y \in$ Ob($\mathcal{C}$) (and X is called **domain** or **source** and Y is called **codomain** or **range** for those morphisms), such that*

1. *For each pair of morphisms $f : X \to Y$ and $g : Y \to Z$, there exists a morphism $g \circ f : X \to Z$, called their **composition**.*
2. *For all $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$, $h \circ (g \circ f) = (h \circ g) \circ f$.*
3. *For each object $Z \in$ Ob($\mathcal{C}$), there is an **identity morphism** $\mathrm{id}_Z : Z \to Z$ such that for each $f : X \to Y$, $\mathrm{id}_Y \circ f = f$ and $f \circ \mathrm{id}_X = f$.*

Definition 2.29. *An **isomorphism** in a category $\mathcal{C}$ is a morphism $f : A \to B$ such that there exists a morphism $g : B \to A$ satisfying $g \circ f = id_A$ and $f \circ g = id_B$. $\cong$ denotes isomorphism in the given category.*

Example: The category **Top** has topological spaces as its objects and continuous functions as its morphisms. Category isomorphisms are homeomorphisms.

Definition 2.30. *A **functor** F from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a mapping that assigns (i) each object $X \in Ob(\mathcal{C})$ to an object $F_X \in Ob(\mathcal{D})$, (ii) each morphism $f : X \to Y$ in $\mathcal{C}$ to a morphism $F_f : F_X \to F_Y$ in $\mathcal{D}$ such that*

1. $F_{\mathrm{id}_X} = \mathrm{id}_{F_X}$ *for every object $X \in \mathcal{C}$.*
2. $F_{g \circ f} = F_g \circ F_f$ *for all morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$.*

Definition 2.31. *A **natural transformation** $\eta : F \Rightarrow G$ between two functors $F, G : \mathcal{C} \to \mathcal{D}$ is a collection of morphisms $\{\eta_X : F_X \to G_X\}_{X \in \mathrm{Ob}(\mathcal{C})}$ of $\mathcal{D}$ such that for every morphism $f : X \to Y$ in $\mathcal{C}$, the following diagram commutes:*

$$\begin{array}{ccc} F_X & \xrightarrow{F_f} & F_Y \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ G_X & \xrightarrow{G_f} & G_Y \end{array}$$

2.5 Auxiliary

Definition 2.32. *A **poset** (partially ordered set) is a set P equipped with a **partial order** $\leq$ satisfying (i) $p \leq p$ for all $p \in P$, (ii) if $p \leq q$ and $q \leq p$, then $p = q$, (iii) if $p \leq q$ and $q \leq r$, then $p \leq r$. A **totally ordered set** is a poset T in which every pair of elements is comparable, i.e., for all $x, y \in T$, either $x \leq y$ or $y \leq x$.*

Proposition 2.5 (Zorn's Lemma). *Let P be a poset. Assume that every totally ordered subset $T \subseteq P$ has an upper bound, i.e., there exists $u \in P$ such that $t \leq u$ for all $t \in T$. Then P contains a **maximal element**, i.e., an element $m \in P$ such that there is no $p \in P$ with $m < p$.*

Remark 2.10. Zorn's Lemma is equivalent to the axiom of choice.

Chapter 3

SIMPLICIAL COMPLEXES

A simplicial complex is a combinatorial object constructed from simplices such as points, edges, triangles, and their higher-dimensional analogues. This section introduces the formal definition of simplicial complexes and discusses their geometric realization. In this chapter, we follow the lecture notes [22].

Definition 3.1. *Let V be a nonempty set. An* ***abstract simplicial complex*** *K on V is a collection of nonempty subsets of V such that (i) each element of K has finite cardinality, (ii) $\{v\} \in K$ for all $v \in V$, (iii) if $\sigma \in K$ and $\tau \subseteq \sigma$ then $\tau \in K$. The elements of V are called* ***vertices****. The elements of K are called* ***simplices****. A simplex with $k+1$ vertices is called a* ***k-simplex****. The* ***dimension*** *of a simplex $\sigma \in K$ is $|\sigma| - 1$, where $|\sigma|$ denotes the cardinality. So, a k-simplex has dimension k. The* ***dimension*** *of the simplicial complex K is the maximum dimension of its simplices. The set of all i-simplices in K is denoted by K_i. Note that K_0 can be taken as V. A* ***face*** *of a simplex σ is any nonempty subset of σ.*

Definition 3.2. *Let K and L be abstract simplicial complexes. A* ***simplicial map*** *$f : K \to L$ is a function $f : K_0 \to L_0$ such that for every simplex $\sigma \in K$, the image $f(\sigma)$ is a simplex in L, i.e., $\forall \sigma \in K, f(\sigma) := \{f(v) \mid v \in \sigma\} \in L$.*

Definition 3.3. *A* ***simplicial isomorphism*** *is a simplicial map $f : K \to L$ that has an inverse simplicial map $g : L \to K$, i.e., $g \circ f = id_K$ and $f \circ g = id_L$.*

Remark 3.1. The **category of simplicial complexes**, denoted by $\mathbf{Simp}$, has abstract simplicial complexes as objects and simplicial maps as morphisms.

Definition 3.4. *Let $x_1, x_2, \ldots, x_k$ be points in $\mathbb{R}^n$.*

- *A* ***linear combination*** *of the points $x_1, x_2, \ldots, x_k$ is a sum of the form $\sum_{i=1}^{k} \lambda_i x_i$ where $\lambda_i \in \mathbb{R}$ for all i.*

- *An* ***affine combination*** *of the points $x_1, x_2, \ldots, x_k$ is a linear combination where the coefficients sum to one, i.e., $\sum_{i=1}^{k} \lambda_i x_i$ with $\sum_{i=1}^{k} \lambda_i = 1$.*
- *A* ***convex combination*** *of the points $x_1, x_2, \ldots, x_k$ is an affine combination with non-negative coefficients, i.e., $\sum_{i=1}^{k} \lambda_i x_i$ with $\sum_{i=1}^{k} \lambda_i = 1$ and $\lambda_i \geq 0$ for all i.*

Definition 3.5. *A set $C \subseteq \mathbb{R}^n$ is called* ***convex*** *if for any two points $x, y \in C$ and any $\lambda \in [0, 1]$, the point $\lambda x + (1 - \lambda)y \in C$.*

Definition 3.6. *The* ***convex hull*** *of a set $S \subseteq \mathbb{R}^n$, denoted by $\operatorname{conv}(S)$, is the smallest convex set containing S. This is equivalent to the set of all convex combinations of points in S, i.e., $\operatorname{conv}(S) := \left\{ \sum_{i=1}^{k} \lambda_i x_i \mid k \geq 1,\, x_i \in S,\, \lambda_i \geq 0,\, \sum_{i=1}^{k} \lambda_i = 1 \right\}$.*

Definition 3.7. *A set of points $\{x_0, x_1, \ldots, x_k\} \subseteq \mathbb{R}^n$ is called* ***affinely independent*** *if the set of vectors $\{x_1 - x_0, x_2 - x_0, \ldots, x_k - x_0\}$ is linearly independent. The* ***convex hull*** *of any finite set of points in $\mathbb{R}^n$ is called a* ***geometric simplex****. A* ***geometric k-simplex*** *is the convex hull of a set of affinely independent points $\{x_0, x_1, \ldots, x_k\} \subseteq \mathbb{R}^n$.*

Definition 3.8. *Let K be an abstract simplicial complex. Let $\varphi : K_0 \to \mathbb{R}^n$ be a function that assigns each vertex of K a point in $\mathbb{R}^n$.*

The ***geometric realization*** *of K* ***with respect to*** *φ, denoted by $|K|_\varphi$, is the union of the geometric simplices spanned by the images of the simplices in K under φ, i.e.,*

$$|K|_\varphi := \bigcup_{\sigma \in K} \operatorname{conv}(\varphi(\sigma)).$$

$\varphi : K_0 \to \mathbb{R}^n$ is called an ***affine embedding*** *if φ maps vertices of each simplex to affinely independent points in $\mathbb{R}^n$. In this case, the geometric realization $|K|_\varphi$ is called the* ***affine geometric realization*** *of K.*

Remark 3.2. If K_0 is infinite, we can define the geometric realization of K inside the space $\mathbb{R}^{K_0}$ as $|K| := \left\{ (x_i)_{i \in K_0} \in \mathbb{R}^{K_0} \;\middle|\; \sum_{i \in K_0} x_i = 1,\, x_i \geq 0 \text{ and } \{i \in K_0 \mid x_i > 0\} \in K \right\}$. This sum is well-defined since every simplex in K has finite cardinality.

Remark 3.3. The topology of $|K|_\varphi$ is independent of the choice of the affine embedding φ. Consequently, we can denote the affine geometric realization of a simplicial complex K by $|K|$, without specifying φ, and $|K|$ is called the **geometric realization**.

Example: The **standard** $(n-1)$**-simplex** Δ^{n-1} is defined as the set of all convex combinations of the standard basis vectors $e_1, e_2, \ldots, e_n$ in $\mathbb{R}^n$, i.e.,

$$\Delta^{n-1} = \left\{ \sum_{i=0}^{n-1} \lambda_i e_{i+1} \mid \lambda_i \geq 0, \sum_{i=0}^{n-1} \lambda_i = 1 \right\}$$

where e_i is the i-th standard basis vector in $\mathbb{R}^n$. A **solid** $(n-1)$**-simplex** is a simplicial complex K whose vertex set is $V = \{v_0, \ldots, v_{n-1}\}$ and whose simplices are all non-empty subsets of V. Define an affine embedding $\varphi : K_0 \to \mathbb{R}^n$ by mapping each vertex v_i to the standard basis vector e_{i+1} for $i = 0, 1, \ldots, n-1$. Then $|K| = |K|_\varphi = \Delta^{n-1}$.

Proposition 3.1. *The* ***geometric realization*** *is a functor* $|-| : \mathbf{Simp} \to \mathbf{Top}$ *defined as*

- ***On Objects****: For a simplicial complex* K*, the functor assigns its geometric realization* $|K|$*.*
- ***On Morphisms****: For a simplicial map* $f : K \to L$ *between simplicial complexes, the functor assigns a continuous map* $|f| : |K| \to |L|$ *as follows: Let* $\phi : K_0 \to \mathbb{R}^{|K_0|}$ *and* $\psi : L_0 \to \mathbb{R}^{|L_0|}$ *be affine embeddings. Then for any point* $x \in |K| = |K|_\phi$*, which can be uniquely written as* $x = \sum_i t_i \cdot \phi(v_i)$ *with* $v_i \in K_0$*,* $t_i \geq 0$*,* $\sum_i t_i = 1$*, we define* $|f|(x) := \sum_i t_i \cdot \psi(f(v_i))$*. This map* $|f| : |K| \to |L|$ *is continuous and is called the* ***geometric realization of*** f*.*

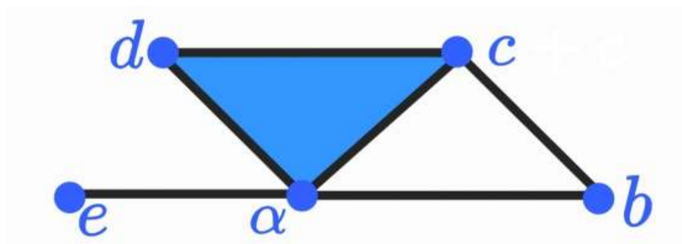

Figure 3.1: The Geometric Realization of The Simplicial Complex K Given Below

Example: Figure 3.1 illustrates the geometric realization of the simplicial complex

$$K := \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{a,e\}, \{a,d\}, \{a,c\}, \{c,d\}, \{a,b\}, \{b,c\}, \{a,c,d\}\}.$$

Definition 3.9. *A* ***triangulation*** *of a topological space* X *is a pair* (K, h)*, where* K *is a simplicial complex and* $h : |K| \to X$ *is a homeomorphism. The space* X *is called* ***triangulable*** *if such a triangulation exists.*

Definition 3.10. *Let K be a finite simplicial complex. The* ***Euler characteristic*** *$\chi(K)$ of K is defined as*

$$\chi(K) = \sum_{k=0}^{\dim K} (-1)^k c_k,$$

where c_k denotes the number of k-simplices in K.

Remark 3.4. The Euler characteristic is a topological invariant, meaning it depends only on the topological structure of the triangulable space X and not on a particular triangulation. In other words, if K and L are two triangulations of the same topological space X, then the Euler characteristic of K is equal to the Euler characteristic of L, i.e., $\chi(K) = \chi(L)$.

Definition 3.11. *The* ***barycentric subdivision*** *of a simplicial complex K, denoted by $\mathrm{Sd}(K)$, is a new simplicial complex whose vertices correspond to the simplices of K. The simplices of $\mathrm{Sd}(K)$ are given by finite sequences of simplices*

$$\sigma_0 \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_k$$

in K, where each $\sigma_i \in K$, and $\subsetneq$ denotes the strict face relation. Each such sequence corresponds to a k-simplex in $\mathrm{Sd}(K)$.

Proposition 3.2. *The geometric realizations of a simplicial complex K and its barycentric subdivision Sd(K) are homeomorphic, i.e., $|K| \cong |Sd(K)|$.*

Chapter 4

NERVE THEOREM

One of the conditions of the classical Nerve theorem is paracompactness. As will be shown in this section, every paracompact Hausdorff space admits a partition of unity, a crucial instrument in proving the theorem. Since manifolds are known to be paracompact, a property that will be demonstrated, and Hausdorff by definition, the conditions of the theorem are satisfied in this setting as well. Hence, the Nerve Theorem holds for manifolds.

4.1 Paracompactness

Definition 4.1. [21] *A collection of sets $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ in a topological space X is called **locally finite** if for every point $x \in X$, there exists an open neighborhood*[1] *$V \subseteq X$ of x such that V intersects only finitely many sets in $\mathcal{U}$, i.e., the set $\{U_\alpha \in \mathcal{U} : V \cap U_\alpha \neq \emptyset\}$ is finite.*

*A cover $\mathcal{V}$ of a space X is a **refinement** of another cover $\mathcal{U}$ of X if for every $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $V \subseteq U$.*

*A topological space X is called **paracompact** if every open cover of X has an open locally finite refinement.*

Remark 4.1. Separable[2] metric spaces are paracompact. By assuming the axiom of choice, all metric spaces are paracompact [3].

Definition 4.2. [21] *Let $\varphi : X \to \mathbb{R}$ be a continuous function. Then **the support of** φ is defined as* $\operatorname{supp}(\varphi) := \overline{\{x \in X \mid \varphi(x) \neq 0\}}$.

---

[1] A set V is a neighborhood of x if there exist an open set U such that $x \in U \subset V$.

[2] A space is separable if the space has a countable subset whose closure gives the whole space.

[3] [21], Theorem 41.4.

Definition 4.3. [21] *Let X be a topological space. A collection of continuous functions $\{\varphi_i\}_{i\in I}$ on X is called a* ***partition of unity*** *if it satisfies the following conditions:*

1. *Each function φ_i takes values in $[0,1]$, i.e., $\varphi_i : X \to [0,1]$.*
2. *The collection of supports $\{\mathrm{supp}(\varphi_i)\}_{i\in I}$ is locally finite, i.e., for every point $x \in X$, there exists an open neighborhood $U \subseteq X$ of x such that U intersects only finitely many of the supports $\mathrm{supp}(\varphi_i)$.*
3. *For each point $x \in X$, $\sum_{i\in I} \varphi_i(x) = 1$.*

Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ be an open cover of X. A partition of unity $\{\varphi_i\}_{i\in I}$ is called ***subordinate*** *to $\mathcal{U}$ if for each $i \in I$, the support of φ_i is contained in some U_α, i.e.,*

$$\mathrm{supp}(\varphi_i) \subseteq U_\alpha \text{ for some } \alpha \in A.$$

Remark 4.2. The sum in (3) is well-defined and continuous because the local finiteness condition (2) ensures that, at each point $x \in X$, only finitely many terms in the sum are nonzero. Some authors omit the local finiteness condition in the definition and instead assume that the sum in (3) is already well-defined and continuous. However, adding condition (2) is harmless and often technically convenient. Indeed, there is a theorem stating that for any collection of continuous functions $\{\psi_i\}$ on X with values in $[0,1]$ such that $\sum_i \psi_i(x) = 1$ for all x, one can construct a locally finite partition of unity $\{\varphi_i\}$ (with the same index set) such that the support of φ_i is contained in the support of ψ_i; that is, the collection $\{\mathrm{supp}(\varphi_i)\}$ is a refinement of $\{\mathrm{supp}(\psi_i)\}$. Our version of the definition, which explicitly includes the local finiteness condition, is often preferred because it simplifies dealing with technical arguments.

Proposition 4.1. *Assume that every open cover of a topological space X admits a partition of unity subordinate to the given cover. Then X is paracompact.*

Proof. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ be an arbitrary open cover of X. Then there exists a partition of unity $\{\varphi_i\}_{i\in I}$ such that for each i, the support $\mathrm{supp}\,\varphi_i = \overline{\{x \in X : \varphi_i(x) \neq 0\}}$ is contained in some member $U_{\alpha(i)} \in \mathcal{U}$. Let $V_i := \{x \in X : \varphi_i(x) > 0\}$. Then

- Each V_i is open.
- The collection $\{V_i\}_{i\in I}$ covers X (At each point $x \in X$, at least one $\varphi_i(x) > 0$).

- Since $V_i \subseteq \mathrm{supp}(\varphi_i)$, and the collection $\{\mathrm{supp}(\varphi_i)\}_{i\in I}$ is locally finite, the collection $\{V_i\}_{i\in I}$ is also locally finite.
- Each $i \in I$, we have $V_i \subseteq \mathrm{supp}(\varphi_i) \subseteq U_{\alpha(i)}$.

Thus, $\{V_i\}_{i\in I}$ is a locally finite open refinement of the open cover $\mathcal{U}$. Since $\mathcal{U}$ is arbitrary, this shows that X is paracompact. □

The existence of a partition of unity is central to the proof of the classical nerve theorem. Proposition 4.1 shows the necessity of the paracompactness condition.

Definition 4.4. [21] *A topological space X is called **normal** if for every pair of disjoint closed sets $A, B \subset X$, there exist disjoint open sets U, V such that $A \subset U$ and $B \subset V$.*

Proposition 4.2. [21, Theorem 41.1.] *Hausdorff paracompact spaces are normal.*

Proof. Let X be a paracompact Hausdorff space. We will first prove that X is regular. Take an arbitrary $x \in X$ and an arbitrary closed set $A \subset X$ not containing x. Regularity means that we will find disjoint open neighborhoods U_x and U_A of x and A, respectively.

Since X is Hausdorff, for each $a \in A$, there exist disjoint open sets O_a containing x and U_a containing a. The collection $\{U_a\}_{a\in A}$ is an open cover of A, and together with $X \setminus A$, it forms an open cover of X.

By paracompactness of X, there exists a locally finite open refinement $\{V_a\}_{a\in A} \cup \{X \setminus A\}$ such that $V_a \subset U_a$ for each a (Without loss of generality, we may assume that the refinement has the same index set as the original cover by grouping the elements of the refinement according to which element of the original cover they are contained in, and taking their unions.). Define $U_A := \bigcup_{a\in A} V_a$, which is an open neighborhood of A. By local finiteness, there exists an open set Y_x containing x and a finite subset $F \subset A$ such that Y_x intersects only finitely many V_a, i.e., $Y_x \cap V_a = \emptyset$ for $a \notin F$. Define $U_x := Y_x \cap \bigcap_{c\in F} O_c$. This is an open set containing x. Also, $U_x \cap U_A = \emptyset$. So, X is regular.

Now we will show that X is normal. Take two disjoint closed subsets $A, B \subset X$. We will find open neighborhoods $U_A \supset A$ and $U_B \supset B$.

By regularity, for each $a \in A$, there exist disjoint open sets U_a containing a and U_{B_a} containing B. The collection $\{U_a\}_{a\in A}$ together with $X \setminus A$ forms an open cover of

X. Using paracompactness, without loss of generality, there is a locally finite refinement $\{V_a\}_{a\in A} \cup \{X \setminus A\}$ such that $V_a \subset U_a$.

Define $U_A := \bigcup_{a\in A} V_A$, which is an open neighborhood of A. By locally finiteness, each $b \in B$ has an open neighborhood Y_b and a finite set $F_b \subset A$ such that $Y_b \cap V_a = \emptyset$ for all $a \notin F_b$. Define $U_b := Y_b \cap \bigcap_{a\in F_b} U_{B_a}$. This is open, it contains b, and avoids U_A. Setting $U_B := \bigcup_{b\in B} U_b$, we get an open set containing B and disjoint from U_A. Hence, X is normal. □

Proposition 4.3. [21, Exercise 36.4.](Shrinking Lemma) *Let X be a normal space.*

1. *Suppose $\{U, U'\}$ is an open cover of X. Then there exists an open set V such that $\overline{V} \subset U$ and $V \cup U' = X$.*
2. *Suppose $\{U_\alpha\}_{\alpha=1}^{\infty}$ is a locally finite open cover of X. Then there exist open sets $\{V_\alpha\}_{\alpha=1}^{\infty}$ such that they cover X and $\overline{V_\alpha} \subset U_\alpha$.*
3. *Suppose $\{U_\alpha\}_{\alpha\in A}$ is a locally finite open cover of X. Assuming the axiom of choice, there exist open sets $\{V_\alpha\}_{\alpha\in A}$ such that they cover X and $\overline{V_\alpha} \subset U_\alpha$.*

Proof. The first item will be used for the proofs of the second and the third items.

1. The closed sets U^c and $(U')^c$ are disjoint since $U \cup U' = X$. By the normality of X, there exist disjoint open sets V and V' such that $(U')^c \subset V$ and $U^c \subset V'$. Then

- $V \cup U' = X$ since every point is either in U' or in $(U')^c \subset V$.
- $V \cap V' = \emptyset$ implies $V \subset (V')^c$.
- $(V')^c$ is closed, so $\overline{V} \subset (V')^c$.
- $\overline{V} \subset U$, since $U^c \subset V'$ implies $(V')^c \subset U$.

Hence, V is open set satisfying $\overline{V} \subset U$ and $V \cup U' = X$.

2. We proceed with the construction of this cover by induction on $\alpha \in \mathbb{N}$. Define $U_1' := \bigcup_{\alpha>1} U_\alpha$. Then $\{U_1, U_1'\}$ is an open cover of X, so by part (1), there exists open $V_1 \subset X$ such that $\overline{V_1} \subset U_1$ and $V_1 \cup U_1' = X$. Suppose $V_1, \dots, V_{n-1}$ have been constructed satisfying $\overline{V_k} \subset U_k$ for $k < n$ and $\{V_1, \dots, V_{n-1}, U_n, U_{n+1} \cdots\}$ covers X. Let

$$U_n' := \left(\bigcup_{\alpha<n} V_\alpha\right) \cup \left(\bigcup_{\alpha>n} U_\alpha\right).$$

Then $\{U_n, U_n'\}$ is an open cover of X, so by part (1), there exists an open set $V_n \subset X$ such that $\overline{V_n} \subset U_n$ and $V_n \cup U_n' = X$. This defines the collection $\{V_\alpha\}_{\alpha=1}^{\infty}$.

To verify that the collection $\{V_\alpha\}_{\alpha=1}^{\infty}$ covers X, fix $x \in X$. Note that local finiteness implies point-finiteness, each point of X lies in only finitely many U_α. Since the original cover $\{U_\alpha\}$ is locally finite, there exists $\beta \in \mathbb{N}$ such that $x \notin U_\alpha$ for all $\alpha > \beta$. Then, by construction of V_β, $\{V_1, \ldots, V_\beta\} \cup \{U_{\beta+1}, U_{\beta+2} \cdots\}$ covers X. Hence, $x \in \bigcup_{\alpha \leq \beta} V_\alpha \subset \bigcup_{\alpha=1}^{\infty} V_\alpha$. Since $x \in X$ is arbitrary, this finishes that $\{V_\alpha\}$ is an open cover of X with $\overline{V_\alpha} \subset U_\alpha$.

3. Define $\mathcal{P}$ to be the set of pairs $(B, \mathcal{W})$, where $B \subset A$ and $\mathcal{W} = \{W_\alpha\}_{\alpha \in A}$ is a collection of open sets satisfying

(i) $\overline{W_\alpha} \subset U_\alpha$ for all $\alpha \in B$,

(ii) $W_\alpha = U_\alpha$ for all $\alpha \notin B$,

(iii) $\{W_\alpha\}_{\alpha \in A}$ is an open cover of X.

Define a partial order on $\mathcal{P}$ by

$$(B_1, \mathcal{W}_1) \leq (B_2, \mathcal{W}_2) \iff B_1 \subseteq B_2 \text{ and } W_{1,\alpha} = W_{2,\alpha} \text{ for all } \alpha \in B_1.$$

Let $T \subset \mathcal{P}$ be a totally ordered subset. Define

$$C := \bigcup_{(B,\mathcal{W}) \in T} B, \text{ and set } Y_\alpha := \begin{cases} W_\alpha & \text{for any } (B, \mathcal{W}) \in T \text{ with } \alpha \in B, \\ U_\alpha & \text{if } \alpha \notin C. \end{cases}$$

This is well-defined since T is totally ordered. We claim $\mathcal{Y} := \{Y_\alpha\}_{\alpha \in A}$ is an open cover of X. By construction, elements of $\mathcal{Y}$ are open. Fix $x \in X$. There are two cases. (i) $x \in U_\alpha$ for some $\alpha \notin C$. So, $x \in Y_\alpha$. (ii) Since $\{U_\alpha\}$ is point-finite too, there exists a neighborhood of x that intersects only finitely many U_α, say for $F_x = \{\alpha_1, \ldots, \alpha_n\}$. Since case (i) doesn't hold, $F_x \subset C$. Since T is totally ordered, there exists $(B', \mathcal{W}') \in T$ such that $F_x \subset B'$. Since W' covers X and $x \notin U_\alpha$ for $\alpha \notin F_x$, it must be $x \in \bigcup_{\alpha \in F_x} W'_\alpha$. So, $x \in \bigcup_{\alpha \in F_x} Y'_\alpha$ by definition of $\mathcal{Y}$. In both cases, $x \in Y_\alpha$ for some $\alpha \in A$.

By Zorn's Lemma, $\mathcal{P}$ has a maximal element $(M, \mathcal{V})$. Suppose $M \neq A$, and pick $\alpha_0 \in A \setminus M$. Define

$$W := \left(\bigcup_{\beta \in M} V_\beta \right) \cup \left(\bigcup_{\beta \in A \setminus (M \cup \{\alpha_0\})} U_\beta \right),$$

which is open. Then $\{U_{\alpha_0}, W\}$ is an open cover of X, so by part (1), there exists an open V'_{α_0} such that $\overline{V'_{\alpha_0}} \subset U_{\alpha_0}$ and $V'_{\alpha_0} \cup W = X$.

Define $\mathcal{V}'$ by setting

$$V'_\alpha := \begin{cases} V_\alpha & \text{if } \alpha \in M, \\ V'_{\alpha_0} & \text{if } \alpha = \alpha_0, \\ U_\alpha & \text{otherwise.} \end{cases}$$

Then $(M \cup \{\alpha_0\}, \mathcal{V}') \in \mathcal{P}$, contradicting maximality. Hence $M = A$, and the corresponding family $\mathcal{V} = \{V_\alpha\}_{\alpha \in A}$ satisfies the desired properties. □

Remark 4.3. Also, (3) $\implies$ (2) $\implies$ (1) $\implies$ (being normal space).

Proposition 4.4. [21, Theorem 33.1.] (Urysohn's Lemma) *Let X be a normal space and $A, B \subset X$ be disjoint closed sets. Then there exists a continuous function $f : X \to [0, 1]$ such that $f_{|A} = 0$ and $f_{|B} = 1$*

Proposition 4.5. *In a second-countable topological space, every locally finite open cover has countably many members.*

Proof. Let X be a second-countable topological space, and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a locally finite open cover of X. Since X is second-countable, it has a countable basis $\{B_n\}_{n \in \mathbb{N}}$.

For each $n \in \mathbb{N}$, define

$$\mathcal{U}_n := \{U \in \mathcal{U} : B_n \cap U \neq \emptyset\}.$$

$$I := \{n \in \mathbb{N} : \mathcal{U}_n \text{ is finite}\}.$$

We claim that $\mathcal{U} = \bigcup_{n \in I} \mathcal{U}_n$. Clearly, for each $n \in I, \mathcal{U}_n \subseteq \mathcal{U}$. Thus, $\bigcup_{n \in I} \mathcal{U}_n \subseteq \mathcal{U}$.

Let $U \in \mathcal{U}$ be an arbitrary member of the cover. Take any point $x \in U$. Since the cover is locally finite, there exists an open neighborhood V of x such that V intersects only finitely many sets in $\mathcal{U}$.

Since $\{B_n\}_{n \in \mathbb{N}}$ is a basis for X, there exists a basis element $B_{n_x} \subseteq V$ that contains x. By construction, $n_x \in I$ because $B_{n_x} \subseteq V$ and V intersects only finitely many sets in the cover, making $\mathcal{U}_{n_x}$ finite. Moreover, since $x \in U \cap B_{n_x}$, we have $B_{n_x} \cap U \neq \emptyset$, which means $U \in \mathcal{U}_{n_x}$. Therefore, $U \in \bigcup_{n \in I} \mathcal{U}_n$.

This shows that $\mathcal{U} = \bigcup_{n \in I} \mathcal{U}_n$, where $I \subseteq \mathbb{N}$ and each $\mathcal{U}_n$ is finite for $n \in I$. Since I is a subset of $\mathbb{N}$, it is countable, and thus $\bigcup_{n \in I} \mathcal{U}_n$ is a countable union of finite sets, making it countable. Hence, the locally finite cover $\mathcal{U}$ has countably many members. □

Proposition 4.6. [21, Theorem 41.7.] *Let X be a paracompact Hausdorff space and take an arbitrary open cover. Then, assuming the axiom of choice, X admits a partition of unity subordinate to the given cover. If X is second-countable, then this is true without the axiom of choice.*

Proof. Let $\mathcal{Y}$ be any open cover of X. Since X is paracompact, there exists a locally finite open refinement $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of $\mathcal{Y}$ such that for each $\alpha \in A$, there is some $Y_\alpha \in \mathcal{Y}$ satisfying $U_\alpha \subset Y_\alpha$.

By proposition 4.2, X is normal, we apply the Shrinking Lemma 4.3 two times (assuming the axiom of choice). For each α, there exist open sets W_α, V_α such that

$$W_\alpha \subset \overline{W_\alpha} \subset V_\alpha \subset \overline{V_\alpha} \subset U_\alpha.$$

and $\{W_\alpha\}_{\alpha \in A}$ and $\{V_\alpha\}_{\alpha \in A}$ are open covers of X.

By Urysohn's Lemma 4.4, there exists a continuous function $f_\alpha : X \to [0,1]$ such that $f_\alpha(x) = 1$ for $x \in \overline{W_\alpha}$ and $f_\alpha(x) = 0$ for $x \in (V_\alpha)^c$.

Since $\{U_\alpha\}$ is locally finite and $\mathrm{supp}(f_\alpha) = \overline{\{f_\alpha \neq 0\}} \subset \overline{V_\alpha} \subset U_\alpha$, the collection $\{\mathrm{supp}(f_\alpha)\}$ is locally finite. Define

$$F(x) = \sum_{\alpha \in A} f_\alpha(x).$$

This sum is well-defined and continuous since at each point only finitely many terms are nonzero. Moreover, for every $x \in X$, there exists some α such that $x \in W_\alpha$, so $f_\alpha(x) = 1$ and hence $F(x) \geq 1$.

Finally, define the functions

$$\varphi_\alpha(x) = \frac{f_\alpha(x)}{F(x)}.$$

Each φ_α is continuous and forms a partition of unity, since for all x,

$$\sum_{\alpha \in A} \varphi_\alpha(x) = \frac{1}{F(x)} \sum_{\alpha \in A} f_\alpha(x) = 1.$$

Moreover, the support of φ_α is contained in $\overline{V_\alpha}$ and hence in Y_α, making the partition of unity subordinate to $\mathcal{Y}$.

Note that if we have a second-countable space, we don't need to assume the axiom of choice. By proposition 4.5, every locally finite open cover has countably many members. So, we can apply the countable case of Shrinking Lemma 4.3. $\square$

4.1.1 Manifolds

Definition 4.5. [19] *A* ***topological manifold*** *M of* ***dimension*** *n is a Hausdorff, second countable topological space such that each point $p \in M$ has an open neighbourhood $U \subseteq M$ and a homeomorphism $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$, where $\varphi(U)$ is an open subset of $\mathbb{R}^n$ (Note that $\varphi(U)$ can be taken as $\mathbb{R}^n$). So, M admits an open cover $\{U_\alpha\}_{\alpha\in A}$, and for each α, there exists a homeomorphism $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$, called a* ***chart****. The pair $(U_\alpha, \varphi_\alpha)$ is called a* ***coordinate chart*** *on M.*

Definition 4.6. [19] *A space X is* ***locally compact*** *if every point $x \in X$ has a compact neighborhood.*

Proposition 4.7. [19, Proposition 4.64.] *Manifolds are locally compact.*

Proof. Let M be a manifold. For all $x \in \mathbb{R}^n$, the closed ball $\overline{B(x,r)}$ is compact by the Heine–Borel theorem (since it is closed and bounded in $\mathbb{R}^n$). So, $\overline{B(x,r)}$ is the compact neighbourhood of x.

Given any point $p \in M$, take a coordinate chart (U, φ) with $\varphi(U) = \mathbb{R}^n$. Choose an open ball $B(\varphi(p), r) \subset \mathbb{R}^n$. So, its closure $\overline{B(\varphi(p), r)}$ is compact. The preimage $W := \varphi^{-1}(B(\varphi(p), r))$ is an open neighborhood of p in U. Since U is open in M, W is open in M. Moreover, since φ^{-1} is continuous and $K := \varphi^{-1}(\overline{B(\varphi(p), r)})$ is compact. Thus, $p \in W \subset K \subset M$. Every point $p \in M$ has a compact neighborhood. □

Definition 4.7. [19] *A subset P of a topological space X is called* ***precompact*** *(or* ***relatively compact****) if the closure $\overline{P}$ is compact.*

Proposition 4.8. [19, Proposition 4.63.] *Let X be a Hausdorff and locally compact topological space. Then X has a basis of precompact open subsets.*

Proof. Take $x \in X$. Then there exists an open set U and a compact set K such that $x \in U \subset K$. Define $\mathcal{B}_x := \{P \in \mathcal{T}_X \mid x \in P \subset U\}$, which is a local basis of x. K is closed since compact subsets of Hausdorff spaces are closed. Take $P \in \mathcal{B}_x$. Then $\overline{P} \subset K$ and $\overline{P}$ is compact since closed subsets of compact spaces are compact. Hence, P is precompact. Also, $\bigcup_{x\in X} \mathcal{B}_x$ is a basis and it consists of precompact sets. □

Definition 4.8. [19] *A topological space X has an* ***exhaustion by compact sets*** *if there exists a sequence of compact subsets $\{K_i\}_{j\in\mathbb{N}}$ such that it covers X and $K_j \subset K_{j+1}^{\circ}$ for all $j \in \mathbb{N}$.*

Proposition 4.9. [19, Proposition 4.76.] *Every second-countable Hausdorff and locally compact topological space X admits an exhaustion by compact sets.*

Proof. Let $\{P_n\}_{n\in\mathbb{N}}$ be a countable basis of X consisting of precompact open sets via proposition 4.8 and second countability. Inductively define $g : \mathbb{N} \to \mathbb{N}$ such that $g(k) > k$ and $g(k)$ is the minimal index for which

$$\overline{P_1} \cup \cdots \cup \overline{P_k} \subseteq P_1 \cup P_2 \cup \cdots \cup P_{g(k)}.$$

(Note that such a function g exists because $\{P_n\}_{n\in\mathbb{N}}$ is a basis and the union of closures $\overline{P_1} \cup \cdots \cup \overline{P_k}$ is compact, so it has a finite subcover.) Set

$$K_j := \bigcup_{i\leq g^{(j)}(1)} \overline{P_i},$$

where $g^{(j)}$ denotes the j-fold composition of g (i.e., $g^{(j)} = g \circ g \circ \cdots \circ g$ j-times). Note that each K_j is compact because it is a finite union of compact sets. Also, $\{K_j\}_{j\in\mathbb{N}}$ covers X. By construction, we have $K_j \subseteq P_1 \cup \cdots \cup P_{g^{(j+1)}(1)}$, which implies $K_j \subseteq K_{j+1}^{\circ}$, since $P_1 \cup \cdots \cup P_{g^{(j+1)}(1)}$ is open and contained in $\overline{P_1} \cup \cdots \cup \overline{P_{g^{(j+1)}(1)}} = K_{j+1}$. Thus, $\{K_j\}_{j\in\mathbb{N}}$ is an exhaustion of X by compact sets. □

Proposition 4.10. [19, Theorem 4.77.] *Let X be a second-countable Hausdorff and locally compact topological space, and $\mathcal{U}$ be an open cover of X. Take a basis $\mathcal{B}$ for the topology of X. Then there exists a countable, locally finite open refinement of $\mathcal{U}$ consisting of elements of $\mathcal{B}$. As a result, X is paracompact.*

Proof. Let $(K_j)_{j=1}^{\infty}$ be an exhaustion of X by compact sets via proposition 4.9. Denote $K_0 = \emptyset$. For every $j \in \mathbb{N}$, define

$$V_j := K_{j+1} - K_j^{\circ} \text{ and } W_j := K_{j+2}^{\circ} - K_{j-1}.$$

Note that V_j is compact because it is a closed subset of a compact set. Also, W_j is open because it is the difference between an open set and a closed set; and compact spaces of

Hausdorff spaces are closed. By construction and properties of nested compact sets, we have $V_j \subset W_j$.

For every $x \in V_j$ there exists $U_x \in \mathcal{U}$. Because $\mathcal{B}$ is a basis, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subset U_x \cap W_j$. So, $\{B_x\}_{x \in V_j}$ is an open cover of V_j. Then, by the compactness of V_j there exists $\mathcal{B}_j = \{B_{x_i} \in \mathcal{B} : x_i \in V_j\}_{i=1}^{n_j}$ a finite cover of V_j and the union of $\mathcal{B}_j$'s elements is contained in W_j. Note that $\bigcup_{j=1}^{\infty} \mathcal{B}_j$ is a countable cover of X and a refinement of $\mathcal{U}$. Also, for every $x \in X$, x s contained in only finitely many W_j by exhaustion. Hence, $\bigcup_{j=1}^{\infty} \mathcal{B}_j$ is a locally finite cover. □

Proposition 4.11. *Manifolds are paracompact.*

Proof. Manifolds are locally compact via proposition 4.7. Since by definition manifolds are second countable and Hausdorff, we can apply proposition 4.10. So, manifolds are paracompact. □

Remark 4.4. Another approach to proving that manifolds are paracompact is by first showing that every manifold is metrizable; since every metrizable space is paracompact, it follows that manifolds are paracompact as well.

4.2 Nerve Theorems

Definition 4.9. [5] *Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a collection of sets indexed by a set A. The **nerve** of $\mathcal{U}$, denoted by $\mathrm{Nerve}(\mathcal{U})$, is an abstract simplicial complex where the vertex set is A and a finite subset $\sigma \subseteq A$ forms a simplex if and only if the intersection of the corresponding sets U_α for $\alpha \in \sigma$ is nonempty, i.e.,*

$$\sigma \in \mathrm{Nerve}(\mathcal{U}) \iff \bigcap_{\alpha \in \sigma} U_\alpha \neq \emptyset.$$

Definition 4.10. [5] *Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a cover of a topological space X. If every nonempty finite intersection of sets in $\mathcal{U}$ is contractible, then $\mathcal{U}$ is called **good cover**.*

Theorem 4.1. [18, Proposition 4G.1.] (Nerve Theorem) *Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a good open cover of a paracompact Hausdorff space X. Then the geometric realization of $\mathrm{Nerve}(\mathcal{U})$ is homotopy equivalent to X, i.e.,*

$$|\mathrm{Nerve}(\mathcal{U})| \simeq X.$$

Remark 4.5. As we will see, a part of the proof relies on the existence of a partition of unity. In the proof, we also follow [5, Theorem 5.9.].

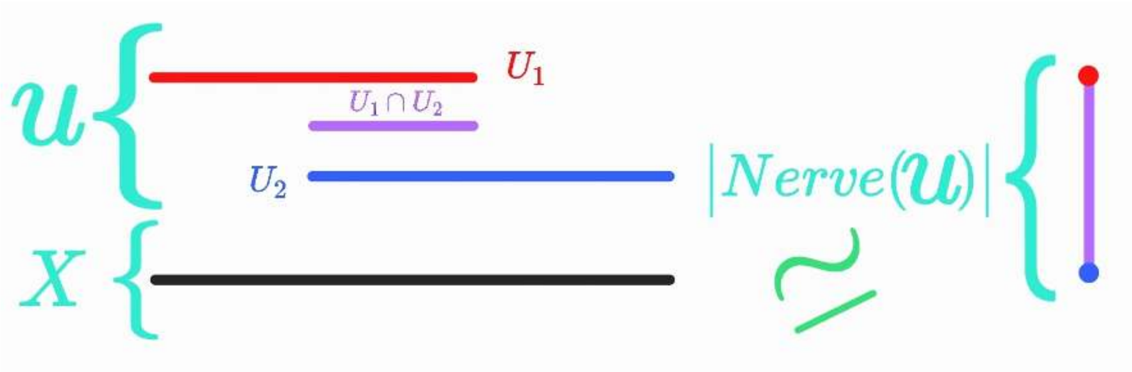

Figure 4.1: Nerve Theorem

Counterexamples: This theorem doesn't hold if we drop paracompactness or Hausdorffness of the space or the goodness condition of its open cover.

Violation for non-paracompact case: [5] Let ω_1 denote the first uncountable ordinal, i.e., the set of all countable ordinals. Define the **long ray** L as follows:

- Consider the set consisting of all pairs (α, t) where $\alpha < \omega_1$ and $t \in [0, 1)$, ordered lexicographically, i.e.,

$$(\alpha, t) < (\beta, s) \iff (\alpha < \beta) \text{ or } (\alpha = \beta \text{ and } t < s).$$

- Topologize this set using the order topology.

The following properties are known: L is Hausdorff. However, L is not paracompact and not contractible. To get a good open cover, for each countable ordinal $s < \omega_1$, define

$$L_s := \{t \in L \mid t < s\}.$$

Each such open set is homeomorphic to the half-open interval $[0, 1)$, and finite intersections of these sets are also homeomorphic to $[0, 1)$. Hence, the family $\mathcal{U} := \{L_s\}_{s\in\omega_1}$ is a good open cover of L. By some advanced machinery and the properties of L, $|\text{Nerve}(\mathcal{U})|$ is contractible. (L is weakly contractible and by results in [5], $|\text{Nerve}(\mathcal{U})|$ is weakly contractible. By Whitehead's theorem, $|\text{Nerve}(\mathcal{U})|$ is contractible.) However, L is not contractible. Contradiction. Therefore, L and $|\text{Nerve}(\mathcal{U})|$ are not homotopy equivalent.

Violation for non-Hausdorff case: [5] Firstly, we define a simplicial complex. Let $K := \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}\}$. It corresponds to the 1-skeleton of a triangle. The geometric realization $|K|$ is homeomorphic to the circle S^1.

Define a finite topological space X whose points correspond to the simplices of K.

$$\begin{aligned} a &:= \{v_1\}, & b &:= \{v_2\}, & c &:= \{v_3\}, \\ d &:= \{v_1, v_2\}, & e &:= \{v_2, v_3\}, & f &:= \{v_1, v_3\}. \end{aligned}$$

Then $X = \{a, b, c, d, e, f\}$. We topologize X by assigning a minimal open set around a simplex σ to the upset (the Alexandrov topology defined by the face poset of K):

$$U_\sigma = \{\tau \in K \mid \tau \supseteq \sigma\}.$$

So, the open sets in this topology are generated by the following:

$$\begin{aligned} \mathcal{O}(a) &= \{a, d, f\}, & \mathcal{O}(b) &= \{b, d, e\}, & \mathcal{O}(c) &= \{c, e, f\}, \\ \mathcal{O}(d) &= \{d\}, & \mathcal{O}(e) &= \{e\}, & \mathcal{O}(f) &= \{f\}. \end{aligned}$$

Note that X is paracompact since it is finite. But X is not Hausdorff since we cannot separate any pair of $\{a, b, c\}$ by disjoint open sets. Define a good open cover $\mathcal{U} = \{U_1, U_2, U_3\}$ of X . For each vertex $v_i \in K$, U_i is the following (the star of v_i):

$$U_1 := \{a, d, f\} \quad U_2 := \{b, d, e\} \quad U_3 = \{c, e, f\}$$

Each U_i is open in X, and the cover $\mathcal{U}$ is a good cover since all finite intersections are either singletons or themselves, which are contractible.

Now compute $\mathrm{Nerve}(\mathcal{U})$. The nerve is a simplicial complex whose vertices are the sets U_i, and whose simplices correspond to nonempty finite intersections.

$$U_1 \cap U_2 = \{d\} \neq \emptyset, \quad U_2 \cap U_3 = \{e\} \neq \emptyset, \quad U_1 \cap U_3 = \{f\} \neq \emptyset.$$

Thus, $\mathrm{Nerve}(\mathcal{U}) \cong K$, the original simplicial complex.

Let $g : X \to S^1$ be a continuous map. Since U_1 is open and connected, the restriction $g_{|U_1}$ is continuous and it has a connected image. The image of U_1 under g is also discrete. As S^1 is Hausdorff, the image is a single point. Therefore, $g(a) = g(d) = g(f)$. Similarly, using U_2 and U_3, we obtain $g(b) = g(d) = g(e)$ and $g(c) = g(e) = g(f)$. Combining these equalities, g is constant on all of X.

We have shown that every continuous map $g : X \to S^1$ is constant. Suppose, for contradiction, that there is a homotopy equivalence between X and S^1, with maps $g : X \to S^1$ and $h : S^1 \to X$ such that $g \circ h \simeq \mathrm{id}_{S^1}$. But since every map g is constant, the composition $g \circ h$ is also constant, hence null-homotopic. This contradicts the fact that id_{S^1} is not null-homotopic. Therefore, no such homotopy equivalence exists, and we conclude X is not homotopy equivalent to $|\mathrm{Nerve}(\mathcal{U})| \simeq |K| \simeq S^1$. This example also shows that the finiteness of cover is not enough to derive the nerve theorem.

Violation for non-good open cover: Let $S^1 \subset \mathbb{R}^2$ be the unit circle. Consider $\mathcal{U} = \{U_1, U_2\}$, where $U_1 = S^1 \setminus \{(-1,0)\}$ and $U_2 = S^1 \setminus \{(1,0)\}$. U_1 and U_2 are open in the subspace topology of S^1, and their union covers the whole circle. The intersection is $U_1 \cap U_2 = S^1 \setminus \{(-1,0),(1,0)\}$. This is not connected and not contractible. Therefore, the cover $\mathcal{U}$ is not a good cover. Now, consider $\mathrm{Nerve}(\mathcal{U})$:

- There are two open sets, so the nerve has two vertices.
- Their intersection is non-empty, so there is an edge connecting the two vertices.

Hence, the nerve is a 1-simplex (an interval), which is contractible. On the other hand, S^1 is not contractible. Hence, S^1 and $\mathrm{Nerve}(\mathcal{U})$ are not homotopy equivalent.

Remark 4.6. We can view poset P as a category $\mathbf{P}$ where

- The objects are the elements of P.
- For each $p \leq q$, there is a unique morphism $p \to q$.

Definition 4.11. [5] *Let P be a poset, and let $F\colon \mathbf{P} \to \mathbf{Top}$ be a functor. The **bar construction** of F, denoted by $\mathrm{Bar}(F)$, is defined as the quotient space*

$$\mathrm{Bar}(F) := \left(\coprod_{(p_0 < p_1 < \cdots < p_n)} F_{p_0} \times \Delta^n \right) / \sim,$$

where the disjoint union runs over all finite non-degenerate chains $(p_0 < p_1 < \cdots < p_n)$ in P, and the equivalence relation $\sim$ is generated by the following identifications:

For each chain $c = (p_0 < p_1 < \cdots < p_n)$ and for each $i = 0, 1, \ldots, n$, let $c_i := (p_0 < \cdots < \widehat{p_i} < \cdots < p_n)$ be the subchain obtained by omitting the i-th element, and let $j_i\colon \Delta^{n-1} \to \Delta^n$ be the i-th face inclusion. Let q_0 denote the first element of the subchain c_i. Then for all $x \in F_{p_0}$ and $z \in \Delta^{n-1}$, we impose the identification

$$(x, j_i(z)) \sim (F_{p_0 \to q_0}(x), z)\,.$$

Proposition 4.12. [5, Proposition 2.6.] *Let P be a poset and $F, G : \mathbf{P} \to \mathbf{Top}$ be functors. Assume $\eta : F \Rightarrow G$ be a natural transformation such that for all $p \in P$, $\eta(p)$ is a homotopy equivalence. Then the induced map $\mathrm{Bar}(F) \to \mathrm{Bar}(G)$ is a homotopy equivalence.*

Definition 4.12. [5] *Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a cover of a topological space X. For each simplex $\sigma \in \mathrm{Nerve}(\mathcal{U})$, define $U_\sigma := \bigcap_{\alpha\in\sigma} U_\alpha$. The **blowup complex** associated to the cover $\mathcal{U}$ is defined as the quotient space*

$$\mathrm{Blowup}(\mathcal{U}) := \left(\coprod_{\sigma \in \mathrm{Nerve}(\mathcal{U})} U_\sigma \times \Delta^{|\sigma|-1} \right) \Big/ \sim .$$

where $\Delta^{|\sigma|-1}$ is the standard simplex of dimension $|\sigma| - 1$ the equivalence relation $\sim$ is generated by the following identifications:

For each $\sigma' \subset \sigma$ Let $j_{\sigma' \subset \sigma} \colon \Delta^{|\sigma'|-1} \to \Delta^{|\sigma|-1}$ is the canonical face inclusion of simplices corresponding to the inclusion of vertex sets, and $i_{\sigma \to \sigma'} : U_\sigma \to U_{\sigma'}$ be the canonical inclusion. Then for all $x \in U_\sigma$ and $z \in \Delta^{|\sigma'|-1}$, we impose the identification

$$(x, j_{\sigma' \subset \sigma}(z)) \sim (i_{\sigma \to \sigma'}(x), z) = (x, z).$$

Remark 4.7. [5] Take poset P as $\mathrm{Nerve}(\mathcal{U})$ with order $\leq$ such that $\sigma \leq \tau \iff \sigma \supseteq \tau$. Let $D_{\mathcal{U}} : \mathbf{P} \to \mathbf{Top}$ be a functor such that $\sigma \mapsto U_\sigma = \bigcap_{\alpha\in\sigma} U_\alpha$ and the corresponding morphisms of comparisons are inclusions of topological spaces. We can construct a homeomorphism $\mathrm{Bar}(D_{\mathcal{U}}) \xrightarrow{\cong} \mathrm{Blowup}(\mathcal{U})$ by assembling natural local homeomorphisms indexed by simplices $\sigma \in \mathrm{Nrv}(\mathcal{U})$. For each such σ, let $|\sigma|$ denote the geometric realization of the simplicial complex generated by the finite set σ, and let $|\mathrm{Sd}(\sigma)|$ denote its barycentric subdivision. Each flag $\sigma = \sigma_0 \supsetneq \sigma_1 \supsetneq \cdots \supsetneq \sigma_n$ in the poset $P_{\mathcal{U}}$ indexes a cell $U_{\sigma_0} \times \Delta^n \subset \mathrm{Bar}(D_{\mathcal{U}})$. These cells glue together over all such flags with the same element $\sigma = \sigma_0$, resulting in a subspace

$$U_\sigma \times |\mathrm{Sd}(\sigma)| \subset \mathrm{Bar}(D_{\mathcal{U}}).$$

Note that there is a homeomorphism $\phi_\sigma \colon |\mathrm{Sd}(\sigma)| \xrightarrow{\cong} |\sigma|$ which collapses the barycentric subdivision back to the original simplex. Taking the identity map on U_σ, we obtain

$$\mathrm{id}_{U_\sigma} \times \phi_\sigma \colon U_\sigma \times |\mathrm{Sd}(\sigma)| \to U_\sigma \times |\sigma|,$$

which lands inside $\mathrm{Blowup}(\mathcal{U})$ by definition. These maps respect the equivalence relations in both $\mathrm{Bar}(D_{\mathcal{U}})$ and $\mathrm{Blowup}(\mathcal{U})$, and hence assemble to a well-defined global homeomorphism. Hence, $\mathrm{Bar}(D_{\mathcal{U}}) \xrightarrow{\cong} \mathrm{Blowup}(\mathcal{U})$.

Now, we have enough tools to prove the nerve theorem 4.1. The strategy of the proof is to construct a sequence of intermediate topological spaces that serve as bridges between the original space X and the nerve of its cover. We will then demonstrate that each of these spaces is homotopy equivalent to the next, ultimately establishing a homotopy equivalence between X and the geometric realization of the nerve. We will use three key constructions: a partition of unity, the blowup complex, and the bar construction.

Proof. Define the projection map $\rho : \mathrm{Blowup}(\mathcal{U}) \to X$ by $\rho([(x, z)]) := x$. This map is well-defined since all identifications in the blowup complex preserve the first coordinate x. For any $x \in X$, define $A_x := \{\alpha \in A \mid x \in U_\alpha\}$. Then

$$\rho^{-1}(x) = \left\{[(x, z)] \mid \sigma \subset A_x \text{ finite}, z \in \Delta^{|\sigma|-1}\right\}$$

$$= \left\{\sum_{i\in\sigma} t_i x_i \mid \sigma \subset A_x \text{ finite and } x_i := x \in U_{\{i\}} \times \Delta^0 \simeq U_i \text{ and } \sum_{i\in\sigma} t_i = 1 \text{ and } t_i \geq 0\right\}.$$

Since X is paracompact and Hausdorff, there exists a partition of unity $\{\psi_\alpha\}_{\alpha\in A}$ subordinate to $\mathcal{U}$ via proposition 4.6 such that $\mathrm{supp}(\psi_\alpha) \subseteq U_\alpha$. Define for each $x \in X$ $\sigma_\psi(x) := \{\alpha \in A \mid \psi_\alpha(x) \neq 0\}$, which is finite by local finiteness. Now define $s_\psi : X \to \mathrm{Blowup}(\mathcal{U})$ by $s_\psi(x) := \left[\left(x, (\psi_\alpha(x))_{\alpha\in\sigma_\psi(x)}\right)\right] = \sum_{\alpha\in A} \psi_\alpha(x)\, x_\alpha$ such that $\left(x, (\psi_\alpha(x))_{\alpha\in\sigma_\psi(x)}\right) \in U_{\sigma_\psi(x)} \times \Delta^{|\sigma_\psi(x)|-1}$.

Note that $(\rho \circ s_\psi)(x) = x$. So, $\rho \circ s_\psi = \mathrm{id}_X$. We want to show that $s_\psi \circ \rho$ is homotopic to $\mathrm{id}_{\mathrm{Blowup}(\mathcal{U})}$. Define a homotopy $H : \mathrm{Blowup}(\mathcal{U}) \times [0, 1] \to \mathrm{Blowup}(\mathcal{U})$ by

$$H(y, t) := [(x, (1 - t)z + t \cdot (\psi_\alpha(x))_{\alpha\in\sigma})]\,,$$

where $x = \rho(y)$ and $z \in \Delta^{|\sigma|-1}$ such that $\sigma_\psi(x) \subseteq \sigma$. This is well-defined since the convex combination of points in a standard simplex remains in the simplex, and compatibility with the equivalence relation is preserved. Also, $H(y, 0) = H([(x, z)], 0) = [(x, z)] = y$ and $H(y, 1) = H([(x, z)], 1) = s_\psi(x) = (s_\psi \circ \rho)(y)$.

Hence H is a homotopy from $\mathrm{id}_{\mathrm{Blowup}(\mathcal{U})}$ to $s_\psi \circ \rho$, and p is a homotopy equivalence with homotopy inverse s_ψ. Thus, $\mathrm{Blowup}(\mathcal{U})$ and X are homotopy equivalent.

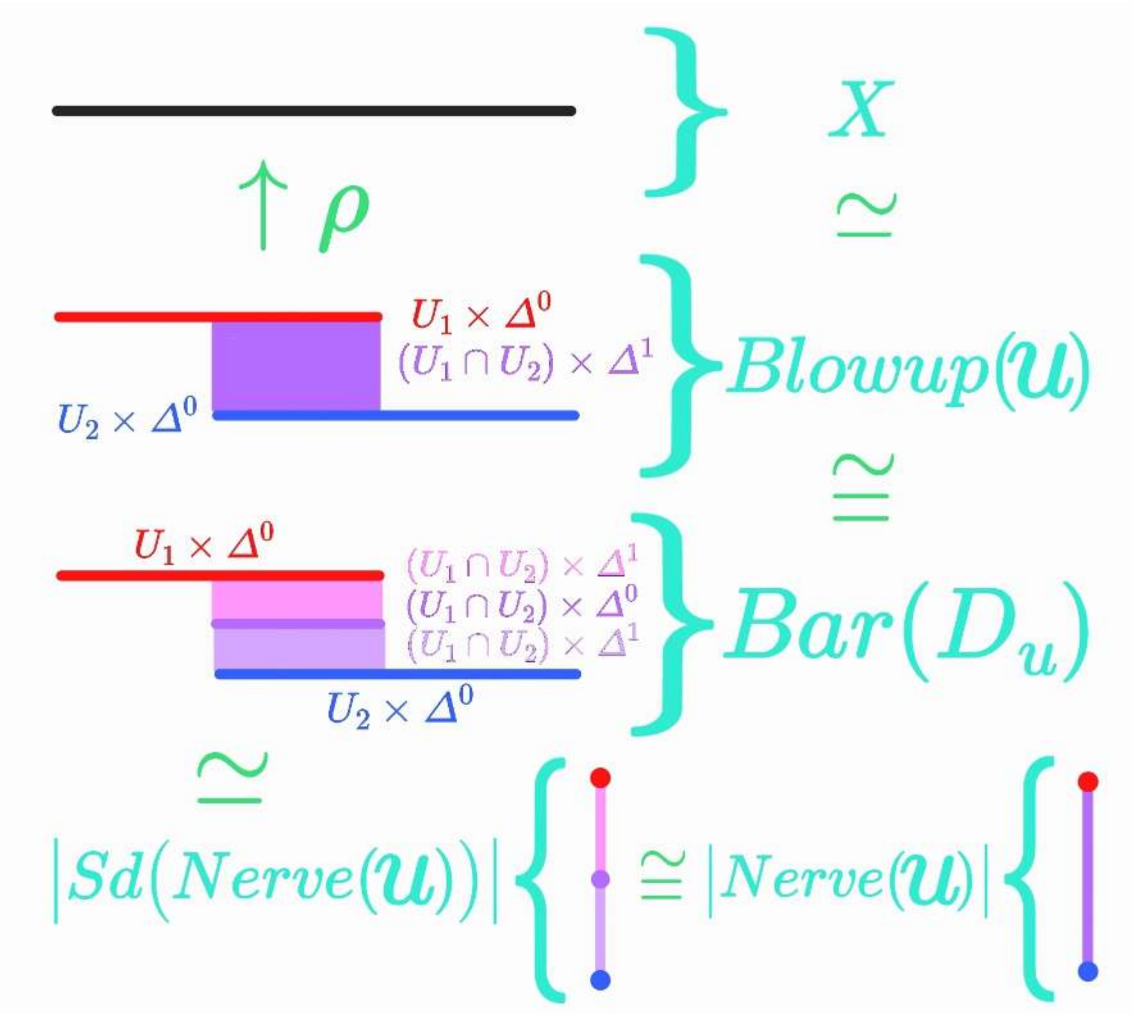

Figure 4.2: Proof Strategy for Nerve Theorem

By remark 4.7, we know that $\mathrm{Blowup}(\mathcal{U})$ and $\mathrm{Bar}(D_{\mathcal{U}})$ are homeomorphic. Now define a functor $F : \mathbf{P} \to \mathbf{Top}$ such that $F_\sigma = \{*\}$ for all $\sigma \in \mathrm{Nerve}(\mathcal{U})$. Since $\mathcal{U}$ is a good cover, U_σ is contractible, hence there exists a homotopy equivalence $\eta_\sigma : U_\sigma \to \{*\}$. Note that η is a natural transformation between $D_{\mathcal{U}}$ and F. By proposition 4.12, we get $\mathrm{Bar}(D_{\mathcal{U}}) \simeq \mathrm{Bar}(F)$. Also, $\mathrm{Bar}(F)$ is homeomorphic to $|\mathrm{Sd}(\mathrm{Nerve}(\mathcal{U}))|$. Thus,

$$\mathrm{Blowup}(\mathcal{U}) \cong \mathrm{Bar}(D_{\mathcal{U}}) \simeq \mathrm{Bar}(F) \cong |\mathrm{Sd}(\mathrm{Nerve}(\mathcal{U}))| \cong |\mathrm{Nerve}(\mathcal{U})|.$$

To sum up, since X is a paracompact Hausdorff space and $\mathcal{U}$ is an open cover, we have a homotopy equivalence between X and $\mathrm{Blowup}(\mathcal{U})$. Also, since $\mathcal{U}$ is a good cover, $\mathrm{Blowup}(\mathcal{U}) \simeq |\mathrm{Nerve}(\mathcal{U})|$. Since both conditions are satisfied, $X \simeq |\mathrm{Nerve}(\mathcal{U})|$. $\square$

There are other versions of the nerve theorem depending on the topological space, its cover, and the index set of the cover. Here, we present two different versions.

Theorem 4.2. [5, Theorem 4.8.] (Nerve Theorem for Simplicial Complexes) *Let $\mathcal{K} = \{K_\alpha\}_{\alpha \in A}$ be a set consisting of subcomplexes of K such that the union of its elements is K and the geometric realizations of all non-empty finite intersections are contractible. Then the geometric realization of* $\mathrm{Nerve}(\mathcal{K})$ *is homotopy equivalent to the geometric realization of K, i.e.,*

$$|\mathrm{Nerve}(\mathcal{K})| \simeq |K|.$$

Theorem 4.3. [5, Theorem 3.9.] (Nerve Theorem for Closed Convex Finite Covers) *Let $\mathcal{C} = \{C_i\}_{i \in [m]}$ be a finite set consisting of closed convex subsets of Euclidean space $\mathbb{R}^n$. Then the geometric realization of* $\mathrm{Nerve}(\mathcal{C})$ *is homotopy equivalent to the union of the sets in $\mathcal{C}$, i.e.,*

$$|\mathrm{Nerve}(\mathcal{C})| \simeq \bigcup_{i \in I} C_i.$$

Chapter 5

HOMOLOGY

Homology is an algebraic concept defined on chain complexes. Although intuitively one might describe a topological space by referring to its "holes", giving a precise mathematical formulation of this idea requires the machinery of homology theory. Among the various approaches, simplicial homology provides a computationally effective method for spaces that admit a triangulation. On the other hand, singular homology is defined for all topological spaces. Importantly, for triangulable spaces, simplicial and singular homology theories are naturally isomorphic, which implies that the simplicial homology is independent of the specific triangulation chosen. This chapter follows a classic textbook [18] on algebraic topology.

Definition 5.1. *Let R be a commutative ring with identity. A **chain complex** (C, ∂) is a sequence of R-modules $\{C_n\}_{n\in\mathbb{Z}}$ together with R-module homomorphisms $\partial_n : C_n \to C_{n-1}$ such that $\partial_{n-1} \circ \partial_n = 0$ for all $n \in \mathbb{Z}$. The elements of C_n are called **chains** of degree n and ∂_n is called the **boundary operator** for all $n \in \mathbb{Z}$. The kernel of ∂_n is denoted by $\operatorname{Ker} \partial_n$ and is called the **cycle module**, which lies in C_n, and the image of ∂_{n+1}, which lies in C_n, denoted by $\operatorname{Im} \partial_{n+1}$, is called the **boundary module**. An element of the cycle module $\operatorname{Ker} \partial_n$ is called a **cycle** of degree n, and an element of the boundary module $\operatorname{Im} \partial_{n+1}$ is called a **boundary** of degree n. Since $\partial_{n-1} \circ \partial_n = 0$ for all $n \in \mathbb{Z}$, $\operatorname{Im} \partial_{n+1} \subseteq \operatorname{Ker} \partial_n$. This means that every boundary of degree n is also a cycle of degree n. Therefore, the quotient module*

$$H_n(C) := \ker \partial_n / \operatorname{im} \partial_{n+1}$$

*is well-defined and it is called the **homology module** of (C, ∂) at degree n. The elements of H_n are called **homology classes**. Two cycles that represent the same homology class are called **homologous**, meaning that their difference is a boundary.*

Definition 5.2. *Let (C, ∂^C) and (D, ∂^D) be two chain complexes over a commutative ring R. A **chain map** $f : C \to D$ is a sequence of R-module homomorphisms $\{f_n : C_n \to D_n\}_{n \in \mathbb{Z}}$ such that for every n, $\partial_n^D \circ f_n = f_{n-1} \circ \partial_n^C$, i.e., the following diagram commutes:*

$$\begin{array}{ccc} C_n & \xrightarrow{\partial_n^C} & C_{n-1} \\ \downarrow f_n & & \downarrow f_{n-1} \\ D_n & \xrightarrow{\partial_n^D} & D_{n-1}. \end{array}$$

Remark 5.1. The chain map f induces a (module) homomorphism on the homology level. $f_* : H_n(C) \to H_n(D)$, defined by $f_*([c]) = [f(c)]$.

Remark 5.2. The **category of chain complexes** consists of chain modules as objects and chain maps as morphisms. H_n is a functor from the category of chain complexes to the category of R-modules, and it is called **homology functor**.

Definition 5.3. *Let $f, g : (C, \partial^C) \to (D, \partial^D)$ be two chain maps between chain complexes. A **chain homotopy** between f and g is a sequence of R-module homomorphisms $\{P_n : C_n \to D_{n+1}\}_{n \in \mathbb{Z}}$ such that for every n,*

$$f_n - g_n = \partial_{n+1}^D \circ P_n + P_{n-1} \circ \partial_n^C.$$

*A chain map $f : C \to D$ between two chain complexes C and D is called a **chain homotopy equivalence** if there exists a chain map $g : D \to C$ such that $f \circ g$ is chain homotopic to the identity map id_D and $g \circ f$ is chain homotopic to the identity map id_C.*

Proposition 5.1. *If f and g are chain homotopic, then they induce the same homomorphism on the homology level, i.e., $f_* = g_* : H_n(C) \to H_n(D)$. If $f : C \to D$ is a chain homotopy equivalence then f induces isomorphisms on homology level, i.e., $H_n(C) \cong H_n(D)$.*

5.1 Singular Homology

Definition 5.4. *Let X be a topological space and $n \in \mathbb{Z}_{\geq 0}$. A **singular n-simplex** in X is a continuous map $\sigma : \Delta^n \to X$ where Δ^n is the standard n-simplex. The set of all singular n-simplices in X is denoted by $S_n(X)$.*

Let R be a commutative ring with identity. The ***singular*** n***-chain module*** $C_n(X;R)$ *is the free R-module generated by $S_n(X)$. An element of $C_n(X;R)$ is called a* ***singular*** n***-chain***. *The* ***boundary operator*** $\partial_{n+1} : C_{n+1}(X;R) \to C_n(X;R)$ *is an R-module homomorphism defined on a singular $(n+1)$-simplex $\sigma : \Delta^{n+1} \to X$ by*

$$\partial_{n+1}(\sigma) = \sum_{j=0}^{n+1}(-1)^j \sigma \circ \delta_j,$$

where $\delta_j : \Delta^n \to \Delta^{n+1}$ is the inclusion of the j-th face of Δ^{n+1}. This definition is extended linearly to all of $C_{n+1}(X;R)$.

Theorem 5.1. *Let X be a topological space and $C_n(X;R)$ be a singular n-chain module for a nonnegative integer n. Then the sequence of R-modules and boundary operators of the singular chain modules*

$$\cdots \xrightarrow{\partial_{n+1}} C_n(X;R) \xrightarrow{\partial_n} C_{n-1}(X;R) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0(X;R) \xrightarrow{\partial_0=0} 0$$

forms a ***chain complex***, *i.e., $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}_{\geq 0}$.*

Remark 5.3. C_n is a functor from the category of topological spaces to the category of chain complexes, and it is called **singular functor**. The quotient module

$$H_n(X;R) := \frac{\ker(\partial_n : C_n(X;R) \to C_{n-1}(X;R))}{\mathrm{im}(\partial_{n+1} : C_{n+1}(X;R) \to C_n(X;R))}.$$

is called the n^{th} **(singular) homology module** of X. $H_n(-;R)$ is a functor from the category of topological spaces to the category of R-modules, and it is called **singular homology functor**.

Remark 5.4. Let X be a topological space, and let $\{X_\alpha\}$ denote the set of path-connected components of X. Then for all $n \geq 0$, $H_n(X;R) \cong \bigoplus_\alpha H_n(X_\alpha;R)$. Specifically, for $n = 0$ we have $H_0(X;R) \cong \bigoplus_\alpha R$.

Proposition 5.2. *Let $f, g : X \to Y$ be continuous maps that are homotopic. Then f_*, g_* are chain homotopic, so $f_* = g_*$ as R-module homomorphisms from $H_n(X;R)$ to $H_n(Y;R)$. Hence, if X and Y are homotopy equivalent spaces, then their homology modules are isomorphic.*

Proposition 5.3. *Let X be a contractible space. Then for any $n \geq 0$ and any commutative ring R, the n-th homology module of X is given by*

$$H_n(X;R) = \begin{cases} R & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

5.2 Simplicial Homology

Definition 5.5. *Let K be a simplicial complex. The **simplicial n-chain module** $C_n(K;R)$ is the free R-module generated by the n-simplices of K. An element of $C_n(K;R)$ is called a **simplicial n-chain**.*

*The **boundary operator** $\partial_n : C_n(K;R) \to C_{n-1}(K;R)$ is an R-module homomorphism defined on an n-simplex $\sigma = [v_0, v_1, \ldots, v_n]$ by*

$$\partial_n([v_0, v_1, \ldots, v_n]) = \sum_{i=0}^{n} (-1)^i [v_0, v_1, \ldots, \hat{v}_i, \ldots, v_n],$$

where $[v_0, v_1, \ldots, \hat{v}_i, \ldots, v_n]$ is the $(n-1)$-simplex obtained by removing the vertex v_i. This definition extends linearly to all of $C_n(K;R)$.

Theorem 5.2. *Let K be a simplicial complex and $C_n(K;R)$ be a simplicial n-chain module for nonnegative integer n. Then the sequence of R-modules and boundary operators of the simplicial chain modules*

$$\cdots \xrightarrow{\partial_{n+1}} C_n(K;R) \xrightarrow{\partial_n} C_{n-1}(K;R) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0(K;R) \xrightarrow{\partial_0 = 0} 0$$

*forms a **chain complex**, i.e., $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}_{\geq 0}$.*

Remark 5.5. C_n is a functor from the category of simplicial complexes to the category of chain complexes, and it is called **simplicial chain functor**. The quotient module

$$H_n^{\Delta}(K;R) := \frac{\ker(\partial_n : C_n(K;R) \to C_{n-1}(K;R))}{\operatorname{im}(\partial_{n+1} : C_{n+1}(K;R) \to C_n(K;R))}.$$

is called the n^{th} **(simplicial) homology module** of the simplicial complex K. $H_n^{\Delta}(-;R)$ is a functor from the category of simplicial complexes to the category of R-modules, and it is called **simplicial homology functor**.

Theorem 5.3. *Let X be a triangulable topological space and let K be a simplicial complex such that $|K|$ is homeomorphic to X. Then the singular homology modules $H_n(X; R)$ and the simplicial homology modules $H_n^\Delta(K; R)$ are isomorphic for all n, i.e.,*

$$H_n(X; R) \cong H_n^\Delta(K; R).$$

Definition 5.6. *Let X be a topological space and F a field. The i^{th}* ***Betti number*** *of X, denoted by $\beta_i(X; F)$, is defined as*

$$\beta_i(X; F) := \dim_F H_i(X; F).$$

Intuitively, β_0 counts the number of path connected components, β_1 for loops, β_2 for two-dimensional voids, and so on.

Proposition 5.4. *Let X be a triangulable topological space, and let K be a finite simplicial complex such that $|K| \cong X$. Then*

$$\chi(X) := \chi(K) = \sum_{i=0}^{\infty} (-1)^i \beta_i(X; F).$$

Remark 5.6. Although the individual Betti numbers may depend on the choice of field F, the Euler characteristic $\chi(X)$ is independent of F (assuming all homology groups are finite-dimensional and vanish beyond some degree).

Chapter 6

PERSISTENCE MODULES

Definition 6.1. [10] *A **product poset** is a poset $P = T_1 \times \cdots \times T_n$, where each T_i is a totally ordered set and its partial order is given for $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$, $p \leq q \iff p_i \leq q_i$ for all $i = 1, \ldots, n$.*

Definition 6.2. [11] *A P**-persistence object** in a category $\mathbf{C}$ is a **functor** $\Phi : \mathbf{P} \to \mathbf{C}$, i.e., (i) a family $\{c_p\}_{p \in P}$ of objects in $\mathbf{C}$ and (ii) morphisms $\phi_{p \leq q} : c_p \to c_q$.*

Definition 6.3. [11] *A **morphism** between two P-persistence objects (functors from $\mathbf{P}$ to $\mathbf{C}$) is a natural transformation between these functors. More precisely, let $\Phi = \{a_p, \phi_{p \leq q}\}$ and $\Psi = \{b_p, \psi_{p \leq q}\}$ be two P-persistence objects in $\mathbf{C}$; a morphism between them is a family (of morphisms of $\mathbf{C}$) $\eta := \{\eta_p : a_p \to b_p\}_{p \in P}$ such that for all $p \leq q$, $\psi_{p \leq q} \circ \eta_p = \eta_q \circ \phi_{p \leq q}$, i.e., the following diagram commutes:*

$$
\begin{array}{ccc}
a_p & \xrightarrow{\phi_{p \leq q}} & a_q \\
\downarrow \eta_p & & \downarrow \eta_q \\
b_p & \xrightarrow{\psi_{p \leq q}} & b_q
\end{array}
$$

*A morphism $f : \Phi \to \Psi$ is called an **isomorphism** if there exists a morphism $g : \Psi \to \Phi$ such that $g \circ f = id_\Phi$ and $f \circ g = id_\Psi$, where id_Φ and id_Ψ are the identity morphisms on Φ and Ψ respectively.*

Definition 6.4. [11] *The **category of** P**-persistence objects** in $\mathbf{C}$, denoted by $\textbf{pers}_P(C)$, consists of objects as functors $\Phi : \mathbf{P} \to \mathbf{C}$ and morphisms as natural transformations between such functors.*

Remark 6.1. If $f : \mathbf{P} \to \mathbf{Q}$ is a **poset map** (order-preserving function), it induces a **functor** between persistence categories:

$$f^* : \textbf{pers}_Q(C) \to \textbf{pers}_P(C), \quad f^*(\Psi) = \Psi \circ f.$$

This means that given a persistence module over Q, we can pull it back along f to get a persistence module over P.

Definition 6.5. [11] *Let R be a commutative ring with unity. A **persistence module** indexed by P over R is a functor $\mathbb{M} : \mathbf{P} \to \boldsymbol{Mod}_R$ where $\boldsymbol{Mod}_R$ denotes the category of R-modules.*

After this point, we consider R as a field F. When $\mathbf{C}$ is the category of vector spaces over a field F, i.e., $\mathbf{Vect}_F$, a **persistence module indexed by** P becomes a functor $\mathbb{V} : \mathbf{P} \to \mathbf{Vect}_F$ $((V_p, \varphi_{p\leq q})_{p,q\in P}$ where V_p is a vector space and $\varphi_{p\leq q}$ is a linear map).

Definition 6.6. [11] *A **homomorphism** of persistence modules from $(\mathbb{V}, \varphi)$ to $(\mathbb{W}, \psi)$ is a natural transformation f, meaning a family of linear maps $f_p : V_p \to W_p$ satisfying for all $p \leq q$, $\psi_{p\leq q} \circ f_p = f_q \circ \varphi_{p\leq q}$, i.e., the following diagram commutes*

$$\begin{array}{ccc} V_p & \xrightarrow{\varphi_{p\leq q}} & V_q \\ \downarrow f_p & & \downarrow f_q \\ W_p & \xrightarrow{\psi_{p\leq q}} & W_q \end{array}$$

*A homomorphism $f : \mathbb{V} \to \mathbb{W}$ is an **isomorphism** if there exists a morphism $g : \mathbb{W} \to \mathbb{V}$ such that $g \circ f = id_{\mathbb{V}}$ and $f \circ g = id_{\mathbb{W}}$, where $id_{\mathbb{V}}$ and $id_{\mathbb{W}}$ are the identity morphisms on $\mathbb{V}$ and $\mathbb{W}$ respectively. It is equivalent to saying that $f_p : V_p \to W_p$ is a bijective linear map for all $p \in P$. We denote $\mathrm{Hom}(\mathbb{V}, \mathbb{W})$ as the set of homomorphism $\mathbb{V} \to \mathbb{W}$ and $\mathrm{End}(\mathbb{V})$ as the set of homomorphism $\mathbb{V} \to \mathbb{V}$.*

Definition 6.7. [14] *Let $\mathbb{V}$ and $\mathbb{W}$ be two persistence modules with linear maps $\varphi^{\mathbb{V}}_{p\leq q} : V_p \to V_q$ and $\varphi^{\mathbb{W}}_{p\leq q} : W_p \to W_q$. The **direct sum** $\mathbb{V} \oplus \mathbb{W}$ of $\mathbb{V}$ and $\mathbb{W}$ is defined as the persistence module where*

$$(V \oplus W)_p := V_p \oplus W_p,$$

and the linear maps

$$\varphi^{\mathbb{V}\oplus\mathbb{W}}_{p\leq q} := \varphi^{\mathbb{V}}_{p\leq q} \oplus \varphi^{\mathbb{W}}_{p\leq q},$$

for all $p \leq q$. It can be generalized for $\bigoplus_{i\in I} \mathbb{V}_i$.

Remark 6.2. The **direct product** of persistence modules can be defined similarly.

Definition 6.8. [14] *A persistence module $\mathbb{W}$ is **indecomposable** if it cannot be expressed as the direct sum of two non-zero persistence modules. In other words, if $\mathbb{W} = \mathbb{U} \oplus \mathbb{V}$, then either $\mathbb{U}$ or $\mathbb{V}$ must be the zero module.*

Definition 6.9. [10] *An **interval** J of a totally ordered set T is its subset such that for $r, s, t \in T$ and $r < s < t$, if $r, t \in J$ then $s \in J$. An **interval** J of a poset P is its subset such that it satisfies the above condition, as well as for all $r, t \in T$ there exists a finite sequence $r =: s_1, s_2, ..., s_k := t$ whose consecutive elements are comparable. An **interval module** $\mathbb{I}^J$ indexed by P defined by*

$$V_t = \begin{cases} F, & \text{if } t \in J \\ 0, & \text{if } t \notin J \end{cases}, \quad \varphi_{s \leq t} = \begin{cases} \mathrm{id}_F, & \text{if } s, t \in J \text{ and } s \leq t \\ 0, & \text{otherwise} \end{cases},$$

which is a persistence module that encodes a feature persisting over the interval $J \subseteq P$ and vanishing elsewhere.

Proposition 6.1. [14, Proposition 1.1.] *Let $\mathbb{I}$ be an interval module. Then* $\mathrm{End}(\mathbb{I}) = F$.

Proof. Let $\mathbb{I} = \mathbb{I}^J$ and f be an endomorphism on $\mathbb{I}$. Then for every $t \in J$ $f_t : I_t \to I_t$, i.e., $f_t : F \to F$ is a scalar multiplication since it is a linear map. And it is the same scalar multiplication over the interval by the commutativity of its diagram and the existence of a finite comparable sequence for each pair. $\square$

Proposition 6.2. [9, Proposition 2.2.] *Interval modules are indecomposable.*

Proof. Let $\mathbb{I}$ be an interval module. Assume $\mathbb{I} = \mathbb{U} \oplus \mathbb{V}$. We want to show that either $\mathbb{U}$ or $\mathbb{V}$ must be the zero module. By decomposition, there exist homomorphisms $\pi_U : \mathbb{I} \to (\mathbb{U}) \subseteq \mathbb{I}$ and $\pi_V : \mathbb{I} \to (\mathbb{V}) \subseteq \mathbb{I}$ such that for every $p \in P$, π_U and π_V are projections $\mathbb{I}_p \to (\mathbb{U}_p \subseteq)\mathbb{I}_p$ and $\mathbb{I}_p \to (\mathbb{V}_p \subseteq)\mathbb{I}_p$ respectively. Hence, they satisfy the idempotent property: $\pi_U \circ \pi_U = \pi_U$ and $\pi_V \circ \pi_V = \pi_V$. By 6.1, $\mathrm{End}(\mathbb{I}) = F$. Since the only idempotent elements in the field (i.e., $a \in F$ where $a^2 = a$) are 0 and 1, only idempotent endomorphisms on an interval module are the zero map or the identity map. If $\pi_U = 0$, then $\mathbb{U} = 0$. If $\pi_U = 1$, then $\mathbb{I} = \mathbb{U}$, so $\mathbb{V} = 0$. Therefore, one of the modules in the decomposition must be the zero module. So, the interval module $\mathbb{I}$ is indecomposable. $\square$

Theorem 6.1. [10, Theorem 4.2.] *Assume that a persistence module $\mathbb{V}$ indexed by a poset P can be expressed as a direct sum of the indecomposables in the following ways:*

$$\mathbb{V} = \bigoplus_{\lambda \in L} \mathbb{V}^{\lambda} = \bigoplus_{\mu \in M} \mathbb{V}^{\mu}.$$

Then there is a bijection $\sigma : L \to M$ such that $\mathbb{V}^{\lambda} \cong \mathbb{V}^{\sigma(\lambda)}$ for all λ.

Definition 6.10. [10] *Let $\mathbb{V}$ be a persistence module indexed by a poset P. $\mathbb{V}$ is called* ***pointwise finite-dimensional persistence module*** *if for each $p \in P$, V_p is finite dimensional vector space.*

Theorem 6.2. [10, Theorem 4.2.] *Assume that a persistence module indexed by a poset P is pointwise finite-dimensional. Then there exists a collection of indecomposables $\{\mathbb{V}^{\lambda}\}_{\lambda \in \mathbb{L}}$ such that*

$$\mathbb{V} = \bigoplus_{\lambda \in L} \mathbb{V}^{\lambda}.$$

Definition 6.11. [10] *A persistence module $\mathbb{V}$ indexed by a poset P is called* ***interval decomposable*** *if it is isomorphic to a direct sum of interval modules, i.e.,*

$$\mathbb{V} = \bigoplus_{\lambda \in L} \mathbb{I}^{J_{\lambda}}.$$

If such a decomposition exists then it is unique (up to isomorphism and permutation of summands), and the multiset $\mathcal{B}(\mathbb{V}) := \{J_{\lambda} : \lambda \in L\}$ is called the ***barcode*** *of $\mathbb{V}$.*

Theorem 6.3. [8, Theorem 1.2.] *Assume that a persistence module $\mathbb{V}$ indexed by a totally ordered set T is pointwise finite-dimensional. Then it is interval decomposable, i.e., there exists a unique multiset consisting of intervals $\mathcal{B}(\mathbb{V})$ such that*

$$\mathbb{V} = \bigoplus_{J \in \mathcal{B}(\mathbb{V})} \mathbb{I}^{J}.$$

Theorem 6.4. [14, Theorem 1.4.] *Let $\mathbb{V}$ be the persistence module indexed by a finite totally ordered set T. Then it is interval decomposable, i.e., there exists a unique multiset consisting of intervals $\mathcal{B}(\mathbb{V})$ such that*

$$\mathbb{V} = \bigoplus_{J \in \mathcal{B}(\mathbb{V})} \mathbb{I}^{J}.$$

Counterexample: [24] For a countable index, $\mathbb{V}$ may not be interval decomposable. Consider the following counterexample: We define $\mathbb{V}$ indexed by $\mathbb{Z}_{\leq 0}$ be as follows:

- Let $V_0 := \mathbb{R}^{\mathbb{N}}$, the space of all real sequences $(x_1, x_2, x_3, \ldots)$.
- For $n \geq 1$, let $V_{-n} := \left\{(x_1, x_2, x_3, \ldots) \in \mathbb{R}^{\mathbb{N}} \mid x_1 = \cdots = x_n = 0\right\}$.
- The structure maps $\varphi_{-m \leq -n} : V_{-m} \to V_{-n}$ for $m \geq n$ as the inclusion maps.

Suppose $\mathbb{V}$ admits an interval decomposition. Since each map $\varphi_{-n-1 \leq -n}$ is injective, the only intervals that can occur are of the form $[-n, 0]$ or $(-\infty, 0]$. Also, $\dim(V_{-n}/V_{-n-1}) = 1$ for all $n \geq 0$. This implies that $\mathbb{V}$ must contain exactly one copy of each interval module $\mathbb{I}^{[-n,0]}$ for $n \geq 0$. Since $\bigcap_{n\in\mathbb{N}} V_{-n} = \{0\}$, no vector persists forever into the past. So the interval $(-\infty, 0]$ cannot appear. Therefore, the decomposition would be

$$\mathbb{V} \cong \bigoplus_{n\geq 0} \mathbb{I}^{[-n,0]}.$$

Each $\mathbb{I}^{[-n,0]}$ contributes one dimension at $t = 0$, so the direct sum contributes countably many dimensions at V_0. But, $V_0 = \mathbb{R}^{\mathbb{N}}$ has uncountable dimension. Thus, the interval decomposition cannot reconstruct V_0, and the module $\mathbb{V}$ is not interval decomposable.

Definition 6.12. [14] *Let $\mathbb{V}$ be a persistence module indexed by $T \subseteq \mathbb{R}$.*

- *$\mathbb{V}$ is called* ***finite*** *if it can be written as the finite direct sum of interval modules.*
- *$\mathbb{V}$ is called* ***locally finite*** *if it can be written as the direct sum of intervals such that every $t \in \mathbb{R}$ has a neighborhood intersecting finitely many of those intervals.*
- *$\mathbb{V}$ is called* ***q-tame*** *for all $s, t \in T$ with $s < t$, rank$(\varphi_{s,t}) < \infty$.*

Remark 6.3. Finite persistence modules are locally finite. Locally finite persistence modules are q-tame. Note that not all q-tame persistence modules are interval decomposable.

Counterexample: [14] The following is an interval decomposable q-tame persistence module, but it is not locally finite: The direct sum of the interval modules $\mathbb{I}^{[0,1/n)}$,

$$\mathbb{V} = \bigoplus_{n\in\mathbb{N}} \mathbb{I}^{[0,1/n)}.$$

Why? For any $p < q \in \mathbb{R}$, only finitely many intervals $[0, 1/n)$ contain both p and q. Thus, the image of $\varphi_{p\leq q}$ lies in the finite direct sum of those summands, so its rank is finite. Hence, $\mathbb{V}$ is q-tame. However, the decomposition has infinitely many intervals intersecting any neighborhood of $t = 0$. Thus, $\mathbb{V}$ is not locally finite.

Counterexample: [14] The following is a q-tame persistence module, but it is not interval decomposable: The direct product of the interval modules $\mathbb{I}^{[0,1/n)}$,

$$\mathbb{V} = \prod_{n\in\mathbb{N}} \mathbb{I}^{[0,1/n)}.$$

For a given $t \in \mathbb{R}$, we have

$$V_t = \prod_{n\in\mathbb{N}} \mathbb{I}_t^{[0,1/n)}$$

- If $t < 0$: Then $t \notin [0, 1/n)$ for any n. Thus, $V_t = \prod_{n\in\mathbb{N}} 0 = 0$.
- If $t = 0$: Then $0 \in [0, 1/n)$ for all n. Thus, $V_0 = \prod_{n\in\mathbb{N}} F$, which is an uncountably infinite-dimensional vector space.
- If $t > 0$: Let $S_t = \{n \in \mathbb{N} \mid t \in [0, 1/n)\}$. This set is finite. Therefore, $V_t = \prod_{n\in S_t} F \times \prod_{n\notin S_t} 0 \cong F^{|S_t|}$, which is a finite-dimensional.

Take $p < q$. Consider the linear map $\varphi_{p\leq q} : V_p \to V_q$. One of these real numbers is not 0, so one of the corresponding vector spaces is finite-dimensional. Hence, the rank of $\varphi_{p\leq q}$ is finite. Thus, $\mathbb{V}$ is q-tame because all its structure maps have finite rank.

Assume that $\mathbb{V}$ is isomorphic to a direct sum of interval modules

$$\mathbb{V} \cong \bigoplus_{i\in I} \mathbb{I}^{J_i}.$$

Then $\mathbb{V}_0$ would be the direct sum of the evaluations at time 0,

$$V_0 = \bigoplus_{i\in I} \mathbb{I}_0^{J_i}.$$

Since $\mathbb{V}_0$ has uncountable dimension, we must have uncountably many summands $\mathbb{I}^{J_i}$ such that $0 \in J_i$. Moreover, to have contributions only at time 0, one would need many copies of the interval module $\mathbb{I}^{[0,0]}$. Now, any summand $\mathbb{I}^{J_i}$ such that $J_i = [0, 0]$ contributes only to V_0 and vanishes for all $t > 0$. Therefore, if $\mathbb{V}$ had any such summands, there would exist nonzero vectors in V_0 that vanish in V_t for all $t > 0$.

However, this is impossible in $\mathbb{V}$. Take any nonzero element $v = (v_1, v_2, v_3, \dots) \in V_0$. Since v is nonzero, there exists some n such that $v_n \neq 0$. But in the interval module $\mathbb{I}^{[0,1/n)}$, v_n persists for all $t \in [0, 1/n)$. Therefore, v remains nonzero in V_t for all $t < 1/n$. This contradicts the presence of any $\mathbb{I}^{[0,0]}$ in a decomposition. $\mathbb{V}$ cannot be written as a direct sum of interval modules.

Proposition 6.3. [14, Proposition 2.21.] *Let $\mathbb{V}$ be a persistence module indexed by $T \subseteq \mathbb{R}$. Then $\mathbb{V}$ is locally finite if and only if each V_t is finite dimensional for all $t \in T$ and there exists locally finite set $F \subseteq T$ such that $\mathbb{V}$ is constant on all closed intervals not intersecting with F.*

Proof. ($\Rightarrow$) Suppose $\mathbb{V}$ is locally finite. Then there exists a multiset $\mathcal{B}$ of intervals in T such that $\mathbb{V} \cong \bigoplus_{J\in\mathcal{B}} \mathbb{I}^J$, and for each $t \in \mathbb{R}$, there is a neighborhood of t intersecting only finitely many intervals in $\mathcal{B}$.

Note that for each $t \in T$ and for each interval $J \in \mathcal{B}$, $\dim(\mathbb{I}^J_t) \in \{0, 1\}$. Also, only finitely many intervals contribute a nonzero dimension because of the locally finiteness of $\mathbb{V}$. Thus, $\dim V_t < \infty$ for all $t \in T$. Let F be the set of endpoints of the intervals in $\mathcal{B}$. Then $\mathbb{V}$ is constant on any closed interval not intersecting F, because the only changes in structure occur at endpoints. Moreover, since every $t \in \mathbb{R}$ has a neighborhood intersecting only finitely many intervals, F is locally finite.

($\Leftarrow$) Assume that each V_t is finite dimensional and that there exists a locally finite subset $F \subseteq T$ such that $\mathbb{V}$ is constant on all closed intervals disjoint from F. Since F is locally finite, we can extend it to a locally finite set $F' = \{t_n\}_{n\in\mathbb{Z}} \subseteq \mathbb{R}$, such that $t_n < t_{n+1}$ for all $n \in \mathbb{N}$ and $F \subseteq \{t_{2n} : n \in \mathbb{Z}\}$. This ensures that for each n, every closed interval $[a, b] \subseteq (t_{2n}, t_{2n+2})$ does not intersect F and hence $\mathbb{V}$ is constant on $\bigcup_{[a,b]\subseteq(t_{2n},t_{2n+2})}[a, b] = (t_{2n}, t_{2n+2})$.

Consider the restriction $\mathbb{V}_{|F'}$, a persistence module indexed by F'. Since $\mathbb{V}$ is pointwise finite dimensional, so is $\mathbb{V}_{|F'}$. By theorem 6.3, $\mathbb{V}_{|F'}$ decomposes as a direct sum of interval modules:

$$\mathbb{V}_{|F'} \cong \bigoplus_{\lambda\in\mathcal{B}} \mathbb{I}^{J'_\lambda}_{|F'}.$$

We now extend this decomposition to a decomposition of $\mathbb{V}$ over all of $T \subseteq \mathbb{R}$. Since $\mathbb{V}$ is constant on each interval (t_{2n}, t_{2n+2}), all structure maps in these intervals are isomorphisms. Thanks to the construction of F', it ensures that we cannot skip any open interval module between two consecutive elements. Therefore, each interval module $\mathbb{I}^{J'_\lambda}_{|F'}$ in the decomposition of $\mathbb{V}_{|F'}$ extends uniquely to an interval module $\mathbb{I}^{J_\lambda}$. Thus, we obtain

$$\mathbb{V} \cong \bigoplus_{\lambda\in\mathcal{B}} \mathbb{I}^{J_\lambda}.$$

To verify local finiteness of this decomposition, fix any $t \in \mathbb{R}$. Since F' is locally finite, there exists a neighborhood U_t of t intersecting at most two of the intervals (t_{2n}, t_{2n+2}). Because $\mathbb{V}$ is constant on each such interval, and each interval module $\mathbb{I}^{J_\lambda}$ that is non-zero at a point contributes at least a one-dimensional summand to V_t, and $\dim V_t < \infty$ by assumption, it follows that only finitely many interval summands $\mathbb{I}^{J_\lambda}$ can be non-zero over U_t. Hence, only finitely many intervals J_λ intersect U_t, and the decomposition is locally finite. □

Definition 6.13. [12] *Let $\mathbb{V}$ be a persistence module indexed by a poset P. The **rank invariant** of $\mathbb{V}$ is the function whose domain set is all comparable pairs $(p \leq q)$ in P and*

$$\rho_{\mathbb{V}}(p \leq q) := \operatorname{rank}(\varphi_{p\leq q}) = \dim \operatorname{im}(V_p \to V_q)\,.$$

Definition 6.14. [12] *An invariant of a persistence module is called **complete** if it determines the persistence module up to isomorphism. This means that if two modules have the same invariant, then they are isomorphic.*

Proposition 6.4. [12, Theorem 12.] *The barcode and the rank invariant are complete invariants for pointwise finite-dimensional persistence modules indexed by a totally ordered set $T \subseteq \mathbb{R}$.*

A multiparameter case refers to the variables of the product poset. The rank invariant is particularly useful for multiparameter persistence modules when a barcode decomposition does not exist. While it serves as an invariant, so different values of the rank invariant imply non-isomorphic persistence modules, it is not a complete invariant in the multiparameter setting. Distinct modules can share the same rank invariant.

6.1 Interleaving Distance

Definition 6.15. *An **extended metric** on a set X is a function $d : X \times X \to [0, \infty]$ satisfying the same properties as a metric, except that the distance between some pairs of points may be infinite. A **pseudometric** on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying all the properties of a metric except that $d(x, y) = 0$ does not necessarily imply $x = y$.*

Definition 6.16. [14, 10] *Let $(\mathbb{U}, \varphi)$ and $(\mathbb{V}, \psi)$ be persistence modules indexed by the product poset $\mathbb{R}^n$, and let $v \in [0, \infty)^n$. A $v-$**shifted homomorphism** is a collection of linear maps $\mathbf{f} = f_p : U_p \to V_{p+v}$ for all $p \in \mathbb{R}^n$, such that the diagram*

$$\begin{tikzcd} U_p \arrow[r, "\varphi_{p\leq q}"] \arrow[d, "f_p"] & U_q \arrow[d, "f_q"] \\ V_{p+v} \arrow[r, "\psi_{p+v\leq q+v}"] & V_{q+v} \end{tikzcd}$$

commutes whenever $p \leq q$. We denote

$$\mathit{Hom}^v(\mathbb{U}, \mathbb{V}) := \{v - \textit{shifted homomorphisms } \mathbb{U} \to \mathbb{V}\},$$

$$\mathit{End}^v(\mathbb{V}) := \{v - \textit{shifted homomorphisms } \mathbb{V} \to \mathbb{V}\}.$$

Remark 6.4. We have $\mathrm{Hom}^{v_2}(\mathbb{V}, \mathbb{W}) \times \mathrm{Hom}^{v_1}(\mathbb{U}, \mathbb{V}) \to \mathrm{Hom}^{v_1+v_2}(\mathbb{U}, \mathbb{W})$. Also, for $\delta \geq 0$, let $\delta^* := (\delta, \delta, \ldots, \delta) \in [0, \infty)^n$. We define the **shift map** $\mathbf{1}_V^{\delta^*}$ as the δ^*-shifted endomorphism on $\mathbb{V}$ given by the family of structure maps

$$\{(\mathbf{1}_{\mathbb{V}}^{\delta^*})_p := \varphi_{p\leq p+\delta^*} : V_p \to V_{p+\delta^*}\}_{p\in\mathbb{R}^n}.$$

Definition 6.17. [14, 10] *Let $\delta \geq 0$ and $\mathbb{U}, \mathbb{V}$ be two persistence modules indexed by $\mathbb{R}^n$. Let $\delta^* := \delta \times (1, 1, ..., 1)$. Assume there are maps*

$$\mathbf{f} \in \mathit{Hom}^{\delta^*}(\mathbb{U}, \mathbb{V}), \quad \mathbf{g} \in \mathit{Hom}^{\delta^*}(\mathbb{V}, \mathbb{U})$$

such that

$$\mathbf{gf} = \mathbf{1}_{\mathbb{U}}^{2\delta^*}, \quad \mathbf{fg} = \mathbf{1}_{\mathbb{V}}^{2\delta^*}.$$

*Then $\mathbb{U}$ and $\mathbb{V}$ are called δ-**interleaved** persistence modules. Expansively, the following diagrams commute for all $p \leq q$:*

$$\begin{tikzcd} U_p \arrow[rr, "\phi_{p\leq q}"] \arrow[dr, "f_p"'] & & U_q \arrow[dr, "f_q"] & \\ & V_{p+\delta^*} \arrow[rr, "\psi_{p+\delta^*\leq q+\delta^*}"] & & V_{q+\delta^*} \end{tikzcd} \qquad \begin{tikzcd} U_{p-\delta^*} \arrow[rr, "\phi_{p-\delta^*\leq p+\delta^*}"] \arrow[dr, "f_{p-\delta^*}"'] & & U_{p+\delta^*} \\ & V_p \arrow[ur, "g_p"'] & \end{tikzcd}$$

$$\begin{tikzcd} V_p \arrow[rr, "\psi_{p\leq q}"] \arrow[dr, "g_p"'] & & V_q \arrow[dr, "g_q"] & \\ & U_{p+\delta^*} \arrow[rr, "\phi_{p+\delta^*\leq q+\delta^*}"] & & U_{q+\delta^*} \end{tikzcd} \qquad \begin{tikzcd} V_{p-\delta^*} \arrow[rr, "\psi_{p-\delta^*\leq p+\delta^*}"] \arrow[dr, "g_{p-\delta^*}"'] & & V_{p+\delta^*} \\ & U_p \arrow[ur, "f_p"'] & \end{tikzcd}$$

Definition 6.18. [10] *Let $\mathbb{U}, \mathbb{V}$ be persistence modules indexed by $\mathbb{R}^n$. The **interleaving distance** $d_i(\mathbb{U}, \mathbb{V})$ is defined by*

$$d_i(\mathbb{U}, \mathbb{V}) := \inf\{\delta \geq 0 \mid \mathbb{U} \text{ and } \mathbb{V} \text{ are } \delta\text{-interleaved}\}.$$

Remark 6.5. Note that if $\mathbb{U}$ and $\mathbb{V}$ are δ-interleaved persistence modules then also $\mathbb{U}$ and $\mathbb{V}$ are $\delta + \epsilon$-interleaved for all $\epsilon > 0$. Also, d_i is an extended pseudometric.

Proposition 6.5. [14, Theorem 4.5.] *Let $\mathbb{U}, \mathbb{V}$ be persistence modules indexed by $\mathbb{R}$. Assume $\mathbb{U} = \bigoplus_{\lambda \in L} \mathbb{U}_\lambda$ and $\mathbb{V} = \bigoplus_{\lambda \in L} \mathbb{V}_\lambda$. Then $d_i(\mathbb{U}, \mathbb{V}) \leq sup(d_i(\mathbb{U}_\lambda, \mathbb{V}_\lambda) : \lambda \in L)$.*

Proof. Let $\delta > 0$. Assume for all $\lambda \in L$, $\mathbb{U}_\lambda$ and $\mathbb{V}_\lambda$ are δ-interleaved. Then there are $\mathbf{f}_\lambda \in \mathrm{Hom}^\delta(\mathbb{U}_\lambda, \mathbb{V}_\lambda)$ and $\mathbf{g}_\lambda \in \mathrm{Hom}^\delta(\mathbb{V}_\lambda, \mathbb{U}_\lambda)$ such that $\mathbf{g}_\lambda \mathbf{f}_\lambda = \mathbf{1}^{2\delta}_{\mathbb{U}_\lambda}$ and $\mathbf{f}_\lambda \mathbf{g}_\lambda = \mathbf{1}^{2\delta}_{\mathbb{V}_\lambda}$.Then the direct sum of maps $\mathbf{f} := \oplus_{\lambda \in L} \mathbf{f}_\lambda \in \mathrm{Hom}^\delta(\mathbb{U}, \mathbb{V})$ and $\mathbf{g} := \oplus_{\lambda \in L} \mathbf{g}_\lambda \in \mathrm{Hom}^\delta(\mathbb{V}, \mathbb{U})$ satisfy $\mathbf{gf} = \mathbf{1}^{2\delta}_{\mathbb{U}}$ and $\mathbf{fg} = \mathbf{1}^{2\delta}_{\mathbb{V}}$. Hence, $d_i(\mathbb{U}, \mathbb{V}) \leq \delta$. By taking the infimum over such λ, we get the desired inequality. □

Proposition 6.6. [14, Theorem 4.19.] *Let $\mathbb{V}$ be a persistence module indexed by $\mathbb{R}$. Then, $\mathbb{V}$ is q-tame if and only if for every $\epsilon > 0$ there exists a locally finite persistence module $\mathbb{W}$ such that $d_i(\mathbb{V}, \mathbb{W}) < \epsilon$.*

6.2 Bottleneck Distance

Definition 6.19. *A **multiset** M is a pair (S, m), where:*

- *S is the underlying set of distinct elements, called the support of the multiset.*
- *$m : S \to \mathbb{N} \cup \{\infty\}$ is a multiplicity function, where $m(x)$ gives the number of occurrences of x in the multiset.*

Definition 6.20. [14] *Let A and B be multisets. A **partial matching** between A and B is a subset $M \subseteq A \times B$ such that for every $x \in A$ there is at most one $y \in B$ with $(x, y) \in M$, and for every $y \in B$ there is at most one $x \in A$ with $(x, y) \in M$. Note that unmatched elements are allowed.*

Definition 6.21. [14] *The **extended upper half plane** is defined as $\overline{H} = \{(b, d) \in (\{-\infty\} \cup \mathbb{R} \cup \{+\infty\})^2 : b \leq d\}$.*

Definition 6.22. [14] *Let $A, B \subset \overline{H}$ be multisets. A δ-**matching** between A and B is a partial matching satisfying:*

1. *For every $(p, q) \in M$, $\|p - q\|_\infty \leq \delta$ ($\|(b, d)\|_\infty := \max\{|b|, |d|\}$).*
2. *For every unmatched point $p \in A$ (and for $q \in B$), we assign it to the diagonal $\Delta = \{(x, x) : x \in \mathbb{R}\}$, so that $d_\infty(p, \Delta) \leq \delta$. ($d_\infty\big((b, d), \Delta\big) = \frac{d-b}{2}$.)*

*For multisets $A, B \subset \overline{H}$, the **bottleneck distance** is defined as*

$$d_b(A, B) := \inf \Big\{ \delta \geq 0 : \text{ there exists a } \delta\text{-matching between } A \text{ and } B \Big\}.$$

*Equivalently, the **bottleneck distance** can be defined by*

$$d_b(A, B) = \inf_{\phi} \sup_{x \in A} \|x - \phi(x)\|_\infty,$$

where ϕ ranges over all bijections between A and B (allowing points to be matched to the diagonal).

Remark 6.6. The bottleneck distance is an extended pseudometric. But for locally finite multisets, it becomes an extended metric because $d_b(A, B) = 0$ if and only if A and B coincide as multisets. If A and B are locally finite (i.e., every bounded subset of $\overline{H}$ contains only finitely many points), then the infimum is attained as a minimum [1].

Definition 6.23. [14] *Let $\mathbb{V}$ be an interval decomposable persistence module indexed by $T \subseteq \mathbb{R}$. Write $M \cong \bigoplus_{\lambda \in \mathcal{B}(\mathbb{V})} \mathbb{I}^{J_\lambda}$, where $\mathcal{B}(\mathbb{V})$ is its barcode. The **persistence diagram** $\mathrm{dgm}(\mathbb{V})$ is the representation of $\mathbb{V}$ recording each interval of $\mathcal{B}(\mathbb{V})$ by the pair $(b, d) \in \mathbb{R}^2$. Then we define **bottleneck distance** for interval decomposable persistence modules $\mathbb{V}, \mathbb{W}$,*

$$d_b(\mathcal{B}(\mathbb{V}), \mathcal{B}(\mathbb{W})) := d_b(\mathrm{dgm}(\mathbb{V}), \mathrm{dgm}(\mathbb{W})).$$

When the indexing set $T \subseteq \mathbb{R}$ is locally finite, we use a representation using half-open intervals of the form $[b, d)$ for finite intervals. In this framework, one can define:

- ***Birth:** The left endpoint b (or $-\infty$ if the feature is present from the beginning).*
- ***Death:** The right endpoint d (or $+\infty$ if the feature never dies).*
- ***Persistence:** The duration $d - b$ (which may be infinite).*

[1] [14, Theorem 4.10.]

We can generalize the bottleneck distance to the multiparameter case.

Definition 6.24. [10] *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be barcodes of two interval decomposable multiparameter persistence modules indexed by the product poset $\mathbb{R}^n$. A δ-**matching** M between $\mathcal{B}_1$ and $\mathcal{B}_2$ is a partial matching satisfying:*

1. *For every interval module $(\mathbb{I}, \mathbb{J}) \in M$, $d_i(\mathbb{I}, \mathbb{J}) \leq \delta$.*
2. *For every unmatched interval module $\mathbb{I} \in \mathcal{B}_1$ (and for $\mathbb{J} \in \mathcal{B}_2$), $d_i(\mathbb{I}, 0) \leq \delta$.*

*The **bottleneck distance** between $\mathcal{B}_1$ and $\mathcal{B}_2$ is defined as*

$$d_b(\mathcal{B}_1, \mathcal{B}_2) := \inf \Big\{ \delta \geq 0 : \textit{ there exists a } \delta\textit{-matching between } \mathcal{B}_1 \textit{ and } \mathcal{B}_2 \Big\}.$$

*Equivalently, the **bottleneck distance** can be defined by*

$$d_b(\mathcal{B}_1, \mathcal{B}_2) = \inf_{\phi} \sup_{\mathbb{I} \in \mathcal{B}_1} d_i(\mathbb{I}, \phi(\mathbb{I})),$$

where ϕ ranges over all bijections between $\mathcal{B}_1$ and $\mathcal{B}_2$ (allowing points to be matched to the 0 persistence module).

Note that this definition coincides with the one-parameter case by propositions 6.7 and 6.8. Let $\mathbb{U}$ and $\mathbb{V}$ be interval decomposable persistence modules indexed by $T \subseteq \mathbb{R}$. It is enough to show that there exists δ-matching between their persistence diagrams if and only if there exists δ-matching between their barcodes.

1. From point-matchings to module-matchings: Suppose we are given a partial matching of points $M_{\text{pts}} \subseteq \text{dgm}(\mathbb{U}) \times \text{dgm}(\mathbb{V})$ such that every matched pair $(p_i, q_i) \leftrightarrow (r_j, s_j)$ satisfies $\|(p_i, q_i) - (r_j, s_j)\|_\infty \leq \delta$, and every unmatched point lies within δ distance of the diagonal. By proposition 6.7, this ensures that the corresponding interval modules satisfy $d_i(\mathbb{I}^{J_i}, \mathbb{I}^{K_j}) \leq \|(p_i, q_i) - (r_j, s_j)\|_\infty \leq \delta$. Likewise by proposition 6.8, each unmatched point (p_i, q_i) corresponds to the interval $\mathbb{I}^{J_i}$, and $d_i(\mathbb{I}^{J_i}, 0) = \frac{q_i - p_i}{2} = d_\infty((p_i, q_i), \Delta) \leq \delta$, ensuring that unmatched intervals are δ-close to the zero persistence module. Hence, we obtain a valid partial δ-matching $M_{\text{mdl}} \subseteq \mathcal{B}(\mathbb{U}) \times \mathcal{B}(\mathbb{V})$ of barcodes by setting $(\mathbb{I}^{J_i}, \mathbb{I}^{K_j})$ for each matched pair $(p_i, q_i) \leftrightarrow (r_j, s_j)$.

2. From module-matchings to point-matchings: Conversely, suppose we are given a partial matching $M_{\text{mdl}} \subseteq \mathcal{B}(\mathbb{U}) \times \mathcal{B}(\mathbb{V})$ such that every matched pair $(\mathbb{I}^{J_i}, \mathbb{I}^{K_j})$ satisfies $d_i(\mathbb{I}^{J_i}, \mathbb{I}^{K_j}) \leq \delta$, and each unmatched interval satisfies $d_i(\mathbb{I}^{J_i}, 0) \leq \delta$ (and $d_i(\mathbb{I}^{K_j}, 0) \leq \delta$).

We may first refine this matching to ensure that for every matched pair $(\mathbb{I}^{J_i}, \mathbb{I}^{K_j})$, the midpoint of J_i lies in the closure of K_j, and vice versa. If this condition fails, we remove the pair from the matching and instead send both intervals to the zero module. Since the original interleaving distance was at most δ, this refinement is still a valid δ-matching of modules because $\delta \geq d_i(\mathbb{I}^{J_i}, \mathbb{I}^{K_j}) = \max(\frac{q_i - p_i}{2}, \frac{s_j - r_j}{2}) = \max(d_i(\mathbb{I}^{J_i}, 0), d_i(\mathbb{I}^{K_j}, 0))$ by propositions 6.7 and 6.8. Now that the midpoint condition holds, we apply proposition 6.7, which ensures that for matched modules $\|(p_i, q_i) - (r_j, s_j)\|_\infty = d_i(\mathbb{I}^{J_i}, \mathbb{I}^{K_j}) \leq \delta$. And for unmatched modules, by proposition 6.8, $d_\infty((p_i, q_i), \Delta) = d_i(\mathbb{I}^{J_i}, 0) = \frac{q_i - p_i}{2} \leq \delta$, so each unmatched point lies within δ distance of the diagonal. Thus, we define a valid partial δ-matching between the persistence diagrams $M_{\text{pts}} \subseteq \mathrm{dgm}(\mathbb{U}) \times \mathrm{dgm}(\mathbb{V})$, by setting $(p_i, q_i) \leftrightarrow (r_j, s_j)$ for each updated matched pair $(\mathbb{I}^{J_i}, \mathbb{I}^{K_j})$.

Proposition 6.7. [14, Proposition 4.6.] *Let $\mathbb{I}_1 = \mathbb{I}^{J_1}, \mathbb{I}_2 = \mathbb{I}^{J_2}$ be two interval modules such that J_1 is one of $(p, q), [p, q), (p, q], [p, q]$ and J_2 is one of $(r, s), [r, s), (r, s], [r, s]$ (endpoints are allowed to be infinite). Then $d_i(\mathbb{I}_1, \mathbb{I}_2) \leq \|(p, q) - (r, s)\|_\infty$. The equality holds if and only if $\frac{p+q}{2} \in [r, s]$ and $\frac{r+s}{2} \in [p, q]$. If the midpoint condition doesn't hold then $d_i(\mathbb{I}_1, \mathbb{I}_2) = \max(\frac{q-p}{2}, \frac{s-r}{2})$.*

Proof. Let $\delta > 0$. Suppose $\mathbb{I}_1$ and $\mathbb{I}_2$ are δ-interleaved. Then there exist morphisms $\mathbf{f} \in \mathrm{Hom}^\delta(\mathbb{I}_1, \mathbb{I}_2)$ and $\mathbf{g} \in \mathrm{Hom}^\delta(\mathbb{I}_2, \mathbb{I}_1)$ such that $\mathbf{gf} = \mathbf{1}^{2\delta}_{\mathbb{I}_1}$, $\mathbf{fg} = \mathbf{1}^{2\delta}_{\mathbb{I}_2}$. This means that for all t, the composition $\mathbb{I}_{1,(t)} \xrightarrow{f_t} \mathbb{I}_{2,(t+\delta)} \xrightarrow{g_{t+\delta}} \mathbb{I}_{1,(t+2\delta)}$ must equal the structure map $\varphi^{\mathbb{I}_1}_{t,t+2\delta}$ (and the other is likewise). For $\delta > \max(|p - r|, |q - s|) = \|(p - r, q - s)\|_\infty$, take $f_t = \mathrm{id}_F$ if both domain and codomain are F; otherwise take $f_t = 0$. Similarly, define g_t for all t. These functions satisfy the desired conditions.

Now, assume the midpoint condition holds. Take $\delta < \max(|p - r|, |q - s|)$. Then, regardless of choices of $\mathbf{f}$ and $\mathbf{g}$, for some t, which is close to one of the endpoints, we have the following diagram, which cannot commute:

$$\begin{tikzcd} F \arrow[rr] \arrow[dr] & & F \\ & 0(= \mathbb{I}_t) \arrow[ur] & \end{tikzcd}$$

Hence, combining with above, we get $d_i(\mathbb{I}_1, \mathbb{I}_2) = \|(p, q) - (r, s)\|_\infty$.

Now, suppose the midpoint condition doesn't hold. Let $\delta > \max(\frac{q-p}{2}, \frac{s-r}{2})$. Take $f_t = \mathrm{id}_F$ if both domain and codomain are F; otherwise take $f_t = 0$. Similarly, define g_t for all t. These functions satisfy the desired conditions. If $\delta < \max(\frac{q-p}{2}, \frac{s-r}{2})$, then regardless of choices of $\mathbf{f}$ and $\mathbf{g}$, for the midpoint, which breaks the condition, we have the following diagram, which cannot commute:

$$\begin{tikzcd}
F \arrow[rr] \arrow[dr] & & F \\
& 0(=\mathbb{I}_{\text{midpoint}}) \arrow[ur] &
\end{tikzcd}$$

Hence, by taking the infimum over possible δ, we have $d_i(\mathbb{I}_1, \mathbb{I}_2) = \max\left(\frac{q-p}{2}, \frac{s-r}{2}\right)$. $\square$

Proposition 6.8. [14, Proposition 4.7.] *Let $\mathbb{I} = \mathbb{I}^J$ be an interval module indexed by $\mathbb{R}$ such that J is one of $(p, q), [p, q), (p, q], [p, q]$ (endpoints are allowed to be infinite). Then $d_i(\mathbb{I}, 0) = \frac{q-p}{2}$ where 0 is the zero persistence module.*

Proof. Let $\delta > 0$. Assume $\mathbb{I}$ and 0 are δ-interleaved. Then there are $\mathbf{f} \in \mathrm{Hom}^\delta(\mathbb{I}, 0)$ and $\mathbf{g} \in \mathrm{Hom}^\delta(0, \mathbb{I})$ such that $\mathbf{gf} = 1_{\mathbb{I}}^{2\delta}$, $\mathbf{fg} = 1_0^{2\delta}$. Note that $\mathbf{g} = 0$ and $1_0^{2\delta} = 0$. So, the second condition is satisfied, and the first condition tells us $1_{\mathbb{I}}^{2\delta} = 0$ independent of all $\mathbf{f}$. So, it is enough to find δ such that $1_{\mathbb{I}}^{2\delta} = 0$.

This means that the linear maps $\varphi^{\mathbb{I}}_{t,t+2\delta} : \mathbb{I}_t \to \mathbb{I}_{t+2\delta}$ must be zero for all t. For the interval module $\mathbb{I}^J$, this is satisfied when the interval J has length less than 2δ, or equivalently, $2\delta > q - p$ and this cannot happen when $2\delta < q - p$. Hence, $d_i(\mathbb{I}, 0)$ is the infimum over all possible δ, i.e., $d_i(\mathbb{I}, 0) = \frac{q-p}{2}$. $\square$

Chapter 7

PERSISTENT HOMOLOGY

Definition 7.1. [10] *Let $(P, \leq)$ be a poset. A **filtration** indexed by P is a functor*

$$\mathcal{F} : \mathbf{P} \to \mathbf{Top},$$

*where we require that for each $p \leq q$ in P, the map $\mathcal{F}_{p \leq q} : \mathcal{F}_p \hookrightarrow \mathcal{F}_q$ is inclusion. An n-**parameter filtration** is a filtration indexed by a product poset $P = T_1 \times \cdots \times T_n$.*

Note that we can see **Simp** as a subcategory of **Top** by taking geometric realization.

Example: [10] Given a set of points Y in a metric space (X, d), the **offset filtration** is the functor $\mathcal{O}(Y) : (0, \infty) \to$ **Top** such that at scale ϵ it is given by

$$\mathcal{O}(Y)_\epsilon = \bigcup_{x \in Y} B(x, \epsilon).$$

Example: [10] Given a set of points Y in a metric space (X, d), the **Čech filtration** is the functor $\check{\mathcal{C}}(Y) : (0, \infty) \to$ **Simp** such that at scale ϵ it gives an abstract simplicial complex where a subset $\sigma \subseteq Y$ is a simplex if and only if the open ϵ-balls around the points in σ intersect non-trivially, i.e.,

$$\sigma \in \check{\mathcal{C}}(Y)_\epsilon \iff \bigcap_{v \in \sigma} B(v, \epsilon) \neq \emptyset.$$

Example: [10] Given a set of points Y in a metric space (X, d), the **Vietoris-Rips filtration** is the functor $\mathcal{VR}(Y) : (0, \infty) \to$ **Simp** such that at scale ϵ it gives an abstract simplicial complex where a subset $\sigma \subseteq Y$ is a simplex if and only if the distance between any pair of points in σ is smaller than 2ϵ, i.e.,

$$\sigma \in \mathcal{VR}(Y)_\epsilon \iff d(v_i, v_j) < 2\epsilon \quad \forall v_i, v_j \in \sigma.$$

While constructing the Čech complex requires examining all $(n+1)$-tuples of points to determine n-simplices, the Vietoris–Rips complex only requires pairwise distance comparisons. Therefore, the Rips complex is computationally more efficient to construct; however, it may not fully capture the underlying topology.

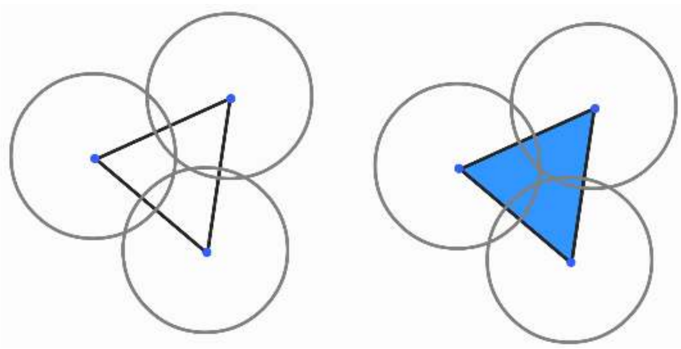

Figure 7.1: The Difference Between a Čech Complex and a Vietoris-Rips Complex

Proposition 7.1. [23, Page 62] *Let Y be a set of points in a metric space $\mathbb{R}^n$ and $r > 0$. Then $\mathcal{VR}(Y)_{r\sqrt{\frac{n+1}{2n}}} \subseteq \check{\mathcal{C}}(Y)_r \subseteq \mathcal{VR}(Y)_r$.*

Proof. (Left Inclusion:) Let $\sigma = \{y_0, \ldots, y_k\} \subset Y$ be a simplex in the Vietoris–Rips complex $\mathcal{VR}(Y)_\rho$, where $\rho = r\sqrt{\frac{n+1}{2n}}$. Then, by definition, the pairwise distances satisfy $\|y_i - y_j\| < 2\rho$ for all i, j. Hence, the diameter of σ is less than 2ρ. By Jung's Theorem, there exists a closed ball of radius at most

$$R = \sqrt{\frac{n}{2(n+1)}} \cdot \operatorname{diam}(\sigma) < \sqrt{\frac{n}{2(n+1)}} \cdot 2\rho = r$$

that contains all the points $y_0, \ldots, y_k$. Therefore, the open ball with the same center and radius r contains all the points, and in particular, the intersection $\bigcap_i B(y_i, r)$ of the open balls is non-empty. Thus, $\sigma \in \check{\mathcal{C}}(Y)_r$, and we have $\mathcal{VR}(Y)_{r\sqrt{\frac{n+1}{2n}}} \subseteq \check{\mathcal{C}}(Y)_r$.

(Right Inclusion:) Now suppose $\sigma = \{y_0, \ldots, y_k\} \in \check{\mathcal{C}}(Y)_r$. Then, by definition, the intersection $\bigcap_i B(y_i, r)$ is non-empty. Let $x \in \bigcap_i B(y_i, r)$. Then for all i, j,

$$\|y_i - y_j\| \leq \|y_i - x\| + \|x - y_j\| < r + r = 2r.$$

Thus, the pairwise distances between the points in σ are all less than $2r$, so $\sigma \in \mathcal{VR}(Y)_r$. Therefore, $\check{\mathcal{C}}(Y)_r \subseteq \mathcal{VR}(Y)_r$. □

Example: [6] Given a metric measure space (X, d, μ), the **measure bifiltration** is the functor $\mathcal{MB}(X, \mu) : (0, \infty)^{op} \times (0, \infty) \to$ **Top** such that at scale k, ϵ it gives

$$\mathcal{MB}(X, \mu)_{k,\epsilon} = \{x \in X : \mu(B(x, \epsilon)) \geq k\}.$$

Example: [6] Given a simplicial complex K, the **subdivision filtration** is the functor $\mathcal{SF}(K) : \mathbb{N}^{op} \to \mathbf{Simp}$ such that at scale k it gives maximal subcomplex of $Sd(K)$ whose vertex set is simplicies of K whose dimension is at least $k-1$. Given a set of points Y in a metric space (X, d), the **subdivision-Rips filtration** is the functor $\mathcal{SR}(Y) : \mathbb{N}^{op} \times (0, \infty) \to \mathbf{Simp}$ such that at scale (k, ϵ) it gives

$$\mathcal{SR}(Y)_{k,\epsilon} = \mathcal{SF}(\mathcal{VR}(Y)_\epsilon)_k.$$

Definition 7.2. *[7] Fix a field of coefficients F. Given a filtration $\mathcal{F} : \mathbf{P} \to \mathbf{Top}$, the **persistent homology module** in degree k is the functor*

$$\mathbb{H}_k(\mathcal{F}; F) : \mathbf{P} \to \mathbf{Vect}_F.$$

Explicitly, for each $p \in P$, we set $\mathbb{H}_k(\mathcal{F}; F)_p := H_k(\mathcal{F}_p; F)$, and for each $p \leq q$, the inclusion $\mathcal{F}_p \hookrightarrow \mathcal{F}_q$ induces the linear map

$$\mathbb{H}_k(\mathcal{F}; F)_{p \leq q} := H_k(\mathcal{F}_p; F) \to H_k(\mathcal{F}_q; F).$$

*The **persistent Betti number** between parameters $p \leq q$ is defined as the rank of this induced map, i.e.,*

$$\beta_k^{p \leq q} := \operatorname{rank}\left(\mathbb{H}_k(\mathcal{F}; F)_{p \leq q}\right).$$

It counts the number of k-dimensional homological features that are born at or before p and persist for at least q.

Example: Consider (the geometric realization of a simplicial) filtration as shown in 7.2 and 7.3. $K_1 \subseteq K_2 \subseteq K_3 \subseteq K_4$. Apply the k-th homology functor with coefficients in a field $\mathbb{R}$: $\mathbb{H}_k(\mathcal{F}; \mathbb{R}) : \{1, 2, 3, 4\} \to \mathbf{Vect}_\mathbb{R}$.

In *degree 0*, we track path-connected components. Initially, each component is isolated. As edges are added, components merge. When two components merge, the older one (i.e., the one that was born earlier) survives. This is called the **elder rule**. In *degree 1*, we track 1-dimensional holes (loops). A loop is born when a cycle of edges is formed, and it dies when a 2-simplex fills it in.

In the barcode representations of these persistent homology modules shown in 7.2 and 7.3, each horizontal arrow represents a homological feature. For example, a blue bar from

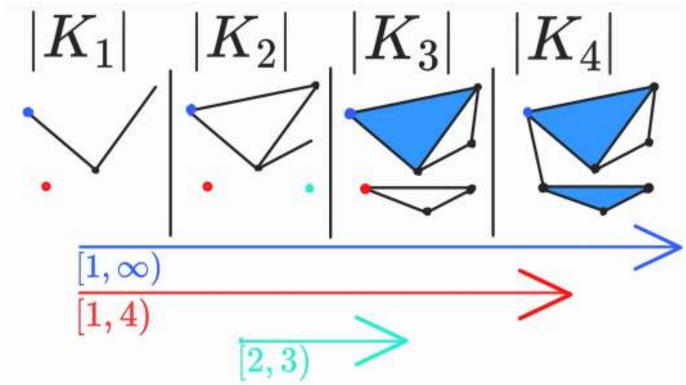

Figure 7.2: Persistent Homology Module of A Filtration in Degree 0

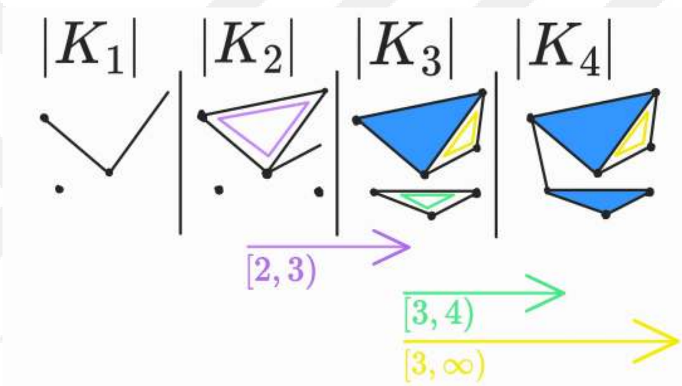

Figure 7.3: Persistent Homology Module of A Filtration in Degree 1

1 to ∞ in 7.2 indicates a path connected component that is born at K_1 and persists forever; and a purple bar from 2 to 3 in 7.3 indicates a loop that is born at K_2 and dies at K_3 when a 2-simplex fills it. Also, figure 7.4 gives a table for this filtration.

	1	2	3	4
1	2	2	2	1
2	-	3	2	1
3	-	-	2	1
4	-	-	-	1

	1	2	3	4
1	0	0	0	0
2	-	1	0	0
3	-	-	2	1
4	-	-	-	1

Figure 7.4: Left: 0^{th} and Right: 1^{st} Persistent Betti Numbers of Above Filtration

In the construction of barcodes for persistent homology indexed by a totally ordered set, the **elder rule** is used to decide which topological feature survives when two features merge. According to this rule, the feature that appears earlier in the filtration — the "elder" — is considered more persistent and therefore survives, while the newer one dies. This choice is necessary because there is no intrinsic labeling of homology class representatives, so we require a consistent convention to assign birth–death pairs. Without the elder rule, the barcode would not be well-defined. For persistent homology indexed by a general poset, there may exist incomparable indices. In such cases, the notions of "birth" and "death" become ill-defined. As a result, persistence diagrams, which rely on assigning a unique birth and death time to each feature, cannot be constructed.

Note that the persistent Betti numbers in degree k coincide with the rank invariant of the k-th persistent homology module, i.e., $\beta_k^{p\leq q} = \rho_{\mathbb{H}_k}(p \leq q)$ for all $p \leq q \in P$. Thus, for pointwise-finite dimensional persistence homology modules indexed by $P = T \subseteq \mathbb{R}$, the collection of all persistent Betti numbers $\{\beta_k^{p\leq q}\}_{p\leq q}$ is complete invariant too. For example, if the k-th persistent homology module is indexed by a locally finite $T \subset \mathbb{R}$, then we can calculate the multiplicity of (b, d) in its persistence diagram. Let $a < b$ and $c < d$ be the greatest elements in T. Then the multiplicity of (b, d) is equal to $\beta_k^{a\leq d} - \beta_k^{b\leq d} + \beta_k^{b\leq c} - \beta_k^{a\leq c}$.

Definition 7.3 ([13]). *Let X be a topological space and let $f : X \to \mathbb{R}^n$ be a continuous function. For each $p \in \mathbb{R}^n$, define the* ***sublevel set*** *as $X_p = f^{-1}(\{x \in \mathbb{R}^n \mid f(x) \leq p\})$. Since for $p \leq q$ (coordinate-wise) we have $X_p \subseteq X_q$, the collection $\{X_p\}_{p\in\mathbb{R}^n}$ forms a filtration, called the* ***sublevel set filtration*** *of (X, f). Applying the homology functor $H_k(-; F)$ with coefficients in a field F to each sublevel set X_p, we define $V_p = H_k(X_p; F)$ for a fixed $k \geq 0$. For $p \leq q$, the inclusion $X_p \hookrightarrow X_q$ induces a linear map $\varphi_{p\leq q} := (H_k)_*(X_p \hookrightarrow X_q) : V_p \to V_q$. Thus, the collection $\mathbb{H}_{k-\mathrm{sub}}(X, f) = \{V_p, \varphi_{p\leq q}\}_{p\in\mathbb{R}^n}$ forms the* ***sublevel set persistent homology module****.*

Proposition 7.2. [14, Theorem 2.22.] *Let X be a triangulable space such that it is a geometric realization of a finite simplicial complex, and $f : X \to \mathbb{R}$ be a continuous function. Then $\mathbb{H}_{k-sub}(X, f)$ is q-tame, i.e., its persistent Betti numbers $\beta_k^{p\leq q}$ are finite for all $p < q$.*

Proof. Let $p < q$. Since X is a finite realization, it is compact. Since f is continuous on a compact space, f is uniformly continuous on X. To get a distance function, we can embed X into an Euclidean space. Hence, for $\varepsilon = q - p > 0$ there exists $\delta > 0$ such that whenever $d(x, y) < \delta$ for $x, y \in X$ we have $|f(x) - f(y)| < q - p$.

By repeatedly applying barycentric subdivision to K, we obtain a new triangulation K' of X in which the diameter of every simplex is less than δ. Consequently, for every simplex σ in K', the oscillation of f on $|\sigma|$ is less than $q - p$, that is,

$$\sup_{x,y\in|\sigma|} |f(x) - f(y)| < q - p.$$

This implies that no simplex σ in K' can intersect both X_p and X_q. Otherwise, there were points $x, y \in |\sigma|$ with $f(x) \leq p$ and $f(y) \geq q$, then $|f(x)-f(y)| \geq q-p$, a contradiction.

Define the subcomplex Y of K' by

$$Y = \bigcup\{\sigma \in K' \mid |\sigma| \cap f^{-1}((-\infty, p]) \neq \emptyset\}.$$

Note that Y is a finite simplicial complex and $Y \subset f^{-1}((-\infty, q))$. So, the inclusion $i_{p\leq q} : X_p \hookrightarrow X_q$ factors as $X_p \hookrightarrow Y \hookrightarrow X_q$. Since the homology groups $H_*(Y; f)$ of a finite simplicial complex are finite-dimensional and homology is functorial, the induce map $i_{(p\leq q)*} : H_*(X_p; F) \to H_*(X_q; f)$ has finite rank. Thus, $\mathbb{H}_{sub}(X, f)$ is q-tame. $\square$

It is possible to generalize this result to a more powerful situation.

Proposition 7.3. [13, Theorem 2.3.] *Let X be a triangulable space and $f : X \to \mathbb{R}^n$ be a continuous function. Then the persistent Betti numbers $\beta_k^{p\leq q}$ of $\mathbb{H}_{k-sub}(X, f)$ are finite for all $p < q$.*

Proposition 7.4. [15, Proposition 5.1.] *Let (X, d) be a metric space. Assume it is totally bounded. Then $\mathbb{H}_k(\check{\mathcal{C}}(X))$ and $\mathbb{H}_k(\mathcal{VR}(X))$ are q-tame.*

Let $X \subset \mathbb{R}^n$ be a bounded subset. Since $\mathbb{R}^n$ is a complete metric space, every bounded subset of $\mathbb{R}^n$ is totally bounded. Therefore, the persistence modules arising from the Čech and Vietoris–Rips filtrations on bounded subsets of $\mathbb{R}^n$ are q-tame. Moreover, when the filtration is built on a finite set of points, the resulting persistence modules are always pointwise-finite dimensional, and hence also q-tame.

7.1 Persistent Nerve Theorems

The Nerve Theorem can be naturally extended to the context of persistent homology through the Persistent Nerve Theorem. It asserts that if at every stage of a filtration the topological space is covered by a good open cover with compatible inclusions, then the persistent homology of the filtration is isomorphic to the persistent homology of the associated filtration of nerve complexes. This result is significant because it allows one to replace complicated geometric filtrations by combinatorial objects without losing topological information at the level of persistent homology. Since nerves are simplicial complexes built from the combinatorics of the covers, they enable efficient and reliable algorithmic computation and analysis.

Theorem 7.1. [20, Corollary 4.3.[1]] (Persistent Nerve Theorem) *Let $X \subseteq X'$ be two paracompact Hausdorff spaces. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ and $\mathcal{U}' = \{U'_\alpha\}_{\alpha \in A}$ be good open covers of X and X' respectively such that $U_\alpha \subseteq U'_\alpha$ for all $\alpha \in A$. Then there exist homotopy equivalences*

$$f : |\mathrm{Nerve}(\mathcal{U})| \to X, \quad g : |\mathrm{Nerve}(\mathcal{U}')| \to X'$$

such that the following diagram commutes:

$$\begin{tikzcd}
H_k(|\mathrm{Nerve}(\mathcal{U})|) \arrow[r] \arrow[d, "f_*"'] & H_k(|\mathrm{Nerve}(\mathcal{U}')|) \arrow[d, "g_*"] \\
H_k(X) \arrow[r] & H_k(X')
\end{tikzcd}$$

Here, the horizontal maps are induced by the inclusions $X \hookrightarrow X'$ and $|\mathrm{Nerve}(\mathcal{U})| \hookrightarrow |\mathrm{Nerve}(\mathcal{U}')|$, the latter arising from the simplicial inclusion $\mathrm{Nerve}(\mathcal{U}) \hookrightarrow \mathrm{Nerve}(\mathcal{U}')$ induced by the inclusions $U_\alpha \subseteq U'_\alpha$. The vertical maps are induced by the homotopy equivalences f and g. (Note that the vertical maps are isomorphisms since homotopy equivalence maps induce isomorphisms at the homology level.)

*This result can be generalized to filtered spaces. Let P be a poset and let $\mathcal{F} : P \to$ **Top** be a filtration assigning each $p \in P$ a paracompact Hausdorff space $X_p =: \mathcal{F}_p$.*

[1]Theorem 4.2 is a generalized version where one can compare two paracompact Hausdorff spaces X, X' via a continuous map $f : X \to X'$ satisfying $f(U_a) \subseteq U'_a$. Since our focus is on filtration, we are using the corollary version.

Suppose that for each $p \in P$, we are given a good open cover $\mathcal{U}_p = \{U_{p,\alpha}\}_{\alpha \in A}$ of X_p such that for all $p \leq q \in P$, we have $U_{p,\alpha} \subseteq U_{q,\alpha}$ for all $\alpha \in A$. Then the nerve construction defines a new filtration

$$\mathcal{N} : \mathbf{P} \to \mathbf{Top}, \quad \mathcal{N}_p := |\mathrm{Nerve}(\mathcal{U}_p)|,$$

where the structure map $\mathcal{N}_{p \leq q} : \mathcal{N}_p \to \mathcal{N}_q$ is the geometric realization of the simplicial inclusion $\mathrm{Nerve}(\mathcal{U}_p) \hookrightarrow \mathrm{Nerve}(\mathcal{U}_q)$, induced by the inclusions $U_{p,\alpha} \subseteq U_{q,\alpha}$. Then for each $k \geq 0$, the persistent homology modules are isomorphic, i.e.,

$$\mathbb{H}_k(\mathcal{F}) \cong \mathbb{H}_k(\mathcal{N}).$$

Remark 7.1. The filtration $\mathcal{N}$ is well defined because the inclusions $U_{p,\alpha} \subseteq U_{q,\alpha}$ for $p \leq q$ directly induce a simplicial map $\mathrm{Nerve}(\mathcal{U}_p) \to \mathrm{Nerve}(\mathcal{U}_q)$. If $\{\alpha_0, \ldots, \alpha_k\}$ forms a simplex in $\mathrm{Nerve}(\mathcal{U}_p)$ (meaning $\bigcap_j U_{p,\alpha_j} \neq \emptyset$), then $\bigcap_j U_{q,\alpha_j}$ is also nonempty (since it contains $\bigcap_j U_{p,\alpha_j}$). Thus, $\{\alpha_0, \ldots, \alpha_k\}$ also forms a simplex in $\mathrm{Nerve}(\mathcal{U}_q)$, confirming the simplicial inclusion.

Remark 7.2. The following diagram may fail to commute as a diagram of continuous maps at the topological level

$$\begin{array}{ccc} |\mathrm{Nerve}(\mathcal{U})| & \longrightarrow & |\mathrm{Nerve}(\mathcal{U}')| \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ X & \longrightarrow & X' \end{array}$$

However, the compositions along both paths in the square are homotopic. Since homotopic maps induce the same homomorphisms in homology, the diagram does commute at the homology level.

Example: The persistent homology of offset filtrations of point clouds can be computed via the Čech complex, which is the nerve of the cover by balls. Observe that for each scale $\epsilon > 0$, the collection of open balls $\mathcal{U}_\epsilon := \{B(x,\epsilon) \mid x \in Y\}$ forms an open cover of the offset space $\mathcal{O}(Y)_\epsilon = \bigcup_{x \in Y} B(x,\epsilon)$. Since each ball is convex and any finite intersection of convex open balls is either empty or convex, which is contractible, the cover $\mathcal{U}_\epsilon$ is good. Moreover, if $0 < \epsilon \leq \delta$, then $B(x,\epsilon) \subseteq B(x,\delta)$ for all $x \in Y$. Therefore, the Persistence Nerve Theorem applies and yields an isomorphism of persistent homology modules, i.e.,

$$\mathbb{H}_k(\mathcal{O}(Y)) \cong \mathbb{H}_k(\check{\mathcal{C}}(Y))$$

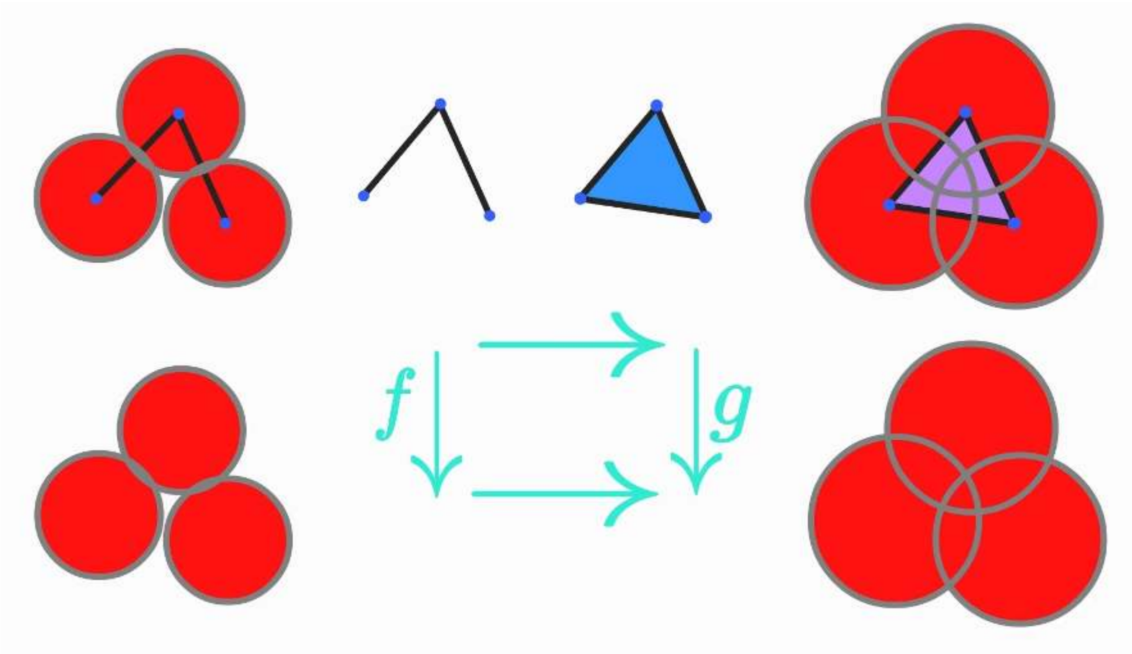

Figure 7.5: Persistent Nerve Theorem-The compositions along both paths are homotopic.

for all homological degrees $k \geq 0$. This means the persistent homology of the geometric offset filtration is completely captured by the combinatorial Čech filtration.

Theorem 7.2. [16, Theorem 26] (Persistent Nerve Theorem for Simplicial Complexes) *Let $\mathcal{K} = \{K_i\}_{i\in[n]}$ and $\mathcal{K}' = \{K'_i\}_{i\in[n]}$ be finite collections of subcomplexes of topological spaces K and K', respectively, such that:*

- $\bigcup_{i\in[n]} K_i = K$*, and* $\bigcup_{i\in[n]} K'_i = K'$*,*
- *For each* $i \in [n]$*, we have* $K_i \subseteq K'_i$*,*
- $\{|K_i|\}_{i\in[n]}$ *is a good cover of* $|K|$ *and* $\{|K'_i|\}_{i\in[n]}$ *is a good cover of* $|K'|$.

Then there exist homotopy equivalences

$$f : |\text{Nerve}(\mathcal{K})| \to |K|, \quad g : |\text{Nerve}(\mathcal{K}')| \to |K'|$$

such that the following diagram commutes:

$$\begin{array}{ccc} H_k(|\text{Nerve}(\mathcal{K})|) & \longrightarrow & H_k(|\text{Nerve}(\mathcal{K}')|) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ H_k(|K|) & \longrightarrow & H_k(|K'|) \end{array}$$

Here, the horizontal maps are induced by the inclusions $K \hookrightarrow K'$ *and* $|\mathrm{Nerve}(\mathcal{K})| \hookrightarrow |\mathrm{Nerve}(\mathcal{K}')|$*, the latter arising from the simplicial inclusion* $\mathrm{Nerve}(\mathcal{K}) \hookrightarrow \mathrm{Nerve}(\mathcal{K}')$ *induced by the inclusions* $K_i \subseteq K'_i$*. The vertical maps are induced by the homotopy equivalences* f *and* g*. (Note that the vertical maps are isomorphisms since homotopy equivalence maps induce isomorphisms at the homology level.)*

This result can be generalized to filtered spaces. Let P *be a poset and let* $\mathcal{F} : P \to$ $\mathbf{Top}$ *be a filtration assigning each* $p \in P$ *a geometric realization* $|K_p| =: \mathcal{F}_p$ *of a simplicial complex* K_p*. Suppose that for each* $p \in P$*, we are given a cover* $\mathcal{K}_p = \{K_{p,i}\}_{i \in [n]}$ *of* K_p *consisting of subcomplexes such that for all* $p \leq q \in P$*, we have* $K_{p,i} \subseteq K_{q,i}$ *for all* $i \in [n]$*. Then the nerve construction defines a new filtration*

$$\mathcal{N} : \mathbf{P} \to \mathbf{Top}, \quad \mathcal{N}_p := |\mathrm{Nerve}(\mathcal{K}_p)|,$$

where the structure map $\mathcal{N}_{p \leq q} : \mathcal{N}_p \to \mathcal{N}_q$ *is the geometric realization of the simplicial inclusion* $\mathrm{Nerve}(\mathcal{K}_p) \hookrightarrow \mathrm{Nerve}(\mathcal{K}_q)$*, induced by the inclusions* $K_{p,i} \subseteq K_{q,i}$*. Then for each* $k \geq 0$*, the persistent homology modules are isomorphic:*

$$\mathbb{H}_k(\mathcal{F}) \cong \mathbb{H}_k(\mathcal{N}).$$

When treated as geometric objects, data can be viewed as lying in a high-dimensional ambient space $\mathbb{R}^n$. However, in practice, the search for patterns often leads us to ignore many of the ambient variables and instead look for a lower-dimensional underlying structure. A manifold locally resembles $\mathbb{R}^m$, and it is commonly assumed that the data is sampled from or near such a manifold for some $m << n$, reflecting the idea that only a small number of intrinsic variables govern the local behavior of the data.

As previously discussed, all manifolds are paracompact and Hausdorff. Also, it is known that all manifolds whose dimension is at most 3 are triangulable. This makes them suitable for combinatorial approaches to topological inference. Both the persistent Nerve Theorem and its variant for simplicial complexes can be applied in the manifold setting. This is particularly relevant when analyzing data arising from physical imaging processes, where the underlying structures exhibit manifold-like behavior, typically of low dimension (e.g., ≤ 3). The following version is helpful too.

Theorem 7.3. [7, Theorem 4.11.] (Persistent Nerve Theorem for Closed Convex Finite Covers) *Let $\mathcal{C} = \{C_i\}_{i\in[m]}$ and $\mathcal{C}' = \{C_i'\}_{i\in[m]}$ be two finite sets consisting of closed convex subsets of Euclidean space $\mathbb{R}^n$ such that for each $i \in [m]$ we have $C_i \subseteq C_i'$. Denote $C := \bigcup_{i\in I} C_i$ and $C' = \bigcup_{i\in I} C_i'$. Then there exist homotopy equivalences*

$$f : |\mathrm{Nerve}(\mathcal{C})| \to C, \quad g : |\mathrm{Nerve}(\mathcal{C}')| \to C'$$

such that the following diagram commutes:

$$\begin{tikzcd} H_k(|\mathrm{Nerve}(\mathcal{C})|) \arrow[r] \arrow[d, "f_*"'] & H_k(|\mathrm{Nerve})(\mathcal{C}')|) \arrow[d, "g_*"] \\ H_k(C) \arrow[r] & H_k(C') \end{tikzcd}$$

Here, the horizontal maps are induced by the inclusions $C \hookrightarrow C'$ and $|\mathrm{Nerve}(\mathcal{C})| \hookrightarrow |\mathrm{Nerve}(\mathcal{C}')|$, the latter arising from the simplicial inclusion $\mathrm{Nerve}(\mathcal{C}) \hookrightarrow \mathrm{Nerve}(\mathcal{C}')$ induced by the inclusions $C_i \subseteq C_i'$. The vertical maps are induced by the homotopy equivalences f and g. (Note that the vertical maps are isomorphisms since homotopy equivalence maps induce isomorphisms at the homology level.)

*This result can be generalized to filtered spaces. Let P be a poset and let $\mathcal{F} : P \to$ **Top** be a filtration assigning each $p \in P$ a set $C_p := \mathcal{F}_p$ in $\mathbb{R}^n$. Suppose that for each $r \in P$, we are given a cover $\mathcal{C}_p = \{C_{p,i}\}_{i\in[m]}$ of C_p consisting of closed convex subsets of $\mathbb{R}^n$ for all $p \leq q \in P$, we have $C_{p,i} \subseteq C_{q,i}$ for all $i \in [m]$. Then the nerve construction defines a new filtration*

$$\mathcal{N} : \mathbf{P} \to \mathbf{Top}, \quad \mathcal{N}_p := |\mathrm{Nerve}(\mathcal{C}_p)|,$$

where the structure map $\mathcal{N}_{p\leq q} : \mathcal{N}_p \to \mathcal{N}_q$ is the geometric realization of the simplicial inclusion $\mathrm{Nerve}(\mathcal{C}_p) \hookrightarrow \mathrm{Nerve}(\mathcal{C}_q)$, induced by the inclusions $C_{p,i} \subseteq C_{q,i}$. Then for each $k \geq 0$, the persistent homology modules are isomorphic

$$\mathbb{H}_k(\mathcal{F}) \cong \mathbb{H}_k(\mathcal{N}).$$

Chapter 8

STABILITY THEOREMS

Proposition 8.1. [14, Example 3.2.] *Let X be a topological space and $f, g : X \to \mathbb{R}$ be functions. Then $d_i(\mathbb{H}_{k-sub}(X, f), \mathbb{H}_{k-sub}(X, g)) \leq ||f - g||_\infty$.*

Proof. Let $\delta := ||f - g||_\infty$. We will show that $\mathbb{H}_{k-sub}(X, f)$ and $\mathbb{H}_{k-sub}(X, g)$ are δ-interleaved. By the definition of δ, we have $f(x) \leq t \Rightarrow g(x) \leq t + \delta$ and $g(x) \leq t \Rightarrow f(x) \leq t + \delta$, for all $x \in X$. This gives us $(X, f)_t \subseteq (X, g)_{t+\delta}$ and $(X, g)_t \subseteq (X, f)_{t+\delta}$, for all t. These inclusions induce natural maps on homology $H_k((X, f)_t) \to H_k((X, g)_{t+\delta})$ and $H_k((X, g)_t) \to H_k((X, f)_{t+\delta})$ which form δ-shifted homomorphisms

$$\mathbf{h}_1 \in \mathrm{Hom}^\delta(\mathbb{H}_{k-sub}(X, f), \mathbb{H}_{k-sub}(X, g)) \text{ and } \mathbf{h}_2 \in \mathrm{Hom}^\delta(\mathbb{H}_{k-sub}(X, g), \mathbb{H}_{k-sub}(X, f)).$$

Note that $\mathbf{h}_2\mathbf{h}_1 = \mathbf{1}^{2\delta}_{\mathbb{H}_{k-sub}(X,f)}$ and $\mathbf{h}_1\mathbf{h}_2 = \mathbf{1}^{2\delta}_{\mathbb{H}_{k-sub}(X,g)}$ since the left hand sides are composition of inclusions and the right hand sides are coming from the persistence module structures and they are inclusions by construction. Hence, $\mathbb{H}_{k-sub}(X, f)$ and $\mathbb{H}_{k-sub}(X, g)$ are δ-interleaved. This gives us the desired inequality. □

Proposition 8.2. [14, Theorem 4.9.] *Assume $\mathbb{U}, \mathbb{V}$ be interval decomposable persistence modules indexed by $T \subseteq \mathbb{R}$. Then, $d_i(\mathbb{U}, \mathbb{V}) \leq d_b(\mathcal{B}(\mathbb{U}), \mathcal{B}(\mathbb{V}))$.*

Proof. Let $\delta > 0$. Write $\mathbb{U} = \bigoplus_{\lambda \in \mathcal{B}(\mathbb{U})} \mathbb{I}_\lambda$ and $\mathbb{V} = \bigoplus_{\lambda \in \mathcal{B}(\mathbb{V})} \mathbb{J}_\lambda$. Assume there exists δ-matching M between $\mathcal{B}(\mathbb{U})$ and $\mathcal{B}(\mathbb{V})$. Then for every interval module $(\mathbb{I}, \mathbb{J}) \in M$, $d_i(\mathbb{I}, \mathbb{J}) \leq \delta$; and for every unmatched interval module $\mathbb{I} \in \mathcal{B}(\mathbb{U})$ (and similarly for $\mathbb{J} \in \mathcal{B}(\mathbb{V})$), $d_i(\mathbb{I}, 0) \leq \delta$.

Let L be the multiset indexing all interval modules after matching, with unmatched intervals paired with 0. Then we may write $\mathbb{U} = \bigoplus_{\lambda \in L} \mathbb{I}_\lambda$ and $\mathbb{V} = \bigoplus_{\lambda \in L} \mathbb{J}_\lambda$. By proposition 6.5, we have $d_i(\mathbb{U}, \mathbb{V}) \leq \sup_{\lambda \in L} d_i(\mathbb{I}_\lambda, \mathbb{J}_\lambda) \leq \delta$.

Since this holds for arbitrary $\delta > d_b(\mathrm{dgm}(\mathbb{U}), \mathrm{dgm}(\mathbb{V}))$, taking the infimum over all such δ yields $d_i(\mathbb{U}, \mathbb{V}) \leq d_b(\mathrm{dgm}(\mathbb{U}), \mathrm{dgm}(\mathbb{V}))$. □

Theorem 8.1. [14, Theorem 4.11.] (The Isometry Theorem) *Let $\mathbb{U}, \mathbb{V}$ be pointwise finite dimensional persistence modules indexed by $T \subseteq \mathbb{R}$. Then,*

$$d_i(\mathbb{U}, \mathbb{V}) = d_b(\mathcal{B}(\mathbb{U}), \mathcal{B}(\mathbb{V})).$$

Remark 8.1. $d_b(\mathcal{B}(\mathbb{U}), \mathcal{B}(\mathbb{V})) \leq d_i(\mathbb{U}, \mathbb{V})$ is known as algebraic stability theorem.

Converse algebraic stability 8.2 is also true in the multiparameter case.

Proposition 8.3. [9, Proposition 2.12] *Let $\mathbb{U}, \mathbb{V}$ be interval decomposable persistence modules indexed by $\mathbb{R}^n$. Then $d_i(\mathbb{U}, \mathbb{V}) \leq d_b(\mathcal{B}(\mathbb{U}), \mathcal{B}(\mathbb{V}))$.*

In the multiparameter setting, we have a result similar to the algebraic stability theorem for nice modules.

Proposition 8.4. [4, Theorem 4.3.] *Let $\mathbb{U}, \mathbb{V}$ be interval decomposable persistence modules indexed by $\mathbb{R}^n$. Assume they are rectangle decomposable, i.e., the intervals are in the form of rectangles in $\mathbb{R}^n$. Then $d_b(\mathcal{B}(\mathbb{U}), \mathcal{B}(\mathbb{V})) \leq (2n-1)d_i(\mathbb{U}, \mathbb{V})$.*

Definition 8.1. [6] *Let (X, d) be a metric space and $K, L \subseteq X$ be non-empty. The **Hausdorff distance** between K and L is defined as*

$$d_H(K, L) := \inf \left\{ \varepsilon \geq 0 \;\middle|\; K \subseteq L^\varepsilon := \bigcup_{x \in L} B(x, \varepsilon) \text{ and } L \subseteq K^\varepsilon := \bigcup_{x \in K} B(x, \varepsilon) \right\}.$$

*Let (X, d_X) and (Y, d_Y) be metric spaces. The **Gromov-Hausdorff distance** between X and Y is defined as*

$$d_{GH}(X, Y) := \inf_Z \inf_{\varphi, \psi} d_H^Z(\varphi(X), \psi(Y)),$$

where the infimum is taken over all metric spaces Z and all isometric embeddings $\varphi : X \to Z$, $\psi : Y \to Z$, and d_H^Z is the Hausdorff distance in Z.

Remark 8.2. d_H is an extended pseudometric. For bounded sets, d_H is a pseudometric. Also, $d_H(K_1, K_2) = 0 \iff \overline{K_1} = \overline{K_2}$. If we take closed bounded subsets of X, then d_H becomes a metric on this space.

Remark 8.3. If X and Y are compact then $d_{GH}(X, Y) = 0$ if and only if X and Y are isomorphic. Hence, d_{GH} becomes a distance on the set of equivalence classes of compact metric spaces up to isometry.

Proposition 8.5. [6, Theorem 1.1.] *(i) Let $K, L \subset \mathbb{R}^n$ be finite subsets and $k \geq 0$. Then*

$$d_b\left(\mathcal{B}(\mathbb{H}_k(\check{\mathcal{C}}(K))),\ \mathcal{B}(\mathbb{H}_k(\check{\mathcal{C}}(L)))\right) \leq d_H(K, L).$$

(ii) Let X and Y be finite metric spaces and $k \geq 0$. Then

$$d_b\left(\mathcal{B}(\mathbb{H}_k(\mathcal{VR}(X))),\ \mathcal{B}(\mathbb{H}_k(\mathcal{VR}(Y)))\right) \leq d_{GH}(X, Y).$$

Čech and Rips filtrations are sensitive to outliers, as data points that lie far from the main structure, often due to sampling noise, can significantly distort the resulting persistent homology, potentially obscuring the genuine topological features of the data. Multi-parameter filtrations can mitigate this effect, providing a more robust analysis that is less vulnerable to such noise.

To explain the related stability theorem for bifiltration examples, we need measure theory [17]. Let X be a topological metric space. The **Borel σ-algebra** on X, denoted by $\mathcal{B}(X)$, is the smallest σ-algebra containing all open subsets of X. A **Borel measure** is a measure on $(X, \mathcal{B}(X))$.

Let $(X, \mathcal{F}, \mu)$ be a measure space, and let $f : X \to Y$ be a function. Construct a measure space on Y using f and μ. Define a collection $\mathcal{G} \subseteq \mathcal{P}(Y)$ by $\mathcal{G} := \{B \subseteq Y \mid f^{-1}(B) \in \mathcal{F}\}$. Then $\mathcal{G}$ is a σ-algebra on Y with respect to f. With this choice, the function $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ becomes measurable and define the **pushforward measure** $f_*\mu : \mathcal{G} \to [0, \infty]$ by $f_*\mu(B) := \mu(f^{-1}(B))$ for all $B \in \mathcal{G}$.

Definition 8.2. [6] *Let (X, d) be a metric space, and μ, ν be Borel measures on X. The **Prohorov distance** between μ and ν is defined as*

$$d_{Pr}(\mu, \nu) := \inf\left\{\varepsilon > 0 \mid \mu(A) \leq \nu(A^\varepsilon) + \varepsilon \,\&\, \nu(A) \leq \mu(A^\varepsilon) + \varepsilon \quad \forall \text{ Borel set } A\right\},$$

where $A^\varepsilon := \{x \in X \mid d(x, A) < \varepsilon\}$ is the open ε-neighborhood of the set $A \subset X$.

*Let (X, d_X, μ_X) and (Y, d_Y, μ_Y) be two metric spaces equipped with Borel measures. The **Gromov–Prohorov distance** between μ_X and μ_Y is defined by*

$$d_{\mathrm{GPr}}(\mu_X, \mu_Y) := \inf_Z \inf_{\varphi, \psi} d^Z_{Pr}\left(\varphi_*\mu_X, \psi_*\mu_Y\right),$$

where the infimum is taken over all metric spaces Z and all isometric embeddings $\varphi : X \hookrightarrow Z$, $\psi : Y \hookrightarrow Z$, and d^Z_{Pr} denotes the Prohorov distance between pushforward measures on Z.

Remark 8.4. For a fixed metric space X, d_{Pr} is an extended metric on the set of all Borel measures on X. For finite Borel measures, it becomes a metric. And, d_{GPr} distance is a pseudometric on the class of separable complete metric probability measure spaces.

Proposition 8.6. [6, Theorem 1.6.] *(i) Let $X, Y \subset \mathbb{R}^n$ be finite subsets, and let $k \geq 0$. Denote by μ_S the counting measure on $\mathbb{R}^n$ for $S \subseteq \mathbb{R}^n$. Then*

$$d_i(\mathcal{B}(\mathbb{H}_k(\mathcal{MB}(X, \mu_X))), \mathcal{B}(\mathbb{H}_k(\mathcal{MB}(Y, \mu_Y)))) \leq d_{Pr}(\mu_X, \mu_Y).$$

(ii) Let X, Y be finite metric spaces and let $k \geq 0$. Denote by ν_Z the counting measure on the metric space Z. Then

$$d_i(\mathcal{B}(\mathbb{H}_k(\mathcal{SR}(X))), \mathcal{B}(\mathbb{H}_k(\mathcal{SR}(Y)))) \leq d_{GPr}(\nu_X, \nu_Y).$$

Identifying which topological features in a filtration are meaningful (i.e., persist longer) and which may be attributed to noise can be effective for their analysis. Information-theoretic tools provide an assessment of the significance of features captured by the persistence diagram of the filtration. Persistent entropy provides a global summary via the distribution of interval lengths. However, reducing a persistence diagram to a single number may be too coarse, as it lacks sensitivity to local appearance. To overcome this, the persistent entropy summary function encodes entropy over time, offering a more expressive alternative. Supported by stability theorems, these summaries can serve as discriminative tools, especially when classification is required rather than precise feature selection.

Definition 8.3. [2] *Given a persistence diagram $D = \{(b_i, d_i)\}_{i=1}^n$, define the lifetime of the i^{th} feature as $\ell_i := d_i - b_i$, and the total lifetime $L = \sum_{i=1}^n \ell_i$. The **persistent entropy** is then given by*

$$H_{\text{pers}}(D) = -\sum_{i=1}^{n} \frac{\ell_i}{L} \log \frac{\ell_i}{L}.$$

*The **persistent entropy summary function** is defined as a function from $\mathbb{R}$ to $\mathbb{R}_{\geq 0}$*

$$H^D_{\text{pers}}(x) := -\sum_{i=1}^{n} \omega_i(x) \frac{\ell_i}{L} \log \frac{\ell_i}{L},$$

where

$$\omega_i(x) = \begin{cases} 1, & \text{if } b_i \leq x < d_i, \\ 0, & \text{otherwise.} \end{cases}$$

Persistent entropy does not track the positions of the intervals in the barcode, whereas the persistent entropy summary function considers this information by assigning weights.

Proposition 8.7. [1, Proposition 1.] *Let D and D' be finite persistence diagrams. Then, for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$d_b(D, D') < \delta \quad \implies \quad |H_{\text{pers}}(D) - H_{\text{pers}}(D')| < \epsilon.$$

Proof. Let $\epsilon > 0$ be given. Since both persistence diagrams are finite, we extend them by adding the diagonal points so that there exists a bijection

$$\phi : D_{\text{ext}} \to D'_{\text{ext}},$$

satisfying

$$d_b(D, D') = \max_{x \in D_{\text{ext}}} \|x - \phi(x)\|_\infty.$$

Label the points in D_{ext} as $x_1, x_2, \ldots, x_n$ and $x_i = (b_i, d_i)$ and $\phi(x_i) = (b'_i, d'_i)$.

Define the function $h(p) = -p \log p$ for $p > 0$ and $h(0) = 0$. Since h is uniformly continuous on $[0, 1]$, there exists $\delta' > 0$ such that for any $p, p' \in [0, 1]$,

$$|p - p'| < \delta' \implies |h(p) - h(p')| < \frac{\epsilon}{n}.$$

For each i, let $\ell_i = d_i - b_i$ and $\ell'_i = d'_i - b'_i$, and define $L = \sum_{i=1}^n \ell_i$, $L' = \sum_{i=1}^n \ell'_i$. Without loss of generality, assume that $L \geq L'$. Choose

$$\delta = \frac{L'\delta'}{2n + 2}.$$

Assume that $d_B(D, D') < \delta$. Then for each i, we have

$$\|x_i - \phi(x_i)\|_\infty = \max\{|b_i - b'_i|, |d_i - d'_i|\} < \delta.$$

Hence, $|\ell_i - \ell'_i| = |(d_i - b_i) - (d'_i - b'_i)| \leq |d_i - d'_i| + |b_i - b'_i| < 2\delta$. It follows that

$$|L - L'| \leq \sum_{i=1}^n |\ell_i - \ell'_i| < 2n\delta.$$

Define the normalized lifetimes by $p_i = \frac{\ell_i}{L}$ and $p'_i = \frac{\ell'_i}{L'}$. Then,

$$|p_i - p'_i| = \left| \frac{\ell_i}{L} - \frac{\ell'_i}{L'} \right| = \frac{|\ell_i L' - \ell'_i L|}{L\, L'}.$$

We have $\left|\ell_i L' - \ell_i' L\right| \leq |\ell_i(L' - L)| + |(\ell_i - \ell_i')L|$. Since $\ell_i \leq L$ and $|L - L'| < 2n\delta$, it follows that

$$|\ell_i(L' - L)| \leq L \cdot 2n\delta = 2n\delta L \quad \& \quad |(\ell_i - \ell_i')L| < 2\delta\, L.$$

Thus, $\left|\ell_i L' - \ell_i' L\right| < (2n+2)\delta L$. Dividing by $L\,L'$ yields

$$|p_i - p_i'| < \frac{(2n+2)\delta}{L'}.$$

By our choice of $\delta = \frac{L'\delta'}{2n+2}$, we have $|p_i - p_i'| < \delta'$. Then, by the uniform continuity of h,

$$|h(p_i) - h(p_i')| < \frac{\epsilon}{n}.$$

Hence, the difference in persistent entropy satisfies

$$\begin{aligned}\left|H_{\text{pers}}(D_{\text{ext}}) - H_{\text{pers}}(D'_{\text{ext}})\right| &= \left|\sum_{i=1}^{n}\Big[h(p_i) - h(p_i')\Big]\right| \\ &\leq \sum_{i=1}^{n} |h(p_i) - h(p_i')| < n \cdot \frac{\epsilon}{n} = \epsilon.\end{aligned}$$

Finally, since the diagonal points do not contribute to the entropy, we have

$$H_{\text{pers}}(D) = H_{\text{pers}}(D_{\text{ext}}) \quad \text{and} \quad H_{\text{pers}}(D') = H_{\text{pers}}(D'_{\text{ext}}).$$

Thus, $|H_{\text{pers}}(D) - H_{\text{pers}}(D')| < \epsilon$. □

Proposition 8.8. [2, Theorem 4.2.] *Let D and D' be finite persistence diagrams. Then, for every $\epsilon > 0$ there exists $\delta > 0$ such that if*

$$d_b(D, D') < \delta \implies \int_{\mathbb{R}} \left|H_{\text{pers}}^{D}(x) - H_{\text{pers}}^{D'}(x)\right| dx < \epsilon.$$

Proof. Let $\epsilon > 0$ be given. Since D and D' are finite, we extend them by including the diagonal so that there exists a bijection

$$\phi : D_{\text{ext}} \to D'_{\text{ext}},$$

satisfying

$$d_b(D, D') = \max_{x \in D_{\text{ext}}} \|x - \phi(x)\|_{\infty}.$$

Label the corresponding bars as (b_i, d_i) in D and (b'_i, d'_i) in D' for $i = 1, \dots, n$. Then,

$$|b_i - b'_i| < \delta \quad \text{and} \quad |d_i - d'_i| < \delta,$$

where we set $d_B(D, D') < \delta$. Later, we will put conditions for δ.

For each i, define the lifetimes $\ell_i = d_i - b_i,\ \ell'_i = d'_i - b'_i$, and the totals $L = \sum_{i=1}^n \ell_i,\ L' = \sum_{i=1}^n \ell'_i$. Define the normalized lifetimes by $p_i = \frac{\ell_i}{L}$ and $p'_i = \frac{\ell'_i}{L'}$.

Define the function $h(p) = -p \log p,\ h(0) = 0$. Since h is uniformly continuous on $[0, 1]$, $\exists \delta' > 0$ such that for any $p, p' \in [0, 1]$,

$$|p - p'| < \delta' \quad \Longrightarrow \quad |h(p) - h(p')| < \frac{\epsilon}{2nL_0},$$

where L_0 is an upper bound on the bar lengths (i.e., $d_i - b_i \le L_0$ for all i). We require that the perturbations in the normalized lifetimes are small; from the proof of the above proposition, this can be done by choosing

$$\delta \le \frac{min\{L, L'\}\delta'}{2n+2},$$

we have $|p_i - p'_i| < \delta'$ for all i.

For each bar i, define the interval

$$I_i = [b_i, d_i) \quad \text{and} \quad I'_i = [b'_i, d'_i).$$

Let

$$I_i^* = I_i \cap I'_i \quad \text{and} \quad I_i^\Delta = (I_i \cup I'_i) \setminus I_i^*.$$

Define

$$F_i(x) = -\omega_i(x)\, p_i \log p_i, \quad F'_i(x) = -\omega'_i(x)\, p'_i \log p'_i,$$

so that

$$[H_{\text{pers}}(D)](x) = \sum_{i=1}^n F_i(x), \quad [H_{\text{pers}}(D')](x) = \sum_{i=1}^n F'_i(x).$$

For a fixed index i, the error of the ith term is

$$E_i = \int_{\mathbb{R}} |F_i(x) - F'_i(x)|\, dx = \int_{I_i^*} |F_i(x) - F'_i(x)|\, dx + \int_{I_i^\Delta} |F_i(x) - F'_i(x)|\, dx.$$

- (On I_i^*:) For $x \in I_i^*$, $\omega_i(x) = \omega_i'(x) = 1$, so

$$|F_i(x) - F_i'(x)| = \left|p_i \log p_i - p_i' \log p_i'\right| = |h(p_i) - h(p_i')|.$$

Thus,

$$\int_{I_i^*} |F_i(x) - F_i'(x)|\, dx = |h(p_i) - h(p_i')|\, \mu(I_i^*),$$

where $\mu(I_i^*) \leq \ell_i \leq L_0$. By the choice of δ',

$$|h(p_i) - h(p_i')| < \frac{\epsilon}{2nL_0},$$

so that

$$\int_{I_i^*} |F_i(x) - F_i'(x)|\, dx < \frac{\epsilon}{2nL_0}\, L_0 = \frac{\epsilon}{2n}.$$

- (On I_i^Δ:) Since $|b_i - b_i'| < \delta$ and $|d_i - d_i'| < \delta$, the measure of the symmetric difference satisfies $\mu(I_i^\Delta) < 2\delta$. Moreover, note that for all x,

$$|F_i(x)| \leq M, \quad |F_i'(x)| \leq M, \quad \text{with } M = \max_{p\in[0,1]} |h(p)|.$$

because the image of compact set is compact and compact sets in $\mathbb{R}$ are bounded. And for all $x \in I_i^\Delta$ either $F_i(x) = 0$ or $F_i'(x) = 0$, we have

$$\int_{I_i^\Delta} |F_i(x) - F_i'(x)|\, dx \leq \int_{I_i^\Delta} M dx = M\mu(I_i^\Delta) < 2M\delta.$$

Combining the two parts, we have

$$E_i < \frac{\epsilon}{2n} + 2M\delta.$$

Summing over all $i = 1, \ldots, n$,

$$\int_{\mathbb{R}} \left|H^D_{\text{pers}}(x) - H^{D'}_{\text{pers}}(x)\right| dx \leq \sum_{i=1}^{n} E_i < n\left(\frac{\epsilon}{2n} + 2M\delta\right) = \frac{\epsilon}{2} + 2nM\delta.$$

To ensure the total error is less than ϵ, we require $\delta > 0$ so that

$$2nM\delta \leq \frac{\epsilon}{2} \quad \Longrightarrow \quad \delta \leq \frac{\epsilon}{4nM}.$$

Thus, to secure

$$\int_{\mathbb{R}} \left|H^D_{\text{pers}}(x) - H^{D'}_{\text{pers}}(x)\right| dx < \frac{\epsilon}{2} + 2nM\delta \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

we set $\delta = \min\left\{\frac{\epsilon}{4nM}, \frac{min\{L, L'\}\delta'}{2n+2}\right\}$, □

These results can be combined with proposition 8.5. Let $X, Y \subset \mathbb{R}^n$ be finite sets. Let D_X and D_Y denote the truncated persistence diagrams of the Čech or Vietoris–Rips filtrations built on X and Y and indexed by a finite set of $\mathbb{R}$, respectively. Then, for every $\epsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that if $d_H(X,Y) < \delta_1$ then $|H_{\text{pers}}(D_X) - H_{\text{pers}}(D_Y)| < \epsilon$, and if $d_H(X,Y) < \delta_2$ then $\int_{\mathbb{R}} \left| H_{\text{pers}}^{D_X}(x) - H_{\text{pers}}^{D_Y}(x) \right| dx < \epsilon$. Truncation here refers to replacing infinite endpoints with a fixed finite value so that entropy is well-defined.

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