ON THOMPSON'S GROUPS

(THOMPSON'IN GRUPLARI ÜZERİNE)

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LIST OF ABBREVIATIONS

$\mathbb{Z}$	: The set of integers
$\mathbb{Q}$	: The set of rational numbers
$\mathbb{R}$	: The set of real numbers
S_n	: Symmetric group
D_n	: Dihedral group
$GL_2(\mathbb{Z})$	: All 2×2 matrices with determinant 1 or -1
$SL_2(\mathbf{Z})$	: All 2×2 matrices with determinant 1
$PSL_2(\mathbf{Z})$	: Modular group
$\mathcal{H}$	: Upper half-plane

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ABSTRACT

In this thesis, we tried to explain the properties of Thompson's T group and its relationship with the tree we created with the group effect on the upper-half plane through continued fractions.

Thompson's T group is a finitely generated simple infinite group. We can achieve this group with the help of fractional linear transformations obtained by applying the movement we define as flip movement on the tree we created with continued fractions. In this study, continued fractions and fractional linear transformations were discussed. Thompson's T group was tried to be obtained with the help of the Modular group using the group effect.

This thesis is composed of three chapters. In the first chapter, Thompson's T group is introduced, and its relationship with the modular group is examined. In the second chapter, group cohomology was examined by introducing the concepts of homology and cohomology. In the last chapter, it is given how the group cohomology, which is introduced in the second chapter, is obtained with the help of group extensions, and we proposed a conjecture concerning the second cohomology of Thompson's group T.

Keywords : modular group, Thomson's group T, continued fractions

ÖZET

Bu tezde, Thompson'ın T grubunun özellikleri ve sürekli kesirler üzerinden üst-yarı düzlem üzerinde grup etkisi ile oluşturduğumuz ağaç ile olan ilişkisi açıklanmaya çalışılmıştır.

Thompson'ın T grubu sonlu olarak üretilmiş basit sonsuz elemanlı gruptur.Bu grubu, sürekli kesirler ile oluşturuduğumuz ağaç üzerinde çevirme hareketi olarak tanımladığımız hareketi uygulayarak elde ettiğimiz kesirli dönüşümler yardımıyla inşa edebiliriz. Bu çalışmada, sürekli kesirler ve kesirli lineer dönüşümler ele alınmış, grup etkisi kullanarak Modüler grup yardımıyla Thompson'ın T grubu elde edilmeye çalışılmıştır.

Bu tez üç bölümden oluşur. Birinci bölümde Thompson'ın T grubu tanıtılmış ve modüler grupla olan ilişkisi incelenmiştir. İkinci bölümde, homoloji ve kohomoloji kavramları tanıtılarak grup kohomolojisi incelenmiştir. Son bölümde ise, ikinci bölümde tanıtılan grup kohomolojisinin grup genişlemeleri yardımıyla nasıl elde edildiği verilmiştir ve Thompson'ın T grubunun ikinci kohomolojisini ilgilendiren bir sanı önerdik.

Anahtar Kelimeler : modüler grup, Thompson'ın T grubu, sürekli kesirler

1 INTRODUCTION

In this chapter, we will first define continued fractions and give examples. We will explain the relationship between continued fractions and the modular group. At the end of the chapter, we define Thompson's groups F and T.

1.1 Continued Fractions

For a real number x, $\lfloor x \rfloor$ is defined as the greatest integer less than x, that is, $\lfloor x \rfloor \leqslant x < \lfloor x \rfloor + 1$ and decimal part of x is defined as $\{x\} = x - \lfloor x \rfloor$. The decimal part lies in the interval $[0, 1)$. Notice that for any $x \in \mathbb{R}$ satisfies $x = \lfloor x \rfloor + \{x\}$, and $x \in \mathbb{Z}$ if and only if $\{x\} = 0$

If $\{x\} \neq 0$, we may rewrite the above equation as:

$$x = \lfloor x \rfloor + \frac{1}{\frac{1}{\{x\}}}$$

Since $\frac{1}{\{x\}}$ is a real number, we write $\frac{1}{\{x\}}$ as a sum of greatest integer and decimal part. We repeat this process for each non-integer decimal part of a real number. At the end of this process, we obtain a sequence of integers.

Definition 1.1. *Let $x \in \mathbb{R}$. The continued fraction associated to x is the sequence of integers obtained as a result of the above process. Such a continued fraction is also written in an abbreviated form $[a_0; a_1, \ldots, a_n, \ldots]$.*

A few remarks are in order. Firstly, $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{Z}_+$ for all $i \geqslant 1$. The sequence:

$$\alpha_k = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}$$

converges to x. We have, $x \in \mathbb{Q}$ if and only if this process terminates, that is, $x = [a_0; a_1, \ldots, a_N] = \alpha_N$. Otherwise, x is a irrational, see Proposition 1.1.

Example 1.1. *Continued fraction expression of the Euler number and the rational number* $\frac{45}{16}$ *is*

$$e \approx 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \ddots}}}}} \qquad \frac{45}{16} = 2 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{3}}}$$

where their abbreviated forms are $[2; 1, 2, 1, 1, 4, 1, 1, 6, \dots]$ *and* $[2; 1, 4, 3]$*, respectively. One can realize that the Euler number has an infinite continued fraction expression while* $\frac{45}{16}$ *has finite.*

Proposition 1.1. *Every rational number can be expressed as a finite continued fraction. Conversely, if a number is expressed by a finite continued fraction, then it is a rational number.*

Proof. Let $\dfrac{p}{q}$ be a rational number with $\gcd(\mathrm{p}, \mathrm{q}) = 1$. By Euclidean algorithm, we have:

$$\begin{aligned} p &= a_0.q + b_0 \\ q &= a_1.b_0 + b_1 \\ b_0 &= a_2.b_1 + b_2 \\ b_1 &= a_3.b_2 + b_3 \\ &\vdots \\ b_{n-2} &= a_n.b_{n-1} + b_n \\ b_{n-1} &= a_{n+1}.1 + 0 \end{aligned}$$

Note that since $\gcd(\mathrm{p}, \mathrm{q}) = 1$, b_n must be 1. So, we may write:

$$\frac{p}{q} = a_0 + \frac{b_0}{q} \tag{1.1}$$

Now, we continue process by writing $a_0 + \dfrac{1}{\frac{q}{b_0}}$ instead of $a_0 + \dfrac{b_0}{q}$. However, we know

from the above that we can write $\frac{q}{b_0}$ as $a_1 + \frac{b_1}{b_0}$. By putting this in Equation 1.1

$$\frac{p}{q} = a_0 + \cfrac{1}{a_1 + \cfrac{b_1}{b_0}}$$

We repeat the same process until we get $b_n = 1$. After reaching $b_n = 1$, our process is completed because if we kept doing the same thing, we would find $b_{n+1} = 0$, which contradicts our definition. Therefore, we finally have:

$$\frac{p}{q} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots\, a_n + \cfrac{1}{b_{n-1}}}}$$

Hence, we get a continued fraction of any irreducible rational number $\frac{p}{q}$, which is $[a_0; a_1, a_2, \ldots, a_n, a_{n+1}]$. The other side of the statement is obvious. □

Example 1.2. *For the rational number* $\frac{p}{q} = \frac{45}{16}$. *Euclidean algorithm gives:*

$$45 = 2 \times 16 + 13$$
$$16 = 1 \times 13 + 3$$
$$13 = 4 \times 3 + 1$$
$$3 = 3 \times 1 + 0$$

Now that we have written the Euclidean algorithm, we can write the rational number $\frac{45}{16}$ *as a continued fraction.*

$$\frac{45}{16} = 2 + \frac{13}{16} = 2 + \cfrac{1}{\cfrac{16}{13}}$$
$$\frac{16}{13} = 1 + \frac{3}{13} = 1 + \cfrac{1}{\cfrac{13}{3}}$$
$$\frac{13}{3} = 4 + \frac{1}{3}$$

Hence, we finally have a continued fraction of the rational number $\frac{45}{16}$ *as*

$$\frac{45}{16} = 2 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{3}}} = [2; 1, 4, 3]$$

1.2 Modular Group

In this section, we will define a general linear group with integer coefficients and special linear group. Then, we define modular group and investigate actions of these groups on $\mathbb{Q} \cup \{\infty\}$.

Definition 1.2. ***General linear group*** *, denoted by* $\mathrm{GL}_2(\mathbf{Z})$*, with integer coefficients is defined as the set of* 2×2 *matrices*

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z},\ ad - bc = \pm 1 \right\}$$

Special linear group $\mathrm{SL}_2(\mathbf{Z})$ *is the subgroup of* $\mathrm{GL}_2(\mathbf{Z})$ *which consists of those matrices of with determinant* 1*. Note that the center of the groups* $\mathrm{SL}_2(\mathbf{Z})$ *and* $\mathrm{GL}_2(\mathbf{Z})$ *are both* $\{I, -I\}$ *where* I *is the identity matrix. Hence, we have the following groups*

$$\mathrm{PSL}_2(\mathbf{Z}) = \mathrm{SL}_2(\mathbf{Z}) \Big/ \{\pm I\}$$

$$\mathrm{PGL}_2(\mathbf{Z}) = \mathrm{GL}_2(\mathbf{Z}) \Big/ \{\pm I\}$$

The latter, namely $\mathrm{PSL}_2(\mathbf{Z})$*, is called the* ***modular group****.*

Theorem 1.1. (*Bell et al.,* 2017) $\mathrm{SL}_2(\mathbf{Z})$ *is generated by* $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ *and* $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$

Remark. *Consider elements* $U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$*,* L*,* T *and* S *of* $\mathrm{GL}_2(\mathbf{Z})$ *where* L *is defined as* $L = TS$*. Then we know the following properties of above matrices:*

i. Order of the matrices S *and* T *in* $\mathrm{GL}_2(\mathbf{Z})$ *is* 4 *and* ∞*, respectively.*

ii. $L = TS$ *has order* 6 *as* $(TS)^3 = -I$

iii. U *has order* 2 *since* $U^2 = I$

Note that the order of S and L are 2 and 3, respectively in $\mathrm{PSL}_2(\mathbf{Z})$.

Corollary 1.1. *$\mathrm{PSL}_2(\mathbf{Z})$ is generated by S and L, freely. That is:*

$$\mathrm{PSL}_2(\mathbf{Z}) \cong {}^{\mathbb{Z}}/_{2\mathbb{Z}} \times {}^{\mathbb{Z}}/_{3\mathbb{Z}}$$

In particular, every element of $\mathrm{PSL}_2(\mathbf{Z})$ can be written as a word in S, L, L^2. We will assume that such a word is minimal, i.e. has no cancellation from now on.

1.2.1 Group Actions

Definition 1.3. *Let G be a group and X be a non-empty set. We say that G **acts on** X if there is a map: $\alpha\colon G \times X \longrightarrow X$ satisfying*

a. $\alpha(e, x) = x$ where e is the identity in G and $x \in X$.

b. $\alpha(g_1, \alpha(g_2, x)) = \alpha(\alpha(g_1, g_2), x)$ where $g_1, g_2 \in G$ and $x \in X$.

We write $g \cdot x$ instead of $\alpha(g, x)$. Therefore, the conditions are:

a. $e \cdot x = x$

b. $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$

Example 1.3. *Let G be the symmetric group S_n, and X be the set containing integers from 1 to n. We can define an action of S_n on X via the map:*

$$\begin{aligned} \alpha\colon S_n \times X &\longrightarrow X \\ (\rho, k) &\mapsto \rho(k) \end{aligned}$$

where $\rho \in S_n$ and $k \in X$.

a. $e \cdot k = e(k) = k$ since e send k to k for all $k \in X$.

b. $\rho_1 \cdot (\rho_2 \cdot k) = \rho_1 \cdot \rho_2(k) = \rho_1(\rho_2(k)) = (\rho_1 \circ \rho_2)(k) = (\rho_1 \circ \rho_2) \cdot k$

where $\rho_1, \rho_2 \in S_n$ and $k \in X$ are arbitrary. Therefore, S_n acts on X via α.

Example 1.4. *Denote* $\mathbb{Q} \cup \{\infty\}$ *as* $\mathbb{Q}_\infty$ *and let* $G = \mathrm{GL}_2(\mathbf{Z})$. $\mathrm{GL}_2(\mathbf{Z})$ *acts on* $\mathbb{Q}_\infty$ *as follows:*

$$\pi \colon \mathrm{GL}_2(\mathbf{Z}) \times \mathbb{Q} \cup \{\infty\} \longrightarrow \mathbb{Q} \cup \{\infty\}$$

$$\left(\begin{pmatrix} p & q \\ r & s \end{pmatrix}, x \right) \mapsto \begin{cases} \begin{pmatrix} p & q \\ r & s \end{pmatrix} \cdot x := \dfrac{px+q}{rx+s} & x \in \mathbb{Q} \\ \begin{pmatrix} p & q \\ r & s \end{pmatrix} \cdot x := \dfrac{p}{r} & x = \{\infty\} \\ \begin{pmatrix} p & q \\ r & s \end{pmatrix} \cdot x := \infty & x = -\dfrac{s}{r} \end{cases}$$

We have:

- $I \cdot x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot x = \dfrac{x+0}{0+1} = x$
- *Let* $g_1 = \begin{pmatrix} p_1 & q_1 \\ r_1 & s_1 \end{pmatrix}$ *and* $g_2 = \begin{pmatrix} p_2 & q_2 \\ r_2 & s_2 \end{pmatrix}$ *be any two elements of* $\mathrm{GL}_2(\mathbf{Z})$ *and* $x \in \mathbb{Q} \cup \{\infty\}$.

$$\begin{aligned} g_1 \cdot (g_2 \cdot x) &= \begin{pmatrix} p_1 & q_1 \\ r_1 & s_1 \end{pmatrix} \cdot \left(\frac{p_2 x + q_2}{r_2 x + s_2} \right) = \frac{(p_1 p_2 + q_1 r_2)x + (p_1 q_2 + q_1 s_2)}{(p_2 r_1 + r_2 s_1)x + (q_2 r_1 + s_1 s_2)} \\ &= \begin{pmatrix} p_1 p_2 + q_1 r_2 & p_1 q_2 + q_1 s_2 \\ p_2 r_1 + r_2 s_1 & q_2 r_1 + s_1 s_2 \end{pmatrix} \cdot x = \left(\begin{pmatrix} p_1 & q_1 \\ r_1 & s_1 \end{pmatrix} \begin{pmatrix} p_2 & q_2 \\ r_2 & s_2 \end{pmatrix} \right) \cdot x \\ &= (g_1 g_2) \cdot x \end{aligned}$$

The reader may verify that the infinite part of this action is also satisfied.

Definition 1.4. *Let* G *be a group acting on a set* X. *The set*

$$G \cdot x := \{g \cdot x \mid g \in G\}$$

is called ***the orbit of*** x.

Definition 1.5. *Let* G *be a group acting on a set* X. *For* $x \in X$, *the set*

$$G_x = \{g \in G \mid g \cdot x = x\}$$

is defined as the ***stabilizer of*** x *in* G.

Note that G_x is a subgroup of G since for arbitrary $g_1, g_2 \in G_x$

$$(g_1)(g_2)^{-1} \cdot x = (g_1)(g_2^{-1} \cdot x) = g_1 \cdot x = x$$
$$e \cdot x = x$$

Hence $g_1 g_2^{-1} \in G_x$ and $e \in G_x$.

Definition 1.6. *Let G be a group acting on a non-empty set X. The kernel of this action is defined as the set*

$$K = \{g \in G \mid g \cdot x = x \ for\ all\ x \in X\}$$

Since we look for elements of G that fix each and every element of X, we can rewrite the kernel of an action as the intersection of all the stabilizers, so

$$\bigcap_{x \in X} G_x = \{g \in G \mid g \cdot x = x \ for\ all\ x \in X\}$$

Since the intersection of subgroups is also a subgroup, K is a subgroup of G. Moreover, for every $g \in G$, we also have $gKg^{-1} = K$ since for any $x \in X$

$$\begin{aligned}(gkg^{-1}) \cdot x &= g \cdot (k \cdot (g^{-1} \cdot x)) \\ &= g \cdot (g^{-1} \cdot x) \text{ since } k \in G_{g^{-1} \cdot x} \\ &= (gg^{-1}) \cdot x \\ &= x\end{aligned}$$

for every $k \in K$. Therefore, K is a normal subgroup of G.

Example 1.5. *We consider the action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathbb{Q}_\infty$. The stabilizer of ∞ is:*

$$\begin{aligned}\mathrm{SL}_2(\mathbf{Z})_\infty &= \{A \in \mathrm{SL}_2(\mathbf{Z}) \mid A \cdot \infty = \infty\} \\ &= \{\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z}) \mid \frac{p}{r} = \infty\} \\ &= \{\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z}) \mid r = 0\} \\ &= \{\left(\begin{smallmatrix} p & q \\ 0 & s \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z}) \mid ps = 1 \ and\ q \in \mathbb{Z}\}\end{aligned}$$

Hence stabilizer of ∞ in $\mathrm{SL}_2(\mathbf{Z})$ is the set $\left\{ \begin{pmatrix} \pm 1 & n \\ 0 & \pm 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \mid \text{where } n \in \mathbb{Z} \right\}$

Proposition 1.2. *Suppose that G is a group so that G acts on a set X and K is the kernel of this action. Then G/K also acts on the set X.*

Proof. Suppose that G acts on a set X with $\alpha : G \times X \to X$ and K is the kernel of this action. Then, define $\pi : {}^{G}/_{K} \times X \to X$ such that $\pi(gK, x) = \alpha(g, x)$. This definition is well-defined since if $gK = hK$, then we know that $g^{-1}h \in K$. Therefore, $g^{-1}h \cdot x = x$ and that gives us $h \cdot x = g \cdot x$ for arbitrary $x \in X$. This also defines a group action since $K \cdot x = x$ by definition of K and

$$g_1K \cdot (g_2K \cdot x) = g_1K \cdot (g_2 \cdot x) = g_1 \cdot (g_2 \cdot x) = (g_1g_2 \cdot x) = (g_1g_2K \cdot x) = (g_1Kg_2K) \cdot x$$

where $g_1K, g_2K \in {}^{G}/_{K}$ and K is identity of ${}^{G}/_{K}$.

□

Proposition 1.3. *Consider the action of $SL_2(\mathbf{Z})$ on $\mathbb{Q}_\infty$. The kernel of this action is the set $\{I, -I\}$ where I is the identity matrix.*

Proof. Let $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ be an element of $SL_2(\mathbf{Z})$ and x be any element of $\mathbb{Q}_\infty$. Then $A \cdot x = x$ if and only if $\dfrac{px + q}{rx + s} = x$ and so we must have $p = s$ and $q = r = 0$. Since $A \in SL_2(\mathbf{Z})$ and $ps - rq = 1 = ps$, it follows that p and s are both 1 or both -1. □

Remark. *As $SL_2(\mathbf{Z})$ acts on $\mathbb{Q} \cup \{\infty\}$, $\mathrm{PSL}_2(\mathbf{Z})$ also acts on $\mathbb{Q}_\infty$ with the same fractional linear transformation by Proposition* 1.2 *and Proposition* 1.3.

We have seen that $\mathrm{GL}_2(\mathbf{Z})$ acts on $\mathbb{Q} \cup \{\infty\}$. Let's consider how some specific elements of $\mathrm{GL}_2(\mathbf{Z})$ acts on $(\mathbb{Q} \cup \{\infty\})$. Let $S, T, L, U \in \mathrm{GL}_2(\mathbf{Z})$ and $x \in \mathbb{Q} \cup \{\infty\}$ then:

- $S \cdot x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot x = -\dfrac{1}{x}$
- $T \cdot x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot x = \dfrac{x + 1}{1} = x + 1$, $T^n \cdot x = x + n$
- $L \cdot x = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \cdot x = \dfrac{x - 1}{x} = 1 - \dfrac{1}{x}$
- $U \cdot x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot x = \dfrac{1}{x}$ and $U \cdot 0 = \infty$

1.3 Continued Fractions and Modular Group

Let $a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots}} = [a_0, a_1, \cdots]$ be any continued fraction. One can realize that:

$$\begin{aligned}
[a_0; a_1, a_2, a_3 \cdots] =& T^{a_0} \cdot \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} \\
&= T^{a_0}.[a_1; a_2, a_3, \ldots] \\
&= T^{a_0}U \cdot \left(a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}} \right) \\
&= T^{a_0}UT^{a_1} \cdot \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}} \\
&= T^{a_0}UT^{a_1}.[a_2; a_3, a_4, \ldots] \\
&= T^{a_0}UT^{a_1}UT^{a_2} \cdot [a_3; a_4, a_5, \ldots]
\end{aligned}$$

This process does not end if the continued fraction is infinite. However, the continued fraction is finite, then it is a finite process. Let $[a_0; a_1, a_2, \ldots, a_n, a_{n+1}]$ be any continued fraction. By applying the above process, we can obtain the following result:

$$[a_0; a_1, a_2, \ldots, a_n, a_{n+1}] = T^{a_0}UT^{a_1}UT^{a_2}U \cdots T^{a_n}UT^{a_{n+1}} \cdot (0)$$

Notice that the matrix U is not an element of neither $\mathrm{SL}_2(\mathbf{Z})$ nor $\mathrm{PSL}_2(\mathbf{Z})$ since $det(U) = -1$. However, we have

$$UTU = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} = L^2S \in \mathrm{PSL}_2(\mathbf{Z})$$

$$(UTU)^n = UT^nU = (L^2S)^n \in \mathrm{PSL}_2(\mathbf{Z})$$

since U^2 is identity. By using the above results, we can write continued fractions in terms of the elements of $\mathrm{PSL}_2(\mathbf{Z})$. Let $[a_0, a_1, a_2, \dots a_n, a_{n+1}]$ be any finite continued fraction. We have:

$$[a_0, a_1, \dots a_n, a_{n+1}] = \begin{cases} (LS)^{a_0}(L^2S)^{a_1} \dots (L^2S)^{a_{n-1}}(LS)^{a_n}(L^2S)^{a_{n+1}} \cdot (\infty), & n \text{ is even} \\ (LS)^{a_0}(L^2S)^{a_1} \dots (LS)^{a_{n-1}}(L^2S)^{a_n}(LS)^{a_{n+1}} \cdot (0), & n \text{ is odd} \end{cases}$$

In fact, when n is even and if we write LS instead of T and L^2S instead of UTU, then we have :

$$(LS)^{a_0}(L^2S)^{a_1} \dots (L^2S)^{a_{n-1}}(LS)^{a_n}UT^{a_{n+1}} \cdot (0)$$

Since there is no more U, we cannot write L^2S again. We fix this problem as follows

$$UT^{n+1} \cdot (0) = UT^{a_{n+1}}U \cdot (\infty) = (L^2S)^{a_{n+1}} \cdot (\infty)$$

Definition 1.7. *To a continued fraction $[a_0; a_1, a_2, \dots]$ (finite or infinite) we associate the word $(LS)^{a_0}(L^2S)^{a_1}(LS)^{a_2} \dots$ in $\mathrm{PSL}_2(\mathbf{Z})$ and call this* ***the word of the continued fraction****.*

Example 1.6. *Let $\dfrac{p}{q} = \dfrac{45}{16}$ and we know from the Example 1.1. that $\dfrac{45}{16} = [2; 1, 4, 3]$ where $n + 1 = 3$ and so $n = 2$ which is even. By using the above formula, we can write $\dfrac{45}{16}$ in terms of elements of $\mathrm{PSL}_2(\mathbf{Z})$ with group action on $\mathbb{Q} \cup \{\infty\}$ as follows:*

$$(LS)^2(L^2S)^1(LS)^4(L^2S)^3 \cdot (\infty)$$

Example 1.7. *Let $\dfrac{p}{q} = \dfrac{83}{19}$. If we write Euclidean algorithm, we have:*

$$83 = 4 \times 19 + 7$$

$$19 = 2 \times 7 + 5$$

$$7 = 1 \times 5 + 2$$

$$5 = 2 \times 2 + 1$$

$$2 = 1 \times 2 + 0$$

where a_0, a_1, a_2, a_3 *and* a_4 *are* $4, 2, 1, 2, 2$ *respectively. Therefore,* $\dfrac{83}{19} = [4; 2, 1, 2, 2]$ *where* $n+1 = 4$ *and so* $n = 3$ *which is odd. Now, we can write this continued fraction in terms of* L *and* S *by using the above formula. Hence, we have the following:*

$$\frac{83}{19} = (LS)^4(L^2S)^2(LS)^1(L^2S)^2(LS)^2 \cdot (0).$$

Notice that $(LS)^4(L^2S)^2(LS)^1(L^2S)^2(LS)^2 = \left(\begin{smallmatrix} 35 & 83 \\ 8 & 19 \end{smallmatrix}\right)$.

Let us summarize the above discussion in the following Theorem.

Theorem 1.2. *For any finite continued fraction* $[a_0; a_1, a_2, \ldots, a_{n+1}]$ *we have:*

$$[a_0; a_1, \ldots, a_{n+1}] = \begin{cases} (LS)^{a_0}(L^2S)^{a_1} \ldots (LS)^{a_n}(L^2S)^{a_{n+1}} \cdot (\infty) & n \text{ is even} \\ (LS)^{a_0}(L^2S)^{a_1} \ldots (L^2S)^{a_n}(LS)^{a_{n+1}} \cdot (0) & n \text{ is odd} \end{cases}$$

Similarly, for any infinite continued fraction $[a_0; a_1, a_2, \ldots]$, *the sequences*

$$P_N = (LS)^{a_0}(L^2S)^{a_1} \ldots (L^2S)^{a_{2N+1}} \cdot (\infty)$$
$$Q_N = (LS)^{a_0}(L^2S)^{a_1} \ldots (LS)^{a_{2N}} \cdot (0)$$

converges to $[a_0; a_1, a_2, \ldots]$.

1.4 Farey Tree and Flip

In this section, we will define the Farey tree and define flip move on the Farey tree. We start with:

Proposition 1.4. *Let* $\mathcal{H} = \{a + ib \mid a \in \mathbb{R}, b \in \mathbb{R}_+\}$ *be the upper half-plane. Then* $\mathrm{PSL}_2(\mathbf{Z})$ *acts on* $\mathcal{H} \cup \mathbb{R} \cup \{\infty\}$ *as fractional linear transformation:*

$$\mathrm{PSL}_2(\mathbf{Z}) \times \mathcal{H} \cup \mathbb{R} \cup \{\infty\} \longrightarrow \mathcal{H} \cup \mathbb{R} \cup \{\infty\}$$
$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \cdot z = \frac{pz + q}{rz + s}$$

where $z \in \mathcal{H} \cup \mathbb{R} \cup \{\infty\}$.

Proof. We have:

- $I \cdot z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot z = \dfrac{z+0}{0+1} = z$

- Let $W_1 = \begin{pmatrix} p_1 & q_1 \\ r_1 & s_1 \end{pmatrix}$ and $W_2 = \begin{pmatrix} p_2 & q_2 \\ r_2 & s_2 \end{pmatrix}$ be any two elements of $\mathrm{PSL}_2(\mathbf{Z})$ and $x \in \mathcal{H} \cup \mathbb{R} \cup \{\infty\}$.

$$\begin{aligned} W_1 \cdot (W_2 \cdot z) &= \begin{pmatrix} p_1 & q_1 \\ r_1 & s_1 \end{pmatrix} \cdot \left(\frac{p_2 z + q_2}{r_2 z + s_2} \right) = \frac{(p_1 p_2 + q_1 r_2) z + (p_1 q_2 + q_1 s_2)}{(p_2 r_1 + r_2 s_1) z + (q_2 r_1 + s_1 s_2)} \\ &= \begin{pmatrix} p_1 p_2 + q_1 r_2 & p_1 q_2 + q_1 s_2 \\ p_2 r_1 + r_2 s_1 & q_2 r_1 + s_1 s_2 \end{pmatrix} \cdot z = \left(\begin{pmatrix} p_1 & q_1 \\ r_1 & s_1 \end{pmatrix} \begin{pmatrix} p_2 & q_2 \\ r_2 & s_2 \end{pmatrix} \right) \cdot z \\ &= (W_1 W_2) \cdot z \end{aligned}$$

□

Example 1.8. *Table* 1.1 *contains the actions of* S, L, L^2, LS *and* L^2S *of* $\mathrm{PSL}_2(\mathbf{Z})$ *on* $0, i$ *and* ∞. *Since the element* S *fixes* i, *this action is not free. Similarly, the element* L *fixes* $e^{\frac{2\pi i}{6}}$.

$\cdot$	0	∞	i
S	∞	0	i
L	∞	1	$i+1$
L^2	1	0	$\frac{i+1}{2}$
LS	1	∞	$i+1$
L^2S	0	1	$\frac{i+1}{2}$

Table 1.1: Elements of $\mathrm{PSL}_2(\mathbf{Z})$ and their action

We label the part of unit circle in $\mathcal{H}$ joining i to $e^{\frac{2\pi i}{6}}$ by I and denote this γ_I. For any $W \in \mathrm{PSL}_2(\mathbf{Z})$, we label $W \cdot \gamma_I$ by γ_W, that is, we consider the orbit of γ_I under the action of $\mathrm{PSL}_2(\mathbf{Z})$. This set is a bipartite ribbon graph, in fact a tree. There are two types of vertices: one is of degree 2, denoted by $\circ$, which can be identified with the W coset of $\langle S \rangle = \{I, S\}$, i.e. $\{W, WS\}$ and the other is of degree 3, denoted by $\bullet$, which can be identified with the W coset of $\langle L \rangle = \{I, L, L^2\}$, i.e. $\{W, WL, WL^2\}$. Vertices of the form $\{W, WS\}$ connect edges labeled with W and WS, and vertices of the form $\{W, WL, WL^2\}$ connect edges labeled with W, WL and WL^2, see Figure 1.1.

Definition 1.8. *The tree defined above is called* ***Farey tree*** *and denoted by* $\mathcal{F}$.

The set of edges of $\mathcal{F}$, denoted $E(\mathcal{F})$, can be identified with elements of $\mathrm{PSL}_2(\mathbf{Z})$, as a result of the construction.

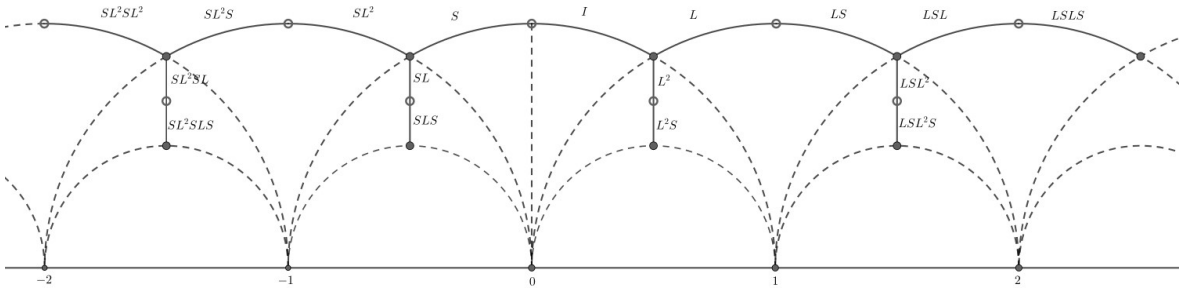


Figure 1.1: Labeled Farey tree $\mathcal{F}$

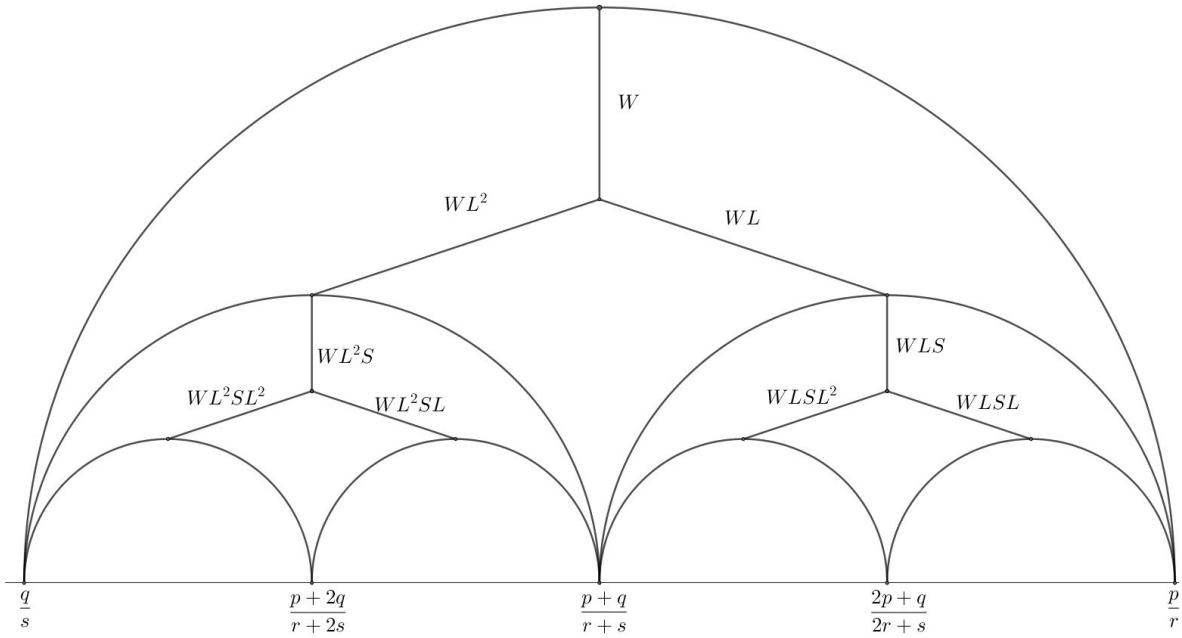


Figure 1.2: Farey tree, locally; where $W = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$

Figure 1.2 can be obtained by considering the image of the line joining 0 to ∞ in $\mathcal{H}$ under the elements $W, WL, WLS, WLSL, WLSL^2, WL^2, WL^2S, WL^2SL$ and WL^2SL^2

Definition 1.9. *Given any* $W \in \mathrm{PSL}_2(\mathbf{Z})$*, the flip move on* $WL \cdot \{I, S\}$ *is defined as:*

$$f_{WL\cdot\{WLS\}} : \mathrm{PSL}_2(\mathbf{Z}) \to \mathrm{PSL}_2(\mathbf{Z})$$
$$X \mapsto f_{WL\cdot\{I,S\}}(X)$$

$$\text{where } f_{WL\cdot\{I,S\}}(X) = \begin{cases} X, & \text{if } X \text{ does not start with } W \\ WL^2SL^2X', & \text{if } X = WL^2X', \text{ for some } X' \text{ in } \mathrm{PSL}_2(\mathbf{Z}) \\ WLX', & \text{if } X = WLSLX' \text{ for some } X' \text{ in } \mathrm{PSL}_2(\mathbf{Z}) \\ WL^2SLX', & \text{if } X = WLSL^2X' \text{ for some } X' \text{ in } \mathrm{PSL}_2(\mathbf{Z}) \end{cases}$$

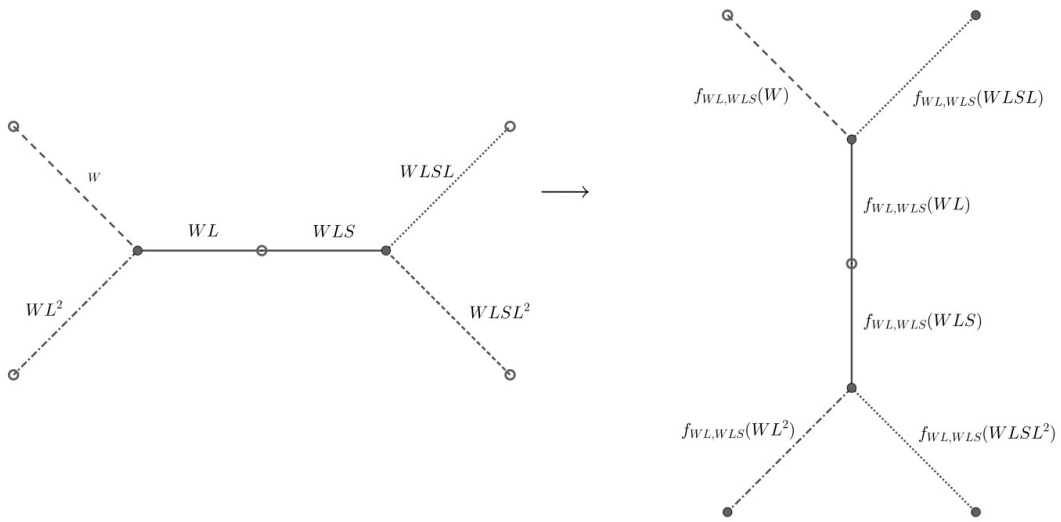


Figure 1.3: The flip $f_{WL,WLS}$

We apply the flip move on WL and WLS, which is shown in red in Figure 1.4 and get the Figure 1.5.

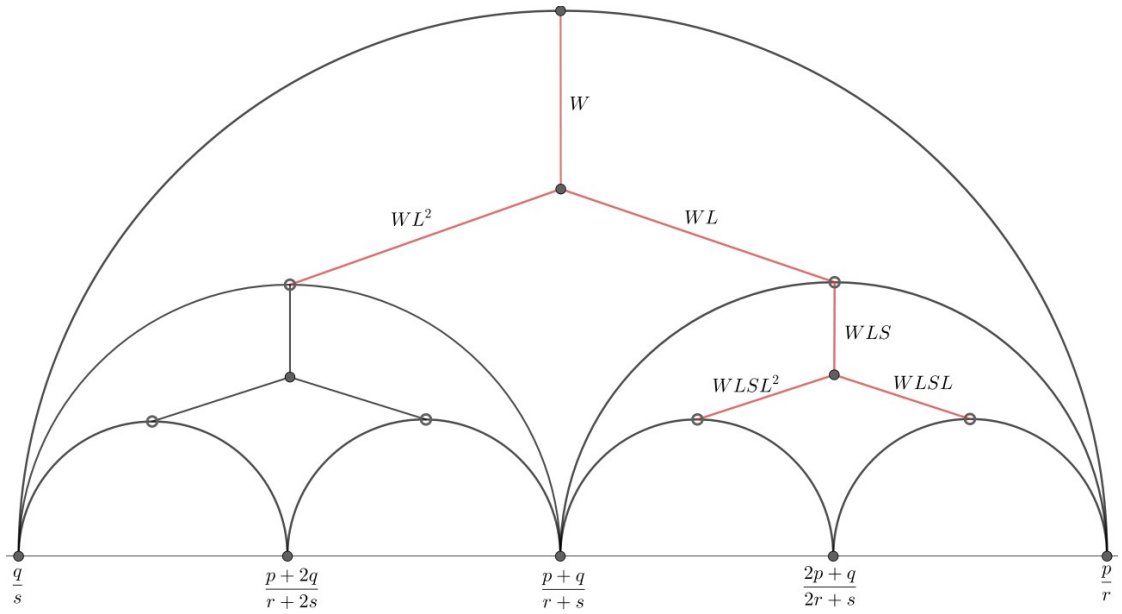


Figure 1.4: Before Flip move

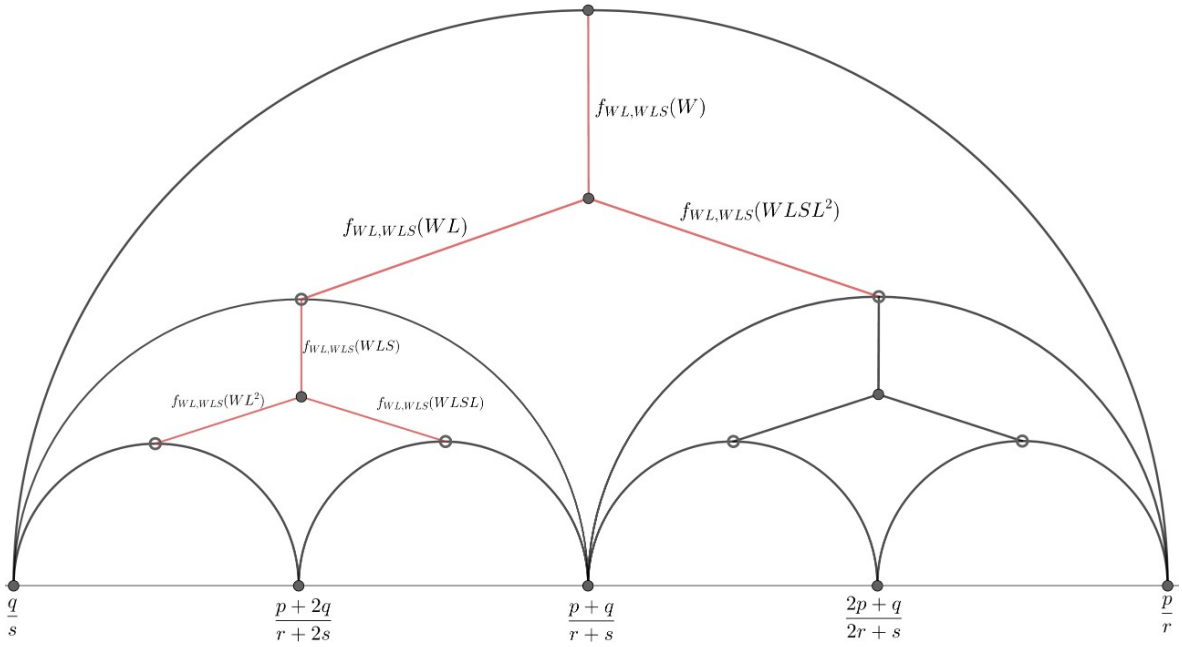


Figure 1.5: After Flip move

We can see from the above figure that the flip move sends the interval $(\frac{q}{s}, \frac{p}{r})$ to itself as follows:

$$\left(\frac{q}{s}, \frac{p+q}{r+s}\right) \longmapsto \left(\frac{q}{s}, \frac{p+2q}{r+2s}\right)$$

$$\left(\frac{p+q}{r+s}, \frac{2p+q}{2r+s}\right) \longmapsto \left(\frac{p+2q}{r+2s}, \frac{p+q}{r+s}\right)$$

$$\left(\frac{2p+q}{2r+s}, \frac{p}{r}\right) \longmapsto \left(\frac{p+q}{r+s}, \frac{p}{r}\right)$$

What we have after applying flip move is actually a piecewise linear function which is identity outside the interval $(\frac{q}{s}, \frac{p}{r})$. Moreover, for $x \in (\frac{q}{s}, \frac{p}{r})$, we have:

$$f_{WL,WLS}(x) = \begin{cases} \begin{pmatrix} 1+qs & -q^2 \\ s^2 & 1-qs \end{pmatrix} \cdot x, & x \in \left(\frac{q}{s}, \frac{p+q}{r+s}\right) \\ \begin{pmatrix} -qs-3qr-pr & q^2+3pq+p^2 \\ -s^2-3rs-r^2 & qs+3ps+pr \end{pmatrix} \cdot x, & x \in \left(\frac{p+q}{r+s}, \frac{2p+q}{2r+s}\right) \\ \begin{pmatrix} 1+pr & -p^2 \\ r^2 & 1-pr \end{pmatrix} \cdot x, & x \in \left(\frac{2p+q}{2r+s}, \frac{p}{r}\right) \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot x, & \textit{otherwise} \end{cases}$$

Consider the flip move on $WLSL, WLSLS$, where $WLS = \left(\begin{smallmatrix} p & p+q \\ r & r+s \end{smallmatrix}\right)$. In a similar fashion $f_{WLSL\cdot\{I,S\}}$ is:

$$\left(\frac{p+q}{r+s}, \frac{2p+q}{2r+s}\right) \longmapsto \left(\frac{p+q}{r+s}, \frac{3p+2q}{3r+2s}\right)$$
$$\left(\frac{2p+q}{2r+s}, \frac{3p+q}{3r+s}\right) \longmapsto \left(\frac{3p+2q}{3r+2s}, \frac{2p+q}{2r+s}\right)$$
$$\left(\frac{3p+q}{3r+s}, \frac{p}{r}\right) \longmapsto \left(\frac{2p+q}{2r+s}, \frac{p}{r}\right)$$

More precisely,

$$f_{WLSL,WLSLS}(x) = \begin{cases} \left(\begin{smallmatrix} 1+(p+q)(r+s) & -(p+q)^2 \\ (r+s)^2 & 1-(p+q)(r+s) \end{smallmatrix}\right) \cdot x, & x \in \left(\frac{p+q}{r+s}, \frac{2p+q}{2r+s}\right) \\ \left(\begin{smallmatrix} -5pr-4qr-ps-qs & 5p^2+5pq+q^2 \\ -5r^2-5rs-s^2 & 5pr+4ps+qr+qs \end{smallmatrix}\right) \cdot x, & x \in \left(\frac{2p+q}{2r+s}, \frac{3p+q}{3r+s}\right) \\ \left(\begin{smallmatrix} 1+pr & -p^2 \\ r^2 & 1-pr \end{smallmatrix}\right) \cdot x, & x \in \left(\frac{3p+q}{3r+s}, \frac{p}{r}\right) \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot x, & otherwise \end{cases}$$

Consider the flip $f_{WL,WLS}$. Before the flip move, the word WL^2 is the label of the half circle joining $\frac{q}{s}$ to $\frac{p+q}{r+s}$. After flip move, WL^2 replaces the word WL^2SL^2 which means that WL^2 attached to the line to which WL^2SL^2 attach. We can say that the flip move $f_{WL,WLS}$ sends WL^2 to WL^2SL^2. If we examine how flip move changes other edges, we will get the following result:

$$WL^2 \longmapsto WL^2SL^2$$
$$WLSL \longmapsto WL^2$$
$$WLSL^2 \longmapsto WL$$

We can get the same transformations while writing the flip $f_{WL,WLS}$ by multiplying matrices to transform the above words on the left to words on the right. For example, $WL^2SW^{-1} \cdot WL^2 = WL^2SL^2$ and notice that

$$WL^2SW^{-1} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} s & -q \\ -r & p \end{pmatrix} = \begin{pmatrix} 1+qs & -q^2 \\ -r^2 & 1-qs \end{pmatrix}$$

Similarly, we can transform $WLSL$ and $WLSL^2$ to WL^2 and WL by multiplying them with $W(L^2S)^2L^2W^{-1}$ and WSL^2W^{-1} respectively, and get the related matrices. Hence, we can write transformation by using words instead of matrices as follows:

$$f_{WL,WLS}(x) = \begin{cases} WL^2SW^{-1} \cdot x, & x \in \left(\frac{q}{s}, \frac{p+q}{r+s}\right) \\ WL^2(SL^2)^2W^{-1} \cdot x, & x \in \left(\frac{p+q}{r+s}, \frac{2p+q}{2r+s}\right) \\ WSL^2W^{-1} \cdot x, & x \in \left(\frac{2p+q}{2r+s}, \frac{p}{r}\right) \end{cases}$$

Similarly, the flip move on $WLSL, WLSL$ can be written as:

$$f_{WLSL,WLSLS}(x) = \begin{cases} WLSLW^{-1} \cdot x, & x \in \left(\frac{p+q}{r+s}, \frac{2p+q}{2r+s}\right) \\ WL(SL^2)^4W^{-1} \cdot x, & x \in \left(\frac{2p+q}{2r+s}, \frac{3p+q}{3r+s}\right) \\ WSL^2W^{-1} \cdot x, & x \in \left(\frac{3p+q}{3r+s}, \frac{p}{r}\right) \end{cases}$$

Definition 1.10. *The **support** of a function $f : J \to J$ is defined as the subset of J on which the function does not fix any point.*

The support of a flip $f_{WL\cdot\{I,S\}}$ can be determined quickly by computing the word WS as a matrix. For example, the matrix $W = LSL^2 = \left(\begin{smallmatrix} -1 & 2 \\ -1 & 1 \end{smallmatrix}\right)$ gives the interval $(2, 1)$ which has no meaning. However, $WS = LSL^2S = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ gives the interval $(1, 2)$. Careful reader will immediately recognize that both the definition and the above discussion for a flip on $\{I, S\}$ is irrelevant. We define the flip $f_{I,S}$ ourselves as follows:

$$f_{I,S}(x) = \begin{cases} L \cdot x, & x \in \left(\frac{-1}{0}, \frac{-1}{1}\right) \\ LS \cdot x, & x \in \left(\frac{-1}{1}, \frac{0}{1}\right) \\ L^2 \cdot x, & x \in \left(\frac{0}{1}, \frac{1}{1}\right) \\ SL \cdot x, & x \in \left(\frac{1}{1}, \frac{1}{0}\right) \end{cases}$$

Note that unlike other flips, $f_{I,S}(x)$ does not fix any point in $\mathbb{R} \cup \{\infty\}$. $f_{I,S}$ is of order 4, see Proposition 1.5 below.

Example 1.9. *The support of $f_{WL,WLS}$ is the interval $(\frac{q}{s}, \frac{p}{r})$ where $W = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$.*

Remark. *The elements we have seen so far are generally in the form $f_{WL,WLS}$, but this just depends on the choice of the word. For example, if we choose $WLSL^2$ instead of W, we will have the element $f_{WLS,WL}$, namely that corresponds to same edges and vertices with W on which we apply the flip move. We can also write the linear fractional transformations defining $f_{WLS,WL}$ by choosing $WLSL^2 = \left(\begin{smallmatrix} -p-q & 2p+q \\ -r-s & 2r+s \end{smallmatrix}\right)$ and get the fractional linear transformation as follows:*

$$f_{WLS,WL}(x) = \begin{cases} W_1 \cdot x & x \in (\frac{2p+q}{2r+s}, \frac{p}{r}) \\ W_2 \cdot x, & x \in (\frac{p}{r}, \infty) \cup (-\infty, \frac{q}{s}) \\ W_3 \cdot x, & x \in (\frac{q}{s}, \frac{p+q}{r+s}) \end{cases}$$

where

$$W_1 = \begin{pmatrix} 1 + 4pq + 2ps + 2rs + qs & -4p^2 - 4pq - q^2 \\ 4r^2 + 4rs + s^2 & 1 - 4pq - 2ps - 2rs - qs \end{pmatrix}$$

$$W_2 = \begin{pmatrix} pr + 3ps + qs & -p^2 - 3pq - q^2 \\ r^2 + 3rs + s^2 & -qs - 3qr - pr \end{pmatrix}$$

$$W_3 = \begin{pmatrix} 1 + pr + ps + qr + qs & -p^2 - 2pq - q^2 \\ r^2 + 2rs + s^2 & 1 - pr - ps - qr - qs \end{pmatrix}$$

Proposition 1.5. *Every flip applied to a pair WL, WLS has order infinity except to the pair I, S. $f_{I,S}$ has order 4.*

Proof. $f_{WL,WLS}$ is a piece-wise linear function and each piece-wise function of $f_{WL,WLS}$ are determined by an element of $\mathrm{PSL}_2(\mathbf{Z})$. We will show that the order of $f_{WL,WLS}$ is infinite by showing that the matrices,which determines piece-wise functions, have order infinity. One can realize that the traces of matrices that determines piece-wise functions of $f_{WL,WLS}$ are either 2 or 3. For example, $tr(WL^2SW^{-1}) = 2$ and $tr(WL^2(SL^2)^2W^{-1}) = 3$. We claim that the order of matrices whose traces are 2 or 3 is infinite. Consider an element W of $\mathrm{PSL}_2(\mathbf{Z})$ with trace 2. Then characteristic polynomial of W is $p(\lambda) = \lambda^2 - 2\lambda + 1$. We also know that W satisfies its characteristic polynomial so we have $W^2 - 2W + I_{2\times 2} = 0$. If we multiply this equation with W

repeatedly and rewrite the equation, we get

$$\begin{aligned} W^3 &= 2W^2 - W \\ &= 2(2W - I_{2\times 2}) - W \\ &= 3W - 2I_{2\times 2} \\ W^4 &= 3W^2 - 2W \\ &= 3(2W - I_{2\times 2}) - 2W \\ &= 4W - 3I_{2\times 2} \\ &\vdots \\ W^n &= nW - (n-1)I_{2\times 2} \end{aligned}$$

One can see from above $W^n = I_{2\times 2}$ or $W^n = -I_{2\times 2}$ if and only if $W^n = I_{2\times 2}$ or $W^n = -I_{2\times 2}$, which contradicts to the fact that W has trace 2 so W has order infinity. Similar process can be applied to when the trace is 3 and it is observed that the matrices with trace 3 have order infinity too. Since matrices corresponding to piecewise functions have order infinite, order of $f_{WL,WLS}$ is infinite as well. □

Proposition 1.6. *The inverse of $f_{WL,WLS}$ is f_{WL^2,WL^2S}.*

Proof. Here we have how the words are replaced by $f_{WL,WLS}$(the left one) and f_{WL^2,WL^2S}(the right one).

$$\begin{aligned} WL^2 &\longmapsto WL^2SL^2 & WL^2SL^2 &\longmapsto WL^2 \\ WLSL &\longmapsto WL^2 & WL^2 &\longmapsto WLSL \\ WLSL^2 &\longmapsto WL & WL &\longmapsto WLSL^2 \end{aligned}$$

Therefore, for any $x \in I$, $(f_{WL^2,WL^2S})(f_{WL,WLS})(x) = (f_{WL,WLS})(f_{WL^2,WL^2S})(x) = x$ □

1.5 Thompson's group F and T

Thompson's group F, T and V were introduced by Richard Thompson in unpublished notes in 1965. Different definitions of these groups have emerged since their first

introduction(in the context of algebraic logic). They all are finitely presented infinite groups. The groups T and V are simple but F is not. These groups have interesting properties. Therefore, these groups, especially F, are widely studied. In this work, we will be interested in F and T both of which will be defined as groups of certain piecewise linear transformations.

Definition 1.11. *Thompson's* F *group is defined as piece-wise linear homeomorphisms of the interval* $[0,1]$ *satisfying*

i. finitely many non-differentiable points, which is also called breakpoints points, are dyadic; that is, they can be written as $\frac{a}{2^n}$ *where* $a \in \mathbb{Z}$ *and* $n \in \mathbb{N}$

ii. any element f *of the group is increasing*

iii. derivative of any point in $f \in$ F *except breakpoints are the integer powers of* 2

Example 1.10. *The function*

$$\alpha(x) = \begin{cases} 2x, & 0 \leqslant x \leqslant \frac{1}{4} \\ x + \frac{1}{4}, & \frac{1}{4} \leqslant x \leqslant \frac{1}{2} \\ \frac{x+1}{2}, & \frac{1}{2} \leqslant x \leqslant 1 \end{cases}$$

is indeed an element of F*. First, the breakpoints of* α *are* $\frac{1}{4}$, $\frac{1}{2}$*, which are dyadic, and* α *is increasing. The derivative at any point, except breakpoints, is of the form* 2^n, $n \in \mathbb{Z}$.

Example 1.11. *Consider two elements of Thompson's group* F

$$f(x) = \begin{cases} \frac{x}{2}, & 0 \leqslant x \leqslant \frac{1}{2} \\ x - \frac{1}{4}, & \frac{1}{2} \leqslant x \leqslant \frac{3}{4} \\ 2x - 1, & \frac{3}{4} \leqslant x \leqslant 1 \end{cases} \qquad g(x) = \begin{cases} x, & 0 \leqslant x \leqslant \frac{1}{2} \\ \frac{x}{2} + \frac{1}{4}, & \frac{1}{2} \leqslant x \leqslant \frac{3}{4} \\ x - \frac{1}{8}, & \frac{3}{4} \leqslant x \leqslant \frac{7}{8} \\ 2x - 1, & \frac{7}{8} \leqslant x \leqslant 1 \end{cases}$$

Since both $f(x)$ *and* $g(x)$ *are increasing functions that have dyadic breakpoints and a derivative to the integer power of* 2 *at each point of the interval* $[0,1]$*, they are elements of Thompson's group* F.

Theorem 1.3. *Belk* (2007) F *is torsion-free.*

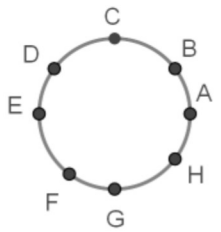
Proof. Let f be a non-identity element and t_0 be such a number that $f(t_0) = t_0$ and $f(t_0 + \epsilon) \neq (t_0 + \epsilon)$ for every $\epsilon > 0$. Therefore, a right-hand derivative of f at t_0 is 2^n so the right-hand derivative of f^n at t_0 must be 2^{nm} for every $m \in \mathbb{N}$ by using the chain rule. This means that all powers of f are different from each other and are not equal to the identity. □

Theorem 1.4. (*Belk*, 2007) *Thompson's group* F *is generated by* $f(x)$ *and* $g(x)$ *where the elements* f *and* g *are defined in the Example* 1.11.

After all the information above about F, we can now give the definition of T. In group F, our elements are functions whose domain and range are the interval $[0, 1]$. Similarly, we define T as a group of piecewise linear transformation on S^1 The remaining conditions are updated accordingly. We interpret S^1 as the quotient, i.e. $[0, 1] \Big/ \{0 \sim 1\}$.

Definition 1.12. *(Cylic Order) A* ***cyclic order*** *on a set* X *is a relation* $\mathrm{R} \subset \mathrm{X}^3$ *, where (a,b,c)* $\in$ R *but we use the notation* $a \to b \to c$*, satisfying the followings:*

1. *Cyclicity: If* $a \to b \to c$ *then* $b \to c \to a$
2. *Anti-symmetry: If* $a \to b \to c$*, then it is not true that* $c \to b \to a$*.*
3. *Transitivity: If* $a \to b \to c$ *and* $a \to c \to d$*, then* $a \to b \to d$*.*
4. *Ir-reflexivity: If* $a \to b \to c$*, they are all distinct.*
5. *Totality: If a,b, and c are distinct, then either* $a \to b \to c$ *or* $c \to b \to a$*.*



Consider the points on the circle given on the right. One can define cyclic order as traveling counterclockwise on this circle. We can order any three elements on the circle. Let's start with the letter A and travel in a counterclockwise direction. We first encounter B, then C. Hence, we can write $A \to B \to C$. If we start from B and use the same letters, we can also write $B \to C \to A$. We cannot write $C \to B \to A$ because after starting the letter C, we reach A then C. We can put in order for any three points that can be randomly selected from the circle.

Definition 1.13. *Thompson's group* T *is defined as piece-wise linear homeomorphisms of* $S^1 = [0, 1] \Big/ \{0 \sim 1\}$ *satisfying*

i. finitely many non-differentiable points, which is also called breakpoints points, are dyadic; that is, elements can be written as $\frac{a}{2^n}$ *where* $a \in \mathbb{Z}$ *and* $n \in \mathbb{N}$

ii. homeomorphisms are orientation-preserving

iii. derivative of any point except breakpoints are the integer powers of 2

Example 1.12. *The only difference between Thompson's group* F *and* T *is the domain; any element* $f \in \mathrm{F}$ *is still an element of* T*. Hence, in Example* 1.10*, the function*

$$\alpha = \begin{cases} 2x, & 0 \leqslant x \leqslant \frac{1}{4} \\ x + \frac{1}{4}, & \frac{1}{4} \leqslant x \leqslant \frac{1}{2} \\ \frac{x+1}{2}, & \frac{1}{2} \leqslant x \leqslant 1 \end{cases}$$

is also an element of T*.*

We can also show this function by using two circles where the inner circle represents the domain of the function and the outer represents the range of the function.

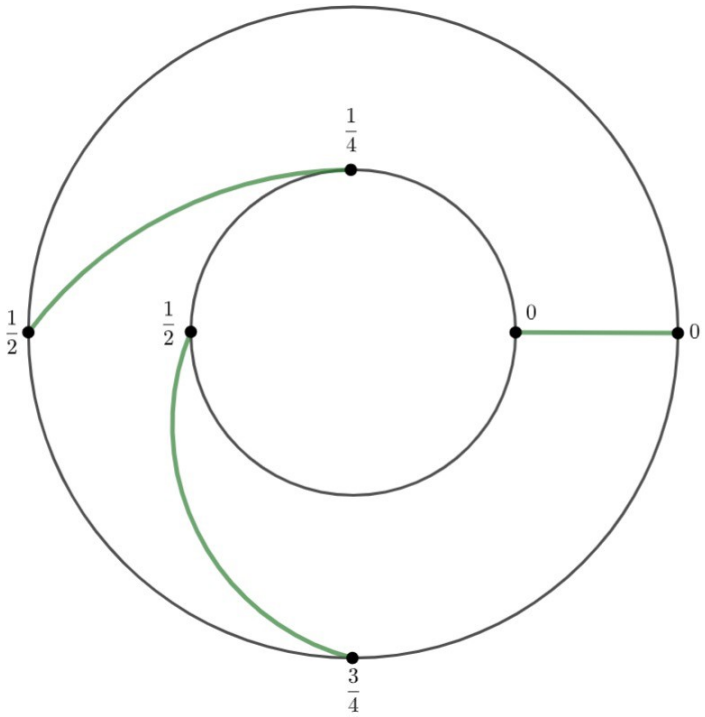


Figure 1.6: Demonstration of the function f in Thompson's group T

Since our elements in Thompson's group T are from S^1 to S^1, $t(0)$ and $t(1)$ do not need to be 0 and 1, respectively as in group F where t is an element of T. In the next example, we will have an example of an element of T which is in T but not in F.

Example 1.13. *Consider a function on unit interval*

$$t(x) = \begin{cases} 2x + \frac{1}{4}, & 0 \leqslant x \leqslant \frac{1}{4} \\ x + \frac{1}{2}, & \frac{1}{4} \leqslant x \leqslant \frac{1}{2} \\ \frac{x}{2} - \frac{1}{4}, & \frac{1}{2} \leqslant x \leqslant 1 \end{cases}$$

whose graph is given as in Figure 1.7. *If we check the intervals the function* t *maps, we get the following:*

i. $t([0, \frac{1}{4}] = [\frac{1}{4}, \frac{3}{4}]$

ii. $t([\frac{1}{4}, \frac{1}{2}] = [\frac{3}{4}, 1]$

iii. $t([\frac{1}{2}, 1] = [0, \frac{1}{4}]$

$t(x)$ *is an orientation-preserving function with rational breakpoints having derivative an integer power of* 2 *which makes* t *an element of group* T. *It seems* $t(x)$ *is also an element of the Thompson's* F *but it is not because* t *does not fix points* 0 *and* 1.

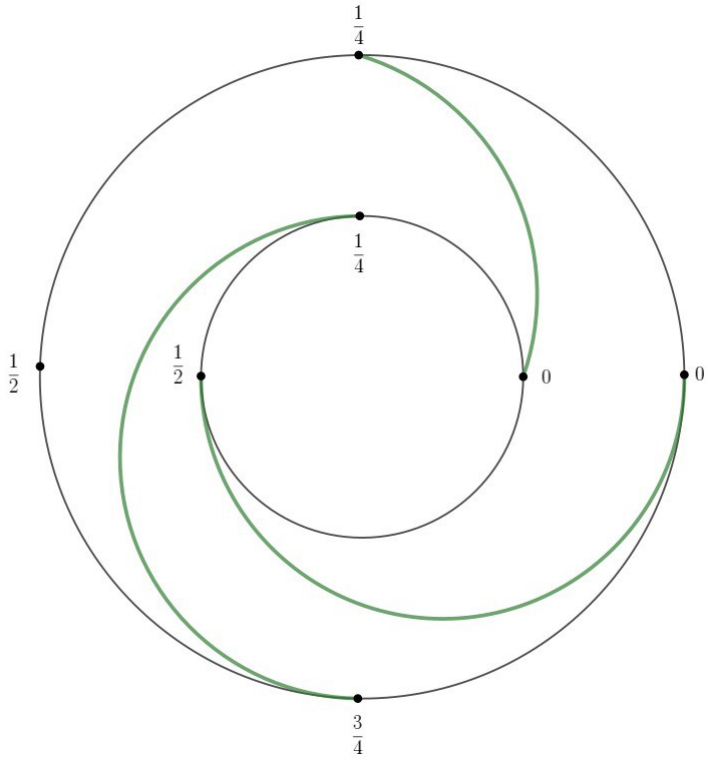


Figure 1.7: Circle diagram of the function t

Example 1.14. *We have mentioned before that the Thompson's group* F *is generated by two elements* $f(x), g(x)$*. Then, consider a function*

$$C(x) = \begin{cases} \frac{x}{2} + \frac{3}{4}, & 0 \leqslant x \leqslant \frac{1}{2} \\ 2x - 1, & \frac{1}{2} \leqslant x \leqslant \frac{3}{4} \\ x - \frac{1}{4}, & \frac{3}{4} \leqslant x \leqslant 1 \end{cases}$$

$C(x)$ *is a function which preserves orientation of* S^1 *with dyadic rational breakpoints and has derivative an integer power of* 2*. Therefore, it is an element of the Thompson's* T *group but it is not an element of Thompson's* F *group since it does not fix the points* 0 *and* 1*.*

Theorem 1.5. (*Cannon et al.*, 1996) *The elements* f, g *and* C *generate the Thompson's group* T *and satisfy the following:*

1. $[fg^{-1}, f^{-1}gf] = 1$
2. $[fg^{-1}, f^{-2}gf^2] = 1$
3. $C = g(f^{-1}Cg)$
4. $(f^{-1}Cg)(f^{-1}gf) = g(f^{-2}Cg^2)$
5. $Cf = (f^{-1}Cg)^2$
6. $C^3 = 1$

Remark. *Every flip is naturally an element of* T*. So we define:*

Definition 1.14. *We define the flip subgroup* P *of* T *as those elements of* T *which can be expressed as a finite sequence of flips.*

Proposition 1.7. *Two elements* $f_{WL,WLS}$*,*$f_{WLSL,WLSLS}$ *of* P *do not commute.*

Proof. We know the all transformations of $f_{WL,WLS}$ and $f_{WLSL,WLSLS}$ from above. If

we look at $f_{WLSL,WLSLS} \circ f_{WL,WLS}$, then

$$f_{WLSL,WLSLS} \circ f_{WL,WLS}(x) = \begin{cases} WL^2SW^{-1} \cdot x, & x \in \left(\frac{q}{s}, \frac{p+q}{r+s}\right) \\ WL^2(SL^2)^2W^{-1} \cdot x, & x \in \left(\frac{p+q}{r+s}, \frac{2p+q}{2r+s}\right) \\ W(LS)^2L^2W^{-1} \cdot x, & x \in \left(\frac{2p+q}{2r+s}, \frac{3p+q}{3r+s}\right) \\ WL(SL^2)^5W^{-1} \cdot x, & x \in \left(\frac{3p+q}{3r+s}, \frac{4p+q}{4r+s}\right) \\ W(SL^2)^2W^{-1} \cdot x, & x \in \left(\frac{4p+q}{4r+s}, \frac{p}{r}\right) \\ x, & otherwise \end{cases}$$

$$f_{WL,WLS} \circ f_{WLSL,WLSLS}(x) = \begin{cases} WL^2SW^{-1} \cdot x, & x \in \left(\frac{q}{s}, \frac{2p+q}{2r+s}\right) \\ WL^2SL(SL^2)^3W^{-1} \cdot x, & x \in \left(\frac{2p+q}{2r+s}, \frac{3p+q}{3r+s}\right) \\ W(SL^2)^2W^{-1} \cdot x, & x \in \left(\frac{3p+q}{3r+s}, \frac{p}{r}\right) \\ x, & otherwise \end{cases}$$

One can see from above that transformations of $f_{WL,WLS} \circ f_{WLSL,WLSLS}(x)$ on the interval $(\frac{p+q}{r+s}, \frac{4p+q}{4r+s})$ are not same with transformations of $f_{WLSL,WLSLS} \circ f_{WL,WLS}$. Therefore, they cannot commute. Since we proved that these two functions do not commute on the interval $(\frac{p+q}{r+s}, \frac{4p+q}{4r+s})$ just by checking transformation, then we can ask the following question. How much the composition of these two functions are close to commute? To measure it, we calculate the commutator of elements on the intervals that functions do not commute. Since transformations are different on the intervals $\left(\frac{p+q}{r+s}, \frac{2p+q}{2r+s}\right)$, $\left(\frac{2p+q}{2r+s}, \frac{3p+q}{3r+s}\right)$ and $\left(\frac{3p+q}{3r+s}, \frac{4p+q}{4r+s}\right)$, respectively, we compute the commutators for each interval. Note that commutator of two element a, b of a group is $a^{-1}b^{-1}ab$ and it is denoted by $[a, b]$. Let's begin with computing the commutator of the transformations of the interval $\left(\frac{p+q}{r+s}, \frac{2p+q}{2r+s}\right)$. $f_{WLSL,WLSLS} \circ f_{WL,WLS}(x) = WL^2SW^{-1}$ and $f_{WLSL,WLSLS} \circ f_{WL,WLS}(x) = WL^2SL^2SL^2W^{-1}$ on the interval $\left(\frac{p+q}{r+s}, \frac{2p+q}{2r+s}\right)$. Therefore, their commutator $[WL^2SW^{-1}, WL^2(SL^2)^2W^{-1}]$ is $WSLSL^2W^{-1}$. Similarly we need to check $[WL^2SL(SL^2)^3W^{-1}, W(LS)^2L^2W^{-1}]$ for the commutator on the interval $\left(\frac{2p+q}{2r+s}, \frac{3p+q}{3r+s}\right)$, and we get $W(LS)^3L^2(SL^2)^2S(LS)^2LW^{-1}$. For the last interval $\left(\frac{2p+q}{2r+s}, \frac{3p+q}{3r+s}\right)$, the commutator $[W(SL^2)^2W^{-1}, WLS(L^2S)^4L^2W^{-1}]$ is $W(LS)^7L^2SLS(L^2S)^4L^2W^{-1}$ □

Remark. *Commutativity in group P depends only on the support of the element. This means that if we look for the commutativity of two elements, it is enough to check*

whether their supports intersect. If they intersect, they do not commute; if they do not, they commute.

Example 1.15. *In this example, we try to decide whether elements* $f_{W(LS)^nL,W(LS)^{n+1}}$ $f_{WL,WLS}$ *commute or not. First, we write the element* $f_{W(LS)^nL,W(LS)^{n+1}}$.

$$f_{W(LS)^nL,W(LS)^{n+1}} = \begin{cases} W(LS)^{n-1}L(SL^2)^nW^{-1}\cdot x, & x\in\left(\frac{np+q}{nr+s},\frac{(n+1)p+q}{(n+1)r+s}\right) \\ W(LS)^nL^2(SL^2)^{n+2}W^{-1}\cdot x, & x\in\left(\frac{(n+1)p+q}{(n+1)r+s},\frac{(n+2)p+q}{(n+2)r+s}\right) \\ WSL^2W^{-1}\cdot x, & x\in\left(\frac{(n+2)p+q}{(n+2)r+s},\frac{p}{r}\right) \end{cases}$$

Since we know the element $f_{WL,WLS}$ *from above, we can obtain* $f_{W(LS)^nL,W(LS)^{n+1}}\circ f_{WL,WLS}$ *and* $f_{WL,WLS}\circ f_{W(LS)^nL,W(LS)^{n+1}}$ *, respectively.*

$$f_{W(LS)^nL,W(LS)^{n+1}}\circ f_{WL,WLS}(x) = \begin{cases} WL^2SW^{-1}\cdot x, & x\in\left(\frac{q}{s},\frac{p+q}{r+s}\right) \\ W(L^2S)^2L^2W^{-1}\cdot x, & x\in\left(\frac{p+q}{r+s},\frac{2p+q}{2r+s}\right) \\ WSL^2W^{-1}\cdot x, & x\in\left(\frac{2p+q}{2r+s},\frac{(n+1)p+q}{(n+1)r+s}\right) \\ W(LS)^{n-1}L(SL^2)^{n+1}L^2W^{-1}\cdot x, & x\in\left(\frac{(n+1)p+q}{(n+1)r+s},\frac{(n+2)p+q}{(n+2)r+s}\right) \\ W(LS)^nL^2(SL^2)^{n+3}W^{-1}\cdot x, & x\in\left(\frac{(n+2)p+q}{(n+2)r+s},\frac{(n+3)p+q}{(n+3)r+s}\right) \\ WSL^2SL^2W^{-1}\cdot x, & x\in\left(\frac{(n+3)p+q}{(n+3)r+s},\frac{p}{r}\right) \\ x, & otherwise \end{cases}$$

$$f_{WL,WLS}\circ f_{W(LS)^nL,W(LS)^{n+1}}(x) = \begin{cases} WL^2SW^{-1}\cdot x, & x\in\left(\frac{q}{s},\frac{p+q}{r+s}\right) \\ W(L^2S)^2L^2W^{-1}\cdot x, & x\in\left(\frac{p+q}{r+s},\frac{2p+q}{2r+s}\right) \\ WSL^2W^{-1}\cdot x, & x\in\left(\frac{2p+q}{2r+s},\frac{np+q}{nr+s}\right) \\ W(LS)^{n-2}L(SL^2)^nW^{-1}\cdot x, & x\in\left(\frac{np+q}{nr+s},\frac{(n+1)p+q}{(n+1)r+s}\right) \\ W(LS)^{n-1}L(SL^2)^nW^{-1}\cdot x, & x\in\left(\frac{(n+1)p+q}{(n+1)r+s},\frac{(n+2)p+q}{(n+2)r+s}\right) \\ WSL^2SL^2W^{-1}\cdot x, & x\in\left(\frac{(n+2)p+q}{(n+2)r+s},\frac{p}{r}\right) \\ x, & otherwise \end{cases}$$

As in the above Proposition 1.7*, we compute the commutators for the intervals that* $f_{WL,WLS}$ *and* $f_{W(LS)^nL,W(LS)^{n+1}}$ *do not have the same transformation to show how close these functions are to commute. One can see that we need to compute commutators of transformations for the intervals* $\left(\frac{np+q}{nr+s},\frac{(n+1)p+q}{(n+1)r+s}\right)$, $\left(\frac{(n+1)p+q}{(n+1)r+s},\frac{(n+2)p+q}{(n+2)r+s}\right)$ *and* $\left(\frac{(n+2)p+q}{(n+2)r+s},\frac{(n+3)p+q}{(n+3)r+s}\right)$, *respectively. For the first interval, one can check the commutator*

$[WSL^2W^{-1}, W(LS)^{n-2}L(SL^2)^nW^{-1}]$ *and get*

$$W(LS)^{n+1}L^2(SL^2)^3L(SL^2)^nW^{-1}$$

Secondly, we compute the commutator for the interval $\left(\frac{(n+1)p+q}{(n+1)r+s}, \frac{(n+2)p+q}{(n+2)r+s}\right)$*,that is,* $[W(LS)^{n-1}L(SL^2)^{n+1}L^2W^{-1}, W(LS)^{n-1}L(SL^2)^nW^{-1}]$ *and we have*

$$WL(LS)^nL(SL^2)^{n+1}L^2(LS)^{n-1}L(SL^2)^nW^{-1}$$

Finally, for the last interval, we need $[W(LS)^nL^2(SL^2)^{n+3}W^{-1}, WSL^2SL^2W^{-1}]$ *for the commutator, and we have*

$$WL(LS)^{n+3}L(LS)^5L(SL^2)^{n-2}W^{-1}$$

We actually know that these two functions do not commute without computing any transformation of intervals because their supports intersect.

Theorem 1.6. *The set of fractional linear transformations getting from the flip move is isomorphic to Thomson's* T *group.*

We will prove Theorem 1.6, but we need more information.

Minkowski's Question Mark Function, denoted $?(x)$, is an order-preserving, continuous function that bijectively maps rational numbers on the unit interval to dyadic rational numbers on the unit interval. There are many ways to define Minkowski's Question Mark Function such as using continued fractions or modified Farey sequences. We define the function by using continued fraction expression of the rational numbers. If x is a rational number on the unit interval, there exists a unique continued fraction expression of $x = [a_0; a_1, a_2, \cdots, a_n]$ for some integer $n \geqslant 1$. Note that a_0 must be 0 for this case due to $x \in [0, 1]$.

Definition 1.15. (*Salem*, 1943) *Minkowski's Question Mark Function* $?(x)$ *is defined as*

$$?(x) = \sum_{i=1}^{n} \frac{(-1)^{i-1}}{2^{(a_1+a_2+\cdots+a_i)-1}}$$

for any rational number x *in* $[0, 1]$*.*

This function can be extended for every element of the unit interval since rational numbers are dense in real numbers and because of the continuity of the function.

In that case, we write ∞ instead of n because the continued fraction expression of irrational numbers does not terminate. What we did till now is not sufficient to prove the theorem. This theorem only allows us to find a bijective function from rationals on the unit interval to dyadic rationals on the unit interval. However, support for each element of the group getting from the flip moves is not limited to rationals on the unit interval. We need to have all rationals in $\mathbb{R}$. Therefore, we need a function that bijectively maps dyadic rationals to $\mathbb{Q}$. Let's denote dyadic rational numbers on the unit interval by $\mathbb{Q}_2$. Such function $\zeta : \mathbb{Q}_2 \longrightarrow \mathbb{Q}_\infty$ is defined as for every $x \in \mathbb{Q}_2$

$$\zeta(x) = \begin{cases} \frac{-1}{?^{-1}(4x)}, & 0 \leqslant x \leqslant \frac{1}{4} \\ -?^{-1}(2-4x), & \frac{1}{4} \leqslant x \leqslant \frac{1}{2} \\ ?^{-1}(4x-2), & \frac{1}{2} \leqslant x \leqslant \frac{3}{4} \\ \frac{1}{?^{-1}(4-4x)}, & \frac{3}{4} \leqslant x \leqslant 1 \end{cases}$$

where $\frac{1}{0}$ and $\frac{-1}{0}$ represents ∞ and $-\infty$,respectively. ζ is a continuous function sending all rational numbers bijectively to dyadic rationals on the unit interval. As we do in Minkowski's Question Mark Function, we can change domain of the function $\zeta : [0,1] \longrightarrow \mathbb{R} \cup \{\infty\}$. (See (*Vepstas*, 2014) for more details)

Remark. *For any* $W \in \mathrm{PSL}_2(\mathbf{Z})/\{I, S\}$ *starting with* L^2, $f_{WL,WLS}$ *always induces an element of the Thompson's group* F.

Now we can give the proof of the Theorem 1.6 as follows.

Proof. Consider the flip $f_{I,S}$ and the sequence of the flips $f_{L,LS}, f_{I,S}, f_{SL,SLS}, f_{I,S}, f_{L,LS}$ and set these flips as α and β, respectively. By (*Lochak and Schneps*, 1997), a homomorphism ϕ is an isomorphism from the group P to T such that

$$\phi(A) = \beta\alpha^2$$
$$\phi(B) = \beta^2\alpha$$
$$\phi(C) = \beta^2$$

□

2 HOMOLOGY AND COHOMOLOGY

In this section, we will define homology and cohomology modules in terms of chain complexes and give examples. Then, we define group cohomology by defining group module structure and investigate some low-dimensional group cohomology modules.

2.1 Homology

Let R be a commutative ring with unity.

Definition 2.1. *Let $\{C_n\}_{n\in\mathbb{Z}}$ be a family of R-modules and let R-module maps $d_n : C_n \mapsto C_{n-1}$ be R-module homomorphisms such that $d_{n-1} \circ d_n = 0$ for every $n \in \mathbb{Z}$. We call such a sequence family a **chain complex** denoted by C_*. Elements of the kernel of d_n are called **n-cycles** and denoted by $Z_n = Z_n(C_*)$. Elements of the image of d_{n+1} are called **n-boundaries** of C_* and denoted by $B_n = B_n(C_*)$*

Example 2.1. *Notice that, since $d_{n-1} \circ d_n = 0$, we have*

$$0 \subseteq B_n \subseteq Z_n \subseteq C_n$$

Observe that Z_n and B_n are R-modules. Therefore, Z_n / B_n is a R-module as well.

Definition 2.2. *The n^{th} homology group of C_* is the quotient $H_n(C_*) = Z_n / B_n$.*

Example 2.2. *Let p be a prime number and $C_n = \mathbb{Z}/p^2\mathbb{Z}$ for $n \geqslant 0$, and $C_n = 0$ for $n < 0$. Consider $d_n : C_n \mapsto C_{n-1}$ defined as $d_n(\bar{t}) = p\bar{t}$ so that we have*

$$\ldots \xrightarrow{d_{n+1}} \mathbb{Z}/p^2 \xrightarrow{d_n} \mathbb{Z}/p^2 \xrightarrow{d_{n-1}} \mathbb{Z}/p^2 \ldots \xrightarrow{d_2} \mathbb{Z}/p^2 \xrightarrow{d_1} \mathbb{Z}/p^2 \xrightarrow{d_0} 0$$

Above sequence is a chain complex since $d_{n-1} \circ d_n(\bar{t}) = d_{n-1}(p\bar{t}) = p^2\bar{t} \equiv 0$ for all $n \in \mathbb{Z}$ Hence, we can compute the homology of the chain complex. Precisely,

$$Z_n = \{x \in \mathbb{Z}/p^2\mathbb{Z} |\ d_n(x) \equiv 0\}$$

$$B_n = \{x \mid d_{n+1}(n) \equiv x \text{ for some } x \in \mathbb{Z}/p^2\mathbb{Z}\}$$

for $n > 0$,

$$H_n(C_*) = Z_n \big/ B_n = \{\bar{0}, \overline{p}, \overline{2p} \dots \overline{(p-1)p}\} \Big/ \{\bar{0}, \overline{p}, \overline{2p} \dots \overline{(p-1)p}\} = \{\bar{0}\}$$

for $n = 0$,

$$H_0(C_*) = Z_0 \big/ B_0 = \mathbb{Z}/p^2\mathbb{Z} \big/ 0 \cong \mathbb{Z} \big/ p^2\mathbb{Z}$$

for $n < 0$, $H_n = \{\bar{0}\}$ *since all* $C_i = \{\bar{0}\}$ *for* $i < 0$

2.2 Cohomology

Again, let R be a commutative ring with unity.

Definition 2.3. *Let* $\{C^n\}_{n\in\mathbb{Z}}$ *be a family of* R*-modules and let* $d^n : C^n \mapsto C^{n+1}$ *be a family of* R*-module homomorphisms so that* $d^{n+1} \circ d^n = 0 \quad for\ all\ n \in \mathbb{Z}$. *Such a sequence is called cochain complex and denoted by* C^*. *Elements of the kernel of* d^n *are called* ***n-cocycles*** *and denoted by* $Z^n = Z^n(C^*)$. *Elements of the image of* d^{n-1} *are called* ***n-coboundaries*** *of* C^* *and denoted by* $B^n = B^n(C_*)$.

Again, we have

$$0 \subseteq B^n \subseteq Z^n \subseteq C^n$$

because $d^{n+1} \circ d^n = 0$.

Definition 2.4. *The* n^{th} *cohomology of* C^* *is the quotient* $H^n(C^*) = Z^n \big/ B^n$.

Example 2.3. *Let* C^n *be the same* $\mathbb{Z}$*-modules in the Example* 2.2. *This time consider* $d^n : C^n \mapsto C^{n+1}$ *where* $d^n(\bar{t}) = p\bar{t}$ *for every* $n \in \mathbb{Z}$. *We get following sequence*

$$\dots \xleftarrow{d^{n+1}} \mathbb{Z} \big/ p^2 \xleftarrow{d^n} \mathbb{Z} \big/ p^2 \xleftarrow{d^{n-1}} \mathbb{Z} \big/ p^2 \dots \xleftarrow{d^2} \mathbb{Z} \big/ p^2 \xleftarrow{d^1} \mathbb{Z} \big/ p^2 \xleftarrow{d^0} \mathbb{Z} \big/ p^2 \xleftarrow{d^{-1}} 0 \dots$$

Since $d^{n-1} \circ d^n(\bar{t}) = d^{n-1}(p^2\bar{t}) = p^2\bar{t} \equiv \bar{0}$, *above sequence is cochain complex. We compute its cohomology as follows:*

for $n > 0$,

$$H_n(C_*) = Z_n \big/ B_n = \{\bar{0}, \overline{p}, \overline{2p} \dots \overline{(p-1)p}\} \Big/ \{\bar{0}, \overline{p}, \overline{2p} \dots \overline{(p-1)p}\} = \{\bar{0}\}$$

for n=0,

$$H_0(C_*) = Z_0 \big/ B_0 = \mathbb{Z}/p^2\mathbb{Z} \big/ 0 \cong \mathbb{Z} \big/ p^2\mathbb{Z}$$

for $n < 0$, $H_n = \{\bar{0}\}$ *since all* $C_i = 0\ for\ i < 0$

2.3 Group Cohomology

Definition 2.5. *Let G be a group. We say that an abelian group M is G-module if there is a group action $\alpha : G \times M \mapsto M$ satisfying*

$$g \cdot (a_1 + a_2) = g \cdot a_1 + g \cdot a_2$$

$$(g_1 g_2) \cdot a = g_1 \cdot (g_2 \cdot a)$$

where $g, g_1, g_2 \in G$ and $a, a_1, a_2 \in M$ are arbitrary

Example 2.4. *Let M be any abelian group and $\alpha : G \times M \mapsto M$ where $g \cdot m = m$ for all $g \in G$ and for all $m \in M$. This action is called the trivial action, and M is called a trivial G-module.*

As in the cohomology of R-modules, we use cochain complexes to define group cohomology.

Let A be a G-module and for $n \geqslant 0$, let $C^n(G, A)$ be the set of functions $f : G^n \to A$, namely

$$C^n(G, A) = \{f \mid f : G^n \longrightarrow A\}$$

We endow $C^n(G, A)$ with pointwise addition so that it becomes an abelian group. Since A is abelian, the set $C^n(G, A)$ is abelian for all n where the zero map which sends everything zero is the identity element of the set. Now, we define G-module homomorphisms to create cochain complex structure as follows:

Definition 2.6. *Let $f \in C^n(G, A)$ and define $d^n : C^n(G, A) \mapsto C^{n+1}(G, A)$ as*

$$d^n(f)(g_1, \ldots, g_{n+1}) = g_1 \cdot f(g_2, g_3 \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_1, g_2, .., g_{i-1}, g_i g_{i+1}, \ldots g_{n+1}) + (-1)^{n+1} f(g_1, g_2, \ldots g_n)$$

Proposition 2.1. *In this setting, $d^{n+1} \circ d^n(f) = 0$ for every $n \in \mathbb{N}$ and as a result, we can obtain the following cochain complex*

$$0 \xrightarrow{d^{-1}} C^0(G^0, A) \xrightarrow{d^0} C^1(G, A) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} C^n(G^n, A) \xrightarrow{d^n} C^{n+1}(G^{n+1}, A) \xrightarrow{d^{n+1}} \cdots$$

Remark. *Since $G^0 = \{1\}$, $C^0(1, A)$ contains all functions from 1 to A and each function sends 1 to different elements of A. Therefore, $C^0(1, A) \cong A$. Therefore:*

$$0 \xrightarrow{d^{-1}} A \xrightarrow{d^0} C^1(G, A) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} C^n(G^n, A) \xrightarrow{d^n} C^{n+1}(G^{n+1}, A) \xrightarrow{d^{n+1}} \cdots$$

Definition 2.7. *Let $Z^n(G,A) = \ker(d^n)$ and $B^n(G,A) = im(d^{n-1})$. Then, n^{th} cohomology group of G with coefficient A is the quotient group*

$$H^n(G,A) = Z^n(G,A) \Big/ B^n(G,A) = \ker(d^n) \Big/ \mathrm{im}(d^{n-1})$$

Example 2.5. ***(Zeroth Cohomology Group)***

From definition, $H^0(G,A) = Z^0(G,A) \Big/ B^0(G,A) = Z^0(G,A)/0 = Z^0(G,A)$. We have mentioned before that we can see $C^0(G,A)$ as A. Therefore,

$$H^0(G,A) = Z^0(G,A) = \{a \in A \mid xa = a\}$$

for every $x \in G$. This $H^0(G,A)$ is also denoted by A^G.

Example 2.6. ***(First Cohomology Group)***

If we write $Z^1(G,A)$ and $B^1(G,A)$ respectively, we get

$$Z^1(G,A) = \{f : G \mapsto A \mid f(gh) = gf(h) + f(g)\}$$
$$B^1(G,A) = \{f : G \mapsto A \mid f(g) = ga - a\} \; for \; some \; a \in A$$

In the case that A is a trivial G-module, $B^1(G,A) = 0$ and $Z^1(G,A) = \{f : G \mapsto A \mid f(gh) = f(h) + f(g)\}$ which means that $Z^1(G,A) = Hom(G,A)$. Therefore, $H^1(G,A) = Z^1(G,A) \Big/ B^1(G,A) = Z^1(G,A)/0 = Hom(G,A)$

3 GROUP EXTENSIONS AND COHOMOLOGY

In this chapter, firstly, we will define group extension and discuss some of their properties. Then, we will classify group extensions via factor system. Finally, we will study the connection between the group extensions and elements of the second cohomology group.

3.1 Group Extensions

Definition 3.1. *Let $\{G_n\}_{n\in\mathbb{Z}}$ be a family of groups with homomorphisms f_i where $i \in \mathbb{Z}$ such that*

$$\dots G_{-2} \xrightarrow{f_{-1}} G_{-1} \xrightarrow{f_0} G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} G_2 \dots G_{n-1} \xrightarrow{f_{n-1}} G^n \xrightarrow{f_n} G^{n+1} \dots$$

*The above sequence is called **exact at** G_n if $\mathrm{im}(f_n) = \ker(f_{n+1})$. The sequence is called exact if it is **exact** for all G_n.*
If the sequence has only five terms, that is of the form

$$1 \longrightarrow N \xrightarrow{\alpha} G \xrightarrow{\beta} Q \longrightarrow 1$$

*and is exact, then it is called **short exact sequence**.*

Remark. *In such a case, we must have $\ker(\alpha) = \mathrm{im}(1 \mapsto N) = 1$, meaning α must be injective. Similarly, since $Q = \ker(Q \mapsto 1) = \mathrm{im}(\beta)$, β must be surjective.*

Example 3.1. *Consider the sequence*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

where f sends x to $2x$ for all $x \in \mathbb{Z}$, and g sends y to $\overline{y}$ for all $y \in \mathbb{Z}$. Our claim is that the above sequence is exact. Since $\ker(f) = \{x \mid f(x) = 2x = 0\} = 0 = \mathrm{im}(0 \mapsto \mathbb{Z})$, the sequence is exact at left $\mathbb{Z}$,and since $\mathrm{im}(g) = \mathbb{Z}/2\mathbb{Z} = \ker(\mathbb{Z}/2\mathbb{Z} \mapsto 0)$, it is exact at $\mathbb{Z}/2\mathbb{Z}$. To prove that the sequence is exact at middle term $\mathbb{Z}$, it is enough to show that $\ker(f) = \mathrm{im}(g)$. Since f sends every x to $2x$, $\mathrm{im}(f) = 2\mathbb{Z}$ and since $g(y) = 0$ whenever $y = 2k$ for some $k \in \mathbb{Z}$, $\ker(g) = 2\mathbb{Z}$. Hence, $\ker(g) = \mathrm{im}(f)$ which gives exactness at right $\mathbb{Z}$ and also gives the exactness of the sequence.

Lemma 3.1. *Consider the short exact sequence*

$$1 \longrightarrow N \xrightarrow{\alpha} G \xrightarrow{\beta} Q \longrightarrow 1$$

Then, the followings hold true:

i. There is a normal subgroup N' of G which is isomorphic to N.

ii. There is a normal subgroup N'' of G such that ${}^{G}/_{N''} \cong Q$

Proof. *i.* Since we have a short exact sequence, α is injective. By exactness, we also know that $\text{im}(\alpha) = \ker(\beta)$. Since $\ker(\beta) \trianglelefteq G$, $\text{im}(\alpha) = \alpha(N)$ is also normal in G. We know $\alpha(N) \cong N$ with map $a \mapsto \alpha(a)$. Hence, we found a normal subgroup of G isomorphic to N.
ii. By the first isomorphism theorem, we know that ${}^{G}\big/_{\ker(\beta)} \cong \text{im}(\beta)$. However, $\text{im}(\beta) = Q$ by surjectivity ,and $\ker(\beta) = \text{im}(\alpha) = \alpha(N)$. Hence, we get ${}^{G}\big/_{\alpha(N)} \cong Q$.
□

In particular, for any normal subgroup N of G, the sequence

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{\pi} {}^{G}/_{N} \longrightarrow 1$$

is a short exact sequence where $i : N \to G$ (resp. $\pi : G \to {}^{G}/_{N}$) defined as $i(x) = x$ (resp. $\pi(x) = x + N$).

Definition 3.2. *Let N and Q be two groups. A group G is said to be an extension of N by Q if*

1. *G has a normal subgroup N' which is isomorphic to N.*
2. *${}^{G}/_{N'}$ is isomorphic to Q.*

In such a case, we have the following short exact sequence

$$1 \longrightarrow N \to G \to {}^{G}/_{N} \longrightarrow 1$$

Example 3.2. *We can extend any two groups via direct product. That is, if we let A and B be any groups, then the sequence*

$$1 \longrightarrow A \xrightarrow{f} A \times B \xrightarrow{g} B \longrightarrow 1$$

where $f(a) = (a, 1)$ for all $a \in A$ and $g(a, b) = b$ for all $(a, b) \in A \times B$, is exact.

Example 3.3. *Consider the sequence*

$$0 \longrightarrow \mathbb{Z}_m \xrightarrow{f} \mathbb{Z}_{mn} \xrightarrow{g} \mathbb{Z}_n \longrightarrow 0$$

where $f(\overline{x}) = m\overline{x}$ *for any* $\overline{x} \in \mathbb{Z}_m$ *and* $g(\overline{y}) = \overline{y}$ *for any* $\overline{y} \in \mathbb{Z}_{mn}$ *Since* $\mathrm{im}(f) = \{\overline{0}, \overline{m}\} = \ker(g)$*, the sequence is exact at* $\mathbb{Z}_{mn}$*. This sequence is exact by injectivity and surjectivity, so we have a group extension. Hence,* $\mathbb{Z}_{mn}$ *is an extension of* $\mathbb{Z}_m$ *by* $\mathbb{Z}_n$*.*

We can also write $\mathbb{Z}_{mn}$ as a group extension of $\mathbb{Z}_n$ by $\mathbb{Z}_m$ since $\mathbb{Z}_n \trianglelefteq \mathbb{Z}_{mn}$ and $\mathbb{Z}_{mn}/\mathbb{Z}_n \cong \mathbb{Z}_m$. However, this change may sometimes not be valid. For example, we can write symmetric group S_3 as an extension of $\mathbb{Z}_3$ by $\mathbb{Z}_2$. However, it is not an extension of $\mathbb{Z}_2$ by $\mathbb{Z}_3$ because S_3 has no normal subgroup which is isomorphic to $\mathbb{Z}_2$. S_3 has only one non-trivial normal subgroup which is A_3 and that is isomorphic to $\mathbb{Z}_3$.

Definition 3.3. *Consider a group extension of* N *by* Q

$$1 \longrightarrow N \xrightarrow{\alpha} G \xrightarrow{\beta} Q \longrightarrow 1$$

A function $\gamma : Q \mapsto G$ *is called* ***transversal function*** *if* $\beta(\gamma(q)) = q$ *for every* $q \in Q$*.*

Notice that γ need not be a group homomorphism. If it is a group homomorphism, the extension is called the **split** and exact sequence is called **splitting exact sequence**.

Example 3.4. *Consider the sequence in Example* 3.3

$$0 \longrightarrow \mathbb{Z}_m \xrightarrow{f} \mathbb{Z}_{mn} \xrightarrow{g} \mathbb{Z}_n \longrightarrow 0$$

We need to find a function $h : \mathbb{Z}_m \mapsto \mathbb{Z}_{mn}$ *such that* $g(h(q)) = q$ *for every* $q \in \mathbb{Z}_n$*. Let* $h(q) = q$ *for every* q*. Then*

$$\begin{aligned} g(h(\overline{0})) &= g(\overline{0}) = \overline{0} \\ g(h(\overline{1})) &= g(\overline{1}) = \overline{1} \\ g(h(\overline{2})) &= g(\overline{2}) = \overline{2} \\ &\vdots \\ g(h(\overline{n-1})) &= g(\overline{n-1}) = \overline{n-1} \end{aligned}$$

Hence, we get $g(h(q)) = q$ *for every* $q \in \mathbb{Z}_n$*. Since* h *is a homomorphism, this is a split extension and the sequence is splitting exact sequence.*

Example 3.5. *Consider the sequence*

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{f} \mathbb{Z}/4\mathbb{Z} \xrightarrow{g} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

where $f(\bar{1}) = \bar{2}$, $g(\bar{1}) = \bar{1}$.

- *Since* $\mathrm{im}(f) = \{\bar{0}, \bar{2}\} = \ker(g)$, *the sequence is exact at* $\mathbb{Z}/4\mathbb{Z}$.
- *Since* $f(\bar{0}) = \bar{0}$ *and* $f(\bar{1}) = \bar{2}$, f *is injective ,and that gives exactness at left* $\mathbb{Z}/2\mathbb{Z}$.
- *For* $\bar{0} \in \mathbb{Z}/2\mathbb{Z}$, $g(\bar{0}) = \bar{0}$ *and for* $\bar{1} \in \mathbb{Z}/2\mathbb{Z}$, $g(\bar{1}) = \bar{1}$ *so* g *is surjective and that gives exactness at right* $\mathbb{Z}/2\mathbb{Z}$.

Hence, we know that $\mathbb{Z}/4\mathbb{Z}$ *is a group extension of* $\mathbb{Z}/2\mathbb{Z}$ *by* $\mathbb{Z}/2\mathbb{Z}$. *Let* $h : \mathbb{Z}/2\mathbb{Z} \mapsto \mathbb{Z}/4\mathbb{Z}$ *be a function such that* $f(\bar{0}) = \bar{0}$ *and* $f(\bar{1}) = \bar{1}$. $g(h(\bar{0}) = g(\bar{0}) = \bar{0}$ *and* $g(h(\bar{1}) = g(\bar{1}) = \bar{1}$ *so* $gh = \mathbb{1}\ |_{\mathbb{Z}/2\mathbb{Z}}$. *However,* h *is not a homomorphism since* $1 = h(z.z) \neq h(z)h(z) = y^2 \neq 1$. *Hence,* h *is just a transversal function, which means that the extension is not split.*

3.2 Classification of Group Extensions

Let N and Q be two groups. We can define many group extension by using N and Q as we have seen above.

Let us first introduce an equivalence relation on the set of group extensions.

Lemma 3.2. (*Hungerford*, 2003) ***(The Short Five Lemma)***

Let A, B, C, A', B' *and* C' *be groups and* i, π, i', π' *be homomorphisms, as shown in the diagram below.*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{\pi} & C & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \beta} & & \\ 1 & \longrightarrow & A' & \xrightarrow{i'} & B' & \xrightarrow{\pi'} & C' & \longrightarrow & 1 \end{array}$$

Suppose that the following diagram is commutative, which means that $i'\alpha(a) = \phi i(a)$ *for every* $a \in A$ *and* $\pi'\phi(b) = \beta\pi(b)$ *for every* $b \in B$. *If the above two short sequences are exact and if* α *and* β *are isomorphism, then* ϕ *is also an isomorphism.*

Definition 3.4. *Let N and Q be two groups. Two group extensions G, G' of N by Q are called* ***equivalent*** *if there is a function $\phi : G \mapsto G'$ which makes the following diagram commutative.*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \xrightarrow{i} & G & \xrightarrow{\pi} & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle id} & & \\ 1 & \longrightarrow & N & \xrightarrow{i'} & G' & \xrightarrow{\pi'} & Q & \longrightarrow & 1 \end{array}$$

Notice that since maps from N to N and from Q to Q are isomorphisms, this makes $\phi : G \mapsto G'$ isomorphism by short five-lemma. Hence, any two equivalent group extensions are always isomorphic.

Example 3.6. *Let $S_3 = \{(1), (12), (13), (23), (123), (132)\}$ be the symmetric group on 3 letters and $D_3 = \{1, a, b, b^2, ab, ab^2\}$ be the dihedral group where a and b are generators of the group satisfying $a^2 = b^3 = (ab)^2 = 1$. We can write both S_3 and D_3 as a group extension of $\mathbb{Z}_3$ by $\mathbb{Z}_2$ with following maps:*
For S_3

$$\begin{gathered} i(\overline{0}) = (1) \quad i(\overline{1}) = (123) \quad i(\overline{2}) = (132) \\ \pi((1)) = \tilde{0} \quad \pi((123)) = \tilde{0} \quad \pi((132)) = \tilde{0} \\ \pi((12)) = \tilde{1} \quad \pi((23)) = \tilde{1} \quad \pi((13)) = \tilde{1} \end{gathered}$$

For D_3

$$\begin{gathered} i'(\overline{0}) = 1 \quad i'(\overline{1}) = b \quad i'(\overline{2}) = b^2 \\ \pi'(1) = \tilde{0} \quad \pi'(b) = \tilde{0} \quad \pi'(b^2) = \tilde{0} \\ \pi'(a) = \tilde{1} \quad \pi'((ab)) = \tilde{1} \quad \pi'(ab^2) = \tilde{1} \end{gathered}$$

Hence, we have the following diagram of extensions of $\mathbb{Z}_3$ by $\mathbb{Z}_2$.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}_3 & \xrightarrow{i} & S_3 & \xrightarrow{\pi} & \mathbb{Z}_2 & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle id} & & \\ 1 & \longrightarrow & \mathbb{Z}_3 & \xrightarrow{i'} & D_3 & \xrightarrow{\pi'} & \mathbb{Z}_2 & \longrightarrow & 1 \end{array}$$

Let ϕ be a function from S_3 to D_3 such that

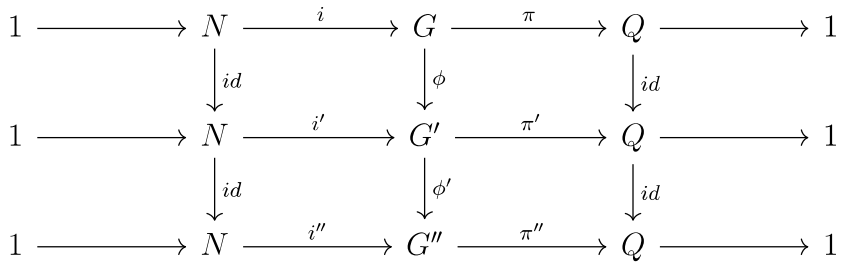
$$\phi((1)) = 1 \quad \phi((123)) = b \quad \phi((132)) = b^2$$
$$\phi((12)) = a \quad \phi((13)) = ab \quad \phi((23)) = ab^2$$

Since $i'(id)(a) = \phi i(a)$ for every $a \in \mathbb{Z}_3$ and $\pi'\phi(b) = id(\pi)(b)$ for every $b \in \mathbb{Z}_2$, the above diagram is commutative. Hence, we find a homomorphism ϕ that makes diagram commutative and that ϕ is isomorphism by short five lemma. Hence, S_3 and D_3 are two equivalent group extensions of $\mathbb{Z}_3$ by $\mathbb{Z}_2$.

Proposition 3.1. *Let R be the set of all group extensions of N by Q and let being equivalent be our binary relation on R. Then "being equivalent" is an equivalence relation on R.*

Proof. We denote elements of R as (G, i, π) ,and binary operation as $(G, i, \pi) \sim (G', i', \pi')$ where i, π, i' and π' are maps that make up related extension. For all $(G, i, \pi)(G', i', \pi')$ and $(G'', i'', \pi'') \in R$:

- Reflexive: It is clear since any group is isomorphic to itself by identity map. Therefore we have such function making diagram commute. Hence, $(G, i, \pi) \sim (G, i, \pi)$
- Symmetric: Let $(G, i, \pi) \sim (G', i', \pi')$. Then there is a $\phi : G \mapsto G'$ where ϕ is a isomorphism. Since ϕ is a isomorphism, its inverse ϕ^{-1} exists. Hence, $(G', i', \pi') \sim (G, i, \pi)$ where ϕ^{-1} makes diagram commute.
- Transitive: Let $(G, i, \pi) \sim (G', i', \pi')$ and $(G', i', \pi') \sim (G'', i'', \pi'')$. Therefore, there are isomorphisms ϕ and ϕ' which make diagrams commute as in the following diagram.



Since $\phi'\phi : G \mapsto G'$ is isomorphism which does not disrupt commutativity of diagram, we can say $(G, i, \pi) \sim (G'', i'', \pi'')$. Hence $\sim$ is an equivalence relation on R. $\square$

Let us introduce factor systems on G-modules.

Remark. *Let N be an abelian group with respect to addition and Q be any group. Suppose that we have a group extension G of N by Q.*

$$1_N \longrightarrow N \xrightarrow{i} G \xrightarrow{p} Q \longrightarrow 1_Q \qquad (\star)$$

We can see N as a Q-module. Let $\Phi : Q \times N \to N$ be a function such that

$$\begin{aligned} \Phi :& Q \times N \to N \\ & (q, n) \mapsto q \cdot n := n' \end{aligned}$$

where $i(n') = g^{-1}i(n)g$ and $p(g) = q$ for every $n, n' \in N$ and for every $q \in Q$. Note that such g always exists because p is surjective but choice of g is not unique. For this reason, we first show that this definition is independent of the choice of g. Suppose that $q \cdot n := g_1^{-1}i(n)g_1$ and $q \cdot n := g_2^{-1}i(n)g_2$ for some $q \in Q$, $g_1, g_2 \in G$ with $p(g_1) = q$, $p(g_2) = q$ and $n \in N$. We have $p(g_2)^{-1}p(g_1) = q^{-1}q = 1 = p(g_2^{-1}g_1)$. Since the sequence $\star$ is a short exact sequence, we know that $\ker(\mathrm{p}) = \mathrm{im}(\mathrm{i})$. *Hence, we get* $g_2^{-1}g_1 \in \mathrm{im}(\mathrm{N})$. *We now check that $g_1^{-1}i(n)g_1 = g_2^{-1}i(n)g_2$. Before checking it, we should point out two things. First, N is isomorphic to $i(N)$ via the map $\phi : N \to i(N)$,where $\phi(n) := i(n)$ and we proved in the Lemma* 3.1 *that $i(N)$ is normal subgroup of G. For this reason, $g^{-1}i(n)g$ is always an element of $i(N)$. Second, since N is an abelian group, $i(N)$ is also abelian group. If we multiply $g_1^{-1}i(n)g_1$ from right with $(g_2^{-1}g_1)^{-1}(g_2^{-1}g_1) = (g_1^{-1}g_2)(g_2^{-1}g_1)$, we get*

$$\begin{aligned} (g_1^{-1}i(n)g_1) &= (g_1^{-1}i(n)g_1)(g_1^{-1}g_2)(g_2^{-1}g_1) \\ &= (g_2^{-1}g_1)(g_1^{-1}i(n)g_1)(g_1^{-1}g_2) \text{ since } i(N) \text{ is abelian} \\ &= g_2^{-1}i(n)g_2 \text{ since } G \text{ is associative.} \end{aligned}$$

Hence, we show that our definition of G-module structure is independent of g. Now that we have proven that choice of g is not important, we can proceed to the proof that N is a Q-module.

- *Let $q_1 \cdot (q_2 \cdot n) = n_1$, $q_1 \cdot n = n_2$ and $q_1 \cdot n_2 = n_1$ where $p(g_1) = q_1$ and $p(g_2) = q_2$ so we know that $g_1^{-1}i(n)q_1 = i(n_2)$ and $g_2^{-1}i(n_2)g_2 = i(n_1)$. If we combine these, we get $g_2^{-1}g_1^{-1}i(n)g_1g_2 = i(n_1)$. Hence $q_1 \cdot (q_2 \cdot n) = n_1 = (q_1q_2) \cdot n$ for every $n, n_1, n_2 \in N$ and for every $q_1, q_2 \in Q$.*

- *Let* $q(n_1 + n_2) = n_3$, $q \cdot n_1 = n'$ *and* $q \cdot n_2 = n''$ *where* $p(g) = q$. *Hence, we can write the following:*

$$\begin{aligned} i(n_3) = g^{-1}i(n_1 + n_2)g &= g^{-1}i(n_1)i(n_2)g \\ &= (g)^{-1}i(n_1)g)(g^{-1}i(n_2)g) \\ &= i(n')i(n'') = i(n' + n'') \end{aligned}$$

- *Since* i *is injective* $n_3 = n' + n'' = q \cdot n_1 + q \cdot n_2$ *for every* $n', n'', n_1, n_2 \in N$ *and for every* $q_1, q_2 \in Q$.

Hence, we can see N *as a* Q*-module.*

Example 3.7. *Consider the following short exact sequence*

$$0 \longrightarrow \mathbb{Z}/3\mathbb{Z} \xrightarrow{f} S_3 \xrightarrow{g} \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

be an short exact sequence with the maps

$$\begin{gathered} f(\bar{0}) = (1) \quad f(\bar{1}) = (123) \quad f(\bar{2}) = (132) \\ g((1)) = \bar{1} \quad g((123)) = \bar{1} \quad g((132)) = \bar{1} \\ g((12)) = \bar{1} \quad g((23)) = \bar{1} \quad g((13)) = \bar{1} \end{gathered}$$

where $\bar{1}$ *and* $\bar{1}$ *are generators of* $\mathbb{Z}/3\mathbb{Z}$ *and* $\mathbb{Z}/2\mathbb{Z}$ *respectively. Now, we try to see* $\mathbb{Z}/3\mathbb{Z}$ *as a* $\mathbb{Z}/2\mathbb{Z}$*-module.*

$1 \cdot \bar{1} = \bar{1}$, $1 \cdot \bar{2} = \bar{2}$ *and* $1 \cdot \bar{0} = \bar{0}$ *are trivial since* $g((1)) = 1$
Since $(12)^{-1}f(\bar{0})(12) = (12)(1)(12) = (1) = f(0)$ *we have* $\bar{2} \cdot \bar{0} = \bar{0}$
Since $(12)^{-1}f(\bar{1})(12) = (12)(123)(12) = (132) = f(\bar{2})$ *we have* $\bar{2} \cdot \bar{1} = \bar{2}$
Since $(12)^{-1}f(\bar{2})(12) = (12)(132)(12) = (123) = f(\bar{1})$ *we have* $\bar{2} \cdot \bar{2} = \bar{1}$

Lemma 3.3. *Let* N *be a* Q*-module and assume that we have the following a splitting exact sequence:*

$$1_N \longrightarrow N \xrightarrow{i} G \xrightarrow{\pi} Q \longrightarrow 1_Q$$

with homomorphism p *satisfying* $\pi(p(q)) = q$ *for all* $q \in Q$ *.Then* $G \cong N \times Q$.

Proof. Before proving , we should note that $p(Q) = \{p(q) : q \in Q\}$ and $i(N) = \{i(n) : n \in N\}$. To proof this, we should first realize that $i(N) \cap p(Q) = \{1\}$. Assume that there is a non-trivial element $g \in i(N) \cap p(Q)$. Therefore, there exists $n \in N$ such that $i(n) = g$ and there exist $q \in Q$ such that $p(q) = g$. Moreover, since our sequence is exact, $\text{im(i)} = \ker(\pi)$ must be satisfied so $g \in \text{im(i)} = \ker(\pi)$. However, $\pi(p(q)) = \pi(g) = 1$. This can be true if and only if $q = 1_Q$ which contradicts to our assumption. Hence, we get $i(N) \cap p(Q) = \{1\}$. Now, we can write a bijection between $N \times Q$ and G.

$$\alpha : N \times Q \longrightarrow G$$
$$(n, q) := i(n)p(q)$$

Secondly, we will show that the above map is actually bijection. Note that identity element of the group $N \times Q$ is $(1_N, 1_Q)$. This map is injective because both i and p are monomorphism and $\alpha(n, q) = i(n)p(q) = 1_G$ it follows that $i(n) = 1_G$ and $p(q) = 1_G$ so this can be true if and only if $n = 1_N$ and $q = 1_Q$. This map is surjective because we can write $g = g(p(\pi(g)))^{-1}p(\pi(g))$ for every $g \in G$. It is obvious that $p(\pi(g)) \in p(Q)$. It is enough to show that $gp(\pi(g))^{-1} \in i(N)$. Since $\pi(gp(\pi(g))^{-1}) = \pi(g)\pi(p(\pi(g))^{-1} = \pi(g)\pi(g)^{-1} = 1$, we conclude that $gp(\pi(g))^{-1} \in \ker(\pi) = \text{im(i)}$ and so $gp(\pi(g))^{-1} \in i(N)$. Therefore, for any $g \in G$, there exist (n, q) such that $\alpha(n, q) = g$. Finally, we will prove that for every $g \in G$, there is a unique n and q such that $g = i(n)p(q)$. Suppose that, representation is not unique. Let's say $g = i(n_1)p(q_1) = i(n_2)p(q_2)$ with $n_1, n_2 \in N$ and $q_1, q_2 \in Q$. If we multiply g with $i(n_1)^{-1}$ from left and $p(q_2)^{-1}$ from right, we finally get $i(n_1)^{-1}gp(q_2)^{-1} = p(q_1)p(q_2)^{-1} = i(n_1)^{-1}i(n_2) \in i(N) \cap p(Q) = 1$. Therefore, we must have $n_1 = n_2$ and $q_1 = q_2$ and this gives us that representation is unique. □

Remark. *Lemma* 3.3 *may not be true if the given exact sequence is not split. For example, consider the exact sequence given in Example* 3.5. *We proved that the sequence is not split. However, we cannot find an isomorphism between* $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ *and* $\mathbb{Z}/4\mathbb{Z}$ *because* $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ *has no elements of order* 4 *while* $\mathbb{Z}/4\mathbb{Z}$ *has.*

Remark. *Let* N *be an abelian group written additively and* Q *be any group. As we mentioned and prove in Remark* 3.2 *that we can see* N *as a* Q*- module. Then we can*

modify what we have in that remark as below:

$$\begin{aligned} i(q \cdot n) &= g^{-1} i(n) g \\ g i(q \cdot n) &= i(n) g \\ g i(n) &= i(q \cdot n) g \end{aligned}$$

where $\pi(g) = q$ for some $g \in G$. We denote this by commutation rule.

From now on, for notational convenience we denote the identity of the Q-module N by 0 instead of 1_N.

Definition 3.5. *Consider a group extension of N by Q determined by the exact sequence*

$$0 \longrightarrow N \xrightarrow{i} G \xrightarrow{\pi} Q \longrightarrow 1 \qquad (3.1)$$

*where N is a Q-module and p is the transversal function from Q to G and $p(1) = 1$. A function $f : Q \times Q \mapsto N$ is called **factor system** if it satisfies the following:*

$$p(x)p(y) = i(f(x, y))p(xy)$$

The function f is measure of p being a group homomorphism. How close f is to be $f(x, y) = 0 \; for \; all \; x, y \in Q$, so is p close to being a group homomorphism.

Theorem 3.1. *Let*

$$0 \longrightarrow N \xrightarrow{i} G \xrightarrow{\pi} Q \longrightarrow 1$$

be a group extension with transversal function p. Then this extension gives rise to a factor system. Conversely, if given any f is a factor system, there is a group extension G of N by Q depending on transversal function p.

Before proving this, we should first define a group law on $N \times Q$. We define group law on $N \times Q$ by using the fact that we have a bijection from $N \times Q$ to G where $\alpha(n, q) = i(n)p(q)$. Before defining, we first calculate $[i(n)p(q)] \cdot [i(m)p(x)]$ this as below:

$$\begin{aligned} i(n)p(q)i(m)\pi(x) &= i(n)i(q \cdot m)p(q)p(x) \\ &= i(n + q \cdot m)i(f(q, x))p(qx) \\ &= i(n + q \cdot m + f(q, x))p(qx) \end{aligned}$$

where f is a factor system. Now, we posses everything to define group law on $N \times Q$. The group law on $N \times Q$ is given by

$$(n, q) \cdot (m, x) = (n + q \cdot m + f(q, x), qx)$$

Note that $(0, 1)$ is identity element of this group since

$$(0, 1) \cdot (m, x) = (0 + 1 \cdot m + f(1, x), x) = (m, x)$$

for every $(m, x) \in N \times Q$. We show later that $f(1, x) = 0$. Moreover, $(n, q) \cdot (m, x) = (0, 1)$ if we choose $q = x^{-1}$ and $n = (q \cdot m + f(q, x))^{-1}$ so we posses that every element has a multiplicative inverse. It is enough to show that above law is associative. Let $(n_1, q_1), (n_2, q_2)$ and (n_3, q_3) be elements of (N, Q). Then

$$\begin{aligned}
(n_1, q_1) \cdot ((n_2, q_2) \cdot (n_3, q_3)) &= (n_1, q_1) \cdot (n_2 + q_2 \cdot n_3 + f(q_2, q_3), q_2 q_3) \\
&= (n_1 + q_1 \cdot (n_2 + q_2 \cdot n_3 + f(q_2, q_3) + f(q_1, q_2 q_3), q_1 q_2 q_3) \\
&= (n_1 + q_1 \cdot n_2 + q_1 q_2 \cdot n_3 + q_1 \cdot f(q_2, q_3) + f(q_1, q_2 q_3), q_1 q_2 q_3) \\
((n_1, q_1) \cdot (n_2, q_2)) \cdot (n_3, q_3) &= (n_1 + q_1 \cdot n_2 + f(q_1, q_2), q_1 q_2) \\
&= (n_1 + q_1 \cdot n_2 + f(q_1, q_2) + q_1 q_2 \cdot n_3 + f(q_1 q_2, q_3), q_1 q_2 q_3)
\end{aligned}$$

Hence $(N \times Q, \cdot)$ is associative if and only if it satisfies

$$q_1 \cdot f(q_2, q_3) + f(q_1, q_2 q_3) = f(q_1, q_2) + f(q_1 q_2, q_3)$$

We actually know that the group $(N \times Q, \cdot)$ is associative because we get law from Lemma 3.3 where associativity of $N \times Q$ comes from associativity of N and Q. Now, we can prove Theorem 3.1.

Proof. Suppose that

$$1_N \longrightarrow N \xrightarrow{i} G \xrightarrow{\pi} Q \longrightarrow 1_Q$$

be any group extension of N by Q and p is a transversal function satisfying $p(1) = 1$. Since

$$\pi(p(x)p(y)p(xy))^{-1} = (\pi p)(x)(\pi p)(y) \cdot (\pi p)((xy)^{-1}) = (xy)(xy)^{-1} = 1$$

then we get $p(x)p(y)p(xy)^{-1} \in \ker(\pi) = \mathrm{im(i)}$. Therefore, there exist a unique element $n \in N$ such that $i(n) = p(x)p(y)p(xy)^{-1}$. If we define $f(x, y) = n$, then we have

$$p(x)p(y) = i(f(x, y))p(xy)$$

for all $x, y \in Q$.

Conversely, let's $N \times Q$ is a group defined as above and denote it by G_f. Since $i(n) = i(n)p(1)$ for any $n \in N$ which gives us an injective function from N to $N \times Q$ and $(n, q) \mapsto q$ which gives us a surjective function from $N \times Q$ to Q, we can write a new group extension where factor system is used

$$0 \longrightarrow N \xrightarrow{i} G_f \xrightarrow{\pi} Q \longrightarrow 1$$

and this extension is equivalent to the extension 3.1 so the group extension gives rise to a factor system under the G-module structure. □

Theorem 3.2. (*Brown*, 2012) *The function $f : Q \times Q \mapsto N$ is a factor system if and only if it satisfies the following*

$$f(x, 1) = 0 = f(1, y) \quad for\ all\ x, y \in Q \tag{3.2}$$

$$x \cdot f(y, z) - f(xy, z) + f(x, yz) - f(x, y) = 0\ for\ all\ x, y, z \in Q. \tag{3.3}$$

Proof. Let f be a factor system and G be a group that we defined on $N \times Q$ as above. Equation 3.2 holds since $p(x)p(1) = i(f(x, 1))p(x)$ and it follows that $f(x, 1) = 0$,and similarly $p(1)p(y) = i(f(1, y))p(y)$ which gives $f(1, y) = 0$ where p is transversal function. Equation 3.3 holds since G is associative.
Conversely, suppose that the equation 3.2 and 3.3 holds on a function f. We define a group G where G is a set $N \times Q$ and the group law is defined as

$$(n_1, q_1)(n_2, q_2) = (n_1 + q_1 \cdot n_2 + f(q_1, q_2), q_1 q_2)$$

Since $N \times 1 \cong N$ and ${}^{G}/_{N \times 1} \cong 0 \times Q \cong Q$, we get a group extension

$$0 \longrightarrow N \to G \to Q \longrightarrow 1$$

The rest is what we do in the proof of the Theorem 3.1 so we guarantee that the function f is the factor system. □

While determining the factor system, we use the homomorphism i and the transversal function p. We will try to answer that what would change if we choose another transversal function p' satisfying $p'(1) = 1$ because this p' can give another factor system f' which is different than f. Now, we try to investigate the connection between f and f'.

Remark. *Let p and p' be two transversal function and both satisfy $p(1) = p'(1) = 1$. We proved in Remark* 3.3 *that we can write every element g of G as a multiplication of $i(n)p(q)$ for some $n \in N$ and $q \in Q$. Therefore, for every $x \in Q$, there must be an element $i(n)$ such that $p'(x) = i(n)p(x)$ which means that $p(x)$ and $p'(x)$ are in the same coset of $i(N)$ for all $x \in Q$. Let $\mu : Q \mapsto N$ be a function such that $i(\mu(x)) = i(n)$ so if we rewrite the equation, then $p'(x) = i(\mu(x))p(x)$.*

Suppose that p and p' are two transversal functions satisfying $p(1) = 1$ and $p'(1) = 1$ and these functions corresponds to the factor system f and f' respectively. We show that there is an element $i(\mu(x)) \in N$ such that $p'(x) = i(\mu(x))p(x)$. Hence we can write the factor system f' as

$$\begin{aligned} i(f'(x,y)) &= p'(x)p'(y)p'(xy)^{-1} \\ &= i(\mu(x))p(x)i(\mu(y))p(y)p(xy)^{-1}i(\mu(xy)^{-1}) \\ &= i(\mu(x)) + p(x)i(\mu(y))p(x)^{-1} + p(x)p(y)p(xy)^{-1} - i(\mu(xy)) \\ &= i(f(x,y)) + i(\mu(x)) - i(\mu(xy)) + i(x \cdot \mu(y)) \end{aligned}$$

Hence we get a relation between f with f' by using injectivity of i

$$\begin{aligned} i(f'(x,y) - f(x,y)) &= i(\mu(x) - \mu(xy) + x \cdot \mu(y)) \\ f'(x,y) - f(x,y) &= \mu(x) - \mu(xy) + x \cdot \mu(y) \end{aligned}$$

Note that, f and f' are two factor systems that give rise to equivalent group extensions.

Definition 3.6. *Let*

$$0 \longrightarrow N \xrightarrow{i} G \xrightarrow{\pi} Q \longrightarrow 1$$

be a group extension and f, f' be two factor systems of this group extension. We say f and f' are equivalent if there is a function $\mu : Q \to N$ so that

$$f'(x,y) - f(x,y) = \mu(x) - \mu(xy) + x \cdot \mu(y).$$

Proposition 3.2. (*Brown*, 2012) *Two equivalent group extensions corresponds to equivalent factor systems.*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \xrightarrow{i} & G & \xrightarrow{\pi} & Q & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle id} & & \\ 1 & \longrightarrow & N & \xrightarrow{i'} & G' & \xrightarrow{\pi'} & Q & \longrightarrow & 1 \end{array}$$

Proof. Let f be a factor system determined by i and p and f' be a factor system determined by i' and p' where we define $p' : Q \to G'$ such that $p'(x) = \phi(p(x))$. Note that, p' is a transversal function since $\pi'(p'(x)) = \pi'(\phi(p(x)) = \pi(p(x)) = x$ by commutativity of diagram. Therefore, we can write

$$\begin{aligned} p'(xy)i'(f'(x,y)) &= p'(x)p'(y) = \phi(p(x))\phi(p(y)) = \phi(p(x)p(y)) \\ &= \phi(p(xy)i(f(x,y)) = \phi(p(xy))\phi(i(f(x,y)) \\ &= p'(xy)i'(f(x,y)) \end{aligned}$$

Since i' is injective, we get $f(x,y) = f'(x,y)$ for every $x, y \in Q$. □

Note that converse of the this proposition is also true.

Corollary 3.1. *There is a one-to-one correspondence between the set of equivalence classes of extensions of N by Q and the set of equivalence classes of factor systems determined by N and Q.*

3.3 Second Cohomology Group and Group Extension

We know that

$$Z^2(G,A) = \{f : G \times G \mapsto A \mid g_1 f(g_2,g_3) - f(g_1g_2,g_3) + f(g_1,g_2g_3) - f(g_1,g_2) = 0\ \}$$
$$B^2(G,A) = \{f : G \times G \mapsto A \mid f(g,h) = xh(y) - h(xy) + h(x)\}$$

Therefore, we can compute $H^2(G,A)$ as $H^2(G,A) = Z^2(G,A) \Big/ B^2(G,A)$.

Theorem 3.3. (*Brown*, 2012) *There is a one-to-one correspondence between the set of equivalence classes of group extensions of A by G and $H^2(G,A)$.*

Proof. We proved before that any factor system f satisfies $g_1 f(g_2,g_3) - f(g_1g_2,g_3) + f(g_1,g_2g_3) - f(g_1,g_2) = 0$ for every $g_1, g_2, g_3 \in G$ so it is clear that $f \in Z^2(G,A)$. Moreover, two factor system f, f' are equivalent if $f' - f = xh(y) - h(xy) + h(x)$ for some function h so $f' - f \in B^2(G,A)$. Hence, if we use the fact that in Theorem 3.1, then we posses that two extensions of A by G are equivalent if and only if there is a class of $H^2(G,A)$. □

3.4 Euler Class as a factor system

We define a function $f : \mathrm{T} \times \mathrm{T} \to \mathbb{Z}$ such that $f(g_1, g_2) = N(g_1 g_2) - N(g_2 g_1)$ where $N(g)$ is the number of breakpoints an element $g \in \mathrm{T}$. We will use the fact we obtained at the end of the Chapter 1, namely that there is a one-to-one correspondence between the Thompson's group T and the group generated by the flip moves to investigate that f is a factor system, that is, we calculate the number of breakpoints of the functions provided by flip moves in the function f instead of the elements of the group T in the function f. Before investigating whether f is actually a factor system, we first need to define module structure of the system. We consider the $\mathbb{Z}$ as a trivial T- module in our case. Now, we need to check that the function f satisfies the following

$$f(g_1, g_2) + f(g_1 g_2, g_3) = f(g_1, g_2 g_3) + f(g_2, g_3)$$

If we rewrite the equation in terms of the number of breakpoints and simplify, we posses the following:

$$N(g_1 g_2) - N(g_2 g_1) - N(g_3 g_1 g_2) = -N(g_2 g_3 g_1) + N(g_2 g_3) - N(g_3 g_2) \qquad (3.4)$$

$$N(g_1 g_2) - N(g_2 g_1) - N(g_3 g_1 g_2) + N(g_2 g_3 g_1) - N(g_2 g_3) + N(g_3 g_2) = 0 \qquad (3.5)$$

where g_i's are of the form $f_{WL,WLS}$ for some $W \in \mathrm{PSL}_2(\mathbf{Z})$. We need some facts about the group P to go further.

Remark. *Consider two elements of the group P, the the followings are true:*

- *Their non-trivial support are totally different, that is, they commute*
- *Their non-trivial support intersect at only one point*
- *One's non-trivial support contain the other's.*

These properties are made inferences with ease just by checking the Figure 1.2.

In light of this information, we will check into whether the equation 3.5 are satisfied by separating into cases:
Case 1: All g_1,g_2 and g_3 commute each other.
Case 2: While one of the three elements non-trivial support contains the non-trivial support of other, the third one commutes with both of them.
Case 3: One's non-trivial support contains the other two's.

In case 1, all details are trivial since $N(gh) = N(g)+N(h) = N(hg)$ whenever $gh = hg$, the equation 3.5 are satisfied. In case 2, there are six situations to prove this but we will examine only three of them.

1. g_1's non-trivial support contains g_2's and g_3 commutes with these two. In this situation, if we rewrite the equation

$$\begin{aligned}
&N(g_1g_2) - N(g_2g_1) - N(g_3g_1g_2) + N(g_2g_3g_1) - N(g_2g_3) + N(g_3g_2) \\
&= N(g_1g_2) - N(g_2g_1) - N(g_3g_1g_2) + N(g_2g_3g_1) \\
&= N(g_1g_2) - N(g_2g_1) - N(g_3) - N(g_1g_2) + N(g_2g_1) + N(g_3) \\
&= 0
\end{aligned}$$

2. g_2's non-trivial support contains g_3's and g_1 commutes with these two. In this situation, if we rewrite the equation

$$\begin{aligned}
&N(g_1g_2) - N(g_2g_1) - N(g_3g_1g_2) + N(g_2g_3g_1) - N(g_2g_3) + N(g_3g_2) \\
&= -N(g_3g_1g_2) + N(g_2g_3g_1) - N(g_2g_3) + N(g_3g_2) \\
&= -N(g_1) - N(g_3g_2) + N(g_2g_3) + N(g_1) - N(g_2g_3) + N(g_3g_2) \\
&= 0
\end{aligned}$$

3. g_3's non-trivial support contains g_1's and g_2 commutes with these two. In this situation, if we rewrite the equation

$$\begin{aligned}
&N(g_1g_2) - N(g_2g_1) - N(g_3g_1g_2) + N(g_2g_3g_1) - N(g_2g_3) + N(g_3g_2) \\
&= -N(g_3g_1g_2) + N(g_2g_3g_1) \\
&= -N(g_3g_1) - N(g_2) + N(g_2) + N(g_3g_1) \\
&= 0
\end{aligned}$$

The rest of the other three situations are just another version of the above. One can check into that they all are verified.

Now we suggest the following conjecture.

Conjecture: There is still some computations to be carried out in order to deduce that the above f is indeed a factor system. Furthermore, we conjecture that the element of $H^2(\mathrm{T}, \mathbb{Z})$ determined by f is the Euler class.

Future Work: The **Case 3** should be proved to show that the function f is a factor system. Then, we will show that the class determined by the function f is indeed the Euler Class.

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