TRIANGULATIONS OF THE SPHERE AFTER THURSTON

by

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ABSTRACT

TRIANGULATIONS OF THE SPHERE AFTER THURSTON

Thurston proved that space of non-negatively curved triangulations of the sphere can be parametrized by a quotient of positive part of a Lorentzian lattice of rank 10 by a group of isomorphisms of the lattice. Also square-norm of a lattice point gives number of triangles in corresponding triangulation.

We extended his result to 2 new families of triangulations. More precisely we prove that there are 2 Eisenstein lattices with a Hermitian product on them such that positive part of each lattice divided by some isomorphism group of the lattice parametrizes space of triangulations with 4 singular points incident to 3 triangles and space of triangulation with 6 singular points incident to 4 triangles, respectively. Also we prove that square-norm of a lattice point gives number of triangles in corresponding triangulation.

In addition we prove a similar result for space of quadrangulations having 4 singular points incident to 2 quadrangles.

Our approach gives another proof of Thurston's theorem and Ayberk Zeytin's theorem which is an analogue of Thurston's result for non-negatively curved quadrangulations.

ÖZET

THURSTON'IN ARDINDAN KÜRE ÜÇGENLEMELERİ

Thurston kürenin negatif olmayan eğrilikli üçgenlemelerinin 10 boyutlu bir Lorentz kafesin pozitif bölgesinin kafesin bir izomorfizma grubuna bölümüyle parametrize edilebileceğini gösterdi. Ayrıca kafesteki pozitif bir noktanın kare normunun ona karşılık gelen üçgenlemedeki üçgen sayısını vereceğini gösterdi.

Biz bu kanıtı iki yeni üçgenleme ailesine genelledik. Üzerinde Hermite çarpımı olan öyle bir Eisenstein kafes vardır ki, kafesin pozitif bölümünün kafesin bir izomorfizma grubuna bölüüü 4 tekil noktalı, her tekil noktanın 3 üçgene komşu olduğu üçgenleme ailesini parametrize eder. Benzer şekilde öyle bir Eisenstein kafes vardr ki, bu kafesin pozitif bölm 6 tekil noktal ve her tekil noktanın 4 üçgene komşu olduğu üçgenleme ailesini parametrize eder. Ayrıca bir kafes noktasının kare normu ona karşılık gelen üçgenlemedeki üçgen sayısını verir.

Bununla birlikte 4 tekil noktalı ve her bir noktanın 2 kareye komşu olduğu karelemelerin uzayı için benzer bir sonuç kanıtladık.

Kullandığımız metod Thurston'ın teoreminin ve Ayberk Zeytin'in negatif olmayan eğrilikli karelemeler için Thurston'ın sonucunun benzeri olan teoreminin yeni bir kanıtını vermektedir.

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Last but not least, I would like to express my gratitude to my family, especially to my mother. They were always patient and helpful during the years that I spent for PhD.

TABLE OF CONTENTS

LIST OF FIGURES

1. INTRODUCTION

To explain our setting let

$$0 < k_1, k_2, \ldots, k_n < 6, \; n > 3, \tag{1.1}$$

be natural numbers so that

$$\sum_{i=1}^{n} k_i = 12 \tag{1.2}$$

There is a finite list of tuples $\vec{k} = (k_1, k_2, ..., k_n)$ satisfying these conditions. Let $\overline{T}(\vec{k}, l)$ be set of labeled triangulations of the sphere having *essentially* n vertices incident to $6 - k_1, ..., 6 - k_n$ triangles and others incident to 6 triangles. Meaning of *essentially* will be explained later. When $\vec{k} = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ Thurston proved that [4]

$$\overline{T}(\vec{k}, l) = E^+/\Gamma, \tag{1.3}$$

where E is lattice in a complex Lorentzian space and Γ is a group of automorphisms and E^+ is positive part of lattice with respect to a Hermitian form. We give a different proof of this result and generalize it. More precisely, we prove that if

$$\vec{k} = (k_0, k_0, \ldots, k_0) \tag{1.4}$$

satisfies conditions of (1.1), (1.2) then

$$\overline{T}(\vec{k}, l) = E(\vec{k})^{+}/\Gamma(\vec{k}), \tag{1.5}$$

where $E(\vec{k})$ is lattice in a complex Lorentzian space and $\Gamma(\vec{k})$ is a group of automorphisms. Here is a list of tuples $\vec{k} = (k_0, k_0, .., k_0)$ satisfying (1.1), (1.2):

$$(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1) \tag{1.6}$$

$$(2, 2, 2, 2, 2, 2) \tag{1.7}$$

$$(3, 3, 3, 3). \tag{1.8}$$

Let

$$0 < q_1, q_2, \ldots, q_n < 4, \ n > 3, \tag{1.9}$$

be natural numbers so that

$$\sum_{i=1}^{n} q_i = 8 \tag{1.10}$$

Similarly set $\vec{q} = (q_1, q_2, \ldots, q_n)$. Let $\overline{Q}(\vec{q}, l)$ be the set of labeled quadrangulations of the sphere having 'essentially' n vertices incident to $4 - q_1, ..., 4 - q_n$ quadrangles and others incident to 4 quadrangles. When $\vec{q} = (1, 1, 1, 1, 1, 1, 1, 1)$ it was proven [3] that

$$\overline{Q}(\vec{q}, l) = G^+/\Gamma', \tag{1.11}$$

where G is lattice in a complex Lorentzian space and Γ' is a group of automorphisms. We give a different proof of this result and generalize it. More precisely we prove that if

$$\vec{q} = (q_0, q_0, ..., q_0) \tag{1.12}$$

satisfies conditions of (1.9), (1.10) then

$$\overline{Q}(\vec{q}, l) = G(\vec{q})^+/\Gamma(\vec{q}), \tag{1.13}$$

where $G(\vec{q})$ is lattice in a complex Lorentzian space and $\Gamma(\vec{q})$ is a group of automorphisms. Here is the list of these kind of tuples:

$$(1, 1, 1, 1, 1, 1, 1, 1, 1) \tag{1.14}$$

$$(2, 2, 2, 2). \tag{1.15}$$

List of arrays $\vec{k}$ (and $\vec{q}$) satisfying (1.1), (1.2) and (1.9), (1.10) can be found in [2].

2. BASIC FACTS ABOUT CONE METRICS, TRIANGULATIONS AND QUADRANGULATIONS

2.1. Cone Metrics

We present some well-known facts about cone metrics. See [5], [8] for more information.
We will represent the unit sphere in 3 dimensional Euclidean space by S^2. Sometimes we will identify it with complex projective line by steographic projection.
For any $\theta > 0$, let

$$V_\theta = \{(r,t) : r \geq 0, t \in \mathbf{R}/\theta\mathbf{Z}\}/_{(0,t)\sim(0,t')}. \tag{2.1}$$

A *cone* with cone angle θ is the set V_θ under the following metric:

$$ds^2 = dr^2 + rdt^2. \tag{2.2}$$

Definition 2.1.1. *A singular Riemannian metric on S^2 is called (Euclidian) cone metric if each point P has a neighborhood U_P which is (diffeomorphically) isometric to a neighborhood of some cone V_{θ_P}, via a map sending P to $(0,0)$.*

θ_P will be called the *cone angle* at P. $\kappa_P = 2\pi - \theta_P$ will be called the *curvature* at P. When $\kappa_P = 0$, we will call P *regular*, else it will be called *singular*.

Theorem 2.1.2 (Gauss-Bonnet). *Let μ be a cone metric on S^2 with singular points*

$x_1, ..., x_n$ of respective curvatures $\kappa_1, \kappa_2, ..., \kappa_n$, then

$$\sum_{i=1}^{n} \kappa_i = 4\pi. \tag{2.3}$$

This theorem enables us to define spaces of cone metrics. Given $\kappa = (\kappa_1, .., \kappa_n)$ such that

$$\sum_{i=1}^{n} \kappa_i = 4\pi, \kappa_i < 2\pi, \kappa_i \neq 0, \tag{2.4}$$

Definition 2.1.3. *$\kappa = (\kappa_1, ..., \kappa_n)$ is called a curvature data if it satisfies (2.4).*

Let $C(\kappa)$ be the set of cone metrics of unit area with n singular points of curvature $\kappa_1, ..., \kappa_n$, up to orientation preserving diffeomorphisms. Similarly let $C(\kappa, l)$ be the set of the labeled cone metrics of unit area, having n singular points of curvature, respectively, $\kappa_1, ..., \kappa_n$, up to orientation preserving diffeomorphisms.
The following theorem gives a nice way to consider $C(\kappa, l)$. For a proof, see [8].

Theorem 2.1.4. *Assume that $\kappa = (\kappa_1, .., \kappa_n)$ satisfies (2.4). Given any smooth Riemannian metric μ on S^2 and a sequence of distinct points $(x_1, ..., x_n)$ of it, there exists a unique conformal cone metric of area 1 on S^2 with singular points x_i's such that curvature at x_i is κ_i. Conversely any cone metric has a n smooth Riemannian metric in its conformal class.*

Now fix the spherical metric on S^2. Let Δ be the big diagonal in $\mathbb{P}^1_{\mathbf{C}} \times \mathbb{P}^1_{\mathbf{C}} \times \cdots \times \mathbb{P}^1_{\mathbf{C}}$. By the theorem above, we get a map from $\mathbb{P}^1_{\mathbf{C}} \times \mathbb{P}^1_{\mathbf{C}} \times \cdots \times \mathbb{P}^1_{\mathbf{C}} - \Delta$ to

$C(\kappa, l)$, sending each $(x_1, ..., x_n)$ to the cone metric mentioned in the theorem. It is not difficult to see this map is surjective and induces a natural bijection:

$$X(n) = \mathbb{P}^1_{\mathbf{C}} \times \mathbb{P}^1_{\mathbf{C}} \times \cdots \times \mathbb{P}^1_{\mathbf{C}} - \Delta / PGL_2(\mathbf{C}) \equiv C(\kappa, l), \tag{2.5}$$

that is, space of labeled cone metrics is the moduli space of n points on the sphere.

Sometimes we will represent a cone metric $\mu \in C(\kappa, l)$ by $(\mu, (x_1, ..., x_n))$ or by just a point in configuration space, $(x_1, ..., x_n)$, by the help of the previous theorem.

2.2. Triangulations and Quadrangulations

Given a curvature data κ, let $C_0(\kappa)$ be the set of cone metrics on S^2 having curvature data κ. Similarly we define $C_0(\kappa, l)$ to be the set of labeled cone metrics on S^2 having curvature data κ. If the curvature data is of the form

$$\kappa = (k_1 \frac{\pi}{3}, k_2 \frac{\pi}{3}, ..., k_n \frac{\pi}{3}), k_i \in \mathbf{Z} \tag{2.6}$$

$$\vec{k} = (k_1, k_2, ..., k_n), \tag{2.7}$$

then we introduce 2 new sets :

$$T_0(\vec{k}) = \{\mu \in C_0(\kappa) : \mu \textit{ is obtained by gluing equilateral triangles of edge length } 1\} \tag{2.8}$$

$$T_0(\vec{k}, l) = \{\mu \in C_0(\kappa, l) : \mu \textit{ is obtained by gluing equilateral triangles of edge length } 1\}. \tag{2.9}$$

Definition 2.2.1. *We say that two cone metrics μ and μ' in $T_0(\vec{k})$ (respectively in $T_0(\vec{k}, l)$) are equivalent, $\mu \sim \mu'$, if they are isometric as cone metrics via an orientation preserving isometry sending edges, vertices triangles of μ to edges vertices and triangles of μ', respectively. We define the unlabeled and labeled spaces of triangulations as*

$$T(\vec{k}) = T_0(\vec{k})/\sim \tag{2.10}$$

$$T(\vec{k}, l) = T_0(\vec{k}, l)/\sim \tag{2.11}$$

Now we define spaces of quadrangulations. Given the curvature data

$$\kappa = (q_1\frac{\pi}{2}, q_2\frac{\pi}{2}, ..., q_n\frac{\pi}{2}), q_i \in \mathbf{Z} \tag{2.12}$$

$$\vec{q} = (q_1, q_2, ..., q_n), \tag{2.13}$$

we introduce 2 new sets :

$$Q_0(\vec{q}) = \{\mu \in C_0(\kappa) : \mu \textit{ is obtained by gluing squares of edge length } 1\} \tag{2.14}$$

$$Q_0(\vec{q}) = \{\mu \in C_0(\kappa, l) : \mu \textit{ is obtained by gluing squares of edge length } 1\}. \tag{2.15}$$

Definition 2.2.2. *We say that two cone metrics μ and μ' in $Q_0(\vec{q})$ (respectively in $Q_0(\vec{q}, l)$) are equivalent, $\mu \sim \mu'$, if they are isometric as cone metrics via an orientation preserving isometry sending edges, vertices squares of μ to edges vertices and squares of μ', respectively. We define the unlabeled and labeled spaces of quadrangulations as*

$$Q(\vec{q}) = Q_0(\vec{q})/\sim \tag{2.16}$$

$$Q(\vec{q}, l) = Q_0(\vec{q}, l)/\sim \tag{2.17}$$

2.3. Alexandrov Unfolding Process

Let μ be a cone metric on S^2 with n $(n > 2)$ singular points of positive curvature. Call these singular points $v_1, v_2, \ldots, v_n$. Let s_i $(2 \leq i \leq n)$ be a length minimizing geodesic joining v_1 to v_i. It follows that s_i and s_j intersect at only v_1 when $i \neq j$. If we cut S^2 along s_i's, then we can unfold it to the plane without overlapping as a polygon with $2n-2$ vertices.

The process described above is called *Alexandrov unfolding*.

Resulting polygon P has $n-1$ vertices coming from v_1 and $n-1$ vertices corresponding to v_i's $(i > 1)$. We can label vertices of the polygon as $\alpha_1, \alpha_2, \ldots, \alpha_{2n-2}$ so that the labeling gives counter clockwise orientation on boundary of the polygon and the vertices with odd index correspond to v_1 and also each of the other vertices corresponds to a unique v_i $(i > 1)$. This polygon has special property that for each $i > 0$ the lengths of the segments $[\alpha_{2i-1}, \alpha_{2i}]$ and $[\alpha_{2i+1}, \alpha_{2i}]$ are equal. Also the angle at α_i is same the angle at the corresponding vertex of S^2. If we glue the edges

$[\alpha_{2i-1}, \alpha_{2i}]$ and $[\alpha_{2i+1}, \alpha_{2i}]$ we get a cone metric on sphere with n singular points. Indeed this metric, after some normalization, is nothing else than μ.

The theorem below summarizes that what we have said about Alexandrov unfolding process.

Theorem 2.3.1 (Alexandrov Unfolding)**.** *Let μ be cone metric with positive curvature data κ. Let $x_1, .., x_n$ be its singular points. Let s_i, $2 \leq i \leq n$, be a length minimizing geodesic joining x_1 to x_i. Then $S^2 - \cup s_i$ can be developed into the plane as polygon without overlapping. Starting metric μ can be obtained from this polygon after glueing its edges in an appropriate way.*

3. TRIANGULATIONS AND CONE METRICS WITH 3 SINGULAR POINTS

3.1. Sphere with 3 Singular Points

We start with triangulations having exactly 3 singular vertices. There are only 3 such families:

$$T(4,4,4), T(5,4,3), T(5,5,2). \tag{3.1}$$

Let **Eis** be the set of Eisenstein integers. **Eis** gives a triangulation of plane, we will obtain triangulations of sphere by this triangulation. Throughout this thesis **Eis** will represent both Eisenstein integers and corresponding triangulation of plane.

3.1.1. $T(4,4,4)$

In this case there are 3 vertices incident to 2 triangles. Given a point p in $\mathbf{Eis} - \{\mathbf{0}\}$. Let L be the line joining p to origin. Draw 2 equilateral triangles having L as an edge. Vertices of the quadrilateral Q, which is union of these two triangles, are in **Eis**. If we glue l_1 with l_1' and l_2 with l_2', we get a triangulation in $T(4,4,4)$. See Figure 3.1.

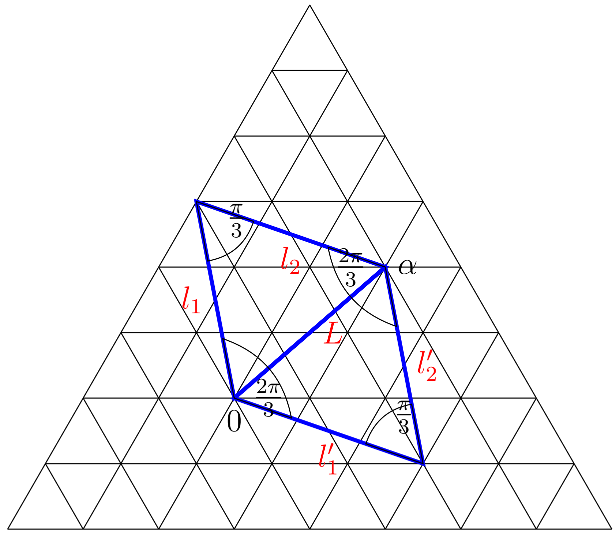

Figure 3.1: Obtaining an element of $T(4,4,4)$ from a lattice point.

Any triangulation in $T((4,4,4))$ arise from this procedure. There are to ways to see this.

Given a triangulation, we associate a metric declaring that each triangle is equilateral of edge length 1. Sphere with this metric admits universal branched cover U which is a triangulated surface, together with a map $P : U \to S^2$, sending triangles to triangles isometricly. If we take a preimage $\overline{v_1}$ of a singular vertex v_1 under P and send one triangle incident to it to $\Delta(0,1,\delta)$,where $\Delta(0,1,\delta)$ is a triangle with vertices 0,1 and δ , $\delta = e^{\frac{2\pi i}{6}}$, while sending singular vertex to origin and keeping orientation, then we can uniquely send universal branched cover to plane by developing map, *dev*. Also developing map sends vertices and triangles in universal cover respectively to vertices and triangles in **Eis**.

Let T be a triangulation in $T(4,4,4)$ and v_1,v_2,v_3 be singular vertices of this triangulation. These points are only singular points in the corresponding (cone) metric and they have cone angle $\frac{2\pi}{3}$. Cut the sphere through a geodesic g_1 joining v_1 to v_2 and a geodesic, g_2, joining v_2 to v_3. It can be shown that g_1 and g_2 intersects at only v_2, therefore $S^2 - \{g_1, g_2\}$ is simply connected Euclidean surface. Therefore $P^{-1}(S^2 - \{g_1, g_2\})$ is a cover for $S^2 - \{g_1, g_2\}$, that consists of isometric copies of $S^2 - \{g_1, g_2\}$. Take one of these copies that contains $\overline{v_1}$ in its closure, call it C. $dev(C)$ is isometric to C. Hence $dev(C)$ is a quadrilateral with vertices in **Eis** and one of them is 0. Also the angle at 0 and at $dev(v_3)$ is $\frac{2\pi}{3}$. Also angle at other 2 vertices is $\frac{\pi}{3}$. Therefore this quadrilateral is 'same' with of Figure 3.1 Gluing l_1, l_1' and l_2, l_2' we get a sphere triangulation same with the original one.

Other method is due to Thurston [4]. Since valencies at each vertex divides 6, quotient of plane by a suitable triangle group is desired triangulation.

Note that we found a correspondence between $T(4,4,4)$ and $\mathbf{Eis} - \{\mathbf{0}\}$. Norm of an element α in $\mathbf{Eis} - \{\mathbf{0}\}$ gives number of triangles in corresponding triangulation. More precisely $2\alpha\overline{\alpha}$ is number of triangles.

This is not a one-to-one correspondence.If you take an element α in $\mathbf{Eis} - \{\mathbf{0}\}$ then α and $\omega\alpha$ give same triangulation, where ω is any sixth root of unity. This is the only case, that is $T(4,4,4)$ is parametrized by $\mathbf{Eis} - \{\mathbf{0}\}/ < \delta >$. Let's prove this.

Assume that a triangulation T is given. We get an element α in $\mathbf{Eis} - \{\mathbf{0}\}$ by above process. We will show that we can get only $\omega\alpha$ by using above process.

While defining developing map we took a triangle incident $\overline{v_1} \in P^{-1}(v_1)$ and sent it $\Delta(0,1,\delta)$. If we take another triangle incident to $\overline{v_1}$ and adjacent to first one and send it to $\Delta(0,1,\delta)$ we get another developing map, $\overline{dev}$, such that

$$\overline{dev} = \delta^{\pm 1} dev. \tag{3.2}$$

Therefore changing developing map in above manner changes α with $\omega\alpha$.

Also if we choose C' adjacent to C, instead of C, the quadrilateral that we get is obtained from the original one by a rotation of angle $\frac{2\pi}{3}$. So we conclude that if we start a region C'' incident to $\overline{v_1}$ we get $\delta^{2k}\alpha$ instead of α.

Next we analyze how does α changes when we start with another $v_1' \in P^{-1}(v_1)$. Let R be the base triangle incident to $\overline{v_1}$, that is the triangle that we sent to $\Delta(0,1,\delta)$. There exists a loop β in $S^2 - \{v_1, v_2, v_3\}$ such that $\tau_{[\beta]}(\overline{v_1}) = v_1'$ and by sending triangle $\tau_{[\beta]}(R)$ to $\Delta(0,1,\delta)$ we get another developing map dev'. We have

$$dev \circ \tau_{\beta}^{-1} = dev'. \tag{3.3}$$

Now changing C with $C' = \tau_{[\beta]}(C)$ and sending C' to plane by dev' we get same quadrangle, therefore α does not change in this case.

Assume that we cut sphere as before through g_1 and g_2 but we change developing map; we start with v_3 and send one of it's inverse images to origin. In that case the quadrilateral Q that we obtained changes with $-Q$, therefore α transfers to

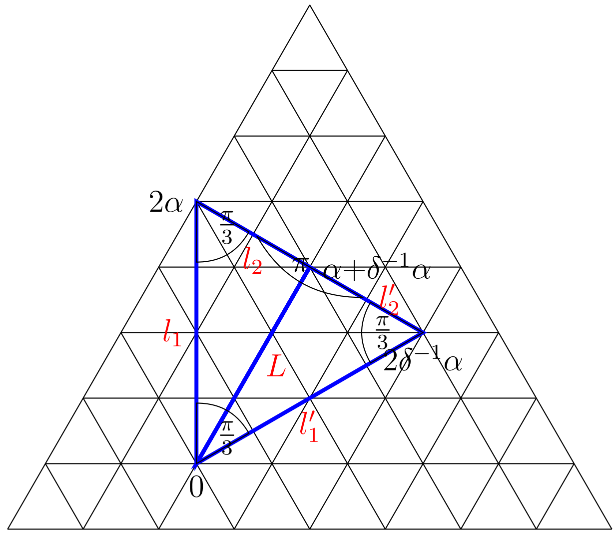

Figure 3.2: Obtaining an element of $T(5,4,3)$ from a lattice point.

$-\alpha = \delta^3\alpha$. Also instead of cutting the sphere through g_1 and g_2 we can cut through a geodesic joining v_1 to v_3 and a geodesic joining v_3 to v_2. In that case the quadrilateral Q rotates by an angle of $\frac{\pi}{3}$, so α changes to $\delta\alpha$. These imply that cutting the sphere through $v_i's$ with different orders just change α by multiplication of a sixth root of unity.

Hence we showed that given a triangulation of desired form, any two numbers obtained from this triangulation by our process differ by a sixth root of unity.

To summarize, we proved that there is a one to one correspondence between $T(4,4,4,4)$ and $\mathbf{Eis}-\{\mathbf{0}\}/<\delta>$. Also norm of an element in later set gives number of triangles in corresponding triangulation.

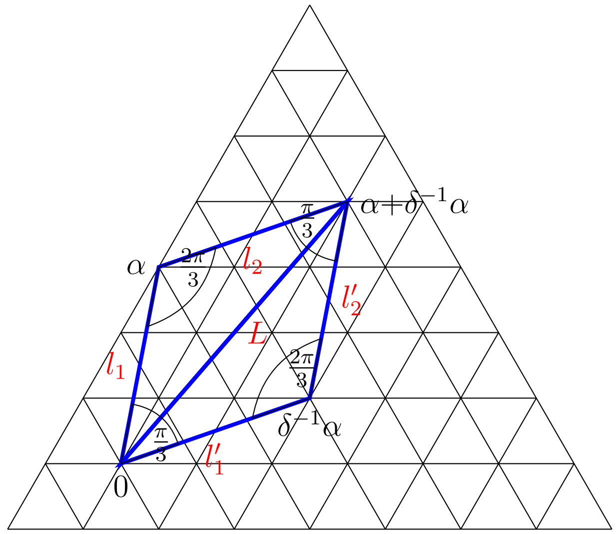

Figure 3.3: Constructing an element of $T(5,5,2)$ from a lattice point.

3.1.2. $T(5,4,3)$

Given $\alpha \in \mathbf{Eis} - \{\mathbf{0}\}$, consider equilateral triangle with vertices 0, 2α, $\delta^{-1}2\alpha$ (Figure 3.2). Call these vertices v_1, v_2, v_3 respectively. Let v be midpoint of the line segment between v_2 and v_3 . Gluing l_1 with l_1' and l_2 with l_2' (see Figure 3.2) we get a triangulation of desired form. As before, all elements in $T((5,4,3))$ can be obtained by this process and $\mathbf{Eis} - \{\mathbf{0}\}/ < \delta >$ parametrizes $T((5,4,3))$. Also observe that $4\alpha\overline{\alpha}$ gives number of triangles in corresponding triangulation.

3.1.3. $T(5,5,2)$

Given $\alpha \in \mathbf{Eis} - \{\mathbf{0}\}$, construct a quadrangle with vertices α, $\delta^{-1}\alpha$, $\alpha + \delta^{-1}\alpha$ (Figure 3.3). Observe that the angle between l_1 and l_2, l_1' andl_2' is $\frac{2\pi}{3}$. Gluing l_1 with l_1', l_2 with l_2', we get a triangulation of sphere of desired form. As before $\mathbf{Eis} - \{\mathbf{0}\}/ < \delta >$ parametrizes $T((5,5,2))$. In this case, $2\alpha\overline{\alpha}$ gives number of

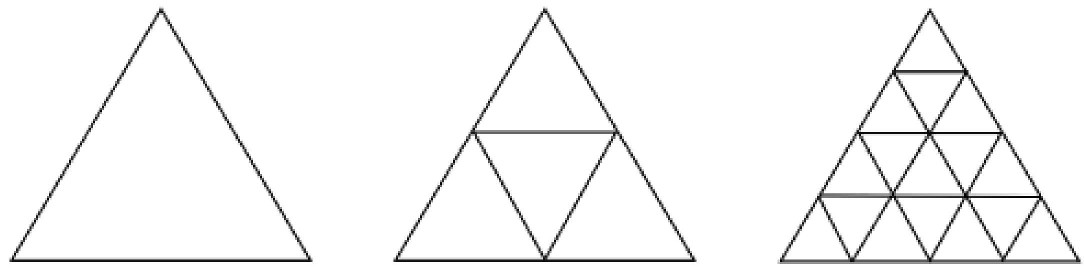

Figure 3.4: Triangle on the left transforms to triangle in the middle when α changes to 2α, Triangle on the left transforms to triangle on the right when when α changes to 4α

triangles in the triangulation.

3.1.4. Summary

Theorem 3.1.1. *Each family of triangulations of sphere with 3 singularities is parametrized by* $\mathbf{Eis} - \{\mathbf{0}\}/ < \delta >$ *. Also norm of an element in* $Eis - \{0\}/ < \delta >$ *gives number of triangles in the corresponding triangulation.*

Also in these cases, if we change α with $k\alpha$, then we change a triangle to a triangle which is subdivided to k^2 triangles accordingly (Figure 3.4). Hence we have the following theorem.

Theorem 3.1.2. *If T is triangulation corresponding to α, then triangulation corresponding to $k\alpha$,where k is a natural number, is obtained by subdividing each triangle in T to k^2 triangles accordingly.*

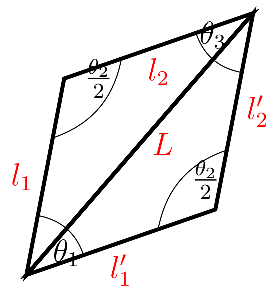

Figure 3.5: Constructing a cone metric with 3 singular points

3.2. Cone Metrics with 3 Singular Points of Positive Curvature

Let S^2 be the sphere with 3 singular points having positive curvature. Let $2\pi > \theta_2 \geq \theta_1 \geq \theta_3 \geq 0$ be 3 cone angles at singular points v_2, v_1, v_3, respectively. Following Gauss-Bonnet equation holds.

$$\sum(2\pi - \theta_i) = 4\pi \tag{3.4}$$

$$\theta_1 + \theta_2 + \theta_3 = 2\pi \tag{3.5}$$

Cut the sphere through a geodesic g_1 joining v_1 to v_2 and g_2 joining v_2 to v_3. Let U be universal branched cover for $S^2 - \{g_1, g_2\}$. Developing map sends U to a quadrangle (Figure 3.5). By identifying $l_1$with l_1' and l_2 with l_2' we can get cone metric back. We proved the following theorem:

Theorem 3.2.1. *Any cone metric on sphere with 3 singular points can be obtained by gluing two isometric copies of some triangle along their boundary.*

4. TRIANGULATIONS AND QUADRANGULATIONS

4.1. An Example: $\kappa = (\pi, \pi, \pi, \pi)$

Let $\kappa = (\pi, \pi, \pi, \pi)$, $\vec{k} = (3, 3, 3, 3)$, $\vec{q} = (2, 2, 2, 2)$. In this section we will explicitly construct $C(\kappa)$, $C(\vec{k}, l)$, $T(\vec{k})$, $T(\vec{k}, l)$, $Q(\vec{q})$, $Q(\vec{q}, l)$. Let

$$H(\kappa) = \mathbf{C}^2 = \{(z_1, z_2) : z_1, z_2 \in \mathbf{C}\}. \tag{4.1}$$

We will call this vector space the *space of parameters corresponding to* κ. Define a Hermitian product on this space :

$$\langle (z_1, z_2), (w_1, w_2) \rangle = \frac{\imath}{4}(z_1 \overline{w_2} - z_2 \overline{w_1}). \tag{4.2}$$

One can regard an element $z = (z_1, z_2)$ as a triangle in $\mathbf{C}$ with vertices $0, z_1, z_2$. Then $\langle z, z \rangle$ is just the signed area of this triangle. This is why this Hermitian product is called the area-Hermitian product. We are mainly interested in the set

$$H^+(\kappa) = \{z \in H(\kappa) : \langle z, z \rangle \ > 0\}, \tag{4.3}$$

which consists of positively oriented triangles in $H(\kappa)$. There is a nice way to obtain cone metrics from these triangles. We first name mid points of edges of a triangle

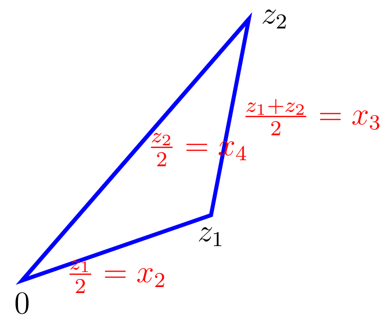

Figure 4.1: Tetrahedrons from triangles in the plane

$z = (z_1, z_2)$ as follows

$$\frac{z_1}{2} = x_2, \frac{z_1 + z_2}{2} = x_3, \frac{z_2}{2} = x_4.$$

Glue the segments $[0, x_2]$ with $[z_1, x_2]$, $[z_1, x_3]$ with $[z_2, x_3]$, $[z_2, x_4]$ with $[0, x_4]$. By this way we get a cone metric with four singular points of curvature π, π, π, π. See Figure 4.1. Thus there is a map

$$H^+(\kappa) \to C(\kappa) \tag{4.4}$$

which induces a map

$$\phi : \mathbb{P}H^+(\kappa) \to C(\kappa), \tag{4.5}$$

where $\mathbb{P}H^+(\kappa)$ is the complex projectifijation of $H^+(\kappa)$. Since signature of area Hermitian form on $H(\kappa)$ is $(1, 1)$, $\mathbb{P}H^+(\kappa)$ is nothing else than $\mathbb{H}^2_{\mathbf{R}} = \mathbb{H}^1_{\mathbf{C}}$.

This map is surjective. Pick a cone metric $\mu \in C(\kappa)$. Choose a singular point, call it x_1. There exists length minimizing geodesics from x_1 to other singular points x_2, x_3, x_4. Call these geodesics g_2, g_3, g_4 and assume that when we look from x_1, g_2, g_3, g_4 are in counter clockwise orientation. Apply Alexandrov unfolding process [1] to this cone metric: Cut S^2 through g_2, g_3, g_4. Resulting object can be embedded to $\mathbf{C}$ without overlapping. Therefore we get an element z in $\mathbb{P}H^+(\kappa)$ and $\phi(z) = \mu$.

This map is not injective. The elements below give rise to the same cone metric:

$$\begin{pmatrix} z_1, & z_2 \end{pmatrix}, \begin{pmatrix} z_1, & z_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} z_1, & z_1 + z_2 \end{pmatrix} \tag{4.6}$$

This is because the latter triangle can be obtained from the former by a cutting and gluing operation. Cut the triangle $(0, z_1, z_2)$ from $[0, x_3]$, glue $[x_3, z_2]$ with $[x_3, z_1]$ to get the triangle $(0, z_1, z_1 + z_2)$. Similarly we see that

$$\begin{pmatrix} z_1 & z_2 \end{pmatrix}, \begin{pmatrix} z_1 & z_2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{4.7}$$

give the same cone metric, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$. Thus ϕ induces a map

$$\mathbb{P}H^+/\mathbb{P}SL_2(\mathbf{Z}) \to C(\kappa) \tag{4.8}$$

One can show that this map is also injective.

One can define a map $\overline{\phi} : \mathbb{P}H^+ \to C(\kappa, l)$ so that the following diagram is commutative:

$$\begin{tikzcd} \mathbb{P}H^+ \arrow[r, "\overline{\phi}"] \arrow[d, "\phi"'] & C(\kappa, l) \arrow[dl, "\pi"'] \\ C(\kappa) & \end{tikzcd}$$

where π is the label-forgetting map; sending $(\mu, (x_1, ..., x_n)))$ to $\mu \in C(\kappa)$. The map $\overline{\phi}$ is defined as follows: take an element $P = (z_1, z_2) \in \mathbb{P}H^+$, glue as before to get a cone metric μ. Let x_1 be the point in singular metric corresponding to points $0, z_1, z_2$ and let x_2, x_3, x_4 be points corresponding to, respectively, the points $\frac{z_1}{2}, \frac{z_1+z_2}{2}, \frac{z_2}{2}$. Define $\overline{\phi}(P) = (\mu, (x_1, x_2, x_3, x_4))$. This map is surjective, but not one to one. Indeed,

$$\overline{\phi}((z_1, z_2)) = \overline{\phi}((z_1 + 2z_2, z_2)) = \overline{\phi}((z_1, 2z_1 + z_2)). \tag{4.9}$$

Thus there is a well defined surjective map

$$\mathbb{P}H^+/\Gamma(2) \to C(\kappa, l), \tag{4.10}$$

where $\Gamma(2)$ is the congruence modular group of level two defined as

$$\Gamma(2) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{P}SL(2, \mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2\}. \tag{4.11}$$

We know that $\Gamma(2)$ has index 6 in $\mathbb{P}SL_2(\mathbf{Z})$. Since there are 6 different ways to label vertices of a generic cone metric with curvature data κ, we see that the map above is indeed a bijection.

Let $\delta = \exp(\frac{2\pi\sqrt{-1}}{6})$. The ring

$$\mathbf{Eis} = \{a + b\delta : a,\ b \in \mathbf{Z}\} \tag{4.12}$$

is called the ring of *Eisenstein integers*. This ring is the ring of algebraic integers in $\mathbf{Q}(\sqrt{-3})$.

Let $\mathbf{E} = 2\mathbf{Eis} \times 2\mathbf{Eis} = \{(2z_1, 2z_2) : z_1, z_2 \in \mathbf{Eis}\}$. Each $z \in \mathbf{E}^+ = \mathbf{E} \cap H^+$ gives us a triangle with vertices and midpoints of its edges are in $\mathbf{Eis}$. The gluing process described above gives us a triangulation. See Figure 4.2. There is a well defined map

$$\psi :\ \mathbf{E}^+ \to T(\vec{k}). \tag{4.13}$$

The Alexandrov unfolding process implies that this map is surjective. It is obvious that

$$\psi((z_1, z_2)) = \psi((\delta z_1, \delta z_2)) = \psi((z_1, z_2) \begin{pmatrix} a & b \\ c & d \end{pmatrix}), \tag{4.14}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$. This implies that there is a well defined map

$$\mathbf{E}^+/SL_2(\mathbf{Z}) \times \mathbf{Z}/6\mathbf{Z} \to T(\vec{k}), \tag{4.15}$$

which is actually a bijection.

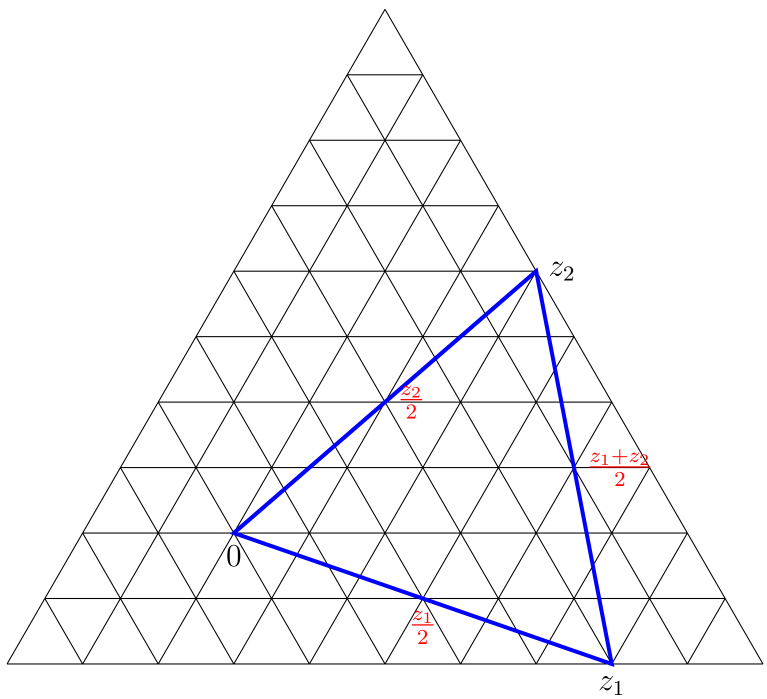

Figure 4.2: Triangulations from Eisenstein lattice

Let $\mathbf{Z}[\imath]$ be set of Gaussian integers, $\mathbf{G} = 2\mathbf{Z}[\imath] \times 2\mathbf{Z}[\imath]$ and $\mathbf{G}^+ = \mathbf{G} \cap H^+$. The same gluing process gives us a mapping $\mathbf{G}^+ \to Q(q)$ and it is true that

$$\mathbf{G}^+/SL_2(\mathbf{Z}) \times \mathbf{Z}/6\mathbf{Z} \equiv Q(\vec{q}). \tag{4.16}$$

Cases of labeled triangulations and quadrangulations can be studied similarly. We have the following results:

$$\mathbf{E}^+/\Gamma(2) \times \mathbf{Z}/6\mathbf{Z} \equiv T(\vec{k}, l), \tag{4.17}$$

$$\mathbf{G}^+/\Gamma(2) \times \mathbf{Z}/6\mathbf{Z} \equiv Q(\vec{q}, l). \tag{4.18}$$

4.2. Space of Parameters

In this section we describe how to obtain elements of $C(\kappa, l)$, where

$$\kappa = (\kappa_1, ..., \kappa_n), \kappa_i \in (0, 2\pi), n > 3 \tag{4.19}$$

$$\sum_{i=1}^{n} \kappa_i = 4\pi \tag{4.20}$$

is a given curvature data. A curvature data satisfying the above conditions will be called positive. Assume that κ is a positive curvature data. Given $\sigma \in S_n$, symmetric group on n letters, let $\sigma\kappa = (\kappa_{\sigma(1)}, ..., \kappa_{\sigma(n)})$. Let $\theta_i = 2\pi - \kappa_i$. We define a parameter space :

$$H(\sigma\kappa) = \{(z_3, ..., z_{2n}) \in \mathbf{C}^{2n-2} : z_3 = 0, e^{\imath\theta_{\sigma(i)}}(z_{2i+1} - z_{2i}) = z_{2i-1} - z_{2i}, 2 \leq i \leq n)\} \tag{4.21}$$

where $z_{2n+1} = 0$. There is an area Hermitian form defined on this parameter spaces: if $z = (z_3, ..., z_{2n})$ and $w = (w_3, ..., w_{2n})$,

$$\langle z, w\rangle = \frac{\imath}{4}\sum_{i=3}^{n}(z_i\overline{w_{i+1}} - z_{i+1}\overline{w_i}) \tag{4.22}$$

Remark 4.2.1. *The signature of this area Hermitian form on $H(\sigma\kappa)$ is $(1, 0, n-3)$. For a proof, see [9].*

Let $H^+(\sigma\kappa) = \{z \in H(\kappa) : \langle z, z\rangle > 0\}$. Since the signature of the area-Hermitian form is $(1, 0, n-3)$, we have

Corollary 4.2.2.

$$\mathbb{P}H^+(\sigma\kappa) \equiv \mathbb{H}_{\mathbf{C}}^{n-3}, \tag{4.23}$$

where $\mathbb{P}H(\sigma\kappa)$ is complex projectification of $H^+(\sigma\kappa)$ and $\mathbb{H}_{\mathbf{C}}^{n-3}$ is $n-3$ dimensional complex hyperbolic space.

Let $U(\sigma\kappa) \subset H^+(\sigma\kappa)$ be the open set consisting of positively oriented simple (non-overlapping) polygons. If z is such a polygon, by gluing edges $[z_{2i+1}, z_{2i}]$ with $[z_{2i-1}, z_{2i}]$ we get a cone metric μ. Label the cone point corresponding to

$z_3, z_5, ..., z_{2n-1}$ as $x_{\sigma(1)}$. Label the cone point corresponding to z_{2i} as $x_{\sigma(i)}$. $(\mu, (x_1, ..., x_n))$, after a normalization, is an element in $C(\kappa, l)$. Thus there are maps

$$\psi_\sigma \ : \mathbb{P}U(\sigma\kappa) \to C(\kappa, l). \tag{4.24}$$

Is every element of $C(\kappa, l)$ in the image of ψ_σ, for some $\sigma \in S_n$? The Alexadrov unfolding process [1] provides an affirmative answer for this question.

4.3. $C(\kappa, l)$ is a Complex Hyperbolic Manifold

In this section we prove that ψ_σ's provide complex hyperbolic charts on $C(\kappa, l)$. These charts are equivalent to Thurston's charts; more precisely they are special examples of Thurston's charts. Therefore we start with a review of Thurston's theory [4][pages 525-527].

Proposition 4.3.1 ([8], Proposition 2). *Every cone metric on a compact surface admits a geodesic triangulation such that singular points are vertices of the triangulation and each edge is incident to two different faces of the triangulation.*

Let $\mu \in C(\kappa, l)$, $v_1, ..., v_n$ be singular points of μ. Let T be a triangulation of μ satisfying properties mentioned in the proposition above. Let $\overline{S^2}$ be universal branched cover of S^2 with respect to μ. Let $\overline{T}$ be associated triangulation on universal branched cover. Choose one of the edges of $\overline{T}$ and map it isometrically on $\mathbf{C}$. This extends to a developing map $D: \ \overline{S^2} \to \mathbf{C}$. If e is a directed edge of $\overline{T}$, let $Z(e)$ be difference of images of end points of e under developing map. These vectors, $Z(e)'s$, satisfy some conditions:

1. The sum of vectors corresponding to oriented boundary of a triangle is 0.

Let $H : \Pi_1(S^2 - \{v_1, ..., v_n\}) \to Isom(\mathbb{E}^2)$ be holonomy of cone metric μ and $H_0 : \Pi_1(S^2 - \{v_1, ..., v_n\}) \to S^1 \subset \mathbf{C}$ be it's orthogonal part.

2. $Z(\tau_\gamma(e)) = H_0(\gamma)Z(e)$, for all $\gamma \in \Pi_1(S^2 - \{v_1, ..., v_n\})$ where τ_γ is covering transformation of $\overline{S^2}$ over S^2 associated to γ.

Now let $H(\mu, T)$ be set of all maps, $e(\overline{T}) \to \mathbf{C}$, satisfying these conditions, where $e(\overline{T})$ is edge set of $\overline{T}$. This space has dimension $n-2$ and has an area Hermitian form on it. Signature of this form is $(1, 0, n-3)$. Therefore $\mathbb{P}H^+(\mu, T) \equiv H_{\mathbf{C}}^{n-3}$.

We describe how cocycles near Z parametrize cone metrics near μ. Take a 'polygon' $F \subset \overline{S^2}$ such that edges of F are in $e(\overline{T})$ and vertices of F are $v(\overline{T})$, set of vertices of $\overline{T}$. Choose F so that interior of F is in one to one correspondence with an open set of S^2 whose closure is S^2. Such a polygon will be called a fundamental domain for μ with respect to T. Develop F to the plane to get a polygon in the plane. We will call this polygon also F. Now assume that F can be chosen so that developing does not overlap. After gluing the edges of F accordingly one can get the cone metric back. Also cocycle conditions imply that angles at vertices of F satisfy some conditions. The polygons close to F and satisfying the same cocycle conditions give a cone metric after appropriately gluing.

Now let $P \in U(\sigma\kappa) \subset H^+(\sigma\kappa)$ be a simple polygon. Let $(\mu, (x_1, ..., x_n)) = \psi_\sigma(P)$. Triangulate P in a way that its vertices are vertices of the triangulation and its edges are in the edge set of triangulation and also this triangulation induces a triangulation on $(\mu, (x_1, ..., x_n)) = \psi_\sigma(P)$. Let T be such a triangulation of P. We will call triangulation induced on μ also T. Our objective is to obtain a complex vector space isomorphism between $H(\mu, T)$ and $H(\sigma\kappa)$ respecting area Hermitian products. This will imply that the maps ψ_σ's, introduced in the previous section give rise to complex hyperbolic charts, which are some examples of Thurston's charts.

In $\overline{S^2}$, universal branched cover of S^2 with respect to μ, there exist a fundamental domain F which is isometric to P, perhaps up to scaling. Let e_i be edge of F corresponding to edge $[z_i, z_{i+1}]$ of P, where $3 \leq i \leq 2n$ and $z_{2n+1} = 0$. Now for each $z' = (z'_3, ..., z'_{2n}) \in H(\sigma\kappa)$, let $y(z') \in H(\mu, T)$ be the cocycle so that $y(z')(e_i) = z'_{i+1} - z'_i$, where $z'_{2n+1} = 0$. This is the desired isomorphism. The following proposition summarizes what we have done in this section.

Proposition 4.3.2. *For each $(\mu, (x_1, ..., x_n)) \in C(\kappa, l) \equiv X(n)$, there exists a permutation $\sigma \in S_n$ and open sets U, U' such that $(\mu, (x_1, .., x_n)) \in U \subset X(n)$, $U' \subset \mathbb{P}H^+(\sigma\kappa)$ and $\psi_\sigma : U' \to U$ is isomorphism of complex manifolds. Furthermore, ψ_σ's provide complex hyperbolic charts; if σ' is another such permutation, transition map is an isomorphism of some open sets of complex hyperbolic spaces $\mathbb{P}H(\sigma\kappa)$ and $\mathbb{P}H(\sigma'\kappa)$. $C(\kappa, l)$ is $n-3$ dimensional complex hyperbolic manifold.*

Definition 4.3.3. *We denote metric completion of $C(\kappa, l)$ with respect to complex* *hyperbolic metric given in above proposition, by* $\overline{C}(\kappa, l)$.

Here we present Thurston's main theorem:

Theorem 4.3.4 ([4], Theorem 0.2)**.** *Let* $\kappa = (\kappa_1, ..., \kappa_n)$ *, be a curvature positive curvature data.* $C(\kappa, l)$ *form a complex hyperbolic manifold whose metric completion,* $\overline{C}(\kappa, l)$ *is a complex hyperbolic cone manifold of finite volume. This cone manifold is an orbifold (that is, the quotient of a discrete group) if and only if for any pair* κ_i, κ_j *whose sum* $s = \kappa_i + \kappa_j$ *that is less than* 2π*, either*

- $(2\pi - s)$ *divides* 2π*, or*
- $\kappa_i = \kappa_j$ *and* $\pi - \kappa_i$ *divides* 2π.

4.4. Reparametrization of $C(\kappa, l)$

In the previous sections we proved that $C(\kappa, l)$ can be parametrized by $\mathbb{P}H^+(\sigma\kappa)$'s. More precisely, we proved that $C(\kappa, l)$ is parametrized by open sets $\mathbb{P}U(\sigma\kappa)$'s. The aim of this section is to prove that $C(\kappa, l)$ can be parametrized by using just an open set of $\mathbb{P}H^+(\kappa)$.
First we define a new vector space of parameters:

Definition 4.4.1. $H_0(\sigma\kappa) := \{(z_1, ..., z_{2n}) \in \mathbf{C}^{2n} : e^{\imath\theta_{\sigma(i)}}(z_{2i+1} - z_{2i}) = z_{2i-1} - z_{2i},\ 1 \leq i \leq n, z_1 = 0\}$,
where $z_{2n+1} = 0$ *and* $\theta_i = 2\pi - \kappa_i$.

We define the area Hermitian product on this space:

Definition 4.4.2. *Let* $z = (z_1, ..., z_{2n}),\ w = (w_1, ..., w_{2n}) \in H_0(\sigma\kappa)$*,* $z_{2n+1} = w_{2n+1} = 0$*.*
$\langle z, w\rangle := \frac{\sqrt{-1}}{4}\sum_{i=1}^{2n}(z_i\overline{w_{i+1}} - z_{i+1}\overline{w_i})$.

Remark 4.4.3. *The signature of the area Hermitian form on* $H_0(\kappa)$ *is* $(1, 1, n-3)$*. A proof of this fact can be found in [9].*

Remark 4.4.4. *The map*

$$I = I_\sigma : (z_3, z_4, ..., z_{2n}) \to (0, 0, z_3, ..., z_{2n}) \tag{4.25}$$

gives an embedding of $H(\sigma\kappa)$ *to* $H_0(\sigma\kappa)$ *which respects area Hermitian form.*

Denote by $N_0(\sigma\kappa)$ *the null vector space in this parameter space. Every element in* $H_0(\sigma\kappa)$ *can be uniquely written as a sum of elements from* $H_0(\kappa)$ *and* $N_0(\kappa)$*. There exists a map*

$$\Pi_\sigma : H_0(\sigma\kappa) \to H(\sigma\kappa) \tag{4.26}$$

such that

1. $\Pi_\sigma \circ I_\sigma = id$ *on* $H(\sigma\kappa)$
2. $Ker(\Pi_\sigma) = N_0(\sigma\kappa)$
3. $\langle z, w\rangle = \langle \Pi_\sigma(z), \Pi_\sigma(w)\rangle$.

Observe that these imply that

$$H_0(\sigma\kappa)/N_0(\sigma\kappa) \equiv H(\sigma\kappa). \tag{4.27}$$

Let

$$U_0(\sigma\kappa) = \Pi_\sigma^{-1}(U(\sigma\kappa)). \tag{4.28}$$

Observe that $\psi_\sigma \circ \Pi_\sigma \restriction_{U_0(\sigma\kappa)}$ gives a map from $U_0(\sigma\kappa)$ to $C(\kappa, l)$. Here is another way of considering this map. Let $z = (z_1, ..., z_{2n}) \in U_0(\sigma\kappa)$ be a simple polygon, necessarily positively oriented. Glue the edge $[z_{2i}, z_{2i+1}]$ with $[z_{2i-1}, z_{2i}]$ for all $1 \leq i \leq n$ to get a cone metric μ. Let $x_{\sigma(i)}$ be the singular point corresponding to z_{2i}, then

$$\psi_\sigma \circ \Pi_\sigma \restriction_{U_0(\sigma\kappa)} (z) = (\mu, (x_1, ..., x_n)). \tag{4.29}$$

Now we define an isomorphism between $H_0(\sigma\kappa)$ and $H_0(\kappa)$ when $\sigma = (i, i+1) \in S_n$ is a transition, $1 \leq i \leq n$.

Take a polygon $z = (z_1, ..., z_{2n}) \in H_0(\sigma\kappa)$. Cut z through the line segment $[z_{2i}, z_{2i+3}]$. Glue the edge $[z_{2i}, z_{2i-1}]$ with the edge $[z_{2i}, z_{2i+1}]$ by rotating the polygon $[z_{2i}, z_{2i+1}, z_{2i+2}, z_{2i+3}, z_{2i}]$ around the z_{2i} by an angle of θ_i in counter-clockwise oriantation to get an element z' in $H_0(\kappa)$. See Figure 4.3.

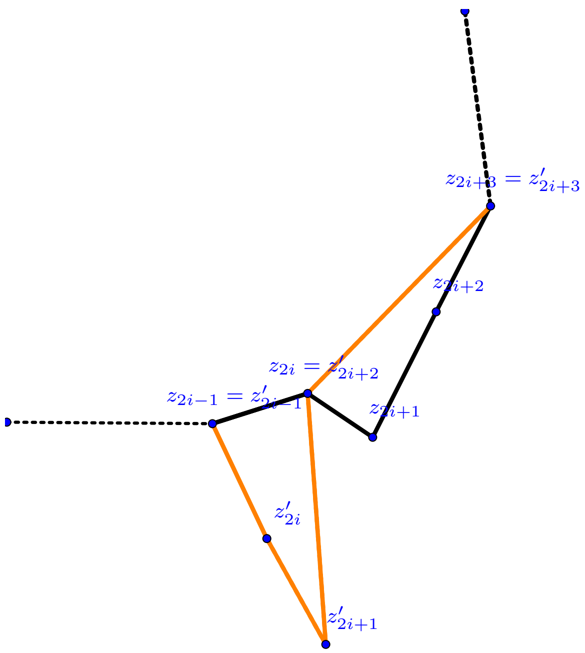

Figure 4.3: Operation $F_{\sigma,0}$

Call this map, which is an isomorphism of complex vector spaces, $F_{\sigma,0}$. Observe that if $z \in U_0(\sigma\kappa)$ and $F_{\sigma,0}(z) \in U_0(\kappa)$ then

$$\psi_\sigma \circ \Pi_\sigma \restriction_{U_0(\sigma\kappa)} (z) = \psi_{id} \circ \Pi_{id} \restriction_{U_0(\kappa)} (F_{\sigma,0}(z)), \tag{4.30}$$

where $id \in S_n$ is the trivial permutation.

$F_{\sigma,0}$ induces a vector space isomorphism

$$F_\sigma : H(\sigma\kappa) \to H(\kappa), \tag{4.31}$$

which respects the area Hermitian form and makes the following diagram commute:

$$\begin{tikzcd} H_0(\sigma\kappa) \arrow[r, "F_{\sigma,0}"] \arrow[d, "\Pi_\sigma"'] & H_0(\kappa) \arrow[d, "\Pi_{id}"] \\ H(\sigma\kappa) \arrow[r, "F_\sigma"'] & H(\kappa) \end{tikzcd}$$

Observe that (4.30) implies that $z \in U(\sigma\kappa)$ and $F_\sigma(z) \in U(\kappa)$ gives the same cone metric :

$$\psi_\sigma(z) = \psi_{id}(F_\sigma(z)) \tag{4.32}$$

Remark 4.4.5. *Let*

$$\kappa = (k_1\frac{\pi}{3}, ..., k_n\frac{\pi}{3}), \;\; k_i \in \mathbb{N}, \tag{4.33}$$

be a given positive curvature data. Let

$$\vec{k} = (k_1, ..., k_n). \tag{4.34}$$

We define two types of Eisenstein lattices:

$$E_0(\sigma\vec{k}) := \{z \in H_0(\sigma\kappa) : z_i \in \mathbf{Eis}\}. \tag{4.35}$$

When $\sigma = (i, i+1)$, $F_{\sigma,0}$ *induces an isomorphism between lattices* $E_0(\sigma\vec{k})$ *and* $E_0(\vec{k})$.

Similarly define

$$E(\sigma\vec{k}) := \{z \in H(\sigma\kappa) : z_i \in \mathbf{Eis}\}. \tag{4.36}$$

When $\sigma = (i, i+1)$, F_σ *induces an isomorphism between lattices* $E(\sigma\vec{k})$ *and* $E(\vec{k})$.

Remark 4.4.6. *Let*

$$\kappa = (q_1\frac{\pi}{2}, ..., q_n\frac{\pi}{2}),\ \ q_i \in \mathbb{N}, \tag{4.37}$$

be a given curvature data. Let

$$\vec{q} = (q_1, ..., q_n). \tag{4.38}$$

We define two types of Gaussian lattices:

$$G_0(\sigma\vec{q}) := \{z \in H_0(\sigma\kappa) : z_i \in \mathbf{Z}[\imath]\}. \tag{4.39}$$

When $\sigma = (i, i+1)$, $F_{\sigma,0}$ *induces an isomorphism between lattices* $G_0(\sigma\vec{q})$ *and* $G_0(\vec{q})$.

Similarly define

$$G(\sigma\vec{q}) := \{z \in H(\sigma\kappa) : z_i \in \mathbf{Z}[i]\}. \tag{4.40}$$

When $\sigma = (i, i+1)$, F_σ induces an isomorphism between lattices $G(\sigma\vec{q})$ and $G(\vec{q})$.

If $\sigma \in S_n$ any permutation, we know that we can write it as a product of transitions. This implies that there are maps $F_{\sigma,0}, F_\sigma$, which are vector space isomorphisms respecting the area-Hermitian form and make the following diagram commute:

$$\begin{tikzcd} H_0(\sigma\kappa) \arrow[r, "F_{\sigma,0}"] \arrow[d, "\Pi_\sigma"'] & H_0(\kappa) \arrow[d, "\Pi_{id}"] \\ H(\sigma\kappa) \arrow[r, "F_\sigma"'] & H(\kappa) \end{tikzcd}$$

Remark 4.4.7. *Since F_σ is a composition of cutting gluing operations it respects cone metrics. More precisely, if $z \in U(\sigma\kappa)$ and $z' \in U(\sigma'\kappa)$ and $F_\sigma(z) = F_{\sigma'}(z')$, then $\psi_\sigma(z) = \psi_{\sigma'}(z')$.*

Remark 4.4.8. *Let*

$$\kappa = (k_1\frac{\pi}{3}, ..., k_n\frac{\pi}{3}),\ \ k_i \in \mathbb{N}, \tag{4.41}$$

be a given positive curvature data. Let

$$\vec{k} = (k_1, ..., k_n). \tag{4.42}$$

Remark 4.4.5 implies that when $\sigma \in S_n$, $F_{\sigma,0}$ induces an isomorphism between lattices $E_0(\sigma\vec{k})$ and $E_0(\vec{k})$.
Also F_σ induces an isomorphism between lattices $E(\sigma\vec{k})$ and $E(\vec{k})$.

Remark 4.4.9. *Let*

$$\kappa = (q_1\frac{\pi}{2}, ..., q_n\frac{\pi}{2}),\ \ q_i \in \mathbb{N}, \tag{4.43}$$

be a given curvature data. Let

$$\vec{q} = (q_1, ..., q_n). \tag{4.44}$$

Remark 4.4.6 implies that when $\sigma \in S_n$, $F_{\sigma,0}$ induces an isomorphism between lattices $G_0(\sigma\vec{q})$ and $G_0(\vec{q})$.
Also F_σ induces an isomorphism between lattices $G(\sigma\vec{q})$ and $G(\vec{q})$.

Since F_σ respects the area Hermitian form, it induces an isomorphism of complex hyperbolic spaces:

$$\bar{F}_\sigma : \mathbb{P}H^+(\sigma\kappa) \to \mathbb{P}H^+(\kappa). \tag{4.45}$$

Let

$$V = V(\kappa) := \cup \bar{F}_\sigma(\mathbb{P}U(\sigma\kappa)) \subset \mathbb{P}H^+(\kappa). \tag{4.46}$$

The following theorem asserts that $C(\kappa, l)$ can be parametrized by the open set V when curvature data κ is nice.

Theorem 4.4.10. *If the curvature data κ is positive, then there is a surjective map*

$$\psi : V \subset \mathbb{P}H^+(\kappa) \to C(\kappa, l) \tag{4.47}$$

such that for any cone metric $\mu \in C(\kappa, l)$ there exists an open set $U \subset V$ with the property that $\mu \in \psi(U) \subset C(\kappa, l)$ is open and $\psi \restriction_U$ induces an isomorphism of complex manifolds U and $\psi(U)$. Moreover if U' is another such open set, the transition map is given by an automorphism of $\mathbb{P}H^+(\kappa)$. $C(\kappa, l)$ is a $\mathbb{P}H^+(\kappa)$ manifold.

Proof. If $z \in V$, there exists $\sigma \in S_n$ and $z' \in \mathbb{P}U(\sigma\kappa)$ such that $\bar{F}_\sigma(z') = z$. Let $\psi(z) = \psi_\sigma(z')$. Remark 4.4.7 implies that ψ is well defined. This map is surjective since images of ψ_σ's cover $C(\kappa, l)$. Since ψ_σ's provide complex hyperbolic charts and $\bar{F}_\sigma$'s are isomorphism of complex hyperbolic spaces, other two assertions follow. □

4.5. Degenerate Cone Metrics : $\bar{C}(\kappa, l)$

In the previous section, we parametrized $C(\kappa, l)$ by $U(\sigma\kappa)'s$. We enlarge $U(\sigma\kappa)'s$ to parametrize $\bar{C}(\kappa, l)$. Assume that $\kappa = (\kappa_1, ..., \kappa_n)$ be a curvature data such that $\kappa_i > 0$ for all $1 \leq i \leq n$. If

$$\kappa = (\kappa_1, \kappa_2, \kappa_3) \tag{4.48}$$

we define

$$U^*(\kappa) = U(\kappa). \tag{4.49}$$

If $\kappa = (\kappa_1, ..., \kappa_n)$, $n > 3$, $U^*(\kappa)$ is defined inductively as follows.

- If $\kappa_1 + \kappa_2 < 2\pi$, then the mapping

$$A(1) : (z_3, ..., z_{2n-2}) \to (0, 0, z_3, ..., z_{2n-2}) \tag{4.50}$$

  is an embedding of $H(\kappa')$ to $H(\kappa)$, where

$$\kappa' = (\kappa_1 + \kappa_2, \kappa_3, .., \kappa_n). \tag{4.51}$$

- If $\kappa_i + \kappa_{i+1} < 2\pi$, $1 < i < n$, the map

$$A(i) : (z_3, .., z_{2i}, ..z_{2n-2}) \to (z_3, .., z_{2i}, z_{2i} + e^{-\imath\theta_i}(z_{2i-1} - z_{2i}), z_{2i}, z_{2i+1}, .., z_{2n}) \tag{4.52}$$

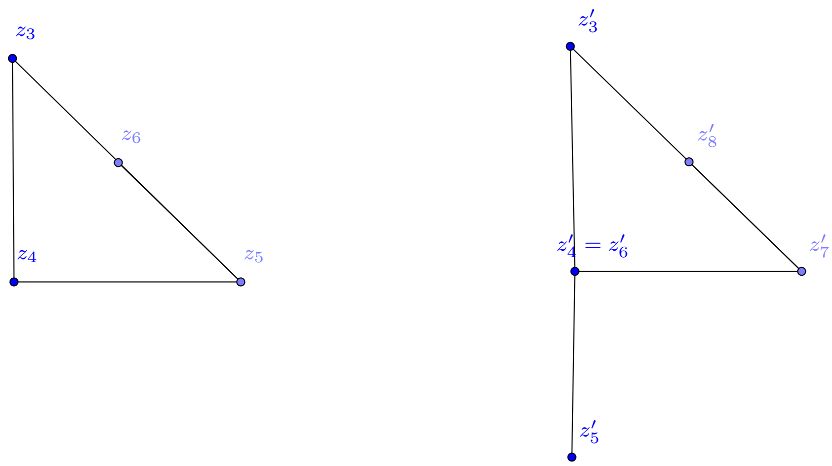

Figure 4.4: When we apply $A(2)$ to the element z in $H(\frac{3\pi}{2}, \pi + \frac{\pi}{2}, \pi)$ (left), we get an element z' in $H(\frac{3\pi}{2}, \pi, \frac{\pi}{2}, \pi)$ (right).

is an embedding of $H(\kappa')$ to $H(\kappa)$, where

$$\kappa' = (\kappa_1, ..\kappa_{i-1}, \kappa_i + \kappa_{i+1}, \kappa_{i+2}, .., \kappa_n). \tag{4.53}$$

Figure 4.4 describes A(2) when $n = 4$ and $\kappa_2 + \kappa_3 < 2\pi$.

Now we define

$$U^*(\kappa) = U(\kappa) \cup \{A(i)(z) : \kappa_i + \kappa_{i+1} < 2\pi, z \in U^*((\kappa_1, .., \kappa_i + \kappa_{i+1}, .., \kappa_n))\}. \tag{4.54}$$

Observe that by gluing edges of elements in U^* as before, we can get degenerate cone metrics.

Proposition 4.5.1. *For all $\sigma \in S_n$, the map ψ_σ extends to a map*

$$\mathbb{P}U^*(\sigma\kappa) \to \bar{C}(\kappa, l) \tag{4.55}$$

such that union of the images of these extension maps cover $\bar{C}(\kappa, l)$.

Proof. Thurston's construction implies that

$$\bar{C}(\kappa, l) = C(\kappa, l) \cup_{\kappa'} \bar{C}(\kappa, l) \tag{4.56}$$

where $\kappa' = (\kappa_1, ..., \kappa_i + \kappa_j, .., \kappa_n)$, whenever $\kappa_i + \kappa_j < 2\pi\ i \neq j$. In our construction, it can be shown that union $U^*(\sigma\kappa)$'s parametrize all elements in $\bar{C}(\kappa, l)$, by an inductive argument. □

Remark 4.5.2. *$\mathbb{P}U^*(\sigma\kappa)$ is in the closure of $\mathbb{P}U(\sigma\kappa)$. The extension given in the proposition is a continuous extension. By abuse of notation we will denote extension of ψ_σ by ψ_σ.*

Let's define

$$V^* = \cup_{\sigma \in S_n} \bar{F}_\sigma(\mathbb{P}U^*(\sigma\kappa)). \tag{4.57}$$

The set V is connected. This follows from the facts that $\mathbb{P}U(\sigma\kappa)$ is connected for all $\sigma \in S_n$ and when $\tau \in S_n$ is a transition, the set $\bar{F}_\tau(\mathbb{P}U(\tau\kappa)) \cap \mathbb{P}U(\kappa)$ is not empty. Now assume that $\kappa = (\kappa_1, ..., \kappa_n)$ is a curvature data that satisfies conditions of Theorem 4.3.4. That is $n > 3$ and for any pair of κ_i, κ_j whose sum $s = \kappa_i + \kappa_j$ is less that 2π, either

$$(2\pi - s) \textit{ divides } 2\pi, \textit{ or} \tag{4.58}$$

$$\kappa_i = \kappa_j, \ \frac{2\pi - s}{2} \textit{ divides } 2\pi. \tag{4.59}$$

Let $U' \subset \mathbb{P}U'(\kappa)$ be an open set such that $\psi \restriction_{U'}$ is a chart. Theorem 4.3.4 implies that there is a surjective map

$$\bar{\psi}: \ \mathbb{P}H^+(\kappa) \to \bar{C}(\kappa, l) \tag{4.60}$$

and a discrete group $\Gamma(\kappa)$ of isomorphisms of $\mathbb{P}H^+(\kappa)$ such that $\bar{\psi}(x) = \bar{\psi}(y)$ if and only if there exists $g \in \Gamma(\kappa)$ with $g(x) = y$. Moreover $\bar{\psi}$ is an extension of $\psi \restriction_{U'}$. Now let $K = \bar{\psi}^{-1}(C(\kappa, l))$. Necessarily

$$\bar{\psi} \restriction_K: \ K \to C(\kappa, l) \tag{4.61}$$

is complex analytic. Since $\bar{\psi}$ coincides with $\psi \restriction_{U'}$ on open set U and since ψ is analytic, V is connected, we have

$$\bar{\psi} \restriction_V = \psi. \tag{4.62}$$

Also observe that continuity of $\bar{\psi}$ and ψ_σ's imply that if $z \in \bar{V}$ and $z = \bar{F}_\sigma(z')$ for some $z' \in \mathbb{P}\bar{U}(\sigma\kappa)$, we have $\bar{\psi}(z) = \psi_\sigma(z')$.

We summarize these results in the following theorem.

Theorem 4.5.3. *Let $\kappa = (\kappa_1, .., \kappa_n)$ be a curvature data satisfying conditions of Theorem 4.3.4. There exists a discrete group $\Gamma(\kappa)$ of isomorphisms of $\mathbb{P}H^+(\kappa)$ and an extension of ψ*

$$\bar{\psi} : \ \mathbb{P}H^+(\kappa) \to \bar{C}(\kappa, l) \tag{4.63}$$

such that

1. *$\bar{\psi}$ is surjective.*
2. *$\bar{\psi}(x) = \bar{\psi}(y)$ if and only if there exists $g \in \Gamma(\kappa)$ such that $gx = y$.*
3. *If $x = \bar{F}_\sigma(y)$ for some $y \in \mathbb{P}\bar{U}(\sigma\kappa)$,then $\bar{\psi}(x) = \psi_\sigma(y)$.*
4. *$\bar{\psi} \restriction_{V^*}: \ V^* \to \bar{C}(\kappa, l)$ is surjective.*

4.6. On The Fundamental Group of $X(n)$

The aim of this section is to review basic facts about braid groups and fundamental group of $X(n)$. Recall that $X(n)$ is moduli space of n points in $\mathbb{P}^1_{\mathbf{C}}$ and $X\{n\} = X(n)/S_n$. Also recall that $X(n) \equiv C(\kappa, l)$.
Let Δ be the main diagonal in $\mathbf{C}^n$. Let $x \in \mathbf{C}^n - \Delta$. Consider the space

$$\mathbf{C}^n - \Delta/S_n \tag{4.64}$$

where the action of the symmetric group is given by permuting coordinates of a given element in the space. Let

$$B_n = \Pi_1(\mathbf{C}^n - \Delta/S_n, x). \tag{4.65}$$

This group is called braid group on n strings. it has $n-1$ generators:

- $\sigma_1,..,\sigma_{n-1}$

The set of relations are generated by the followings:

1. $\sigma_i\sigma_j = \sigma_j\sigma_i$ for all $|i-j| > 1$,
2. $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for all $1 \leq i \leq n-2$.

The quotient map

$$\mathbf{C}^n - \Delta \to \mathbf{C}^n - \Delta/S_n \tag{4.66}$$

is a Galois covering map with S_n as the deck transformation group. Therefore we have a natural inclusion

$$C_n := \Pi_1(\mathbf{C}^n - \Delta, x) \to \Pi_1(\mathbf{C}_n - \Delta/S_n, x) \tag{4.67}$$

This means that C_n can be thought as a subgroup of B_n. This group is called the *pure braid group*. For any $i < j$, let

$$A_{ij} = \sigma_{j-1}^{-1}\sigma_{j-2}^{-1}...\sigma_{i+1}^{-1}\sigma_i^2\sigma_{i+1}...\sigma_{j-2}\sigma_{j-1}. \tag{4.68}$$

$\{A_{ij} : 1 \leq i < j \leq n\}$ is a set of generators for C_n. Relations for C_n are well known, but we do not put them here since we do not need them.

It is well known that fundamental (orbifold) group of $X\{n\}$ is a quotient of B_n. By abuse of notation we represent generators of this group by

- $\sigma_1,..,\sigma_{n-1}$.

Thus

$$A_{ij} = \sigma_{j-1}^{-1}\sigma_{j-2}^{-1}...\sigma_{i+1}^{-1}\sigma_i^2\sigma_{i+1}...\sigma_{j-2}\sigma_{j-1}. \tag{4.69}$$

represents a generator for fundamental group of $X(n)$.

Figure 4.5 shows how to represent a generator σ_i of fundamental group of $X\{n\}$.

4.7. On Developing Map and Holonomy

In this section we we give a review of the theory of the developing map and holonomy representation corresponding to a manifold M with an analytic (G, X)

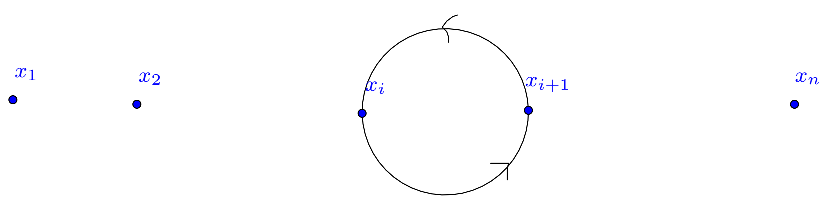

Figure 4.5: The path σ_i

structure on it. Our main reference is [10].

Let X be a manifold and G be a group of diffeomorphisms of X. A (G, X) structure on a manifold M is a collection of open sets U_i and diffeomorphisms $\phi_i : U_i \to X$ to open sets of X, such that transition maps are given by restrictions of elements in G. The pair (G, X) is called analytic if for all $g_1, g_2 \in G$ that coincide on a non-empty open set we have $g_1 = g_2$.

Let (G, X) be an analytic structure on M. Let $\phi_1 : U_1 \subset M \to X$ be a coordinate chart. If $\phi_2 : U_2 \subset M \to X$ is another coordinate chart such that $U_1 \cap U_2$ is non-empty and connected, there exists $g \in G$ such that $g \circ \phi_2 = \phi_1$ on $U_1 \cap U_2$. Moreover, g is unique with respect to this property. Thus we can extend ϕ_1 to a map defined on $U_1 \cup U_2$. Now take a point $x \in U_1$ and let γ be a path starting at x. By above analytic continuation process we can extend ϕ_1 along γ. The map obtained near end point of γ depends only of homotopy class of γ. Therefore we have a map

$$dev : \ M^{\sim} \to X \tag{4.70}$$

where $M^{\sim}$ is the universal cover of M. If $\eta \in \Pi_1(M, x)$ any deck transformation, there exists a unique $h(\eta) \in G$ such that

$$dev \circ \eta = h(\eta) \circ dev. \tag{4.71}$$

The map

$$h: \ \Pi_1(M) \to G \tag{4.72}$$

is a homomorphism and called holonomy representation. We state a well known result relating holonomy and orbifolds.

Theorem 4.7.1 ([10],Theorem 2.26)**.** *Let Q be a (G, X) orbifold where (G, X) is analytic and G is a group of isometries of X. Let $Q - \Sigma(Q)$ be its regular points. If Q is complete and X is simply connected, image of holonomy representation h : $\Pi_1(Q - \Sigma(Q)) \to G$ is discrete subgroup Γ of G which acts properly and discontinuously on X. Q is isometric to X/Γ.*

Corollary 4.7.2. *Assume that κ is a curvature data satisfying conditions of Theorem 4.3.4. Let*

$$hol: \ \Pi_1(C(\kappa, l)) \to Isom(\mathbb{P}H^+(\kappa)) \tag{4.73}$$

be holonomy representation.

$$hol(\Pi_1(C(\kappa, l))) = \Gamma(\kappa) \tag{4.74}$$

Proof. All the requirements of above theorem are satisfied. $\bar{C}(\kappa, l)$ is a complete

$(Isom\mathbb{P}H^+(\kappa), \mathbb{P}H^+(\kappa))$ orbifold. $(Isom\mathbb{P}H^+(\kappa), \mathbb{P}H^+(\kappa))$ is analytic. $\mathbb{P}H^+(\kappa)$ is simply connected. $\square$

4.8. An Explicit Construction of Holonomy Representation When All Cone Angles are Equal

Assume that curvature data $\kappa = (\kappa_1, ..., \kappa_n)$ satisfies conditions of Theorem 4.3.4 and $\kappa_1 = \kappa_2 = ... = \kappa_n$. We have a surjection

$$\bar{\psi} : \ \mathbb{P}H^+(\kappa) \to \bar{C}(\kappa, l) \tag{4.75}$$

and a discrete group $\Gamma(\kappa)$ of isomorphisms of $\mathbb{P}H^+(\kappa)$ such that

$$\mathbb{P}H^+(\kappa)/\Gamma(\kappa) \equiv \bar{C}(\kappa, l) \tag{4.76}$$

See Theorem 4.5.3. The restriction of this map on V^* is surjective and given by the gluing operation defined before. Corollary 4.7.2 says that holonomy representation can be constructed in a way that its image is just $\Gamma(\kappa)$.

The aim of this section is to explicitly construct the generators of $\Gamma(\kappa)$. More precisely, we will construct element $hol(A_{ij})$ where A_{ij} is a generator of $\Pi_1(C(\kappa, l)) \equiv \Pi_1(X(n))$. Our strategy is first to define the holonomy corresponding to half twist σ_i and then construct it.

S_n acts on $C(\kappa, l) \equiv X(n)$ by permuting coordinates of a given point. If $x = (x_1, .., x_n) \in X(n)$ and $\tau \in S_n$,

$$\tau x := (x_{\tau(1)}, ..., x_{\tau(n)}) \in X(n). \tag{4.77}$$

This action induces an action of S_n on paths in $X(n)$. Let τ_i, $1 \leq i \leq n-1$, be the transition $(i, i+1) \in S_n$. Fix $x = (x_1, .., x_n) \in X(n)$.

Let σ_i be the path in the Figure 4.5 joining $(x_1, ..., x_i, x_{i+1}, ..., x_n)$ with $(x_1, ..., x_{i+1}, x_i, ..., x_n)$. One can define products of σ_i's by using composition of paths and action of S_n on paths in $X(n)$. For example

$$\sigma_1\sigma_2 := \sigma_2 \cdot (\tau_2\sigma_1), \tag{4.78}$$

where $\tau_2\sigma_1$ is the path obtained by action of τ_2 applied to σ_1 and $\sigma_2 \cdot (\tau_2\sigma_1)$ is product of paths σ_2 and $\tau_2\sigma_1$. We can write the loop A_{ij} as a product of loops $\sigma_i's$ as follows:

$$A_{ij} = \sigma_{j-1}^{-1}\sigma_{j-2}^{-1}...\sigma_{i+1}^{-1}\sigma_i^2\sigma_{i+1}...\sigma_{j-2}\sigma_{j-1}. \tag{4.79}$$

Each σ_i induces a deck transformation. If γ is any path starting at x, then

$$\sigma_i \cdot (\tau_i\gamma) \tag{4.80}$$

is another path starting at x. Thus there is a deck transformation

$$\sigma_i \ : X(n)^{\sim} \to X(n)^{\sim} \tag{4.81}$$

where $X(n)^{\sim}$ is the universal cover of $X(n)$. There is an element $hol(\sigma_i) \in Isom(\mathbb{P}H^+(\kappa))$ such that

$$dev \circ \sigma_i = hol(\sigma_i) \circ dev. \tag{4.82}$$

This implies that

$$hol(A_{ij}) = hol(\sigma_{j-1})^{-1} hol(\sigma_{j-2})^{-1} ... hol(\sigma_{i+1})^{-1} hol(\sigma_i^2) hol(\sigma_{i+1}) ... hol(\sigma_{j-2}) hol(\sigma_{j-1}). \tag{4.83}$$

Here is another description of $hol(\sigma_i)$. Let x' be and point of σ_i. Take a chart

$$\phi_1 \ : U \to \mathbb{P}H^+(\kappa) \tag{4.84}$$

such that $x \in U$. The analytic continuation of ϕ_1 along σ_1 gives us a map ϕ in a neighborhood of σ_i. Let U' be neighborhood of x' such that ϕ is defined. Let

$$U'' = \{y \in U : \tau_i y \in U'\} \tag{4.85}$$

$$\tau_i U'' = \{\tau_i z : z \in U''\}. \tag{4.86}$$

The map

$$\phi(U'') \to \phi(\tau_i U'') \tag{4.87}$$

given by

$$z \to \phi\tau_i\phi^{-1}(z) \tag{4.88}$$

is a restriction of an element of $\mathbb{P}H^+(\kappa)$ and this element is nothing more than $hol(\sigma_i)$. This is the construction that we will use.

Let $1 \leq i < n$. Consider the parameter space $H_0(\kappa)$. Take a non-overlapping positively oriented element $z = (z_1, ..., z_{2n}) \in H_0(\kappa)$. Cut this polygon z through $[z_{2i+3}, z_{2i}]$. Glue the edge $[z_{2i}, z_{2i+1}]$ with the edge $[z_{2i}, z_{2i-1}]$. By this process we get an element z' in $H_0(\kappa)$. See Figure 4.6.

If the cone metric that z is represented by $(x_1, ..., x_i, x_{i+1}, ...x_n)$, then the cone metric that z' gives is represented by the element $(x_1, .., x_{i+1}, x_i, .., x_n)$. This cutting and gluing operation gives us an isomorphism

$$\alpha_{i,0}: \ H_0(\kappa) \to H_0(\kappa) \tag{4.89}$$

which induces an isomorphism α_i making the following diagram commutative:

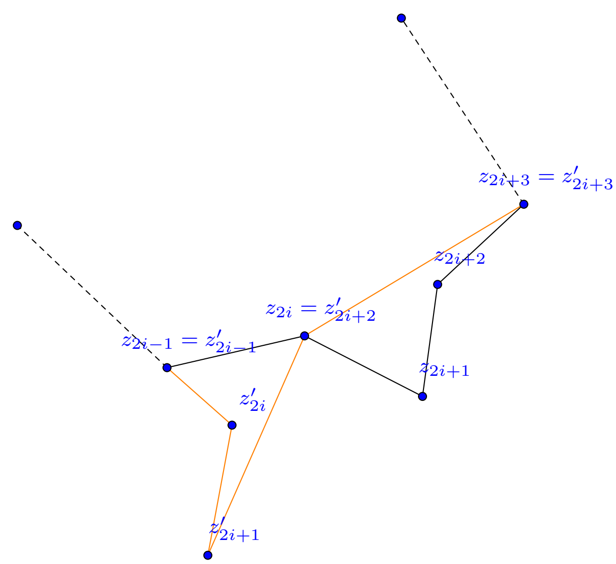

Figure 4.6: A geometric description of $\alpha_{0,i}$

$$\begin{array}{ccc} H_0(\kappa) & \xrightarrow{\alpha_{i,0}} & H_0(\kappa) \\ \Pi_{id} \Big\downarrow & & \Big\downarrow \Pi_{id} \\ H(\kappa) & \xrightarrow[\alpha_i]{} & H(\kappa) \end{array}$$

From the map α_i we obtain an isomorphism

$$\bar{\alpha}_i \ : \mathbb{P}H^+(\kappa) \to \mathbb{P}H^+(\kappa) \tag{4.90}$$

with the property that if $\bar{\psi}(z) = (x_1, , , , x_i, x_{i+1}, .., x_n)$ then $\bar{\psi}(\bar{\alpha}_i(z)) = (x_1, , , , x_{i+1}, x_i, .., x_n)$.

Remark 4.8.1. *If $\vec{k}$ is one of the following tuples*

- $(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$
- $(2, 2, 2, 2, 2, 2, 2)$
- $(3, 3, 3, 3)$

then α_i induces an isomorphism of the lattice $E(\vec{k})$. Similarly if $\vec{q}$ is the one of the following tuples

- $(1, 1, 1, 1, 1, 1, 1, 1, 1)$
- $(2, 2, 2, 2)$

then α_i induces an isomorphism of the lattice $G(\vec{q})$.

If $x \in \mathbb{P}H^+(\kappa)$ than we can find a loop $\gamma_i \in \bar{\phi}^{-1}(X(n)) \subset \mathbb{P}H^+(\kappa)$ starting at

x and ending at $\bar{\alpha}_i(x)$ such that $\bar{\phi}(\gamma_i)$ is homotopic to σ_i. This proves that

$$hol(\sigma_i) = \alpha_i. \tag{4.91}$$

Thus we have the following corollary:

Corollary 4.8.2. *If* $i < j$*,*

$$hol(A_{ij}) = \bar{\alpha}_{j-1}^{-1}\bar{\alpha}_{j-2}^{-1}...\bar{\alpha}_{i+1}^{-1}\bar{\alpha}_i^2\bar{\alpha}_{i+1}...\bar{\alpha}_{j-2}\bar{\alpha}_{j-1}. \tag{4.92}$$

When $i < j$, let

$$A_{ij}^* := \alpha_{j-1}^{-1}\alpha_{j-2}^{-1}...\alpha_{i+1}^{-1}\alpha_i^2\alpha_{i+1}...\alpha_{j-2}\alpha_{j-1} \tag{4.93}$$

A_{ij}^* is an isomorphism of $H(\kappa)$.

4.9. Triangulations

Now assume that the curvature data is of the below form:

$$\kappa = (k_1\frac{\pi}{3}, ..., k_n\frac{\pi}{3}),\ \ k_i = 1, 2, 3, 4, 5. \tag{4.94}$$

Let $\vec{k} = (k_1, ..k_n)$ and

$$E(\vec{k}) = \{z = (z_3, .., z_{2n}) \in H(\kappa) : z_i \in \mathbf{Eis}\} \tag{4.95}$$

Consider the set

$$\cup (E(\sigma\vec{k}) \cap U^*(\sigma\kappa)). \tag{4.96}$$

By gluing edges of an element z in this set as before one can get a labeled cone metric on sphere. Since coordinates of z are in **Eis**, this gluing process gives a labeled triangulation of S^2. Denote the set of triangulations obtained this way by

$$\bar{T}_0(\vec{k}, l). \tag{4.97}$$

Definition 4.9.1. *Two triangulations t_1 and t_2 in $\bar{T}_0(\vec{k}, l)$ are equivalent, $t_1 \sim t_2$, if there exists an orientation preserving isometry of corresponding cone metrics which respects labeling and sends vertices, edges, triangles of t_1 to vertices, edges, triangles of t_2, respectively. Set*

$$\bar{T}(\vec{k}, l) := \bar{T}_0(\vec{k}, l)/\sim . \tag{4.98}$$

The Alexandrov unfolding process implies that any sphere triangulation with appropriate curvature data is equivalent to some triangulation in $\bar{T}_0(\vec{k}, l)$. This is the motivation behind the previous definition.

Now assume that $\vec{k}$ is one of the following tuples:

- (1,1,1,1,1,1,1,1,1,1,1,1)
- (2,2,2,2,2,2)
- (3,3,3,3)

Take any element z in $E^+(\vec{k})$. Applying A^*_{ij}'s appropriately to z, one can obtain an element z' in $E(\vec{k})\cap(F_\sigma U^*(\sigma\kappa))$. The gluing process gives us an element $\bar{T}(\vec{k},l)$. Since α_i's and F_σ's are cutting gluing operations the triangulation that we obtained does not depend on z'. More precisely, if z'' is another such element then the triangulations that z' and z induces are same. Thus there exists a surjective map

$$\bar{\psi}(\vec{k}) : E(\vec{k})^+ \to \bar{T}(\vec{k},l). \tag{4.99}$$

Let $\Gamma(\vec{k})$ be the group of automorphisms of $E(\vec{k})$ generated by A^*_{ij}'s and δ, where δ is a primitive sixth root of unity. Action of δ is given by multiplications of element in $E(\vec{k})$ by δ.

Theorem 4.9.2. *If $\vec{k}$ is one of the followings:*

- *(1,1,1,1,1,1,1,1,1,1,1,1)*
- *(2,2,2,2,2,2)*
- *(3,3,3,3)*

then

$$\bar{T}(\vec{k}, l) \equiv E(\vec{k})^+/\Gamma(\vec{k}). \tag{4.100}$$

If $z \in E(\vec{k})^+$, *then number of triangles in the corresponding trianglulation is* $\frac{4}{\sqrt{3}} < z, z >$.

Proof. By construction

$$\bar{\psi}(\vec{k})(A^*_{ij}(z)) = \bar{\psi}(\vec{k})(z) \tag{4.101}$$

and obviously

$$\bar{\psi}(\vec{k})(\delta z) = \bar{\psi}(\vec{k})(z) \tag{4.102}$$

for all $z \in E^+(\vec{k})$. Thus $\bar{\psi}(\vec{k})$ induces a surjective map

$$E^+(\vec{k})/\Gamma \to \bar{T}(\vec{k}, l). \tag{4.103}$$

We show that this map is injective. Assume that two lattice points z'' and z' give same triangulation. This implies that they give same cone metric. Thus there exists $g \in \Gamma(\vec{k})$ such that g is a composition of A^*_{ij}s and

$$gz' = e^{\imath\beta} z''. \tag{4.104}$$

This implies that $e^{\imath\beta}z''$ and z'' gives same triangulation. Therefore $\beta = l\frac{\pi}{3}$, $l \in \mathbf{Z}$. Thus $hz' = z''$ for some $h \in \Gamma(\vec{k})$. This implies injectivity of the given map.
Second assertion follows from the fact that Hermitian form considered gives actually the area of the polygon corresponding to an element in the parameter space. $\square$

4.10. Quadrangulations

Assume that curvature data is of the below form:

$$\kappa = (q_1\frac{\pi}{3}, ..., q_n\frac{\pi}{3}),\ \ q_i = 1, 2, 3. \tag{4.105}$$

Let $\vec{q} = (q_1, ..q_n)$ and

$$G(\vec{q}) = \{z = (z_3, .., z_{2n}) \in H(\kappa) : z_i \in \mathbf{Z}[\imath]\} \tag{4.106}$$

Consider the set

$$\cup(G(\sigma\vec{q}) \cap U^*(\sigma\kappa)). \tag{4.107}$$

By gluing the edges of an element z in this set as before one can get a labeled cone metric on sphere. Since the coordinates of z are in $\mathbf{Z}[i]$, this gluing process gives a labeled quadrangulation of S^2. Denote the set of quadrangulations obtained by this way by

$$\bar{Q}_0(\vec{q}, l). \tag{4.108}$$

Definition 4.10.1. *Two quadrangulations t_1 and t_2 in $\bar{Q}_0(\vec{q}, l)$ are equivalent, $t_1 \sim t_2$, if there exists an orientation preserving isometry of corresponding cone metrics which respects labeling and sends vertices, edges, quadrangles of t_1 to vertices, edges, quadrangles of t_2, respectively. Let*

$$\bar{Q}(\vec{q}, l) := \bar{Q}_0(\vec{q}, l)/\sim . \tag{4.109}$$

Now assume that $\vec{q}$ is one of the following tuples:

- (1,1,1,1,1,1,1,1)
- (2,2,2,2)

Take any element z in $G^+(\vec{q})$. Applying A^*_{ij}'s appropriately to z, one can obtain an element z' in $G(\vec{q}) \cap (\cup F_\sigma(U^*(\sigma\kappa)))$. The gluing process gives us an element $\bar{T}(k, l)$. Since α_i's and F_σ's are cutting gluing operations the triangulation that we obtained does not depend on z'. More precisely, if z'' is another such element then the triangulations that z' and z'' induces are same. Thus there exists a surjective map

$$\bar{\psi}(q) : G(\vec{q})^+ \to \bar{Q}(\vec{q}, l). \tag{4.110}$$

Let $\Gamma(\vec{q})$ be the group of automorphisms of $G(\vec{q})$ generated by A^*_{ij}'s and $\imath = \sqrt{-1}$. Action of $\imath$ is given by multiplications of element in $G(\vec{q})$ by $\imath$.

Theorem 4.10.2. *If $\vec{q}$ is one of the followings:*

- *(1,1,1,1,1,1,1,1)*
- *(2,2,2,2)*

then

$$\bar{Q}(\vec{q}, l) \equiv G(\vec{q})^{+}/\Gamma(\vec{q}). \tag{4.111}$$

If $z \in G(\vec{q})^{+}$, *then number of quadrangles in the corresponding quadrangulation is* $< z, z >$.

Proof. Proof of this theorem is similar to proof of previous theorem. $\square$

APPENDIX A: More Information about Cone Metrics

In this appendix we will give detailed information about cone metrics defined on arbitrary compact surfaces. Our main references will be [5], [8].
We start with generalizing definition of cone metrics.

Definition A.0.3. *A triangulated surface is a surface S and a set of pairs $\mathbf{T} = \{(T_\alpha, f_\alpha)\}_{\alpha \in A}$ where each T_α is a compact subset of S and $f_\alpha : T_\alpha \to \mathbb{R}^2$ on a non-degenerate euclidean triangle such that*

- *T_α's cover S.*
- *If $\alpha \neq \beta$ then intersection of T_α and T_β is either empty or edge or a vertex.*
- *If $T_\alpha \cap T_\beta$ is not empty then there is an element $g_{\alpha\beta} \in E(2)$ (the group of isometries Euclidean plane) such that $f_\alpha = g_{\alpha\beta} f_\beta$.*

We define triangle edge and vertex accordingly.

Definition A.0.4. *A piecewise flat surface is a triangulated surface.*

Definition A.0.5. *A cone metric is a metric on a piecewise flat surface obtained by using given triangulation.*

Observe that for each point on a piecewise flat surface S there is well defined notion of angle θ_p for each point p.

Theorem A.0.6. *(Gauss-Bonnet) For any compact piecewise linear surface S without boundary we have*

$$\sum_{p \in S} (2\pi - \theta_p) = \chi(S) \tag{A.1}$$

where $\chi(S)$ is Euler charateristics of S.

A.1. Universal Branched Cover, Developing Map and Holonomy

A.1.1. Universal Branched Cover

Let S be a compact piecewise flat surface. Let $x_1 \dots, x_n$ be its singular points. Let

$$S' = S - \{x_1 \dots, x_n\}. \tag{A.2}$$

Let U' be the universal cover of S'. There is an induced metric μ on U'.

Definition A.1.1. *Metric completion of U', U, with respect to μ is called universal branched cover of S.*

A.1.2. Developing Map

Consider a piecewise linear surface S. Take a triangle T on it. There is triangle, $\bar{T}$ on U which maps isometrically to T by the covering map. There is a unique extension of this map to U which is local isometry.

Definition A.1.2. *Above extension is called developing map and will be denoted as*

$$dev : U \to \mathbb{R}^2. \tag{A.3}$$

A.1.3. Holonomy

Let S be a piecewise flat, compact, closed surface with singular points $x_1, \ldots, x_n$. Let

$$S' = S - \{x_1 \ldots, x_n\} \tag{A.4}$$

and also

$$\Pi := \Pi_1(S', p) \tag{A.5}$$

fundamental group based at p of S'.
Denote by τ_α deck transformation corresponding to $\alpha \in \Pi$. Observe that τ_α acts on U, universal branched cover of S, as an isometry. Let dev be a developing map. There is an element $hol(\alpha)$ in $E(2)$ such that

$$dev\tau_\alpha = hol(\alpha)dev \tag{A.6}$$

Lemma A.1.3. *The map*

$$hol : \Pi \to E(2) \tag{A.7}$$

obtained as above is a group homomorphism.

Definition A.1.4. *The map hol is called holonomy associated to developing map dev.*

A.2. Piecewise Flat Surfaces and Riemann Surfaces

Each piecewise flat, closed, oriantable surface can be considered as a Riemann surface. Indeed the metric gives natural coordinates at each point and change of coordinate maps are conformal. Observe that singularities disappear from conformal view of point. There is a theorem about converse direction.

Theorem A.2.1. *Let S be a closed, connected surface without boundary. Fix n distinct points $p_1 \ldots, p_n \in S$ and n reel numbers $\kappa_1, \ldots, \kappa_n \in (-\infty, 2\pi)$ such that*

$$\sum_{i=1}^{n} \kappa_i = 2\pi\chi(S). \tag{A.8}$$

Then, up to homothety, there is unique conformal cone metric having angle $2\pi - \kappa_i$ at p_i.

A.3. Hopf-Rinow Theorem, Arzela-Ascoli Theorem and Generalized Gauss-Bonnet Theorem

In this section we give statements of Hopf-Rinow and Arzela-Ascoli theorem. We also state another useful theorem.

Theorem A.3.1. *(Hopf-Rinow) Let M be a complete connected, piecewise flat surface. Then any two points in M can be joined by a shortest geodesic in M. Also given a vector $v \in T_pM$) (tangent cone at p) there is a unique geodesic with initial tangent vector v and which is defined in a neighborhood of p.*

Theorem A.3.2. *Let X be a metric space. If a sequence of continuos curves γ_n : $[a, b] \to X$is uniformly bounded and equicontinuous, then it contains a uniformly convergent subsequence.*

Theorem A.3.3. *Let M be a compact path metric space. If a sequence of constant speed curves $\gamma_n : [a, b] \to M$ has uniformly bounded lengths then the sequence contains a uniformly convergent subsequence.*

We present a generalization of Gauss-Bonnet theorem for surfaces with boundary.

Theorem A.3.4. *(Generalized Gauss-Bonnet) Let M be a piecewise flat surface. Let F be a compact region of it. Assume that the interior of F contains n singular points of angles $\theta_1 \ldots, \theta_n$ and boundary of F consists of geodesics with corner angles $\alpha_1, \ldots, \alpha_k$. We have*

$$\sum_{i=1}^{n}(2\pi - \theta_i) + \sum_{i=1}^{k}(\pi - \alpha_k) = 2\pi\chi(F). \tag{A.9}$$

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