TOEPLITZ MATRICES AND THEIR NUMERICAL SPECTRAL PROPERTIES

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THESIS SUBMITTED TO

THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

OF

THE ABANT İZZET BAYSAL UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

IN

THE DEPARTMENT OF MATHEMATICS

JULY 2008

Approval of the Graduate School of Natural and Applied Sciences.

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ABSTRACT

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July 2008, 45 pages

In this study , we investigate the eigenvalues of a non-hermitian Toeplitz matrix A. These are usually highly sensitive to perturbations,having condition numbers that increase exponentially with the dimension N. An equivalent statement is that the resolvent $(zI - A)^{-1}$ of a Toeplitz matrix may be much larger in norm than the eigenvalues alone would suggest-exponentially large as a function of N, even when z is far from the spectrum. Because of these facts, the meaningfulness of the eigenvalues of non-hermitian Toeplitz matrices for any but the most theoretical purposes should be considered suspect. In many applications it is more meaningful to investigate the $\varepsilon - pseudoeigenvalues$: the complex numbers z with $\left\|(zI - A)^{-1}\right\| \geq \varepsilon^{-1}$.

In the second part of study we investigate the pseudospectra of Linear Operators and analyzes the pseudospectra of Toeplitz matrices, and in particular relates them to the symbols of the matrices and thereby to the spectra of the General Toeplitz matrices. Our results are reasonably complete in the triangular case, and preliminary in the cases of non-triangular Toeplitz matrices with smoothly varying coefficients.

This study presents computed examples of pseudospectrum for the General Toeplitz matrices with using Mathematica and Matlab programming, and applications in numerical analysis.

Keywords: Eigenvalue, eigenvector, pseudoeigenvalue, spectrum, pseudospectrum, Toeplitz matrix.

ÖZET

TOEPLİTZ MATRİSLER VE ONLARIN NÜMERİK SPEKTRAL ÖZELLİKLERİ

Sarıhüseyinoğlu, Sanem

Master Tezi, Matematik Bölümü

Tez Yöneticisi: Yrd. Doç. Dr. Yusuf Cesur

Temmuz 2008, 45 sayfa

Bu çalışmada, hermitian olmayan bir A Toeplitz matrisinin özdeğerlerini araştırıyoruz. Genellikle bunlar küçük ötelemelere karşı aşırı duyarlıdırlar ve koşul sayıları N boyutuna bağlı olarak üstel olarak artmaktadır. Buna eşdeğer bir ifade ise bir Toeplitz matrisine ait özdeğerleri $(zI - A)^{-1}$ rezolvent fonksiyonu ile incelenebilmesidir. Rezolventin, spektruma ait z noktalarına çok yakın noktalarda N'e bağlı olarak üstel olarak hızla norm değerleri artar. Bu nedenle hermitian olmayan Toeplitz matrislerine ait birçok teorik çalışmada rezolventten yararlanılarak çalışılır. Bu bakış altında birçok uygulamada ε-ötelenmiş özdeğerleri araştırma işlemlerinde rezolventten yararlanılır. Kısaca bir z kompleks sayısında $\left\|(zI - A)^{-1}\right\| \geq \varepsilon^{-1}$ yardımıyla ε-ötelenmiş özdeğerleri incelenebilmektedir.

Çalışmanın ikinci kısmında ise Lineer Operatörlerin pseudospektrumu araştırılmaktadır. Burada ayrıca Toeplitz matrislerin spektrumlarına ait özellikleri semboller yardımıyla da açıklanmaktadır. Yaptığımız çalışmada elde ettiğimiz sonuçlar, üçgen ve üçgen olmayan Toeplitz matrislerinde ve değişken katsayılı durumlarda da elde edilmiştir.

Bu çalışmada Genel Toeplitz matrislerinin pseudospektrumu ile ilgili örneklerin araştırılmasında Mathematica ve Matlab programlarından ve nümerik analiz tekniklerin-

den yararlanılmıştır.

Anahtar kelimeler: Özdeğer, özvektör, ötelenmiş özdeğer, spektrum, pseudospektrum, Toeplitz matris.

ACKNOWLEDGEMENTS

I would like to acknowledge my indebtedness to my supervisor Assistant Prof. Dr. Yusuf Cesur who has guided, encouraged and believed in me throughout this thesis. I am grateful to him for a critical reading of my studies and valuable suggestions in correcting my errors. I also like to thank to the other members of my Committee, Associate Prof. Dr. Soner Durmuş and Assistant Prof. Dr. Erol Yılmaz for their assistance, and also to B.A.P for their financing help of our thesis.

It is also a pleasure to acknowledge a special dept of my gratitude to all the professors who gave me help in my master studies.

Finally, I would like to thank to my family for their un-vanishing support and encouragement.

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CHAPTER 1

INTRODUCTION

1.1 History of Pseudospectra

In this section we attempt to survey the history of the subject of pseudospectra. In the context of Hermitian and near-Hermitian systems, mathematical physicists have for years spoken of quasimodes, which are the same as what we would call pseudomodes or pseudo-eigenvectors.

In the nonnormal context, the earliest definition of pseudospectra we have encountered is given in J. M. Varah's 1967, The Computation of Bounds for the Invariant Subspaces of a General Matrix Operator [1]. Varah introduced the notion of an ε-pseudo-eigenvalue under the name r-approximate eigenvalue, as shown in Definition 1.1. Motivated by his analysis of the accuracy of eigenpairs produced by computer implementations of the inverse iteration algorithm, Varah incorporated a parameter, η_1, in this definition to describe floating-point precision.

Definition 1.1 $\{ \begin{matrix} \lambda \\ y \end{matrix} \}$ *is an r-approximate* $\{ \begin{matrix} \textit{eigenvalue} \\ \textit{eigenvector} \end{matrix} \}$ *of A if there exists a matrix E with* $\|E\|_2 = r.\eta_1$ *such that* $\{ \begin{matrix} \lambda \\ y \end{matrix} \}$ *is an exact* $\{ \begin{matrix} \textit{eigenvalue} \\ \textit{eigenvector} \end{matrix} \}$ *of* $A + E$*.*

This definition is first definition from Varah's unpublished 1967 thesis [1].

Landau applied this concept to the theory of Teoplitz matrices and associated integral operators.

Definition 1.2 *λ is an ε-approximate eigenvalue of A, if there exists $\varphi \in D(rQ)$, with $||\varphi|| = 1$, such that $||A_r\varphi - \lambda\varphi|| \leq \varepsilon$. We call φ an ε-approximate eigenfunction corresponding to λ.*

This definition is first published definition of pseudospectra from Landau, 1975 [2].

In Novosibirsk, S. K. Godunov and his colleagues conducted research related to pseudospectra throughout the 1980s. Those involved included A. G. Antonov, A. Y. Bulgakov (later Haydar Bulgak), O. P. Kirilyuk, V. I. Kostin, A. N. Malyshev, and S. I. Razzakov. Godunov, together with Ryabenkii and others, had made significant contributions in the 1960s to the study of how nonnormality affects the numerical stability of discretized differential equations. In fact, in their 1962 monograph, Godunov and Ryabenkii introduce the spectrum of the family of operators $\{R_h\}$, indexed by a mesh parameter h [3]. A point $z \in \mathbb{C}$ is in this set if, for every $\varepsilon > 0$, z is an ε-pseudo-eigenvalue of R_h for all sufficiently small h.

The Novosibirsk group defined the ε-spectrum by relative rather than absolute perturbations:

$$\sigma_\varepsilon(A) = \{z \in \mathbb{C} : \left\|(z - A)^{-1}\right\| \geqslant (\varepsilon\,||A||)^{-1}\}$$

In [4] and [5], computed plots of pseudospectra were presented and called ”spectral portraits of matrices” containing various ”patches of spectrum” , and computations based on both contour plotting and curve tracing were discussed. A sketch of two pseudospectra had appeared earlier in a 1985 paper by Kostin and Razzakov [6].

Pseudospectra were investigated in several papers by J. W. Demmel in the mid-1980s, appearing with the labels $S(A, \varepsilon)$ in [7] and $\sigma(\varepsilon, A)$ in [8]. The former paper contains the first published computer plot.

Beginning in the mid-1980s, D. Hinrichsen and A.J. Pritchard wrote a number of papers on the stability radius of a matrix, i.e., the distance to the set of unstable matrices. In a 1992 paper [9] they introduced the term spectral value set and notation $\sigma(A,\rho)$ to denote the real structured ε-pseudospectrum of a nonnormal matrix. Another paper by Hinrichsen and Kelb in 1993 extended these ideas to complex perturbations and also to more general structured perturbations [10].

Trefethen's first publications that mentioned pseudospectra were [11] and [12] in 1990 (the first uses the term ε-approximate eigenvalues). These were an outgrowth of a 1987 paper by Trefethen and Trummer, who found eigenvalues that were extraordinarily sensitive to perturbations but failed fully to appreciate their significance [13]. A crucial collabarator in this work was Trefethen's student Satish Reddy, who began to work with pseudospectra in 1998 and made many contributions after that date. Reddy's early work on these topics was summarized in his 1991 thesis at MIT [14].

In 1992 Trefethen's paper 'Pseudospectra of Matrices' appeared, which presented the idea of pseudospectra and exhibited thirteen examples [15]. It was after this point that the idea began to be widely known. A later companion paper, 'Pseudospectra of Linear Operators' , presented ten examples involving operators on infinite-dimensional spaces [16].

The papers by Demmel and Wilkinson mentioned above cite each other, and they both cite the paper of Varah [17]. Apart from these cases, none of the papers discussed here published before 1990 cite any of the others. These data suggest that pseudospectra have been invented at least five times:

J.M. Varah 1967 r-approximate eigenvalues , 1979 ε-spectrum

H.J. Landau 1975 ε-approximate eigenvalues

S.K. Godunov et al. 1982 spectral portrait

L.N. Trefethen 1990 ε-pseudospectrum

D. Hinrichsen and A.J. Pritchard 1992 spectral value set

1.2 Introduction

Many of the non-hermitian matrices that arise in applications have eigenvalues that are highly sensitive to perturbations. Indeed, for a family of matrices with variable dimension N, the sensitivity of the eigenvalues often increases exponentially as $N \to \infty$. In such circumstances eigenvalue analysis may lead to misleading conclusions, and the reasons lie deeper than the possibility of rounding errors on a computer. Highly sensitive eigenvalues are a reflection of the fact that the basis of eigenvectors is highly ill conditioned, and when this is the case, it is unlikely that there is good reason for working with that basis. Yet any statement about eigenvalues depends ultimately on the properties of the associated eigenvectors, whether or not they are mentioned explicitly.

We believe that in problems like this, a fruitful alternative is the analysis of pseudo-eigenvalues. In particular, the purpose of this thesis is to investigate the pseudospectra of various kinds of Toeplitz matrices, finite-difference equations, matrix iterations, spline approximation, signal processing, and other problems.

Traditional analysis of linear models is based upon eigenvalues, and for many problems across mathematics, science, and engineering, such analysis is successful. This is most notably true for self-adjoint matrices and operators, which possess a basis of orthogonal eigenvectors. Areas of successful applications of eigenvalue techniques include acoustics, structural analysis, quantum mechanics, low-Reynolds-number fluid mechanics, and numerical analysis.

In recent decades, recognition has grown that one must proceed with greater caution when a matrix or operator lacks an orthogonal basis of eigenvectors. Such operators are called non-normal, and this property can lead to a rich variety of behavior. For example, non-normality can be associated with transient behavior that differs entirely from the asymptotic behavior suggested by eigenvalues. Such transients may manifest themselves in slow convergence of iterative processes, in nearness to instability, and in the transition to turbulence in fluid flow. Numerous tools have been suggested to describe non-normality and analyze its effects. These include classical tools of matrix and operator theory, such as the numerical range, the angles between invariant subspaces, and the condition numbers of eigenvalues.

This thesis is devoted to describing and illustrating pseudospectra, a further tool that has proved useful in a variety of circumstances.

In the simplest context of matrices, the definitions are as follows. Let A be an $N \times N$ matrix with real or complex coefficients ; we write $A \in \mathbb{C}^{N \times N}$. Let v be a nonzero real or complex column vector of length N, and let z be a real or complex scalar; we write $v \in \mathbb{C}^N$ and $\lambda \in \mathbb{C}$. Then v is an eigenvector of A, and $\lambda \in \mathbb{C}$ is its corresponding eigenvalue, if

$$Av = \lambda v. \tag{1.1}$$

The set of all the eigenvalues of A is the spectrum of A, a nonempty subset of the complex plane $\mathbb{C}$ that we denote by $\sigma(A)$.

$$\sigma(A) = \{\lambda \in \mathbb{C} : |\lambda I - A| = 0\}$$

The spectrum can also be defined as the set of points $\lambda \in \mathbb{C}$ where the resolvent matrix,

$$(\lambda - A)^{-1}$$

does not exist. $(\lambda - A)$ is shorthand for $(\lambda I - A)$, where I is the identity.[18]

If λ is an eigenvalue of A, then by convention we define the norm of $(\lambda - A)^{-1}$ to be infinity. But what if $\left\|(\lambda - A)^{-1}\right\|$ is finite but very large? This pattern of thinking leads to Definition 1.3 (*iii*) of pseudospectra.

Let $\|.\|$ denote the 2-norm. The definition of pseudo-eigenvalues is as follows:

Definition 1.3 *Given $\varepsilon > 0$, the number $\lambda \in \mathbb{C}$ is an ε-pseudo-eigenvalue of A if any of the following equivalent conditions is satisfied:*

(i) λ is an eigenvalue of $A + E$ for some $E \in \mathbb{C}^{N\times N}$ with $\|E\| \leqslant \varepsilon$;

(ii) $\exists u \in \mathbb{C}^N, \|u\| = 1,$ such that $\|(A - \lambda)u\| \leqslant \varepsilon$;

(iii) $\left\|(\lambda - A)^{-1}\right\| \geqslant \varepsilon^{-1}$;

(iv) $\sigma_N(\lambda - A) \leqslant \varepsilon$.

The set of all ε-pseudo-eigenvalues of A, ε-pseudospectrum, is denoted by $\sigma_\varepsilon(A)$ or $\Lambda_\varepsilon(A)$ or simply Λ_ε.

Equivalently, the ε-pseudospectrum can be defined in terms of eigenvalues of perturbed matrices.

If λ is in the ε-pseudospectrum, it is an ε-pseudo-eigenvalue of A. With each pseudo-eigenvalue we can associate a pseudo-eigenvector (non-unique, in general), and these quantities lead to a second definition of pseudospectra.

For any matrix $A \in \mathbb{C}^{N\times N}$, the definitions (*i*), (*ii*) and (*iii*) are equivalent.

If A is an operator of infinite dimension instead of a matrix, we take condition (*iii*) to be the definition of $\Lambda_\varepsilon(A)$. Equivalents of the other definitions can be obtained with slight modifications to ensure that the sets are closed. Our definitions generalize readily from matrices to operators, and also to other norms, but we shall not pursue such generalizations here; see [19].

If the matrix or operator A is normal (i.e., it has an orthogonal basis of eigenvectors), then its 2-norm ε-pseudospectrum consists of closed balls of radius ε surrounding the eigenvalues. For non-normal matrices, the pseudospectra can be much larger, and thus much more interesting.

More generally (in any norm), trouble arises if the basis of eigenvectors is ill-conditioned. If a vector is written in terms of this basis, the expansion coefficients may be huge relative to the size of the vector itself. The ”physics” of the system may then be dominated by the evolving pattern of cancellation of these coefficients, rather than by the behavior of individual eigenvalues.

Consider a simple example, the 2-norm pseudospectra of a matrix A of dimension 30, illustrated in Figure 1.1. The eigenvalues are plotted as black dots on the complex plane, and colored lines mark the boundaries of various pseudospectra. The color bar on the right indicates the $log10$ of each boundary, so in Figure 1.1 we draw the boundaries of the ε-pseudospectra for $\varepsilon = 10^{-0.5}$, 10^{-1}, $10^{-1.5}$, 10^{-2} and $10^{-2.5}$. Notice that for some values of ε, the pseudospectrum is connected, while for smaller ε it can consist of disjoint sets. Sometimes the pseudospectral boundary about an eigenvalue is too small to be clearly visible on our plots. In Figure 1.1, this is the case for the $10^{-2.5}$ contour about the eigenvalue with largest imaginary part.

$$A = \begin{bmatrix} 0 & 2 & \cdots & \cdots & \\ 0 & 0 & 2 & \cdots & \cdots \\ 2 & 0 & 0 & 2 & \cdots \\ \cdots & 2 & 0 & 0 & 0 \\ & \cdots & 2 & 0 & 0 \end{bmatrix} \tag{1.2}$$

with the symbol;

$$f(z) = 2z^{-2} + 2z \tag{1.3}$$

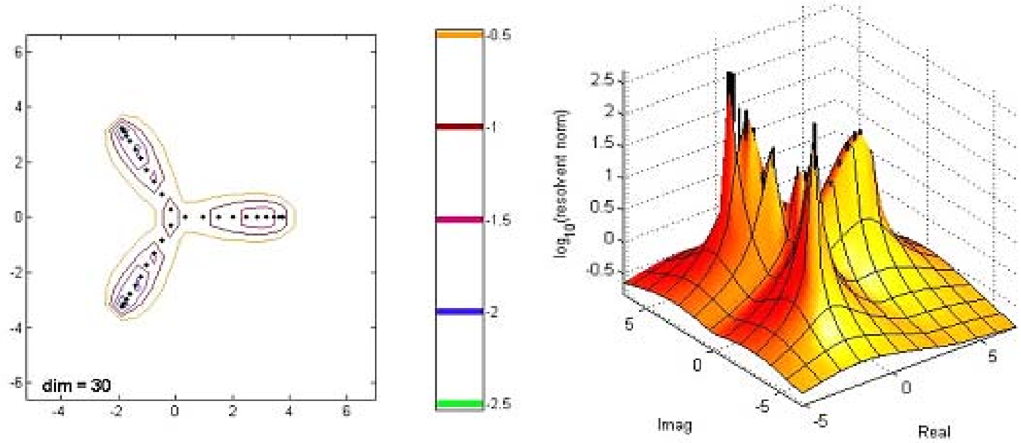

Figure 1.1: Spectrum (black dots) and boundaries (colored lines) of five ε-pseudospectra for a matrix of dimension 30.

Areas where non-normality is important include: hydrodynamic instability, matrix iterations, meteorology, Markov chains, control theory, and analysis of high-powered lasers.[20]

Toeplitz matrices provide one of the most compelling applications of pseudospectra. A Toeplitz matrix has constant entries on each diagonal, and the corresponding infinite-dimensional Toeplitz operator is a singly-infinite matrix. The constants on the diagonals are the Laurent coefficients of the symbol, a complex-valued function whose domain is the unit circle, S. The spectrum of the Toeplitz operator is determined by the symbol, f. If f is continuous, then the spectrum is the $f(S)$ together with all points this curve encloses with non-zero winding number. The eigenvalues of finite Toeplitz matrices are very different, typically falling on curves in the complex plane for arbitrarily large (but finite) dimensions. It is well known that these eigenvalues are typically difficult to compute accurately.

It turns out that while the eigenvalues of finite Toeplitz matrices do not generally converge to the spectrum of the corresponding infinite-dimensional operator, the pseudospectra of the finite matrices do converge to the operator pseudospectra for a broad class of symbols.

CHAPTER 2

PSEUDOSPECTRA

2.1 Basic Definitions

Definition 2.1 *Normed Space, Banach Space :*

A normed space X is a vector space with a norm defined on it. A Banach space is a complete normed space (complete in the metric defined by the norm; see (2.1)*, below). Here a norm on a (real or complex) vector space X is a real-valued function on X whose value at an* $x \in X$ *is denoted by* $\|x\|$ *and which has the properties;*

$(N1)$ $\|x\| \geqslant 0$

$(N2)$ $\|x\| = 0 \Leftrightarrow x = 0$

$(N3)$ $\|\alpha x\| = |\alpha|\,\|x\|$

$(N4)$ $\|x + y\| \leqslant \|x\| + \|y\|$ *(Triangle Inequality);*

here x and y are arbitrary vectors in X and α *is any scalar.*

A norm on X defines a metric d on X which is given by

$$d(x, y) = \|x - y\| \qquad (\text{ x, y} \in \text{X }) \tag{2.1}$$

and is called the metric induced by the norm. The normed space just defined is denoted by $(X, \|.\|)$ *or simply by X.*

Definition 2.2 *The Euclidean norm of a vector* $x = [x_1, x_2, ..., x_n] \in \mathbb{C}^N$ *is :*

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2} \tag{2.2}$$

Definition 2.3 *The Euclidean norm of a matrix $A_{m\times n}$ is :*

$$||\mathrm{A}||_2 = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}\left|a_{ij}\right|^2} \tag{2.3}$$

where a_{ij} is the i-th row and j-th column of the matrix A.

Definition 2.4 *Linear Operator:*

A linear operator A is an operator such that

(i) the domain $D(A)$ of A is a vector space and the range $R(A)$ lies in a vector space over the same field,

(ii) for all $x, y \in D(A)$ and scalars α,

$$A(x+y) = Ax + Ay$$

$$A(\alpha x) = \alpha Ax. \tag{2.4}$$

Definition 2.5 *Bounded Linear Operator:*

Let X and Y be normed spaces and $A : D(A) \to Y$ a linear operator,where $D(A) \subset X$. The operator A is said to be bounded if there is a real number c such that for all $x \in D(A)$.

$$||Ax|| \leq c.\,||x|| \tag{2.5}$$

Definition 2.6 *(Eigenvalue, Eigenvectors, Spectrum and Resolvent set of a matrix in finite dimensional spaces):*

An eigenvalue of a square matrix $A = (a_{ij})$ is a number λ such that

$$Av = \lambda v$$

has a solution $v \neq 0$. This v is called an eigenvector of A corresponding to that eigenvalue λ. The eigenvectors corresponding to that eigenvalue λ and the zero vector *form a vector subspace of X which is called the eigenspace of A corresponding to that eigenvalue λ.*

The set $\sigma(A)$ of all eigenvalues of A is called the spectrum of A. Its complement $\rho(A) = \mathbb{C} - \sigma(A)$ in the complex plane is called resolvent set of A.

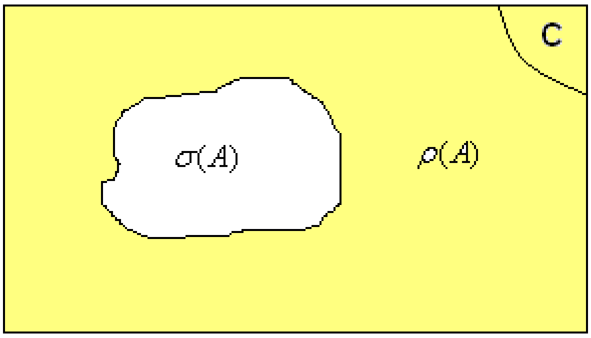

Figure 2.1: where $\rho(A) = \mathbb{C} - \sigma(A)$.

Definition 2.7 *(Regular value, Resolvent set and Spectrum in infinite dimensional spaces):*

Let $X \neq \{0\}$ be a complex normed space and $A : D(A) \to X$ a linear operator with domain $D(A) \subset X$. A **regular value z** *of A is a complex number such that;*

$(R1)$: *$R_z(A) = (zI - A)^{-1}$ exists.*

$(R2)$: *$R_z(A)$ is bounded.*

$(R3)$: *$R_z(A)$ is defined on a set which is dense in X.*

where the matrix $R_z(A) = (zI - A)^{-1}$ is known as the resolvent of A at z.

The resolvent set $\rho(A)$ of A is the set of all regular values z of A.

It's complement $\sigma(A) = \mathbb{C} - \rho(A)$ in the complex plane is called the spectrum of operator A, and a $z \in \sigma(A)$ is called spectral value of A.

Point Spectrum: *The point spectrum or discrete spectrum $\sigma_p(A)$ is the set such that $R_z(A)$ doesn't exist. A $z \in \sigma_p(A)$ is called an eigenvalue of A.*

Continuous Spectrum: *The continuous spectrum $\sigma_c(A)$ is the set such that $R_z(A)$ exists and satisfies (R3) but are not (R2), that is , $R_z(A)$ is bounded.*

Residual Spectrum: *The residual spectrum $\sigma_r(A)$ is the set such that $R_z(A)$ exists (and may be bounded or not) but does not satisfy (R3) that is the domain of $R_z(A)$ is not dense on X.*

If X is infinite dimensional, then A can have spectral values which are not eigenvalues.

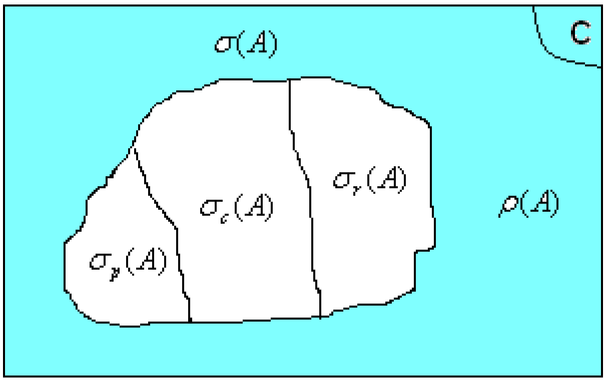

Figure 2.2: where $\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A)$ and $\rho(A) = \mathbb{C} - \sigma(A)$.

Definitions of Pseudospectra:

Definition 2.8 *The $\sigma_\varepsilon(A)$ is the set of $z \in \mathbb{C}$ such that*

$$z \in \sigma(A + E) \tag{2.6}$$

for some $E \in \mathbb{C}^{NxN}$ with $\|E\| < \varepsilon$.

The ε-pseudospectrum is the set of numbers that are eigenvalues of some perturbed matrix $A + E$ with $\|E\| < \varepsilon$.

Definition 2.9 *$\sigma_\varepsilon(A)$ is the set of $z \in \mathbb{C}$ such that*

$$\|(z - A)v\| < \varepsilon \tag{2.7}$$

for some $v \in \mathbb{C}^N$ with $\|v\| = 1$.

The number z in (2.7) *(or equivalently in any of our definitions) is an ε-pseudo-eigenvalue of A, and v is a corresponding ε-pseudo-eigenvector. In words, the ε-pseudospectrum is the set of ε-pseudo-eigenvalues.*

Definition 2.10 *Let $A \in \mathbb{C}^{NxN}$ and $\varepsilon > 0$ be arbitrary.*

The ε-pseudospectrum $\sigma_\varepsilon(A)$ of A is the set of $z \in \mathbb{C}$ such that

$$\sigma_\varepsilon(A) = \{z : \left\|(zI - A)^{-1}\right\| > \varepsilon^{-1}, z \in \mathbb{C}\} \tag{2.8}$$

Definition 2.11 *For $\|.\| = \|.\|_2$, $\sigma_\varepsilon(A)$ is the set of $z \in \mathbb{C}$ such that*

$$s_{\min}(z - A) < \varepsilon. \tag{2.9}$$

From $\left\|(z - A)^{-1}\right\|_2 = [s_{\min}(z - A)]^{-1}$ it is clear that (2.9) *is equivalent to* (2.8) *and therefore also to our other characterizations of pseudospectra.*

Definition 2.12 *A Laurent operator A is a bounded linear operator on $l_2(\mathbb{Z})$ with the property that the matrix of A with respect to the standard orthonormal basis $\{e_j\}_{j=-\infty}^{\infty}$ of $l_2(\mathbb{Z})$ is of the form*

$$\begin{bmatrix} \ddots & \ddots & & & \\ \ddots & a_0 & a_{-1} & a_{-2} & \\ & a_1 & a_0^* & a_{-1} & \\ & a_2 & a_1 & a_0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}$$

Here a_0^* denotes the entry a_0 located in the zero row zero column position. In other words, a bounded linear operator A on $l_2(\mathbb{Z})$ is a Laurent operator if and only if $< Ae_k, e_j >$ depends on the difference $j - k$ only.

Definition 2.13 *Let A be a bounded linear operator on l_2. In the sequel the formula*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \\ \vdots & \vdots & & \ddots \end{bmatrix} \tag{2.10}$$

means that the matrix in the right hand side of (2.10) *is the matrix corresponding to A and the standard orthonormal basis $e_1, e_2, ...$ in l_2. We say that A is a Toeplitz operator if*

$$A = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \cdots \\ a_2 & a_1 & a_0 & \\ \vdots & \vdots & & \ddots \end{bmatrix} \tag{2.11}$$

In this case we refer to the right hand side of (2.11) *as the standard matrix representation of A. Recall that $e_j = (0, ..., 0, 1, 0, ...)$, where the one appears in the j-th entry. Thus A is a Toeplitz operator if and only if $< Ae_k, e_j >$ depends only on the difference $j - k$.*

Definition 2.14 *Formally, a function f is complex analytic on an open set D in the complex line if for any z_0 in D one can write*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + ...$$

in which the coefficients $a_0, a_{1,}...$ are complex numbers and the series is convergent for z in a neighborhood of z_0.

Alternatively, an analytic function is an infinitely differentiable function such that the Taylor series at any point z_0 in its domain

$$A(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

is convergent for z close enough to z_0 *and its value equals* $f(z)$.

2.2 Pseudospectra of Linear Operators

In this section we set down the fundamentals of the theory of pseudospectra for linear operators in Banach space. Fortunately, the technical details do not matter for many applications, and we use a language closer to linear algebra than functional analysis.

Let X be a complex Banach space, that is, a complete normed vector space over the complex field $\mathbb{C}$ with norm $\|.\|$. We shall consider linear operators mapping X into itself. Such an operators A has a domain denoted by $D(A) \subseteq X$, which may or may not be all of X. We denote by $B(X)$ the set of bounded operators on X ; for $A \in B(X)$, we assume without loss of generality $D(A) = X$. We denote by $C(X)$ the set of closed operators on X. (An operators A is closed provided that if $\{u_k\}$ is a sequence in $D(A)$ converging to a limit $u \in X$ and if $\{Au_k\}$ converges to a limit $v \in X$, then $u \in D(A)$ and $Au = v$.) An unbounded closed operator will necessarily have $D(A) \neq X$, although many such operators are densely defined, meaning that the closure of $D(A)$ is X.

For $A \in C(X)$ and $E \in B(X)$, $A+E$ is also in $C(X)$, with domain $D(A+E) = D(A)$. A special case is the situation where E is a multiple of the identity. Thus perturbations of closed operators by bounded operators or shifts introduce no technical difficulties.

Given $A \in C(X)$, a bounded inverse is an operator $A^{-1} \in B(X)$ such that AA^{-1} is the identity on X and $A^{-1}A$ is the identity on $D(A)$. This is the only kind of inverse we shall be concerned with, and when an expression like A^{-1} or $(z - A)^{-1}$ appears, it should always be interpreted as a bounded inverse, defined on all of X.

Invertibility and perturbation of closed operators:

Theorem 2.15 *Suppose $A \in C(X)$ has a bounded inverse A^{-1}. Then for any $E \in B(X)$ with*

$$\|E\| < \frac{1}{\|A^{-1}\|}$$

$A + E$ has a bounded inverse $(A + E)^{-1}$ satisfying

$$\|(A + E)^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|E\| \, \|A^{-1}\|} \tag{2.12}$$

Conversely, for any $\mu > \frac{1}{\|A^{-1}\|}$ there exists $E \in B(X)$ with $\|E\| < \mu$ such that $(A + E)u = 0$ for some nonzero $u \in X$.

The theory of resolvents, spectra, and pseudospectra is derived by applying Theorem 2.15 to shifted operators $z - A$, where z is a complex constant. Given $A \in C(X)$ and $z \in \mathbb{C}$, the resolvent of A at z is the operator $(z - A)^{-1} \in B(X)$, if this exists. The resolvent set $\rho(A)$ is the set of numbers $z \in \mathbb{C}$ for which $(z - A)^{-1}$ exists. Theorem 2.15 implies that $\rho(A)$ is open. In fact, $(z - A)^{-1}$ is an analytic function of $z \in \rho(A)$, which implies that $\|(z - A)^{-1}\|$ is an unbounded continuous subharmonic function of $z \in \rho(A)$ and that it satisfies the maximum principle for $z \in \rho(A)$.[21].

The *spectrum* of $A \in C(X)$ is the complement of the resolvent set in the complex plane: $\sigma(A) = \mathbb{C} - \rho(A)$. Since $\rho(A)$ is open, $\sigma(A)$ is closed. For $A \in B(X)$, $\sigma(A)$ is bounded and nonempty. For $A \in C(X)$, $\sigma(A)$ may be unbounded or empty.

For any $A \in C(X)$ and $z \in \rho(A)$, we have

$$\|(z - A)^{-1}\| \geq \frac{1}{dist(z, \sigma(A))}$$

Thus $\|(z - A)^{-1}\|$ approaches ∞ as z approaches the spectrum. We now introduce a notational convention.

Convention: If $z \in \sigma(A)$, we write $\left\|(z - A)^{-1}\right\| = \infty$.

Thus for $z \in \sigma(A)$, we shall use the notation $\left\|(z - A)^{-1}\right\|$ even though , $(z - A)^{-1}$ itself does not exist. If A is a matrix on a finite-dimensional space, this usage is very natural since any $z \in \sigma(A)$ must be an eigenvalue, so $(z - A)^{-1}$ is indeed '$infinite$'. For operators, it is more artificial, since there may be points in the spectrum that are not eigenvalues. This convention gives us the ability to derive a wide range of results about resolvents and perturbations from a single fact: $\left\|(z - A)^{-1}\right\|$ is a continuous function from the entire complex plane to $(0, \infty]$.

Norm Of The Resolvent:

Theorem 2.16 *Given $A \in C(X)$, and with $\left\|(z - A)^{-1}\right\|$ defined as ∞ for $z \in \sigma(A)$, the norm of resolvent $\left\|(z - A)^{-1}\right\|$ is a function from $z \in \mathbb{C}$ to $(0, \infty]$ with the following properties. It is continuous and unbounded and takes the value ∞ precisely on $\sigma(A)$. For $z \notin \sigma(A)$ it is subharmonic and satisfies the maximum principle as well as the bound*

$$\left\|(z - A)^{-1}\right\| \geq \frac{1}{dist(z, \sigma(A))} \tag{2.13}$$

If $z \notin \sigma(A)$, then $z \notin \sigma(A + E)$ for any $E \in B(X)$ that satisfies

$$||E|| \leq \frac{1}{\left\|(z - A)^{-1}\right\|}$$

conversely, for any $\mu > \left\|(z - A)^{-1}\right\|^{-1}$, there exists $E \in B(X)$ with $||E|| < \mu$ such that $(A + E)u = zu$ for some nonzero $u \in X$.

According to Theorem 2.16 every z that is not in $\sigma(A)$ is also not in $\sigma(A + E)$ for sufficiently small $||E||$.

We defined ε-pseudospectrum of a matrix A in three equivalent ways. The same three definitions apply for linear operators in Banach space. Statements of aspects of this equivalence in various contexts can be found in a number of publications, including [22,23,24].

Three equivalent definitions of pseudospectra:

Let $A \in C(X)$ and $\varepsilon > 0$ be arbitrary. The ε-pseudospectrum $\sigma_\varepsilon(A)$ of A is the set of $z \in \mathbb{C}$ defined equivalently by any of the conditions

$$\left\|(z - A)^{-1}\right\| > \varepsilon^{-1}, \tag{2.14}$$

$$z \in \sigma(A + E) \text{ for some } E \in B(X) \text{ with } \|E\| < \varepsilon, \tag{2.15}$$

$$z \in \sigma(A) \text{ or } \|(z - A)u\| < \varepsilon \text{ for some } u \in D(A),\ \|u\| = 1. \tag{2.16}$$

If $\|(z - A)u\| < \varepsilon$, then z is an ε-pseudo-eigenvalue of A and u is a corresponding ε-pseudo-eigenvector (or pseudoeigenfunction or pseudomode).

From the material above we obtain the following collection of facts about pseudospectra in Banach space Figure 2.3 gives a schematic view.

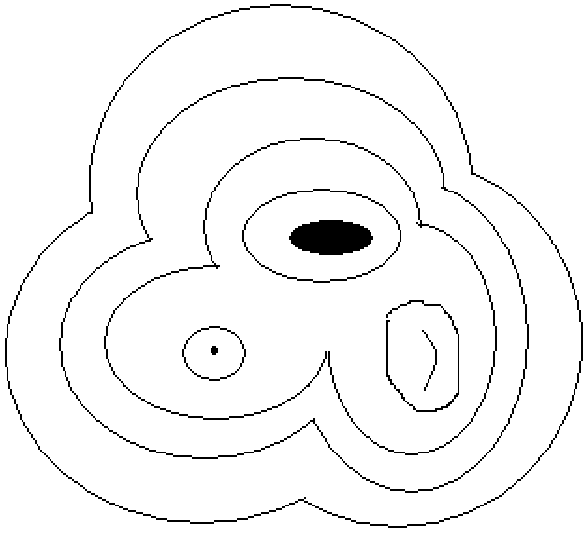

Figure 2.3: Schematic view of the geometry of pseudospectra for an operator in Banach space. The black region and the thick arc indicate components of the spectrum that are not just points. For z in the interior of an open region of spectrum, though z is in the ε-pseudospectrum for any $\varepsilon > 0$, eigenvectors and ε-pseudo-eigenvectors may or may not exist. This sketch corresponds to a bounded operator. For an unbounded operator, though the spectrum may still be bounded, all the pseudospectra will be unbounded.

Properties of pseudospectra:

Theorem 2.17 *Given $A \in C(X)$, the pseudospectra $\{\sigma_\varepsilon(A)\}_{\varepsilon>0}$ have the following properties. They can be defined equivalently by any of the conditions* (2.14) – (2.16)*. Each $\sigma_\varepsilon(A)$ is a nonempty open subset of $\mathbb{C}$, and any bounded connected component of $\sigma_\varepsilon(A)$ has a nonempty intersection with $\sigma(A)$. The pseudospectra are strictly nested supersets of the spectrum:*

$$\cap_{\varepsilon>0}\sigma_\varepsilon(A) = \sigma(A),$$

and conversely, for any $\delta > 0$,

$$\sigma_{\varepsilon+\delta}(A) \supseteq \sigma_\varepsilon(A) + \Delta_\delta,$$

where Δ_δ *is the open disk of radius* δ*.*

Proof.

The equivalence of the three definitions follows from Theorem 2.16. The condition $\sup_{z\in\rho(A)} \left\|(z-A)^{-1}\right\| = \infty$ implies that $\sigma_\varepsilon(A)$ is nonempty, and the maximum principle for $\left\|(z-A)^{-1}\right\|$ implies that any bounded complement of $\sigma_\varepsilon(A)$ intersects the spectrum. (Thus if $\sigma(A)$ is empty, $\sigma_\varepsilon(A)$ is unbounded for all $\varepsilon > 0$.) The statement about $\cap\sigma_\varepsilon(A)$ follows from the upper-semicontinuity of the spectrum. □

CHAPTER 3

TOEPLITZ MATRICES

3.1 General Toeplitz Matrices

Definition 3.1 *Any $n \times n$ matrix A of the form*

$$A = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ a_{-1} & a_0 & \ddots & \vdots & \vdots \\ a_{-2} & a_{-1} & \ddots & a_1 & a_2 \\ \vdots & \vdots & \ddots & a_0 & a_1 \\ a_{1-n} & a_{2-n} & \cdots & a_{-1} & a_0 \end{bmatrix} \tag{3.1}$$

is a Toeplitz Matrix. If the i , j element of A is denoted A_{ij} , then we have $A_{ij} = a_{j-i}$. Let $f(z)$ denote the symbol

$$f(z) = \sum_{k=-\infty}^{\infty} a_k.z^k = ... + a_{-1}.z^{-1} + a_0.z^0 + a_1.z^1 + ... \tag{3.2}$$

with

$$\sum_{k=-\infty}^{\infty} |a_k| < \infty. \tag{3.3}$$

Theorem 3.2 *Let A and f be as describe above.*

(i) If A is a Laurent operator, $\Lambda(A) = f(S)$.

(ii) If A is a Toeplitz operator, $\Lambda(A) = f(\Delta)$.

(iii) If A is a Toeplitz matrix, $\Lambda(A) = f(\{0\}) = \{a_0\}$.

In the case of Toeplitz or Laurent operators, the spectrum Λ of A is fully understood:

Theorem 3.3 *Let A and f be as described above.*

(i) If A is a Laurent operator, $\Lambda(A) = f(S)$.

(ii) If A is Toeplitz operator, $\Lambda(A) = f(S) \cup \{\lambda \in \mathbb{C} : I(f(S), \lambda) \neq 0\}$.

Here and in the theorem below, $I(f(S), \lambda)$ denotes the winding number or index of the continuous curve $f(S)$ about the point $\lambda \in \mathbb{C}$, and where S is the unit circle.

$$I(f(S), \lambda) = \frac{1}{2\pi i} \int_S \frac{f'(z)}{f(z) - \lambda} dz, \qquad \lambda \notin \mathrm{f(S)}.$$

If $\lambda \in f(S)$, then $I(f(S), \lambda)$ is undefined.

Theorem 3.3 (*ii*) asserts that the spectrum of a Toeplitz operator can be divided into three parts, and these can be interpreted as follows:

$$\begin{aligned} I(f(S), \lambda) > 0 &: \text{geometrically decreasing right eigenvectors,} \\ I(f(S), \lambda) < 0 &: \text{geometrically decreasing left eigenvectors,} \\ \lambda \in f(S) &: \text{approximate eigenvectors with no geometric} \\ &\quad \text{increase or decrease.} \end{aligned} \tag{3.4}$$

These interpretations will be the key to our analysis of pseudospectra below.

Proof. As mentioned in the proof of Theorem 3.2 (*i*) carries over to the present case unchanged. The result for Laurent operators was first proved by Toeplitz in 1911 [25]. As for (*ii*), this result was first proved independently around 1958 by Krein [26] and Calderon, Spitzer, and Widom [27], with the regularity assumption (3.3), and generalized to arbitrary continuous symbols by Devinatz in 1964 [28]. A proof based on (3.3) goes as follows. Without loss of generality consider $\lambda = 0$, and suppose first $\lambda \notin f(S) \cup \{\lambda \in \mathbb{C} : I(f(S), \lambda) \neq 0\}$, that is, $I(f(S), \lambda) = 0$. Then $f(z)$ has a continuous logarithm $L(z)$ on S which is also in the Wiener class, and if we divide the Laurent series of $L(z)$ into analytic and coanalytic parts with respect to S, $L(z) = L_+(z) + L_-(z)$, then $L_+(z)$ and $L_-(z)$ are also in the Wiener class. This gives us a factorization of the symbol,

$$f(z) = e^{L(z)} = e^{L_+(z)+L_-(z)} = e^{L_+(z)}e^{L_-(z)} = f_+(z)f_-(z),$$

which corresponds to factorization $A = A_+A_-$ of the Toeplitz operator into upper and lower triangular Toeplitz operators of the kind considered in Theorem 3.2 (*ii*). Since $f_+(z)$ and $f_-(z^{-1})$ are nonzero in Δ, where $\Delta = D \cup S$, D is the open unit disk in the complex plane, S is the unit circle, and $f(z)$ is analytic in D and continuous in Δ, that theorem implies that A_+ and A_- are both invertible, and therefore the same is true for their product A.

Conversely suppose $\lambda \in f(S) \cup \{\lambda \in \mathbb{C} : I(f(S), \lambda) \neq 0\}$. If $\lambda \in f(S)$, then as indicated in (3.4), A may have no eigenvectors, but ε-pseudo-eigenvectors can be constructed by multiplying vectors $(1, z, z^2, ...)^T$ for $z \in S$ by a smooth envelope to reduce edge effects. By making the envelope sufficiently smooth, ε can be made arbitrarily small, and this implies $\lambda \in \Lambda$. On the other hand, suppose $I(f(S), \lambda) \neq 0$, or, without loss of generality (by symmetry), $I(f(S), \lambda) = n > 0$. The arguments below will show that A has geometrically decreasing ε-pseudo-eigenvectors for arbitrarily small ε, so again, $\lambda \in \Lambda$. A more standard proof notes that by the argument of the last paragraph, the Toeplitz operator defined by the symbol $z^{-n}f(z)$ is invertible. If A were invertible too, this would imply the invertibility of the Toeplitz operator defined by the symbol $z^{-n}f(z)/f(z) = z^{-n}$, namely the n-th power of a shift operator, which would be a contradiction. □

If N is large and ε is small, then in analogy to (3.4), Λ_ε looks approximately like the union of three sets;

$$\sigma_\varepsilon = \Lambda_\varepsilon = \Omega_r \cup \Omega^R \cup (\Lambda + \Delta_\varepsilon) \tag{3.5}$$

We must explain this notation. For any $r < 1$ and $R > 1$, the sets Ω_r and Ω^R are defined by

$$\Omega_r = \{z \in \mathbb{C} : I(f(S_r), z) > 0\}, \qquad \Omega^R = \{z \in \mathbb{C} : I(f(S_R), z) < 0\}. \tag{3.6}$$

where S_r and S_R are circles. In (3.5), appropriate values are

$$r = (\varepsilon/c)^{1/N}, \qquad R = (\varepsilon/C)^{-1/N}, \qquad c_N = \sum_{k=1}^{N-1} |a_k|. \tag{3.7}$$

for some constants c and C analogous to c_N. (In the figures below, c and C are taken equal to 1.) The sets Ω_r and Ω^R correspond to geometrically decreasing right and left pseudo-eigenvectors of A, respectively. Equivalently, they correspond to geometrically decreasing and increasing right pseudo-eigenvectors. Finally, the set $\Lambda + \Delta_\varepsilon$ in (3.5) consists of the union of the ε-balls about the eigenvalues of A, and corresponds to pseudo-eigenvectors with no geometric increase or decrease.

3.2 Matrix Examples

To illustrate (3.5), let us begin with the tridiagonal matrix

$$A = \begin{pmatrix} 0 & 3 & & & \\ 1 & 0 & 3 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 3 \\ & & & 1 & 0 \end{pmatrix} \tag{3.8}$$

of dimension N. Since A can be symmetrized by a similarity transformation involving the matrix $D = diag(1, 3^{1/2}, 3^{1}, ..., 3^{N-1/2})$, its eigenvalues are real:

$$\sigma(A) = \Lambda = \{2\sqrt{3}\cos\frac{\pi j}{N+1}, \quad 1 \leqslant j \leqslant N\} \tag{3.9}$$

The condition number of D is exponentially large as a function of N, however, suggesting that the exact eigenvalues of A are unlikely to be very meaningful. Figure (3.1) shows that in fact the ε-pseudo-eigenvalues of A with $\varepsilon = 10^{-2}$ and $N = 100$ lie approximately along an ellipse. To explain this distribution, note that the symbol for this example is

$$f(z) = 3z + z^{-1} \tag{3.10}$$

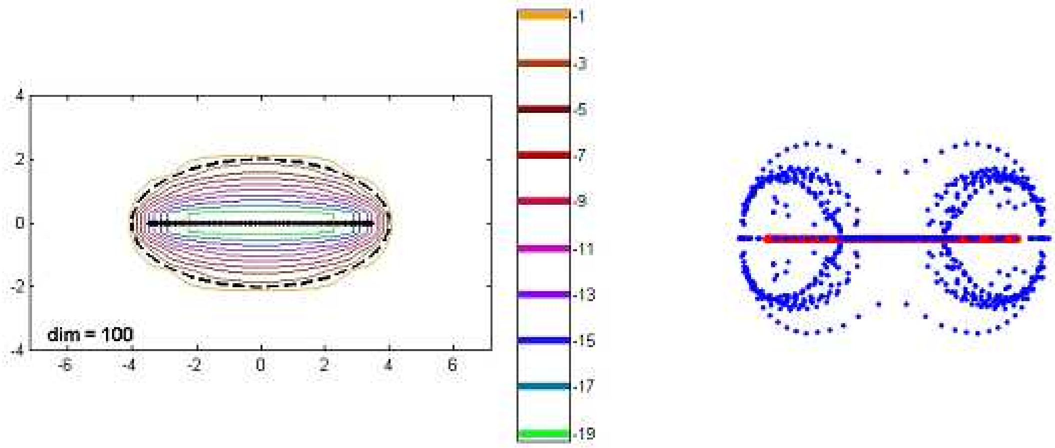

Figure 3.1: The left plot shows the eigenvalues of matrix A with MatLab, where A tridiagonal matrix (3.8) of dimension $N = 100$. The eigenvalues of A are real. The dash curve is $f(S)$, the boundary of the spectrum of the associated Toeplitz operator. The solid curve is the boundary of the set $\Omega_r \cup \Omega^R$ of (3.5), an approximation to the boundary of the ε-pseudospectrum, $\varepsilon = 10^{-1}$, 10^{-3}, 10^{-5},...,10^{-19}. The eigenvalues of A are also ploted by Mathematica and they are marked by red points at the right side. Eigenvalues of $A + E$ (3.8) are marked by blue points, $\|E\| \approx \varepsilon = 10^{-2}$; result from ten random perturbations are superimposed.

The dashed curve in the figure is the ellipse is $f(S)$, which is the boundary of the spectrum of the associated Toeplitz operator. The solid curve is $f(S_r)$ with r defined by (3.7). Evidently Λ_ε is closely approximated by the set Ω_r of (3.5). In the example Ω_r is the only term in (3.5) that matters, because Λ is contained in Ω_r, and Ω^R is the empty set.

Obviously the matrix (3.8) is very special. To illustrate (3.5) more fully, consider the more "generic" example

$$B = \begin{bmatrix} 0 & 0 & 1 & 0.8 & & \\ 2i & 0 & 0 & 1 & 0.8 & \\ & 2i & 0 & 0 & 1 & 0.8 \\ & & 2i & 0 & 0 & 1 \\ & & & 2i & 0 & 0 \\ & & & & 2i & 0 \end{bmatrix} \tag{3.11}$$

of dimension N. The symbol of this matrix is

$$f(z) = 2iz^{-1} + z^2 + 0.8z^3 \tag{3.12}$$

The curve in Figure (3.2) shows that this function maps S onto a rather complicated shape that might be a rendering by Picasso of the head of a bull. The two "horns" are enclosed by $f(S)$ with winding number +1, the "face" with winding number −1, and according to Theorem 3.3, the spectrum of the Toeplitz operator is the union of both of these regions together with the dashed boundary curve. The pseudospectrum of the Toeplitz matrix, however, is smaller, and consists of three parts. Within each horn is a region Ω_r enclosed by $f(S_r)$ with winding number +1 (solid curve). Within the face is a quite disjoint region Ω^R enclosed by $f(S_R)$ with winding number −1. Connecting these sets are chains of eigenvalues that are evidently insensitive to perturbations.

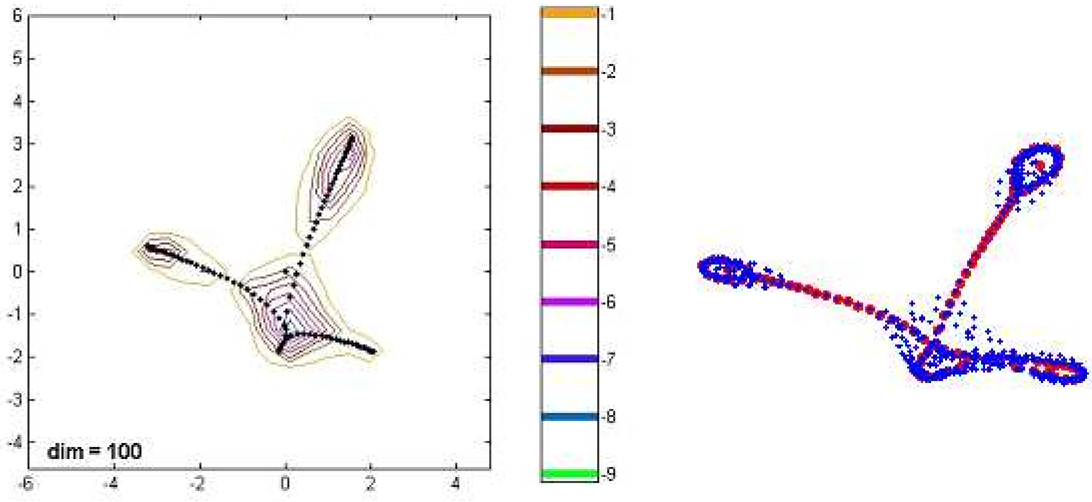

Figure 3.2: Same as Figure (3.1), but for the matrix B of (3.11), $N = 100$. The two regions of pseudo-eigenvalues within the "horns" are enclosed by $f(S_r)$ with winding number $I = +1$, and the region of pseudo-eigenvalues in the "face" is enclosed by $f(S_R)$ with winding number $I = -1$. The eigenvalues of B are also plotted by Mathematica and they are marked by red points at the right side. Eigenvalues of $B + E$ (3.11) are marked by blue points, $||E|| \approx \varepsilon = 10^{-2}$; result from ten random perturbations are superimposed.

For a more orderly pair of examples, Figure 3.3 and Figure 3.11 are analogous plot for the two 100×100 matrices,

$$C_2 = \begin{bmatrix} 0 & 2 & & & \\ 0 & 0 & 2 & & \\ 2 & 0 & 0 & 2 & \\ & 2 & 0 & 0 & 2 \\ & & 2 & 0 & 0 \end{bmatrix} \qquad C_{0.8} = \begin{bmatrix} 0 & 2 & & & \\ 0 & 0 & 2 & & \\ 0.8 & 0 & 0 & 2 & \\ & 0.8 & 0 & 0 & 2 \\ & & 0.8 & 0 & 0 \end{bmatrix} \tag{3.13}$$

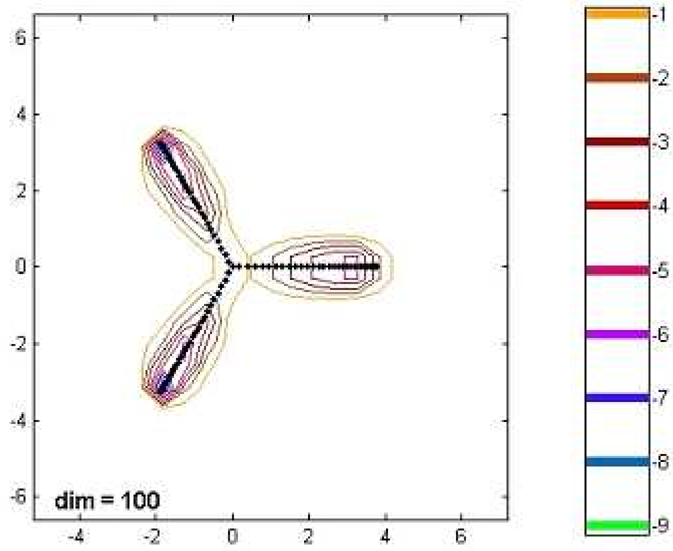

Figure 3.3: The matrix C_2 of (3.13), $N = 100$. The three lobes of pseudoeigenvalues are enclosed by $f(S_R)$ with winding number $I = -1$.

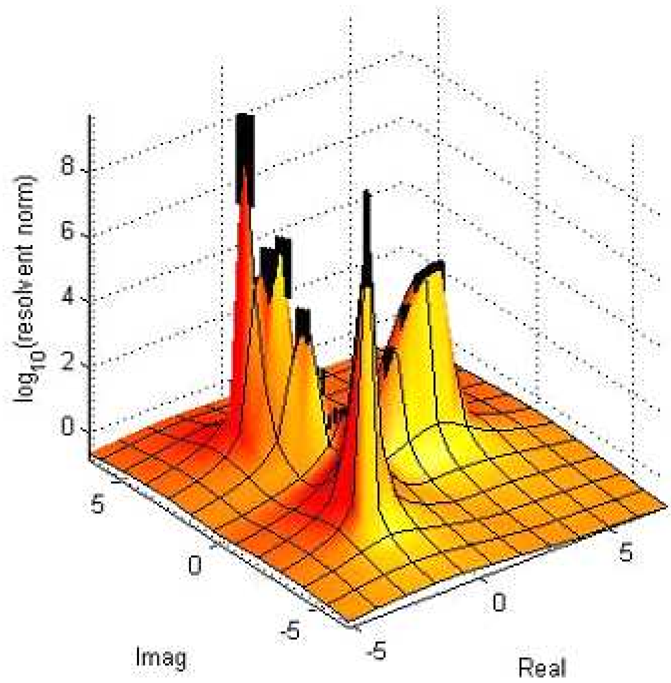

Figure 3.4: The resolvent norm as a function of $z \in \mathbb{C}$ for the matrix of C_2. The spikes occur at eigenvalues of C_2; in principle they extend to infinity.

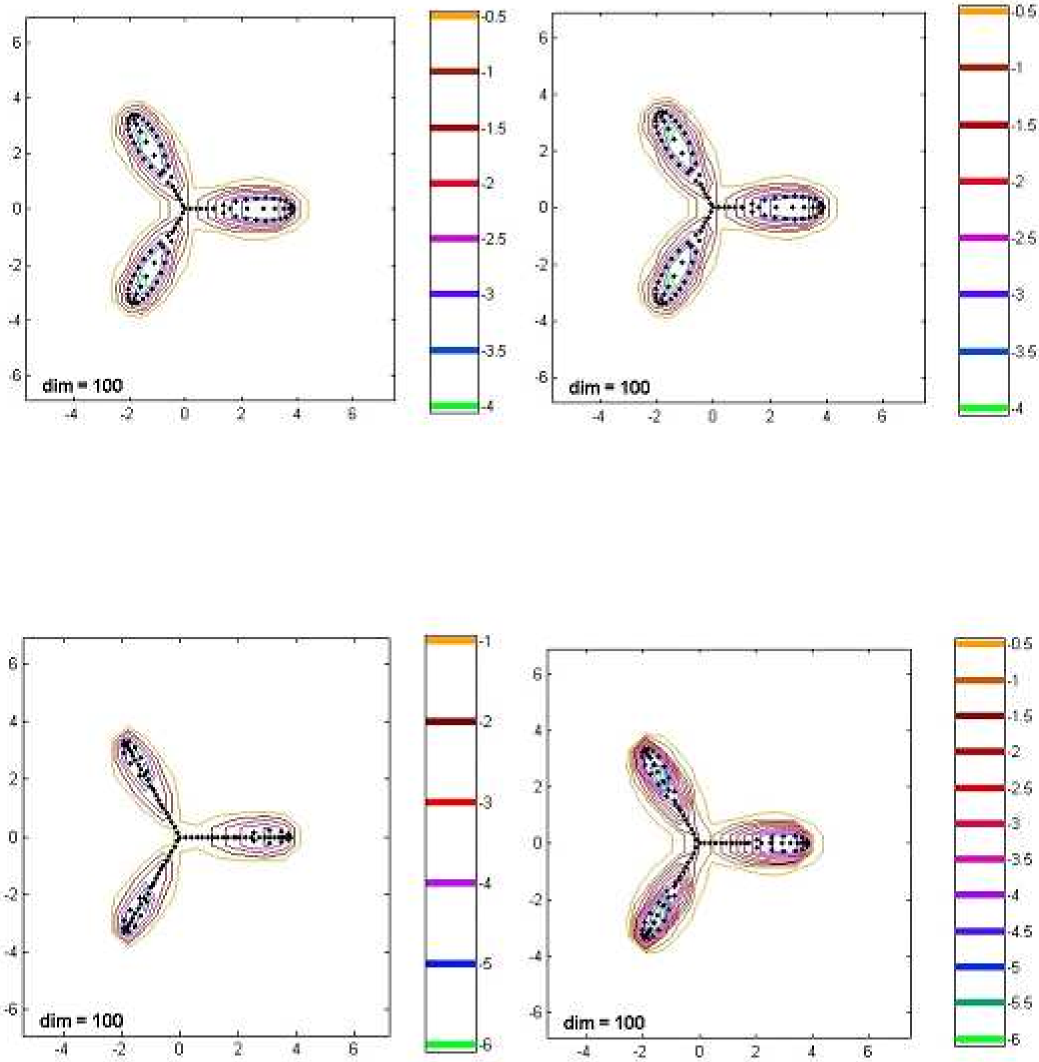

Figure 3.5: Four plots corresponding to $N = 100$ and $\|E\| = 10^{-2}$ show eigenvalues of 4 randomly perturbed matrices $C_2 + E$.

```
n = 100;
C2 = DiagonalMatrix[Table[0, {n}]];
For[i = 1, i ≤ n - 2, i++, C2[[i+2,i]] = 2 ;];
For[i = 1, i ≤ n - 1, i++, C2[[i,i+1]] = 2 ;];
Print[MatrixForm[C2]];
In = IdentityMatrix[n];
M = C2 - λ In;
p[λ_] = Det[C2 - λ In];
solset = Solve[p[λ] == 0, λ];
For[i = 1, i ≤ n, i++,
  λi = solset[[i,1,2]];] ;
Print[      C2 = , MatrixForm[C2]];
Print[C2 - λ, In,  = , MatrixForm[C2],  - λ, MatrixForm[In]];
Print[C2 - λ, In,  = , MatrixForm[M]];
Print[The characteristic polynomial is];
Print[p[λ] = |C2-λ In|];
Print[p[λ] = , p[λ]];
q[λ_] = Factor[p[λ]];
If[Not[p[λ] === q[λ]], Print[p[λ] = , q[λ]]];
Print[To find the eigenvalues of the matrix C2];
Print[Solve  , p[λ] == 0];
Print[Get];
For[i = 1, i ≤ n, i++,
  Print[  , λi,  = , λi,  = , Chop[N[λi]]] ];
Print[{(xk,yk)} = , nokta = Table[{ComplexExpand[Re[Chop[N[λi]]]],
    ComplexExpand[Im[Chop[N[λi]]]]}, {i, n}]]
Needs[Graphics`Colors`];
dots = ListPlot[nokta, PlotStyle → { Red , PointSize[0.025]},
   DisplayFunction → Identity];
Show[dots, PlotRange → All, Axes → False, AxesLabel → {X, Y},
  DisplayFunction → $DisplayFunction];
```

Figure 3.6: Program for plotting $\sigma(C_2)$ and $\sigma(C_2 + E)$ with Mathematica.

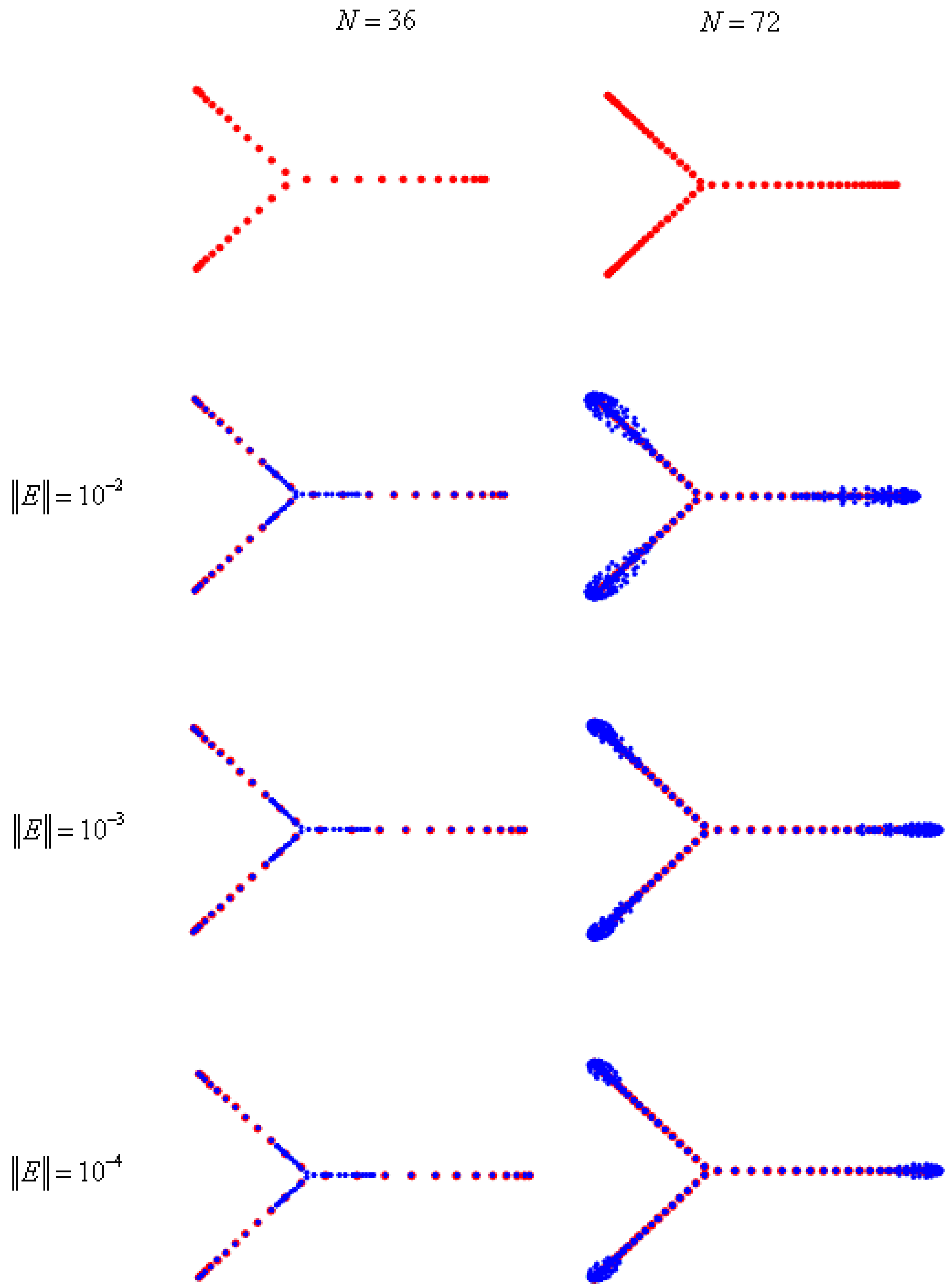

Figure 3.7: Eight plots corresponding to $N = 36, 72$, the eigenvalues of C_2 are marked by red points and $||E|| = 10^{-2}, 10^{-3}, 10^{-4}$. Each plot shows eigenvalues of 10 randomly perturbed matrices $C_2 + E$ and this eigenvalues are marked by blue points plotted by Mathematica.

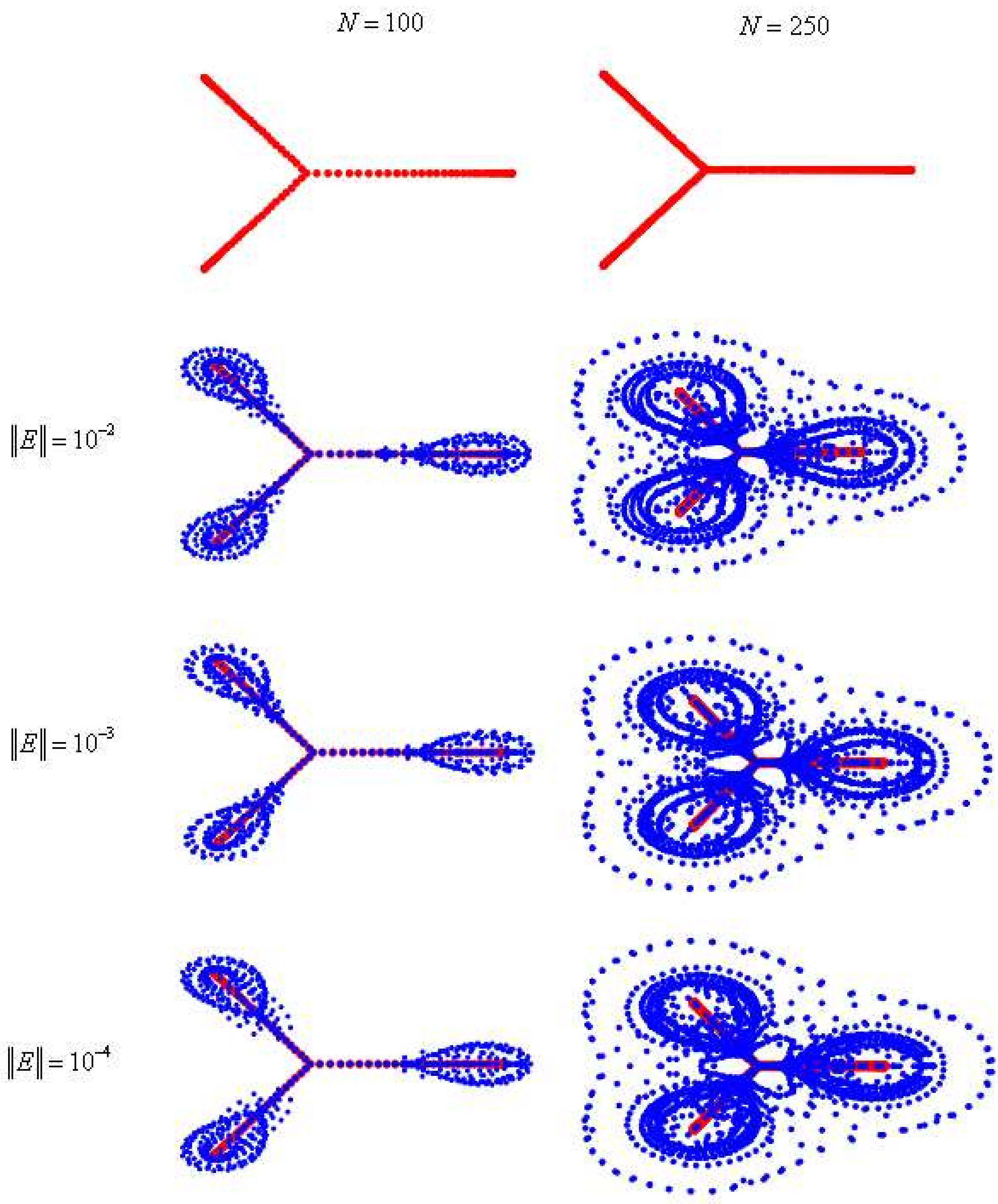

Figure 3.8: Eight plots corresponding to N=100,250, the eigenvalues of C_2 are marked by red points and $||E|| = 10^{-2}, 10^{-3}, 10^{-4}$. Each plot shows eigenvalues of 10 randomly perturbed matrices $C_2 + E$ plotted by Mathematica.

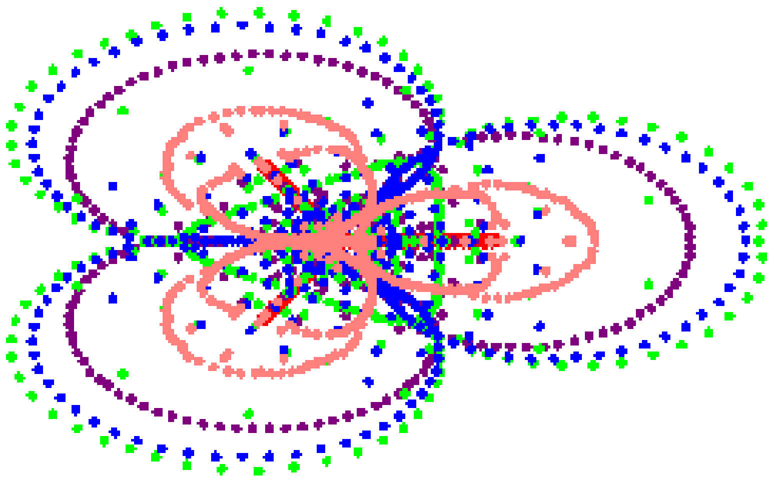

Figure 3.9: The plot corresponding to $N = 500$, $\|E\| = 10^{-2}$ and shows eigenvalues of 4 randomly perturbed matrices $C_2 + E$.

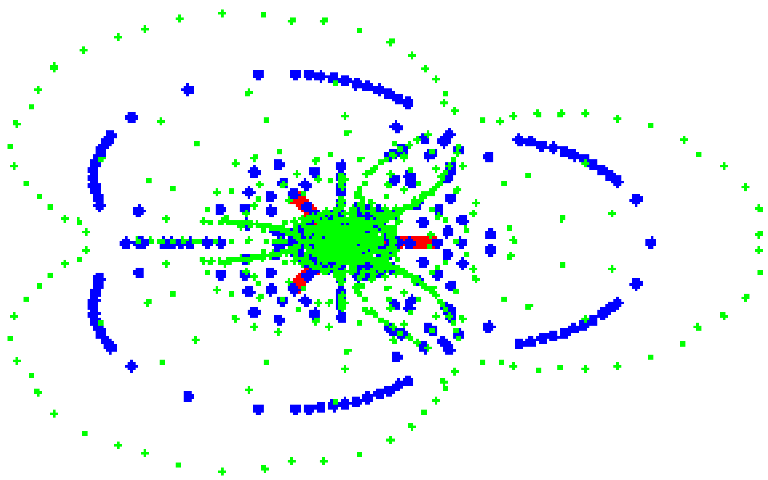

Figure 3.10: The plot corresponding to N=1000 , $\|E\| = 10^{-2}$ and shows eigenvalues of 2 randomly perturbed matrices $C_2 + E$.

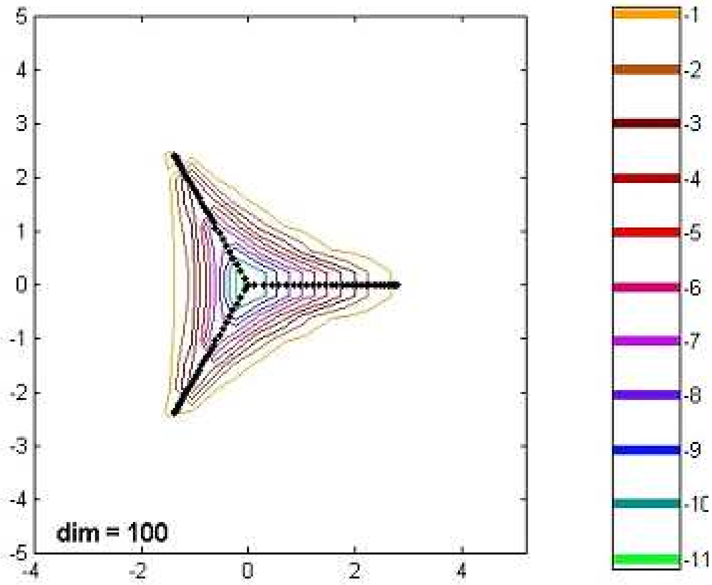

Figure 3.11: The matrix $C_{0.8}$ of (3.14), $N = 100$. The triangular region of pseudo-eigenvalues is enclosed by $f(S_r)$ with winding number $I = +1$.

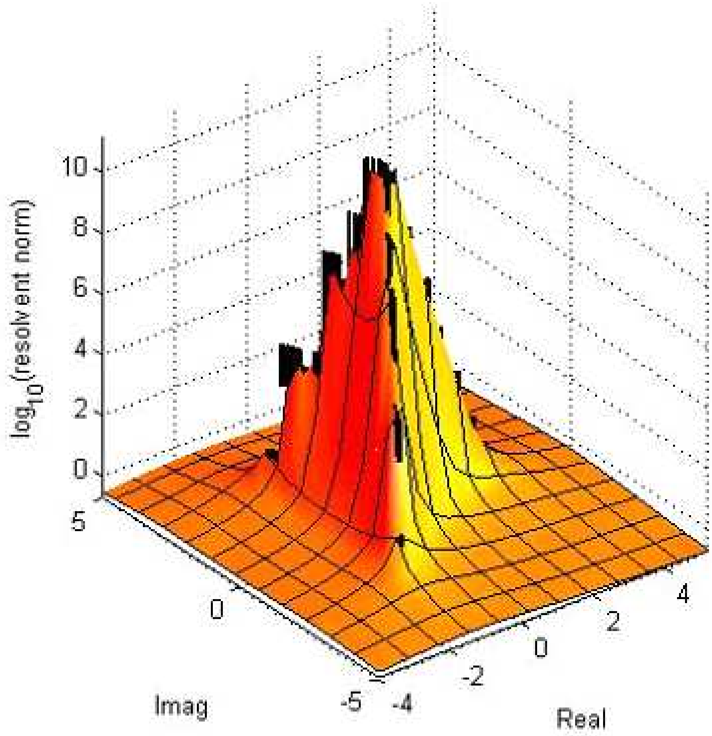

Figure 3.12: The resolvent norm as a function of $z \in \mathbb{C}$ for the matrix of $C_{0.8}$. The spikes occur at eigenvalues of $C_{0.8}$; in principle they extend to infinity.

In both cases, the ε-pseudospectrum for $\varepsilon = 10^{-2}$ is a three-fold symmetric shape about the origin consisting of the union of three spikes and a region with significant interior. The spikes would have lengths approximately 3,9 in Figure 3.3 and approximately 2,9 in Figure 3.11 in the limit $N \to \infty$. The regions with interior are the contributions Ω_r and Ω^R. To see where they come from, note that C_2 and $C_{0.8}$ are special cases of a matrix C_γ with symbol

$$f(z) = z + \gamma z^{-2} \tag{3.14}$$

For $|\gamma| \leqslant 0.8$, $f(S)$ has positive or zero index with respect to all points $\gamma \in C - f(S)$, for $|\gamma| \geqslant 2$ the index is negative or zero, and for $0.8 < |\gamma| < 2$ there are points λ of both positive and negative index. These observations associated Toeplitz operators. For the pseudo-eigenvalues of the Toeplitz matrices, other curves $f(S_r)$ and $f(S_R)$ become relevant. In Figure 3.3, with $\gamma = 2$, Ω_r is empty and only Ω^R contributes to the pseudospectrum. In Figure 3.11, with $\gamma = 0.8$ is empty and only Ω_r appears.

Theorem 3.4 *Let A be a banded Toeplitz operator with bandwith l, i.e.,* $a_k = 0$ *for* $|k| > l$. *Let* $f(z)$ *be the symbol of A, and let* A_N *denote the* $N \times N$ *Toeplitz matrix defined by* $a_{1-N}, ..., a_{N-1}$. *Then for any* $r < 1$ *and* $\rho > r$ *we have*

$$\Omega_r \cup \Omega^{1/r} \cup (\Lambda(A) + \Delta_\varepsilon) \subseteq \Lambda_\varepsilon(A_N) \qquad for \qquad \varepsilon = C^N_\rho, \tag{3.15}$$

where C is a constant that depends on r , ρ *and* $a_{-1}, ..., a_1$ *but not on* N.

Proof. First of all we note that the inclusion $\Lambda(A) + \Delta_\varepsilon \subseteq \Lambda_\varepsilon(A_N)$ is a traviality, valid for any matrix or operator. Second, by symmetry, $\Omega^{1/r}$ must satisfy an estimate of the type (3.16) if Ω^r does. Thus all that we really have to prove is $\Omega^r \subseteq \Lambda_\varepsilon(A_N)$.

Given any $r < 1$, let $\lambda \in \Omega_r$ be arbitrary. Assume without loss of generality that $a_{-1} \neq 0$. Then $f(z)$ has a pole or order exactly l at $z = 0$, and since $I(f(S_r), \lambda) \geqslant 1$, it follows by the argument principle that the equation $f(z) = \lambda$ has at least $l + 1$ solutions

$z \in D_r$, counted with multiplicity. Let $z_0, z_1, ..., z_l$ be any $l+1$ of these solutions; if there are more, we do not need them. Assume for the moment that the z_j are distinct. Corresponding to each z_j is a vector $u_j = (1, z_j, z_j^2, ..., z_j^{N-1})^T$ that satisfies,

$$(\lambda I - A_N)u_j = z_j^N \begin{pmatrix} a_l & & \\ \vdots & \ddots & \\ a_1 & \cdots & a_l \end{pmatrix} u_j + v_j, \tag{3.16}$$

where v_j is defined via an $N \times N$ matrix with nonzero entries only in the upper-right $l \times l$ triangle:

$$v_j = z_j^{-N} \begin{pmatrix} & a_{-l} & \cdots & a_{-1} \\ & & \ddots & \vdots \\ & & & a_{-l} \end{pmatrix} u_j.$$

Now let $u = \sum_{j=0}^{l} c_j u_j$ be a nonzero linear combination of these $l+1$ vectors with the property that the last l entries of $\sum_{j=0}^{l} z_j^{-N} c_j u_j$ are 0. Then the contributions involving v_j in (3.17) cancel, and we have

$$(\lambda I - A_N)u = \begin{pmatrix} a_l & & \\ \vdots & \ddots & \\ a_1 & \cdots & a_l \end{pmatrix} \sum_{j=0}^{l} z_j^N c_j u_j. \tag{3.17}$$

We need to relate the norm of the right-hand side of this equation to $||u||$. To do this, write $u = Uc$, where U is the $N \times (l+1)$ Vandermonde matrix whose columns are the vectors u_j, and c is an $l+1$-vector. Let L be the lower triangular matrix above, and let D be the diagonal matrix with elements $z_0^N, ..., z_l^N$. Then we have

$$||UD_c|| = \left\|UDU^{+}U_c\right\| = \left\|UDU^{+}u\right\| \leqslant \kappa(U)\, ||D||\, ||u||\,,$$

where U^+ denotes the pseudoinverse of U and $\kappa(U) = \sigma_0/\sigma_l$ is its condition number. Since $\|D\| \leqslant r^N$, it follows that (3.18) implies

$$\frac{\|(\lambda I - A)u\|}{\|u\|} \leqslant r^N \kappa(U)\,\|L\|\,, \tag{3.18}$$

by condition and therefore, by condition (*ii*) of the definition in the Introduction, $\lambda \in \Lambda_\varepsilon(A_N)$ for $\varepsilon = r^N \kappa(U)\,\|L\|\,.$

This completes the proof except for two points: the constant C in (3.16) is required to be independent of $\lambda \in \Omega_r$ and N, and we have assumed that the roots $z_0, ..., z_l$ are distinct. These matters can be dealt with as follows. If the roots $z_0, ..., z_l$ are distinct for all $\lambda \in \overline{\Omega}_r$, then the function

$$\sup_{N \geqslant 1} \frac{\sigma_N(\lambda I - A_N)}{r^N} = \sup_{N \geqslant 1} \frac{\left\|(\lambda I - A_N)^{-1}\right\|^{-1}}{r^N}$$

is a continuous function of λ on the compact set $\overline{\Omega}_r$; its maximum provides the constant C in (3.16), and we can take $\rho = r$. On the other hand, if some of the roots $z_0, ..., z_l$ are confluent at some points $\lambda \in \overline{\Omega}_r$, then an analysis involving confluent Vandermonde matrices yields a bound analogous to (3.19) except with r^N replaced by an algebraically growing factor at worst $N^l r^N$. In this case we need to take $\rho > r$. Doing so yields the conclusion that

$$\sup_{N \geqslant 1} \frac{\sigma_N(\lambda I - A_N)}{\rho^N} = \sup_{N \geqslant 1} \frac{\left\|(\lambda I - A_N)^{-1}\right\|^{-1}}{\rho^N}$$

is bounded at every point $\lambda_\varepsilon \in \overline{\Omega}_r$, and since this supremum is easily shown to be continuous, this establishes (3.16) in the confluent case. □

Theorem 3.5 *Let A be an arbitrary Toeplitz operator with absolutely summable entries, and let A_N denote its $N \times N$ Toeplitz matrix section. Then*

$$\lim_{N\to\infty} \Lambda_\varepsilon(A_N) = \Lambda_\varepsilon(A) \tag{3.19}$$

for each $\varepsilon > 0$, and therefore

$$\lim_{\varepsilon\to 0} \lim_{N\to\infty} \Lambda_\varepsilon(A_N) = \Lambda(A). \tag{3.20}$$

To close this section, we present a final example from numerical analysis. Let

$$A = L + D + U \tag{3.21}$$

be the symmetric tridiagonal matrix with entries 1, −2, 1, with L, D, and U denoting the subdiagonal, diagonal, and superdiagonal parts. Such a matrix would arise in the discrete solution by finite differences of a one-dimensional constant-coefficient diffusion equation. When a linear system $Ax = B$ is solved iteratively by the Gauss-Seidel iteration, the errors $e^{(n)}$ satisfies $e^{(n)} = G^n e^{(0)}$, where

$$G = -(L + D)^{-1} U$$

is the *Gauss-Seidel iteration matrix* . The spectrum of this matrix has been of interest for many years, because it determines the asymptotic convergence rate of the Gauss-Seidel iteration. The asymptotic convergence factor is the spectral radius $\rho(G)$, which is equal to the largest of these eigenvalues and of size $1 - O(N^{-2})$.

It is readily shown that except for a first column which is zero, G is an upper Hessenberg Toeplitz matrix:

$$G = \begin{bmatrix} 0 & \frac{1}{2} & & & & \\ 0 & \frac{1}{4} & \frac{1}{2} & & & \\ 0 & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & & \\ 0 & \frac{1}{16} & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & \\ 0 & \frac{1}{32} & \frac{1}{16} & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{64} & \frac{1}{32} & \frac{1}{16} & \frac{1}{8} & \frac{1}{4} \end{bmatrix} \qquad (N \times N). \tag{3.22}$$

If one ignores the first column, the symbol of this matrix in the limit $N \to \infty$ is the quotient of the symbols of $-U$ and $L + D$:

$$f(z) = \frac{z}{2 - z^{-1}}.$$

For $N = 100$, the corresponding curve $f(\Omega_r)$ and some computed pseudo-eigenvalues are shown in Figure 3.13.

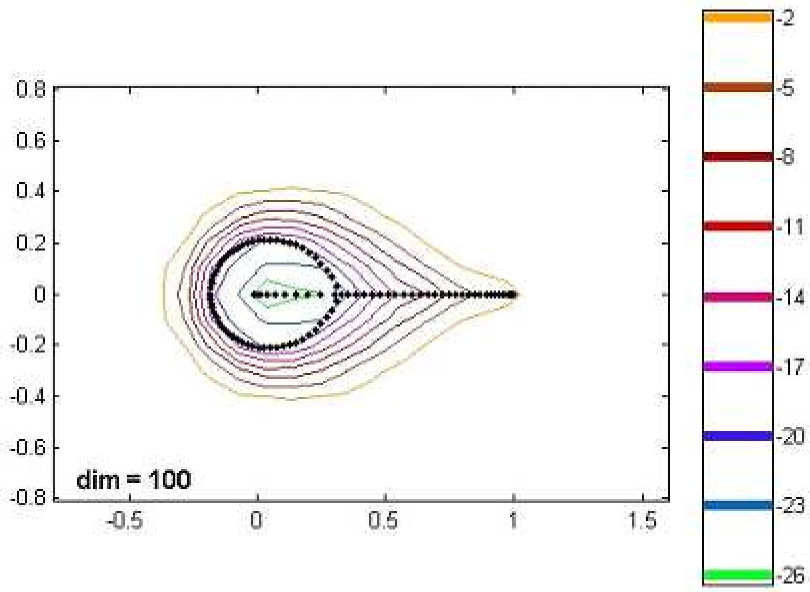

Figure 3.13: Eigenvalues of $G + E$, $\|E\| = \varepsilon = 10^{-2}, \ldots, 10^{-26}$ are in the curves, where G is the Gauss-Seidel iteration matrix of (3.23), with $N = 100$. As shown by Frankel in 1950, the eigenvalues of G are real and complex numbers in $\approx (-0.2, 0.3)$ and complex numbers in $\approx (0.3, 1)$, marked by black points.

Why was the sensitivity to perturbations of the eigenvalues of the Gauss-Seidel iteration matrix not discovered in the 1950? The reason is that the *largest* eigenvalue of G is insensitive; in fact, one can show that its condition number is asymptotic to 1 as $N \to \infty$. Thus the distinction between eigenvalues and pseudo-eigenvalues has no effect on the observed convergence rate for the Gauss-Seidel iteration applied to the matrix A of (3.20). If A is taken to be nonsymmetric, however, as occurs in the discretization of convection-diffusion problems, the picture changes utterly. The pseudospectral radius of G may be much closer to 1 than the spectral radius.

REFERENCES

[1] J. M. Varah. The compitation of bounds for the invariants subspaces of a general matrix operators. Technical Report CS 66, Computer Sicience Department, Stanford University, 1967.

[2] H. J. Landau. On Szegeö's eigenvalue distribution theorem and non-Hermitian kernels. *J. d'Analyse Math.*, 28:335-357, 1975.

[3] S. K. Godunov and V. S. Ryabenkii. *Theory of Difference Schemes:* An Introduction. North-Holland, Amsterdam, 1964. Russian original published by Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1962.

[4] S. K. Godunov, O. P. Kirilyuk, and V. I. Kostin. Spectral portraits of matrices. Technical Report Preprint 3, Inst. of Math., Sib. Branch of USSR Acad. Sci., 1990. In Russian.

[5] V. I. Kostin. On definition of matrices' spectra. In M. Durand and F. E. Dabaghi, editors, *High Performance Computing II*. North-Holland, msterdam, 1991.

[6] V. I. Kostin and S. I. Razzakov. On convergence of the power orthogonal method of spectrum computing. *Trans. Inst. Math. Sib. Branch Acad. Sci.*, 6:55-84, 1985. In Russian.

[7] J. W. Demmel. A counterexample for two conjectures about stability. *IEEE Trans. Auto. Control*, AC-32:340-343, 1987.

[8] J. W. Demmel. Nearest defective matrices and the geometry of ill-conditioning. In M. G. Cox. and S. Hammarling, editors, *Reliable Numerical Computatiton*, pages 35-55. Oxford University Press, Oxford, 1990.

[9] D. Hinrichsen and A. J. Pritchard. On spectral variations under bounded real matrix perturbations. *Numer. Math.,* 60:500-524, 1992.

[10] D. Hinrichsen and B. Kelb. Spectral value set: A graphical tool for robustness analysis. *Sys. Control Lett.*, 21:127-136, 1993.

[11] L. N. Trefethen. Approximation theory and numerical linear algebra. In J. C. Mason and M. G. Cox, editors, *Algorithms for Approximation II.* Chapman and Hall, London, 1990.

[12] S. C. Reddy and L. N. Trefethen. Lax-stability of fully discrete spectral methods via stability regions and pseudo-eigenvalues. *Comp. Methods Appl. Mech. Eng.,* 80:147-164, 1990.

[13] L. N. Trefethen and M. R. Trummer. An instability pheonomenon in spectral methods. *SIAM J. Numer. Anal.,* 24:1008-1023, 1987.

[14] S. C. Reddy. *Pseudospectra of Operators and Discretization Matrices and an Application to Stability of the Method of Lines.* Ph.D. thesis, MIT, Cambridge, MA. 1991.

[15] L. N. Trefethen. Pseudospectra of matrices. In D. F. Griffiths and G. A. Watson, editors, *Numerical Analysis 1991,* pages 234-266. Longman Scientific and Technical, Harlow, Essex, UK, 1992.

[16] L. N. Trefethen. Pseudospectra of linear operators. *SIAM Review,* 39:383-406, 1997.

[17] J. M. Varah. On the separation of two matrices. *SIAM J. Numer. Anal.,* 16:216-222, 1979.

[18] M. Gasanov and Y. Cesur. ”On a nonlinear eigenvalue problem”, Integral Equations and Operator Theory. 491-500, Birkhauser Verlag, Basel,1997.

[19] L. N. Trefethen. *Non-normal Matrices and Pseudospectra,* thesis in introduction.

[20] G. F. Jónsson and L. N. Trefethen. *A numerical analyst looks at the "cutoff phenomenon" in card shuffling and other Markov chains.* In D. F. Griffiths, D. J. Higham and G. A. Watson, eds., *Numerical Analysis 1997 (Dundee, 1997)*, 150-178. Longman Sci. Tech., Harlow, Essex, UK, 1998.

[21] E. Hille and R. S. Phillips. *Functional Analysis and Semi-Groups.* American Mathematical Society, Providence, RI, revised edition, 1957. First edition by Hille, 1948.

[22] A. Böttcher. Pseudospectra and singular values of large convolution operators. *J. Int. Eqs. Appl.*, 6:267-301, 1994.

[23] A. Böttcher and B. Silbermann. *Introduction to Large Truncated Toeplitz Matrices.* Springer-Verlag, New York, 1999.

[24] F. Chaitin-Chatelin and A. Harrabi. About definitions of pseudospectra of closed operators in Banach spaces. Tecnichal Report TR/PA/98/08, CERFACS, August 1998.

[25] O. Toeplitz, Zur Theorie der quadratischen und bilinearen Formen von unendlichvielen Veranderlichen, *Math. Ann.* 70:351-376, 1911.

[26] M. G. Krein. Integral equations on a half-line whose kernel depends on the difference of its arguments, *Uspekhi Mat. Nauk.* 13:3-120, 1950; English transl., *Amer. Math. Soc. Transl. Ser. 2* 22:163-288, 1962.

[27] A. Calderon, F. Spitzer, and H. Widom. Inversion of Toeplitz matrices, *Illinois J. Math.* 3:490-498, 1959.

[28] A. Devinatz. Toeplitz operators on H^2 spaces, *Trans. Amer. Math. Soc.* 112:304-317, 1964.