ASSORTMENT PLANNING IN TRANSSHIPMENT SYSTEMS

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By

Hilal DAĞ

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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Assortment planning, i.e., determining the set of products to offer to customers is a challenging task with immediate effects on profitability, market share and customer service. In this thesis, we study a multiple location assortment planning problem in a make-to-order environment. Each location has the flexibility to access others' assortments by transshipping products he/she does not keep. This allows them to offer higher variety and increase sales without increasing costs associated with assortment. Customer behavior is defined using exogenous demand model where each arriving customer to a location chooses a product with an exogenous probability among all possible options. In our multiple location setting, we assume that the customer has access to the complete assortment in all locations. If a customer's requested product is not available in that customer's assigned location but available in another location, the firm ships the product to the customer at the same price and incurs a transshipment cost. If his/her first choice product is not offered by any of the locations then he/she switches to a substitute product, which can be either satisfied from customer's assigned location, or by transshipment. Otherwise, it is lost. The problem is then to determine the assortment in each location such that the total expected profit is maximized.

We first show that the optimal assortments are nested, i.e., the assortment of a location with a smaller market share is a subset of the assortment of a location with a larger market share. We then show that while the common assortment is in the popular set (i.e., some number of products with highest purchase probabilities), the individual assortments do not necessarily have this property. We also derive a sufficient condition for each assortment to be in the popular set. In the final part of the thesis, we conduct an extensive numerical study to understand the effects of various parameters such as assortment cost and transshipment cost on optimal assortments and effects of allowing transshipments on resulting assortments compared to a no-transshipment system. Finally, we introduce an approximation algorithm that benefits from the structural properties obtained in this study and also test its performance with extensive numerical analyses.

Keywords: assortment planning, product variety, transhipment, inventory sharing.

ÖZET

TRANSFER-SATIŞ KULLANAN SİSTEMLERDE ÜRÜN ÇEŞİTLİLİĞİ PLANLAMASI

Hilal DAĞ
Endüstri Mühendisliği, Yüksek Lisans
Tez Danışmanı: Doç. Dr. Alper ŞEN ve Yrd. Doç. Dr. Nagihan ÇÖMEZ DOLGAN
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Ürün çeşitliliği planlaması, diğer bir deyişle, tüketiciye sunulacak ürünlerin çeşitlerini belirlemek; karlılığı, pazar payını ve müşteri hizmetlerini doğrudan etkileyen ve gayret gerektiren bir iştir. Bu tezde, sipariş üzerine üretim yapılan bir sisteminde, birden fazla noktada ürün çeşitliliği planlaması problemi incelenmiştir. Her bir nokta, kendi çeşitliliğinde olmayan ürünler için diğer noktaların ürün çeşitliliğine transfer-satış yoluyla erişebilmektedir. Böylece daha fazla çeşitlilik sunulmasına ve çeşitliliğin artmasına bağlı olarak maliyetleri yükseltmeden satışların artışına imkan verilecektir. Müşteri davranışı, sisteme gelen bir müşterinin tüm olası ürünler arasından, dış kaynaklı olasılık ile bir ürünü seçtiği harici talep modeli olarak tanımlanmıştır. Müşterilerin, bölgelerin tümünde bulunan toplam çeşitliliğe erişimi olduğu varsayılmaktadır. Müşterinin istediği ürün kendi noktası dışında bir noktada mevcut ise, firma ürünü müşteriye aynı fiyata satar ve nakliye ücretini kendisi karşılar. Eğer müşterinin ilk seçtiği ürün hiç bir noktasında sunulmuyorsa, müşteri alternatif bir ürüne yönelmektedir. Bu ürün müşterinin noktasından temin edilebileceği gibi farklı bir noktadan transfer-satış yoluyla da karşılanabilir. Diğer durumda ise satış kaybedilir. Problem toplam gelirin maksimize edilmesi için, her noktadaki ürün çeşitliliğine karar vermek olarak tanımlanmıştır. İlk olarak optimal çeşitliliğin iç içe kümeler olduğu gösterilmiştir. Yani, pazar payı küçük bir noktadaki çeşitlilik, pazar payı daha büyük bir noktadaki çeşitliliğinin bir alt kümesidir. Daha sonra ortak çeşitliliğin popüler küme (satın alınma olasılığı en yüksek olan ürünlerden bazıları) olduğu; her bir noktadaki çeşitliliğin ise kendi başına bu özelliğe sahip olmayabileceği gösterilmektedir. Ayrıca, her çeşitliliğin popüler küme olması için yeterli bir koşul türetilmiştir. Tezin son bölümünde, ürün tutma ve transfer-satış maliyeti gibi çeşitli parametrelerin ve transfer-satışa izin vermenin optimal ürün çeşitliliği

üzerindeki etkilerini anlamak amacıyla geniş çaplı sayısal bir çalışma yapılmıştır. Son olarak, bu çalışmada elde edilmiş olan yapısal özelliklerden yararlanılarak sezgisel bir algoritma geliştirilmiş ve sayısal analizler ile bu algoritmanın performansı kapsamlı olarak test edilmiştir.

Anahtar sözcükler: ürün çeşitliliği planlaması, transfer satış, stok paylaşımı.

To my beloved mother

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

Supply chain management, being the backbone of operational excellence, serves the purpose of satisfying end customers with the right product in the right place at the right time and at the right price. Globalization has increased the dependency of many companies on supply chain management for building their business strategy. Today's competitive market expects even more from supply chain management. Especially in centrally managed companies, integrated flow of information from downstream to upstream levels is a vital element of an efficient supply chain.

In this study, we focus our attention on the very downstream part where the end customer is offered a portfolio of products in the distribution channel. According to a survey by McKinsey & Company [1], retailers and consumer goods producers rated the "optimization of product portfolio and category management" as the most important task for achieving operational performance goals. Customers look at the product variety the firms offer and attempt to make best decisions for themselves based on various parameters such as price, quality, brand, availability of a product and so on. The more variety the firm offers typically leads to more demand it captures. However, often offering the entire set of possible products in stores is neither feasible nor profitable due to space constraints and/or the fixed cost of keeping products in stock. Therefore, firms should truly understand what the customer or market demands and align their decisions with these expectations. Assortment is defined as the specific set of products each firm has in its store. Assortment planning is determining the optimal set of products to be offered to the customers. Assortment planning is crucial for a company's competitiveness and success.

Assortment has three levels: (*i*) breadth, also called variety -number of different variants/categories the firm holds- (*ii*) depth -number of stock keeping unit (SKU) the firm holds in each category- and (*iii*) amount of inventory for a specific SKU. Offering a high variety product line gives a broad set of product options to the customer and eliminates the customer's search in competitors' stores [2]. Consumers look for variety in the first place and since each individual has his/her particular taste, there is a significant potential of boosting revenue.

On the other hand, there are some studies, which present positive experimental results on assortment reduction. According to their conclusions, decreasing product variety does not necessarily end up with decrease in sales when excessive levels of variety is the case. In other words, eliminating number of products in assortment help customer reduce their effort in making decision and hence increase the probability of purchase [3, 4]. From the psychological point of view, when there are too many options for the customers, their expectations become much higher that the anxiety level rises due to the fear of not being able to choose the best product among all available alternatives. Therefore, they may end up buying nothing because of not being able to decide what to pick from a broad assortment. Schwartz [5] describe this phenomenon as the paradox of choice. Boatwright and Nunes [6] also reveal that assortment reduction up to 54% increases the average sales by 11%. In addition, by decreasing number of variants companies can take advantage of the economies of scale more. Wider product line increases the inventory, shipping and other related costs. In addition space constraints in stores and warehouses also limit the number of products a firm can offer. Therefore, decision makers should find an efficient way to optimize their level of product variety in their assortments.

As is the case with variety, firms' product depth also requires assortment decisions. The number of SKUs in each product category plays an important role for both sides. Customers' preferences depend on various attributes such as price, design, quality, size, color and so forth. Based on their tastes and choice criteria when determining the product to purchase, they may be willing to search more alternatives as they cannot find their desired item in a particular outlet. In order to respond to this customer tendency, companies such as Toys "R" Us, Media Markt and Ikea offer very deep assortments in their fields of specialty.

The type of production strategy used by a firm is closely related with its assortment planning. Make-to-order systems have more advantages over make-to-stock systems to provide a broader assortment. In make-to-order systems, product supply is delayed until the realization of demand. While this creates an additional waiting time for the customer, in contrast to a make-to-stock firm, the make-to-order firm does not have to stock the product, rather request from a supplier or produce whenever a customer order is placed. Thus, the make-to-order firm can offer a broader range of products that can be better suited for customers' preferences by enjoying the benefits of lower inventory levels and hence lower holding and operating costs than a make-to-stock firm. Dell Computer Corp. is a pioneering company that manages most of its supply as a make-to-order system. This helps the company to offer customized products to its customers compared to competitive computer producers.

In contrast, assortment planning in make-to-stock systems is hard to manage because of its dependance on demand forecasts to determine inventory levels in addition to assortment choice. Alptekinoğlu and Grasas [7] express that in make-to-stock environment, supply decisions are riskier since the firm may over-stock or under-stock each product in the assortment set. They also add that modeling substitution in a stock-out situation, where a consumer may switch from her most preferred product that is not in stock to a different one in stock, would be considerably more complex and in many cases intractable. Gupta and Benjaafar [8] emphasize that when there is substantial increase in product variety, firms switch from make-to-stock strategy to make-to-order since producing large amounts for a broad range of products is quite costly.

According to Akçay and Tan [9], in a make-to-order environment, firms have pressure from buyers on pricing and maintaining high service levels can be quite challenging. In order to deal with this situation, cooperation with other firms through sharing inventory may play an important role. Firms can cooperate by sharing their on hand inventory with same echelon players either up to some level or resort to complete sharing. The sharing inventory means shipment of products from requested firm to the requesting one who is in need to satisfy customer demand. This type of cooperation is called as virtual pooling or lateral transshipment in the literature. In the case of a stock out, it can be more efficient to use lateral transshipment and supply from a neighboring base that has stock on hand rather than the upstream level when it is geographically distant from the lower levels of the supply chain. An illustration can be seen in Figure 1.1.

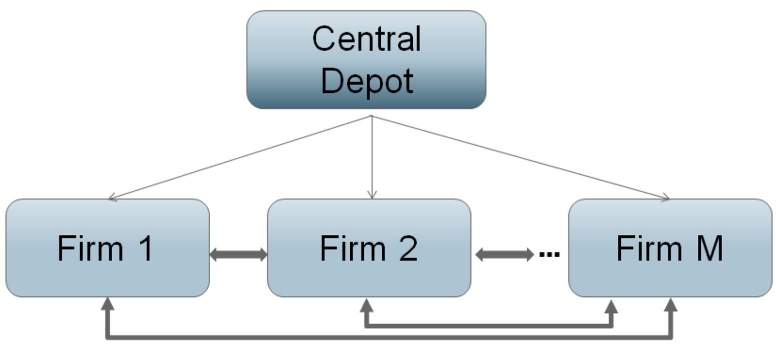

Figure 1.1: Lateral transshipments

Transshipments are broadly investigated in the literature mostly in terms of their effects on regular inventory replenishment policies and resulting inventory levels. Most studies analyzing lateral transshipments limit their attention to slow-moving and critical items such as the repairable service parts. For instance, failure of only one component of an aircraft or an automobile can bring the entire system to a downtime. It is hard to maintain a high service performance in these systems, since the failure rate is hard to predict and replacement of an item must be done very quickly. For this reason, the part can replenished from a nearby base instead of a main depot so that the lead time is shortened. Consequently, lateral transshipment reduces the response time of the backordered demand. In fact, it is a risk pooling method by giving flexibility to the firm against demand uncertainty in the market. Sharing limited resources enables high capacity utilization, lower inventory levels, and accordingly cost reduction. Kranenburg [10] show that a semiconductor company could save around 50% of its inventory related costs by using lateral transshipments. While the effects of the use of transshipments on inventory decisions is analyzed in the literature, availability of transshipments on the choice of assortments is not studied before, to our knowledge. Akcay and Tan [9] aim to determine the characteristics of firms that should cooperate through transshipment with their given assortments. However, they don't investigate the optimal choice of assortments under transshipment option in case of an assortment related stock-out.

When firms make assortment, inventory decisions, and/or transshipment decisions, knowing customer choice behavior is crucial. Because, although transshipments can help to decrease stock-outs by overcoming supply-demand match that result from either assortment or inventory choices, still transshipments may not always be available to cover all stock-outs. Then demand substitution can arise as an additional option to match supply with the demand. Failing to understand customers' substitution behavior can cause misinterpretation of the total demand of a product, since the total demand for a product is composed of not only the direct demand for that product but also the substituted demand when desired product demand cannot be fulfilled. Substitution is the switch from the desired product to another product that the firm can offer. This switch may occur due to the unavailability of the desired product for one of two reasons; the firm can be out of stock for this product or it does not carry the product in its product portfolio. These two forms of substitution are formally called as stock-out based substitution and assortment based substitution, respectively.

Stock-out based substitution is the substitution in the temporary absence of the desired product because of stock shortage. In this case, customers buy the product periodically and alter their choices only when the product is not available at the time of purchase. The assortment based substitution, which is the one used in this study, occurs when customers make their choice from a given assortment without knowing product availability a-priori and may substitute their first choice with another one when the first choice is not offered. Both types of substitutions describe how customers react to the assortment decision of the firm. In related literature, assortment based substitution is known as *static* substitution. Stock-out based substitution or the combination of stock-out based and assortment based substitution is referred to as *dynamic* substitution [11, 12].

In the light of aforementioned discussion, our study addresses the assortment planning problem of a centrally managed company in a single period with the presence of transshipment option between its locations. Through transshipment, each location shares its complete assortment with others in a way that customers can choose the products from a combined assortment of these locations. We assume a make-to-order environment and therefore exclude the inventory decision from the scope of this study. We adopt an exogenous demand model with substitution option to explain the consumer choice behavior. Each arriving customer to a location has a favorite product. If this product is not available in the visited location, but available in another location, it is transshipped by incurring a transshipment cost. If the customer's favorite product is not available in any of the company's locations, the customer can switch to a substitute product with a certain probability. The substitute product can be directly satisfied from the visited firm's assortment if available or may be shipped from another location at a transshipment cost. If the substitute product is not available in any of the locations, then no further substitution occurs and the customer demand is lost. The objective of the company is to determine the optimal assortment for each location such that the total expected profit, which is equal to revenues from sales minus the transshipment and assortment costs, is maximized.

In this thesis, using the model described above, we study the structure of optimal assortments benefiting from the definition of a popular set under the allowance of transshipments in a make-to-order environment. We prove that the assortments of firms using transshipments should be nested sets, i.e., the assortment of a firm is a subset of another one with an equal or larger market share. Moreover, keeping a less popular product in the assortment of a firm can be more profitable when substitution probability is high enough. These structural properties obtained on optimal assortments are shown to decrease the computational complexity, i.e., increase the efficiency of obtaining the optimal assortment, quite significantly. By using these optimality properties, close-to-optimal solutions are shown to exist even in much more efficient way. Moreover, the effects of transshipments on the assortment decisions of the company are investigated by comparing to assortment planning problem of a no-transshipment system.

The remainder of the thesis is organized as follows. Chapter 2 reviews and classifies the related studies in both the assortment planning and the transshipment literatures. Chapter 3 describes the problem and presents the details of the analytical model. Structural results on optimal assortments are also derived in this chapter. Chapter 4 presents the results of numerical analyses to support and complement the analytical findings. Finally Chapter 5 summarizes the thesis and suggests avenues for future research.

Chapter 2

Literature Review

Our study is a bridge that aims to combine two well-studied problems; assortment planning and inventory sharing (transshipment) models. While assortment planning is a strategic-level decision that determines which products will be offered by a firm, transshipments are mostly ad hoc decisions that help firms to increase their customer service level when strategic-level assortment and inventory decisions do not lead to an exact match with supply and random customer demand.

In this chapter, we first review previous work in each of assortment planning and transshipment studies. In Chapter 2.1, past studies on assortment planning are reviewed by categorizing according to customer demand model used. In Chapter 2.2, some of the past transshipment studies are summarized to outline the basic model settings and the findings. Finally, in Chapter 2.3, limited previous work on simultaneous planning of multiple assortments is reviewed that is related to our model of incorporation of transshipment option in assortment planning.

2.1 Assortment Planning Studies

Assortment is the specific set of products each firm has in its store and assortment planning is obtaining the optimal set of products to be offered to customers. Assortment planning might include the inventory level decisions for the to-be-offered products as well. Li [13], Smith and Agrawal [14], Yücel *et al.* [15], Kök and Fisher [16] are among examples. Assortment planning literature is rich and handled by both operations management and marketing researchers. Kök *et al.*[11], Pentico [17], and Mantrala *et al.* [18] provide extensive reviews of past studies on assortment planning.

Past studies differ in terms of the way they handle several model characteristics such as consumer demand model, demand substitution pattern, inclusion of inventory level decisions, and existence of assortment capacity or not. In following, we categorize assortment planning studies according to the demand model used in these studies. Exogenous demand model and Multinomial Logit (MNL) are the most commonly used consumer choice models. Although Generalized Attraction Model (GAM) is newly introduced to the literature, it has gathered attention and hence is summarized in the following Chapter 2.1.3. Among different demand models, the resulting customer substitution behavior mainly determines the model choice. While we review each of the demand models, the type of substitution pattern used is also discussed.

2.1.1 Exogenous Demand Model

In an exogenous demand model, the demand for each product is ex-ante specified for all possible products. There is no consumer behavior model defining demand rates and the initial demand rates do not depend on selected assortment. When the most favorite product of a customer is not available, either because of stock-out or being out of assortment, with a pre-determined probability, the demand is substituted with the second favorite product, which is not necessarily within the available assortment. The number of substitutions can be fixed to a certain number or it can continue until an available product can be reached. Smith and Agrawal [14] and Kok *et al.* [16] discuss the difference between allowing only a single substitution for analytical tractability or more. Substitution rates to each possible product can be specified by a substitution mechanism. Kok *et al.* [16] defines several different mechanisms such as random substitution, where each product has the same probability of being a substitute, adjacent substitution, where a product can be substituted with only closest ones, and proportional substitution, where each product has a certain probability of being a substitute proportional to its demand rate.

One of the seminal papers on assortment planning utilizing an exogenous demand model is Smith and Agrawal [14]. They emphasis the effect of substitution on product variety while solving a profit maximizing inventory problem. Products are replenished with a fixed cycle typically on a weekly basis in an apparel industry. The newsvendor or base stock model is used in determining the inventory levels and both assortment based and stock-out based substitutions are considered. Accordingly, assortment fill rate and inventory fill rates are presented as performance parameters of their proposed demand model. Illustrative examples are given over 5 substitutable products. Finally they are able to show that the substitution has significant effect on the profit function and it can decrease the optimal assortment size even the fixed costs are set to zero. Moreover, they prove that it is not always optimal to carry the most popular products when substitution behavior is incorporated.

Kök *et al.* [16] develop a method to obtain the best assortment for multiple stores. They present an estimation technique for the parameters of product substitution and demand and then apply their methodology on a supermarket chain data operating with over 1000 stores in the Netherlands. Periodic review inventory model with stochastic demand and fixed lead time is adopted. They resort first to assortment based substitution to model customer choice behavior and then add stocking levels to include stock-out based substitution. They include shelf space constraints and solve a profit maximizing nonlinear optimization model using an iterative heuristic via knapsack problem. The empirical results suggests only 0.05 % gap from the optimal solution. They show that products having high demand, high margin or smaller case sizes should be assigned to the shelves first. Then products with lower demand variability and smaller case sizes should be covered as well. Kök *et al.* [16] fill a gap in the assortment planning literature by integrating their theoretical model with a real world problem with actual data. Their assortment planning solution promises 50% increase in profits of the supermarket chain compared to their current assortment practice.

Caro and Gallien [19] examine the retail assortment problem of a single store with a dynamic approach including demand learning. Inventory replenishment system is assumed to have no stock-outs or lost sales. As for the parameters, profit margin of products is the same throughout the season in the finite time horizon. Both switching cost due to substitution and inventory holding cost are ignored. Yet, these exclusions are covered as extensions of the model. In each period, firm has to determine the assortment either by looking at the current data, referred as "exploitation" or by waiting for more demand data to become available, referred as "exploration". The authors propose two heuristic methods and show that their performances are near optimal.

Yücel *et al.* [15] propose a practical hybrid model that considers supplier selection, customer-driven multi-level demand substitution and inventory planning. After suppliers announce their product variety and the order quantity, the retailers make their selection based on the parameters of the model, such as substitution, market share related to each product and all other associated costs, ie., purchasing cost, inventory holding cost, substitution cost. The model assumes both assortment based and stock-out based substitution. It identifies which and how much product to offer while maximizing the expected profit. While presenting a useful scheme for assortment planning, the authors show that ignorance of customer substitution or supplier selection decisions in assigning product portfolio decreases the profit. Similar conclusions are made when analyzing the effect of shelf space. Moreover, when demand variability increases, depth of the product assortment also increases and retailers may need to expand the width of their assortments by collaborating with more suppliers. Their integrated model can also be used in determining new product entries.

Fadıloğlu *et al.* [20] develop a nonlinear mixed integer optimization model in the context of FMCG industry. The model helps retailers improve the product mix on their shelves by eliminating product pollution. They make use of the assortment based substitution. They establish lower and upper bounds for the objective function after decomposing the problem and linearize the model to attain the optimal solution efficiently. They perform a case study for one of the FMCG companies in Turkey for the assortment of shampoo products. Their analysis suggests that SKU's which have higher quality / higher price continue to be included in the assortment under the elimination scheme by means of their higher share in total profit.

2.1.2 Multinomial Demand Model

Multinomial Logit model is a logistic regression model for unordered multiple choices. It was first proposed by McFadden [21]. Since then MNL has been widely studied and gave rise to ample amount of research in the literature and in business practice as well. Basically it refers to the single decision among two or more discrete options and is used to model customer purchase behavior. Customers making decision from a set of products are utility maximizing individuals. The critical characteristic of MNL model is the *independence from irrelevant alternatives* (IIA). This property states that, for the ith customer, the ratio of choice probabilities j and k does not depend on other choices in the overall set. It means each item in the set is differentiated or irrelevant from each other such that omitting it from the model will not change the parameter estimates of the remaining items [22]. However, in the case of similar alternatives, which can be partitioned into subsets such as colors or sizes, IIA fails to hold. Because adding or removing one item from a subset might eliminate or escalate the choice probabilities of the specific subset. For this reason, IIA is not appealing from the perspective of customer behavior since it omits the relation of similar alternatives. Yet, it is commonly applied in the area of market research, economics, logistics etc. because of its robustness on estimation. Moreover, one of the main advantages of MNL model over an exogenous demand model is that it allows easy incorporation of other decisions such as pricing into the customer utility model.

One of the seminal papers using MNL as customer demand model is van Ryzin and Mahajan [12]. Their study is the first to merge MNL with newsvendor inventory model. There are two hierarchical objectives they try to attain in a single period. The first one is finding the optimal assortment set from the set of available variants, and then determining the stocking level of each variant. They assume that the selling price and unit cost are the same for all products and they only allow assortment based substitution. They prove that the optimal assortment is composed of some of the most profitable variants. They show that the higher the price of the category, the more variety that the store should carry. In addition, when no-purchase option goes down (or equivalently the purchase probabilities increase across the board), the firm can either incline to lower the level of variety supposing a noncompetitive market or vice versa. Finally, the authors study the effect of economies of scale using two different models on how consumers make decisions.

Cachon *et al.* [23] analyze the impact of consumer search in retail context using the MNL model. They present an explicit model for consumer search since there are some drawbacks in the implicit search model. For example, when a consumer leaves the system with no-purchase, whether he/she continues to search for a better product or gives up the purchase decision cannot be identified. Another skeptical point of the implicit model is its independency of the no-purchase decision from the retailer's assortment. Here, the aim of the consumer search is to find a better product rather than a product with a lower price. They establish three different assortment search models; independent model, overlapping model and no-search model. In the first one, retailers have more sophisticated variants which makes the search option independent from the retailer's assortment since the customers are expected to face new variants by searching. In the second model, adding new variants to the retailer's assortment might reduce the consumer search since there occurs some common products in the other retailers' assortments. The last one, no-search model is akin to the model of van Ryzin and Mahajan [12] which assumes that the consumer product choice is independent from the other variants. Their findings indicate that independent model and no-search model give better results when the number of variants is very large. However, no-search model does not perform well when there are overlapping product assortments. In the end, they conclude that there might be significant profit loss by using no-search model, in other words, ignoring the consumer search in assortment decision.

2.1.3 Generalized Attraction Demand Model

Generalized Attraction Demand Model (GAM) has been introduced by Gallego *et al.* [24] as a customer demand model in the assortment decision of network revenue management problems. GAM stands between the Basic Attraction Model which is based on Luce's choice axiom [25] (as cited in Davis *et al.*,2014) stating that customer chooses the product i with probability:

$$P(i) = \frac{w_i}{\sum_j w_i}$$

where w_i is a weight of a product attribute and Independent Demand Model which assumes that a customer chooses the product i with probability :

$$P_i(S) = \begin{cases} \alpha_i & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$$

where S is the choice set [26, 27]. GAM includes both Basic Attraction Model and Independent Demand Model as its two extreme cases. Choice probabilities may come from the MNL model which is a special case of Basic Attraction Model. Shifts from the original demand can be either spill or recaptured with other available alternatives. Basic Attraction Model ignores the consumer search option when the first choice is unavailable and this yields to the overestimation of recaptured demand. On the contrary, Independent Demand Model ignores the switching option from direct demand -*the substitution*- and hence, yielding underestimation of overall demand. GAM is a more flexible method and captures demand dependencies more realistically by surmounting the shortcomings of the former two. Another major contribution is that the optimal choice of the assortment set S can be found in polynomial time. They proposed a sales based linear program which is equivalent to choice based linear program but does not need column generation and hence much easier to solve with polynomial number of variables.

As an extension of the GAM formulation, Wang [28] analyzes the assortment problem with capacity constraints. The preceding research of Rusmevichientong *et al.* [29] established a ground for this study. Wang (2013) studies the same problem under GAM and proposes a new nonrecursive polynomial time algorithm. With the idea of the optimal set being in efficient sets, the solution space is downgraded to polynomial size from the initial combinatorial search space. When capacity constraint is incorporated, although the nested structure no longer exists, problem complexity is preserved at polynomial size.

2.1.4 Studies Using Different Substitution Models

Locational Choice Model is another demand model that is often utilized in assortment studies. It has been firstly proposed by Lancaster [30, 31] as an extended study of Hotelling [32]. Later, Gaur and Honhon [33] advanced this previous study by generalizing some of its characteristics. In this type of model, products are regarded as bunches of attributes and consumer choices are defined according to these attributes. The specific attribute that identifies each product is its location. Consumers are supposed to prefer the products in the nearest location and substitute to the products in the second nearest location and so forth. Although its concept in modeling the demand is the same as MNL, i.e. the utility based approach, it does not require IIA property. Besides, substitution can be controlled by the retailers. Both static and dynamic substitutions are covered in the model. Since the products have horizontal differentiation, prices and costs are assumed identical. In static substitution, optimal product set is equally located such that substitution does not occur between the products regardless of the demand amount or demand distribution. Dynamic substitution yields higher variety in assortment when compared to static substitution. In contrast to MNL model, the findings state that the most popular product does not have to be in the optimal set due to fragmentation of demand by creating diseconomies of scale for other products in the assortment.

2.2 Transshipment Planning Studies

To mitigate customer dissatisfaction, firms need to find ways to deal with the excess demand when they are out of stock or when they do not have the customer's desired product in their assortment. Customer search may also be in consideration, however in order to ensure high service levels, inventory sharing becomes an ideal solution in centralized or decentralized system. In the literature, inventory sharing is also called as *virtual pooling* or *transshipment* between same echelon players. In an inventory sharing system, inventories are stocked in several locations, yet can be linked through an information system such that partners in the distribution network can access the necessary stock information and request a transshipment from an appropriate location in case of out-of-stock situation. The parties in the system share their on hand inventory either up to some level or completely. This is a more advantageous way in some respects than physical pooling, which keeps a common inventory physically in a single centralized location. Transshipment studies can be classified into two according to their use of pre-determined policies or search for optimal policies.

2.2.1 Pre-determined Transshipment Policies

The benefits of transshipments and effects on regular inventory replenishment policies are often investigated by using pre-determined transshipment policies, which are not necessarily optimal. Anupindi *et al.* [34] introduce a decentralized system facing exogenous demand with several retailers and warehouses. Inventories in the system are carried both locally at each facility and in a given central location as pooled inventory without capacity constraints. There are two sequential decisions in their analysis; inventory and transshipment decisions, respectively made before and after the realization of demand. The inventory levels are established unilaterally by each retailer in a competitive manner. Given the inventory levels and the realized demand, the transshipment decision is made to decide whether to share the excess inventory or not. This requires a cooperative action on allocation of surplus profits resulting from transshipment. In this respect, a new type of game theoretical solution is presented with dual allocation mechanism with the objective of attaining centralized profits.

Zhao and Atkins [35] consider transshipment planning between two competing retailers selling substitutable products. They extend the transshipment game by including price competition between retailers. Although the products offered by the retailers are the same, consumers associate them with service, location, delivery or with other facilities and regard them as imperfect substitutes. The existence of pure-strategy Nash equilibrium in retail price and safety stocks has been proved for both games. Transshipment cost is integrated into the transshipment price which varies between zero and selling price of a product and a stochastic linear demand model is chosen. According to their findings low transshipment price and high degree of competition lead retailers to allow consumers substitution and high transshipment price and low degree of competition lead retailers to transshipment. As competition rises then transshipment becomes less attractive specifically at low transshipment prices. Moreover, given exogenous transshipment price, transshipment occurs only when the retailers' profit with transshipment dominates the profit with substitution.

2.2.2 Optimal Transshipment Policies

Zhao *et al.* [36] focus on minimizing inventory costs in decentralized transshipment network. Independent dealers have the flexibility of choosing an inventory sharing policy. The authors first set a demand classification, i.e., demand from direct customer of dealers has the high priority whereas demand from another dealer is considered as low priority. Therefore, overall demand is identified endogenously in their model. Infinite time horizon with continuous review system of two retailers also implies multiperiod inventory sharing. There exist two dimensional strategy (S, K) between two dealers such that S represents base stock level and K inventory rationing level. In addition to this strategy, the manufacturer plays an important role so as to promote inventory sharing between dealers through offering incentives to the dealer sharing its inventory or subsidizing the sharing cost paid by requesting dealer. Some of the interesting outcomes of this model include that an inventory pooling scheme in a decentralized system does not necessarily reduce expected backorders. For example, increasing incentive does decrease the backorders. Because dealers want to share more inventory in order to benefit from the incentive and not raising its base stock levels but rather lowering rationing levels. On the other hand, subsidizing cost of sharing causes dealers to keep less inventory, and it can boost the expected backorders and hence low service level. To refrain this undesired situation, manufacturers are proposed to charge dealers transshipment processing costs.

Comez *et al.* [37] consider an inventory pooling system of two retailers by allowing multiple transshipments between replenishments, so that the backorder costs of a retailer facing the actual demand and the inventory holding costs of another retailer making the transaction could be minimized. An inventory manager establishes a transshipment policy which is based on hold-back levels and whether the inventory at that time is above the hold-back level or not. Also he/she constructs a replenishment policy to set the base stock levels. In their study, it is shown that holding back inventory is optimal. Moreover, sensitivity of the transshipment model parameters like transshipment time and demand probabilities is examined. Finally if replenishment time is positive since the problem gets more complicated, then a heuristic solution is presented whose optimality gap averages at 1.1%.

Comez *et al.* [38] review the transshipment studies on decentralized system of retailers and introduce a dynamic policy that facilitates shipment during the season, specifically right after the demand realization. In this decentralized system, retailers make their own decisions to fulfill each transshipment request. The sales season is assumed to have N short periods such that only one request can arrive within a period. The authors show that the marginal benefit of keeping a product in stock is nondecreasing in the number of remaining periods and accordingly the holdback levels. That means retailers are inclined to respond affirmatively to the transshipment demand of the other retailers towards the end of the season. Their findings also indicate distinctive properties such as remotely situated retailers take more advantage of transshipment on the basis of relatively low customer overflow probabilities in their proposed model.

2.3 Assortment Cooperation

One of the most important issues in inventory sharing is establishing a cooperation scheme among firms. Akçay and Tan [9] investigate various questions which arise in cooperation process, such as, which firms should cooperate, under which circumstances joining a cooperation is profitable, what would be the policy of sharing the total benefit among members. In their analysis, cooperation is based on the firms' combined assortment which they call assortment based cooperation and is managed with discounted price contracts. It is further assumed that firms operate under make-to-order policy and assumed to have infinite capacity so that stock out based cooperation never occurs. They show that for symmetric firms with regard to their product market shares, sizes, and product profit margins, cooperation is always beneficial.

By analyzing the profit function for two asymmetric single product firms with identical product market shares, it can be seen that cooperation is beneficial when the product of smaller firm has a greater profit margin or the product of larger firm has greater profit margin and a threshold value is fulfilled. With equal sizes, cooperation is beneficial when the product which has larger market share is more profitable than the product which has smaller market share, or the product which has smaller market share is more profitable than the product which has greater market share and a threshold value is fulfilled. With equal profit margins, cooperation is beneficial when the product of smaller firm has greater market share or, the product of smaller firm has smaller market share and a threshold value is fulfilled. Another conclusion is increasing number of overlapping products negatively affects the total profit of cooperation. As for identifying the cooperation structure and its discount parameters, a mixed integer linear model has been proposed. Computations show that cooperation is advantageous if the number of firms in the market increase and not advantageous if the number of products in the assortment increase.

Akçay and Tan [39] extend their above model [9], by introducing production capacities. Both centralized and decentralized assortment-based cooperation between two make-to-stock firms each of which produces only one product are studied. In the centralized system, firms jointly replenish their inventory while in the decentralized scheme they operate independently. Cooperation is made through a discount-based contract like in their previous study. The authors aim to find in which situations assortment-based cooperation is always beneficial and the effects of the parameters i.e., firm size, substitution, product market share, on the benefit of cooperation. It is also important to note that their model incorporates the customer substitution into the make-to-stock environment. The results of their analysis can be summarized as follows: Assortment-based cooperation is always profitable for both centralized and decentralized systems if the firms are symmetrical (have identical parameters). Its effectiveness is proven for small-sized firms selling the products with small market shares. Discount-based contract is beneficial under appropriately chosen parameters. In a decentralized scheme, firms need to have excess capacity to gain from the assortment-based cooperation.

Chapter 3

Optimal Assortments in Transshipment Systems

3.1 Problem Definition

We consider multiple firms that operate under a make-to-order environment and are managed by a central agent. Each firm has a certain market share in the system and specializes on certain products, whose assortment is determined centrally. Consumers in the market have certain demand rates for the products offered by the whole system. When a potential customer of a firm asks for a product, which is not within the assortment of the visited firm, then the product can be supplied from another firm in the system by a transshipment if it is offered by that firm. Otherwise, the customer may switch from his most favorite product to a second favorite product, which can be either satisfied directly by the visited firm, or by transshipment from another firm, or lost due to unavailability in the system. We consider assortment-based and one-time substitution. As the firms operate under a make-to-order environment, we do not consider inventory stocking decisions for the products in the assortment. Thus, substitutions resulting from inventory shortage (stockout-based substitution) are not considered in this study as in Akçay and Tan [9], Kök and Fisher [16], Caro and Gallien [19], Besbes and Saure [40], Yücel et al. [15]. Consideration of at most one substitution opportunity is mainly due to analytical tractability, which is also shown to be a good approximation when an appropriate substitution probability is selected. (See [9, 14, 16].)

Our consumer choice model is the exogenous demand model (also known as the independent demand model [28]). λ denotes the total expected number of customers in the market. Firm m has a market share of q_m among all M firms in the system such that $0 < q_m \leq 1$ and $\sum_{m=1}^{M} q_m = 1$. The set of all products that can be offered by the firm is denoted by S. Each product i has a certain probability of being the first choice of a visiting customer as α_i denoted by $0 < \alpha_i \leq 1$ and $\sum_{i \in S} \alpha_i \leq 1$. There is a cost of κ for keeping each product at a firm's assortment which is called as the assortment cost. Assortment cost can be due to fixed costs affiliated with material handling, labor cost for merchandize presentation, record keeping and reordering as in [14] or due to warehousing, monitoring, personnel and computer time as in [41, 42]. We assume identical cost and margins for the products. Alptekinoğlu and Grasas [7] also model products with equal cost and profit margin. Exactly like our assumption, products only differ in their attractiveness. They also consider make-to order systems in addition to make-to-stock. They do not consider substitution, but assume the availability of an emergency order from the manufacturer.

Let firm m's assortment be $N_m \subseteq S$. The overall set of all products that will be offered by the set of firms is called as *union assortment* and is denoted by N such that $N = (N_1 \cup N_2 \cup ... \cup N_M)$. The set of all products that will be offered by each and every firm is called as *common assortment* and is denoted by $\bar{N}$ such that $\bar{N} = (N_1 \cap N_2 \cap ... \cap N_M)$.

An arriving customer of firm m pays r to the firm to buy the product, if his most favorite product i is in the assortment of the firm m, i.e., $i \in N_m$. If product i is not offered by firm m, but offered by firm k, then by incurring a transshipment cost t, the demand can be satisfied through firm k, for $k \in \{1, 2, ..., M\}$. As it is a centrally managed system, it does not matter which firm the transshipment cost is incurred by. If product i is not available in the union assortment N, then the customer may substitute to another product with probability θ. Given that the customer decided to substitute his first choice with a second one, the probability that the customer substitutes product i with product j is given by

$$\delta_{ij} = \frac{\alpha_j}{1 - \alpha_i}.$$

This definition is the same as the proportional substitution matrix in Kök & Fisher's model [16]. According to this expression, substitution probabilities for a 4-product firm are calculated in the following form:

$$\begin{bmatrix} 0 & \theta\alpha_2/(1-\alpha_1) & \theta\alpha_3/(1-\alpha_1) & \theta\alpha_4/(1-\alpha_1) \\ \theta\alpha_1/(1-\alpha_2) & 0 & \theta\alpha_3/(1-\alpha_2) & \theta\alpha_4/(1-\alpha_2) \\ \theta\alpha_1/(1-\alpha_3) & \theta\alpha_2/(1-\alpha_3) & 0 & \theta\alpha_4/(1-\alpha_3) \\ \theta\alpha_1/(1-\alpha_4) & \theta\alpha_2/(1-\alpha_4) & \theta\alpha_3/(1-\alpha_4) & 0 \end{bmatrix}$$

Here, the summation of each matrix row means the total amount of substitution *from* product i to other products when product i is not carried in the assortment of any firm in the system. The summation of each matrix column represents the total amount of substitution *to* product j when any of the other products is not included in any firm's assortment.

If firm m has the second favorite product j, then the customer gets it directly from firm m. Otherwise, if it is available by any other firm k, then the demand is satisfied via transshipment. Finally, if the product j is not available in the overall assortment N, the customer leaves the firm without a purchase. The notation used in the paper is summarized in Table 3.1.

The total expected system profit of M firms for a given set of firm assortments

Table 3.1: Notation.

Parameters	
S	The set of all possible products
M	Total number of firms
λ	Total expected number of customers in the market
q_m	Fraction of customers visiting firm $m, m = 1..M$
α_i	Probability that product i is a customer's first choice, $i \in S$
θ	Probability of a customer substituting his/her first choice
δ_{ij}	Substitution probability of product i with product j, $\delta_{ii} = 0\ \forall i$
r	Profit margin for one unit of product
t	Transshipment cost per unit
κ	Cost of keeping a product in the assortment of a firm
Variables	
N_m	Assortment of firm $m, m = 1..M$
x_{im}	1, if product i is in the assortment of firm m 0, otherwise.
y_{ijm}	1, if product i is substituted with product j in firm m 0, otherwise.
z_{ij}	1, if product i is substituted with product j 0, otherwise.
u_i	1, if product i is available in the overall assortment 0, otherwise.

$(N_1, N_2, ..., N_M)$ is denoted by $\Pi(N_1, N_2, ..., N_M)$ and calculated as follows.

$$\begin{aligned}\Pi(N_1, N_2, ..., N_M) = {} & \lambda \sum_{m=1}^{M} q_m \sum_{i \in N_m} \big(\alpha_i r + \theta r \sum_{j \notin N} \alpha_j \delta_{ji}\big) \\ & + \lambda \sum_{m=1}^{M} q_m \sum_{i \in N/N_m} \big(\alpha_i (r-t) + \theta (r-t) \sum_{j \notin N} \alpha_j \delta_{ji}\big) \\ & - \kappa \sum_{m=1}^{M} |N_m|, \end{aligned} \tag{3.1}$$

where $|X|$ denotes the cardinality of a set X.

The profit function is composed of the expected revenue from the direct sales of visited firms for customers' first choice products, expected revenue from the substitute products from the visited firm's assortment, expected revenue from the transshipment for the first choice products, and the expected revenue from the substitute products from another firm's assortment via transshipment. The last term in (3.1) indicates the total assortment cost for all the products in the assortments of M firms. The objective of the central manager is to determine the optimal assortment for each firm m, N_m, to maximize the total expected profit of the system.

Linear model of the problem can be given as follows:

$$max\ \Pi = \lambda \sum_{m=1}^{M} q_m \sum_{i=1}^{|S|} \left[\alpha_i r x_{im} + \sum_{j=1}^{|S|} \theta r \alpha_i \frac{\alpha_j}{1-\alpha_i} y_{ijm} \right]$$
$$+ \lambda \sum_{m=1}^{M} q_m \sum_{i=1}^{|S|} \left[\alpha_i (r-t)(u_i - x_{im}) + \sum_{j=1}^{|S|} \theta (r-t) \alpha_i \frac{\alpha_j}{1-\alpha_i} (z_{ij} - y_{ijm}) \right]$$
$$- \kappa \sum_{m=1}^{M} \sum_{i=1}^{|S|} x_{im} \tag{3.2}$$

s.t.

$$\sum_{m=1}^{M} x_{im} \geq u_i \quad \forall i \in S \tag{3.3}$$

$$u_i \geq x_{im} \qquad \forall i \in S, m = 1..M \tag{3.4}$$

$$z_{ij} \geq u_j - u_i \qquad \forall i, j \in S \tag{3.5}$$

$$z_{ij} \leq \frac{(1-u_i) + u_j}{2} \qquad \forall i, j \in S \tag{3.6}$$

$$y_{ijm} \geq x_{jm} - x_{im} \quad \forall i, j \in S, m = 1..M \tag{3.7}$$

$$y_{ijm} \leq \frac{(1-x_{im}) + x_{jm}}{2} \qquad \forall i, j \in S, m = 1..M \tag{3.8}$$

$$x_{im}, y_{ij}, z_{ijm}, u_i \in \{0, 1\} \tag{3.9}$$

The objective function in (3.2) is clearly a translated form of (3.1). The first constraint (3.3) states that for each product i, if product i is included in overall assortment, then it has to be available in at least one of the firms. The second constraint (3.4) ensures that if product i is included in assortment of firm m, then it has to be available in the overall assortment. Third constraint (3.5) ensures that if product j is available in the overall assortment but not product i, then substitution occurs from product i to product j. Fourth constraint (3.6) states that if either product i is available or product j is not available in the overall assortment, then substitution does not occur. If product i is not available but product j is, then third and fourth constraints (3.5,3.6) force z_{ij} to be 1 so that the substitution occurs. The fifth constraint (3.7) is specific version of the third constraint (3.5) which guarantees that if product i is not available in firm m but product j is present, substitution occurs within firm m. Similarly, the sixth constraint (3.8) is a version of fourth constraint (3.6) specifical to firm m which defines the condition of substitution from product i to product j within firm m. Finally, we define each of $x_{im}, y_{ijm}, z_{ij}, u_i$ as binary variables. In Section 3.2, we investigate the properties of the optimal assortments by analyzing the expected profit function.

3.2 Optimal Assortments

The optimal assortment for each firm can be found with full enumeration over all possible assortments at all firms. Since each assortment is a subset of the complete product set with $|S|$ elements, each firm may have as many as 2^S different assortments including the null assortment. Then for M firms, a total of $2^{M\,|S|}$ possible assortments should be enumerated. For instance, for a system of 4 retailers and 6 candidate products, 16,777,216 possible assortment combinations should be evaluated and compared. In this section of the study, we investigate the existence of some structural properties of the optimal assortments that can lead us to develop algorithms to reach the optimal solution more efficiently than the full enumeration. Moreover, some optimality properties that exist under restricted situations can be extended to more generic settings to obtain close-to-optimal solutions.

Firstly, we analyze the effect of market size on the assortment of a firm to understand firm-specific assortments. We can show that if firm k has a larger market share than firm m, then it is more profitable to add a product to the assortment of firm k rather than firm m. Thus, the assortment of a firm will be subset of the assortment of a relatively larger firm.

Theorem 1. *In any optimal assortment, if $q_k \geq q_m$, then $N_k \supseteq N_m$. Thus, if firms are indexed in a descending order of their market shares such that $q_1 \geq q_2 \geq ... \geq q_M$, then any optimal assortment satisfies $N_1 \supseteq N_2 \supseteq ... \supseteq N_M$.*

Proof. Assume the contrary that the assortment $A = (N_1, .., N_k, .., N_m, ..., N_M)$ is optimal with $N_k \not\supseteq N_m$. Then construct a new assortment $A' = (N_1, .., (N_k \cup N_m), .., (N_k \cap N_m), .., N_M)$. The total profit of this new assortment set can be written as:

$$\begin{aligned}\Pi(A') = \Pi(A) &+ \lambda(q_k - q_m)r \sum_{i \in N_m \setminus N_k} \alpha_i + \lambda(q_k - q_m)\theta r \sum_{i \notin N} \sum_{j \in N_m \setminus N_k} \alpha_i \delta_{ij} \\ &- \lambda(q_k - q_m)(r - t) \sum_{i \in N_m \setminus N_k} \alpha_i - \lambda(q_k - q_m)\theta(r - t) \sum_{i \notin N} \sum_{j \in N_m \setminus N_k} \alpha_i \delta_{ij} \\ &= \Pi(A) + \lambda(q_k - q_m)t \sum_{i \in N_m \setminus N_k} \alpha_i + \lambda(q_k - q_m)\theta t \sum_{j \notin N} \sum_{i \in N_m \setminus N_k} \alpha_j \delta_{ji}. \end{aligned} \quad (3.10)$$

Because $q_k \geq q_m$, it follows that $\Pi(A') \geq \Pi(A)$. Since the assortment A' leads to a higher expected profit, A cannot be optimal. $\square$

In fact, Theorem 1 also holds with product-specific product margins r_i (see Appendix A for the proof). Thus, even if the products differ in their profit margins, the optimal firm assortments are such that a firm's assortment is always a subset of another firm with an equal or higher market share. The illustration of the assortment sets can be seen in Figure 3.1. S is the set of all possible products. Firm 1 has the highest proportion of customer, thus N_1 has the largest assortment and each successive assortment set spans the remaining ones up to firm M.

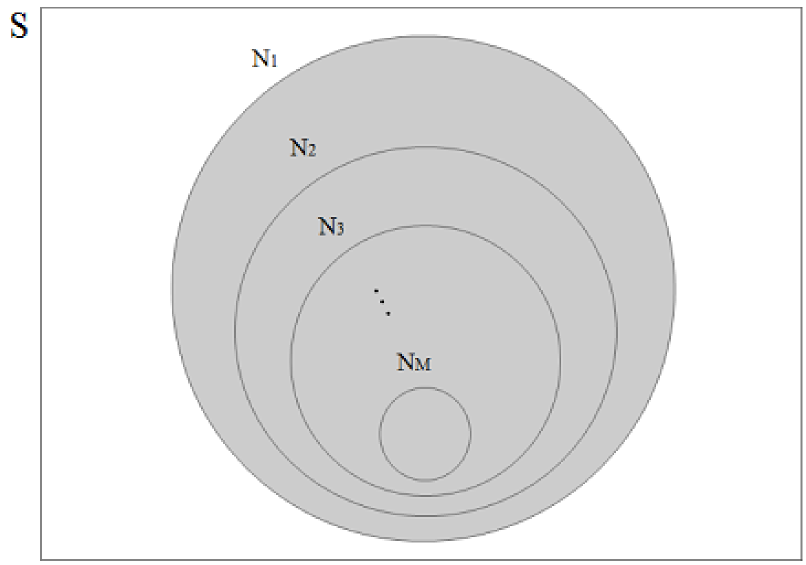

Figure 3.1: Assortment Sets of M Firms

Next, we can show that the common assortment for all firms include a set of most popular products called as the *popular set* (Kok and Xu [43]). Popular assortment set is the set of products in descending order according to their purchase probabilities in our model. The most popular product i, having the largest α_i is indexed by $i = 1$ and accordingly remaining ones are sorted in decreasing order of α_i. Popular set also includes the null set in an extreme case. Let P denote the popular set. Then, $P = \big\{\{\}, \{1\}, \{1, 2\}, .., \{1, 2, .., |S|\}\big\}$.

Theorem 2. *There does not exist any products x and y such that $x \notin \bar{N}$ and $y \in \bar{N}$ with $\alpha_x > \alpha_y$, where $\bar{N} = (N_1 \cap N_2 \cap .. \cap N_M)$ is the common assortment of all firms. Common assortment $\bar{N}$ is in the popular set.*

Proof. Let $(N_1, N_2, .., N_M)$ be an existing assortment and $x, y \notin N$. Consider including product x to every firm's assortment. The total profit of this new assortment is as follows.

$$
\begin{aligned}
\Pi(N_1 \cup \{x\}, .., N_M \cup \{x\}) = \Pi(N_1, .., N_M) &+ \lambda\alpha_x r \sum_{m=1}^{M} q_m + \lambda\theta r \sum_{m=1}^{M} q_m \sum_{i \notin N} \alpha_i \delta_{ix} \\
&- \lambda\theta r \sum_{m=1}^{M} q_m \sum_{i \in N_m} \alpha_x \delta_{xi} - \lambda\theta(r-t) \sum_{m=1}^{M} q_m \sum_{i \in N \setminus N_m} \alpha_x \delta_{xi} \\
&- \kappa M. \qquad (3.11)
\end{aligned}
$$

Now consider including product y to every firm's assortment.

$$
\begin{aligned}
\Pi(N_1 \cup \{y\}, .., N_M \cup \{y\}) = \Pi(N_1, .., N_M) &+ \lambda\alpha_y r \sum_{m=1}^{M} q_m + \lambda\theta r \sum_{m=1}^{M} q_m \sum_{i \notin N} \alpha_i \delta_{iy} \\
&- \lambda\theta r \sum_{m=1}^{M} q_m \sum_{i \in N_m} \alpha_y \delta_{yi} - \lambda\theta(r-t) \sum_{m=1}^{M} q_m \sum_{i \in N \setminus N_m} \alpha_y \delta_{yi} \\
&- \kappa M. \qquad (3.12)
\end{aligned}
$$

The difference between (3.11) and (3.12) is

$$
\begin{aligned}
&\Pi(N_1 \cup \{x\}, .., N_M \cup \{x\}) - \Pi(N_1 \cup \{y\}, .., N_M \cup \{y\}) \\
&= \lambda(\alpha_x - \alpha_y) r \sum_{m=1}^{M} q_m + \lambda\theta r \sum_{m=1}^{M} q_m \sum_{i \notin N} (\alpha_i \delta_{ix} - \alpha_i \delta_{iy}) \\
&\quad - \lambda\theta r \sum_{m=1}^{M} q_m \sum_{i \in N_m} \big(\alpha_x \delta_{xi} - \alpha_y \delta_{yi}\big) - \lambda\theta(r-t) \sum_{m=1}^{M} q_m \sum_{i \in N \setminus N_m} \big(\alpha_x \delta_{xi} - \alpha_y \delta_{yi}\big).
\end{aligned}
\qquad (3.13)
$$

If we replace δ_{ij} with its open form $\frac{\alpha_j}{1-\alpha_i}$ and divide the expression by $\lambda\theta(\alpha_x - \alpha_y)r$, then (3.13) can be rewritten as follows.

$$
\begin{aligned}
\sum_{m=1}^{M} q_m \Big[\frac{1}{\theta} + \sum_{i \notin N} \frac{\alpha_i}{1-\alpha_i} - \frac{1}{(1-\alpha_x)(1-\alpha_y)} \Big(\sum_{i \in N_m} \alpha_i + \sum_{i \in N \setminus N_m} \alpha_i \Big) \\
+ \frac{t}{(1-\alpha_x)(1-\alpha_y) r} \sum_{i \in N \setminus N_m} \alpha_i \Big].
\end{aligned}
\qquad (3.14)
$$

There are four main terms in square brackets in (3.14). The second and fourth terms are clearly positive. Now consider the first and third terms together.

$$
\frac{1}{\theta} - \frac{1}{(1-\alpha_x)(1-\alpha_y)} \sum_{i \in N} \alpha_i. \qquad (3.15)
$$

In order to show that (3.15) is non-negative, we need to show that

$$1 \geq \frac{\theta \sum\limits_{i \in N} \alpha_i}{(1-\alpha_x)(1-\alpha_y)}. \tag{3.16}$$

By rearranging the terms in (3.16), we can obtain

$$1 + \alpha_x \alpha_y \geq \theta \sum_{i \in N} \alpha_i + \alpha_x + \alpha_y. \tag{3.17}$$

By definition, it holds that $\theta \leq 1$ and $\sum\limits_{i \in N} \alpha_i + \alpha_x + \alpha_y \leq 1$ as $\sum\limits_{i \in S} \alpha_i = 1$. Thus, inequality (3.17) holds. □

Next, we investigate whether the complete assortment of a firm is in the popular set. We find that if the substitution probability θ is small enough, a firm's optimal assortment should be in the popular set.

Theorem 3. *If firms are indexed such that $q_1 \geq q_2 \geq .... \geq q_M$, then the optimal assortment of firm m is in the popular set if*

$$\theta \leq \frac{r - t \sum\limits_{k=m+1}^{M} q_k}{r}, \tag{3.18}$$

for $m \in \{1, 2, ..., M\}$.

Proof. Consider an existing assortment $(N_1, N_2, .., N_M)$ and adding product x or y to the assortment of firm m. From Theorem 1, we know that at optimality, it should hold that $N_1 \supseteq N_2 \supseteq ... \supseteq N_M$. Thus, if a product is added to the assortment of firm m, it should also be added to the assortments of firms, 1 through $m-1$.

Thus, when product x is added to the assortments of firms 1 through m, the total profit of the new assortment $(N_1 \cup \{x\}, \ldots, N_m \cup \{x\}, N_{m+1}, \ldots, N_M)$ can be rewritten as follows.

$$\begin{aligned}
&\Pi(N_1 \cup \{x\}, \ldots, N_m \cup \{x\}, N_{m+1}, \ldots, N_M) \\
&= \Pi(N_1, \ldots, N_m, N_{m+1}, \ldots, N_M) + \lambda\alpha_x r \sum_{k=1}^{m} q_k + \lambda\alpha_x (r-t) \sum_{k=m+1}^{M} q_k \\
&\quad + \lambda\theta r \sum_{k=1}^{m} q_k \sum_{i \notin N \cup \{x\}} \alpha_i \delta_{ix} + \lambda\theta(r-t) \sum_{k=m+1}^{M} q_k \sum_{i \notin N \cup \{x\}} \alpha_i \delta_{ix} \\
&\quad - \lambda\theta r \sum_{k=1}^{M} q_k \sum_{i \in N_k} \alpha_x \delta_{xi} - \lambda\theta(r-t) \sum_{k=1}^{M} q_k \sum_{i \in N \setminus N_k} \alpha_x \delta_{xi}. \qquad (3.19)
\end{aligned}$$

Given that $\sum_{m=1}^{M} q_m = 1$, (3.19) can be rewritten as,

$$\begin{aligned}
&\Pi(N_1, \ldots, N_m, \ldots, N_M) + \lambda\alpha_x (r - t \sum_{k=m+1}^{M} q_k) + \lambda\theta(r - t \sum_{k=m+1}^{M} q_k) \sum_{i \notin N \cup \{x\}} \alpha_i \delta_{ix} \\
&- \lambda\theta(r - t \sum_{k=m+1}^{M} q_k) \sum_{i \in N} \alpha_x \delta_{xi} + \lambda\theta t \sum_{k=1}^{m} q_k \sum_{i \in N} \alpha_x \delta_{xi} - \lambda\theta t \sum_{k=1}^{M} q_k \sum_{i \in N_k} \alpha_x \delta_{xi}.
\end{aligned}$$

When product y is added to the assortments of firms 1 through m, the total profit of the assortment $(N_1 \cup \{y\}, \ldots, N_k \cup \{y\}, N_{k+1}, \ldots, N_M)$ can be written similarly. Then the difference in profits is as follows.

$$
\begin{aligned}
&\Pi(N_1 \cup \{x\}, .., N_m \cup \{x\}, N_{m+1}, .., N_M) - \Pi(N_1 \cup \{y\}, .., N_m \cup \{y\}, N_{m+1}, .., N_M) \\
&= \lambda(\alpha_x - \alpha_y)(r - t \sum_{k=m+1}^{M} q_k) \\
&\quad + \lambda\theta(r - t \sum_{k=m+1}^{M} q_k)\Big[\sum_{i \notin N \cup \{x,y\}} \frac{\alpha_i(\alpha_x - \alpha_y)}{1-\alpha_i} + \frac{\alpha_y \alpha_x}{1-\alpha_y} - \frac{\alpha_x \alpha_y}{1-\alpha_x}\Big] \\
&\quad - \lambda\theta(r - t \sum_{k=m+1}^{M} q_k)\Big[\alpha_x\Big(\frac{1 - \alpha_x - \alpha_y - \sum_{i \notin N \cup \{x,y\}} \alpha_i}{1-\alpha_x}\Big) \\
&\qquad\qquad - \alpha_y\Big(\frac{1 - \alpha_x - \alpha_y - \sum_{i \notin N \cup \{x,y\}} \alpha_i}{1-\alpha_y}\Big)\Big] \\
&\quad + \lambda\theta t \sum_{k=1}^{m} q_k \sum_{i \in N} \alpha_i \big(\frac{\alpha_x}{1-\alpha_x} - \frac{\alpha_y}{1-\alpha_y}\big) \\
&\quad - \lambda\theta t \sum_{k=1}^{M} q_k \sum_{i \in N_k} \alpha_i \big(\frac{\alpha_x}{1-\alpha_x} - \frac{\alpha_y}{1-\alpha_y}\big).
\end{aligned} \tag{3.20}
$$

(3.20) can be rearranged to obtain the equivalent equation below.

$$
\begin{aligned}
&\lambda(\alpha_x - \alpha_y)(r - t \sum_{k=m+1}^{M} q_k) \\
&\quad + \lambda\theta(r - t \sum_{k=m+1}^{M} q_k)\Big[\sum_{i \notin N \cup \{x,y\}} \frac{\alpha_i(\alpha_x - \alpha_y)}{1-\alpha_i} + \frac{\alpha_y \alpha_x}{1-\alpha_y} - \frac{\alpha_x \alpha_y}{1-\alpha_x}\Big] \\
&\quad - \lambda\theta(r - t \sum_{k=m+1}^{M} q_k)\Big[(\alpha_x - \alpha_y) + \frac{\alpha_y \alpha_x}{1-\alpha_y} - \frac{\alpha_x \alpha_y}{1-\alpha_x} \\
&\qquad\qquad - \frac{(\alpha_x - \alpha_y)}{(1-\alpha_x)(1-\alpha_y)} \sum_{i \notin N \cup \{x,y\}} \alpha_i\Big] \\
&\quad + \lambda\theta t \frac{(\alpha_x - \alpha_y)}{(1-\alpha_x)(1-\alpha_y)}\Big[\sum_{k=1}^{m} q_k \sum_{i \in N} \alpha_i - \sum_{k=1}^{M} q_k \sum_{i \in N_k} \alpha_i\Big].
\end{aligned} \tag{3.21}
$$

Dividing (3.21) by $\lambda(\alpha_x - \alpha_y)$ and simplifying further, we get

$$\begin{aligned}
&(r - t\sum_{k=m+1}^{M} q_k) + \theta(r - t\sum_{k=m+1}^{M} q_k)\sum_{i\notin N\cup\{x,y\}}\frac{\alpha_i}{1-\alpha_i} \\
&- \theta(r - t\sum_{k=m+1}^{M} q_k) + \frac{\theta(r - t\sum\limits_{k=m+1}^{M} q_k)}{(1-\alpha_x)(1-\alpha_y)}\sum_{i\notin N\cup\{x,y\}}\alpha_i \\
&+ \frac{\theta t}{(1-\alpha_x)(1-\alpha_y)}\Big[\sum_{k=1}^{m} q_k\sum_{i\in N}\alpha_i - \sum_{k=1}^{M} q_k\sum_{i\in N_k}\alpha_i\Big].
\end{aligned} \tag{3.22}$$

The second and the fourth terms in (3.22) are clearly positive. Since $N_k \subseteq N$, the expression within square brackets in the last term is clearly larger than $-\sum_{k=m+1}^{M} q_k \sum_{i\in N_k}\alpha_i$. Therefore (3.22) is larger than or equal to

$$(r - t\sum_{k=m+1}^{M} q_k) - \theta(r - t\sum_{k=m+1}^{M} q_k) - \frac{\theta t}{(1-\alpha_x)(1-\alpha_y)}\sum_{k=m+1}^{M} q_k\sum_{i\in N_k}\alpha_i. \tag{3.23}$$

Consider the term $\frac{\sum\limits_{i\in N_k}\alpha_i}{(1-\alpha_x)(1-\alpha_y)}$. This can be rewritten as

$$\frac{1-\alpha_x-\alpha_y-\sum\limits_{i\notin N_k\cup\{x,y\}}\alpha_i}{1-\alpha_x-\alpha_y+\alpha_x\alpha_y},$$

which is clearly lower than 1. Therefore (3.23) is larger than

$$(r - t\sum_{k=m+1}^{M} q_k) - \theta(r - t\sum_{k=m+1}^{M} q_k) - t\theta\sum_{k=m+1}^{M} q_k. \tag{3.24}$$

For (3.24) to be non-negative, it should hold

$$\theta \le \frac{r - t\sum_{k=m+1}^{M} q_k}{r}.$$

This completes the proof. □

Note that the condition (3.18) is a sufficient condition, not necessary. Thus, optimal assortment of a firm can be in the popular set under more general settings.

From Theorem 1 and Theorem 2, we know that the optimal assortment of the smallest firm is also equal to the common assortment $N_M = \bar{N}$ and it is in the popular set. Theorem 3 also confirms this result as for $m = M$, (3.18) turns into $\theta \leq 1$, which holds by definition. It is also clear that as $q_1 \geq q_2 \geq .... \geq q_M$, if N_k is in the popular set, then N_m is also in the popular set for $q_k \geq q_m$. This can be better seen with Corollary 1.

Corollary 1. *If firms are indexed such that $q_1 \geq q_2 \geq .... \geq q_M$, the optimal assortment of firm m is in the popular set if*

$$\theta \leq 1 - \frac{t(M-m)}{rM}.$$

Proof. Since $q_1 \geq q_2 \geq \ldots \geq q_{M-1} \geq q_M$, $\sum_{k=m+1}^{M} q_k \leq \frac{M-m}{M}$. Using this information, we get the desired result. $\square$

From Corollary 1, for a given θ, as m increases (so, q_m decreases), it is more probable for the firm m to have an assortment in the popular set. In other words, it is more likely for the larger firms to have assortments that are not in the popular set, i.e., to include less popular products in their assortments, while excluding some more popular products. This result is mainly due to the allowance of transshipments in case of a need, which has a non-zero transshipment cost. When $t = 0$, then (3.18) turns into $\theta \leq 1$, which holds by definition. The transshipment cost has a greater effect on the assortment of a larger firm as larger firms keep more products by Theorem 1. So they are prone to more transshipment costs if they keep some popular products, which are not kept by smaller firms. By letting them to keep some less popular products, while more popular products are kept out of their assortments, customers are pushed to substitute their first-choice demands with second choices that can be available at visited stores. Thus, some transshipment cost can be avoided.

3.3 Notes on Complexity

As we state in Section 3.2, the optimal assortment for each firm can be found by fully enumerating $2^{M|S|}$ possible assortments for $|S|$ available products and M firms. Next, we show how the structural properties obtained in Section 3.2 can decrease the number of necessary iterations to obtain optimal assortments.

By Theorem 1, at optimality $N_1 \supseteq N_2 \supseteq .... \supseteq N_M$ for $q_1 \geq q_2 \geq \ldots \geq q_M$. We call this a *nested* assortment.

Theorem 4. *The number of nested assortments is*

$$\sum_{i=0}^{M} \binom{M+1}{i} \sum_{j=0}^{M+1-i} (-1)^{M+1-i-j} \binom{M+1-i}{j} j^{|S|}. \tag{3.25}$$

Proof. Since we have $N_1 \supseteq N_2 \supseteq .... \supseteq N_M$, each possible assortment is a partition of the set S into $M+1$ sets $N_M, N_{M-1} \setminus N_M, \ldots, N_1 \setminus N_2, N \setminus N_1$. The number of ways to partition a set of x labeled objects into y nonempty unlabeled subsets is known as the Stirling number of the second kind and is denoted by

$$\left\{ \begin{array}{c} x \\ y \end{array} \right\} = \frac{1}{y!} \sum_{j=0}^{y} (-1)^{y-j} \binom{y}{j} j^{\,x}. \tag{3.26}$$

In our case, the partitions are labeled and we also allow for empty subsets. The number of ways to partition a set of x labeled objects into y nonempty labeled subsets is given by

$$\sum_{j=0}^{y} (-1)^{y-j} \binom{y}{j} j^{\,x}. \tag{3.27}$$

y subsets can be labeled in $y!$ different ways. Up to $y-1$ of these partitioned sets can be empty sets. If there are i empty sets, these can be assigned to y spots in $\binom{y}{i}$ ways and there are still x objects to be partitioned into $y-i$ nonempty subsets. Therefore the number of ways to partition a set of x labeled objects into

y (possibly empty) labeled subsets is

$$\sum_{i=0}^{y} \binom{y}{i} \sum_{j=0}^{y-i} (-1)^{y-i-j} \binom{y-i}{j} j^n. \tag{3.28}$$

As $y = M + 1$ and $x = |S|$ in our problem, we get the desired result. □

By Theorem 2, the common assortment $\bar{N}$, which is also equal to the assortment of the smallest firm, firm M, is in the popular set. This result reduces the number of candidate assortments further.

Theorem 5. *The number of nested assortments where firm M's assortment is in the popular set is*

$$\sum_{k=0}^{|S|} \sum_{i=0}^{M-1} \binom{M}{i} \sum_{j=0}^{M-i} (-1)^{M-i-j} \binom{M-i}{j} j^{|S|-k}.$$

Proof. If firm M's assortment is in the popular set that includes the most popular k products for $k \in \{0, 1, \ldots, |S|\}$, then we are left with a problem with $M - 1$ firms and $|S| - k$ products. □

If a firm's assortment is in the popular set besides being nested, we call this a *nested* and *popular* assortment. If the condition on θ in (3.18) is satisfied for all M firms, then the number of candidate assortments decreases further.

Theorem 6. *The number of nested and popular assortments is*

$$\binom{M + |S|}{M}.$$

Proof. Each possible assortment is a partition of the set S into $M + 1$ sets $N_M, N_{M-1} \backslash N_M, \ldots, N_1 \backslash N_2, N \backslash N_1$. However since each assortment is in the popular set, this is equivalent to partitioning a sequence of $|S|$ numbers into $M+1$ subsequences. This can be also represented as an integer partitioning problem where an integer x is to be represented sum of y numbers, i.e., $X_1 + X_2 + \ldots + X_y = x$. If all X_i's are positive integers, these are called compositions and there are $\binom{x-1}{y-1}$ combinations. If X_i can also take the value of zero, these are called weak compositions and there are $\binom{x+y-1}{y-1}$ combinations. In our case, we have $y = M+1$ and $x = |S|$ and we obtain the desired result. $\square$

Next, we compare the number of candidate assortments to be evaluated to obtain optimal assortments with full enumeration and under the use of various optimality properties obtained in the study. Table 3.2 reports results for different number of firms and available product types.

Table 3.2: The Number of Candidate Assortments to be Evaluated To Obtain Optimal Solution.

M	$\lvert S\rvert$	Full Enumeration	Nested	Nested & Common in Popular	Nested & All in Popular
3	5	32,768	1,024	364	56
3	6	262,144	4,096	1,093	84
4	5	1,048,576	3,125	1,365	126
4	6	16,777,216	15,625	5,461	210
5	5	33,554,432	7776	3906	252
5	6	1,073,741,824	46,656	19,531	462

According to Table 3.2, for a relatively small system of $M = 3$ firms and 5 product types, by only knowing that each firm's assortment is a subset of another with a larger market size, thus optimal assortments will be nested, the number of potential optimal assortments can be decreased by 96,9% from 32,768 to 1,024. For a relatively large system with $M = 5$ and $|S| = 6$, the number of nested assortments to be evaluated is only 0.004 % of those in full enumeration. This result is especially important given that nested assortments exist even if the product margins are product-specific r_i. Otherwise, for the same product margins, we know that the common assortment is in the popular set, which decreases the number of candidate assortments to 0.0018% of the full enumeration for the problem with $M = 5$ and $|S| = 6$. If θ is small enough that all firm assortments are in the popular set, the number of possible assortments can be decreased to 0.000043% of those in full enumeration.

Chapter 4

Computational Results

In this chapter, we present the results of numerical analyses that aim to support and complement the analytical results obtained. In §4.1, we investigate the sensitivity of the optimal assortments to model inputs. When performing this analysis, we generate 200 problem instances of which we change the value of one paramater and keep the rest of them the same. We report the variations in some metrics through Table 4.2 to 4.6. Then, in §4.2, we compare the results of full enumeration to the case when there is no transshipment. In this section, we present a condition under which allowing transshipment is more profitable along with contradictory examples where not allowing transshipment is superior. Finally, in §4.3, we introduce a heuristic solution and test its performance in comparison to the optimal policy of the system. We calculate the gaps between heuristic and full enumeration over the results of the non-popular optimal sets. Finally, in order to give more insight we provide some illustrative examples.

Table 4.1: Uniform Distributions of Randomly Generated Parameters

q_j	r_i	α_{ij}	λ	θ	t	v	β
U(0,1)	U(1,5)	U(0,1)	100	U(0.3,1)	U(0,1)	0.16	U(0.2,0.6)

During numerical analyses, we use a broad range of randomized parameters to increase the robustness of the results. Each random parameter is selected from Uniform distrubiton with the ranges summarized in Table 4.1. The assortment cost is calculated according to the following formula;

$$\kappa = \lambda v \left(\frac{1}{n}\right)^{\beta}.$$

Regarding the assortment cost, the formula we used here is almost similar to the formula in Cachon *et al.* [23] for calculating the operational cost, $c(q_i)$ associated with including a product in the assortment. Their cost function is defined as; $c(q_i) = vq_i^{\beta}$. The difference between two formulations is that our preference of $\left(\frac{1}{n}\right)$ instead of q_i which denotes the share of demand for each product. Since we assume identical assortment costs across products, we replace q_i with the average demand rate, $\left(\frac{1}{n}\right)$, where n represents the total number of products. The ranges for v and β in Table 4.1 are also analogous with Cachon *et al.* [23].

4.1 Sensitivity Analysis

In this section, we investigate how the system inputs substitution rate θ, transshipment cost t, profit margin r, assortment cost κ and the asymmetry in market shares of firms affect the optimal assortments and the related system metrics. All the metrics are obtained with the optimal assortments. Particularly, we use following metrics to measure the effects of system inputs.

- Percent change in total profit ($\Delta\%$),
- Number of products in total assortment N,
- Number of different products in total assortment N,
- Number of products in the common assortment $\bar{N}$,
- Percentage of sales through transshipment in total revenue, both for the first and second-choice demands ($TS\%$),
- Percentage of sales for the substituted demand from stock in total revenue ($SS\%$),

- Percentage of direct sales from stock in total revenue ($DS\%$),
- Percentage of assortment cost in total revenue ($AC\%$).

For a 2-firm system, the number of products in total assortment is equal to the sum of number of different products and the number of products in the common assortment. Total revenue is composed of direct customer sales for the customers' first-choice demands, sales through transshipment for the first-choice demand, sales of substituted demand from stock, and the sales of substituted demand through transshipment. We define percentage of sales for the substituted demand in total revenue ($SS\%$) to cover only the substitution sales from firms' own stock. The substitution sales that is satisfied from another firm through transshipment is measured within $TS\%$. Thus, $TS\% + SS\% + DS\% = 100\%$. Total profit, Π, is equal to total revenue minus the assortment cost. We can associate above definitions with the objective function as follows.

$$\text{Transshipment Sales} : \lambda \sum_{m=1}^{M} q_m \sum_{i \in N/N_m} \alpha_i (r-t) + \theta (r-t) \sum_{j \notin N} \alpha_j \delta_{ji}$$
$$\text{Substitution Sales} : \lambda \sum_{m=1}^{M} q_m \sum_{i \in N_m} \theta r \sum_{j \notin N} \alpha_j \delta_{ji}$$
$$\text{Direct Sales} : \lambda \sum_{m=1}^{M} q_m \sum_{i \in N_m} \alpha_i r$$
$$\text{Assortment Cost} : \kappa \sum_{m=1}^{M} |N_m|$$

To test the effect of each parameter, we used 200 problem instances for a 2-firm system with 5 possible products, i.e., $M = 2$ and $|S| = 5$. For each problem instance, we set one of the parameters to some certain value and generate the rest of the parameters randomly from continuous Uniform distributions with the ranges summarized in Table 4.1. For the same fixed parameter, we generate 10 problem instances, in each of which the remaining parameters are randomly generated. By taking the average of performance measures over each of 10 problem instances, we indicate the average performance under this fixed parameter. For example, average percent change in total profit is calculated by taking the average of total optimal profits over 10-problem instances for each 10-problem set among 200 problems. Then by accepting the average total profit of the first 10-problem set as the basis, for the other 10-problem set profit averages, the percent deviation from the first-set average is calculated. For the other metrics, not the relative, but the absolute average performance is calculated. By reporting the average metrics over 10 problem instances, the effect of the fixed parameter on performance metrics can be evaluated more rigorously by eliminating the bias selection effects of other parameters.

We first test the effects of substitution probability θ. Table 4.2 shows that as θ increases, although the number of products in the total assortment decreases, total profit increases steadily. While keeping less number of products saves from assortment cost, the percentage of direct sales from stock in total revenue decreases from 83.24% to 70.69%. On the other hand, as the substitution probability increases, the contribution of substitutions to total revenue increases its share from 0.26% to 13.90%. The effect of θ on the share of transshipments does not seem to be significant.

Next we check how the transshipment cost t affect the optimal solution. As t increases, percentage of revenue from transshipped products decreases significantly since it becomes more costly to transship products. This is achieved by having more products in common and having more products carried in each location, thus avoiding the need for transshipment. Having a larger assortment increases direct sales and substitution sales as well. Because of both higher transshipment cost and assortment cost, as expected, total profit decreases while t is increased.

We next study the effect of the product profit margin r in Table 4.4. It comes as no surprise that total profit increases significantly. The number of products in total assortment increases, while the size of common assortment, which is equivalent to N_2 in a 2-firm setting by Theorem 1, is stable. Thus, as the product margin increases, it is preferred to add new products to the assortment of larger firm to transfer them to smaller firm in case of a need. Hence, it can be seen that the share of transshipment sales increases steadily with r, so substitution need decreases. Although more products are kept as r increases, the share of assortment cost seems to decrease because of significantly increased total revenue.

Table 4.2: Sensitivity of optimal solution to θ in a 2-firm system with $\mid S \mid = 5$

θ	Δ% in Avg Π	Avg # of Prod. in N	Avg # of Diff. Prod. in N	Avg # of Prod. in $\bar{N}$	TS%	SS%	DS%	AC%
0.300	0.00	4.80	4.70	0.10	16.49	0.26	83.24	16.57
0.337	0.05	4.70	4.60	0.10	16.49	0.42	83.08	16.31
0.374	0.14	4.50	4.40	0.10	16.49	1.25	82.26	15.37
0.411	0.23	4.50	4.40	0.10	16.49	1.37	82.14	15.33
0.448	0.32	4.50	4.40	0.10	16.49	1.49	82.02	15.29
0.485	0.41	4.50	4.40	0.10	16.49	1.60	81.90	15.26
0.522	0.51	4.50	4.40	0.10	16.49	1.72	81.79	15.22
0.559	0.61	4.30	4.20	0.10	16.44	2.79	80.77	14.31
0.596	0.74	4.30	4.20	0.10	16.44	2.96	80.60	14.26
0.633	0.87	4.30	4.20	0.10	16.44	3.12	80.44	14.21
0.670	0.99	4.30	4.20	0.10	16.44	3.28	80.28	14.16
0.707	1.12	4.30	4.20	0.10	16.44	3.44	80.12	14.11
0.744	1.25	4.10	4.00	0.10	16.44	4.81	78.75	13.52
0.781	1.50	3.90	3.80	0.10	16.44	6.12	77.44	13.05
0.818	1.84	3.60	3.50	0.10	16.34	8.55	75.11	12.25
0.855	2.27	3.50	3.40	0.10	16.34	9.79	73.87	11.68
0.892	2.73	3.50	3.40	0.10	16.34	10.14	73.51	11.60
0.929	3.26	3.40	3.30	0.10	16.34	11.47	72.19	11.39
0.966	3.80	3.40	3.30	0.10	16.34	11.85	71.81	11.31
1.000	4.35	3.40	3.20	0.20	15.41	13.90	70.69	11.17

In Table 4.5 we illustrate the effect of assortment cost κ. κ affects the value of β through the function $\kappa = \lambda v(\frac{1}{n})^{\beta}$. Therefore, in order to vary κ, β is varied from 0.6 to 0.2. We observe that increasing assortment cost lowers the system profit. Less number of products are kept by both firms 1 and 2. Thus, more substitution is initiated and less direct sales are made.

Lastly, we investigate the effect of market share asymmetry, $|q_1 - q_2|$. In Table 4.6, we observe that total profit declines as the difference between firm sizes decreases. Apparently, the central agent does not benefit from the balance among its firms. In addition, as the difference between the firm sizes decreases, the percentage of direct sales decreases noticeably although total number of products increases. This is because of the increase in proportion of transshipment sales.

Table 4.3: Sensitivity of optimal solution to t in a 2-firm system with $\mid S \mid = 5$

t	Δ% in Avg Π	Avg # of Prod. in N	Avg # of Diff. Prod. in N	Avg # of Prod. in $\bar{N}$	TS%	SS%	DS%	AC%
0.00	0.00	4.10	4.10	0.00	26.54	4.84	68.62	12.99
0.05	-0.49	4.10	4.10	0.00	25.94	4.89	69.16	13.10
0.11	-0.98	4.10	4.10	0.00	25.56	4.93	69.51	13.20
0.16	-1.47	4.10	4.10	0.00	25.18	4.97	69.86	13.31
0.21	-1.97	4.10	4.10	0.00	24.78	5.00	70.22	13.43
0.26	-2.46	4.10	4.10	0.00	24.37	5.04	70.58	13.55
0.32	-2.95	4.10	4.10	0.00	23.96	5.08	70.96	13.67
0.37	-3.42	4.20	4.10	0.10	21.36	5.36	73.28	14.36
0.42	-3.87	4.20	4.10	0.10	20.98	5.40	73.63	14.46
0.47	-4.32	4.20	4.10	0.10	20.59	5.43	73.98	14.56
0.53	-4.75	4.20	4.00	0.20	17.91	6.47	75.62	14.30
0.58	-5.10	4.40	4.00	0.40	14.56	7.08	78.37	15.50
0.63	-5.40	4.50	4.00	0.50	12.76	7.12	80.12	15.77
0.68	-5.69	4.50	4.00	0.50	12.51	7.14	80.35	15.82
0.74	-5.97	4.70	4.00	0.70	10.32	7.28	82.40	16.27
0.79	-6.21	4.80	4.00	0.80	9.01	7.49	83.50	16.79
0.84	-6.43	4.70	4.90	0.80	8.15	8.72	83.13	16.34
0.89	-6.63	4.70	3.90	0.80	7.98	8.73	83.29	16.37
0.95	-6.82	4.80	3.90	0.90	7.23	8.74	84.03	16.62
1.00	-7.00	4.90	3.90	1.00	6.57	8.75	84.68	16.86

When there are more customers for the second firm and the system still benefits from keeping some products in the assortment of only one of the firms (which should be firm 1 by Theorem 1), more sales of the second firm are satisfied through transshipment. Hence, although the size of the common assortment is increasing, transshipment percentage rises above 35%.

4.2 Benefits of Transshipments

Next, we inspect the benefits of transshipment by comparing the performance of a system with transshipments to a system which is not utilizing transshipments.

Table 4.4: Sensitivity of optimal solution to r in a 2-firm system with $\mid S \mid= 5$

r	Δ% in Avg Π	Avg # of Prod. in N	Avg # of Diff. Prod. in N	Avg # of Prod. in $\bar{N}$	TS%	SS%	DS%	AC%
1.00	0.00	3.10	3.00	0.10	12.47	12.27	75.26	31.07
1.21	32.85	3.30	3.20	0.10	13.53	10.03	76.44	26.50
1.42	66.01	3.30	3.20	0.10	14.25	9.95	75.79	22.38
1.63	99.59	3.60	3.50	0.10	14.90	8.07	77.02	20.60
1.84	133.47	3.80	3.70	0.10	15.31	6.67	78.02	19.07
2.05	167.79	3.90	3.80	0.10	15.63	6.16	78.21	17.49
2.26	202.27	4.00	3.90	0.10	15.89	5.72	78.39	16.08
2.47	236.77	4.00	3.90	0.10	16.10	5.71	78.19	14.68
2.68	271.36	4.20	4.10	0.10	16.28	4.25	79.48	14.22
2.89	306.17	4.20	4.10	0.10	16.43	4.24	79.33	13.17
3.11	342.65	4.30	4.20	0.10	16.66	3.57	79.76	12.41
3.32	377.54	4.30	4.20	0.10	16.78	3.57	79.65	11.61
3.53	412.43	4.30	4.20	0.10	16.88	3.56	79.56	10.90
3.74	447.32	4.30	4.20	0.10	16.97	3.56	79.47	10.28
3.95	482.22	4.40	4.30	0.10	17.04	3.13	79.83	9.92
4.16	517.18	4.40	4.30	0.10	17.11	3.12	79.76	9.41
4.37	552.20	4.50	4.40	0.10	17.18	2.94	79.89	9.15
4.58	587.25	4.50	4.40	0.10	17.24	2.93	79.83	8.73
4.79	622.30	4.50	4.40	0.10	17.29	2.93	79.78	8.34
5.00	657.34	4.50	4.40	0.10	17.34	2.93	79.73	7.99

In both systems we obtain optimal assortments. In the *symmetric case* where all firms are of the same size ($q_1 = q_2 = ... = q_M$), we show that it is always optimal to allow transshipments. In the *asymmetric case* however, in which firms have non-identical market shares, it is not always true that allowing transshipment yields higher profits over no-transshipment case.

4.2.1 Case of Identical Firms

Let the total profit for the system where transshipments are allowed is denoted by Π and the total profit for the system without transshipments is denoted by

Table 4.5: Sensitivity of optimal solution to κ in a 2-firm system with $\mid S \mid= 5$

κ	Δ% in Avg Π	Avg # of Prod. in N	Avg # of Diff. Prod. in N	Avg # of Prod. in $\bar{N}$	TS%	SS%	DS%	AC%
6.09	0.00	4.30	4.20	0.10	16.34	4.44	79.22	9.48
6.30	-0.31	4.30	4.20	0.10	16.34	4.44	79.22	9.80
6.52	-0.63	4.30	4.20	0.10	16.34	4.44	79.22	10.14
6.74	-0.95	4.30	4.20	0.10	16.34	4.44	79.22	10.49
6.98	-1.29	4.30	4.20	0.10	16.34	4.44	79.22	10.85
7.22	-1.65	4.30	4.20	0.10	16.34	4.44	79.22	11.22
7.46	-2.01	4.20	4.20	0.10	16.34	4.62	79.04	11.40
7.72	-2.37	4.10	4.00	0.10	16.34	5.25	78.41	11.59
7.99	-2.74	4.10	4.00	0.10	16.34	5.25	78.41	11.99
8.26	-3.12	4.10	4.00	0.10	16.34	5.25	78.41	12.40
8.55	-3.52	4.10	4.00	0.10	16.34	5.25	78.41	12.83
8.84	-3.93	4.00	3.90	0.10	16.34	6.21	77.45	12.86
9.15	-4.34	4.00	3.90	0.10	16.34	6.21	77.45	13.31
9.46	-4.77	4.00	3.90	0.10	16.34	6.21	77.45	13.76
9.79	-5.22	4.00	3.90	0.10	16.34	6.21	77.45	14.24
10.13	-5.68	4.00	3.90	0.10	16.34	6.21	77.45	14.73
10.48	-6.15	4.00	3.90	0.10	16.34	6.21	77.45	15.24
10.84	-6.63	3.80	3.80	0.00	17.40	6.88	75.72	14.92
11.21	-7.11	3.70	3.70	0.00	17.40	7.70	74.90	14.58
11.60	-7.60	3.70	3.70	0.00	17.40	7.70	74.90	15.08

$\Pi^{w/o}$.

Lemma 1. *When $q_i = 1/M$ for all i, allowing transshipments is always beneficial, i.e.,$\Pi^* \geq \Pi^{w/o*}$.*

Proof. In a no-transshipment system, assortment of each firm is obtained independently. If $q_1 = q_2 = ... = q_M = 1/M$, then we should have $N_1^{w/o*} = N_2^{w/o*} = ... = N_M^{w/o*}$. Now consider the system with transshipment that has the assortment $(N_1^{w/o*}, N_2^{w/o*}, ..., N_M^{w/o*})$. Since the assortments are same, there will be no transshipments even when the transshipment is allowed. Therefore, we should have

$$\Pi(N_1^{w/o*}, ..., N_M^{w/o*}) = \Pi^{w/o}(N_1^{w/o*}, ..., N_M^{w/o*}) = \Pi^{w/o*},$$

Table 4.6: Sensitivity of optimal solution to market share asymmetry in a 2-firm system with $\mid S \mid= 5$

q_1	q_2	Δ% in Avg Π	Avg # of Prod. in N	Avg # of Diff. Prod. in N	Avg # of Prod. in $\bar{N}$	TS%	SS%	DS%	AC%
0.90	0.10	0.00	4.00	4.00	0.00	8.48	6.60	84.92	13.10
0.88	0.12	-0.34	4.00	4.00	0.00	10.30	6.47	83.22	13.15
0.86	0.14	-0.68	4.10	4.00	0.10	11.66	6.41	81.94	13.52
0.84	0.16	-0.99	4.10	4.00	0.10	13.42	6.29	80.29	13.56
0.82	0.18	-1.30	4.10	4.00	0.10	15.20	6.17	78.63	13.60
0.79	0.21	-1.60	4.10	4.00	0.10	16.98	6.05	76.97	13.65
0.77	0.23	-1.91	4.10	4.00	0.10	18.78	5.93	75.29	13.69
0.75	0.25	-2.22	4.10	4.00	0.10	20.58	5.81	73.61	13.73
0.73	0.27	-2.52	4.10	4.00	0.10	22.40	5.69	71.91	13.78
0.71	0.29	-2.81	4.20	4.00	0.20	23.72	5.63	70.65	14.13
0.69	0.31	-3.10	4.20	4.00	0.20	25.52	5.51	68.97	14.17
0.67	0.33	-3.39	4.20	4.00	0.20	27.33	5.39	67.28	14.21
0.65	0.35	-3.67	4.10	3.90	0.20	28.85	6.15	65.00	13.94
0.63	0.37	-3.95	4.20	3.90	0.30	29.71	6.03	64.25	14.14
0.61	0.39	-4.21	4.20	3.90	0.30	31.47	5.91	62.61	14.18
0.58	0.42	-4.48	4.20	3.90	0.30	33.25	5.79	60.96	14.22
0.56	0.44	-4.73	4.40	3.90	0.50	32.84	5.67	61.49	14.58
0.54	0.46	-4.97	4.40	3.90	0.50	34.51	5.55	59.94	14.61
0.52	0.48	-5.21	4.40	3.90	0.50	36.20	5.42	58.38	14.65
0.50	0.50	-13.5	4.50	3.90	0.60	35.33	5.96	58.72	16.18

But, by definition of optimality, we have $\Pi^{w*} = \Pi(N_1^{w*}, ..., N_M^{w*}) \geq \Pi(N_1^{w/o*}, ..., N_M^{w/o*})$ leading to the desired result. $\square$

4.2.2 Case of Non-Identical Firms

By using an example, we can show that allowing transshipments may decrease total expected profits with optimal assortments when firms are of different sizes. Consider 2 example problem settings.

For the first example, let $|S| = 5$ and $M = 2$. Also Let $\lambda = 100$,

$q_j = \{0.77, 0.23\}$, $\theta = 0.91$, $t = 0.95$, $r = 1.4$, $\kappa = 8.12$ and $\alpha_i = \{0.4876, 0.3332, 0.0818, 0.0561, 0.0413\}$.

For the second example, let $|S| = 5$ and $M = 4$. Other parameters are set such that $\lambda = 100$, $q_j = \{0.46, 0.27, 0.24, 0.03\}$, $\theta = 0.94$, $t = 0.82$, $r = 1.02$, $\kappa = 6.45$ and $\alpha_i = \{0.2539, 0.2334, 0.2148, 0.2042, 0.0936\}$.

Table 4.7: Summary of Results for the Examples of Non-identical Firms

	with transshipment	Π	no-transshipment	$\Pi^{w/o}$
1	$N_1^* = \{1,2\}$ $N_2^* = \{1,2\}$	102.49	$N_1^{w/o*} = \{1,2\}$ $N_2^{w/o*} = \{1\}$	105.16
	$N_1 = \{1,2\}$ $N_2 = \{1\}$	102.14	$N_1^{w/o} = \{1,2\}$ $N_2^{w/o} = \{1,2\}$	102.49
2	$N_1^* = N_2^* = N_3^* = \{1,2\}$, $N_4^* = \emptyset$	38.44	$N_1^{w/o*} = \{1,2,3\}$, $N_2^{w/o*} = N_3^{w/o*} = \{1,2\}$, $N_4^{w/o*} = \emptyset$	39.07
	$N_1 = \{1,2,3\}$, $N_2 = N_3 = \{1,2\}$, $N_4 = \emptyset$	36.32	$N_1^{w/o} = N_2^{w/o} = N_3^{w/o} = \{1,2\}$, $N_4^{w/o} = \emptyset$	38.44

Table 4.7 summarizes optimal solutions of two different problem settings. For each problem setting, while first rows give optimal solutions, second rows show the corresponding solutions of optimal assortment sets crosswise. As can be seen in Table 4.7, optimal solutions of no-transshipment cases (denoted by superscript $^{w/o*}$) for 2 examples perform better than the optimal solutions when transshipments are allowed (denoted by superscript *). In Example 1, optimal solution by allowing transshipments indicates that product 1 and 2 should be included in both firm assortments. On the other hand, when transshipments are not allowed, it is more profitable to keep only product 1 in second firm's assortment. This way, even if the revenue from direct sales of total products decreases, total profit increases. The reason is excluding product 2 from the assortment increases the revenue from substitution and decreases the assortment costs at the same time. If we use the optimal assortment $N_1 = \{1, 2\}, N_2 = \{1\}$ of the no-transshipment

case and use it in transshipment system, we observe that total profit further decreases. Since the product variety in both firms are exactly the same, revenue from direct sales and the assortment costs are obviously equal. However, when transshipments are allowed, the potential revenue that can be obtained from substitution is lost to transshipment of products due to their availability in other firm. In this specific example, since the revenue coming from transshipment does not compensate the decrease in revenue from substitution, it is more profitable for the company to hinder any transshipment and let the customers substitute.

In the latter example, optimal assortments of the first three firms include the same products when transshipment is allowed. In this situation, since each firm carries the same assortment, there will not be any transshipment between them. In a no-transshipment system, it becomes more profitable to add the third product to the assortment of the first firm. It is obvious that adding a product increases direct sales. In this case, the increase in direct sales is not high enough to justify the additional cost of transshipments from first firm to other firm. Therefore adding this product to the assortment is not optimal in a system with transshipments.

The above result is mainly due to not allowing optional transshipments, but setting default transshipments in case of a need. When transshipments are optional between firms by comparing potential costs and benefits of transshipment, then the result can be different. Our aim with this study is to show how predetermined transshipment policies affect the assortment policy. Optional transshipments are not within the scope of current study, but can be the subject of a follow-up work.

4.3 Performance of Heuristic Solution

In this section, we introduce a heuristic algorithm which assumes that the assortments of all firms are in the popular set according to the product choice probabilities, α_i. With this assumption, the property (3.18) given in Theorem 3 is ignored. When the property is satisfied for all firms $m \in \{1, ..., M\}$, we know that all firm assortments are in the popular set at optimality, thus heuristic procedure finds the optimal solution. However, if the property is not satisfied for some firms, then some of the assortments may not be in the popular set yielding a gap between the profits from heuristic and full enumeration solutions.

Table 4.8 summarizes the results from 200 randomized problem instances for each firm & product combinations generated by using the parameter ranges in Table 4.1. For each set of 200 problems, the table reports the number of problem instances in which the property (3.18) is violated for all firms and the number of instances in which (3.18) is violated only for Firm 1 (largest firm). As a matter of fact, we give the number of instances where the property is violated for all firms only for informative purposes. We evaluate the results over the property violation for Firm 1, because Firm has the highest probability of violating (3.18) compared to other (smaller) firms.

Table 4.8: Performance of the heuristic solution with respect to optimal solution

#of Firms	#of Products	#of Instances violating property for firm 1	#of Instances violating property for all firms	#of Instances with non-popular sets in optimal assortment	Avg %gap
2	5	20	20	2	0.17
3	5	28	10	3	0.28
4	5	39	3	2	0.14
5	5	38	1	2	0.31
5	2	32	4	0	0.00
5	3	34	3	11	1.30
5	4	5	0	0	0.00

Table 4.8 shows that for a system of 2 firms and 5 products, out of 200 randomly problem settings, there exist 20 instances, in which θ does not satisfy (3.18) for Firm 1. Out of these 20 settings, only 2 of them does not have the popular assortments in optimal solution. Even if the property is not satisfied in the remaining 18 instances, the optimal assortments are in the popular set. This confirms that (3.18) is not a necessary but a sufficient condition as stated previously. In the last column, gap stands for the average percent difference between total profits of

heuristic and optimal solutions calculated only for the number of instances which have non-popular sets in optimal assortments, i.e., there is a non-zero gap. In a system of 5 firms and 3 products, among 34 instances out of 200, we get 11 assortments which are not in the popular set. Still, we attain an average profit gap of 1.30 % by the use of the heuristic solution over the optimal one. We infer from these results that with even relatively high substitution probability (with an average of $\theta = 0.89$ among $7 * 200 = 1400$ instances), the optimal assortments happen to be in the popular set. Hence, assuming that all assortments are in the popular set, we can find close-to-optimal solutions by considering a very small number of solutions.

Chapter 5

Conclusion

In this thesis, the assortment planning problem for multiple locations in a make-to-order system is examined. Inventory sharing through transshipments, which is a way of demand/risk pooling, is allowed such that each location can access to other locations' assortments in case that the desired product by the customer of the visited location is not kept by this location. Customer demand arrives with exogenous demand model. Demand substitutions are also incorporated to explain the consumer behavior when a demanded product can neither be satisfied from the assigned location's nor any other location's assortment through transshipment. We investigate how the optimal assortments of all locations can be determined to maximize the total expected system profit. We did not include any inventory considerations since we model a make-to-order environment.

Our analytical investigations show that firms have a nested form of optimal assortment sets such that the assortment of a smaller firm is a subset of the assortment of a larger firm. Using the knowledge of having nested assortments at optimality, the number of iterations required to find optimal assortments for a 5-firms 6-products problem is only 0.004 % of those in full enumeration. Moreover, we prove that the intersection set of individual assortments is in the popular set, i.e., it is composed of some number of most demanded products. This property decreases the number of iterations for the optimal solution to 0.0018% of full enumeration. Then, we obtain a upper limit on the substitution probability for each firm, which is a function of the profit margin, transshipment cost, and the firm size, to ensure that the optimal firm assortment is in the popular set. Otherwise, when the substitution probability of customers is sufficiently high, the firm can be better off by keeping a less popular item in its assortment instead of higher demanded ones.

We conduct extensive numerical analyses to demonstrate the effects of the system inputs such as profit margin, assortment cost, transshipment cost, and firm size asymmetry on optimal assortments and profit. We also investigate the results of allowing transshipments by comparing to a system where the transshipments among firms are omitted. Numerical examples show that allowing transshipments may yield inferior system profits when firms are in different sizes. Finally, we introduce a heuristic solution procedure by using the optimality properties obtained on assortments. Numerical comparison of the performance of this heuristic procedure to optimal solution obtained through full enumeration shows that the percent deviation from the optimal solution is significantly low.

To our knowledge, this thesis is the first attempt to investigate optimal assortments of multiple locations that can cooperate through transshipments. Given that the optimal assortment planning problem is challenging even for a single firm system, the analytical findings obtained in this study help to decrease the challenge in multi-location assortment planning problem quite significantly. The study is supported by numerical analyses to illustrate the sensitivity of optimal assortments to system parameters and also effects of allowing transshipments compared to a traditional no-transshipment system.

The current study can be further detailed and extended in several directions. One would be allowing non-identical parameters for products such as profit margins and assortment costs, as well as non-identical parameters for the firms such as firm-specific demand probabilities. We prove in the thesis that some of our results are also valid for product-specific profit margins, but all results might not be. Another interesting, but challenging extension can be allowing optional transshipments. It is challenging because, it is shown in the transshipment literature that for a given single product problem, finding the optimal transshipment policy, i.e., when to ask and how to accept/reject transshipment requests, is quite complex. Thus, investigating optimal assortment planning by also allowing optional transshipments will bring additional complexity. However, it would be interesting to see how the characteristics of the resulting assortments can change in this more comprehensive approach.

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Appendix A

Proof of Theorem 1 with Product-Specific Margins (r_i):

Assume the contrary that the assortment $A = (N_1, .., N_k, .., N_m, ..., N_M)$ is optimal with $N_k \not\supseteq N_m$. Then construct a new assortment $A' = (N_1, .., (N_k \cup N_m), .., (N_k \cap N_m), .., N_M)$. The total profit of this new assortment set can be written as:

$$\begin{aligned} \Pi(A') &= \Pi(A) + \lambda(q_k - q_m) \sum_{i \in N_m \backslash N_k} \alpha_i r_i + \lambda(q_k - q_m)\theta \sum_{i \notin N} \sum_{j \in N_m \backslash N_k} \alpha_i \delta_{ij} r_i \\ &\quad -\lambda(q_k - q_m) \sum_{i \in N_m \backslash N_k} \alpha_i (r_i - t) - \lambda(q_k - q_m)\theta \sum_{i \notin N} \sum_{j \in N_m \backslash N_k} \alpha_i \delta_{ij} (r_i - t) \\ &= \Pi(A) + \lambda(q_k - q_m)t \sum_{i \in N_m \backslash N_k} \alpha_i + \lambda(q_k - q_m)\theta t \sum_{j \notin N} \sum_{i \in N_m \backslash N_k} \alpha_j \delta_{ji}. \end{aligned}$$

Because $q_k \geq q_m$, it follows that $\Pi(A') \geq \Pi(A)$. Since the assortment A' leads to a higher expected profit, A cannot be optimal. $\square$

Appendix B

Code : 2-Firm System with Transshipments

```
#include <stdio.h>
#include <math.h>
#include <iostream.h>
#include <stdlib.h>
#include <string.h>
#include <time.h>
#define M 5  //number of products
#define R 2  //number of retailers
#define Z 200
double q[(const int) R];
double r[(const int) M];
double k[(const int) M];
double alpha[(const int) R][(const int) M];
double delta[(const int) R][(const int) M][(const int) M];
double subs[(const int) R][(const int) M];
double Pi[2][2][2][2][2][2][2][2][2][2];
double PiO[2][2][2][2][2][2][2][2][2][2];
double PiT[2][2][2][2][2][2][2][2][2][2];
```

```
double PiOS[2][2][2][2][2][2][2][2][2][2];
double PiTS[2][2][2][2][2][2][2][2][2][2];
double C[2][2][2][2][2][2][2][2][2][2];
double PiownB[10];
double PitransB[10];
double PiownsubsB[10];
double PitranssubsB[10];
double CostB[10];
int I[(const int) R][(const int) M]={0};
int Inone[(const int) M]={0};
int Istar[10][(const int) R][(const int) M]={0};
double Piown,Pitrans,Piownsubs,Pitranssubs,Cost,Pii,PiB;
double lambda,theta,tau,V;
int n,m,i,j,l,a,b,z;
FILE *data,*wrt;

generate()
{
wrt = fopen("optimal-assortments.data", "a+");
PiB=-100000;

for(n=0; n<R; n++)
 {
  for(i=0; i<M; i++)
  {
   for(j=0; j<M; j++)
   {
        delta[n][i][j]=(theta*alpha[n][j])/(1-alpha[n][i]);
        if(i==j)
                delta[n][i][j]=0;
   }// j loop
  }// i loop
 }// n loop
```

```
  for(I[0][0]=0; I[0][0]<2; I[0][0]++)
 {
  for(I[0][1]=0; I[0][1]<2; I[0][1]++)
 {
  for(I[0][2]=0; I[0][2]<2; I[0][2]++)
 {
  for(I[0][3]=0; I[0][3]<2; I[0][3]++)
 {
  for(I[0][4]=0; I[0][4]<2; I[0][4]++)
 {
  for(I[1][0]=0; I[1][0]<2; I[1][0]++)
 {
  for(I[1][1]=0; I[1][1]<2; I[1][1]++)
 {
  for(I[1][2]=0; I[1][2]<2; I[1][2]++)
 {
  for(I[1][3]=0; I[1][3]<2; I[1][3]++)
 {
  for(I[1][4]=0; I[1][4]<2; I[1][4]++)
 {

for(i=0;i<M;i++)
 Inone[i]=1; // The case where no retailer keeps the product i

Piown=0;
Pitrans=0;
Piownsubs=0;
Pitranssubs=0;
Cost=0;

for(i=0;i<M;i++)
{
```

```
  for(n=0;n<R;n++)
  {  if(I[n][i]>0)
  Inone[i]=0; // If at least one retailer keeps the product i,
              //then this product is present in the assortment
  }
}

for(n=0;n<R;n++) // Calculate the expected revenue from own
                 // sales of retailers
{
        V=0;
        for(i=0;i<M;i++)
                V=V+I[n][i]*alpha[n][i]*r[i];
        Piown=Piown+q[n]*V;
}
Piown=lambda*Piown;

for(n=0;n<R;n++) // Calculate the expected revenue
                 //from transshipment
{
        V=0;
        for(i=0;i<M;i++)
         V=V+(1-I[n][i])*(1-Inone[i])*alpha[n][i]*(r[i]-tau);
        Pitrans=Pitrans+q[n]*V;
}
Pitrans=lambda*Pitrans;

for(n=0;  n<R;  n++)
  {
    for(j=0;  j<M;  j++)
    {
    subs[n][j]=0;
      for(i=0;  i<M;  i++)
```

```
    subs[n][j]=subs[n][j]+alpha[n][i]*delta[n][i][j]*Inone[i];

  }// j loop
 }// n loop

for(n=0;n<R;n++) //Calculate the expected revenue from
                 //own substitution
{
        V=0;
        for(i=0;i<M;i++)
                V=V+I[n][i]*subs[n][i]*r[i];
        Piownsubs=Piownsubs+q[n]*V;
}
Piownsubs=lambda*Piownsubs;

for(n=0;n<R;n++) //Calculate the expected revenue from
                 //transshipment for substitution
{
        V=0;
        for(i=0;i<M;i++)
                V=V+(1-I[n][i])*(1-Inone[i])*subs[n][i]*(r[i]-tau);
        Pitranssubs=Pitranssubs+q[n]*V;
}
Pitranssubs=lambda*Pitranssubs;

for(n=0;n<R;n++) // Calculate the assortment cost
{
        V=0;
        for(i=0;i<M;i++)
                V=V+I[n][i]*k[i];
        Cost=Cost+V;
}
Pii=Piown+Pitrans+Piownsubs+Pitranssubs-Cost;//Calculate the total
```

```
                                              //expected profit
Pi[I[0][0]][I[0][1]][I[0][2]][I[0][3]][I[0][4]]
    [I[1][0]][I[1][1]][I[1][2]][I[1][3]][I[1][4]]=Pii;
PiO[I[0][0]][I[0][1]][I[0][2]][I[0][3]][I[0][4]]
    [I[1][0]][I[1][1]][I[1][2]][I[1][3]][I[1][4]]=Piown;
PiT[I[0][0]][I[0][1]][I[0][2]][I[0][3]][I[0][4]]
    [I[1][0]][I[1][1]][I[1][2]][I[1][3]][I[1][4]]=Pitrans;
PiOS[I[0][0]][I[0][1]][I[0][2]][I[0][3]][I[0][4]]
    [I[1][0]][I[1][1]][I[1][2]][I[1][3]][I[1][4]]=Piownsubs;
PiTS[I[0][0]][I[0][1]][I[0][2]][I[0][3]][I[0][4]]
    [I[1][0]][I[1][1]][I[1][2]][I[1][3]][I[1][4]]=Pitranssubs;
C[I[0][0]][I[0][1]][I[0][2]][I[0][3]][I[0][4]]
    [I[1][0]][I[1][1]][I[1][2]][I[1][3]][I[1][4]]=Cost;

if(Pii>=PiB)
        PiB=Pii;
 }
 }
 }
 }
 }
 }
 }
 }
 }
 }
j=0;
  for(I[0][0]=0; I[0][0]<2; I[0][0]++)
 {
  for(I[0][1]=0; I[0][1]<2; I[0][1]++)
 {
  for(I[0][2]=0; I[0][2]<2; I[0][2]++)
 {
```

```
for(I[0][3]=0; I[0][3]<2; I[0][3]++)
{
for(I[0][4]=0; I[0][4]<2; I[0][4]++)
{
for(I[1][0]=0; I[1][0]<2; I[1][0]++)
{
for(I[1][1]=0; I[1][1]<2; I[1][1]++)
{
for(I[1][2]=0; I[1][2]<2; I[1][2]++)
{
for(I[1][3]=0; I[1][3]<2; I[1][3]++)
{
for(I[1][4]=0; I[1][4]<2; I[1][4]++)
{
if(Pi[I[0][0]][I[0][1]][I[0][2]][I[0][3]][I[0][4]]
    [I[1][0]][I[1][1]][I[1][2]][I[1][3]][I[1][4]]-PiB>-0.00001)
{printf("inside if\n");
PiownB[j]=PiO[I[0][0]][I[0][1]][I[0][2]][I[0][3]][I[0][4]]
    [I[1][0]][I[1][1]][I[1][2]][I[1][3]][I[1][4]];
PitransB[j]=PiT[I[0][0]][I[0][1]][I[0][2]][I[0][3]][I[0][4]]
    [I[1][0]][I[1][1]][I[1][2]][I[1][3]][I[1][4]];
PiownsubsB[j]=PiOS[I[0][0]][I[0][1]][I[0][2]][I[0][3]][I[0][4]]
    [I[1][0]][I[1][1]][I[1][2]][I[1][3]][I[1][4]];
PitranssubsB[j]=PiTS[I[0][0]][I[0][1]][I[0][2]][I[0][3]][I[0][4]]
    [I[1][0]][I[1][1]][I[1][2]][I[1][3]][I[1][4]];
CostB[j]=C[I[0][0]][I[0][1]][I[0][2]][I[0][3]][I[0][4]]
    [I[1][0]][I[1][1]][I[1][2]][I[1][3]][I[1][4]];
for(n=0; n<R; n++)
 {
  for(i=0; i<M; i++)
    Istar[j][n][i]=I[n][i];
  }
j++;
```

```
}
 }
 }
 }
 }
 }
 }
 }
 }
 }
 }
printf("j %d\n",j);
fprintf(wrt,"PiB %f ",PiB);

for (z=0; z<j; z++)
{
 fprintf(wrt,"PiownB[%d] %f PitransB[%d] %f PiownsubsB[%d]
    %f PitranssubsB[%d] %f CostB[%d] %f ",z,PiownB[z],z,
    PitransB[z],z,PiownsubsB[z],z,PitranssubsB[z],z,CostB[z]);
        for(n=0;n<R;n++)
        {
         for(i=0;i<M;i++)
         {
          fprintf(wrt," I[%d][%d][%d] %d ",z,n,i,Istar[z][n][i]);
         }
        }
}
fprintf(wrt,"\n");

}//close generate

int main()
{
```

```
float temp1;
data = fopen("parameters.txt", "r");
wrt = fopen("optimal-assortments.data", "w");

for (l =0; l < Z; l++)
 {printf("set %d\n",l+1);
  a=0;
  b=0;
  for (j =0; j < R+R*M+2*M+3; j++)
    {fscanf(data, "%f", &temp1);
        //printf("a[j]=%f\n",temp);
    for(i=0;i<R;i++)
          {
          if(j==i)
           {q[i]= temp1;
           printf("q[%d] %f\n",i,q[i]);
           }
          }
    for(i=R;i<R+M;i++)
          {
          if(j==i)
           {r[i-R]= temp1;
           printf("r[%d] %f\n",i-R,r[i-R]);
           }
          }
    for(i=R+M;i<R+2*M;i++)
          {
          if(j==i)
           {k[i-R-M]= temp1;
           printf("k[%d] %f\n",i-R-M,k[i-R-M]);
           }
          }
    for(i=R+2*M;i<R+2*M+R*M;i++)
```

```
          {
          if(j==i)
           {
           if(b<M)
            {alpha[a][b]= temp1;
                 printf("alpha[%d][%d] %f\n",a,b,alpha[a][b]);
            }
             else
                 {
                  a++;
                  b=0;
                  alpha[a][b]= temp1;
                  printf("alpha[%d][%d] %f\n",a,b,alpha[a][b]);
                 }
            b++;
            }
           }
           if(j==R+2*M+R*M)
                  {lambda=temp1;
                  }
           if(j==R+2*M+R*M+1)
                  {theta=temp1;
                  }
           if(j==R+2*M+R*M+2)
                  {tau=temp1;
                  }
     }//end of j loop
    generate();
   }// end of l loop
}//end of main
```