Generic Non-Uniqueness of Complete H-Surfaces Embedded in $\mathbb{H}^3$

by

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Koç University

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This is to certify that I have examined this copy of a doctoral dissertation by

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This thesis is dedicated to my wife, my family and my friends for their endless support.

ABSTRACT

In this thesis, we prove that, given $|H| < 1$, a generic simple closed curve embedded in the asymptotic boundary of $\mathbb{H}^3$ bounds more than one complete surface embedded in $\mathbb{H}^3$ with constant mean curvature H.

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TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

INTRODUCTION

The term *minimal surface* may refer to a surface with boundary which minimizes area among competitive surfaces with the same boundary or, more generally, it may refer to a surface which corresponds to a critical point of the area functional on neighboring surfaces with the same boundary. A less intuitive but precise definition is that it is a surface which has mean curvature zero at all of its points.

Minimal surfaces can be observed in nature in the form of thin layers of some liquid substances. A physicist Joseph Plateau(1801-1883), a physicist studied *soap films* in various forms through experiments. When one dips a wire frame -for example, in the shape of a simple closed curve- into a soapy water and takes it out, a thin layer of soapy water is formed which spans the wire frame. Such surfaces are examples of minimal surfaces. Plateau has predicted many facts regarding soap films and these predictions motivated mathematicians to develop theories formulating his predictions. One of the most famous conjecture of Plateau is called the *classical*

Plateau problem by mathematicians and it basically asks the existence of minimal surfaces with a prescribed boundary. First results in this direction are obtained by Jesse Douglas [11] and Tibor Rado [17] in 1930s and Douglas had won Fields medal in 1936 with his work. Solutions they had obtained were restricted to embedded or self-intersecting disks which are necessarily orientable. Their solutions minimize area among all disks with the same boundary, although there may still exist other surfaces with different topology and less area. However, soap films obtained from the experiments cannot have self-intersections, they might have positive genus, they might be nonorientable, and they have always the least area among all surfaces due to the principle of least energy. Further attempts have been made to formulate the problem of Plateau where the candidate surfaces contain all possible soap films which might occur in nature. These attempts have flourished many areas of mathematics such as calculus of variations and geometric measure theory among many others.

Naturally, mathematical theory of minimal surfaces started to be studied in Euclidean space. Later, the interest has extended to other ambient spaces, especially to other space forms of constant curvature: the sphere and the hyperbolic space. In this thesis we are interested in minimal surfaces in three dimensional hyperbolic space. The interest in the study of minimal surfaces in hyperbolic space increased after the work of Michael T. Anderson in 1982 [2] in which he gives a complete solution to the so-called *asymptotic Plateau problem.*

Hyperbolic space is the unique simply connected and complete Riemannian manifold of constant sectional curvature -1. The asymptotic Plateau problem asks the following:

- Given a closed submanifold Γ embedded in the asymptotic boundary of the hyperbolic space does there exist a (necessarily complete) minimal surface which is asymptotic to Γ?

From now on we will denote the three dimensional hyperbolic space by $\mathbb{H}^3$ and its asymptotic boundary by $\partial_\infty \mathbb{H}^3$. Also all surfaces will be assumed to be orientable. In [2], Anderson proved the existence of minimal surfaces for every closed submanifold Γ embedded in $\partial_\infty \mathbb{H}^3$. This result also holds for simple closed curves. Solutions he obtained are *absolutely area minimizing*, i.e. any compact subsurface minimizes area among all surfaces with the same boundary. In this thesis we are going to prove the following result:

- Given a generic simple closed curve Γ there are more than one complete, embedded minimal surfaces with asymptotic boundary Γ.

By the result of Anderson we know that given a simple closed curve Γ there is always a complete embedded minimal surface which is realized as an absolutely area minimizing surface, the topology of the solution being unknown. However, it is also possible to solve the asymptotic Plateau problem so that we know the topology of the solution. A complete surface Σ embedded in $\mathbb{H}^3$ is said to be a *least area plane* if any compact subdisk minimizes area among all disks with the same boundary. In [3] Anderson also solved the asymptotic Plateau problem for disk type surfaces and proved that any simple closed curve is the asymptotic boundary of a least area plane which is complete and embedded.

In short, given a simple closed curve $\Gamma \subset \partial_\infty \mathbb{H}^3$, we can always construct two complete and embedded minimal surfaces one of which is homeomorphic to a disk. We do not know the topology of the other however, so two solutions obtained by Anderson might be different or identical. We are going to prove that generically these two solutions are different, i.e. generically the least area plane solution is not an absolutely area minimizing solution.

One can find examples of explicit curves such that both solutions are the same. It is known that a simple closed curve which is the boundary of a star-shaped domain in $\partial_\infty \mathbb{H}^3$ is the asymptotic boundary of a unique complete minimal surface in $\mathbb{H}^3$([2], [12]) so in this case both solutions must coincide. On the other hand, there exist simple closed curves for which any absolutely area minimizing surface must have positive genus ([3], [13]) so that least area plane solution cannot be an absolutely area minimizing solution.

Now we are going to give a sketch for the proof of our generic non-uniqueness result. Let A_0 denote the space of all simple closed curves embedded in $\partial_\infty \mathbb{H}^3$ with respect to the C^0-topology and A_1 be the subset of A_0 consisting of curves for which any absolutely area minimizing surface has genus greater than or equal to one. A_1 is not empty as we noted in the previous paragraph. Then the following holds:

Proposition. *A_1 is an open and dense subset of A_0.*

From this and the existence of least area planes, it follows that generically a simple closed curve bounds more than one complete embedded minimal surface. An analogous result exists for the space of simple closed curves embedded in the boundary of mean convex 3-manifolds [4] and here we follow those ideas.

In order to show that A_1 is open, we take a sequence of curves $\Gamma_n \in A_0 \setminus A_1$ which converges to a simple closed curve Γ. Then for each n, by definition of the sets A_0 and A_1, we can always find an absolutely area minimizing surface Σ_n which has genus zero and asymptotic boundary Γ_n. By the standard convergence arguments already established in the minimal surface theory we can extract a subsequence of Σ_n which converges smoothly on compact sets to an absolutely area minimizing surface Σ. Then Σ must have genus zero too and has asymptotic boundary Γ. It follows that Γ belongs to $A_0 \setminus A_1$. Hence $A_0 \setminus A_1$ is closed so that A_1 is open. In other words, if a simple closed curve does not bound an absolutely area minimizing surface of genus zero then the same is also true for all curves in a sufficiently small neighborhood of Γ.

In order to show the density of A_1 we use a *bridge principle* developed by Francisco Martin and Brian White [15]. This principle allows us to attach tiny handles to an absolutely area minimizing surface Σ with smooth boundary Γ (remember that we are interested in simple closed curves which are not necessarily smooth), short edges of the handles being attached to Γ. By attaching handles successively we can construct a new absolutely area minimizing surface Σ' such that $\text{genus}(\Sigma') = \text{genus}(\Sigma) + 1$. Moreover, we can make the construction in such a way that the asymptotic boundary Γ' of Σ' stays in a sufficiently small neighborhood of Γ (with respect to the C^0-topology). At this point we should be careful because the bridge principe obtained by Martin and White can be applied only to an absolutely area minimizing surface Σ with smooth boundary Γ for which Σ is the unique absolutely area minimizing surface with asymptotic boundary Γ. We overcome this by using a generic uniqueness result obtained by Coskunuzer [8]:

- Given an arbitrary simple closed curve Γ embedded in $\partial_\infty \mathbb{H}^3$, in every neighborhood of Γ, we can find a smooth simple closed curve Γ' for which there is a unique absolutely area minimizing surface with asymptotic boundary Γ'.

Therefore, in order to prove the density of A_1, we take an arbitrary simple closed curve Γ and by the generic uniqueness result stated above we find a smooth simple closed curve Γ' for which there is only one absolutely area minimizing surface Σ with asymptotic boundary Γ'. If genus$(\Sigma) > 0$ then $\Gamma' \in A_1$ because Σ is the unique absolutely area minimizing surface with asymptotic boundary Γ. Otherwise, we apply bridge principle to Σ to obtain a new absolutely area minimizing surface Σ' with genus$(\Sigma') = 1$ and $\partial_\infty \Sigma'$ stays close to Γ as much as we want. Now $\partial_\infty \Sigma' \in A_1$ so the density of A_1 follows.

Analogous tools exist also for constant mean curvature surfaces and our main theorem can be stated as follows:

Theorem. *Given $|H| < 1$, the set of simple closed curves $\Gamma \subset \partial_\infty \mathbb{H}^3$ with one smooth point and which bounds more than one complete H-surface embedded in $\mathbb{H}^3$ contains an open and dense subset of the space of all simple closed curves with respect to* C^0*-topology.*

In the next chapter we give the related background for our work. In the third chapter we first prove our result for complete embedded minimal surfaces. In the fourth chapter we generalize it to H-surfaces. In the last chapter we will comment on some questions that naturally arise; what happens for the space of C^1 curves and for the curves with more than one component?

Chapter 2

PRELIMINARIES

2.1 *Minimal Surfaces*

In this section we will give a short background on the theory of minimal surfaces. Let (M, g) be a three dimensional Riemannian manifold where g denotes the Riemannian metric on M. Let Σ be a two dimensional submanfiold of M, and ∇ denote the Levi-Civita connection associated to g. Let X and Y be local vector fields on Σ and η is a unit normal vector field on Σ. Then the second fundamental form associated with Σ is a symmetric, bilinear tensor field which is defined as follows:

$$B(X, Y) = g(\nabla_X Y, \eta)$$

At a point $p \in \Sigma$, $B(X, Y)$ depends only on the values $X(p)$ and $Y(p)$ so there is an associated symmetric linear map, namely the Shape operator $S : T_p\Sigma \to T_p\Sigma$ which satisfies

$$B(X(p), Y(p)) = g(S(X(p)), Y(p))$$

The mean curvature $H(p)$ of a surface at a point $p \in \Sigma$ is defined to be $H(p) := \frac{1}{2}\mathrm{trace}(S)$. If the mean curvature is equal to 0 for all $p \in \Sigma$, then we call Σ a minimal surface.

Minimal surfaces can also be realized as critical points of area functional with respect to compactly supported variations. Let $f : \Sigma \to \mathbb{R}$ be a compactly supported function on Σ and $F : \Sigma \times (-\epsilon, \epsilon) \to M$ be the compactly supported variation which is defined as

$$F(p,t) = \exp_p(tf(p)\eta(p))$$

where $\exp_p$ denotes the exponential map of M at $p \in M$ and η is the unit normal vector filed on Σ. Then

$$\left.\frac{\mathrm{dA}(t)}{\mathrm{d}t}\right|_{t=0} = -2\int_{\Sigma} f \cdot H$$

where A denotes the area of the surface. Loosely speaking, we can see the space of all compactly supported normal fields as the tangent space at Σ in the space of submanifolds of M. If we put the L^2 inner product metric on this space then the above formula can be written as follows:

$$\left.\frac{\mathrm{dA}(t)}{\mathrm{d}t}\right|_{t=0} = \langle \nabla \mathrm{A}, f\eta \rangle_{L^2}.$$

where the gradient of the area functional ∇A is to be read as $--2H\eta$. From this it follows that the mean curvature vector $H\eta$ gives the direction in which the area decreases most. Two direct implications of the above formula are as follows:

- A surface is minimal if and only if it is the critical point of the area functional with respect to all compactly supported variations.

- If a surface with boundary minimizes area among all surfaces with the same boundary then it has to be a minimal surface.

It is also natural to study the second variation area of a minimal surface since it might give information about whether the surface is a local minimum for area or not. The second variation formula for a minimal surface is as follows:

$$\left.\frac{\mathrm{d}^2\mathrm{A}(t)}{\mathrm{d}t^2}\right|_{t=0} = -\int_{\Sigma} f \cdot L(f)$$

where L is the well known *stability operator* which acts on compactly supported functions on Σ. It is given by

$$L(f) = \Delta f + |B|^2 f + \mathrm{Ric}(\eta) f.$$

Here Δ is the laplacian operator on Σ which is equal to the divergence of the gradient, $|B|^2 = \lambda_1^2 + \lambda_2^2$ (λ_{i} being the eigenvalues of S), namely the squared norm of the second fundamental form and $\mathrm{Ric}(\eta)$ is the Ricci curvature in the direction of the unit normal η.

We say that a compact surface Σ with boundary is a *local minimum of area* if $\mathrm{A}(0) \leq \mathrm{A}(t)$ for all $t \in (-\epsilon, \epsilon)$ and for all compactly supported variations. So if

Σ is a local minimum of area then by the second derivative test $\left.\frac{\mathrm{d}^2\mathrm{A}(t)}{\mathrm{d}t^2}\right|_{t=0} \geq 0$ for all compactly supported variatons. In this case we call Σ a *stable* minimal surface. Conversely, if $\left.\frac{\mathrm{d}^2\mathrm{A}(t)}{\mathrm{d}t^2}\right|_{t=0} > 0$ for all compactly supported variations, then Σ is a local minimum of area.

We know from the classical theory of elliptic operators that eigenfunctions of the stability operator generates the space of compactly supported smooth functions on Σ. From this it follows that Σ is a stable surface if and only if the eigenvalues of the operator $-L$ are all nonnegative. Also, it follows that Σ is a local minimum of area if all eigenvalues are positive. If the domain of the operator is sufficiently small, it can be shown using Poincaré's inequality that the first eigenvalue of $-L$ is positive. This implies the following important fact:

- Any sufficiently small portion of a minimal surface is a local minimum of area.

2.2 Asymptotic Plateau Problem

In order to find examples of compact minimal surfaces one may try to solve area minimization problems of different type. However, finding examples of noncompact complete minimal surfaces is a very different problem. For example, until 1980s it was conjectured that the plane, the helicoid and the catenoid were the only complete minimal surfaces of finite topological type in Euclidean space (Figure 2.1 and 2.2; pictures taken from http://www.indiana.edu/ minimal/archive/Classical/Classical/Helicoid/web/index.htm Later, in 1982 this conjecture was disproved by Celso Costa who found a new example of a complete embedded minimal surface with positive genus and this has led to new directions in the theory of minimal surfaces.

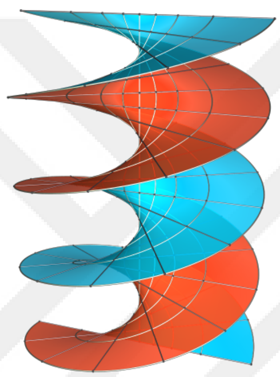

Figure 2.1: Helicoid is the only nontrivial minimal ruled surface.

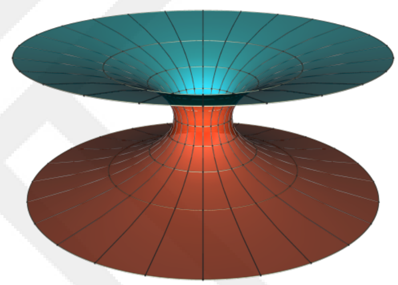

Figure 2.2: Catenoid is the only nontrivial minimal surface of revolution

Hyperbolic space on the other hand is quite different in this aspect of the theory. One can find plenty of examples of complete minimal surfaces in hyperbolic space through solving the asymptotic Plateau problem. In fact, recently it has been proved by Martin and White [15] that any open surface of arbitrary topology can be embedded into $\mathbb{H}^3$ as a minimal surface. Before we state the asymptotic Plateau problem and related problems we now briefly summarize the necessary concepts.

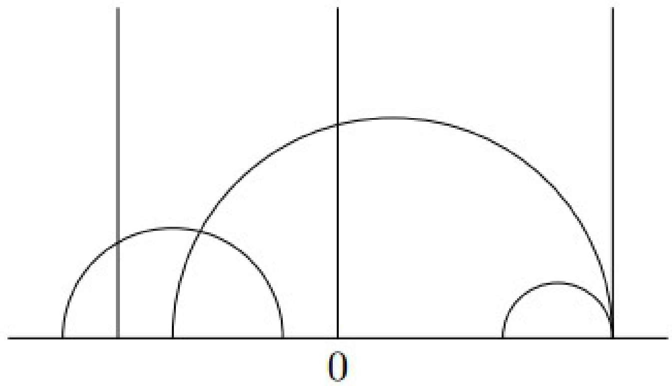

Figure 2.3: Geodesics in upper half space model

Hyperbolic space is the unique simply connected Riemannian manifold with constant sectional curvature -1 and it can be modelled on different subsets of Euclidean space. One of the most common models is the *upper half space model* which resides on the set $\{(x, y, z) \in \mathbb{R}^3 : z > 0\}$ with the following Riemannian metric:

$$g_h(p)(v, w) := \frac{1}{z^2} g_0(v, w)$$

Here g_h denotes the hyperbolic metric, g_0 denotes the standard Euclidean metric and v, w are elements of the tangent space at (x, y, z). Isometry group of this model is generated by rotations around z-axis, dilations and the inversions with respect to the spheres which intersect the xy-plane $\{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ orthogonally. Geodesics in this model are half circles and half lines intersecting the xy-plane orthogonally. Hyperbolic space can be associated with a so-called *asymptotic boundary* which is defined as follows. Let S be the set of all geodesic half-lines parametrized by arc length on $[0, \infty)$. Define an equivalence relation on S such that for two half geodesics $\gamma_1, \gamma_2 \in S$

$$\gamma_1 \, R \, \gamma_2 \in S \iff \sup d(\gamma_1(t), \gamma_2(t)) < \infty$$

where d denotes the distance in hyperbolic space. The set S/R is called the asymptotic boundary of $\mathbb{H}^3$ which we denote by $\partial_\infty \mathbb{H}^3$. A geodesic half-line determines a unique point on the boundary of the upper half space, the point where geodesic converges in the usual topology. Also two half-geodesics determine the same point on the boundary if and only if they stay at a bounded distance. Hence, we can write a bijection

$$\partial_\infty \mathbb{H}^3 \longleftrightarrow \{(x, y, z) \in \mathbb{R}^3 : z = 0\} \cup \{\infty\}$$

It is possible to define a topology on $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$ such that the induced topology on $\mathbb{H}^3$ is the usual topology and the bijection defined in the previous paragraph is a homeomorphism. This implies that $\partial_\infty \mathbb{H}^3$ is homemorphic to the two dimensional sphere. We will denote $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$ by $\overline{\mathbb{H}^3}$. For a subset $A \subset \mathbb{H}^3$, $\overline{A}$ will denote the closure of A in $\overline{\mathbb{H}^3}$. We will call the set $\overline{A} \cap \partial_\infty \mathbb{H}^3$ the *asymptotic boundary of* A which will be denoted by $\partial_\infty A$. Now we can state the asymptotic Plateau problem:

- Given a simple closed curve Γ embedded in $\partial_\infty \mathbb{H}^3$ does there exist a minimal surface Σ with $\partial_\infty \Sigma = \Gamma$?

This problem is very different from the classical Plateau problem in that such surfaces must have infinite area. So it is subtle to solve the minimization problem in this case. Anderson [2] has solved the problem realizing the solutions as *absolutely area minimizing surfaces.*

Definition 2.2.1. A compact or noncompact surface Σ is said to be an absolutely area minimizing surface if every compact subsurface $K \subset \Sigma$ minimizes area among all surfaces with boundary ∂K. If for a simple closed curve $\Gamma \subset \partial_\infty \mathbb{H}^3$ there exists a unique absolutely area minimizing surface asymptotic to Γ, then we will call Γ a *uniquely minimizing curve* and Σ a *uniquely minimizing surface*

Anderson obtained the following result. We will state the part necessary for our purposes:

Theorem 2.2.1 ([2], [3])**.** *Let Γ be a simple closed curve embedded in $\partial_\infty \mathbb{H}^3$. Then there exists an absolutely area minimizing surface with asymptotic boundary Γ.*

The proof of this theorem depends on the techniques from geometric measure theory and holds for general dimensions and codimensions. However, in the general case one should be careful about regularity of the solutions.

Note that solutions given by the above theorem does not give information about the topology of the surface. In [3], Anderson also solved the asymptotic Plateau problem of disk type. Before we state the theorem, we first give the following definition:

Definition 2.2.2. A complete embedded plane P is called a *least area plane* if any compact subdisk D of P minimizes area among all disks with boundary ∂D.

Theorem 2.2.2 ([3])**.** *Let $\Gamma \subset \partial_\infty \mathbb{H}^3$ be a simple closed curve. Then there exists a complete embedded least area plane with asymptotic boundary Γ.*

Before we finish this section we state a density result for uniquely minimizing surfaces which we will need later.

Theorem 2.2.3 ([8])**.** *Let Γ be a simple closed curve. Then there exists a sufficiently small $\epsilon > 0$ such that, if $\{\Gamma_t : t \in (-\epsilon, \epsilon)\}$ is a set of simple closed curves foliating an annulus around Γ, then except a countable subset of $(-\epsilon, \epsilon)$ Γ_t bounds a unique absolutely area minimizing surface.*

2.3 Bridge Principle

Mathematical formulation of the classical *bridge principle* is motivated by soap film experiments. Suppose that WE have two wire frames Γ_1 and Γ_2 and let Σ_1, Σ_2 be soap films spanning Γ_1 and Γ_2. Now suppose that we construct a new wire frame Γ by omitting small arcs on each wire and connecting the newly created endpoints of one

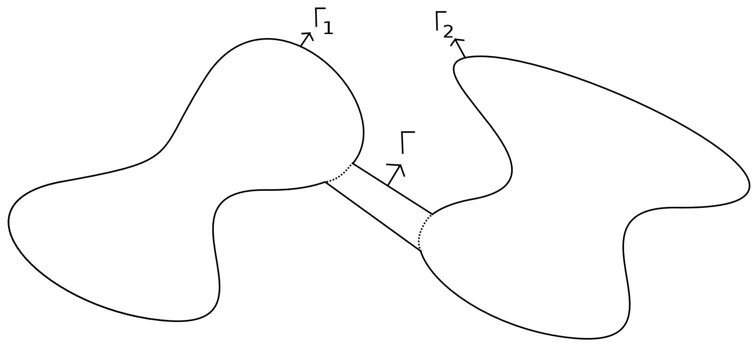

Figure 2.4: Γ_1 and Γ_2 represent disjoint wire frames (with the dashed arcs). Γ is the union of Γ_1, Γ_2, and the two parallel arcs (omitting dashed arcs).

curve with the other by parallel arcs which are close to each other (Figure 2.4). Let Σ be the soap film spanning Γ. It turns out that Σ has to be homeomorphic to the union of Σ_1 and Σ_2 with a tiny rectangle attaching them and the complement of the rectangle in Σ is close to the union of Σ_1 and Σ_2. Roughly speaking one can create a new minimal surface from the old ones by attaching them with a tiny rectangle. In this section we will state a *bridge principle at infinity* for uniquely minimizing surfaces in $\mathbb{H}^3$. Although this resembles the classical bridge principle which relies on the possibility of connecting two compact minimal surfaces with a rectangle, in the case of the bridge principle at infinity the 'boundaries' of the minimal surfaces lie in the asymptotic boundary putting the problem in a more abstract setting.

For the statement of the bridge principle at infinity we first need the following setup. Let Γ be a finite disjoint union of smooth simple closed curves embedded in

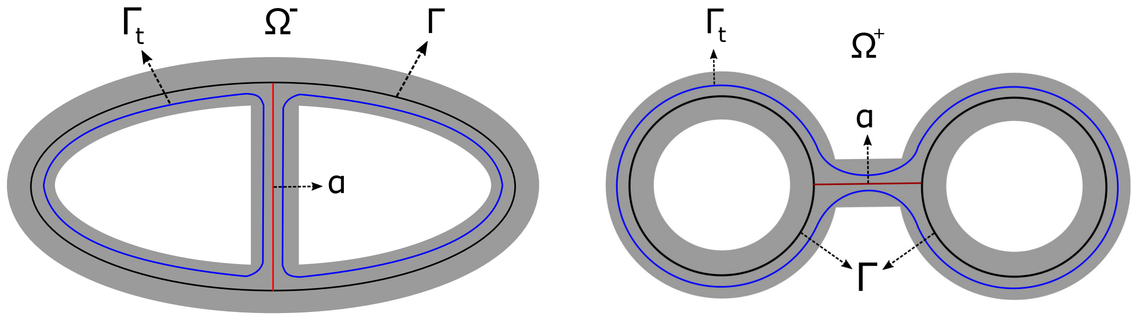

Figure 2.5: Γ is connected on the left and Ω^+ is the region enclosed by Γ. On the right Γ is disconnected and Ω^- is the region enclosed. The gray region is the set $A_\epsilon(\Gamma \cup \alpha)$.

$\partial_\infty \mathbb{H}^3$ and α be a line segment joining any two distinct points $p,\ q \in \Gamma$ such that $\alpha \cap \Gamma = \{p, q\}$ and $\alpha \perp \Gamma$ (Figure 2.5). For $\epsilon > 0$ define a neighborhood of the set $\Gamma \cup \alpha$ by $A_\epsilon(\Gamma \cup \alpha) = \{x \in \partial_\infty \mathbb{H}^3 : \mathrm{dist}(x, \Gamma \cup \alpha) < \epsilon\}$. Note that Γ separates the region into two (possibly disconnected) parts Ω^+ and Ω^- with $\partial\Omega^+ = \partial\Omega^- = \Gamma$. Let Ω^+ be the region containing α and $\{\Gamma_t : 0 < t < \epsilon\}$ be a foliation of $\Omega^+ \cap A_\epsilon(\Gamma \cup \alpha)$ such that Γ_t converges to $\Gamma \cup \alpha$ as t goes to 0. Lastly, we call the closure of the set $P_\alpha := \Omega^+ \cap A_\epsilon(\alpha)$ a *bridge along* α. The following theorem is obtained in [15].

Theorem 2.3.1 (Bridge Principle)**.** *Let Γ be as above and bound a unique absolutely area minimizing surface Σ. Then (with the notation above) there exists a sequence $t_n \in (0, \epsilon)$ converging to 0 such that $\Gamma_{t_n} \subset \Omega^+ \cap A_\epsilon(\Gamma \cup \alpha)$ bounds a unique absolutely area minimizing surface Σ_n and that Σ_n is homeomorphic to $\overline{\Sigma} \cup \overline{P_\alpha}$ (closures being taken in $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$).*

2.4 H-surfaces

Constant mean curvature surfaces are surfaces whose mean curvature is constant for all points on the surface. As in the case of minimal surfaces, we can also realize H-surfaces as a critical point of some functional. For this, let Σ and Σ_0 be compact surfaces with boundary Γ in an ambient Riemannian manifold M, η be the unit normal vector field on Σ and for $-\epsilon < t < \epsilon$, $\Sigma_t = \{\exp_p tf(p)\eta(p) : p \in \Sigma\}$ be a compactly supported variation of Σ with respect to a compactly supported function $f : \Sigma \to \mathbb{R}$. For a real number H_0 define the following functional

$$I_{H_0}(\Sigma_t) = A(\Sigma_t) + 2H_0\mathrm{Vol}(\Omega_t)$$

where A denotes the area and $\mathrm{Vol}(\Omega_t)$ denotes the volume of the region Ω_t enclosed by Σ_t and Σ_0. Then

$$\left.\frac{\mathrm{dA}(\Sigma_t)}{\mathrm{d}t}\right|_{t=0} = -2\int\limits_{\Sigma} f \cdot H$$

where H is the mean curvature function on $\Sigma_0 = \Sigma$ and

$$\left.\frac{\mathrm{dVol}(\Omega_t)}{\mathrm{d}t}\right|_{t=0} = \int\limits_{\Sigma} f$$

Hence,

$$\left.\frac{\mathrm{d}I_{H_0}(\Sigma_t)}{\mathrm{d}t}\right|_{t=0} = 2\int\limits_{\Sigma} f(H_0 - H).$$

It follows that if Σ is a critical point of the functional I_{H_0}, for example, a minimizer, then the above equality holds for all compactly supported variations. Hence, $H(p) =$

H_0 for all $p \in \Sigma$ so that Σ is a surface of constant mean curvature H_0. The second variation of the functional I_{H_0} in the direction of a compactly supported variation $f : \Sigma \to \mathbb{R}$ is given by the same formula as in the case of second variation of area at a minimal surface:

$$\frac{\mathrm{d}^2 \mathrm{A}(t)}{\mathrm{d}t^2}_{t=0} = -\int_\Sigma f \cdot (L(f)).$$

where L is the stability operator

$$L(f) = \Delta f + |B|^2 f + \mathrm{Ric}(\eta) f.$$

We call that Σ is a stable H-surface if all eigenvalues of L are nonnegative.

2.5 Convergence of H-Surfaces

There are different notions of convergence for manifolds; smooth convergence, flat convergence, Hausdorff convergence etc. For our purposes smooth convergence is the most important and it is the strongest one. Flat convergence is a weaker notion and essential from the perspective of geometric measure theory which is also used in the proof of asymptotic Plateau problem (Theorem 2.2.1). Now we give the precise definition of smooth convergence.

Definition 2.5.1. [16] Let Σ_n be a sequence of surfaces in a Riemannian manifold M. We say that Σ_n smoothly converges on compact sets to a surface Σ with finite multiplicity k if Σ is the accumulation set for $\bigcup_{n=1}^{\infty} \Sigma_n$ and the following holds:

- For all $p \in \Sigma$ there is a neighborhood W around p such that for n sufficiently large $\Sigma_n \cap W$ consists of a finite number of normal graphs over M; in other words, there exist functions $u_n^i : \Sigma \cap W \to \mathbb{R}$, $i \in \{1, 2, .., k\}$ such that $\Sigma_n \cap W = \bigcup_{i=1}^{i=k} \exp(u_n^i(p)\eta(p))$($\eta$ is the unit normal vector field on Σ). Moreover, u_n^i converges to 0 in the C^m topology for all $m \in \mathbb{N}$.

We will need the following compactness theorem for H-surfaces later.

Theorem 2.5.1. *[16] Let Σ_n be a sequence of H-surfaces embedded in a complete Riemannian manifold of bounded sectional curvature. If the norms of the second fundamendal forms corresponding to Σ_n's and the area of Σ_n's are bounded on every compact domain, then there exists a subsequence of Σ_n which converges smoothly on compact sets to an H-surface Σ with finite multiplicity.*

Before we finish this section, we will state a theorem which gives a uniform global bound on second fundamental forms of stable H-surfaces. This allows us, by the theorem above, to obtain convergence for complete H-surfaces with only local area bounds.

Theorem 2.5.2. *[18] Let M be a complete three dimensional Riemannian manifold with bounded sectional curvature $K > 0$. Then there exists a constant $C > 0$ such* *that for all stable H-surfaces Σ and for all $p \in \Sigma$*

$$|B(p)| \leq \frac{C}{\min\left(d(p, \partial\Sigma), \frac{\pi}{2\sqrt{C}}\right)}.$$

Chapter 3

GENERIC NON-UNIQUENESS OF MINIMAL SURFACES

In this chapter we prove our main theorem for minimal surfaces. In the next chapter we will generalize it to the constant mean curvature surfaces. Before stating the theorem we give a definition:

Definition 3.0.1.

- Let $\Gamma \in \partial_\infty \mathbb{H}^3$ be a simple closed curve. Then Γ is the image of a continuous embedding $f : S^1 \to \partial_\infty \mathbb{H}^3$. We define an open neighborhood $N_\epsilon(\Gamma)$ around Γ as the set of all simple closed curves Γ' for which there exists a one-to-one continuous map $f' : S^1 \to \partial_\infty \mathbb{H}^3$ with $f'(S^1) = \Gamma'$ such that $\|f(x) - f'(x)\| < \epsilon$ for all $x \in S^1$. We are going to denote the space of simpile closed curves by A_0.

- Let $g \in \mathbb{N}$. We define A_g as the subset of A_0 consisting of simple closed curves for which any absolutely area minimizing surface has positive genus.

Remark 3.0.1. When $\epsilon > 0$ is sufficiently small (relative to the diameter of Γ) another useful characterization of $N_\epsilon(\Gamma)$ can be given as follows: Let $A_\epsilon(\Gamma) = \{x \in \partial_\infty \mathbb{H}^3 : \text{dist}(x, \Gamma) < \epsilon\}$be an annulus around Γ. Then $N_\epsilon(\Gamma)$ consists of curves $\Gamma' \subset A_\epsilon(\Gamma)$ which are homotopic to Γ in $A_\epsilon(\Gamma)$.

Theorem 3.0.1. *The set of simple closed curves which bound more than one complete embedded minimal surface contains an open and dense subset of A_0.*

Proof. We know that every simple closed curve bounds a least area plane by Theorem 2.2.2. From this it follows that for all $\Gamma \in A_1$ there exist at least two complete embedded minimal surfaces with asymptotic boundary Γ. Hence, if we can show that A_1 is an open and dense subset of A_0, the proof will be complete. □

In the next two lemmas we are going to prove that A_1 is an open and dense subset of A_0 but first we need a classical convergence theorem for a sequence of area minimizing surfaces.

Lemma 3.0.2. *Let Γ_n be a sequence in A_0 which converges to some $\Gamma \in A_0$. Let Σ_n be a sequence of absolutely area minimizing surfaces with $\partial_\infty \Sigma_n = \Gamma_n$. Then there exists a subsequence of Σ_n which converges smoothly on compact sets to an absolutely area minimizing surface Σ with $\partial_\infty \Sigma = \Gamma$.*

Proof. By Theorem 2.5.1 it is enough to find local area bounds and local bounds on the second fundamental form of Σ_n. Note that any absolutely area minimizing surface is a stable minimal surface. Also since Σ_n's are complete by Theorem 2.5.2 there is a uniform global bound on the second fundamental form of Σ_n. Hence, it is enough to find local area bounds for the sequence Σ_n to conclude that Σ_n converges to a minimal surface Σ with finite multiplicity. For this let Ω_n be one of the connected open sets separated by Σ_n in $\mathbb{H}^3$. Then for any compact domain K, since Σ_n is absolutely area minimizing, $A(\Sigma_n \cap K) \leq A(\partial K \cap \Omega_n)$ where A denotes the area. The latter has always less area than ∂K so that $A(\partial K)$ serves as a local area bound. Finally, the limit of a sequence of area minimizing surfaces is again area minimizing so that $\Sigma \cap K$ is area minimizing for all K. In this case the convergence has to be with multiplicity one. □

Corollary 3.0.3. *A_1 is an open subset of the space of all simple closed curves.*

Proof. We will show that $A_0 \setminus A_1$ is closed. Let Γ_n be a sequence in $A_0 \setminus A_1$ which converges to some $\Gamma \in A_0$ and Σ_n be a sequence of absolutely area minimizing surfaces with $\partial_\infty \Sigma_n = \Gamma_n$. By definition of the sets A_n we can choose Σ_n to be homeomorphic to a plane for each n. Then by the above lemma we can choose a subsequence of Σ_n which converges smoothly on compact sets to an absolutely area minimizing surface Σ with asymptotic boundary Γ. From this it follows that for any compact domain K and for a sufficiently large n, $\Sigma_n \cap K$ can be written as a normal graph over $\Sigma \cap K$ so that both surfaces are homeomorphic. Since this holds for any compact domain and Σ_n is of genus 0, Σ must have genus 0 too. Now we have $\partial_\infty \Sigma = \Gamma$ and $\Gamma \in A_0 \setminus A_1$, so the proof is complete. □

Proposition 3.0.4. *A_1 is dense in the space of all simple closed curves.*

Proof. Let Γ be a simple closed curve and $\epsilon > 0$. By Lemma 2.2.3 there exists a smooth simple closed curve $\Gamma_u \in N_{\epsilon/2}(\Gamma)$ (Definition 3.0.1) which bounds a unique absolutely area minimizing surface Σ. Suppose that $\Gamma_u \notin A_1$, i.e. Σ is a plane. Since Γ_u is a uniquely minimizing curve we can apply the bridge principle.

For this let α be a line segment joining two distinct points of Γ_u satisfying the hypothesis of Theorem 2.3.1 and such that $\Gamma_u \cup \alpha$ lies in the annulus $A_{\epsilon/2}(\Gamma_u)$ around Γ_u. Then by the bridge principle we can find a curve $\Gamma'_u \subset A_{\epsilon/2}(\Gamma_u)$ which is a disjoint union of two simple closed curves and bounds a unique absolutely area minimizing surface Σ' (Figure 3.1). Furthermore $\overline{\Sigma'}$ is homeomorphic to $\overline{\Sigma} \cup \overline{P_\alpha}$ where $\overline{P_\alpha}$ is a *bridge along* α. Since Σ is a plane, Σ' is homeomorphic to an annulus.

Now we can apply the bridge principle to Σ' because it is also a uniquely area minimizing surface. For this let α' be a line segment connecting two distinct points of Γ'_u which lies on distinct components of Γ'_u and such that $\Gamma'_u \cup \alpha'$ still lies inside $A_{\epsilon/2}(\Gamma_u)$. Then by the bridge principle again, we can find a new curve $\Gamma''_u \subset A_{\epsilon/2}(\Gamma_u)$ which is a simple closed curve which bounds a unique absolutely area minimizing surface Σ''. Furthermore, $\overline{\Sigma''}$ is homeomorphic to $\overline{\Sigma'} \cup \overline{P_{\alpha'}}$ where $P_{\alpha'}$ is a bridge

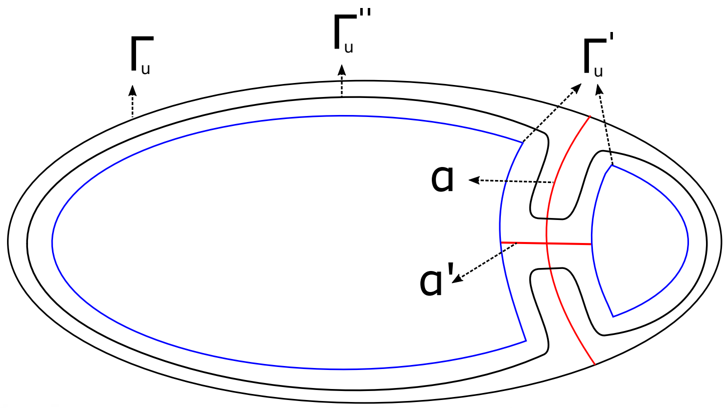

Figure 3.1: We can apply the bridge principle successively to a uniquely minimizing curve Γ_u successively in a neighborhood.

along α'. Since Σ' is an annulus and the bridge $\overline{P_{\alpha'}}$ connects distinct components of $\partial\overline{\Sigma'}$, Σ'' is homeomorphic to a surface of genus one with a disk removed. Since Γ_u'' is a simple closed curve lying inside $A_{\epsilon/2}(\Gamma_u)$ and homotopic to Γ_u, Γ_u'' belongs to $N_{\epsilon/2}(\Gamma_u)$. So it also belongs to $N_\epsilon(\Gamma)$. Lastly, Γ_u'' bounds a unique absolutely area minimizing surface of genus one so that $\Gamma_u'' \in A_1$. This completes the proof. □

Remark 3.0.2. Let $g > 0$ be an integer and let A_g denote the set of all simple closed curves for which all absolutely area minimizing surfaces have genus greater than or equal to g. Then the previous proofs can be generalized to show that A_g is open and dense for every $g \in \mathbb{N}$. This gives us the stratification of the space of simple closed curves as $A_0 \supset A_1 \supset A_2 \supset ... \supset A_g \supset ...$ where A_{g+1} is an open and dense subset of A_g for every $g \in \mathbb{N}$.

Chapter 4

GENERIC NON-UNIQUENESS OF H-SURFACES

In this section we generalize our result to H-surfaces, where $|H| < 1$. Following the formulation of [19] and the terminology of [7], we first state the generalization of asymptotic Plateau problem for H-surfaces. For this we will use the upper half-space model for $\mathbb{H}^3 = \{(x, y, z) \in \mathbb{R}^3 | z > 0\}$. So $\partial_\infty \mathbb{H}^3$ can be identified with the one-point compactification of $\mathbb{R}^2 \times \{0\}$.

Definition 4.0.1. Let $\Sigma \subset \mathbb{H}^3$ be a complete embedded surface with asymptotic boundary Γ. Then Γ separates $\mathbb{R}^2 \times \{0\}$ into two domains one of which is bounded. Let $O \subset \mathbb{R}^2 \times \{0\}$ be the bounded part and $\Omega_\Sigma \subset \mathbb{H}^3$ be the region enclosed by $\Sigma \cup O$. We say that Σ is an *H-minimizing* surface if for any compact domain $K \subset \mathbb{H}^3$, Σ minimizes the functional $I_K^H = A(\Sigma \cap K) + 2HV(\Omega_\Sigma \cap K)$ where A denotes the area and V denotes the volume. We will also say that Σ is *uniquely H-minimizing* if it is the unique H-minimizing surface with its asympotic boundary. Lastly, if Σ is a complete embedded plane which minimizes the functional among all disks on each compact domain, then we will call Σ an *H-minimizing plane*.

Lemma 4.0.1 ([1], [19]). *Let $\Gamma \subset \partial_\infty \mathbb{H}^3$ be C^1 curve and $|H| < 1$. Then there exists a complete embedded H-minimizing surface (with respect to the normal field pointing into Ω_Σ) with asymptotic boundary Γ.*

Remark 4.0.1. As noted in [19] it is enough to assume O to be a set of locally finite perimeter which in turn is rectifiable. Any simple closed curve can be approximated by rectifiable sets which allows us to extend the theorem to simple closed curves.

Claim. *The previous theorem still holds if we take Γ to be a simple closed curve.*

Proof. Let Γ be a simple closed curve. Then there exists a sequence of inscribed polygons Γ_n which converges to Γ. By the above theorem there exists a sequence of H-minimizing surfaces Σ_n with asymptotic boundary Γ_n. As we did in Proposition 3.0.3 we can extract a subsequence of Σ_n which converges smoothly on compact sets to an H-minimizing surface Σ with asymptotic boundary Γ. Since H-minimizing surfaces are stable and complete in this case (see Section 2.4) global uniform bounds onΣ_n exist by Theorem 2.5.2. So it is enough to find local area bounds. Let K be a compact domain in $\mathbb{H}^3$. Then $A(\Sigma_n \cap K) \leq A(\partial K \cap \Omega_{\Sigma_n})$ because otherwise we could replace $\Sigma_n \cap K$ with $\partial K \cap \Omega_{\Sigma_n}$ decreasing the total value of the functional I_K^H as the second term disappears. So $A(\partial K)$ serves as a local area bound. Hence, by Theorem 2.5.1 there exist a subsequence which converge to a H-minimizing surface Σ with asymptotic boundary Γ. □

Recently, the asymptotic Plateau problem for H-minimizing disks is also obtained.

Lemma 4.0.2 ([9]). *Given $|H| < 1$ and any simple closed curve $\Gamma \subset \partial\mathbb{H}^3$ with at least one smooth point there exists an embedded minimizing H-plane Σ with asymptotic boundary Γ.*

Because of the restriction of this theorem we will restrict ourselves to the space of simple closed curves which has at least one smooth point. So let A_0^* be the space of simple closed curves with at least one smooth point and A_1^H be the subset of A_0^* consisting of curves for which any H-minimizing surface has positive genus.

Proposition 4.0.3. *For each $|H| < 1$, A_1^H is an open and dense subset of A_0^*.*

Proof. The proof follows similarly as in the case of minimal surfaces. Let $\Gamma_n \in A_0 \backslash A_1$ be a sequence converging to a curve $\Gamma \in A_0$. By definition, there exists H-minimizing surfaces Σ_n which are planes and asymptotic to Γ_n. Then as in the proof of the previous claim we can extract a subsequence of Σ_n which converges smoothly on compact sets to some H-minimizing plane Σ with asymptotic boundary Γ. The limit surface Σ is a plane which shows that $A_0 \setminus A_1$ is closed, so A_1^H is open.

For the density we note that the bridge principle for absolutely area minimizing surfaces (Lemma 2.3.1) applies to H-minimizing surfaces[5]. Also, Lemma 2.2.3 also extends to uniquely minimizing H-surfaces ([6]). Then the density of A_1^H follows exactly the same way as in Proposition 3.0.4.

□

Corollary 4.0.4. *Given $|H| < 1$, the space of simple closed curves (with at least one smooth point) which bounds more than one complete embedded H-surface contains an open and dense subset.*

Proof. By Lemma 4.0.2 every simple closed curve $\Gamma \in A_1^H$ bounds an H-minimizing plane so the result follows. □

Chapter 5

CONCLUDING REMARKS

In this section we will comment on two natural extensions of our result. Firstly, we will ask what happens for the space of C^1 simple closed curves. Then we consider the case for curves with more than one component.

5.0.1 *The space of C^1 simple closed curves*

We have obtained the generic non-uniqueness result for the space of simple closed curves endowed with the C^0-topology. A natural question is what happens for C^1 simple closed curves? We know that a simple closed curve $\Gamma \subset \partial_\infty \mathbb{H}^3$ which bounds a star-shaped domain bounds a unique complete embedded minimal surface which in turn has to be a plane ([12]). Following lemma allows us to conclude that there exist C^1-open neighborhoods consisting of simple closed curves which bound a unique complete embedded minimal surface.

We consider the upper half-space model for $\mathbb{H}^3$ (see the beginning of Section 4). We will use the following characterization of star-shaped domains: A bounded simply-connected domain $\Omega \subset \mathbb{R}^2$ with a C^1 boundary is star-shaped with respect to a point $x \in \Omega$ if and only if $(y - x) \cdot \nu(y) \leq 0$ for all $y \in \partial\Omega$ where ν is the inward pointing normal on $\partial\Omega$.

Lemma 5.0.1. *Let $\Gamma \subset \mathbb{R}^2$ be a convex simple closed curve. Then there exists a C^1-neighbourhood of Γ consisting only of curves which bound a star-shaped domain.*

Proof. Let Γ be a convex curve such that the origin is contained in the bounded region enclosed by Γ. We claim that in a sufficiently small C^1-neighborhood of Γ every curve bounds a star-shaped region with respect to the origin. Otherwise, there exists a sequence of curves Γ_n which are not star-shaped with respect to the origin and the sequence converges to Γ in C^1. Then, by the characterization of star-shaped domains given above, there exists a sequence of points $x_n \in \Gamma_n$ with $x_n \cdot \nu_n(x_n) \geq 0$ where ν_n is the inward pointing normal vector field to Γ_n. By passing to a subsequence we can assume that x_n converges to a point $x \in \Gamma$ and the tangent lines at x_n to Γ_n converge to the tangent line of Γ passing through x. In particular $\nu_n(x_n)$ converges to $\nu(x)$. Hence, we find a point $x \in \Gamma$ such that $x \cdot \nu(x) \geq 0$. On the other hand since Γ is also star-shaped with respect to the origin, we have $x \cdot \nu(x) \leq 0$ so that $x \cdot \nu(x) = 0$. Therefore the tangent line L of Γ passing through x coincides with the line passing through the origin and x. Hence, L intersects the bounded open set enclosed by Γ which contradicts with the convexity of Γ. $\square$

By the above lemma and the uniqueness results for curves which bound star-shaped domains ([12]) we have the following corollary.

Corollary 5.0.2. *There exist* C^1*-open subsets of the space of simple closed curves in which every curve bounds a unique complete embedded minimal surface.*

Hence, generic non-uniqueness does not hold for the space of C^1 simple closed curves.

5.0.2 Curves with more than one component

Given a closed curve $\Gamma \subset \partial\mathbb{H}^3$ with more than one connected component, in general there is no connected minimal surface with asymptotic boundary Γ. For instance, if Γ is a disjoint union of two round circles and if Σ is a connected minimal surface, then Σ has to be a surface of revolution ([14]). However, there exist pairs of round circles which cannot bound a surface of revolution ([20]). On the other hand solving the asymptotic Plateau problem in mean convex domains of $\mathbb{H}^3$ ([10]) we can obtain plenty of connected minimal surfaces with two boundary components.

Of course, if we don't restrict to connected minimal surfaces, existence problem is still solvable by absolutely area minimizing surfaces. Then we can apply the bridge principle as we did before to show that A_1 is an open and dense subset of A_0. Hence, generic non-uniqueness follows as any curve with more than one component bounds a disjoint union of least area planes. More precisely the following generalization of our main theorem holds.

Theorem 5.0.3. *Let $A_{0,k}$ be the space of curves with k connected components. Then the subset of $A_{0,k}$ consisting of curves which bound more than one complete embedded minimal surface (not necessarily connected) contains an open and dense subset of $A_{0,k}$.*

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