SUPERCONFORMAL SIGMA MODELS IN THREE DIMENSIONS

by

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ABSTRACT

SUPERCONFORMAL SIGMA MODELS IN THREE DIMENSIONS

In this thesis, we study superconformal sigma models in three dimensions [1, 2]. These models are formulated as field theories [3] of smooth maps from a three-dimensional world-volume into a target space, subject to supersymmetry and conformal invariance. The geometric constraints imposed by these symmetries on the world-volume and on the target space are analyzed in detail [1, 4–6].

It is established that conformal invariance forces the target space to be a Riemannian cone [4], regardless of whether supersymmetry is present [1]. The minimal supersymmetric extension of the model does not impose additional restrictions on the target space geometry. Instead, supersymmetry invariance constrains the geometry of the three-dimensional world-volume, since it requires the supersymmetry parameter to be a conformal Killing spinor [2].

Three-dimensional manifolds admitting local conformal Killing spinors have been classified in both Euclidean and Lorentzian signatures [2, 5, 6]. Regardless of the metric signature, the conformal Killing spinor equation admits four independent local solutions on a three-dimensional manifold if the manifold is locally conformally flat. In the Euclidean case, no solutions exist otherwise. In contrast, in Lorentzian signature, if the manifold is not locally conformally flat but is locally conformally equivalent to a *pp*-wave metric, then exactly one conformal Killing spinor may exist.

ÖZET

ÜÇ BOYUTTA SÜPERKONFORMAL SİGMA MODELLER

Bu tezde, üç boyutta süperkonformal sigma modelleri incelenmektedir [1,2]. Bu modeller, üç boyutlu bir zar-hacminden hedef uzaya tanımlı türevlenebilir fonksiyonların oluşturduğu, süpersimetri ve konformal değişmezliğe tabi olan alan kuramları olarak formüle edilmektedir [3]. Bu simetrilerin zar-hacmi ve hedef uzay üzerinde dayattığı geometrik kısıtlamalar ayrıntılı biçimde analiz edilmiştir [1,4–6].

Süpersimetrinin varlığından bağımsız olarak, yalnızca konformal değişmezliğin, hedef uzayın bir Riemann konisi olmasını zorunlu kıldığı ortaya konmuştur [1,4]. Modelin asgari süpersimetri ile genişletilmesi, hedef uzay geometrisi üzerinde ek kısıtlamalar getirmez. Bunun yerine, süpersimetri değişmezliği süpersimetri parametresini bir konformal Killing spinoru olmaya zorlar ve üç boyutlu zar-hacminin geometrisini kısıtlar [2].

Yerel olarak konformal Killing spinor denkleminin çözümünün olduğu üç boyutlu çok katlılar hem Öklidyen hem de Lorentz durumlarında sınıflandırılmıştır [2, 5, 6]. Hem Öklidyen hem de Lorentz durumlarında, eğer üç boyutlu çok katlı yerel olarak konformal düz ise, konformal Killing spinor denklemi dört bağımsız yerel çözüm kabul eder. Öklidyen durumda, aksi takdirde hiçbir çözüm yoktur. Buna karşılık, Lorentz durumunda, üç boyutlu yerel olarak konformal düz olmayan çok katlılar üzerinde ancak yerel olarak bir *pp*-dalga metriğine konformal olarak denkse, yalnızca bir adet konformal Killing spinoru var olabilir.

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LIST OF SYMBOLS

$A, B, C, \ldots$	Coordinate indices for $(d+1)$-dimensional Target space
$a, b, c, \ldots$	Coordinate indices for d-dimensional Target space
C	Charge conjugation matrix
$C_{\mu\nu\rho}$	The Cotton tensor
$\tilde{\mathcal{D}}$	The Levi-Civita connection on $(d+1)$-dimensional target space
$\mathcal{D}$	The Levi-Civita connection on d-dimensional target space
$\not{D}$	Dirac operator
${e^i}_\mu$	A vielbein
${e_i}^\mu$	An inverse vielbein
$\mathcal{F}$	The real superpotential
$\tilde{G}_{AB}$	$(d+1)$-dimensional target space metric tensor
G_{ab}	d-dimensional target space metric tensor
$G_{\mu\nu}(\phi)$	Induced metric tensor
$g_{\mu\nu}$	World-volume metric tensor
$h_{\mu\nu}$	Fixed world-volume metric tensor
i, j, k	Tangent indices
J	The jacobian determinant of a world-volume coordinate transformation
k^a	Components of a target space Killing vector field
$\tilde{M}$	$(d+1)$-dimensional Target space
M	d-dimensional Target space
N_v	Norm of the vector field v^A
R_g	Ricci scalar associated to g
R_h	Ricci scalar associated to h
$R_{\mu\nu}$	Ricci tensor
$R_{\mu\nu\alpha\beta}$	Riemann curvature tensor
r	Radial coordinate on a Riemannian cone

S_{NG}	The Nambu-Goto Action
S_P	The Polyakov Action
$S_{\mu\nu}$	The Schouten tensor
T	String tension
T_p	p-brane tension
$U(\phi)$	d-dimensional target space potential
$\tilde{U}(\tilde{\phi}), \tilde{V}(\tilde{\phi})$	$(d+1)$-dimensional target space potentials
V^μ	Vector field associated a conformal Killing spinor
$\mathcal{V}^\mu$	The vector field obtained from the Cotton tensor
v^A	Homothetic vector field on $(d+1)$-dimensional target space
X	World-volume
X^A	The unit vector field along the r-direction
x^μ	Coordinates on the world-volume
y^a	Coordinates on d-dimensional target space
y^A	Coordinates on $(d+1)$-dimensional target space
$\alpha, \beta, \mu, \nu, \ldots$	Coordinate indices for $(p+1)$-dimensional world-volume space, curved-indices
$\Gamma^\sigma{}_{\mu\nu}$	Components of Levi-civita connection on the world-volume
$\tilde{\Gamma}^A{}_{BC}$	Components of Levi-civita connection on $(d+1)$-dimensional target space
$\Gamma^a{}_{bc}$	Components of Levi-civita connection on d-dimensional target space
γ_i	A gamma matrix
γ_μ	A local gamma matrix
δ_C	Rigid conformal transformation
$\delta_{int.}$	Infinitesimal internal transformation
δ_{SUSY}	Supersymmetry transformation
ϵ	Conformal Killing spinor
$\bar{\epsilon}$	Majorana conjugate of the spinor ϵ
$\eta_{ij}, \eta_{\mu\nu}$	Minkowski metric tensor

Λ_p	Cosmological Constant
$\Lambda^{\hat{i}}{}_{i}$	Local Lorentz transformation
ξ	Conformal Killing vector field
Φ	A sigma model map
ϕ	Represents scalar fields ϕ^a
$\tilde{\phi}$	Represents scalar fields ϕ^A
ϕ^A	Component fields of a sigma model map into $(d+1)$-dimensional target space
ϕ^a	Component fields of a sigma model map into d-dimensional target space
ϕ^0	Dilaton field
ϕ^r	Dilaton field written in terms of radial r-coordinates
ψ^A	Fermionic fields
ω	The proportionality function of conformal Killing vector field
$\omega_\mu{}^i{}_j$	The spin connection
∇	Levi-Civita connetion on the world-volume

1. INTRODUCTION

String theory is a candidate framework for unifying quantum mechanics and general relativity into a consistent theory of quantum gravity by replacing point particles with one dimensional extended objects, called *strings* [7–11]. The fundamental theory for a bosonic free string is formulated by considering the area of the surface traced out by the string during its motion in space-time. This area is computed through the integral of the area element constructed from the induced metric tensor, leading to the *Nambu-Goto action* [12]. However, the presence of field derivatives inside a square root in the Nambu-Goto action makes the quantization process technically challenging. An equivalent and more tractable formalism is provided by the *Polyakov action* [13]. The Polyakov action requires an independent metric tensor on the world-sheet, which results in the fact that it exhibits Weyl symmetry [14] in addition to the diffeomorphism symmetry. As a consequence, the Polyakov theory of a string possesses *conformal symmetry*. Conformal symmetry refers to invariance of the action under transformations that preserve angles while allowing local scaling of distances (see for example [15, 16]). The requirement of conformal invariance on the two-dimensional world-sheet ensures the consistency of the theory by constraining the dynamics of the string and determining the critical space-time dimensions where the string can consistently propagate [9].

Additionally, *supersymmetry*, which relates bosonic and fermionic degrees of freedom (see [17]), is an equally fundamental symmetry in string theory. By associating particles of different spin statistics, supersymmetry naturally stabilizes quantum corrections and yields theories that are both mathematically consistent and elegantly structured. There appears five consistent supersymmetric string theories in ten dimensions which are believed to be different limits of a single unifying theory known as the *M-theory* [18]. At low energy limit, M-theory reduces to *eleven-dimensional supergravity*, which admits a particular stable solution called the *M2-brane*, where a brane can be understood as higher dimensional generalizations of a string (for further details, see [19,20]). Recently, it has been realized that the coincident M2-brane theory may be described by a superconformal field theory in three-dimensions involving scalar fields in a suitable representation of certain Yang-Mills groups and Chern Simons interactions [21–25]. In most studies of this model, the manifold parametrized by the scalar fields, referred to as the *target space*, is assumed to be the Euclidean space, while the three-dimensional space-time is taken to be the Minkowski space. The aim of this thesis is to generalize this approach by considering a non-flat target space and an arbitrarily curved three-dimensional space-time, thereby formulating the theory as a sigma model. The resulting theory is known as the *superconformal sigma model in three dimensions* [1, 2].

The sigma model is originally introduced in the 1960s by Gell-Mann and Lévy [26]. Its name *sigma model* originates from the fact that one of the fields in their work was denoted by the Greek letter σ. It was then reformulated into the nonlinear sigma model [27]. In the 1970s, it gained attention for its asymptotic freedom in two dimensions and became a prototype for studying non-perturbative effects in quantum field theory [28–34]. In the 1980s, the nonlinear sigma model became central in string theory [35–37], as the two-dimensional quantum field theory on a string's world-sheet is a sigma model whose target space is the space-time itself. Additionally, supersymmetric extensions [38–41] of nonlinear sigma models were introduced around the same time. Specifically in three dimensions, locally supersymmetric nonlinear sigma models and their gauged extensions are introduced in [42] and [43], respectively. The sigma model had also a lasting impact on mathematics, particularly through its connections to harmonic maps/wave maps, moduli spaces, and mirror symmetry [44–46].

From the field theory perspective, a sigma model can be regarded as a theory of smooth maps between differentiable manifolds [3, 47]. The domain is generally taken to be a Lorentzian manifold, modeling spacetime, and is commonly referred to as the *world-volume* in string theory literature [7–11]. The codomain is typically a Riemannian manifold, known as the *target space*, which, particularly in supergravity contexts, is often realized as a coset space (for example, see chapter 4 of [48]). Sigma models play a pivotal role in describing the dynamics of branes, as they provide a systematic framework to encode how branes propagate through and interact with the ambient spacetime, represented by the target space. A particularly important class of sigma models, known as *conformal sigma models*, consists of those that exhibit conformal invariance [1]. One significant consequence of imposing conformal symmetry is that the geometry of the target space becomes strongly constrained [4]. When conformal symmetry is extended to include supersymmetry, the resulting theories are referred to as *superconformal sigma models*. The enhanced symmetry structure imposed by superconformal invariance further restricts the allowed target space geometries, with the specific constraints depending on the number of supersymmetries present. Moreover, the presence of supersymmetry also imposes constraints on the geometry of the world-volume.

This thesis is devoted to the study of superconformal sigma models in three dimensions, with a primary focus on the geometric constraints imposed by these symmetries on the world-volume and target space, following [1, 2, 4–6] and using [49–51] for the fundamental results.

Chapter 2 defines the sigma model [3, 47], presents the fundamental aspects of sigma model field theory, and discusses its symmetries. It is emphasized that the symmetries of the sigma model is divided into two categories: space-time symmetries discussed in Section 2.1.1 and internal symmetries explained in Section 2.1.2 [49].

Chapter 3 explores the applications of sigma models in brane mechanics [7–11,52]. It begins with a brief discussion of the point particle in Section 3.1, then proceeds to the bosonic free string in Section 3.2. Subsequently, the string is generalized to a p-brane, and the role of sigma models in brane dynamics is examined in Section 3.3.

Chapter 4 introduces the conformal sigma model [1,2]. This model is analyzed in detail, demonstrating that its target space must be a Riemannian cone [4]. Additionally, the application of conformal sigma models with a one-dimensional target space to the study of conformally coupled scalar fields is briefly discussed in Section 4.1 [53–59].

Chapter 5 focuses on the three-dimensional case and discusses supersymmetric extensions of the conformal sigma models [1, 2]. It begins by presenting the necessary formalism for spinors on curved manifold and introducing our spinor conventions [49] in Section 5.1. Subsequently, the minimal supersymmetric extensions of the conformal sigma models are discussed in Section 5.2, where it is addressed that the supersymmetry parameter used to construct the supersymmetry transformations must be a conformal Killing spinor [1, 2]. Next, Section 5.3 examines various properties of conformal Killing spinors: In particular, Theorem 5.1 establishes that a conformal Killing spinor gives rise to a conformal vector field [60], Theorem 5.2 proves that the number of solutions to the conformal Killing spinor equation depends solely on the conformal class of the metric tensor [2, 5, 6] and Theorem 5.3 derives the restrictions imposed on a manifold by the existence of a conformal Killing spinor [5, 61]. As a consequence, we conclude that any three-dimensional locally conformally flat manifold admits four independent local conformal Killing spinors and that in Lorentzian signature, a non-zero conformal Killing spinor exists on a three-dimensional manifold that is not locally conformally flat if and only if it is conformally equivalent to a *pp*-wave metric [2, 5, 6].

Finally, the thesis is concluded with remarks on extended supersymmetric conformal sigma models, emphasizing that the target space of the $\mathcal{N} = 2$ superconformal sigma model is a Kähler cone whose base is necessarily a Sasakian manifold [2, 4, 62].

2. SIGMA MODEL

A *sigma model* is defined by a triple (X, M, Φ), where Φ represents a smooth map, referred to as the *sigma model map*, from a $p+1$ dimensional Lorentzian manifold (X, g) with metric tensor signature $(-, +, \dots, +)$ into a d-dimensional Riemannian manifold (M, G), where g and G represent metric tensor fields, respectively [3, 47]. The domain, X, of the sigma model is called the *world-volume* (in particular, *world-line* if $dim X = 1$ and *world-sheet* if $dim X = 2$) while the co-domain, M, is called the *target space*. The reason for this terminology is that it fits best for the brane mechanics [7–11]. Specifically, a sigma model is called a *linear sigma model* if the target space happens to be a vector space. Otherwise, it is said *nonlinear sigma model* (see Section 2.1).

The notation for coordinates is schematically illustrated in Figure 2.1. Specifically, local coordinates on the world-volume X and the target space M are denoted by x^μ for $\mu = 0, \dots, p$ and y^a for $a = 1, \dots, d$, respectively. Accordingly, the components of the Levi-Civita connections ∇ on the world-volume and $\mathcal{D}$ on the target space are denoted by ${\Gamma^\alpha}_{\mu\nu}$ and ${\Gamma^c}_{ab}$, respectively. Since the coordinates, y^a, are real functions on the target space, the compositions

$$\phi^a \equiv y^a \circ \Phi \tag{2.1}$$

yield d real functions on the world-volume, which plays the role of the scalar fields of a field theory, called the *sigma model field theory.* As it can be seen in the sigma model literature, particularly in [3, 7–11, 47, 48], the most basic level sigma model field theory action is expressed as

$$S[\phi^a] = \int_X d^{p+1}x \sqrt{-g}\, g^{\mu\nu} G_{ab}(\phi) \partial_\mu \phi^a \partial_\nu \phi^b, \tag{2.2}$$

where $\sqrt{-g}$ denotes the square root of the absolute value of the determinant of the world-volume metric tensor $g_{\mu\nu}$. The minus sign is introduced to ensure that the expression under the square root is positive, since the determinant is negative due to the mostly plus Lorentzian signature of the world-volume metric.

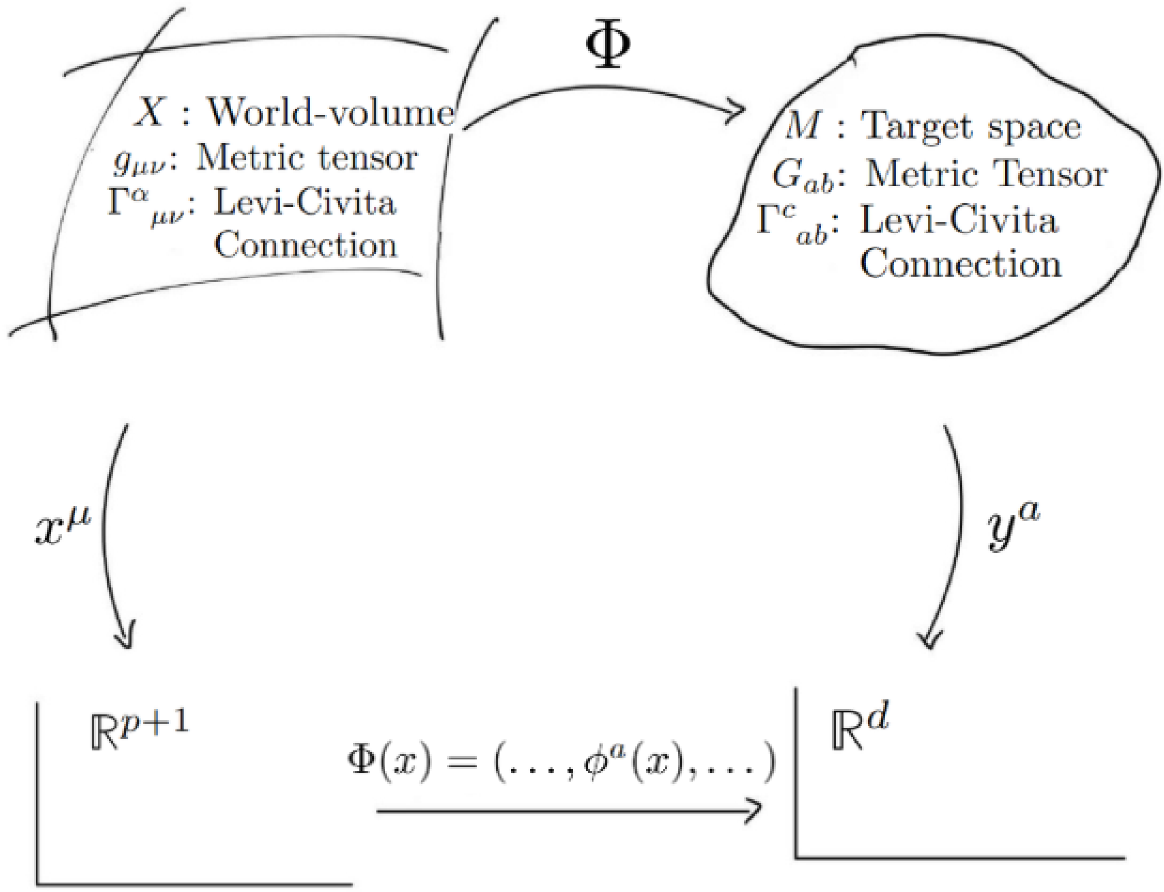

Figure 2.1. Illustration of a sigma model in local coordinates

2.1. Symmetries

The sigma model field theory possesses both *space-time (orexternal) symmetries* and *internal symmetries*. The space-time symmetry refers to the invariance of the action governing the dynamics of the scalar fields (2.2) under a world-volume diffeomorphism. On the other hand, the internal symmetry corresponds to transformations of the scalar fields without transforming the world-volume coordinates. As we will show in Section 2.1.2, internal symmetries of the sigma model (2.2) is formed by the isometries of the target space, that is, the metric preserving diffeomorphisms on the target space.

2.1.1. Space-time Symmetries

Given a diffeomorphism on the world-volume sending $x^\mu \mapsto \hat{x}^{\hat{\mu}}(x)$ with the inverse $\hat{x}^{\hat{\mu}} \mapsto x^\mu(\hat{x})$, the world-volume metric tensor and scalar fields transform as

$$g_{\mu\nu}(x) \mapsto \hat{g}_{\hat{\mu}\hat{\nu}}(\hat{x}) = g_{\alpha\beta}(x(\hat{x}))\frac{\partial x^\alpha}{\partial \hat{x}^{\hat{\mu}}}\frac{\partial x^\beta}{\partial \hat{x}^{\hat{\nu}}}, \quad \phi^a(x) \mapsto \hat{\phi}^a(\hat{x}) = \phi^a(x(\hat{x})). \tag{2.3}$$

These induce the transformations of the term including the target space indices and the inverse world-volume metric as

$$G_{ab}(\phi)\frac{\partial \phi^a}{\partial x^\mu}\frac{\partial \phi^b}{\partial x^\nu} \mapsto G_{ab}(\hat{\phi})\frac{\partial \hat{\phi}^a}{\partial \hat{x}^{\hat{\mu}}}\frac{\partial \hat{\phi}^b}{\partial \hat{x}^{\hat{\nu}}} = G_{ab}(\hat{\phi})\frac{\partial \phi^a}{\partial \hat{x}^{\hat{\mu}}}\frac{\partial \phi^b}{\partial \hat{x}^{\hat{\nu}}} = G_{ab}(\hat{\phi})\frac{\partial \phi^a}{\partial x^\alpha}\frac{\partial \phi^b}{\partial x^\beta}\frac{\partial x^\alpha}{\partial \hat{x}^{\hat{\mu}}}\frac{\partial x^\beta}{\partial \hat{x}^{\hat{\nu}}}, \tag{2.4}$$

$$g^{\mu\nu}(x) \mapsto \hat{g}^{\hat{\mu}\hat{\nu}}(\hat{x}) = g^{\alpha\beta}(x(\hat{x}))\frac{\partial \hat{x}^{\hat{\mu}}}{\partial x^\alpha}\frac{\partial \hat{x}^{\hat{\nu}}}{\partial x^\beta}, \tag{2.5}$$

where equation (2.5) is obtained by $\hat{g}^{\hat{\mu}\hat{\sigma}}(\hat{x})\hat{g}_{\hat{\sigma}\hat{\nu}}(\hat{x}) = \delta^{\hat{\mu}}{}_{\hat{\nu}}$. Moreover, the volume element transforms trivially which can be verified by observing the transformations

$$d^{p+1}x \mapsto d^{p+1}\hat{x} = J d^{p+1}x, \quad \sqrt{-g} \mapsto \sqrt{-\hat{g}} = \frac{\sqrt{-g}}{J^{-1}}, \tag{2.6}$$

where J is the Jacobian determinant of coordinate transformation. Consequently, transformation (2.4) is compensated by (2.5), leaving the action (2.2) invariant.

Invariance under world-volume diffeomorphisms is essential from the perspective of field theory: In string theory, the world-volume is typically taken to be Minkowski space with the corresponding Minkowski metric tensor denoted by $\eta_{\mu\nu} = \mathrm{diag}(-, +, \ldots, +)$. Consequently, the Lorentz symmetry of the action (2.2) can be directly verified by considering a world-volume diffeomorphism of the following form: $x^\mu \mapsto \hat{x}^{\hat{\mu}}(x) = \Lambda^{\hat{\mu}}{}_\mu x^\mu$, where $\Lambda^{\hat{\mu}}{}_\mu$ is a Lorentz transformation, i.e., it satisfies $\Lambda^{\hat{\mu}}{}_\mu \Lambda^{\hat{\nu}}{}_\nu \eta_{\hat{\mu}\hat{\nu}} = \eta_{\mu\nu}$.

2.1.2. Internal Symmetries

Similarly, one can consider a diffeomorphism on the target space space, sending $y^a \mapsto \hat{y}^{\hat{a}}(y)$. It implies transformations of scalar fields as

$$\phi^a(x) \mapsto \hat{\phi}^{\hat{a}}(\phi(x)). \tag{2.7}$$

We emphasize that the transformation (2.7) of the scalar fields is an internal transformation as the coordinates x^μ remain fixed. Infinitesimal version of the internal transformations can be described by considering a flow on the target space instead of an arbitrary diffeomorphism and the infinitesimal internal transformations of the scalar fields is expressed as [49]

$$\delta_{\text{int.}}\phi^a = k^a(\phi), \tag{2.8}$$

where the k^a are the components of the infinitesimal generator of the flow. Under the internal transformation, only fields in the integral (2.2) that carry target space indices are transformed, namely,

$$\begin{aligned}
\delta_{\text{int.}}\left(G_{ab}(\phi)\partial_\mu\phi^a\ \partial_\nu\phi^b\right) &= \frac{\partial G_{ab}}{\partial\phi^d}k^d\partial_\mu\phi^a\partial_\nu\phi^b + G_{ab}\partial_\mu k^a\partial_\nu\phi^b + G_{ab}\partial_\mu\phi^a\partial_\nu k^b \\
&= \left(\frac{\partial G_{ab}}{\partial\phi^d}k^d + G_{cb}\frac{\partial k^c}{\partial\phi^a} + G_{ac}\frac{\partial k^c}{\partial\phi^b}\right)\partial_\mu\phi^a\partial_\nu\phi^b \\
&= \left(G_{cb}\left[\frac{\partial k^c}{\partial\phi^a} + \varGamma^c{}_{da}\,k^d\right] + G_{ac}\left[\frac{\partial k^c}{\partial\phi^b} + \varGamma^c{}_{db}\,k^d\right]\right)\partial_\mu\phi^a\partial_\nu\phi^b \\
&= \left(G_{cb}\mathcal{D}_a k^c + G_{ac}\mathcal{D}_b k^c\right)\partial_\mu\phi^a\partial_\nu\phi^b \\
&= \left(\mathcal{D}_a k_b + \mathcal{D}_b k_a\right)\partial_\mu\phi^a\partial_\nu\phi^b,
\end{aligned} \tag{2.9}$$

for which we use the metric compatibility condition, that is

$$0 = \mathcal{D}_d G_{ab} = \frac{\partial G_{ab}}{\partial\phi^d} - \varGamma^c{}_{da}\,G_{cb} - \varGamma^c{}_{db}\,G_{ac}. \tag{2.10}$$

The transformation (2.9) leaves the action (2.2) invariant if and only if

$$\mathcal{D}_a k_b + \mathcal{D}_b k_a = 0. \tag{2.11}$$

A vector field satisfying the condition (2.11) is called a *Killing vector field*, the flow of which is an *isometry*. Therefore, only target space isometries induce internal symmetries. Generators of the Lie algebra of the isometry group are target space vector fields with components $k^a_I(\phi)$ where I is the Lie algebra index. Thus, a Killing vector field can be expressed as $k^a = \theta^I k^a_I$, with constants θ^I, so that the infinitesimal transformation (2.8) reads $\delta_{\text{int.}}\phi^a = \theta^I k^a_I(\phi)$. Promoting θ^I to x-dependent functions on the world-volume gives the gauged sigma model. For further details, we refer [1, 2, 63].

Finally, we clarify the terminological distinction between linear and nonlinear sigma models: When the target space is a vector space, the isometries correspond to linear transformations. In contrast, if the target space is an arbitrary smooth manifold, the isometries may be highly nonlinear.

3. SIGMA MODELS IN BRANE MECHANICS

Sigma models occupy a central role in theoretical physics, particularly in the formulation of the dynamics of p-branes [8, 9]. The present chapter aims to provide a motivation for employing sigma models in the study of p-brane dynamics. We begin with the motion of a free point particle, which can be regarded as a zero-brane, i.e., $p = 0$. We then proceed to the case of a free bosonic string, which is defined as a one-brane, i.e., $p = 1$. Finally, we generalize the bosonic string action to an extended object known as a *p-brane*.

3.1. A Free Point Particle

The trajectory of a free point particle moving in a curved space-time (M, G) is described by a smooth curve $\Phi\colon I \mapsto M$, where the interval I is called the *world-line* with coordinate τ. The trajectory is expected to be a geodesic in space-time, which is defined as a solution to the geodesic equation given by [51]

$$\frac{d^2\phi^a}{d\tau^2} + \Gamma^a_{\ bc}(\phi)\frac{d\phi^b}{d\tau}\frac{d\phi^c}{d\tau} = 0, \tag{3.1}$$

where $\phi^a(\tau)$'s are components of the curve in coordinates given in Figure 2.1. The associated action yielding the equations of motion (3.1) is given in terms of the length of the trajectory as [8]

$$S[\phi^a] = \int_I d\tau \sqrt{-G_{ab}(\phi)\frac{d\phi^a}{d\tau}\frac{d\phi^b}{d\tau}}, \tag{3.2}$$

where m represents the mass of the particle. Additionally, introducing an auxiliary non-zero field $e(\tau)$ on the interval I, the motion of a free particle can be also formulated through the action [8]

$$S[e, \phi^a] = \frac{1}{2}\int_I d\tau \left(e^{-1}(\tau)G_{ab}(\phi)\frac{d\phi^a}{d\tau}\frac{d\phi^b}{d\tau} - e(\tau)\right), \tag{3.3}$$

which also yields the geodesic equation (3.1) as equations of motion so that action (3.3) is on-shell equivalent to the action (3.2).

3.2. A Free Bosonic String

A *string* is a fundamental generalization of a point particle to a one-dimensional object propagating through a space-time (M, G). As it propagates, it traces out a two-dimensional surface in the space-time. In string theory, this surface is usually required to be a 2-dimensional embedded submanifold of M. Hence, dynamics of a string is encoded by a smooth embedding $\varPhi\colon X \mapsto M$, where X represents a 2-dimensional manifold X, known as *world-sheet.* Then, the surface drawn by the motion of the string corresponds to the image of X under the map $\varPhi$. Analogous to the free point particle case, the theory governing the free motion of a bosonic string is constructed based on the area of the surface traced out by the string in space-time [7–11,52]. Using the notation introduced in Figure 2.1, the components of the embedding map are denoted by the real functions given in equation (2.1) so that the action characterizing the motion of the string, known as the *Nambu-Goto action*, is given by

$$S_{NG}[\phi^a] = T \int_X \sqrt{-G(\phi)} d^2x, \tag{3.4}$$

where T is a constant, known as the *string tension* and $G(\phi)$ denotes the determinant of the *induced metric* which is given as

$$G_{\mu\nu}(\phi) \equiv G_{ab}\partial_\mu \phi^a \partial_\nu \phi^b, \tag{3.5}$$

so that $\sqrt{-G(\phi)}d^2x$ becomes the area element of the surface traced out by the string.

There exists yet another action, known as the *Polyakov action*, describing the free motion of a bosonic string. The Polyakov action requires an independent metric tensor, denoted by g, on the world-sheet X and is given by

$$S_P[g_{\mu\nu}, \phi^a] = \frac{T}{2} \int_X d^2x \sqrt{-g} g^{\mu\nu} G_{\mu\nu}(\phi) \tag{3.6}$$

which has the same form as the sigma model action (2.2), except that the Polyakov action (3.6) admits the world-sheet metric tensor g as a dynamical field, in contrast to the sigma model action given in equation (2.2). The Polyakov action provides an equivalent formulation of the motion of a free bosonic string as the Nambu–Goto action. This statement is explicitly proven in Theorem 3.1.

Theorem 3.1. *The Polyakov action is on-shell equivalent to the Nambu-Goto action.*

Proof. The variation of the Polyakov action (3.6) with respect to $g^{\mu\nu}$ yields

$$\delta S_P = \int d^2x\sqrt{-g}\left(G_{\mu\nu}(\phi) - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}G_{\alpha\beta}(\phi)\right)\delta g^{\mu\nu}, \tag{3.7}$$

which results in the equations of motion

$$G_{\mu\nu}(\phi) = \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}G_{\alpha\beta}(\phi). \tag{3.8}$$

That is, the induced metric tensor $G_{\mu\nu}(\phi)$ is proportional to the world-sheet metric tensor $g_{\mu\nu}$ with proportionality factor given by the scalar function $g^{\alpha\beta}G_{\alpha\beta}(\phi)$. As a result, evaluating the determinants and inverses of both sides of equation (3.8) yields

$$\sqrt{-G(\phi)} = \frac{g^{\alpha\beta}G_{\alpha\beta}(\phi)}{2}\sqrt{-g}, \quad G^{\mu\nu} = \frac{2}{g^{\alpha\beta}G_{\alpha\beta}(\phi)}g^{\mu\nu}, \tag{3.9}$$

where the first equality follows from the fact that $\dim X = 2$, while the second equality uses the observation that $g^{\alpha\beta}G_{\alpha\beta}(\phi) \neq 0$ since otherwise, equation (3.8) would imply $G_{\mu\nu}(\phi) = 0$, which contradicts the requirement that $G_{\mu\nu}(\phi)$ be a metric tensor. Consequently, the overall factors cancel each other so that we end up with

$$\sqrt{-G(\phi)}G^{\mu\nu} = \sqrt{-g}g^{\mu\nu}. \tag{3.10}$$

Finally, we conclude the proof by inserting (3.10) into (3.6), resulting in

$$S_P|_{\text{on shell}} = \frac{T}{2}\int_X d^2x\sqrt{-G(\phi)}G^{\mu\nu}G_{\mu\nu}(\phi) = T\int_X d^2x\sqrt{-G(\phi)} = S_{NG}|_{\text{on shell}}, \tag{3.11}$$

since $G^{\mu\nu}G_{\mu\nu}(\phi) = 2$. □

The beauty of the Polyakov action (3.6) lies in the fact the the partial derivatives of the scalar fields ϕ^a do not appear under a square root, in contrast to the Nambu–Goto action (3.4), which simplifies canonical quantization. Furthermore, the Polyakov action possesses Weyl symmetry, which is absent in the Nambu–Goto action. A *Weyl transformation* is defined as a rescaling of the world-sheet metric tensor $g_{\mu\nu}$ without a transformation of the coordinates x^μ, that is,

$$g_{\mu\nu}(x) \mapsto e^{2f(x)}g_{\mu\nu}(x) \tag{3.12}$$

for some real-valued smooth function f defined on X. We emphasize that a Weyl transformation is more general than a conformal transformation that we will study in the next chapter [15,16]: In a Weyl transformation, the smooth function f in equation (3.12) is arbitrary. Although a conformal transformation can also be written in the form of equation (3.12), in that case the function f is not arbitrary; rather, it is determined by a coordinate transformation and thus strongly depends on the chosen coordinate system. The invariance of the Polyakov action under a Weyl transformation can be readily verified by evaluating

$$\sqrt{-g} \mapsto e^{2f}\sqrt{-g}, \quad g^{\mu\nu} \mapsto e^{-2f} g^{\mu\nu}, \tag{3.13}$$

where the x-dependence has been suppressed. As a result, the overall factors cancel out, and the Polyakov action (3.6) remains unchanged. The emergence of Weyl symmetry is closely to the fact that introducing an independent dynamical world-sheet metric tensor in the Polyakov action does not introduce additional dynamics beyond those already present in the Nambu–Goto action since the dynamical world-sheet metric tensor g in the Polyakov action can always be gauged away by a suitable Weyl transformation as follows: Firstly, it is a well-known fact that any two-dimensional manifold is locally conformally flat (see for example [50]), meaning that, around any point, there exists a coordinate system in which the metric tensor is conformally related to the flat Minkowski metric tensor $\eta_{\mu\nu} = \mathrm{diag}(-,+,\ldots,+)$. That is,

$$g_{\mu\nu}(x) = e^{-2F(x)}\eta_{\mu\nu}, \tag{3.14}$$

where F is a real-valued smooth function defined on the world-sheet and strongly depends on the chosen coordinate system x^μ. Consequently, the Weyl symmetry provides the opportunity of locally scaling the world-sheet metric tensor with e^{2F} without disturbing the action, resulting in a symmetry of the Polyakov action given by

$$g_{\mu\nu} \mapsto \eta_{\mu\nu}. \tag{3.15}$$

Consequently, one can always locally use the Minkowski metric tensor field.

3.3. Bosonic p-brane

A p-brane is the natural generalization of a string to a p-dimensional extended object, where $p > 1$. Analogously, it propagates through space-time (M, G) and sweeps out a $(p+1)$-dimensional volume which is supposed to be an embedded submanifold of the space-time and its dynamics is also modeled by an embedding $\Phi : X \mapsto M$, where

X is a $(p+1)$-dimensional Lorentzian manifold and Φ is an embedding. In the brane mechanics literature [7–11], the Lorentzian manifold X is referred to as the *world-volume*, thereby justifying our terminology. The action characterizing the dynamics of a free bosonic p-brane is formulated in terms of the volume swept out by the p-brane during its evolution through space-time. Thus, it is given as

$$S[\phi^a] = T_p \int_X d^{p+1}x \sqrt{-G(\phi)} \tag{3.16}$$

where the constant T_p is called the *p-brane tension* [8] and $G(\phi)$ is the induced metric described in equation (3.5). As in the case of string, there exists an on-shell equivalent formulation of the dynamics of a free bosonic p-brane, explained in Theorem 3.2.

Theorem 3.2. *The free p-brane action* (3.16) *is equivalent to the action*

$$S[g_{\mu\nu}, \phi^a] = \frac{T_p}{2} \int_X d^{p+1}x \sqrt{-g} \left(g^{\mu\nu} G_{\mu\nu}(\phi) + \Lambda_p\right), \tag{3.17}$$

where the cosmological constant is defined as $\Lambda_p \equiv 1 - p$.

Proof. Following the same procedure used to derive equation (3.8), variation of the action (3.17) with respect to $g^{\mu\nu}$ gives

$$G_{\mu\nu}(\phi) = \frac{1}{2} g_{\mu\nu} \left(g^{\alpha\beta} G_{\alpha\beta}(\phi) + \Lambda_p\right). \tag{3.18}$$

Multiplying both sides of equation (3.18) by $g^{\mu\nu}$ and contracting indices gives

$$g^{\mu\nu} G_{\mu\nu}(\phi) = \frac{p+1}{2} \left(g^{\alpha\beta} G_{\alpha\beta}(\phi) + \Lambda_p\right). \tag{3.19}$$

Hence, solving equation (3.19) for $g^{\mu\nu} G_{\mu\nu}(\phi)$, we obtain

$$g^{\mu\nu} G_{\mu\nu}(\phi) = -\frac{p+1}{p-1} \Lambda_p. \tag{3.20}$$

Substituting equation (3.20) back into equation (3.18), we find that

$$G_{\mu\nu}(\phi) = g_{\mu\nu}, \tag{3.21}$$

showing that the world-volume metric is equal to the induced metric. Finally, inserting equation (3.21) back into the action (3.17), we end up with

$$S|_{\text{on shell}} = \frac{T_p}{2} \int_X d^{p+1}x \sqrt{-G(\phi)} \left(G^{\mu\nu} G_{\mu\nu}(\phi) + \Lambda_p\right) = T_p \int_X d^{p+1}x \sqrt{-G(\phi)}. \qquad \square$$

As a consequence, free p-brane can be equivalently formulated as a sigma model field theory with dynamical world-volume metric tensor. Since the bosonic p-brane action (3.17) treats the world-volume metric tensor g as a dynamical field, it is natural to include the Einstein–Hilbert term, namely $\sqrt{-g}R_g$, where R_g is the Ricci scalar associated with g, into the action (3.17) to promote it to a genuine theory of gravity. Moreover, the cosmological constant Λ_p in (3.17) can be replaced by a target space potential $U(\phi)$ to incorporate more general interactions of the p-brane. Consequently, the action for a bosonic p-brane may be written as

$$S[g_{\mu\nu},\phi^a] = -\frac{1}{2}\int_X d^{p+1}x\sqrt{-g}\left(R_g + g^{\mu\nu}G_{ab}(\phi)\partial_\mu\phi^a\partial_\nu\phi^b + U(\phi)\right), \tag{3.22}$$

where we fix constant $T_p = -1$ as in [2]. A possible solution for the world-volume metric tensor in (3.22) may be obtained by help of an ansatz

$$g = e^{2\phi^0}h, \tag{3.23}$$

where h is a fixed metric tensor on the world-volume and ϕ^0 is a scalar field on the world-volume, known as the *dilaton field*. Hence the dilaton field, ϕ^0, replaces the dynamics of the world-volume metric tensor $g_{\mu\nu}$ and so we aim to write the action (3.22) in terms of the dilaton field. Firstly, using the ansatz (3.23), a relation between volume forms, inverse metric tensors and associated Ricci scalars corresponding to g and h are found as (see Appendix D of [51])

$$g^{\mu\nu} = e^{-2\phi^0}h^{\mu\nu}, \quad \sqrt{-g} = e^{(p+1)\phi^0}\sqrt{-h}, \tag{3.24}$$

$$R_g = e^{-2\phi^0}\left[R_h - p(p-1)h^{\mu\nu}\partial_\mu\phi^0\partial_\nu\phi^0 - 2p\nabla^h_\mu\partial^\mu\phi^0\right], \tag{3.25}$$

where ∇^h is the Levi-Civita connection associated to the fixed metric tensor h. Combination of (3.24) and (3.25) simplifies the first term in the action (3.22) as

$$\begin{aligned}
&\sqrt{-g}R_g = \sqrt{-h}\left(e^{(p-1)\phi^0}\left[R_h - p(p-1)h^{\mu\nu}\partial_\mu\phi^0\partial_\nu\phi^0\right] + 2pe^{(p-1)\phi^0}\nabla^h_\mu\partial^\mu\phi^0\right)\\
&= \sqrt{-h}\left(e^{(p-1)\phi^0}\left[R_h - p(p-1)h^{\mu\nu}\partial_\mu\phi^0\partial_\nu\phi^0\right] + 2p\nabla^h_\mu(e^{(p-1)\phi^0}\partial^\mu\phi^0) - 2p\partial^\mu\phi^0\partial_\mu e^{(p-1)\phi^0}\right)\\
&= \sqrt{-h}\left(e^{(p-1)\phi^0}\left[R_h - p(p-1)h^{\mu\nu}\partial_\mu\phi^0\partial_\nu\phi^0\right] - 2p(p-1)e^{(p-1)\phi^0}\partial^\mu\phi^0\partial_\mu\phi^0\right)\\
&= \sqrt{-h}e^{(p-1)\phi^0}\left[R_h + p(p-1)h^{\mu\nu}\partial_\mu\phi^0\partial_\nu\phi^0\right],
\end{aligned} \tag{3.26}$$

where the divergence term, namely $\nabla^h_\mu\left(e^{(p-1)\phi^0}\partial^\mu\phi^0\right)$, disappears at the third step since its integral is converted into a boundary integral, which is assumed to vanish under the condition that the fields vanish at the boundary. Subsequently, the equalities (3.24) helps us to write second and third terms of action (3.22) in terms of dilaton field as

$$\sqrt{-g}g^{\mu\nu}G_{ab}(\phi^a)\partial_\mu\phi^a\partial_\nu\phi^b = e^{(p-1)\phi^0}\sqrt{-h}h^{\mu\nu}G_{ab}(\phi)\partial_\mu\phi^a\partial_\nu\phi^b, \tag{3.27}$$

$$\sqrt{-g}U(\phi^a) = e^{(p+1)\phi^0}\sqrt{-h}U(\phi^a). \tag{3.28}$$

As a result, by substituting the expressions (3.26), (3.27), and (3.28) into the action (3.22), we obtain it in the form

$$\begin{aligned} S[\phi^0,\phi^a] &= \int_X d^{p+1}x\sqrt{-h}e^{(p-1)\phi^0}\left(R_h + p(p-1)h^{\mu\nu}\left[\partial_\mu\phi^0\partial_\nu\phi^0 + G_{ab}(\phi)\partial_\mu\phi^a\partial_\nu\phi^b\right] + U(\phi^a)\right) \\ &= \int_X d^{p+1}x\sqrt{-h}e^{(p-1)\phi^0}\left(R_h + h^{\mu\nu}G_{AB}(\phi)\partial_\mu\phi^A\partial_\nu\phi^B + U(\phi^a)\right) \end{aligned} \tag{3.29}$$

where the capital Latin indices $A = 0, 1, \ldots, d$ are introduced to distinguish between ϕ^0 and ϕ^a and we have defined

$$G_{AB}(\phi) \equiv \begin{cases} G_{ab}(\phi) \;\text{ for }\; A = a, B = b \\ p(p-1) \;\text{ for }\; A = 0, B = 0 \\ 0 \;\text{ for } A \neq 0,\; B = 0. \end{cases} \tag{3.30}$$

The action (3.29) also describes the motion of a string and is written in the so-called *string frame*, whereas the action (3.17) is expressed in the *Einstein frame*. For further details on the distinction between these two frames, we refer the reader to Section 1.8.1 of [9]. Additionally, the string frame action of a bosonic p-brane, given in equation (3.29), can be expressed more compactly as [2]

$$S[\phi^0,\phi^a] = -\frac{1}{2}\int_X d^{p+1}x\,\sqrt{-h}\left(e^{(p-1)\phi^0}R_h + \tilde{G}_{AB}(\tilde{\phi})h^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^B + \tilde{U}(\tilde{\phi})\right), \tag{3.31}$$

where we have defined (3.29)

$$\tilde{U}(\tilde{\phi}) \equiv e^{(p+1)\phi^0}U(\phi), \quad \tilde{G}_{AB}(\tilde{\phi}) \equiv \begin{cases} \tilde{G}_{ab}(\tilde{\phi}) = e^{(p-1)\phi^0}G_{ab}(\phi) \\ \tilde{G}_{00}(\tilde{\phi}) = p(p-1)e^{(p-1)\phi^0} \\ \tilde{G}_{A0}(\tilde{\phi}) = 0 \end{cases} \tag{3.32}$$

where $\tilde{\phi}$ represents $\phi^A = (\phi^0, \phi^a)$ while ϕ stands for only ϕ^a's. The action (3.32) will be used in the next chapter to introduce conformal sigma model field theory.

4. CONFORMAL SIGMA MODEL

In this section, we assume $p > 1$ unless stated otherwise. The *conformal sigma model field theory* is obtained by generalizing the string frame action of a p-brane written in the form given in equation (3.31). Specifically, we promote the factor $e^{(p-1)\phi^0}$ multiplying the Einstein-Hilbert term in (3.31) to an arbitrary target space potential, denoted by $\tilde{V}(\tilde{\phi})$, and treat the target space metric tensor $\tilde{G}_{AB}(\tilde{\phi})$ and the target space potential $\tilde{U}(\tilde{\phi})$ as arbitrary functions of the scalar fields. Consequently, we end up with an action given by [1]

$$\tilde{S}[\phi^0, \phi^a] = -\frac{1}{2}\int_X d^{p+1}x\,\sqrt{-h}\left(\tilde{V}(\tilde{\phi})R_h + \tilde{G}_{AB}(\tilde{\phi})h^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^B + \tilde{U}(\tilde{\phi})\right), \tag{4.1}$$

in which we emphasize that the metric tensor h is considered as a fixed background metric tensor. As previously mentioned, the action (4.1) describes a sigma model for which the world-volume, target space, and a sigma model map are denoted by (X, h), $(\tilde{M}, \tilde{G})$, and $\tilde{\Phi}$, respectively. We emphasize that the target space $(\tilde{M}, \tilde{G})$ is considered as a $(d+1)$-dimensional Riemannian manifold with coordinates $y^A = (y^0, y^a)$ so that the components of a sigma model map is given by $\tilde{\phi} = (\phi^A) = (\phi^0, \phi^a)$ while we denote $\phi = (\phi^a)$ as in (3.31). The conformal sigma model field theory is obtained by assuming an additional ingredient on the world-volume: The world-volume metric tensor is assumed to admit a conformal Killing vector field ξ, which is defined in Definition 4.2.

Definition 4.1. *A vector field ξ is called a conformal Killing vector field if*

$$\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = 4\omega h_{\mu\nu}, \tag{4.2}$$

where $\nabla_\mu\xi_\nu = \partial_\mu\xi_\nu - \mathit{\Gamma}^\alpha{}_{\mu\nu}\,\xi_\alpha$ and ω is a smooth real function given as[1]

$$\omega = \frac{1}{2(p+1)}\nabla_\mu\xi^\mu. \tag{4.3}$$

A Killing vector field is a special case of a conformal Killing vector field, defined by equation (4.2) with $\omega = 0$.

[1]This can be verified by observing that $4(p+1)\omega = 4\omega h^{\mu\nu}h_{\mu\nu} = h^{\mu\nu}(\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu) = 2\nabla_\mu\xi^\mu$.

Finally, the conformal sigma model field theory is defined by action (4.1) and the condition that it is invariant under the *rigid conformal transformation* given by [1]

$$\delta_C \phi^A = \xi^\mu \partial_\mu \phi^A + \omega v^A(\tilde{\phi}) \tag{4.4}$$

where ξ is a conformal Killing vector field satisfying (4.2) and v^A's are real functions on the target space $\tilde{M}$. Obviously, the rigid conformal transformation (4.4) reduces to the Lie derivative of the scalar fields when $\omega = 0$. The transformation (4.4) is described as a rigid transformation because it leaves the world-volume metric tensor h unchanged, that is, $\delta_C h_{\mu\nu} = 0$. The conformal transformations (4.4) constitutes the closed conformal algebra, as proven in Theorem 4.1.

Theorem 4.1. *The Lie bracket of any two conformal transformation defined in* (4.4) *is again a conformal transformation.*

Proof. It is easy to show that given conformal Killing vector fields ξ_1 and ξ_2 with proportionality functions ω_1 and ω_2 respectively, their Lie bracket, denoted by $\xi_3 \equiv [\xi_2, \xi_1]$, is a conformal Killing vector field with the proportionality function $\omega_3 \equiv \xi_2^\mu \partial_\mu \omega_1 - \xi_1^\mu \partial_\mu \omega_2$. Thus, if the conformal transformations (4.4) corresponding to the conformal vector fields ξ_1 and ξ_2 are denoted by δ_{C_1} and δ_{C_2}, the calculation

$$\begin{aligned}
\delta_{C_1}\delta_{C_2}\phi^A &= \delta_{C_1}(\xi_2^\mu \partial_\mu \phi^A + \omega_2 v^A) = \xi_2^\mu \partial_\mu(\delta_{C_1}\phi^A) + \omega_2 \frac{\partial v^A}{\partial \phi^B}(\delta_{C_1}\phi^B) \\
&= \xi_2^\mu(\partial_\mu \xi_1^\nu)\partial_\nu \phi^A + \xi_1^\nu \xi_2^\mu \partial^2_{\nu\mu}\phi^A + \xi_2^\mu \partial_\mu(\omega_1 v^A) + \omega_2 \xi_1^\mu \partial_\mu v^A + \omega_1 \omega_2 v^B \frac{\partial v^A}{\partial \phi^B}
\end{aligned} \tag{4.5}$$

directly reveals the bracket to be

$$\begin{aligned}
[\delta_{C_1}, \delta_{C_2}]\phi^A &= \xi_2^\mu(\partial_\mu \xi_1^\nu)\partial_\nu \phi^A + \xi_1^\nu \xi_2^\mu \partial^2_{\nu\mu}\phi^A + \xi_2^\mu \partial_\mu(\omega_1 v^A) + \omega_2 \xi_1^\mu \partial_\mu v^A + \omega_1 \omega_2 v^B \frac{\partial v^A}{\partial \phi^B} \\
&\quad - \left(\xi_1^\mu(\partial_\mu \xi_2^\nu)\partial_\nu \phi^A + \xi_2^\nu \xi_1^\mu \partial^2_{\nu\mu}\phi^A + \xi_1^\mu \partial_\mu(\omega_2 v^A) + \omega_1 \xi_2^\mu \partial_\mu v^A + \omega_2 \omega_1 v^B \frac{\partial v^A}{\partial \phi^B} \right) \\
&= \xi_2^\mu(\partial_\mu \xi_1^\nu)\partial_\nu \phi^A - \xi_1^\mu(\partial_\mu \xi_2^\nu)\partial_\nu \phi^A + \xi_2^\mu v^A \partial_\mu \omega_1 + \xi_1^\mu v^A \partial_\mu \omega_2 \\
&= \left[\xi_2^\mu(\partial_\mu \xi_1^\nu) - \xi_1^\mu(\partial_\mu \xi_2^\nu)\right] \partial_\nu \phi^A + \left[\xi_2^\mu \partial_\mu \omega_1 + \xi_1^\mu \partial_\mu \omega_2\right] v^A \\
&= \xi_3^\nu \partial_\nu \phi^A + \omega_3 v^A(\tilde{\phi}) \\
&= \delta_{C_3}\phi^A
\end{aligned} \tag{4.6}$$

which implies that $[\delta_{C_1}, \delta_{C_2}] = \delta_{C_3}$ where the reversal of order is emphasized [1]. $\square$

The target space of the conformal sigma model is kept as arbitrary as possible, satisfying the requirement that the conformal sigma model action is invariant under the conformal transformation. The infinitesimal variation of the action (4.1) with respect to the conformal transformation (4.4) is calculated [1] as

$$\begin{aligned}\delta_C \tilde{S}[\phi^A] = -\frac{1}{2}\int_X d^{p+1}x\sqrt{-h}\Big[\omega\partial_\mu\phi^A\partial^\mu\phi^B\left(\tilde{\mathcal{D}}_A v_B + \tilde{\mathcal{D}}_B v_A - 2(p-1)\tilde{G}_{AB}\right)\\ +\omega\left(v^A\frac{\partial\tilde{U}}{\partial\phi^A} - 2(p+1)\tilde{U}(\tilde{\phi})\right) + \omega R_h\left(v^A\frac{\partial\tilde{V}}{\partial\phi^A} - 2(p-1)\tilde{V}(\tilde{\phi})\right)\\ -2\partial_\mu\omega\partial^\mu\phi^B\left(v_B - 2p\frac{\partial\tilde{V}}{\partial\phi^B}\right)\Big],\end{aligned} \tag{4.7}$$

where $\tilde{\mathcal{D}}$ represents the Levi-Civita connection of the target space $\tilde{M}$ with components $\tilde{\Gamma}^C{}_{AB}$ so that

$$\tilde{D}_A v_B = \partial_A v_B - \tilde{\Gamma}^C{}_{AB}\, v_C. \tag{4.8}$$

The infinitesimal variation (4.7) is vanishing if

$$\tilde{\mathcal{D}}_A v_B + \tilde{\mathcal{D}}_B v_A = 2(p-1)\tilde{G}_{AB}, \tag{4.9}$$

$$v_B = 2p\partial_B\tilde{V}, \tag{4.10}$$

$$v^A\partial_A\tilde{V} = 2(p-1)\tilde{V}, \tag{4.11}$$

$$v^A\partial_A\tilde{U} = 2(p+1)\tilde{U}. \tag{4.12}$$

Equation (4.10) shows that the v^A's are components of a vector field that is *hypersurface orthogonal* to the level surfaces of $\tilde{V}$. Additionally, any vector field satisfying equation (4.9) is known as a *homothetic vector field* [4].

The constraint equations (4.9)–(4.12) impose strong restrictions on the geometry of the target space, and our objective is to classify the Riemannian manifolds that satisfy these conditions. A concrete example that solves these constraints is provided by the Euclidean space equipped with the Kronecker delta metric tensor, that is, $\tilde{M} = \mathbb{R}^{d+1}$ with $\tilde{G}_{AB} = \delta_{AB}$, for which a solution is obtained by setting [1]

$$v^A = (p-1)y^A, \quad \tilde{V} = \frac{p-1}{4p}\delta_{AB}y^A y^B \tag{4.13}$$

and U is an arbitrary homogeneous function of degree $\frac{2(p+1)}{p-1}$, meaning that it satisfies $U(ky^0,\ldots,ky^d) = k^{\frac{2(p+1)}{p-1}}U(y^0,\ldots,y^d)$ for any real constant k as a consequence of

Euler's homogeneous function theorem [64]. This example fits into a more general framework: as we prove in Theorem 4.2, the target space must be a Riemannian cone, following [1, 4]. Since Euclidean space can be regarded as a Riemannian cone by choosing suitable coordinates, it is therefore expected that a solution exists for a Euclidean target space.

Definition 4.2. *The Riemannian cone over a Riemannian manifold* (M, G) *is the smooth manifold* $\tilde{M} = \mathbb{R}^+ \times M$ *equipped with the Riemannian metric*

$$\tilde{G} = dr^2 + r^2 G, \tag{4.14}$$

where r *denotes the coordinates on* $\mathbb{R}^+$*. The metric* (4.14) *is known as the cone metric and* (M, G) *is called the base manifold.*

Theorem 4.2. *The target space of the conformal sigma model is a Riemannian cone.*

Proof. We begin by demonstrating that v^A is a non-zero vector field, which is equivalent to the statement that its norm defined as

$$N_v \equiv v^A v_A \equiv \tilde{G}_{AB} v^A v^B, \tag{4.15}$$

is never zero as the signature of $\tilde{G}$ is $(+, \dots, +)$. It can be seen as follows: The second constraint equation (4.10) implies that

$$\tilde{\mathcal{D}}_A v_B = 2p(\partial_A \partial_B \tilde{V} - \tilde{\Gamma}^C{}_{AB}\, \partial_C \tilde{V}) = 2p(\partial_B \partial_A \tilde{V} - \tilde{\Gamma}^C{}_{BA}\, \partial_C \tilde{V}) = \tilde{\mathcal{D}}_B v_A \tag{4.16}$$

so that the first equation (4.9) boils down to

$$\tilde{\mathcal{D}}_A v_B = (p-1)\tilde{G}_{AB}. \tag{4.17}$$

As a consequence, the covariant derivative of the norm is evaluated as

$$\tilde{\mathcal{D}}_A N_v = 2(\tilde{\mathcal{D}}_A v_B) v^B = 2(p-1) v_A, \tag{4.18}$$

where we use metric compatibility condition, i.e., $\tilde{\mathcal{D}}_A \tilde{G}_{BC} = 0$, and equation (4.17). Thus, metric tensor is found to be proportional to second derivative of norm since

$$\tilde{\mathcal{D}}_A \tilde{\mathcal{D}}_B N_v = 2(p-1)\tilde{\mathcal{D}}_A v_B = 2(p-1)^2 \tilde{G}_{AB}, \tag{4.19}$$

which proves that the vector field v cannot have vanishing norm, that is,

$$N_v \neq 0, \tag{4.20}$$

since $N_v = 0$ implies that $\tilde{G}_{AB} = 0$, yielding a contradiction as the metric tensor cannot be zero. Next, since the norm N_v is a real function, its covariant derivative is equal to its exterior derivative, that is $\partial_A N_v = \tilde{\mathcal{D}}_A N_v$. Combining this fact with the equations (4.10) and (4.18) yields the differential equation

$$\partial_A N_v = 4p(p-1)\partial_A \tilde{V} \tag{4.21}$$

for which a solution can be constructed as

$$N_v = 4p(p-1)\tilde{V} \tag{4.22}$$

up to an integration constant. The solution (4.22) provides a method for constructing a unit vector field in the direction of the vector field v as

$$X \equiv \frac{v}{\sqrt{N_v}}. \tag{4.23}$$

Using equation (4.10), the vector field X can be written in components as

$$X_A = \frac{v_A}{\sqrt{4p(p-1)\tilde{V}}} = \sqrt{\frac{p}{(p-1)\tilde{V}}}\partial_A \tilde{V} = \partial_A r, \tag{4.24}$$

where we have defined a smooth real function r on the target space $\tilde{M}$ as

$$r \equiv \frac{\sqrt{N_v}}{p-1} = \sqrt{\frac{4p}{(p-1)}\tilde{V}} \neq 0. \tag{4.25}$$

We emphasize that the real function r is globally defined on the entire target space, as potential $\tilde{V}$ is globally defined. Thus, X turns out to be a unit vector field that is hypersurface orthogonal to r, which implies that there exists a coordinate system (r, y^a) on $\tilde{M}$ such that the metric tensor field can be expressed as (Corollary 6.42 in [50])

$$\tilde{G} = dr^2 + \tilde{G}_{ab}(r, y^a)dy^a dy^b, \tag{4.26}$$

in particular, $\tilde{G}_{A0} = \delta_{A0}$ and $\tilde{G}^{A0} = \delta^{A0}$ where $y^A = (y^0 \equiv r, y^a)$ [4]. In these coordinates, X becomes unit vector field along the r-direction. That is, in these coordinates, equation (4.24) becomes[2] $X_A = \delta_{A0}$, so that $X^A = \tilde{G}^{AB}X_B = \tilde{G}^{A0} = \delta^{A0}$, implying

$$X = X^A \partial_A = \partial_0 = \partial_r. \tag{4.27}$$

Therefore, the definitions given in (4.23) and (4.25) imply that

$$X_A = \frac{v_A}{(p-1)r}, \tag{4.28}$$

so that we can express v^A's in the coordinate system $y^A = (r, y^a)$ as

$$v^A = (p-1)r\,\delta^{A0}, \quad \text{that is,} \quad v = (p-1)r\,\partial_r. \tag{4.29}$$

[2] In fact, the X_A are the components of the 1-form dual to the vector field X^A. In the basis formed by coordinates y^A, this 1-form can be written as $X_A dy^A = \partial_A r dy^A = dr$.

Finally, the homothety condition (4.9) turns out to be [4]

$$
\begin{aligned}
2(p-1)\tilde{G}_{AB} &= v^C\partial_C\tilde{G}_{AB} + \tilde{G}_{AC}\partial_B v^C + \tilde{G}_{BC}\partial_A v^C \\
&= (p-1)\left(r\partial_0\tilde{G}_{AB} + \tilde{G}_{A0}\partial_B r + \tilde{G}_{B0}\partial_A r\right) \\
&= (p-1)\left(r\partial_0\tilde{G}_{AB} + \tilde{G}_{A0}\delta^0{}_B + \tilde{G}_{B0}\delta^0{}_A\right),
\end{aligned}
\tag{4.30}
$$

where we have used the fact that $\tilde{\mathcal{D}}_A v_B + \tilde{\mathcal{D}}_B v_A$ corresponds to the Lie derivative of $\tilde{G}_{AB}$ along the vector field v, and hence it is equal to the right-hand side of the first line of equation (4.30). In particular, setting $A = a$ and $B = b$ in equation (4.30) implies

$$
2\tilde{G}_{ab}(r, y^a) = r\partial_r\tilde{G}_{ab}(r, y^a) \tag{4.31}
$$

which can be solved by separation of variables as

$$
\tilde{G}_{ab}(r, y^a) = r^2 G_{ab}(y^a) \tag{4.32}
$$

where the functions G_{ab} are independent of the r-coordinate. Finally, substituting the expression (4.32) into (4.26) yields

$$
\tilde{G}(r, y^a) = dr^2 + r^2 G_{ab}(y^a)\, dy^a dy^b, \tag{4.33}
$$

which explicitly reveals the fact that the target space $(\tilde{M}, \tilde{G})$ is a Riemannian cone. The base manifold is a sub-manifold $M \subseteq \tilde{M}$, determined by fixing $r =$ constant (or equivalently determined by the level sets of the target space potential $\tilde{V}$ due to the definition (4.25)). In this setting, the coordinates y^a parametrize the base manifold M, which is equipped with the Riemannian metric tensor $G = G_{ab}(y^a)dy^a dy^b$. $\square$

The appearance of the Riemannian cone structure of the target space enables us to write the conformal sigma model (4.1) as a sigma model field theory with a Riemannian cone target space, following Section 2 of [1], as follows: The definition of the r-coordinate given in (4.25) can be reversed to find the target space potential as

$$
\tilde{V}(r) = \frac{p-1}{4p} r^2, \tag{4.34}
$$

that is, it turns out that the target space potential is a function of only r. Additionally, equation (4.12) for target space potential $\tilde{U}$ on the coordinates (r, y^a) is reduced to

$$
v^A\partial_A\tilde{U} = (p-1)r\partial_r\tilde{U} = 2(p+1)\tilde{U} \tag{4.35}
$$

due to equation (4.29). To solve the differential equation (4.35), we apply separation of variables method and write $\tilde{U}(y^A) = U_r(r)U(y^a)$, yielding a solution

$$\tilde{U}(r, y^a) = r^{\frac{2(p+1)}{p-1}} U(y^a) \tag{4.36}$$

for some arbitrary function $U(y^a)$ that is independent of r. Consequently, substituting expressions (4.33), (4.34), and (4.36) into the conformal sigma model action $\tilde{S}[\phi^r, \phi^a]$ given by (4.1) yields the Lagrangian density of the action on the (r, y^a)-coordinates of the target space as

$$\tilde{\mathcal{L}} = \sqrt{-h}(\phi^r)^2 \left(\tfrac{p-1}{4p} R_h + h^{\mu\nu} \left[\partial_\mu \phi^r \partial_\nu \phi^r + G_{ab}(\phi) \partial_\mu \phi^a \partial_\nu \phi^b \right] + (\phi^r)^{\frac{4}{p-1}} U(\phi) \right), \tag{4.37}$$

where $\phi^r \equiv \phi^0 = r \circ \tilde{\Phi}$ and, as usual, $\phi = (\phi^1, \ldots, \phi^d)$. Thus, the scalar field ϕ^r plays a role analogous to the dilaton field appearing in the string frame p-brane action (3.29), in the sense that the procedure used to derive the action (3.29) from the action (3.22) can be reversed. This is achieved by introducing a world-volume metric defined as

$$g_{\mu\nu} = (\phi^r)^{\frac{4}{p-1}} h_{\mu\nu}, \tag{4.38}$$

where $g_{\mu\nu}$ becomes a dynamical field reflecting the dynamics of ϕ^r, while $h_{\mu\nu}$ remains a fixed background metric tensor. The identification (4.38) immediately yields that

$$\sqrt{-h} = (\phi^r)^{-\frac{2(p+1)}{p-1}} \sqrt{-g}, \quad h^{\mu\nu} = (\phi^r)^{\frac{4}{p-1}} g^{\mu\nu} \tag{4.39}$$

so that one directly derives the identity

$$\sqrt{-h} h^{\mu\nu} = \frac{1}{(\phi^r)^2} \sqrt{-g} g^{\mu\nu}. \tag{4.40}$$

Additionally, equation (4.38) relates the Ricci scalars associated to g and h as (for details, see equation D.9 in [51])

$$R_h = (\phi^r)^{\frac{4}{p-1}} \left(R_g + \frac{4p}{(p-1)} \nabla^g_\mu \left(\frac{\partial^\mu \phi^r}{\phi^r} \right) - \frac{4p}{p-1} \frac{1}{(\phi^r)^2} g^{\mu\nu} \partial_\mu \phi^r \partial_\nu \phi^r \right), \tag{4.41}$$

where ∇^g represents the Levi-Civita connection associated to the metric tensor g. Therefore, multiplying equation (4.41) by $\frac{p-1}{4p} \sqrt{-h} (\phi^r)^2$ yields the expression

$$\begin{aligned} \frac{p-1}{4p} (\phi^r)^2 \sqrt{-h} R_h &= \sqrt{-g} \left(\frac{p-1}{4p} R_g - \frac{1}{(\phi^r)^2} g^{\mu\nu} \partial_\mu \phi^r \partial_\nu \phi^r \right) + \underbrace{\sqrt{-g}\, \nabla^g_\mu \left(\frac{\partial^\mu \phi^r}{\phi^r} \right)}_{\text{divergence term}} \\ &= \sqrt{-g} \left(\frac{p-1}{4p} R_g - \frac{1}{(\phi^r)^2} g^{\mu\nu} \partial_\mu \phi^r \partial_\nu \phi^r \right), \end{aligned} \tag{4.42}$$

where we have used equations (4.39) and (4.41), and we have integrated out the total divergence term by the same reasoning applied in equation (3.26). As a result, by inserting the expressions (4.39), (4.40), and (4.42) into the Lagrangian density (4.37), the action (4.1) can be rewritten as

$$S[g_{\mu\nu}, \phi^a] = -\frac{1}{2}\int_X d^{p+1}x\,\sqrt{-g}\left\{\frac{p-1}{4p}R_g + g^{\mu\nu}G_{ab}(\phi)\,\partial_\mu\phi^a\,\partial_\nu\phi^b + U(\phi)\right\} \tag{4.43}$$

which closely resembles the action (3.22), where the space-time (M, G) corresponds to the base manifold of $(\tilde{M}, \tilde{G})$. This correspondence reveals a connection between the conformal sigma model and general relativity, as the action (4.43) effectively describes scalar fields coupled to gravity with a potential.

4.1. Conformally Coupled Scalar Field

A particularly interesting case arises when the target space of the conformal sigma model (4.1) is taken to be a one-dimensional manifold, with the single scalar field denoted by $\tilde{\phi}$. In this case, the target space is trivially a Riemannian cone, which allows us to identify $\tilde{\phi}$ with ϕ^r in (4.37). Specializing to a three-dimensional world-volume, i.e., setting $p = 2$, the action takes the form

$$S[\tilde{\phi}] = -\frac{1}{2}\int d^3x\,\sqrt{-h}\left(\frac{1}{8}\tilde{\phi}^2 R_h + h^{\mu\nu}\partial_\mu\tilde{\phi}\,\partial_\nu\tilde{\phi} + 2\alpha\tilde{\phi}^6\right), \tag{4.44}$$

where the potential term $U(\phi)$ in (4.37) is taken to be constant 2α. This is necessary because there are no additional dimensions, so that the solution given in (4.36) implies that $U(y^a)$ must be a constant. Accordingly, the ansatz (3.23) becomes

$$g_{\mu\nu} = \tilde{\phi}^4 h_{\mu\nu}, \tag{4.45}$$

which implies that the action (4.44) can be expressed as

$$S[g_{\mu\nu}] = -\frac{1}{2}\int d^3x\sqrt{-g}\,(R_g + 2\alpha)\,. \tag{4.46}$$

This expression follows directly from the procedure used to derive (4.43) from (4.1). The action (4.44) is invariant under a conformal transformation that is defined in terms of a smooth real valued function Ω as

$$h_{\mu\nu} \mapsto e^{2\Omega}h_{\mu\nu}, \quad \tilde{\phi} \mapsto e^{-\frac{\Omega}{2}}\tilde{\phi}, \tag{4.47}$$

since transformation (4.47) does not change ansatz (4.45). This model, known as the *conformally coupled scalar field*, has been studied in detail in the literature [53–59].

5. SUPERCONFORMAL SIGMA MODELS IN THREE DIMENSIONS

Focusing on a three-dimensional world-volume, corresponding to the membrane case ($p = 2$), the *superconformal sigma model in three dimensions* [1,2] is defined as the supersymmetric extension of the conformal sigma model (4.1). Since the introduction of supersymmetry requires the existence of spinors on the world-volume (X, h), we begin by explaining how spinors can be defined on a curved space-time and by introducing our spinor conventions.

5.1. Majorana Spinors in Three Dimensions

A spinor on a curved space can be defined through the *vielbein (or tetrad) formalism* (see for example [65]). The fundamental field in the vielbein formalism is called the vielbeins that carry two different types of indices; the indices $\mu, \nu, \rho = 0, 1, 2$, which are the coordinate indices and sometimes called *curved indices* and the indices $i, j, k = 0, 1, 2$ that are referred to as *tangent (or flat) indices.* A vielbein is denoted as $e^i{}_\mu(x)$ and is defined through the condition that

$$h_{\mu\nu} = e^i{}_\mu e^j{}_\nu \eta_{ij}, \tag{5.1}$$

There also exists *inverse vielbeins*, denoted as $e_i{}^\mu$, which are defined as the matrix inverses of the vielbeins, that is,

$$e_i{}^\mu e^i{}_\nu = \delta^\mu{}_\nu, \quad \delta^i{}_j = e^i{}_\mu e_j{}^\mu. \tag{5.2}$$

Since vielbein carries an upper tangent index and a lower curved index, it transforms as a one-form under coordinate transformations while under a change of vielbeins, its transformation law is given by

$$e^i{}_\mu(x) \mapsto \hat{e}^{\hat{i}}{}_\mu(x) = \Lambda^{\hat{i}}{}_i(x) e^i{}_\mu(x), \tag{5.3}$$

where $\Lambda^{\hat{i}}{}_i(x)$ is called a *local Lorentz transformation*, satisfying

$$\Lambda^{\hat{i}}{}_i(x) \Lambda^{\hat{j}}{}_j(x) \eta_{\hat{i}\hat{j}} = \eta_{ij}. \tag{5.4}$$

Accordingly, covariant derivative of a vielbein, denoted by $\nabla_\mu e^i{}_\mu$, is defined so as to be compatible with both coordinate transformations and local Lorentz transformations.

It is given by (equation (7.111) in [49])

$$\nabla_\mu e^i{}_\nu = \partial_\mu e^i{}_\nu + \omega_\mu{}^i{}_j e^j{}_\nu - \Gamma^\sigma{}_{\mu\nu}\, e^i{}_\sigma, \tag{5.5}$$

where $\omega_\mu{}^k{}_j \equiv \eta^{ik}\omega_{\mu ij}$ are the components of the *spin connection*. Components of the spin connection are obtained by the *vielbein postulate*[3] given as

$$\nabla_\mu e^i{}_\nu = 0, \tag{5.6}$$

which results in

$$\omega_{\mu ij} = e_{i\alpha}\left(\partial_\mu e_j{}^\alpha + e_j{}^\nu \Gamma^\alpha{}_{\mu\nu}\right), \tag{5.7}$$

where $e_{i\alpha} = \eta_{ik}e^k{}_\alpha$. Although it is not apparent in (5.7), components of the spin connection are anti-symmetric in (i, j) indices which can be verified by observing that

$$\begin{aligned} e_{i\alpha}\partial_\mu e_j{}^\alpha &= \eta_{ik}e^k{}_\alpha \partial_\mu e_j{}^\alpha = \eta_{ik}e^k{}_\alpha \partial_\mu e_j{}^\alpha = \eta_{ik}[\partial_\mu\left(e^k{}_\alpha e_j{}^\alpha\right) - e_j{}^\alpha \partial_\mu e^k{}_\alpha] \\ &= -e_j{}^\alpha \partial_\mu e_{i\alpha} = -e_j{}^\alpha \partial_\mu\left(h_{\alpha\beta}e_i{}^\beta\right) = -e_j{}^\alpha e_i{}^\beta \partial_\mu h_{\alpha\beta} - e_j{}^\alpha h_{\alpha\beta}\partial_\mu e_i{}^\beta \\ &= -e_j{}^\alpha e_i{}^\beta\left(\nabla_\mu h_{\alpha\beta} + \Gamma^\theta{}_{\mu\alpha}\, h_{\theta\beta} + \Gamma^\theta{}_{\mu\beta}\, h_{\alpha\theta}\right) - e_{j\beta}\partial_\mu e_i{}^\beta \\ &= -e_j{}^\alpha e_{i\theta}\Gamma^\theta{}_{\mu\alpha} - e_{j\theta}e_i{}^\beta \Gamma^\theta{}_{\mu\beta} - e_{j\theta}\partial_\mu e_i{}^\theta \end{aligned} \tag{5.8}$$

implying

$$\begin{aligned} \omega_{\mu ij} &= -e_j{}^\alpha e_{i\theta}\Gamma^\theta{}_{\mu\alpha} - e_{j\theta}e_i{}^\beta \Gamma^\theta{}_{\mu\beta} - e_{j\theta}\partial_\mu e_i{}^\theta + e_{i\theta}e_j{}^\alpha \Gamma^\theta{}_{\mu\alpha} \\ &= -e_{j\theta}\left(\partial_\mu e_i{}^\theta + e_i{}^\beta \Gamma^\theta{}_{\mu\beta}\right) = -\omega_{\mu ji}. \end{aligned} \tag{5.9}$$

Next, a *Clifford algebra* on the world-volume is defined as an algebra generated by the gamma matrices γ_i satisfying

$$\{\gamma_i, \gamma_j\} \equiv \gamma_i\gamma_j + \gamma_j\gamma_i = 2\eta_{ij} \tag{5.10}$$

where η_{ij} is the Minkowski metric represented in tangent indices. The tangent indices of gamma matrices are raised and lowered by the Minkowski metric, such as

$$\gamma^i \equiv \eta^{ij}\gamma_j. \tag{5.11}$$

Here, we emphasize that the gamma matrices are constant 2×2 complex matrices that constitute the irreducible Hermitian representation of the Clifford algebra, that is, each γ^i satisfies the condition that [49]

$$(\gamma_i)^\dagger = \gamma_0\gamma_i\gamma_0, \tag{5.12}$$

where $(\gamma_i)^\dagger$ is defined as the complex conjugate of the transpose of matrix γ_i.

---

[3] One can show that the vielbein postulate (5.6) is equivalent to the metric compatibility condition, i.e., $\nabla_\sigma h_{\mu\nu} = 0$.

Using gamma matrices and vielbeins, the *local gamma matrices* are defined as

$$\gamma_\mu \equiv e^i{}_\mu \gamma_i \;\text{ and }\; \gamma^\mu \equiv e^\mu{}_i \gamma^i \implies \gamma^\mu = h^{\mu\nu}\gamma_\nu. \tag{5.13}$$

One immediate implication of the definition (5.13) is that

$$\gamma^\mu\gamma_\mu = h^{\mu\nu} e^i{}_\nu e^j{}_\mu \gamma_i\gamma_j = \eta^{ij}\gamma_i\gamma_j = \frac{1}{2}\eta^{ij}\left(\gamma_i\gamma_j + \gamma_j\gamma_i\right) = \eta_{ij}\eta^{ij} = 3, \tag{5.14}$$

since the world-volume (X, h) is three-dimensional manifold. Moreover, the definition (5.13) allows us to compute the anti-commutators of local gamma as

$$\{\gamma_\mu, \gamma_\nu\} = e^i{}_\mu e^j{}_\nu \{\gamma_i, \gamma_j\} = e^i{}_\mu e^j{}_\nu \eta_{ij} = 2h_{\mu\nu}. \tag{5.15}$$

The anti-commutator of the gamma matrices, defined by

$$\gamma_{ij} \equiv \frac{1}{2}[\gamma_i, \gamma_j] = \frac{1}{2}(\gamma_i\gamma_j - \gamma_j\gamma_i), \tag{5.16}$$

generates a Lie algebra that is isomorphic to the Lorentz algebra. This isomorphism yields a representation of the Lorentz algebra known as the *spinor representation.* Finally, a spinor field $\epsilon(x)$ is defined as an object that transforms under local Lorentz transformations according to the spinor representation of the Lorentz group. That is,

$$\epsilon(x) \mapsto e^{-\frac{1}{4}\lambda^{ij}(x)\gamma_{ij}}\epsilon(x). \tag{5.17}$$

Accordingly, the covariant derivative of a spinor is defined such that it transforms in the same way as the spinor itself under local Lorentz transformations, namely,

$$\nabla_\mu \epsilon(x) \equiv \left(\partial_\mu + \frac{1}{4}\omega_{\mu ij}(x)\gamma^{ij}\right)\epsilon(x) \tag{5.18}$$

which implies a significant commutator identity given as

$$[\nabla_\mu, \nabla_\nu]\epsilon = \frac{1}{4}R_{\mu\nu ij}\gamma^{ij}\epsilon = \frac{1}{4}R_{\mu\nu\alpha\beta}\gamma^{\alpha\beta}. \tag{5.19}$$

Here, $R_{\mu\nu ij} = R_{\mu\nu\alpha\beta}e^\alpha{}_i e^\beta{}_j$, where $R_{\mu\nu\alpha\beta}$ is the Riemann curvature tensor and

$$\gamma^{\alpha\beta} = e^\alpha{}_i e^\beta{}_j \gamma^{ij} = \frac{1}{2}\left[e^\alpha{}_i\gamma^i, e^\beta{}_j\gamma^j\right] = \frac{1}{2}\left[\gamma^\alpha, \gamma^\beta\right]. \tag{5.20}$$

In three dimensions, it is possible to impose an additional reality condition on spinors as follows: For the 2-dimensional complex irreducible Hermitian representation of the Clifford algebra in three dimensions, there exists a unitary matrix, denoted by C and called the *charge conjugation matrix*, which satisfies

$$C^T = -C \;\text{ and }\; \gamma_i^T = -C\gamma_i C^{-1}, \tag{5.21}$$

in addition to *unitarity* that is defined as

$$C^\dagger = C^{-1}. \tag{5.22}$$

The existence of the charge conjugation matrix allows us to construct symmetric matrices from the gamma matrices as

$$(C\gamma_i)^T = \gamma_i^T C^T = (-C\gamma_i C^{-1})(-C) = C\gamma_i, \tag{5.23}$$

due to equations (5.21). Moreover, equipped with the charge conjugation matrix C, one defines the *Majorana conjugate* of a spinor ϵ, denoted by $\bar{\epsilon}$, as

$$\bar{\epsilon} \equiv \epsilon^T C. \tag{5.24}$$

One can show that $\bar{\epsilon}\epsilon$ transforms as a scalar under local Lorentz transformations (see Chapter 7 of [49]), which implies that $\partial_\mu(\bar{\epsilon}\epsilon) = \nabla_\mu(\bar{\epsilon}\epsilon)$ yielding

$$\nabla_\mu\bar{\epsilon} = (\nabla_\mu\epsilon)^T C = \overline{\nabla_\mu\epsilon}. \tag{5.25}$$

Finally, the *Majorana spinor field* is defined as a spinor field ϵ satisfying

$$\epsilon^* = B\epsilon, \quad \text{where} \quad B \equiv C\gamma_0. \tag{5.26}$$

Throughout this thesis, all spinor fields are taken to be Majorana spinors.

5.2. $\mathcal{N} = 1$ Superconformal Sigma Model

The minimal supersymmetric extension of the conformal sigma model (4.1) is obtained by introducing the supermultiplet of fields (ϕ^A, ψ^A), where the ϕ^A denote scalar fields, i.e., spin-0 fields, as before and the ψ^A denote fermionic fields that are spin-$\frac{1}{2}$ fields. The corresponding action is given by [2]

$$S_{\text{SUSY}} = S_B + S_F, \tag{5.27}$$

where S_B is the bosonic part defined in (4.1), for which $\tilde{U}(\tilde{\phi})$ is expressed in terms of a real function, $\mathcal{F}$, on the target space $\tilde{M}$, referred to as the *real superpotential*, via

$$U(\tilde{\phi}) = \tilde{G}^{AB}(\tilde{\phi})\,\partial_A\mathcal{F}(\tilde{\phi})\,\partial_B\mathcal{F}(\tilde{\phi}) \tag{5.28}$$

and, for convenience [2], $\tilde{V}(\tilde{\phi})$ is multiplied by a factor of $\frac{1}{4}$. On the other hand, the term S_F corresponds to the fermionic part and is given by

$$S_F = -\frac{1}{2}\int_X d^3x\sqrt{-h}\left(\bar{\psi}^A\gamma^\mu D_\mu\psi^B\tilde{G}_{AB}(\tilde{\phi}) + \tilde{D}_A\partial_B\mathcal{F}(\tilde{\phi})\bar{\psi}^A\psi^B\right), \tag{5.29}$$

where $D_\mu\psi^A \equiv \nabla_\mu\psi^A + \tilde{\Gamma}^A{}_{BC}\,\partial_\mu\phi^B\psi^C$ and $\nabla_\mu\psi^A$ as in (5.18). The action given in (5.27) characterizes a superconformal sigma model as it is invariant under both supersymmetry transformations and rigid conformal transformations. The rigid conformal transformation of the scalar fields is given in (4.4), while conformal transformation of the spinor fields is given by

$$\delta_C \psi^A = \xi^\mu \nabla_\mu \psi^A + \frac{1}{4}(\nabla_\mu \xi_\nu)\gamma^{\mu\nu}\psi^A + 2\omega\psi^A - \omega v^B \tilde{\Gamma}^A{}_{BC}\,\psi^C, \tag{5.30}$$

where ξ, ω are defined as in (4.2) and v^A is given as in equation (4.4). Supersymmetry transformations are given in terms of supersymmetry parameters ϵ and λ as

$$\delta_{SUSY}\phi^A = \bar{\epsilon}\psi^A, \quad \delta_{SUSY}\psi^A = \partial_\mu \phi^A \gamma^\mu \epsilon - \tilde{G}^{AB}\partial_B \mathcal{F}\epsilon + \frac{1}{2}v^A \lambda. \tag{5.31}$$

As already shown in Section 4, the bosonic part S_B is invariant under the conformal transformation (4.4) if and only if the target space is a Riemannian cone. Similarly, as demonstrated in [1], the Riemannian cone structure of the target space is also a sufficient condition for the fermionic part S_F to be invariant under the conformal transformation of the spinor fields (5.30). However, supersymmetry invariance (5.31) of the action (5.27) requires a relation between ϵ and λ, which is given as

$$\nabla_\mu \epsilon = \frac{1}{2}\gamma_\mu \lambda. \tag{5.32}$$

Hence, the spinor field λ can be written in terms of ϵ as

$$\gamma^\mu \nabla_\mu \epsilon = \frac{1}{2}\gamma^\mu \gamma_\mu \lambda = \frac{3}{2}\lambda \implies \lambda = \frac{2}{3}\gamma^\mu \nabla_\mu \epsilon = \frac{2}{3}\not{D}\epsilon, \tag{5.33}$$

where $\not{D} \equiv \gamma^\mu \nabla_\mu$ is the *Dirac operator*. Thus, equation (5.32) turns out to be

$$\nabla_\mu \epsilon = \frac{1}{3}\gamma_\mu \not{D}\epsilon, \tag{5.34}$$

and we conclude that ϵ is a conformal Killing spinor as defined in Definition 5.1.

Definition 5.1. *A spinor field ϵ in a general $p+1$-dimensions satisfying*

$$\nabla_\mu \epsilon = \frac{1}{p+1}\gamma_\mu \not{D}\epsilon, \tag{5.35}$$

is called a conformal Killing spinor [2]. When λ is proportional to ϵ, i.e., $\lambda = c\epsilon$ for some complex constant c, the spinor ϵ is called a Killing spinor.

As a result, the minimal supersymmetric extension of the conformal sigma model is called the $\mathcal{N} = 1$ *superconformal sigma model*, where $\mathcal{N}$ denotes the number of independent conformal Killing spinors used as supersymmetry parameters to construct the supersymmetry transformations. We emphasize that $\mathcal{N}$ does not represent the number of solutions to equation (5.32).

5.3. Conformal Killing Spinor

The existence of a conformal Killing spinor on a three-dimensional manifold is not guaranteed a priori. In this chapter, we classify three-dimensional manifolds that admit local conformal Killing spinors, following [2,5]. Since the signature of the metric tensor h plays no role in the analysis, results presented here are valid for arbitrary metric signatures. We begin showing the relationship between conformal Killing spinors and conformal Killing vector fields.

Theorem 5.1. *A conformal Killing spinor* (5.34) *induces a conformal Killing vector field* (4.2) *given as*

$$V^\mu \equiv \bar{\epsilon}\gamma^\mu\epsilon. \tag{5.36}$$

Proof. The covariant derivative of V_μ is evaluated as

$$\nabla_\mu V_\nu = (\nabla_\mu\bar{\epsilon})\gamma_\nu\epsilon + \bar{\epsilon}\gamma_\nu(\nabla_\mu\epsilon) = \frac{1}{2}\left(\bar{\epsilon}\gamma_\nu\gamma_\mu\lambda - \bar{\lambda}\gamma_\mu\gamma_\nu\epsilon\right) \tag{5.37}$$

where we use $\nabla_\mu\gamma_\nu = \gamma_i\nabla_\mu e^i{}_\mu = 0$ and equation (5.25) to derive that

$$\nabla_\mu\bar{\epsilon} = \overline{\nabla_\mu\epsilon} = (\nabla_\mu\epsilon)^T C = \frac{1}{2}\lambda^T\gamma_\mu^T C = -\frac{1}{2}\lambda^T C\gamma_\mu = -\frac{1}{2}\bar{\lambda}\gamma_\mu. \tag{5.38}$$

Consequently, equation (5.37) gives the symmetrization of $\nabla_\mu V_\nu$ in (μ,ν) indices as

$$\begin{aligned}\nabla_\mu V_\nu + \nabla_\mu V_\mu &= \frac{1}{2}\left(\bar{\epsilon}\{\gamma_\nu,\gamma_\mu\}\lambda - \bar{\lambda}\{\gamma_\mu,\gamma_\nu\}\epsilon\right) = h_{\mu\nu}\left(\bar{\epsilon}\lambda - \bar{\lambda}\epsilon\right)\\ &= \frac{2}{3}h_{\mu\nu}\left(\bar{\epsilon}\gamma^\sigma\nabla_\sigma\epsilon + (\nabla_\sigma\bar{\epsilon})\gamma^\sigma\epsilon\right) = \frac{1}{3}h_{\mu\nu}\nabla_\sigma\left(\bar{\epsilon}\gamma^\sigma\epsilon\right)\\ &= \frac{2}{3}h_{\mu\nu}\nabla_\sigma V^\sigma\end{aligned} \tag{5.39}$$

where we use equation (5.33) and multiply equation (5.38) from the left by γ^μ to derive that $\bar{\lambda} = -\frac{2}{3}\nabla_\mu\bar{\epsilon}$. We conclude the proof by recalling equation (5.39) is the definition of a conformal Killing vector field given in (4.2). □

Our aim is to find local solutions to the conformal Killing spinor equation (5.32), provided a world-volume metric tensor h. As it is proved in Theorem 5.2, number of solutions to the conformal Killing spinor equation depends only on the conformal class of the world-volume metric tensor.

Theorem 5.2. *Let h and $\hat{h}$ be two world-volume metric tensors, satisfying*

$$\hat{h}_{\mu\nu} = e^{2\Omega} h_{\mu\nu}, \tag{5.40}$$

for some smooth real-valued function Ω. If (ϵ, λ) is a solution to (5.32) with respect to $h_{\mu\nu}$, then $(\hat{\epsilon}, \hat{\lambda}) \equiv \left(e^{\Omega/2}\epsilon, e^{\Omega/2}\left(\lambda + \frac{1}{2}\gamma^\sigma \partial_\sigma \Omega \epsilon\right)\right)$ is a solution with respect to $\hat{h}_{\mu\nu}$.

Proof. Equation (5.40) gives the relation between inverse metric tensors as

$$\hat{h}^{\mu\nu} = e^{-2\Omega} h^{\mu\nu}. \tag{5.41}$$

Then, definition of vielbeins (5.1) yields that

$$\hat{h}_{\mu\nu} = e^{2\Omega} h_{\mu\nu} = e^{2\Omega} e^i{}_\mu e^j{}_\nu \eta_{ij} = \hat{e}^i{}_\mu \hat{e}^j{}_\mu \eta_{ij}. \tag{5.42}$$

As a result, the objects defined by $\hat{e}^i{}_\mu \equiv e^\Omega e^i{}_\mu$ constitute a set of vielbein fields associated with the metric $\hat{h}$. Accordingly, inverses are found by definition (5.2) as

$$\hat{e}_i{}^\mu = e^{-\Omega} e_i{}^\mu. \tag{5.43}$$

Subsequently, spin connection $\hat{\omega}_{\mu ij}$ associated with $\hat{h}$ is expressed by formula (5.7) as

$$\hat{\omega}_{\mu ij} = \hat{e}_{i\nu}\left(\hat{\Gamma}^\nu{}_{\sigma\mu} \hat{e}_j{}^\sigma + \partial_\mu \hat{e}_j{}^\nu\right) \tag{5.44}$$

where $\hat{\Gamma}^\nu{}_{\sigma\mu}$'s are components of the Levi-Civita connection associated to $\hat{h}$. These are related to the components of the Levi-Civita connection $\Gamma^\nu{}_{\sigma\mu}$ associated with the metric tensor h via the relation [51]

$$\hat{\Gamma}^\nu{}_{\sigma\mu} = \Gamma^\nu{}_{\sigma\mu} + \delta^\nu{}_\sigma \partial_\mu \Omega + \delta^\nu{}_\mu \partial_\sigma \Omega - h_{\sigma\mu} h^{\nu\tau} \partial_\tau \Omega. \tag{5.45}$$

Consequently, we evaluate the spin connection associated with $\hat{h}$ as

$$\begin{aligned}
\hat{\omega}_{\mu ij} &= e^\Omega e_{i\nu}\left\{\left[\Gamma^\nu{}_{\sigma\mu}(h) + \delta^\nu{}_\sigma \partial_\mu \Omega + \delta^\nu{}_\mu \partial_\sigma \Omega - h_{\sigma\mu} h^{\nu\tau}\partial_\tau \Omega\right] e^{-\Omega} e_j{}^\sigma + \partial_\mu\left(e^{-\Omega} e_j{}^\nu\right)\right\} \\
&= e_{i\nu}\left\{e_j{}^\sigma \Gamma^\nu{}_{\sigma\mu}(h) + e_j{}^\nu \partial_\mu \Omega + \delta^\nu{}_\mu e_j{}^\sigma \partial_\sigma \Omega - e_j{}^\sigma h_{\sigma\mu} h^{\nu\tau}\partial_\tau \Omega + \partial_\mu e_j{}^\nu - e_j{}^\nu \partial_\mu \Omega\right\} \\
&= \omega_{\mu ij} + \left(e_j{}^\sigma e_{i\mu} - e_i{}^\sigma e_{j\mu}\right)\partial_\sigma \Omega
\end{aligned}$$

where $e_i{}^\tau \equiv h^{\nu\tau} e_{i\nu}$. Accordingly, covariant derivative of ϵ with respect to $\hat{h}$ is given by

$$\begin{aligned}
\nabla^{\hat{h}}_\mu \epsilon &= \partial_\mu \epsilon + \frac{1}{4}\omega_{\mu ij}\gamma^{ij}\epsilon + \frac{1}{2}\partial_\sigma \Omega e_j{}^\sigma e_{i\mu}\gamma^{ij}\epsilon = \nabla^h_\mu \epsilon + \frac{1}{2}\left(\partial_\sigma \Omega \gamma_\mu \gamma^\sigma - \partial_\mu \Omega\right)\epsilon \\
&= \frac{1}{2}\gamma_\mu\left(\lambda + \gamma^\sigma \partial_\sigma \Omega \epsilon\right) - \frac{1}{2}\partial_\mu \Omega \epsilon
\end{aligned} \tag{5.46}$$

where we use that $e_j{}^\sigma e_{i\mu}\gamma^{ij} = h_{\mu\nu} e_i{}^\nu e_j{}^\sigma(\gamma^i\gamma^j - \eta^{ij}) = \gamma_\mu \gamma^\sigma - \delta^\sigma{}_\mu$. Therefore,

$$\nabla^{\hat{h}}_\mu \hat{\epsilon} = e^{\Omega/2}\left(\frac{1}{2}\partial_\mu \Omega \epsilon + \nabla^{\hat{h}}_\mu \epsilon\right) = \frac{1}{2}\gamma_\mu e^{\Omega/2}\left(\lambda + \gamma^\sigma \partial_\sigma \Omega \epsilon\right) = \frac{1}{2}\gamma_\mu \hat{\lambda}. \tag{5.47}$$

□

Moreover, existence of a solution of the conformal Killing spinor equation imposes a condition, known as *integrability condition*, on the *Cotton tensor* [66] defined as

$$C_{\mu\nu\rho} := \nabla_\rho S_{\nu\mu} - \nabla_\nu S_{\rho\mu}, \tag{5.48}$$

where the *Schouten tensor* [67] $S_{\mu\nu}$ is given as

$$S_{\mu\nu} = R_{\mu\nu} - \frac{R}{4} h_{\mu\nu} \implies h^{\mu\nu} S_{\mu\nu} = \frac{1}{4} R, \tag{5.49}$$

with $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$ and $R = h^{\mu\nu} R_{\mu\nu}$ denoting the Ricci tensor and Ricci scalar associated to the metric tensor h, respectively. The details[4] are discussed in Theorem 5.3.

Lemma 5.1. *For any dimension $p + 1$, we have the identity*

$$\not{D}^2 \epsilon = \nabla^\mu \nabla_\mu \epsilon - \frac{1}{4} R, \tag{5.50}$$

known as the Lichnerowicz formula [68, 69]

Proof. Riemann curvature tensor in three dimensions can be written as [50]

$$R_{\mu\nu\alpha\beta} = h_{\mu\alpha} S_{\nu\beta} - h_{\nu\alpha} S_{\mu\beta} - h_{\mu\beta} S_{\nu\alpha} + h_{\nu\beta} S_{\mu\alpha}, \tag{5.51}$$

which, in turn, implies that

$$R_{\mu\nu\alpha\beta} \gamma^\nu \gamma^{\alpha\beta} = 2 \left(h_{\mu\alpha} S_{\nu\beta} \gamma^\nu \gamma^{\alpha\beta} + h_{\nu\beta} S_{\mu\alpha} \gamma^\nu \gamma^{\alpha\beta} \right) = -\frac{R}{2} \gamma_\mu - 2 S_{\mu\alpha} \gamma^\alpha. \tag{5.52}$$

Here, we use the fact that product of two gamma matrices can be decomposed into symmetric and antisymmetric parts[5] as $\gamma^\mu \gamma^\nu = h^{\mu\nu} + \gamma^{\mu\nu}$, yielding

$$\begin{aligned} h_{\mu\alpha} S_{\nu\beta} \gamma^\nu \gamma^{\alpha\beta} &= \frac{1}{2} h_{\mu\alpha} S_{\nu\beta} \gamma^\nu \left(\gamma^\alpha \gamma^\beta - \gamma^\beta \gamma^\alpha \right) = \frac{1}{2} h_{\mu\alpha} S_{\nu\beta} \gamma^\nu \left(2h^{\alpha\beta} - 2\gamma^\beta \gamma^\alpha \right) \\ &= S_{\nu\mu} \gamma^\nu - S_{\nu\beta} h^{\nu\beta} \gamma_\mu = S_{\nu\mu} \gamma^\nu - \frac{R}{4} \gamma_\mu, \end{aligned} \tag{5.53}$$

where in the third equality we have used the symmetry of the Schouten tensor, and the identity that

$$h_{\nu\beta} \gamma^\nu \gamma^{\alpha\beta} = \frac{1}{2} h_{\nu\beta} \gamma^\nu \left(2h^{\alpha\beta} - 2\gamma^\beta \gamma^\alpha \right) = \gamma^\alpha - \gamma^\nu \gamma_\nu \gamma^\alpha = -2\gamma^\alpha. \tag{5.54}$$

As a result, we calculate that

$$R_{\mu\nu\alpha\beta} \gamma^{\mu\nu} \gamma^{\alpha\beta} = R_{\mu\nu\alpha\beta} h^{\mu\nu} + \gamma^\mu \left(R_{\mu\nu\alpha\beta} \gamma^\nu \gamma^{\alpha\beta} \right) = -\frac{R}{2} \gamma^\mu \gamma_\mu - 2 S_{\mu\alpha} h^{\mu\alpha} = -2R \tag{5.55}$$

since $R_{\mu\nu\alpha\beta}$ is anti-symmetric in (μ, ν) indices.

---

[4] Although the Lichnerowicz formula holds in any dimensions, we provide a proof here restricted to three dimensions.

[5] More precisely, $\gamma^\mu \gamma^\nu = \frac{1}{2} \left(\{\gamma^\mu, \gamma^\nu\} + [\gamma^\mu, \gamma^\nu] \right) = h^{\mu\nu} + \gamma^{\mu\nu}$.

Consequently, we end up with

$$\begin{aligned}\not{D}^2\epsilon &= \gamma^\mu\nabla_\mu\left(\gamma^\nu\nabla_\nu\epsilon\right) = \gamma^\mu\gamma^\nu\nabla_\mu\nabla_\nu\epsilon = h^{\mu\nu}\nabla_\mu\nabla_\nu\epsilon + \frac{1}{2}\gamma^{\mu\nu}[\nabla_\mu,\nabla_\nu]\epsilon \\ &= \nabla^\mu\nabla_\mu\epsilon + \frac{1}{8}R_{\mu\nu\alpha\beta}\gamma^{\mu\nu}\gamma^{\alpha\beta}\epsilon = \nabla^\mu\nabla_\mu\epsilon - \frac{1}{4}R\epsilon.\end{aligned} \tag{5.56}$$

□

Lemma 5.2. *A conformal Killing spinor ϵ satisfies $\not{D}^2\epsilon = -\frac{3}{8}R\epsilon$.*

Proof. For a conformal Killing spinor (5.34), we have

$$\nabla^\mu\nabla_\mu\epsilon = \frac{1}{3}h^{\mu\alpha}\nabla_\alpha(\gamma_\mu\gamma^\nu\nabla_\nu\epsilon) = \frac{1}{3}h^{\mu\alpha}\gamma_\mu\nabla_\alpha(\gamma^\nu\nabla_\nu\epsilon) = \frac{1}{3}\gamma^\alpha\nabla_\alpha(\gamma^\nu\nabla_\nu\epsilon) = \frac{1}{3}\not{D}^2\epsilon$$

so that we end up with

$$\not{D}^2\epsilon = -\frac{3}{8}R\epsilon. \tag{5.57}$$

□

Theorem 5.3. *Existence of a conformal Killing spinor ϵ on (X,h) implies*

$$C_{\sigma\nu\mu}\gamma^\sigma\epsilon = 0. \tag{5.58}$$

Proof. Firstly, we begin by calculating

$$\begin{aligned}\frac{1}{4}R_{\mu\nu\alpha\beta}\gamma^\nu\gamma^{\alpha\beta}\epsilon &= \gamma^\nu[\nabla_\mu,\nabla_\nu]\epsilon = \frac{1}{3}\left(\gamma^\nu\gamma_\nu\nabla_\mu\not{D}\epsilon - \gamma^\nu\gamma_\mu\nabla_\nu\not{D}\epsilon\right) \\ &= \nabla_\mu\not{D}\epsilon - \frac{1}{3}\left(2h_{\mu\nu} - \gamma_\mu\gamma_\nu\right)\nabla^\nu\not{D}\epsilon \\ &= \frac{1}{3}\left(\nabla_\mu\not{D}\epsilon + \gamma_\mu\not{D}^2\epsilon\right) \\ &= \frac{1}{3}\nabla_\mu\not{D}\epsilon - \frac{R}{8}\gamma_\mu\epsilon.\end{aligned} \tag{5.59}$$

Inserting equation (5.52) into equation (5.59) and then solving for $\nabla_\mu\not{D}\epsilon = \frac{3}{2}\nabla_\mu\lambda$ gives us the result

$$\nabla_\mu\lambda = -S_{\mu\alpha}\gamma^\alpha\epsilon. \tag{5.60}$$

Accordingly, the second covariant derivative of λ is found as

$$\nabla_\nu\nabla_\mu\lambda = -\left(\nabla_\nu S_{\mu\alpha}\right)\gamma^\alpha\epsilon - S_{\mu\alpha}\gamma^\alpha\nabla_\nu\epsilon = -\left(\nabla_\nu S_{\mu\alpha}\right)\gamma^\alpha\epsilon - \frac{1}{2}S_{\mu\alpha}\gamma^\alpha\gamma_\nu\lambda. \tag{5.61}$$

Anti-symmetrization of equation (5.61) in (μ,ν) indices yields the identity

$$[\nabla_\mu,\nabla_\nu]\lambda = C_{\alpha\mu\nu}\gamma^\alpha\epsilon + \frac{1}{2}\left(S_{\mu\alpha}\gamma^\alpha\gamma_\nu - S_{\nu\alpha}\gamma^\alpha\gamma_\mu\right)\lambda, \tag{5.62}$$

the left-hand side of which is computed via a method analogous to that used in equation (5.52), yielding

$$\begin{aligned}[\nabla_\mu, \nabla_\nu]\lambda &= \frac{1}{4}R_{\mu\nu\alpha\beta}\gamma^{\alpha\beta}\lambda = \frac{1}{2}\left(h_{\mu\alpha}S_{\nu\beta} - h_{\nu\alpha}S_{\mu\beta}\right)\gamma^{\alpha\beta}\lambda \\ &= \frac{1}{2}\left(S_{\nu\beta}(\delta^\beta{}_\mu - \gamma^\beta\gamma_\mu) - S_{\mu\beta}(\delta^\beta{}_\nu - \gamma^\beta\gamma_\nu)\right)\lambda \\ &= \frac{1}{2}\left(S_{\nu\mu} - S_{\nu\beta}\gamma^\beta\gamma_\mu - S_{\mu\nu} + S_{\mu\beta}\gamma^\beta\gamma_\nu\right)\lambda \\ &= \frac{1}{2}\left(S_{\mu\alpha}\gamma^\alpha\gamma_\nu - S_{\nu\alpha}\gamma^\alpha\gamma_\mu\right)\lambda\end{aligned} \tag{5.63}$$

since

$$h_{\mu\alpha}\gamma^{\alpha\beta} = h_{\mu\alpha}(h^{\alpha\beta} - \gamma^\beta\gamma^\alpha) = \delta^\beta{}_\mu - \gamma^\beta\gamma_\mu. \tag{5.64}$$

Finally, the integrability condition (5.58) is obtained by inserting equation (5.63) into equation (5.62). □

Next, we classify three-dimensional Euclidean and Lorentzian manifolds admitting conformal Killing spinors.

Definition 5.2. *A smooth manifold is called locally conformally flat if there exists a coordinate chart on which the metric tensor can be written as*

$$h_{\mu\nu} = e^{2\Omega}\eta_{\mu\nu}, \tag{5.65}$$

for some real-valued smooth function Ω.

Conformally flat three-dimensional manifolds are characterized by Theorem 5.4. For the proof, we refer Theorem 7.37 of [50].

Theorem 5.4. *A three-dimensional manifold is locally conformally flat if and only if its Cotton tensor vanishes.*

Therefore, the integrability condition (5.58) is trivially satisfied for any locally conformally flat manifold regardless of the signature of the metric tensor. This implies that every locally conformally flat manifold, such as S^3 in Euclidean signature and AdS_3

in Lorentzian signature, admits four linearly independent, non-zero local conformal Killing spinors[6] . In flat space, equation (5.32) reduces to

$$\partial_\mu \epsilon_\eta = \frac{1}{3}\gamma_\mu \gamma^\nu \partial_\nu \epsilon_\eta, \tag{5.66}$$

for which the general solution is found in terms of constant spinors ϵ_1 and ϵ_2 as [2]

$$\epsilon_\eta = x^\mu \gamma_\mu \epsilon_1 + \epsilon_2, \tag{5.67}$$

where x^μ's represent the global coordinates on the flat space. One can easily confirm that this constitutes a solution by observing that

$$\frac{1}{3}\gamma_\mu \gamma^\nu \partial_\nu \epsilon_\eta = \frac{1}{3}\gamma_\mu \gamma^\nu \gamma_\nu \epsilon_1 = \gamma_\mu \epsilon_1 = \partial_\mu \epsilon_\eta, \tag{5.68}$$

since $\partial_\mu x^\nu = \delta^\nu{}_\mu$ and $\partial_\mu \gamma_\nu = \nabla^\eta_\mu \gamma_\nu = 0$. Therefore, Theorem 5.2 gives those four independent local conformal Killing spinors on a locally conformally flat manifold as

$$\epsilon = e^{\Omega/2}\epsilon_\eta. \tag{5.69}$$

Now, we suppose that the manifold is not locally conformally flat, i.e., $C_{\mu\nu\rho} \neq 0$.

Lemma 5.3. *Given arbitrary vector fields X^μ, Y^μ, the vector field*

$$\mathcal{V}^\mu \equiv C^\mu{}_{\nu\rho} X^\nu Y^\rho \tag{5.70}$$

is null if there exists a non-zero local conformal Killing spinor.

Proof. Using the vector field (5.70), we construct 2×2 matrix $\mathcal{V}_\mu \gamma^\mu$ that annihilates ϵ due to equation (5.58). Therefore, existence of a local conformal Killing spinor ϵ implies that

$$0 = (\mathcal{V}_\mu \gamma^\mu)^2 \epsilon = \mathcal{V}_\mu \mathcal{V}_\nu \gamma^\mu \gamma^\nu \epsilon = \mathcal{V}_\mu \mathcal{V}_\nu h^{\mu\nu} \epsilon = \mathcal{V}^\mu \mathcal{V}_\mu \epsilon. \tag{5.71}$$

Therefore, if $\epsilon \neq 0$, then $\mathcal{V}^\mu \mathcal{V}_\mu = 0$. □

In Euclidean signature, a null vector must vanish identically, i.e., $\mathcal{V}^\mu = 0$. This occurs if and only if the Cotton tensor is zero, since the vector fields X^μ and Y^μ in equation (5.70) are arbitrary. Hence, in Euclidean signature, a conformal Killing spinor exists if and only if the manifold is locally conformally flat, for which there are four independent solutions.

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[6] We have two independent spinors, each with two free components, yielding a total of four linearly independent solutions to the conformal Killing spinor equation.

On the other hand, in Lorentzian signature, the vector $\mathcal{V}^\mu$ can be non-zero even if it is null. In this case, the matrix $\mathcal{V}_\mu\gamma^\mu$ is not the zero matrix since the gamma matrices are linear independent, while its square is zero. Such a matrix has precisely one zero eigenvalue, implying that on a Lorentzian manifold which is not locally conformally flat, there exists exactly one non-zero local conformal Killing spinor. Furthermore, as shown in [5], any such manifold is locally conformally equivalent to a *pp*-wave metric. For further details, we refer [2, 5, 6]

5.4. Comments on $\mathcal{N} = 2$ Case

$\mathcal{N} = 2$ superconformal sigma model in three dimensions is discussed in [1, 2]. As is generally the case in supersymmetric theories, $\mathcal{N} = 2$ supersymmetry requires the target space of a general sigma model to be a Kähler Manifold [70,71]. Additionally, the presence of conformal symmetry necessitates that the target space be a Riemannian cone. Consequently, the target space of the $\mathcal{N} = 2$ superconformal sigma model in three dimensions must be a Kähler cone. A crucial consequence of this result is that base manifold is necessarily a Sasakian manifold [62, 72].

6. CONCLUSION

In this thesis, we have undertaken a detailed study of three-dimensional superconformal sigma models. A central result established is that conformal invariance alone requires the target space to be a Riemannian cone [4], irrespective of whether supersymmetry is present [1], illustrating the strong geometric constraints imposed by conformal symmetry. Specializing to three dimensions, we investigated the minimal supersymmetric extension of the conformal sigma model and realized that supersymmetry transformations can be consistently defined only when the supersymmetry parameter is a conformal Killing spinor [2], ensuring compatibility between supersymmetry and conformal invariance. We further emphasized that, just as rigid conformal symmetry constrains the target space geometry, the existence of a conformal Killing spinor imposes nontrivial restrictions on the world-volume geometry. To analyze this, we classified three-dimensional Euclidean and Lorentzian manifolds admitting conformal Killing spinors, using the integrability condition and exploiting the fact that the number of solutions depends solely on the conformal class of the metric. We showed that, for both signatures, there exist four independent local conformal Killing spinors when the three-manifold is locally conformally flat. In the Euclidean case, no solutions exist otherwise. In contrast, for Lorentzian manifolds that are not locally conformally flat but are locally conformally equivalent to a *pp*-wave metric, exactly one conformal Killing spinor exists. These results provide a deeper understanding of the geometric and symmetry structures underlying superconformal sigma models in three dimensions, which are particularly significant for understanding coincident M2-branes and related aspects of M-theory and supergravity. Nevertheless, several open directions remain for future research.

First, while models with $\mathcal{N} = 1, 2$ supersymmetry have been explicitly constructed in [1, 2], the formulation of models with $\mathcal{N} = 4$ supersymmetry remains an open problem.

Second, the framework of superconformal sigma models can be extended to incorporate higher-curvature terms on the world-volume, in analogy with $f(R)$ theories. For instance, adding a term of the form $\sqrt{-h}\, f(R_h)$, defined purely on the world-volume and independent of the scalar fields, preserves invariance under rigid conformal transformations. Consequently, the key result that the target space must necessarily be a Riemannian cone remains valid. However, such modifications could affect the supersymmetry transformations.

Another promising direction concerns extending the present framework beyond three dimensions. In this work, we focused on M2-branes by restricting attention to three-dimensional world-volumes. However, eleven-dimensional supergravity, the low-energy limit of M-theory, also admits another stable solution, the *M5-brane* [20, 73], which may be expected to be described by six-dimensional superconformal sigma models. Extending the present analysis to this higher-dimensional setting could provide valuable new insights into the geometry and dynamics of M-theory.

Additionally, the internal symmetries of a sigma model, arising from target space isometries, can be promoted to local symmetries by gauging them [47]. Gauged versions of superconformal sigma models have already been analyzed in detail in [2, 63], and further exploration could deepen our understanding of their structure and applications.

Finally, an alternative and elegant approach involves constructing superconformal sigma models directly within the superspace formalism as in [74], which may yield a more unified and geometrically transparent framework.

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