

**ZONGULDAK BÜLENT ECEVİT ÜNİVERSİTESİ**  
**FEN BİLİMLERİ ENSTİTÜSÜ**

**ULTRAHİPERBOLİK SCHRÖDINGER DENKLEMİ İÇİN BİR TERS PROBLEM**

**MATEMATİK ANABİLİM DALI**

**YÜKSEK LİSANS TEZİ**

**SABRİYE GÖZDE KİRLİ**

**AĞUSTOS 2019**

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**DANIŞMAN: Doç. Dr. Fikret GÖLGELEYEN**

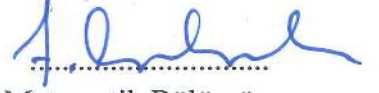
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**KABUL:**

Sabriye Gözde KİRLİ tarafından hazırlanan “Ultrahiperbolik Schrödinger Denklemi İçin Bir Ters Problem” başlıklı bu çalışma jürimiz tarafından değerlendirilerek Zonguldak Bülent Ecevit Üniversitesi, Fen Bilimleri Enstitüsü, Matematik Anabilim Dalında Yüksek Lisans Tezi olarak oybirliğiyle kabul edilmiştir. 23/08/2019

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**ONAY:**

Yukarıdaki imzaların, adı geçen öğretim üyelerine ait olduğunu onaylarım.

....../....../2019



Prof. Dr. Ahmet ÖZARSLAN

Fen Bilimleri Enstitüsü Müdürü

*“Bu tezdeki tüm bilgilerin akademik kurallara ve etik ilkelere uygun olarak elde edildiğini ve sunulduğunu; ayrıca bu kuralların ve ilkelerin gerektirdiği şekilde, bu çalışmadan kaynaklanmayan bütün atıfları yaptığımı beyan ederim.”*

Sabriye Gözde KİRLİ

## ÖZET

Yüksek Lisans Tezi

### ULTRAHİPERBOLİK SCHRÖDINGER DENKLEMİ İÇİN BİR TERS PROBLEM

Sabriye Gözde KİRLİ

Zonguldak Bülent Ecevit Üniversitesi

Fen Bilimleri Enstitüsü

Matematik Anabilim Dalı

Tez Danışmanı: Doç. Dr. Fikret GÖLGELEYEN

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Bu tezde, bir ultrahiperbolik Schrödinger denklemi sınırsız bir bölgede Cauchy başlangıç şartları ile birlikte ele alınmıştır. Başlangıç anında çözüm hakkında verilen bir ek bilgi yardımıyla denklemin sağ tarafındaki bir bilinmeyen fonksiyonun belirlenmesi ters probleminin çözümünün tekliği araştırılmıştır. Bu kapsamda tezin birinci bölümünde, diğer bölümlerde ihtiyaç duyulan bazı temel tanım ve teoremler verilmiştir. İkinci bölümde bazı yardımcı önermeler yer almıştır. Son bölümde ise ele alınan problemin çözümünün tekliğine ilişkin bir teorem sunulmuştur.

**Anahtar Kelimeler:** Ultrahiperbolik Schrödinger denklemi, ters problem, çözümün tekliği.

**Bilim Kodu:** 403.06.00.



## **ABSTRACT**

**M. Sc. Thesis**

# **AN INVERSE PROBLEM FOR THE ULTRAHYPERBOLIC SCHRÖDINGER EQUATION**

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In this thesis, an ultrahyperbolic Schrödinger equation is considered in an unbounded domain with Cauchy initial conditions. By using the given additional data at the initial time for the solution, uniqueness of solution of an inverse problem of determining a function in the right-hand side of the equation is investigated. In this context, in the first chapter, some basic definitions and theorems which are needed in the other chapters are given. In the second chapter, some auxiliary lemmata are given. In the last chapter, a theorem devoted to the uniqueness of solution of the problem is presented.

**Keywords:** Ultrahyperbolic Schrödinger equation, inverse problem, uniqueness of solution.

**Science Code:** 403.06.00.



## TEŞEKKÜR

Tezimin tüm aşamalarında bilgi ve birikimi ile bana destek olan, değerli vaktini esirgemeyen, kullandığı her kelimeyle bakış açımı geliştiren, danışman hocam sayın Doç. Dr. Fikret GÖLGELEYEN'e teşekkürlerimi sunarım.

Bu süreç boyunca desteğini, ilgisini ve sabrını hiç esirgemeyen değerli aileme ve arkadaşlarıma teşekkür ederim.



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## SİMGELER VE KISALTMALAR DİZİNİ

### SİMGELER

|                 |                                                                                                      |
|-----------------|------------------------------------------------------------------------------------------------------|
| $Q$             | : Verilen bir bölge                                                                                  |
| $Q_\gamma$      | : Oluşturulan alt bölge                                                                              |
| $\partial Q$    | : $Q$ bölgesinin sınırı                                                                              |
| $\bar{Q}$       | : $Q$ bölgesinin kapanışı                                                                            |
| $h$             | : Planck sabiti                                                                                      |
| $L^1(Q)$        | : $Q$ bölgesi üzerinde Lebesgue ölçülebilir ve integrallenebilir fonksiyonlar uzayı                  |
| $L^2(Q)$        | : $Q$ bölgesi üzerinde Lebesgue ölçülebilir ve karesi integrallenebilir fonksiyonlar uzayı           |
| $C^k(Q)$        | : $Q$ bölgesinde tanımlı $k$ . mertebeye kadar sürekli kısmi türevlere sahip fonksiyonlar uzayı      |
| $H^k(Q)$        | : Kendisi ve $k$ . mertebeye kadar tüm genelleşmiş türevleri $L^2(Q)$ ya ait olan fonksiyonlar uzayı |
| $\lambda, \nu$  | : Büyük Parametreler                                                                                 |
| $d_i$           | : Divergent terimler, $i = 1, 2, 3, 4$                                                               |
| $\hat{w}$       | : $w$ fonksiyonunun Fourier dönüşümü                                                                 |
| $u_t$           | : $u$ fonksiyonunun $t$ değişkenine göre kısmi türevi; $u_t = \frac{\partial u}{\partial t}$         |
| $u_{x_i}$       | : $u$ fonksiyonunun $x_i$ değişkenine göre kısmi türevi; $u_{x_i} = \frac{\partial u}{\partial x_i}$ |
| $u_{y_j}$       | : $u$ fonksiyonunun $y_j$ değişkenine göre kısmi türevi; $u_{y_j} = \frac{\partial u}{\partial y_j}$ |
| $\nabla_x u$    | : $u$ fonksiyonunun $x$ vektörüne göre gradienti; $\nabla_x u = (u_{x_1}, u_{x_2}, \dots, u_{x_n})$  |
| $\nabla_y u$    | : $u$ fonksiyonunun $y$ vektörüne göre gradienti; $\nabla_y u = (u_{y_1}, u_{y_2}, \dots, u_{y_m})$  |
| $i = \sqrt{-1}$ | : Sanal birim                                                                                        |



## BÖLÜM 1

### ÖN BİLGİLER

#### 1.1 GİRİŞ

Bu çalışmada bir

$$Q = \{(x, y, t) : x \in D \subset \mathbb{R}^n, y \in G \subset \mathbb{R}^m, t \in \mathbb{R}\}$$

bölgesinde

$$\begin{aligned} Lu &\equiv iu_t + u_{x_1x_1} + \sum_{s=2}^n a_{ss}(x, y)u_{x_sx_s} - \sum_{j=1}^m b_{jj}(x, y)u_{y_jy_j} \\ &\quad + \sum_{r=1}^n a_r(x, y, t)u_{x_r} + \sum_{l=1}^m b_l(x, y, t)u_{y_l} + a_0(x, y, t)u \\ &= f(x, y, t)g(x, y) \end{aligned} \tag{1.1}$$

ultrahiperbolik Schrödinger denklemi

$$u(0, 'x, y, t) = u_{x_1}(0, 'x, y, t) = 0 \tag{1.2}$$

koşulları ile birlikte ele alınacaktır. Burada  $x = (x_1, x_2, \dots, x_n)$ ,  $'x = (x_2, x_3, \dots, x_n) \in \mathbb{R}^{n-1}$ ,  $y = (y_1, y_2, \dots, y_m)$ ,  $\frac{\partial^2 u}{\partial x_1^2} = u_{x_1x_1}$ ,  $\frac{\partial^2 u}{\partial x_s^2} = u_{x_sx_s}$ ,  $\frac{\partial^2 u}{\partial y_j^2} = u_{y_jy_j}$ ,  $\frac{\partial u}{\partial x_r} = u_{x_r}$ ,  $\frac{\partial u}{\partial y_l} = u_{y_l}$  gösterimleri kullanılmıştır. Verilen bir ek bilgi yardımıyla (1.1) denkleminin sağ tarafındaki bilinmeyen fonksiyonun bulunması ters probleminin çözümünün tekliği araştırılmıştır.

Bilindiği üzere kuantum mekanikinde, bir parçacık  $\psi(r, t)$  dalga fonksiyonuyla karakterize edilir. Bu fonksiyon,  $t$  anında parçacığın uzaysal konumu hakkında bilgi verir. Şimdi  $m$  kütleli ve  $V(r, t)$  potansiyelinin etkisi altındaki bir parçacığı düşünelim. Bu durumda dalga fonksiyonu

$$i\hbar \frac{\partial \psi(r, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(r, t) + V(r, t)\psi(r, t) \tag{1.3}$$

Schrödinger denklemini sağlar. Burada  $\hbar = \frac{h}{2\pi}$  olup  $h$  Planck sabitidir. Ayrıca  $\nabla^2$  Laplace operatörünü göstermektedir. Bu denklemin iki önemli özelliği vardır:

i) Schrödinger denklemi  $\psi$  ye göre lineer ve homojen bir denklemdir.

ii) Schrödinger denklemi zamana göre birinci mertebeden bir denklemdir. Bu nedenle  $t_0$  anındaki durum bütün zamanlardaki alt durumları belirlemektedir (Peleg et al. 1998).

(1.1) denklemi genelleşmiş Schrödinger denklemi olarak da adlandırılır. (1.3) denklemi üzerine pek çok çalışma mevcut iken (1.1) denklemi ile ilgili sınırlı sayıda çalışma yapılmıştır. (1.3) denklemi için çeşitli ters problemler Baudouin ve Puel (2002), Mercado vd. (2008), Amirov ve Yamamoto (2008), Bellassoued ve Choulli (2009), Yuan ve Yamamoto (2010), Cristofol ve Soccorsi (2011), Kian vd. (2015), Triggiani ve Zhang (2015) de ele alınmıştır. Diğer yandan (1.1) denklemi için bir başlangıç değer probleminin çözülebilirliği Kenig vd. (1998, 2006) tarafından araştırılmıştır. Gölgeleyen ve Kaytmaz (2019), (1.1) denklemi için bir ters problemin çözümünün kararlılığını göstermiştir. Bu tür denklemlerin başta su dalgası problemleri olmak üzere farklı alanlarda önemli uygulamaları vardır (Davey and Stewartson 1974, Djordjevic and Redekopp 1977, Escauriaza et al. 2011, Ichinose 1990, Zakharov and Schulman 1980, Zakharov and Kuznetsov 1986). Ayrıca kuantum kinetik teorisinde de bu denklemler karşımıza çıkmaktadır.

Bu tezin birinci bölümünde temel tanım ve teoremler verilmiştir, ikinci bölümünde bazı yardımcı önermeler yer almıştır. Son bölümde ise ele alınan problemin çözümünün tekliğine ilişkin bir teorem sunulmuştur.

## 1.2 TEMEL TANIM VE TEOREMLER

**Tanım 1.1 (Direkt Problem)** *Matematiksel fizikte denklem, bölge ve koşullar verildiğinde denklemi ve koşulları sağlayan bir çözümün bulunmasına direkt problem denir (Amirov 2001).*

**Tanım 1.2 (Ters Problem)** *Pratikte karşılaşılan öyle problemler vardır ki, bunların çözümleri için ayrıca ek bilgiye gerek duyulur. Verilen bu ek bilgiye göre denklemin bir veya birkaç katsayısının veya sağ tarafının ya da sınır koşullarından bir veya birkaçını denklemin çözümü ile birlikte bulmak gerekir. Böyle problemlere ters problem adı verilir (Amirov 2001).*

**Tanım 1.3 (Fourier Dönüşümü)**  $f \in L^1(\mathbb{R})$  olmak üzere  $f$  fonksiyonunun Fourier dönüşümü ve eşlenik Fourier dönüşümü sırasıyla

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x) dx,$$

$$\overline{\mathcal{F}}f(\xi) = \int_{\mathbb{R}} e^{i\xi x} f(x) dx$$

şeklinde tanımlanır (Gasquet and Witomski 1999). Eğer  $\mathcal{F}[f] \in L^1(\mathbb{R})$  ise  $\overline{\mathcal{F}}$  operatörü,  $\mathcal{F}$ 'nin tersi olur ( $\mathcal{F}^{-1} = \overline{\mathcal{F}}$ ). O halde  $f$  ve  $\hat{f}$  fonksiyonları  $L^1(\mathbb{R})$  uzayına ait olduğunda, hemen hemen her yerde  $\overline{\mathcal{F}}\hat{f}(t) = f(t)$  yazılabilir.

**Tanım 1.4 ( $C^k(Q)$  Uzayı)**  $Q, \mathbb{R}^n$  uzayında bir bölge olsun. Her negatif olmayan  $k$  tam sayısı için  $|\alpha| \leq k$  olmak üzere  $D^\alpha \varphi$  kısmi türevleri  $Q$  bölgesinde sürekli olacak şekilde tüm  $\varphi$  fonksiyonlarının oluşturduğu vektör uzayı  $C^k(Q)$  ile gösterilir (Adams and Fournier 2003). Bu uzayda norm aşağıdaki şekilde tanımlanır (Mikhailov 1978):

$$\|f\|_{C^k(\overline{Q})} = \sum_{|\alpha| \leq k} \max_{x \in \overline{Q}} |D^\alpha f(x)|.$$

**Tanım 1.5 ( $L^1(Q), L^2(Q)$  Uzayları)**  $L^1(Q), Q$  üzerinde Lebesgue ölçülebilir ve integrallenebilir tüm fonksiyonların oluşturduğu cümle;  $L^2(Q), Q$  üzerinde Lebesgue ölçülebilir ve modülünün karesi integrallenebilir tüm fonksiyonların oluşturduğu cümledir. Bu kümelere ait bazı önemli özellikler aşağıda verilmiştir (Mikhailov 1978):

i)  $L^1(Q)$  ve  $L^2(Q)$  lineer uzaydır ve  $Q$  sınırlı bir bölge ise

$$C(\overline{Q}) \subset L^2(Q) \subset L^1(Q),$$

ii)  $L^1(Q)$ , üzerinde tanımlanan

$$\|f\|_{L^1(Q)} = \int_Q |f(x)| dx$$

normuna göre bir Banach uzaydır,

iii)  $L^2(Q)$ , üzerinde tanımlanan

$$(f_1, f_2)_{L^2(Q)} = \int_Q f_1(x) \overline{f_2(x)} dx$$

iç çarpımına göre bir Hilbert uzayıdır, bu iç çarpım ile tanımlanan norm

$$\|f\|_{L^2(Q)} = \left( \int_Q |f(x)|^2 dx \right)^{1/2}$$

biçimindedir.

**Tanım 1.6 ( $H^k(Q)$  Uzayı)**  $H^k(Q)$ , kendisi ve  $k$ . mertebeye kadar tüm genelleşmiş türevleri  $L^2(Q)$  uzayına ait olan tüm fonksiyonların oluşturduğu cümledir. Bu cümleye ait bazı özellikler aşağıda verilmiştir:

i)  $H^k(Q)$  lineer uzayıdır ve  $H^0(Q) = L^2(Q)$  olarak tanımlanır.

ii)  $H^k(Q)$  üzerinde tanımlanan

$$(f_1, f_2)_{H^k(Q)} = \sum_{|\alpha| \leq k} \int_Q D^\alpha f_1 \overline{D^\alpha f_2} dx$$

iç çarpımına göre bir Hilbert uzayıdır, bu iç çarpım ile tanımlanan norm

$$\|f\|_{H^k(Q)} = \left( \sum_{|\alpha| \leq k} \int_Q |D^\alpha f|^2 dx \right)^{1/2}$$

biçimindedir (Mikhailov 1978).

**Teorem 1.1** Eğer  $f \in C^2(\mathbb{R})$  ve  $f, f', f''$  fonksiyonları integrallenebilirse  $\widehat{f}$  integralenebilirdir (Gasquet and Witomski 1999).

**Teorem 1.2 (Plancherel)**  $\mathbb{R}$ 'nin Lebesgue ölçümü sonsuz olduğundan  $L^2(\mathbb{R})$  kümesi  $L^1(\mathbb{R})$ 'nin alt kümesi değildir. Bu nedenle yukarıda verilen Fourier dönüşümü tanımı her  $f \in L^2(\mathbb{R})$  fonksiyonu için direkt olarak uygulanabilir değildir. Ancak, eğer  $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  ise yukarıdaki tanımın kolayca uygulanabileceği açıktır. Ayrıca bu durumda  $\widehat{f} \in L^2(\mathbb{R})$  dir. Aslında  $\|\widehat{f}\|_{L^2} = \|f\|_{L^2}$  olup bu bağıntıya Plancherel dönüşümü adı verilir. Böylece  $L^2$  uzayında  $f$  ve  $\widehat{f}$  aynı rolü oynar (Rudin 1987).

**Teorem 1.3** Her  $f \in L^2$  fonksiyonuna, aşağıdaki özellikleri sağlayan bir  $\widehat{f} \in L^2$  fonksiyonu karşılık gelir:

i) Eğer  $f \in L^1 \cap L^2$  ise  $\widehat{f}$  daha önce tanımlandığı şekliyle  $f$ 'nin Fourier dönüşümüdür.

ii) Her  $f \in L^2$  ise  $\|\widehat{f}\|_{L^2} = \|f\|_{L^2}$ .

iii)  $f \rightarrow \widehat{f}$  dönüşümü  $L^2 \rightarrow L^2$  ye olan bir Hilbert uzayı izomorfizmidir (Rudin 1987).





## BÖLÜM 2

### BAZI YARDIMCI ÖNERMELER

Ele alınan ters problemin çözümünün tekliğini araştırmak için öncelikle bazı lemmalar vereceğiz. Denklemleri daha uygun bir hale getirmek için yeni bir değişken dönüşümü tanımlayalım:

$$\tilde{x}_1 = \sqrt{2x_1} - \delta_0, \quad x_1 = \frac{1}{2}(\tilde{x}_1 + \delta_0)^2, \quad 2\delta_0 \leq \min\{\alpha_0, \gamma\}, \quad \delta_0 > 0.$$

Buna göre

$$\begin{aligned} u(x, y, t) &= u\left(\frac{1}{2}(\tilde{x}_1 + \delta_0)^2, x_2, \dots, x_n, y, t\right) \equiv \tilde{u}(\tilde{x}_1, x_2, \dots, x_n, y, t), \\ u_{x_1} &= \tilde{u}_{\tilde{x}_1} \frac{d\tilde{x}_1}{dx_1} = \tilde{u}_{\tilde{x}_1} \frac{1}{\tilde{x}_1 + \delta_0}; \quad u_{x_1 x_1} = \tilde{u}_{\tilde{x}_1 \tilde{x}_1} (\tilde{x}_1 + \delta_0)^{-2} - \tilde{u}_{\tilde{x}_1} (\tilde{x}_1 + \delta_0)^{-3} \end{aligned}$$

olduğu görülür. Bu durumda (1.1) denklemi aşağıdaki formlarda yazılabilir:

$$\begin{aligned} & i\tilde{u}_t + \tilde{u}_{\tilde{x}_1 \tilde{x}_1} (\tilde{x}_1 + \delta_0)^{-2} - \tilde{u}_{\tilde{x}_1} (\tilde{x}_1 + \delta_0)^{-3} + \sum_{s=2}^n \tilde{a}_{ss} \tilde{u}_{x_s x_s} - \sum_{j=1}^m \tilde{b}_{jj} \tilde{u}_{y_j y_j} \\ & + (\tilde{a}_1 (\tilde{x}_1 + \delta_0) + (\tilde{x}_1 + \delta_0)^{-2}) \tilde{u}_{\tilde{x}_1} \frac{1}{\tilde{x}_1 + \delta_0} + \sum_{r=2}^n \tilde{a}_r \tilde{u}_{x_r} + \sum_{l=1}^m \tilde{b}_l \tilde{u}_{y_l} + \tilde{a}_0 \tilde{u} \\ & = \tilde{f}(x, y, t) \tilde{g}(x, y), \\ & i\tilde{u}_t + \tilde{u}_{\tilde{x}_1 \tilde{x}_1} (\tilde{x}_1 + \delta_0)^{-2} + \tilde{u}_{\tilde{x}_1} (\tilde{a}_1 + (\tilde{x}_1 + \delta_0)^{-3} - (\tilde{x}_1 + \delta_0)^{-3}) + \sum_{s=2}^n \tilde{a}_{ss} \tilde{u}_{x_s x_s} \\ & - \sum_{j=1}^m \tilde{b}_{jj} \tilde{u}_{y_j y_j} + \sum_{r=2}^n \tilde{a}_r \tilde{u}_{x_r} + \sum_{l=1}^m \tilde{b}_l \tilde{u}_{y_l} + \tilde{a}_0 \tilde{u} \\ & = \tilde{f}(x, y, t) \tilde{g}(x, y), \\ & (\tilde{x}_1 + \delta_0) i\tilde{u}_t + (\tilde{x}_1 + \delta_0)^{-1} \tilde{u}_{\tilde{x}_1 \tilde{x}_1} + (\tilde{x}_1 + \delta_0) \tilde{a}_1 \tilde{u}_{\tilde{x}_1} + (\tilde{x}_1 + \delta_0) \sum_{s=2}^n \tilde{a}_{ss} \tilde{u}_{x_s x_s} \\ & - (\tilde{x}_1 + \delta_0) \sum_{j=1}^m \tilde{b}_{jj} \tilde{u}_{y_j y_j} + (\tilde{x}_1 + \delta_0) \sum_{r=2}^n \tilde{a}_r \tilde{u}_{x_r} + (\tilde{x}_1 + \delta_0) \sum_{l=1}^m \tilde{b}_l \tilde{u}_{y_l} + (\tilde{x}_1 + \delta_0) \tilde{a}_0 \tilde{u} \\ & = (\tilde{x}_1 + \delta_0) \tilde{f}(x, y, t) \tilde{g}(x, y), \end{aligned}$$

$$\begin{aligned}
& (\tilde{x}_1 + \delta_0)^{-1} \tilde{u}_{\tilde{x}_1 \tilde{x}_1} + (\tilde{x}_1 + \delta_0) \left( \sum_{s=2}^n \tilde{a}_{ss} \tilde{u}_{x_s x_s} - \sum_{j=1}^m \tilde{b}_{jj} \tilde{u}_{y_j y_j} + i \tilde{u}_t \right) \\
& + (\tilde{x}_1 + \delta_0) \left( \sum_{r=1}^n \tilde{a}_r \tilde{u}_{x_r} + \sum_{l=1}^m \tilde{b}_l \tilde{u}_{y_l} + \tilde{a}_0 \tilde{u} \right) \\
& = (\tilde{x}_1 + \delta_0) \tilde{f}(x, y, t) \tilde{g}(x, y).
\end{aligned}$$

Burada

$$\begin{aligned}
\tilde{a}_{ss} &= a_{ss} \left( \frac{1}{2} (\tilde{x}_1 + \delta_0)^2, x_2, \dots, x_n, y \right); \quad s = 2, \dots, n, \\
\tilde{a}_r &= a_r \left( \frac{1}{2} (\tilde{x}_1 + \delta_0)^2, x, y, t \right); \quad r = 2, \dots, n, \\
\tilde{a}_0 &= a_0 \left( \frac{1}{2} (\tilde{x}_1 + \delta_0)^2, x, y, t \right), \\
\tilde{a}_1 &= a_1 \left( \frac{1}{2} (\tilde{x}_1 + \delta_0)^2, x, y, t \right) (\tilde{x}_1 + \delta_0)^{-1} - (\tilde{x}_1 + \delta_0)^{-3}, \\
\tilde{b}_{jj} &= b_{jj} \left( \frac{1}{2} (\tilde{x}_1 + \delta_0)^2, x_2, \dots, x_n, y \right); \quad j = 1, \dots, m, \\
\tilde{b}_l &= b_l \left( \frac{1}{2} (\tilde{x}_1 + \delta_0)^2, x, y, t \right); \quad l = 1, \dots, m, \\
\tilde{f} &= f \left( \frac{1}{2} (\tilde{x}_1 + \delta_0)^2, x, y, t \right), \\
\tilde{g} &= g \left( \frac{1}{2} (\tilde{x}_1 + \delta_0)^2, x, y \right)
\end{aligned}$$

şeklinde alınmıştır. Basitlik açısından  $\tilde{x}_1, \tilde{x}_s, \tilde{x}_r, \tilde{y}_j, \tilde{y}_l, \tilde{u}, \tilde{a}_{ss}, \tilde{a}_r, \tilde{b}_{jj}, \tilde{b}_l$  gösterimlerini

$x_1, x_s, x_r, y_j, y_l, u, a_{ss}, a_r, b_{jj}, b_l$  gösterimleri ile değiştirelim:

$$\begin{aligned}
\tilde{L}u &\equiv (x_1 + \delta_0)^{-1} u_{x_1 x_1} + (x_1 + \delta_0) \left( \sum_{s=2}^n a_{ss} u_{x_s x_s} - \sum_{j=1}^m b_{jj} u_{y_j y_j} + i u_t \right) \\
&+ (x_1 + \delta_0) \left( \sum_{r=1}^n a_r u_{x_r} + \sum_{l=1}^m b_l u_{y_l} + a_0 u \right) \\
&= (x_1 + \delta_0) f(x, y, t) g(x, y).
\end{aligned} \tag{2.1}$$

Şimdi yeni bir bilinmeyen fonksiyon tanımlayalım:

$$w = u e^{-kt^2},$$

$k \geq \max\{d_u, d_f\}$  olup  $d_u, d_f$  sayıları üçüncü bölümde tanımlanmıştır. Buradan

$$\begin{aligned}
u &= w e^{kt^2}, \quad u_t = w_t e^{kt^2} + 2kt w e^{kt^2}, \\
u_{x_1} &= w_{x_1} e^{kt^2}, \quad u_{x_1 x_1} = w_{x_1 x_1} e^{kt^2}, \\
u_{x_s} &= w_{x_s} e^{kt^2}, \quad u_{x_s x_s} = w_{x_s x_s} e^{kt^2}, \\
u_{y_j} &= w_{y_j} e^{kt^2}, \quad u_{y_j y_j} = w_{y_j y_j} e^{kt^2}
\end{aligned}$$

yazılabilir. O halde (2.1) denkleminde

$$\begin{aligned}
& (x_1 + \delta_0)^{-1} w_{x_1 x_1} e^{kt^2} \\
& + (x_1 + \delta_0) \left( \sum_{s=2}^n a_{ss} w_{x_s x_s} e^{kt^2} - \sum_{j=1}^m b_{jj} w_{y_j y_j} e^{kt^2} + i(w_t e^{kt^2} + 2ktw e^{kt^2}) \right) \\
& + (x_1 + \delta_0) \left( \sum_{r=1}^n a_r w_{x_r} e^{kt^2} + \sum_{l=1}^m b_l w_{y_l} e^{kt^2} + a_0 w e^{kt^2} \right) \\
= & (x_1 + \delta_0) f(x, y, t) g(x, y),
\end{aligned}$$

$$\begin{aligned}
& 2(x_1 + \delta_0) iktw e^{kt^2} + (x_1 + \delta_0) i w_t e^{kt^2} + (x_1 + \delta_0)^{-1} w_{x_1 x_1} e^{kt^2} \\
& + (x_1 + \delta_0) \left( \sum_{s=2}^n a_{ss} w_{x_s x_s} e^{kt^2} - \sum_{j=1}^m b_{jj} w_{y_j y_j} e^{kt^2} \right) \\
& + (x_1 + \delta_0) \left( \sum_{r=1}^n a_r w_{x_r} e^{kt^2} + \sum_{l=1}^m b_l w_{y_l} e^{kt^2} + a_0 w e^{kt^2} \right) \\
= & (x_1 + \delta_0) f(x, y, t) g(x, y)
\end{aligned}$$

elde edilir. Diğer bir şekilde

$$\tilde{L}w + 2(x_1 + \delta_0) iktw = (x_1 + \delta_0) F(x, y, t) g(x, y), \quad (2.2)$$

$$w(0, 'x, y, t) = w_{x_1}(0, 'x, y, t) = w(x, y, 0) = 0 \quad (2.3)$$

olarak yazılabilir. Burada

$$F(x, y, t) = f(x, y, t) e^{-kt^2}$$

dir. Diğer yandan

$$\begin{aligned}
\tilde{L}w = & (x_1 + \delta_0) i w_t + (x_1 + \delta_0)^{-1} w_{x_1 x_1} + (x_1 + \delta_0) \left( \sum_{s=2}^n a_{ss} w_{x_s x_s} - \sum_{j=1}^m b_{jj} w_{y_j y_j} \right) \\
& + (x_1 + \delta_0) \left( \sum_{r=1}^n a_r w_{x_r} + \sum_{l=1}^m b_l w_{y_l} + a_0 w \right)
\end{aligned}$$

olduğu göz önünde bulundurularak  $t = 0$  için (2.2) denkleminde

$$\begin{aligned}
g(x, y) &= \frac{1}{f(x, y, 0)} (i w_t(x, y, 0)), \\
g(x, y) &= -\frac{1}{2\pi f(x, y, 0)} \int_{-\infty}^{\infty} \xi \hat{w}(x, y, \xi) d\xi
\end{aligned} \quad (2.4)$$

olarak bulunur. Burada

$$2\pi f(x, y, 0) = f_0, \quad \mathcal{F}(w_t) = i\xi \hat{w}, \quad \mathcal{F}^{-1} \mathcal{F}(w_t) = \mathcal{F}^{-1}(i\xi \hat{w}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} i\xi \hat{w} e^{i\xi t} d\xi$$

gösterimleri kullanılmıştır.

(2.2) denklemine ve (2.3) koşullarına  $t$ -ye göre Fourier dönüşümü uygulanır ve  $i = \sqrt{-1}$  olduğu göz önünde bulundurulursa

$$\begin{aligned}
& -(x_1 + \delta_0)\xi\widehat{w} + (x_1 + \delta_0)^{-1}\widehat{w}_{x_1x_1} + (x_1 + \delta_0)\left(\sum_{s=2}^n a_{ss}\widehat{w}_{x_sx_s} - \sum_{j=1}^m b_{jj}\widehat{w}_{y_jy_j}\right) \\
& + (x_1 + \delta_0)\left(\sum_{r=1}^n \widehat{a_r w_{x_r}} + \sum_{l=1}^m \widehat{b_l w_{y_l}} + \widehat{a_0 w}\right) - 2(x_1 + \delta_0)k\widehat{w}_\xi \\
& = -(x_1 + \delta_0)\widehat{F}\frac{1}{f_0}\int_{-\infty}^{\infty}\xi\widehat{w}d\xi
\end{aligned} \tag{2.5}$$

ve

$$\widehat{w}(0, 'x, y, \xi) = \widehat{w}_{x_1}(0, 'x, y, \xi) = 0 \tag{2.6}$$

elde edilir. Diğer taraftan  $\widehat{w} = \varphi_1(x, y, \xi) + i\varphi_2(x, y, \xi)$ ,  $\widehat{F} = f_1(x, y, \xi) + if_2(x, y, \xi)$  şeklinde yazılabildiği kabul edilerek aşağıdaki gösterimi tanımlayalım:

$$\begin{aligned}
L_0\varphi & \equiv -(x_1 + \delta_0)\xi\varphi + (x_1 + \delta_0)^{-1}\varphi_{x_1x_1} \\
& + (x_1 + \delta_0)\left(\sum_{s=2}^n a_{ss}\varphi_{x_sx_s} - \sum_{j=1}^m b_{jj}\varphi_{y_jy_j}\right) - 2(x_1 + \delta_0)k\varphi_\xi \\
& = (x_1 + \delta_0)^{-1}\varphi_{x_1x_1} + (x_1 + \delta_0)\left(\sum_{s=2}^n a_{ss}\varphi_{x_sx_s} - \sum_{j=1}^m b_{jj}\varphi_{y_jy_j} - 2k\varphi_\xi - \xi\varphi\right).
\end{aligned} \tag{2.7}$$

Burada  $\varphi$ ,  $\widehat{w}$  fonksiyonunun reel veya sanal kısmını göstermektedir. Daha açık olarak  $\widehat{w} = \varphi_1(x, y, \xi) + i\varphi_2(x, y, \xi)$  olmak üzere  $\varphi = \varphi_1$  veya  $\varphi = \varphi_2$  dir.

Aşağıdaki gösterimleri tanımlayalım:

$$Q_\gamma = \{(x, y) : x \in \mathbb{R}^n, y \in \mathbb{R}^m, x_1 > 0, 0 < \delta x_1 + \frac{1}{2}\sum_{j=2}^n (x_j - x_j^0)^2 + \frac{1}{2}\sum_{s=1}^m (y_s - y_s^0)^2 < \gamma\},$$

$$\psi(x, y) = \delta x_1 + \frac{1}{2}\sum_{j=2}^n (x_j - x_j^0)^2 + \frac{1}{2}\sum_{s=1}^m (y_s - y_s^0)^2;$$

$$\chi = e^{\lambda\psi^{-\nu}}, \quad a_0 > 0, \quad \gamma + a_0 = \eta < 1, \quad a_0 < \psi(x, y) < \eta.$$

## 2.1 LEMMA 2.1

**Lemma 2.1** *Kabul edelim ki  $\|a_{ii}\|_{C^3(\overline{D \times G})} \leq M$ ,  $2 \leq i \leq n$ ,  $\|b_{jj}\|_{C^3(\overline{D \times G})} \leq M$ ,  $1 \leq j \leq m$  ve  $0 < \gamma < \min\{\frac{2\sqrt{2\varepsilon_0}}{3M\sqrt{n+m-1}}, 1\}$  olsun. Burada  $\varepsilon_0$  bir sabit olup  $0 < \varepsilon_0 < \frac{\alpha_1}{4}$ ,  $0 < \varepsilon_0 < \frac{\alpha_2}{4}$  şeklindedir. Bu durumda yeterince büyük*

$$\delta_* = \delta_*(\alpha_1, \alpha_2, M, n, m, \gamma), \quad \nu_* = \nu_*(\alpha_1, \alpha_2, M, \delta, n, m, \gamma), \quad \lambda_* = \lambda_*(\alpha_1, \alpha_2, M, \nu, n, m, \gamma)$$

sayıları vardır öyle ki eğer  $\delta > \delta_*$ ,  $\nu > \nu_*$  ise tüm  $\varphi \in C^2(\overline{Q}_\gamma)$  ve her  $\lambda > \lambda_*$  için aşağıdaki eşitsizlik geçerlidir:

$$\begin{aligned}
& \psi^{\nu+1}(L_0\varphi)^2\chi^2 + 2\lambda\nu(x_1 + \delta_0)\chi^2\Lambda\varphi(L_0\varphi) \\
\geq & 2\lambda\nu\varphi_{x_1}^2\chi^2 + \lambda\nu(x_1 + \delta_0)\chi^2 \sum_{i=2}^n \varphi_{x_i}^2 + \lambda\nu(x_1 + \delta_0)\chi^2 \sum_{j=1}^m \varphi_{y_j}^2 \\
& + 4\lambda^3\nu^3\psi^{-2\nu-3}\varphi^2\chi^2 + \sum_{i=1}^4 d_i(\varphi). \tag{2.8}
\end{aligned}$$

Lemma 2.1'i ispatlamak için aşağıda Lemma 2.2 ve Lemma 2.3 verilecektir.

## 2.2 LEMMA 2.2

**Lemma 2.2** *Lemma 2.1'in koşulları sağlansın. Eğer  $\lambda$ ,  $\nu$  ve  $\delta$  parametreleri bir pozitif sabitten büyük ise bu durumda her  $\varphi \in C^2(\overline{Q}_\gamma)$  aşağıdaki eşitsizliği sağlar:*

$$\begin{aligned}
\psi^{\nu+1}(L_0\varphi)^2\chi^2 \geq & 2\lambda\nu\gamma^{-3}\delta\varphi_{x_1}^2\chi^2 + \lambda\nu(x_1 + \delta_0)\delta\alpha_1\chi^2 \sum_{i=2}^n \varphi_{x_i}^2 \\
& + \lambda\nu(x_1 + \delta_0)\delta\alpha_2\chi^2 \sum_{j=1}^m \varphi_{y_j}^2 + \lambda^3\nu^4\delta^4\psi^{-2\nu-3}\varphi^2\chi^2 \\
& + 2\lambda\nu\xi(x_1 + \delta_0)^2 \left( \sum_{i=2}^n (a_{ii}\psi_{x_i})_{x_i} - \sum_{j=1}^m (b_{jj}\psi_{y_j})_{y_j} \right) \varphi^2\chi^2 + \sum_{j=1}^3 d_j(\varphi). \tag{2.9}
\end{aligned}$$

**İspat.** İlk olarak yeni bir bilinmeyen fonksiyon tanımlayalım:

$$\vartheta = \chi\varphi.$$

Bu durumda

$$\begin{aligned}
\varphi &= \vartheta\chi^{-1}, \quad \varphi_\xi = \vartheta_\xi\chi^{-1}, \\
\varphi_{x_i} &= \chi^{-1}(\vartheta_{x_i} + \lambda\nu\psi^{-\nu-1}\psi_{x_i}\vartheta),
\end{aligned}$$

$$\begin{aligned}
\varphi_{x_i x_i} &= \chi^{-1}(\vartheta_{x_i x_i} + \lambda\nu\psi^{-\nu-1}\psi_{x_i}\vartheta_{x_i} + \lambda\nu\psi^{-\nu-1}\psi_{x_i}\vartheta_{x_i} \\
&+ (\lambda^2\nu^2\psi^{-2\nu-2}\psi_{x_i}^2 - \lambda\nu(\nu+1)\psi^{-\nu-2}\psi_{x_i}^2 + \lambda\nu\psi^{-\nu-1}\psi_{x_i x_i})\vartheta) \\
&= \chi^{-1}(\vartheta_{x_i x_i} + 2\lambda\nu\psi^{-\nu-1}\psi_{x_i}\vartheta_{x_i} + (\lambda^2\nu^2\psi^{-2\nu-2}\psi_{x_i}^2 - \lambda\nu(\nu+1)\psi^{-\nu-2}\psi_{x_i}^2 \\
&+ \lambda\nu\psi^{-\nu-1}\psi_{x_i x_i})\vartheta),
\end{aligned}$$

$$\begin{aligned}
\varphi_{x_1 x_1} &= \chi^{-1}(\vartheta_{x_1 x_1} + 2\lambda\nu\psi^{-\nu-1}\psi_{x_1}\vartheta_{x_1} + (\lambda^2\nu^2\psi^{-2\nu-2}\psi_{x_1}^2 - \lambda\nu(\nu+1)\psi^{-\nu-2}\psi_{x_1}^2 \\
&\quad + \lambda\nu\psi^{-\nu-1}\psi_{x_1 x_1})\vartheta), \\
\varphi_{y_j y_j} &= \chi^{-1}(\vartheta_{y_j y_j} + 2\lambda\nu\psi^{-\nu-1}\psi_{y_j}\vartheta_{y_j} + (\lambda^2\nu^2\psi^{-2\nu-2}\psi_{y_j}^2 - \lambda\nu(\nu+1)\psi^{-\nu-2}\psi_{y_j}^2 \\
&\quad + \lambda\nu\psi^{-\nu-1}\psi_{y_j y_j})\vartheta)
\end{aligned}$$

eşitlikleri yazılabilir. Yukarıdaki eşitlikler dikkate alınır ve gerekli hesaplamalar yapılırsa,

$$\begin{aligned}
\psi^{\nu+1}(L_0\varphi)^2\chi^2 &= \psi^{\nu+1}((x_1 + \delta_0)^{-1}\varphi_{x_1 x_1} + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\varphi_{x_i x_i} - \sum_{j=1}^m b_{jj}\varphi_{y_j y_j} \\
&\quad - 2k\varphi_\xi - \xi\varphi))^2\chi^2 \\
&= \chi^2\psi^{\nu+1}((x_1 + \delta_0)^{-1}\chi^{-1}(\vartheta_{x_1 x_1} + 2\lambda\nu\psi^{-\nu-1}\psi_{x_1}\vartheta_{x_1} \\
&\quad + (\lambda^2\nu^2\psi^{-2\nu-2}\psi_{x_1}^2 - \lambda\nu(\nu+1)\psi^{-\nu-2}\psi_{x_1}^2 + \lambda\nu\psi^{-\nu-1}\psi_{x_1 x_1})\vartheta) \\
&\quad + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\chi^{-1}(\vartheta_{x_i x_i} + 2\lambda\nu\psi^{-\nu-1}\psi_{x_i}\vartheta_{x_i} \\
&\quad + (\lambda^2\nu^2\psi^{-2\nu-2}\psi_{x_i}^2 - \lambda\nu(\nu+1)\psi^{-\nu-2}\psi_{x_i}^2 + \lambda\nu\psi^{-\nu-1}\psi_{x_i x_i})\vartheta) \\
&\quad - \sum_{j=1}^m b_{jj}\chi^{-1}(\vartheta_{y_j y_j} + 2\lambda\nu\psi^{-\nu-1}\psi_{y_j}\vartheta_{y_j} \\
&\quad + (\lambda^2\nu^2\psi^{-2\nu-2}\psi_{y_j}^2 - \lambda\nu(\nu+1)\psi^{-\nu-2}\psi_{y_j}^2 + \lambda\nu\psi^{-\nu-1}\psi_{y_j y_j})\vartheta) \\
&\quad - 2k\vartheta_\xi\chi^{-1} - \xi\vartheta\chi^{-1}))^2 \\
&= \psi^{\nu+1}(z_1 + z_2 + z_3 + z_4 + z_5)^2 \\
&= \psi^{\nu+1}(z_1^2 + (z_2 + z_4 + z_5)^2 + z_3^2 + 2z_1z_2 + 2z_1z_3 \\
&\quad + 2z_1(z_4 + z_5) + 2z_2z_3 + 2z_3(z_4 + z_5))
\end{aligned} \tag{2.10}$$

bulunur. Burada

$$\begin{aligned}
z_1 &= 2k(x_1 + \delta_0)\vartheta_\xi, \\
z_2 &= -((x_1 + \delta_0)^{-1}\vartheta_{x_1 x_1} + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\vartheta_{x_i x_i} - \sum_{j=1}^m b_{jj}\vartheta_{y_j y_j})), \\
z_3 &= -2\lambda\nu\psi^{-\nu-1}(\psi_{x_1}(x_1 + \delta_0)^{-1}\vartheta_{x_1} + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j})), \\
z_4 &= -E_0\vartheta, \\
z_5 &= \xi(x_1 + \delta_0)\vartheta,
\end{aligned}$$

$$\begin{aligned}
E_0 &= (x_1 + \delta_0)^{-1} (\lambda^2 \nu^2 \psi^{-2\nu-2} \psi_{x_1}^2 - \lambda \nu (\nu + 1) \psi^{-\nu-2} \psi_{x_1}^2) \\
&\quad + (x_1 + \delta_0) \left( \sum_{i=2}^n a_{ii} (\lambda^2 \nu^2 \psi^{-2\nu-2} \psi_{x_i}^2 - \lambda \nu (\nu + 1) \psi^{-\nu-2} \psi_{x_i}^2 + \lambda \nu \psi^{-\nu-1} \psi_{x_i x_i}) \right. \\
&\quad \left. - \sum_{j=1}^m b_{jj} (\lambda^2 \nu^2 \psi^{-2\nu-2} \psi_{y_j}^2 - \lambda \nu (\nu + 1) \psi^{-\nu-2} \psi_{y_j}^2 + \lambda \nu \psi^{-\nu-1} \psi_{y_j y_j}) \right)
\end{aligned}$$

olarak tanımlıdır. Şimdi (2.10) eşitliğindeki terimleri tek tek değerlendirelim. Burada

$$\begin{aligned}
\psi_{x_i x_s} &= \begin{cases} 1, & i = s \\ 0, & i \neq s \end{cases} \quad (2 \leq i, s \leq n); \quad \psi_{y_j y_k} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases} \quad (1 \leq j, k \leq m), \\
\psi_{x_i y_j} &= 0, \quad \psi_{x_1} = \delta, \quad \psi_{x_1 x_1} = 0
\end{aligned}$$

olduğuna dikkat edilmelidir.

İlk olarak

$$\begin{aligned}
2\psi^{\nu+1} z_1 z_2 &= 2\psi^{\nu+1} 2k(x_1 + \delta_0) \vartheta_\xi(-(x_1 + \delta_0)^{-1} \vartheta_{x_1 x_1} \\
&\quad -(x_1 + \delta_0) \left( \sum_{i=2}^n a_{ii} \vartheta_{x_i x_i} - \sum_{j=1}^m b_{jj} \vartheta_{y_j y_j} \right)) \\
&= -4k\psi^{\nu+1} (x_1 + \delta_0) \vartheta_\xi((x_1 + \delta_0)^{-1} \vartheta_{x_1 x_1} \\
&\quad + (x_1 + \delta_0) \left( \sum_{i=2}^n a_{ii} \vartheta_{x_i x_i} - \sum_{j=1}^m b_{jj} \vartheta_{y_j y_j} \right)) \\
&= -4k\psi^{\nu+1} \vartheta_\xi \vartheta_{x_1 x_1} - 4k\psi^{\nu+1} (x_1 + \delta_0)^2 \vartheta_\xi \sum_{i=2}^n a_{ii} \vartheta_{x_i x_i} \\
&\quad + 4k\psi^{\nu+1} (x_1 + \delta_0)^2 \vartheta_\xi \sum_{j=1}^m b_{jj} \vartheta_{y_j y_j} \\
&= -2k(2\psi^{\nu+1} \vartheta_\xi \vartheta_{x_1 x_1}) - 2k(x_1 + \delta_0)^2 \sum_{i=2}^n 2\psi^{\nu+1} a_{ii} \vartheta_\xi \vartheta_{x_i x_i} \\
&\quad + 2k(x_1 + \delta_0)^2 \sum_{j=1}^m 2\psi^{\nu+1} b_{jj} \vartheta_\xi \vartheta_{y_j y_j}
\end{aligned}$$

yazılabilir. Aşağıdaki işlemlerde  $\psi$ ,  $a_{ii}$  ve  $b_{jj}$  fonksiyonlarının  $\xi$  değişkeninden bağımsız olduğu unutulmamalıdır.

$$\begin{aligned}
2\psi^{\nu+1} \vartheta_\xi \vartheta_{x_1 x_1} &= (\psi^{\nu+1} \vartheta_\xi \vartheta_{x_1})_{x_1} + (\psi^{\nu+1} \vartheta_\xi \vartheta_{x_1})_{x_1} - (\psi^{\nu+1} \vartheta_{x_1}^2)_\xi \\
&\quad - (\nu + 1) \psi^\nu \psi_{x_1} \vartheta_\xi \vartheta_{x_1} - (\nu + 1) \psi^\nu \psi_{x_1} \vartheta_\xi \vartheta_{x_1} + (\psi^{\nu+1})_\xi \vartheta_{x_1}^2 \\
&= 2(\psi^{\nu+1} \vartheta_\xi \vartheta_{x_1})_{x_1} - 2(\nu + 1) \psi^\nu \psi_{x_1} \vartheta_\xi \vartheta_{x_1} - (\psi^{\nu+1} \vartheta_{x_1}^2)_\xi,
\end{aligned}$$

$$\begin{aligned}
2\psi^{\nu+1}a_{ii}\vartheta_\xi\vartheta_{x_ix_i} &= (\psi^{\nu+1}a_{ii}\vartheta_\xi\vartheta_{x_i})_{x_i} + (\psi^{\nu+1}a_{ii}\vartheta_\xi\vartheta_{x_i})_{x_i} - (\psi^{\nu+1}a_{ii}\vartheta_{x_i}^2)_\xi \\
&\quad - (\psi^{\nu+1}a_{ii})_{x_i}\vartheta_\xi\vartheta_{x_i} - (\psi^{\nu+1}a_{ii})_{x_i}\vartheta_\xi\vartheta_{x_i} + (\psi^{\nu+1}a_{ii})_\xi\vartheta_{x_i}^2 \\
&= 2(\psi^{\nu+1}a_{ii}\vartheta_\xi\vartheta_{x_i})_{x_i} - (\psi^{\nu+1}a_{ii}\vartheta_{x_i}^2)_\xi - 2(\psi^{\nu+1}a_{ii})_{x_i}\vartheta_\xi\vartheta_{x_i} \\
&= 2(\psi^{\nu+1}a_{ii}\vartheta_\xi\vartheta_{x_i})_{x_i} - (\psi^{\nu+1}a_{ii}\vartheta_{x_i}^2)_\xi \\
&\quad - 2(\nu+1)\psi^\nu\psi_{x_i}a_{ii}\vartheta_\xi\vartheta_{x_i} - 2\psi^{\nu+1}(a_{ii})_{x_i}\vartheta_\xi\vartheta_{x_i},
\end{aligned}$$

$$\begin{aligned}
2\psi^{\nu+1}b_{jj}\vartheta_\xi\vartheta_{y_jy_j} &= (\psi^{\nu+1}b_{jj}\vartheta_\xi\vartheta_{y_j})_{y_j} + (\psi^{\nu+1}b_{jj}\vartheta_\xi\vartheta_{y_j})_{y_j} - (\psi^{\nu+1}b_{jj}\vartheta_{y_j}^2)_\xi \\
&\quad - (\psi^{\nu+1}b_{jj})_{y_j}\vartheta_\xi\vartheta_{y_j} - (\psi^{\nu+1}b_{jj})_{y_j}\vartheta_\xi\vartheta_{y_j} + (\psi^{\nu+1}b_{jj})_\xi\vartheta_{y_j}^2 \\
&= 2(\psi^{\nu+1}b_{jj}\vartheta_\xi\vartheta_{y_j})_{y_j} - (\psi^{\nu+1}b_{jj}\vartheta_{y_j}^2)_\xi - 2(\psi^{\nu+1}b_{jj})_{y_j}\vartheta_\xi\vartheta_{y_j} \\
&= 2(\psi^{\nu+1}b_{jj}\vartheta_\xi\vartheta_{y_j})_{y_j} - (\psi^{\nu+1}b_{jj}\vartheta_{y_j}^2)_\xi \\
&\quad - 2(\nu+1)\psi^\nu\psi_{y_j}b_{jj}\vartheta_\xi\vartheta_{y_j} - 2\psi^{\nu+1}(b_{jj})_{y_j}\vartheta_\xi\vartheta_{y_j}
\end{aligned}$$

eşitlikleri yerine yazılırsa

$$\begin{aligned}
2\psi^{\nu+1}z_1z_2 &= -4k(\psi^{\nu+1}\vartheta_\xi\vartheta_{x_1})_{x_1} + 4k(\nu+1)\psi^\nu\psi_{x_1}\vartheta_\xi\vartheta_{x_1} + 2k(\psi^{\nu+1}\vartheta_{x_1}^2)_\xi \\
&\quad - 2k(x_1 + \delta_0)^2 \sum_{i=2}^n (2(\psi^{\nu+1}a_{ii}\vartheta_\xi\vartheta_{x_i})_{x_i} - (\psi^{\nu+1}a_{ii}\vartheta_{x_i}^2)_\xi \\
&\quad - 2(\nu+1)\psi^\nu\psi_{x_i}a_{ii}\vartheta_\xi\vartheta_{x_i} - 2\psi^{\nu+1}(a_{ii})_{x_i}\vartheta_\xi\vartheta_{x_i}) \\
&\quad + 2k(x_1 + \delta_0)^2 \sum_{j=1}^m (2(\psi^{\nu+1}b_{jj}\vartheta_\xi\vartheta_{y_j})_{y_j} - (\psi^{\nu+1}b_{jj}\vartheta_{y_j}^2)_\xi \\
&\quad - 2(\nu+1)\psi^\nu\psi_{y_j}b_{jj}\vartheta_\xi\vartheta_{y_j} - 2\psi^{\nu+1}(b_{jj})_{y_j}\vartheta_\xi\vartheta_{y_j}) \\
&= 4k(\nu+1)\psi^\nu\psi_{x_1}\vartheta_\xi\vartheta_{x_1} + 4k(\nu+1)(x_1 + \delta_0)^2\psi^\nu \sum_{i=2}^n \psi_{x_i}a_{ii}\vartheta_\xi\vartheta_{x_i} \\
&\quad + 4k(x_1 + \delta_0)^2\psi^{\nu+1} \sum_{i=2}^n (a_{ii})_{x_i}\vartheta_\xi\vartheta_{x_i} - 4k(\nu+1)(x_1 + \delta_0)^2\psi^\nu \sum_{j=1}^m \psi_{y_j}b_{jj}\vartheta_\xi\vartheta_{y_j} \\
&\quad - 4k(x_1 + \delta_0)^2\psi^{\nu+1} \sum_{j=1}^m (b_{jj})_{y_j}\vartheta_\xi\vartheta_{y_j} + d_2(\vartheta) \\
&= 2\psi^{\nu+1}z_1((\nu+1)\psi^{-1}(x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (\nu+1)\psi^{-1}(x_1 + \delta_0) \sum_{i=2}^n \psi_{x_i}a_{ii}\vartheta_{x_i} \\
&\quad + (x_1 + \delta_0) \sum_{i=2}^n (a_{ii})_{x_i}\vartheta_{x_i} - (\nu+1)\psi^{-1}(x_1 + \delta_0) \sum_{j=1}^m \psi_{y_j}b_{jj}\vartheta_{y_j} \\
&\quad - (x_1 + \delta_0) \sum_{j=1}^m (b_{jj})_{y_j}\vartheta_{y_j}) + d_2(\vartheta) \\
&= 2\psi^{\nu+1}z_1z_6 + d_2(\vartheta)
\end{aligned}$$

olduğu görülür. Burada

$$\begin{aligned}
z_6 = & (\nu + 1)\psi^{-1}((x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j})) \\
& + (x_1 + \delta_0)(\sum_{i=2}^n (a_{ii})_{x_i}\vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j}\vartheta_{y_j})
\end{aligned}$$

dir. O halde, (2.10) eşitliğinde bulunan birinci, üçüncü, dördüncü ve beşinci terimler için  $2ab \geq -a^2 - b^2$  kullanılarak

$$\begin{aligned}
& \psi^{\nu+1}(z_1^2 + z_3^2 + 2z_1z_3 + 2z_1z_2) \\
= & \psi^{\nu+1}(z_1^2 + z_3^2) + \psi^{\nu+1}2z_1z_3 + \psi^{\nu+1}2z_1z_2 \\
= & \psi^{\nu+1}(z_1^2 + z_3^2) + \psi^{\nu+1}2z_1z_3 + \psi^{\nu+1}2z_1z_6 + d_2(\vartheta) \\
= & \psi^{\nu+1}(z_1^2 + z_3^2 + 2z_1z_3 + 2z_1z_6) + d_2(\vartheta) \\
= & \psi^{\nu+1}(z_1^2 + z_3^2 + 2z_1(z_3 + z_6)) + d_2(\vartheta) \\
\geq & \psi^{\nu+1}(z_1^2 + z_3^2 - z_1^2 - (z_3 + z_6)^2) + d_2(\vartheta) \\
= & \psi^{\nu+1}(z_3^2 - z_3^2 - 2z_3z_6 - z_6^2) + d_2(\vartheta) \\
= & \psi^{\nu+1}(-2z_3z_6 - z_6^2) + d_2(\vartheta) \\
= & \psi^{\nu+1}(-2z_3((\nu + 1)\psi^{-1}((x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} \\
& - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j})) + (x_1 + \delta_0)(\sum_{i=2}^n (a_{ii})_{x_i}\vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j}\vartheta_{y_j})) \\
& - ((\nu + 1)\psi^{-1}((x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} \\
& - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j})) + (x_1 + \delta_0)(\sum_{i=2}^n (a_{ii})_{x_i}\vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j}\vartheta_{y_j}))^2) + d_2(\vartheta)
\end{aligned}$$

$$\begin{aligned}
&= \psi^{\nu+1}(-2z_3(\nu+1)\psi^{-1}((x_1+\delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (x_1+\delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} \\
&\quad - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j})) - 2z_3(x_1+\delta_0)(\sum_{i=2}^n (a_{ii})_{x_i}\vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j}\vartheta_{y_j}) \\
&\quad - (\nu+1)^2\psi^{-2}((x_1+\delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (x_1+\delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j}))^2 \\
&\quad - 2(\nu+1)\psi^{-1}((x_1+\delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (x_1+\delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j})) \\
&\quad \times (x_1+\delta_0)(\sum_{i=2}^n (a_{ii})_{x_i}\vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j}\vartheta_{y_j}) \\
&\quad - (x_1+\delta_0)^2(\sum_{i=2}^n (a_{ii})_{x_i}\vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j}\vartheta_{y_j})^2 + d_2(\vartheta) \\
&= \psi^{\nu+1}(-2z_3(\nu+1)\psi^{-1}((x_1+\delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (x_1+\delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j})) \\
&\quad - (\nu+1)^2\psi^{-2}((x_1+\delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (x_1+\delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j}))^2 \\
&\quad - 2z_3(x_1+\delta_0)(\sum_{i=2}^n (a_{ii})_{x_i}\vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j}\vartheta_{y_j}) \\
&\quad - 2(\nu+1)\psi^{-1}((x_1+\delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (x_1+\delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j})) \\
&\quad \times (x_1+\delta_0)(\sum_{i=2}^n (a_{ii})_{x_i}\vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j}\vartheta_{y_j}) \\
&\quad - (x_1+\delta_0)^2(\sum_{i=2}^n (a_{ii})_{x_i}\vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j}\vartheta_{y_j})^2 + d_2(\vartheta)
\end{aligned}$$

elde edilir. Ayrıca

$$z_3 = -2\lambda\nu\psi^{-\nu-1}((x_1+\delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (x_1+\delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j}))$$

olduğu göz önünde bulundurularak

$$\begin{aligned}
& \psi^{\nu+1}(z_1^2 + z_3^2 + 2z_1z_3 + 2z_1z_2) \\
\geq & \psi^{\nu+1}(-2(-2\lambda\nu\psi^{-\nu-1}((x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j}))) \\
& \times (\nu + 1)\psi^{-1}((x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j})) \\
& - (\nu + 1)^2\psi^{-2}((x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j}))^2 \\
& - 2(-2\lambda\nu\psi^{-\nu-1}((x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j}))) \\
& \times (x_1 + \delta_0)(\sum_{i=2}^n (a_{ii})_{x_i}\vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j}\vartheta_{y_j}) \\
& - 2(\nu + 1)\psi^{-1}((x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j})) \\
& \times (x_1 + \delta_0)(\sum_{i=2}^n (a_{ii})_{x_i}\vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j}\vartheta_{y_j}) \\
& - (x_1 + \delta_0)^2(\sum_{i=2}^n (a_{ii})_{x_i}\vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j}\vartheta_{y_j})^2 + d_2(\vartheta) \\
= & \psi^{\nu+1}(4\lambda\nu(\nu + 1)\psi^{-\nu-2}((x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} \\
& + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j}))^2 \\
& - (\nu + 1)^2\psi^{-2}((x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j}))^2 \\
& + 4\lambda\nu(x_1 + \delta_0)\psi^{-\nu-1}((x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} \\
& + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j}))(\sum_{i=2}^n (a_{ii})_{x_i}\vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j}\vartheta_{y_j}) \\
& - 2(\nu + 1)(x_1 + \delta_0)\psi^{-1}((x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} \\
& + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j}))(\sum_{i=2}^n (a_{ii})_{x_i}\vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j}\vartheta_{y_j}) \\
& - (x_1 + \delta_0)^2(\sum_{i=2}^n (a_{ii})_{x_i}\vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j}\vartheta_{y_j})^2 + d_2(\vartheta)
\end{aligned}$$

$$\begin{aligned}
&= \psi^{\nu+1}((4\lambda\nu(\nu+1)\psi^{-\nu-2} - (\nu+1)^2\psi^{-2}) \\
&\quad \times ((x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j}))^2 \\
&\quad + (4\lambda\nu\psi^{-\nu-1} - 2(\nu+1)\psi^{-1}) \\
&\quad \times ((x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j})) \\
&\quad \times (x_1 + \delta_0)(\sum_{i=2}^n (a_{ii})_{x_i}\vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j}\vartheta_{y_j}) \\
&\quad - (x_1 + \delta_0)^2(\sum_{i=2}^n (a_{ii})_{x_i}\vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j}\vartheta_{y_j})^2) + d_2(\vartheta)
\end{aligned}$$

elde edilir. Aşağıda  $2ab \geq -\sigma_0 a^2 - \frac{b^2}{\sigma_0}$ ,  $\sigma_0 = (\nu+1)\psi^{-1} > 0$  eşitsizliği kullanılırsa,

$$\begin{aligned}
&2((x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j})) \\
&\quad \times (x_1 + \delta_0)(\sum_{i=2}^n (a_{ii})_{x_i}\vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j}\vartheta_{y_j}) \\
&\geq -\sigma_0((x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j}))^2 \\
&\quad - \frac{1}{\sigma_0}(x_1 + \delta_0)^2(\sum_{i=2}^n (a_{ii})_{x_i}\vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j}\vartheta_{y_j})^2
\end{aligned}$$

bulunur. Böylece

$$\begin{aligned}
&\psi^{\nu+1}(z_1^2 + z_3^2 + 2z_1z_3 + 2z_1z_2) \\
&\geq \psi^{\nu+1}((4\lambda\nu(\nu+1)\psi^{-\nu-2} - (\nu+1)^2\psi^{-2}) \\
&\quad \times ((x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j}))^2 \\
&\quad + (2\lambda\nu\psi^{-\nu-1} - (\nu+1)\psi^{-1}) \\
&\quad \times (-\sigma_0((x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j}))^2 \\
&\quad - \frac{1}{\sigma_0}(x_1 + \delta_0)^2(\sum_{i=2}^n (a_{ii})_{x_i}\vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j}\vartheta_{y_j})^2) \\
&\quad - (x_1 + \delta_0)^2(\sum_{i=2}^n (a_{ii})_{x_i}\vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j}\vartheta_{y_j})^2) + d_2(\vartheta)
\end{aligned}$$

$$\begin{aligned}
&= \psi^{\nu+1}((4\lambda\nu(\nu+1)\psi^{-\nu-2} - (\nu+1)^2\psi^{-2} - 2\lambda\nu\sigma_0\psi^{-\nu-1} + (\nu+1)\sigma_0\psi^{-1}) \\
&\quad \times ((x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j}))^2 \\
&\quad - \frac{1}{\sigma_0}(2\lambda\nu\psi^{-\nu-1} - (\nu+1)\psi^{-1})(x_1 + \delta_0)^2(\sum_{i=2}^n (a_{ii})_{x_i}\vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j}\vartheta_{y_j})^2 \\
&\quad - (x_1 + \delta_0)^2(\sum_{i=2}^n (a_{ii})_{x_i}\vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j}\vartheta_{y_j})^2 + d_2(\vartheta) \\
&= \psi^{\nu+1}(2\lambda\nu\psi^{-\nu-1}\sigma_0((x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j}))^2 \\
&\quad - \frac{1}{\sigma_0}2\lambda\nu\psi^{-\nu-1}(x_1 + \delta_0)^2(\sum_{i=2}^n (a_{ii})_{x_i}\vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j}\vartheta_{y_j})^2 + d_2(\vartheta) \\
&\geq -2\lambda\nu\psi(x_1 + \delta_0)^2(\sum_{i=2}^n (a_{ii})_{x_i}\vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j}\vartheta_{y_j})^2 + d_2(\vartheta)
\end{aligned} \tag{2.11}$$

elde edilir.

İkinci olarak, (2.10) eşitliğinde altıncı terim için

$$\begin{aligned}
2z_1(z_4 + z_5)\psi^{\nu+1} &= 2(2k(x_1 + \delta_0)\vartheta_\xi)(-E_0\vartheta + \xi\vartheta(x_1 + \delta_0))\psi^{\nu+1} \\
&= -4k(x_1 + \delta_0)E_0\vartheta\vartheta_\xi\psi^{\nu+1} + 4k(x_1 + \delta_0)^2\xi\vartheta\vartheta_\xi\psi^{\nu+1} \\
&= (-2k(x_1 + \delta_0)\psi^{\nu+1}E_0\vartheta^2)_\xi + (2k(x_1 + \delta_0)^2\xi\psi^{\nu+1}\vartheta^2)_\xi \\
&\quad - 2k(x_1 + \delta_0)^2\psi^{\nu+1}\vartheta^2
\end{aligned} \tag{2.12}$$

yazılabilir.

Üçüncü olarak:

$$\begin{aligned}
2z_3z_5\psi^{\nu+1} &= 2(-2\lambda\nu\psi^{-\nu-1}((x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} \\
&\quad + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j})))\xi\vartheta(x_1 + \delta_0)\psi^{\nu+1} \\
&= -4\lambda\nu\xi\vartheta(x_1 + \delta_0)((x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} \\
&\quad + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j})) \\
&= -4\lambda\nu\xi\psi_{x_1}\vartheta\vartheta_{x_1} - 4\lambda\nu\xi\vartheta(x_1 + \delta_0)^2(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j})
\end{aligned}$$

$$\begin{aligned}
&= -2\lambda\nu(\xi\psi_{x_1}\vartheta^2)_{x_1} - 2\lambda\nu(x_1 + \delta_0)^2 \sum_{i=2}^n (a_{ii}\psi_{x_i}\xi\vartheta^2)_{x_i} \\
&\quad + 2\lambda\nu(x_1 + \delta_0)^2 \xi\vartheta^2 \sum_{i=2}^n (a_{ii}\psi_{x_i})_{x_i} + 2\lambda\nu(x_1 + \delta_0)^2 \sum_{j=1}^m (b_{jj}\psi_{y_j}\xi\vartheta^2)_{y_j} \\
&\quad - 2\lambda\nu(x_1 + \delta_0)^2 \xi\vartheta^2 \sum_{j=1}^m (b_{jj}\psi_{y_j})_{y_j}
\end{aligned} \tag{2.13}$$

olur. Burada  $\psi_{x_1x_1} = 0$  olduğu göz önünde bulundurulmuştur.

Dördüncü olarak, (2.10) eşitliğindeki yedinci terim için

$$\begin{aligned}
2z_2z_3\psi^{\nu+1} &= 2(-(x_1 + \delta_0)^{-1}\vartheta_{x_1x_1} - (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\vartheta_{x_ix_i} - \sum_{j=1}^m b_{jj}\vartheta_{y_jy_j})) \\
&\quad \times (-2\lambda\nu\psi^{-\nu-1}((x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} \\
&\quad + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j})))\psi^{\nu+1} \\
&= 4\lambda\nu((x_1 + \delta_0)^{-1}\vartheta_{x_1x_1} + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\vartheta_{x_ix_i} - \sum_{j=1}^m b_{jj}\vartheta_{y_jy_j})) \\
&\quad \times ((x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j}))
\end{aligned}$$

elde edilir. Yukarıdaki eşitlikte çarpma işleminden dolayı ortaya çıkan dört terim aşağıda değerlendirilmiştir:

Birinci terim:

$$4\lambda\nu(x_1 + \delta_0)^{-2}\psi_{x_1}\vartheta_{x_1}\vartheta_{x_1x_1} = 2\lambda\nu\psi_{x_1}((x_1 + \delta_0)^{-2}\vartheta_{x_1}^2)_{x_1} + 4\lambda\nu\psi_{x_1}(x_1 + \delta_0)^{-3}\vartheta_{x_1}^2$$

olarak yazılabilir.

İkinci terim için

$$\begin{aligned}
&4\lambda\nu\vartheta_{x_1x_1}(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j}) \\
&= 2\lambda\nu \sum_{i=2}^n 2\psi_{x_i}a_{ii}\vartheta_{x_i}\vartheta_{x_1x_1} - 2\lambda\nu \sum_{j=1}^m 2\psi_{y_j}b_{jj}\vartheta_{y_j}\vartheta_{x_1x_1}
\end{aligned}$$

eşitliğinde

$$\begin{aligned}
2\psi_{x_i}a_{ii}\vartheta_{x_i}\vartheta_{x_1x_1} &= (\psi_{x_i}a_{ii}\vartheta_{x_i}\vartheta_{x_1})_{x_1} + (\psi_{x_i}a_{ii}\vartheta_{x_i}\vartheta_{x_1})_{x_1} - (\psi_{x_i}a_{ii}\vartheta_{x_1}^2)_{x_i} \\
&\quad - (\psi_{x_i}a_{ii})_{x_1}\vartheta_{x_i}\vartheta_{x_1} - (\psi_{x_i}a_{ii})_{x_1}\vartheta_{x_i}\vartheta_{x_1} + (\psi_{x_i}a_{ii})_{x_i}\vartheta_{x_1}^2 \\
&= 2(\psi_{x_i}a_{ii}\vartheta_{x_i}\vartheta_{x_1})_{x_1} - (\psi_{x_i}a_{ii}\vartheta_{x_1}^2)_{x_i} \\
&\quad - 2(\psi_{x_i}a_{ii})_{x_1}\vartheta_{x_i}\vartheta_{x_1} + (\psi_{x_i}a_{ii})_{x_i}\vartheta_{x_1}^2
\end{aligned}$$

ve

$$\begin{aligned}
2\psi_{y_j} b_{jj} \vartheta_{y_j} \vartheta_{x_1 x_1} &= (\psi_{y_j} b_{jj} \vartheta_{y_j} \vartheta_{x_1})_{x_1} + (\psi_{y_j} b_{jj} \vartheta_{y_j} \vartheta_{x_1})_{x_1} - (\psi_{y_j} b_{jj} \vartheta_{x_1}^2)_{y_j} \\
&\quad - (\psi_{y_j} b_{jj})_{x_1} \vartheta_{y_j} \vartheta_{x_1} - (\psi_{y_j} b_{jj})_{x_1} \vartheta_{y_j} \vartheta_{x_1} + (\psi_{y_j} b_{jj})_{y_j} \vartheta_{x_1}^2 \\
&= 2(\psi_{y_j} b_{jj} \vartheta_{y_j} \vartheta_{x_1})_{x_1} - (\psi_{y_j} b_{jj} \vartheta_{x_1}^2)_{y_j} \\
&\quad - 2(\psi_{y_j} b_{jj})_{x_1} \vartheta_{y_j} \vartheta_{x_1} + (\psi_{y_j} b_{jj})_{y_j} \vartheta_{x_1}^2
\end{aligned}$$

terimleri yerine yazılırsa

$$\begin{aligned}
&4\lambda\nu \vartheta_{x_1 x_1} \left( \sum_{i=2}^n a_{ii} \psi_{x_i} \vartheta_{x_i} - \sum_{j=1}^m b_{jj} \psi_{y_j} \vartheta_{y_j} \right) \\
&= 2\lambda\nu \sum_{i=2}^n (2(\psi_{x_i} a_{ii} \vartheta_{x_i} \vartheta_{x_1})_{x_1} - (\psi_{x_i} a_{ii} \vartheta_{x_1}^2)_{x_i} - 2(\psi_{x_i} a_{ii})_{x_1} \vartheta_{x_i} \vartheta_{x_1} + (\psi_{x_i} a_{ii})_{x_i} \vartheta_{x_1}^2) \\
&\quad - 2\lambda\nu \sum_{j=1}^m (2(\psi_{y_j} b_{jj} \vartheta_{y_j} \vartheta_{x_1})_{x_1} - (\psi_{y_j} b_{jj} \vartheta_{x_1}^2)_{y_j} - 2(\psi_{y_j} b_{jj})_{x_1} \vartheta_{y_j} \vartheta_{x_1} + (\psi_{y_j} b_{jj})_{y_j} \vartheta_{x_1}^2)
\end{aligned}$$

bulunur.

Üçüncü terim için

$$\begin{aligned}
&4\lambda\nu \left( \sum_{i=2}^n a_{ii} \vartheta_{x_i x_i} - \sum_{j=1}^m b_{jj} \vartheta_{y_j y_j} \right) \psi_{x_1} \vartheta_{x_1} \\
&= 2\lambda\nu \sum_{i=2}^n 2\psi_{x_1} a_{ii} \vartheta_{x_1} \vartheta_{x_i x_i} - 2\lambda\nu \sum_{j=1}^m 2\psi_{x_1} b_{jj} \vartheta_{x_1} \vartheta_{y_j y_j}
\end{aligned}$$

eşitliğinde

$$\begin{aligned}
2\psi_{x_1} a_{ii} \vartheta_{x_1} \vartheta_{x_i x_i} &= (\psi_{x_1} a_{ii} \vartheta_{x_1} \vartheta_{x_i})_{x_i} + (\psi_{x_1} a_{ii} \vartheta_{x_1} \vartheta_{x_i})_{x_i} - (\psi_{x_1} a_{ii} \vartheta_{x_i}^2)_{x_1} \\
&\quad - (\psi_{x_1} a_{ii})_{x_i} \vartheta_{x_1} \vartheta_{x_i} - (\psi_{x_1} a_{ii})_{x_i} \vartheta_{x_1} \vartheta_{x_i} + (\psi_{x_1} a_{ii})_{x_1} \vartheta_{x_i}^2 \\
&= 2(\psi_{x_1} a_{ii} \vartheta_{x_1} \vartheta_{x_i})_{x_i} - (\psi_{x_1} a_{ii} \vartheta_{x_i}^2)_{x_1} \\
&\quad - 2(\psi_{x_1} a_{ii})_{x_i} \vartheta_{x_1} \vartheta_{x_i} + (\psi_{x_1} a_{ii})_{x_1} \vartheta_{x_i}^2
\end{aligned}$$

ve

$$\begin{aligned}
2\psi_{x_1} b_{jj} \vartheta_{x_1} \vartheta_{y_j y_j} &= (\psi_{x_1} b_{jj} \vartheta_{x_1} \vartheta_{y_j})_{y_j} + (\psi_{x_1} b_{jj} \vartheta_{x_1} \vartheta_{y_j})_{y_j} - (\psi_{x_1} b_{jj} \vartheta_{y_j}^2)_{x_1} \\
&\quad - (\psi_{x_1} b_{jj})_{y_j} \vartheta_{x_1} \vartheta_{y_j} - (\psi_{x_1} b_{jj})_{y_j} \vartheta_{x_1} \vartheta_{y_j} + (\psi_{x_1} b_{jj})_{x_1} \vartheta_{y_j}^2 \\
&= 2(\psi_{x_1} b_{jj} \vartheta_{x_1} \vartheta_{y_j})_{y_j} - (\psi_{x_1} b_{jj} \vartheta_{y_j}^2)_{x_1} \\
&\quad - 2(\psi_{x_1} b_{jj})_{y_j} \vartheta_{x_1} \vartheta_{y_j} + (\psi_{x_1} b_{jj})_{x_1} \vartheta_{y_j}^2
\end{aligned}$$

eşitlikleri kullanılarak

$$\begin{aligned}
& 4\lambda\nu\left(\sum_{i=2}^n a_{ii}\vartheta_{x_i x_i} - \sum_{j=1}^m b_{jj}\vartheta_{y_j y_j}\right)\psi_{x_1}\vartheta_{x_1} \\
&= 2\lambda\nu\sum_{i=2}^n (2(\psi_{x_1}a_{ii}\vartheta_{x_1}\vartheta_{x_i})_{x_i} - (\psi_{x_1}a_{ii}\vartheta_{x_i}^2)_{x_1} - 2(\psi_{x_1}a_{ii})_{x_i}\vartheta_{x_1}\vartheta_{x_i} + (\psi_{x_1}a_{ii})_{x_1}\vartheta_{x_i}^2) \\
&\quad - 2\lambda\nu\sum_{j=1}^m (2(\psi_{x_1}b_{jj}\vartheta_{x_1}\vartheta_{y_j})_{y_j} - (\psi_{x_1}b_{jj}\vartheta_{y_j}^2)_{x_1} - 2(\psi_{x_1}b_{jj})_{y_j}\vartheta_{x_1}\vartheta_{y_j} + (\psi_{x_1}b_{jj})_{x_1}\vartheta_{y_j}^2)
\end{aligned}$$

elde edilir.

Dördüncü terim için

$$\begin{aligned}
& 4\lambda\nu(x_1 + \delta_0)^2\left(\sum_{s=2}^n a_{ss}\vartheta_{x_s x_s} - \sum_{k=1}^m b_{kk}\vartheta_{y_k y_k}\right)\left(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j}\right) \\
&= 2\lambda\nu(x_1 + \delta_0)^2\sum_{s,i=2}^n 2a_{ss}a_{ii}\psi_{x_i}\vartheta_{x_i}\vartheta_{x_s x_s} - 2\lambda\nu(x_1 + \delta_0)^2\sum_{s=2}^n\sum_{j=1}^m 2a_{ss}b_{jj}\psi_{y_j}\vartheta_{y_j}\vartheta_{x_s x_s} \\
&\quad - 2\lambda\nu(x_1 + \delta_0)^2\sum_{k=1}^m\sum_{i=2}^n 2b_{kk}a_{ii}\psi_{x_i}\vartheta_{x_i}\vartheta_{y_k y_k} + 2\lambda\nu(x_1 + \delta_0)^2\sum_{k,j=1}^m 2b_{kk}b_{jj}\psi_{y_j}\vartheta_{y_j}\vartheta_{y_k y_k}
\end{aligned}$$

eşitliğinde yer alan terimleri aşağıda değerlendirelim:

$$\begin{aligned}
2a_{ss}a_{ii}\psi_{x_i}\vartheta_{x_i}\vartheta_{x_s x_s} &= (a_{ss}a_{ii}\psi_{x_i}\vartheta_{x_i}\vartheta_{x_s})_{x_s} + (a_{ss}a_{ii}\psi_{x_i}\vartheta_{x_i}\vartheta_{x_s})_{x_s} - (a_{ss}a_{ii}\psi_{x_i}\vartheta_{x_s}^2)_{x_i} \\
&\quad - (a_{ss}a_{ii}\psi_{x_i})_{x_s}\vartheta_{x_i}\vartheta_{x_s} - (a_{ss}a_{ii}\psi_{x_i})_{x_s}\vartheta_{x_i}\vartheta_{x_s} + (a_{ss}a_{ii}\psi_{x_i})_{x_i}\vartheta_{x_s}^2 \\
&= 2(a_{ss}a_{ii}\psi_{x_i}\vartheta_{x_i}\vartheta_{x_s})_{x_s} - (a_{ss}a_{ii}\psi_{x_i}\vartheta_{x_s}^2)_{x_i} \\
&\quad - 2(a_{ss}a_{ii}\psi_{x_i})_{x_s}\vartheta_{x_i}\vartheta_{x_s} + (a_{ss}a_{ii}\psi_{x_i})_{x_i}\vartheta_{x_s}^2,
\end{aligned}$$

$$\begin{aligned}
2a_{ss}b_{jj}\psi_{y_j}\vartheta_{y_j}\vartheta_{x_s x_s} &= (a_{ss}b_{jj}\psi_{y_j}\vartheta_{y_j}\vartheta_{x_s})_{x_s} + (a_{ss}b_{jj}\psi_{y_j}\vartheta_{y_j}\vartheta_{x_s})_{x_s} - (a_{ss}b_{jj}\psi_{y_j}\vartheta_{x_s}^2)_{y_j} \\
&\quad - (a_{ss}b_{jj}\psi_{y_j})_{x_s}\vartheta_{y_j}\vartheta_{x_s} - (a_{ss}b_{jj}\psi_{y_j})_{x_s}\vartheta_{y_j}\vartheta_{x_s} + (a_{ss}b_{jj}\psi_{y_j})_{y_j}\vartheta_{x_s}^2 \\
&= 2(a_{ss}b_{jj}\psi_{y_j}\vartheta_{y_j}\vartheta_{x_s})_{x_s} - (a_{ss}b_{jj}\psi_{y_j}\vartheta_{x_s}^2)_{y_j} \\
&\quad - 2(a_{ss}b_{jj}\psi_{y_j})_{x_s}\vartheta_{y_j}\vartheta_{x_s} + (a_{ss}b_{jj}\psi_{y_j})_{y_j}\vartheta_{x_s}^2,
\end{aligned}$$

$$\begin{aligned}
2b_{kk}a_{ii}\psi_{x_i}\vartheta_{x_i}\vartheta_{y_k y_k} &= (b_{kk}a_{ii}\psi_{x_i}\vartheta_{x_i}\vartheta_{y_k})_{y_k} + (b_{kk}a_{ii}\psi_{x_i}\vartheta_{x_i}\vartheta_{y_k})_{y_k} - (b_{kk}a_{ii}\psi_{x_i}\vartheta_{y_k}^2)_{x_i} \\
&\quad - (b_{kk}a_{ii}\psi_{x_i})_{y_k}\vartheta_{x_i}\vartheta_{y_k} - (b_{kk}a_{ii}\psi_{x_i})_{y_k}\vartheta_{x_i}\vartheta_{y_k} + (b_{kk}a_{ii}\psi_{x_i})_{x_i}\vartheta_{y_k}^2 \\
&= 2(b_{kk}a_{ii}\psi_{x_i}\vartheta_{x_i}\vartheta_{y_k})_{y_k} - (b_{kk}a_{ii}\psi_{x_i}\vartheta_{y_k}^2)_{x_i} \\
&\quad - 2(b_{kk}a_{ii}\psi_{x_i})_{y_k}\vartheta_{x_i}\vartheta_{y_k} + (b_{kk}a_{ii}\psi_{x_i})_{x_i}\vartheta_{y_k}^2,
\end{aligned}$$

$$\begin{aligned}
2b_{kk}b_{jj}\psi_{y_j}\vartheta_{y_j}\vartheta_{y_k y_k} &= (b_{kk}b_{jj}\psi_{y_j}\vartheta_{y_j}\vartheta_{y_k})_{y_k} + (b_{kk}b_{jj}\psi_{y_j}\vartheta_{y_j}\vartheta_{y_k})_{y_k} - (b_{kk}b_{jj}\psi_{y_j}\vartheta_{y_k}^2)_{y_j} \\
&\quad - (b_{kk}b_{jj}\psi_{y_j})_{y_k}\vartheta_{y_j}\vartheta_{y_k} - (b_{kk}b_{jj}\psi_{y_j})_{y_k}\vartheta_{y_j}\vartheta_{y_k} + (b_{kk}b_{jj}\psi_{y_j})_{y_j}\vartheta_{y_k}^2 \\
&= 2(b_{kk}b_{jj}\psi_{y_j}\vartheta_{y_j}\vartheta_{y_k})_{y_k} - (b_{kk}b_{jj}\psi_{y_j}\vartheta_{y_k}^2)_{y_j} \\
&\quad - 2(b_{kk}b_{jj}\psi_{y_j})_{y_k}\vartheta_{y_j}\vartheta_{y_k} + (b_{kk}b_{jj}\psi_{y_j})_{y_j}\vartheta_{y_k}^2.
\end{aligned}$$

Buradan dördüncü terim

$$\begin{aligned}
&4\lambda\nu(x_1 + \delta_0)^2 \left( \sum_{s=2}^n a_{ss}\vartheta_{x_s x_s} - \sum_{k=1}^m b_{kk}\vartheta_{y_k y_k} \right) \left( \sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j} \right) \\
= &2\lambda\nu(x_1 + \delta_0)^2 \sum_{s,i=2}^n (2(a_{ss}a_{ii}\psi_{x_i}\vartheta_{x_i}\vartheta_{x_s})_{x_s} - (a_{ss}a_{ii}\psi_{x_i}\vartheta_{x_s}^2)_{x_i} \\
&- 2(a_{ss}a_{ii}\psi_{x_i})_{x_s}\vartheta_{x_i}\vartheta_{x_s} + (a_{ss}a_{ii}\psi_{x_i})_{x_i}\vartheta_{x_s}^2) \\
&- 2\lambda\nu(x_1 + \delta_0)^2 \sum_{s=2}^n \sum_{j=1}^m (2(a_{ss}b_{jj}\psi_{y_j}\vartheta_{y_j}\vartheta_{x_s})_{x_s} - (a_{ss}b_{jj}\psi_{y_j}\vartheta_{x_s}^2)_{y_j} \\
&- 2(a_{ss}b_{jj}\psi_{y_j})_{x_s}\vartheta_{y_j}\vartheta_{x_s} + (a_{ss}b_{jj}\psi_{y_j})_{y_j}\vartheta_{x_s}^2) \\
&- 2\lambda\nu(x_1 + \delta_0)^2 \sum_{k=1}^m \sum_{i=2}^n (2(b_{kk}a_{ii}\psi_{x_i}\vartheta_{x_i}\vartheta_{y_k})_{y_k} - (b_{kk}a_{ii}\psi_{x_i}\vartheta_{y_k}^2)_{x_i} \\
&- 2(b_{kk}a_{ii}\psi_{x_i})_{y_k}\vartheta_{x_i}\vartheta_{y_k} + (b_{kk}a_{ii}\psi_{x_i})_{x_i}\vartheta_{y_k}^2) \\
&+ 2\lambda\nu(x_1 + \delta_0)^2 \sum_{k,j=1}^m (2(b_{kk}b_{jj}\psi_{y_j}\vartheta_{y_j}\vartheta_{y_k})_{y_k} - (b_{kk}b_{jj}\psi_{y_j}\vartheta_{y_k}^2)_{y_j} \\
&- 2(b_{kk}b_{jj}\psi_{y_j})_{y_k}\vartheta_{y_j}\vartheta_{y_k} + (b_{kk}b_{jj}\psi_{y_j})_{y_j}\vartheta_{y_k}^2)
\end{aligned}$$

olarak bulunur. Böylece

$$\begin{aligned}
&2z_2 z_3 \psi^{\nu+1} \\
= &2\lambda\nu\psi_{x_1}((x_1 + \delta_0)^{-2}\vartheta_{x_1}^2)_{x_1} + 4\lambda\nu\psi_{x_1}(x_1 + \delta_0)^{-3}\vartheta_{x_1}^2 \\
&+ 2\lambda\nu \left( \sum_{i=2}^n (2(\psi_{x_i}a_{ii}\vartheta_{x_i}\vartheta_{x_1})_{x_1} - (\psi_{x_i}a_{ii}\vartheta_{x_1}^2)_{x_i} - 2(\psi_{x_i}a_{ii})_{x_1}\vartheta_{x_i}\vartheta_{x_1} + (\psi_{x_i}a_{ii})_{x_i}\vartheta_{x_1}^2) \right. \\
&- \sum_{j=1}^m (2(\psi_{y_j}b_{jj}\vartheta_{y_j}\vartheta_{x_1})_{x_1} - (\psi_{y_j}b_{jj}\vartheta_{x_1}^2)_{y_j} - 2(\psi_{y_j}b_{jj})_{x_1}\vartheta_{y_j}\vartheta_{x_1} + (\psi_{y_j}b_{jj})_{y_j}\vartheta_{x_1}^2) \\
&+ \sum_{i=2}^n (2(\psi_{x_1}a_{ii}\vartheta_{x_1}\vartheta_{x_i})_{x_i} - (\psi_{x_1}a_{ii}\vartheta_{x_i}^2)_{x_1} - 2(\psi_{x_1}a_{ii})_{x_i}\vartheta_{x_1}\vartheta_{x_i} + (\psi_{x_1}a_{ii})_{x_1}\vartheta_{x_i}^2) \\
&\left. - \sum_{j=1}^m (2(\psi_{x_1}b_{jj}\vartheta_{x_1}\vartheta_{y_j})_{y_j} - (\psi_{x_1}b_{jj}\vartheta_{y_j}^2)_{x_1} - 2(\psi_{x_1}b_{jj})_{y_j}\vartheta_{x_1}\vartheta_{y_j} + (\psi_{x_1}b_{jj})_{x_1}\vartheta_{y_j}^2) \right)
\end{aligned}$$

$$\begin{aligned}
& +2\lambda\nu(x_1 + \delta_0)^2 \left( \sum_{s,i=2}^n (2(a_{ss}a_{ii}\psi_{x_i}\vartheta_{x_i}\vartheta_{x_s})_{x_s} - (a_{ss}a_{ii}\psi_{x_i}\vartheta_{x_s}^2)_{x_i} \right. \\
& - 2(a_{ss}a_{ii}\psi_{x_i})_{x_s}\vartheta_{x_i}\vartheta_{x_s} + (a_{ss}a_{ii}\psi_{x_i})_{x_i}\vartheta_{x_s}^2) \\
& - \sum_{s=2}^n \sum_{j=1}^m (2(a_{ss}b_{jj}\psi_{y_j}\vartheta_{y_j}\vartheta_{x_s})_{x_s} - (a_{ss}b_{jj}\psi_{y_j}\vartheta_{x_s}^2)_{y_j} \\
& - 2(a_{ss}b_{jj}\psi_{y_j})_{x_s}\vartheta_{y_j}\vartheta_{x_s} + (a_{ss}b_{jj}\psi_{y_j})_{y_j}\vartheta_{x_s}^2) \\
& - \sum_{k=1}^m \sum_{i=2}^n (2(b_{kk}a_{ii}\psi_{x_i}\vartheta_{x_i}\vartheta_{y_k})_{y_k} - (b_{kk}a_{ii}\psi_{x_i}\vartheta_{y_k}^2)_{x_i} \\
& - 2(b_{kk}a_{ii}\psi_{x_i})_{y_k}\vartheta_{x_i}\vartheta_{y_k} + (b_{kk}a_{ii}\psi_{x_i})_{x_i}\vartheta_{y_k}^2) \\
& + \sum_{k,j=1}^m (2(b_{kk}b_{jj}\psi_{y_j}\vartheta_{y_j}\vartheta_{y_k})_{y_k} - (b_{kk}b_{jj}\psi_{y_j}\vartheta_{y_k}^2)_{y_j} \\
& - 2(b_{kk}b_{jj}\psi_{y_j})_{y_k}\vartheta_{y_j}\vartheta_{y_k} + (b_{kk}b_{jj}\psi_{y_j})_{y_j}\vartheta_{y_k}^2) \Big) \tag{2.14}
\end{aligned}$$

elde edilir.

Beşinci olarak:

$$\begin{aligned}
& 2z_3z_4\psi^{\nu+1} \\
& = 2(-2\lambda\nu\psi^{-\nu-1}((x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j}))) \\
& \quad \times (-E_0\vartheta)\psi^{\nu+1} \\
& = 4\lambda\nu E_0\vartheta((x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} - \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j})) \\
& = 4\lambda\nu E_0\vartheta(x_1 + \delta_0)^{-1}\psi_{x_1}\vartheta_{x_1} + 4\lambda\nu E_0\vartheta(x_1 + \delta_0) \sum_{i=2}^n a_{ii}\psi_{x_i}\vartheta_{x_i} \\
& \quad - 4\lambda\nu E_0\vartheta(x_1 + \delta_0) \sum_{j=1}^m b_{jj}\psi_{y_j}\vartheta_{y_j} \\
& = 2\lambda\nu\psi_{x_1}(E_0(x_1 + \delta_0)^{-1}\vartheta^2)_{x_1} - 2\lambda\nu\psi_{x_1}(E_0(x_1 + \delta_0)^{-1})_{x_1}\vartheta^2 \\
& \quad + 2\lambda\nu \sum_{i=2}^n (E_0(x_1 + \delta_0)a_{ii}\psi_{x_i}\vartheta^2)_{x_i} - 2\lambda\nu \sum_{i=2}^n (E_0(x_1 + \delta_0)a_{ii}\psi_{x_i})_{x_i}\vartheta^2 \\
& \quad - 2\lambda\nu \sum_{j=1}^m (E_0(x_1 + \delta_0)b_{jj}\psi_{y_j}\vartheta^2)_{y_j} + 2\lambda\nu \sum_{j=1}^m (E_0(x_1 + \delta_0)b_{jj}\psi_{y_j})_{y_j}\vartheta^2 \\
& = 2\lambda\nu\psi_{x_1}((x_1 + \delta_0)^{-1}E_0\vartheta^2)_{x_1} - 2\lambda\nu\psi_{x_1}((x_1 + \delta_0)^{-1}E_0)_{x_1}\vartheta^2 \\
& \quad + 2\lambda\nu(x_1 + \delta_0)(\sum_{i=2}^n ((a_{ii}\psi_{x_i}E_0\vartheta^2)_{x_i} - (a_{ii}\psi_{x_i}E_0)_{x_i}\vartheta^2) \\
& \quad - \sum_{j=1}^m ((b_{jj}\psi_{y_j}E_0\vartheta^2)_{y_j} - (b_{jj}\psi_{y_j}E_0)_{y_j}\vartheta^2)) \tag{2.15}
\end{aligned}$$

bulunur.

(2.11)-(2.15) kullanılarak,

$$\begin{aligned}
& \psi^{\nu+1}(z_1^2 + (z_2 + z_4 + z_5)^2 + z_3^2 + 2z_1z_2 + 2z_1z_3 \\
& + 2z_1(z_4 + z_5) + 2z_2z_3 + 2z_3(z_4 + z_5)) \\
\geq & -2\lambda\nu\psi(x_1 + \delta_0)^2 \left( \sum_{i=2}^n (a_{ii})_{x_i} \vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j} \vartheta_{y_j} \right)^2 + d_2(\vartheta) \\
& - (2k(x_1 + \delta_0)\psi^{\nu+1}E_0\vartheta^2)_\xi + (2k(x_1 + \delta_0)^2\xi\psi^{\nu+1}\vartheta^2)_\xi - 2k(x_1 + \delta_0)^2\psi^{\nu+1}\vartheta^2 \\
& - 2\lambda\nu(\xi\psi_{x_1}\vartheta^2)_{x_1} - 2\lambda\nu(x_1 + \delta_0)^2 \sum_{i=2}^n (a_{ii}\psi_{x_i}\xi\vartheta^2)_{x_i} \\
& + 2\lambda\nu(x_1 + \delta_0)^2\xi\vartheta^2 \sum_{i=2}^n (a_{ii}\psi_{x_i})_{x_i} + 2\lambda\nu(x_1 + \delta_0)^2 \sum_{j=1}^m (b_{jj}\psi_{y_j}\xi\vartheta^2)_{y_j} \\
& - 2\lambda\nu(x_1 + \delta_0)^2\xi\vartheta^2 \sum_{j=1}^m (b_{jj}\psi_{y_j})_{y_j} \\
& + 2\lambda\nu\psi_{x_1}((x_1 + \delta_0)^{-2}\vartheta_{x_1}^2)_{x_1} + 4\lambda\nu\psi_{x_1}(x_1 + \delta_0)^{-3}\vartheta_{x_1}^2 \\
& + 2\lambda\nu \left( \sum_{i=2}^n (2(\psi_{x_i}a_{ii}\vartheta_{x_i}\vartheta_{x_1})_{x_1} - (\psi_{x_i}a_{ii}\vartheta_{x_1}^2)_{x_i} - 2(\psi_{x_i}a_{ii})_{x_1}\vartheta_{x_i}\vartheta_{x_1} + (\psi_{x_i}a_{ii})_{x_i}\vartheta_{x_1}^2) \right. \\
& \left. - \sum_{j=1}^m (2(\psi_{y_j}b_{jj}\vartheta_{y_j}\vartheta_{x_1})_{x_1} - (\psi_{y_j}b_{jj}\vartheta_{x_1}^2)_{y_j} - 2(\psi_{y_j}b_{jj})_{x_1}\vartheta_{y_j}\vartheta_{x_1} + (\psi_{y_j}b_{jj})_{y_j}\vartheta_{x_1}^2) \right. \\
& \left. + \sum_{i=2}^n (2(\psi_{x_1}a_{ii}\vartheta_{x_1}\vartheta_{x_i})_{x_i} - (\psi_{x_1}a_{ii}\vartheta_{x_i}^2)_{x_1} - 2(\psi_{x_1}a_{ii})_{x_i}\vartheta_{x_1}\vartheta_{x_i} + (\psi_{x_1}a_{ii})_{x_1}\vartheta_{x_i}^2) \right. \\
& \left. - \sum_{j=1}^m (2(\psi_{x_1}b_{jj}\vartheta_{x_1}\vartheta_{y_j})_{y_j} - (\psi_{x_1}b_{jj}\vartheta_{y_j}^2)_{x_1} - 2(\psi_{x_1}b_{jj})_{y_j}\vartheta_{x_1}\vartheta_{y_j} + (\psi_{x_1}b_{jj})_{x_1}\vartheta_{y_j}^2) \right) \\
& + 2\lambda\nu(x_1 + \delta_0)^2 \left( \sum_{s,i=2}^n (2(a_{ss}a_{ii}\psi_{x_i}\vartheta_{x_i}\vartheta_{x_s})_{x_s} - (a_{ss}a_{ii}\psi_{x_i}\vartheta_{x_s}^2)_{x_i} \right. \\
& \left. - 2(a_{ss}a_{ii}\psi_{x_i})_{x_s}\vartheta_{x_i}\vartheta_{x_s} + (a_{ss}a_{ii}\psi_{x_i})_{x_i}\vartheta_{x_s}^2) \right. \\
& \left. - \sum_{s=2}^n \sum_{j=1}^m (2(a_{ss}b_{jj}\psi_{y_j}\vartheta_{y_j}\vartheta_{x_s})_{x_s} - (a_{ss}b_{jj}\psi_{y_j}\vartheta_{x_s}^2)_{y_j} \right. \\
& \left. - 2(a_{ss}b_{jj}\psi_{y_j})_{x_s}\vartheta_{y_j}\vartheta_{x_s} + (a_{ss}b_{jj}\psi_{y_j})_{y_j}\vartheta_{x_s}^2) \right. \\
& \left. - \sum_{k=1}^m \sum_{i=2}^n (2(b_{kk}a_{ii}\psi_{x_i}\vartheta_{x_i}\vartheta_{y_k})_{y_k} - (b_{kk}a_{ii}\psi_{x_i}\vartheta_{y_k}^2)_{x_i} \right. \\
& \left. - 2(b_{kk}a_{ii}\psi_{x_i})_{y_k}\vartheta_{x_i}\vartheta_{y_k} + (b_{kk}a_{ii}\psi_{x_i})_{x_i}\vartheta_{y_k}^2) \right. \\
& \left. + \sum_{k,j=1}^m (2(b_{kk}b_{jj}\psi_{y_j}\vartheta_{y_j}\vartheta_{y_k})_{y_k} - (b_{kk}b_{jj}\psi_{y_j}\vartheta_{y_k}^2)_{y_j} \right. \\
& \left. - 2(b_{kk}b_{jj}\psi_{y_j})_{y_k}\vartheta_{y_j}\vartheta_{y_k} + (b_{kk}b_{jj}\psi_{y_j})_{y_j}\vartheta_{y_k}^2) \right)
\end{aligned}$$

$$\begin{aligned}
& +2\lambda\nu\psi_{x_1}((x_1 + \delta_0)^{-1}E_0\vartheta^2)_{x_1} - 2\lambda\nu\psi_{x_1}((x_1 + \delta_0)^{-1}E_0)_{x_1}\vartheta^2 \\
& +2\lambda\nu(x_1 + \delta_0)\left(\sum_{i=2}^n((a_{ii}\psi_{x_i}E_0\vartheta^2)_{x_i} - (a_{ii}\psi_{x_i}E_0)_{x_i}\vartheta^2)\right) \\
& - \sum_{j=1}^m((b_{jj}\psi_{y_j}E_0\vartheta^2)_{y_j} - (b_{jj}\psi_{y_j}E_0)_{y_j}\vartheta^2)) \\
= & -2\lambda\nu\psi(x_1 + \delta_0)^2\left(\sum_{i=2}^n(a_{ii})_{x_i}\vartheta_{x_i} - \sum_{j=1}^m(b_{jj})_{y_j}\vartheta_{y_j}\right)^2 - 2k(x_1 + \delta_0)^2\psi^{\nu+1}\vartheta^2 \\
& +2\lambda\nu(x_1 + \delta_0)^2\xi\vartheta^2\sum_{i=2}^n(a_{ii}\psi_{x_i})_{x_i} - 2\lambda\nu(x_1 + \delta_0)^2\xi\vartheta^2\sum_{j=1}^m(b_{jj}\psi_{y_j})_{y_j} \\
& +4\lambda\nu\psi_{x_1}(x_1 + \delta_0)^{-3}\vartheta_{x_1}^2 \\
& -2\lambda\nu\left(\sum_{i=2}^n(2(\psi_{x_i}a_{ii})_{x_1}\vartheta_{x_i}\vartheta_{x_1} - (\psi_{x_i}a_{ii})_{x_i}\vartheta_{x_1}^2 + 2(\psi_{x_1}a_{ii})_{x_i}\vartheta_{x_1}\vartheta_{x_i} - (\psi_{x_1}a_{ii})_{x_1}\vartheta_{x_i}^2)\right) \\
& - \sum_{j=1}^m(2(\psi_{y_j}b_{jj})_{x_1}\vartheta_{y_j}\vartheta_{x_1} - (\psi_{y_j}b_{jj})_{y_j}\vartheta_{x_1}^2 + 2(\psi_{x_1}b_{jj})_{y_j}\vartheta_{x_1}\vartheta_{y_j} - (\psi_{x_1}b_{jj})_{x_1}\vartheta_{y_j}^2)) \\
& -2\lambda\nu(x_1 + \delta_0)^2\left(\sum_{s,i=2}^n(2(a_{ss}a_{ii}\psi_{x_i})_{x_s}\vartheta_{x_i}\vartheta_{x_s} - (a_{ss}a_{ii}\psi_{x_i})_{x_i}\vartheta_{x_s}^2)\right) \\
& - \sum_{s=2}^n\sum_{j=1}^m(2(a_{ss}b_{jj}\psi_{y_j})_{x_s}\vartheta_{y_j}\vartheta_{x_s} - (a_{ss}b_{jj}\psi_{y_j})_{y_j}\vartheta_{x_s}^2) \\
& - \sum_{k=1}^m\sum_{i=2}^n(2(b_{kk}a_{ii}\psi_{x_i})_{y_k}\vartheta_{x_i}\vartheta_{y_k} - (b_{kk}a_{ii}\psi_{x_i})_{x_i}\vartheta_{y_k}^2) \\
& + \sum_{k,j=1}^m(2(b_{kk}b_{jj}\psi_{y_j})_{y_k}\vartheta_{y_j}\vartheta_{y_k} - (b_{kk}b_{jj}\psi_{y_j})_{y_j}\vartheta_{y_k}^2)) \\
& -2\lambda\nu(\psi_{x_1}((x_1 + \delta_0)^{-1}E_0)_{x_1} + (x_1 + \delta_0)\left(\sum_{i=2}^n(a_{ii}\psi_{x_i}E_0)_{x_i} - \sum_{j=1}^m(b_{jj}\psi_{y_j}E_0)_{y_j}\right))\vartheta^2 \\
& +d_1(\vartheta) + d_2(\vartheta) \\
= & 4\lambda\nu\delta(x_1 + \delta_0)^{-3}\vartheta_{x_1}^2 \\
& -2\lambda\nu\left(\sum_{i=2}^n(2(\psi_{x_i}a_{ii})_{x_1}\vartheta_{x_i}\vartheta_{x_1} - (a_{ii})_{x_i}\psi_{x_i}\vartheta_{x_1}^2 + \delta(2(a_{ii})_{x_i}\vartheta_{x_1}\vartheta_{x_i} - (a_{ii})_{x_1}\vartheta_{x_i}^2))\right) \\
& - \sum_{j=1}^m(2(\psi_{y_j}b_{jj})_{x_1}\vartheta_{y_j}\vartheta_{x_1} - (b_{jj})_{y_j}\psi_{y_j}\vartheta_{x_1}^2 + \delta(2(b_{jj})_{y_j}\vartheta_{x_1}\vartheta_{y_j} - (b_{jj})_{x_1}\vartheta_{y_j}^2))) \\
& +2\lambda\nu\sum_{i=2}^na_{ii}\psi_{x_ix_i}\vartheta_{x_1}^2 - 2\lambda\nu\sum_{j=1}^mb_{jj}\psi_{y_jy_j}\vartheta_{x_1}^2 + I
\end{aligned}$$

$$\begin{aligned}
& -2(x_1 + \delta_0)^2 (\lambda \nu \psi (\sum_{i=2}^n (a_{ii})_{x_i} \vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j} \vartheta_{y_j})^2 \\
& + (k \psi^{\nu+1} - \lambda \nu \xi (\sum_{i=2}^n (a_{ii} \psi_{x_i})_{x_i} - \sum_{j=1}^m (b_{jj} \psi_{y_j})_{y_j})) \vartheta^2) \\
& - 2\lambda \nu (\delta ((x_1 + \delta_0)^{-1} E_0)_{x_1} + (x_1 + \delta_0) (\sum_{i=2}^n (a_{ii} \psi_{x_i} E_0)_{x_i} - \sum_{j=1}^m (b_{jj} \psi_{y_j} E_0)_{y_j})) \vartheta^2 \\
& + d_1(\vartheta) + d_2(\vartheta)
\end{aligned} \tag{2.16}$$

elde edilir. Burada  $\psi_{x_1} = \delta$  olduğu dikkate alınmıştır ve

$$\begin{aligned}
I = & -2\lambda \nu (x_1 + \delta_0)^2 (\sum_{s,i=2}^n (2(a_{ss} a_{ii} \psi_{x_i})_{x_s} \vartheta_{x_i} \vartheta_{x_s} - (a_{ss} a_{ii} \psi_{x_i})_{x_i} \vartheta_{x_s}^2) \\
& - \sum_{s=2}^n \sum_{j=1}^m (2(a_{ss} b_{jj} \psi_{y_j})_{x_s} \vartheta_{y_j} \vartheta_{x_s} - (a_{ss} b_{jj} \psi_{y_j})_{y_j} \vartheta_{x_s}^2) \\
& - \sum_{k=1}^m \sum_{i=2}^n (2(b_{kk} a_{ii} \psi_{x_i})_{y_k} \vartheta_{x_i} \vartheta_{y_k} - (b_{kk} a_{ii} \psi_{x_i})_{x_i} \vartheta_{y_k}^2) \\
& + \sum_{k,j=1}^m (2(b_{kk} b_{jj} \psi_{y_j})_{y_k} \vartheta_{y_j} \vartheta_{y_k} - (b_{kk} b_{jj} \psi_{y_j})_{y_j} \vartheta_{y_k}^2)),
\end{aligned}$$

$$\begin{aligned}
d_1(\vartheta) = & -(2k(x_1 + \delta_0) \psi^{\nu+1} E_0 \vartheta^2)_\xi + (2k(x_1 + \delta_0)^2 \xi \psi^{\nu+1} \vartheta^2)_\xi \\
& - 2\lambda \nu (\xi \psi_{x_1} \vartheta^2)_{x_1} - 2\lambda \nu (x_1 + \delta_0)^2 (\sum_{i=2}^n (a_{ii} \psi_{x_i} \xi \vartheta^2)_{x_i} - \sum_{j=1}^m (b_{jj} \psi_{y_j} \xi \vartheta^2)_{y_j}) \\
& + 2\lambda \nu \psi_{x_1} (((x_1 + \delta_0)^{-2} \vartheta_{x_1}^2)_{x_1} + ((x_1 + \delta_0)^{-1} E_0 \vartheta^2)_{x_1})
\end{aligned}$$

$$\begin{aligned}
& + 2\lambda \nu (\sum_{i=2}^n (2(\psi_{x_i} a_{ii} \vartheta_{x_i} \vartheta_{x_1})_{x_1} - (\psi_{x_i} a_{ii} \vartheta_{x_1}^2)_{x_i} \\
& + 2(\psi_{x_1} a_{ii} \vartheta_{x_1} \vartheta_{x_i})_{x_i} - (\psi_{x_1} a_{ii} \vartheta_{x_i}^2)_{x_1}) \\
& - \sum_{j=1}^m (2(\psi_{y_j} b_{jj} \vartheta_{y_j} \vartheta_{x_1})_{x_1} - (\psi_{y_j} b_{jj} \vartheta_{x_1}^2)_{y_j} \\
& + 2(\psi_{x_1} b_{jj} \vartheta_{x_1} \vartheta_{y_j})_{y_j} - (\psi_{x_1} b_{jj} \vartheta_{y_j}^2)_{x_1}))
\end{aligned}$$

$$\begin{aligned}
& +2\lambda\nu(x_1 + \delta_0)^2 \left( \sum_{s,i=2}^n (2(a_{ss}a_{ii}\psi_{x_i}\vartheta_{x_i}\vartheta_{x_s})_{x_s} - (a_{ss}a_{ii}\psi_{x_i}\vartheta_{x_s}^2)_{x_i}) \right. \\
& - \sum_{s=2}^n \sum_{j=1}^m (2(a_{ss}b_{jj}\psi_{y_j}\vartheta_{y_j}\vartheta_{x_s})_{x_s} - (a_{ss}b_{jj}\psi_{y_j}\vartheta_{x_s}^2)_{y_j}) \\
& - \sum_{k=1}^m \sum_{i=2}^n (2(b_{kk}a_{ii}\psi_{x_i}\vartheta_{x_i}\vartheta_{y_k})_{y_k} - (b_{kk}a_{ii}\psi_{x_i}\vartheta_{y_k}^2)_{x_i}) \\
& + \sum_{k,j=1}^m (2(b_{kk}b_{jj}\psi_{y_j}\vartheta_{y_j}\vartheta_{y_k})_{y_k} - (b_{kk}b_{jj}\psi_{y_j}\vartheta_{y_k}^2)_{y_j}) \\
& \left. + 2\lambda\nu(x_1 + \delta_0) \left( \sum_{i=2}^n (a_{ii}\psi_{x_i}E_0\vartheta^2)_{x_i} - \sum_{j=1}^m (b_{jj}\psi_{y_j}E_0\vartheta^2)_{y_j} \right) \right),
\end{aligned}$$

$$\begin{aligned}
d_2(\vartheta) &= -4k(\psi^{\nu+1}\vartheta_\xi\vartheta_{x_1})_{x_1} + 2k(\psi^{\nu+1}\vartheta_{x_1}^2)_\xi \\
& - 2k(x_1 + \delta_0)^2 \left( \sum_{i=2}^n (2(\psi^{\nu+1}a_{ii}\vartheta_\xi\vartheta_{x_i})_{x_i} - (\psi^{\nu+1}a_{ii}\vartheta_{x_i}^2)_\xi) \right. \\
& \left. - \sum_{j=1}^m (2(\psi^{\nu+1}b_{jj}\vartheta_\xi\vartheta_{y_j})_{y_j} - (\psi^{\nu+1}b_{jj}\vartheta_{y_j}^2)_\xi) \right)
\end{aligned}$$

olarak tanımlıdır. Diğer taraftan  $\psi_{x_i} = (x_i - x_i^0)$ ,  $|\psi_{x_i}| = |x_i - x_i^0| \leq \sqrt{2\gamma}$ ,  $2 \leq i \leq n$  ve  $\psi_{y_j} = (y_j - y_j^0)$ ,  $|\psi_{y_j}| = |y_j - y_j^0| \leq \sqrt{2\gamma}$ ,  $1 \leq j \leq m$  dir. Gerçekten de

$$\begin{aligned}
\psi(x, y) &= \delta x_1 + \frac{1}{2} \sum_{i=2}^n (x_i - x_i^0)^2 + \frac{1}{2} \sum_{j=1}^m (y_j - y_j^0)^2 < \gamma, \\
|x_i - x_i^0|^2 &< \gamma, \quad |x_i - x_i^0| < \sqrt{2\gamma}, \\
|\psi_{x_i}| &= |x_i - x_i^0| \leq \sqrt{2\gamma}, \\
|y_j - y_j^0|^2 &< \gamma, \quad |y_j - y_j^0| \leq \sqrt{2\gamma}, \\
|\psi_{y_j}| &= |y_j - y_j^0| \leq \sqrt{2\gamma}
\end{aligned}$$

olduğu kolayca görülebilir. O halde yukarıdaki eşitsizlikler,  $\|a_{ii}\|_{C^1(\overline{D \times G})} \leq M$  ( $2 \leq i \leq n$ ),  $\|b_{jj}\|_{C^1(\overline{D \times G})} \leq M$  ( $1 \leq j \leq m$ ),  $\forall \varepsilon_0 > 0$  için  $-2ab \geq -\frac{1}{\varepsilon_0(x_1 + \delta_0)}a^2 - \varepsilon_0(x_1 + \delta_0)b^2$  eşitsizliği ve

$$\begin{aligned}
\psi_{x_i x_s} &= \begin{cases} 1, & i = s \\ 0, & i \neq s \end{cases} \quad (2 \leq i, s \leq n); \quad \psi_{y_j y_k} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases} \quad (1 \leq j, k \leq m); \\
\psi_{x_i y_j} &= 0, \quad \psi_{y_j x_1} = 0, \quad \psi_{x_i x_1} = 0
\end{aligned}$$

yardımla

$$\begin{aligned}
-4\lambda\nu\delta\sum_{i=2}^n(a_{ii})_{x_i}\vartheta_{x_1}\vartheta_{x_i} &\geq -4\lambda\nu\delta\sum_{i=2}^n|(a_{ii})_{x_i}||\vartheta_{x_1}\vartheta_{x_i}| \\
&= 2\lambda\nu\delta\sum_{i=2}^n(-2|(a_{ii})_{x_i}||\vartheta_{x_1}\vartheta_{x_i}|) \\
&\geq -2\lambda\nu\delta\sum_{i=2}^n((x_1+\delta_0)\varepsilon_0\vartheta_{x_i}^2 + \frac{M^2}{\varepsilon_0(x_1+\delta_0)}\vartheta_{x_1}^2) \\
&= -2\lambda\nu\delta((x_1+\delta_0)\varepsilon_0\sum_{i=2}^n\vartheta_{x_i}^2 + \frac{M^2}{\varepsilon_0(x_1+\delta_0)}(n-1)\vartheta_{x_1}^2), \quad (2.17)
\end{aligned}$$

$$2\lambda\nu\delta\sum_{i=2}^n(a_{ii})_{x_1}\vartheta_{x_i}^2 \geq 2\lambda\nu(x_1+\delta_0)\delta\alpha_1\sum_{i=2}^n\vartheta_{x_i}^2, \quad (2.18)$$

$$\begin{aligned}
4\lambda\nu\delta\sum_{j=1}^m(b_{jj})_{y_j}\vartheta_{x_1}\vartheta_{y_j} &\geq -4\lambda\nu\delta\sum_{j=1}^m|(b_{jj})_{y_j}||\vartheta_{x_1}\vartheta_{y_j}| \\
&= 2\lambda\nu\delta\sum_{j=1}^m(-2|(b_{jj})_{y_j}||\vartheta_{x_1}\vartheta_{y_j}|) \\
&\geq -2\lambda\nu\delta\sum_{j=1}^m((x_1+\delta_0)\varepsilon_0\vartheta_{y_j}^2 + \frac{M^2}{\varepsilon_0(x_1+\delta_0)}\vartheta_{x_1}^2) \\
&= -2\lambda\nu\delta((x_1+\delta_0)\varepsilon_0\sum_{j=1}^m\vartheta_{y_j}^2 + \frac{M^2}{\varepsilon_0(x_1+\delta_0)}m\vartheta_{x_1}^2), \quad (2.19)
\end{aligned}$$

$$-2\lambda\nu\delta\sum_{j=1}^m(b_{jj})_{x_1}\vartheta_{y_j}^2 \geq 2\lambda\nu(x_1+\delta_0)\delta\alpha_2\sum_{j=1}^m\vartheta_{y_j}^2, \quad (2.20)$$

$$\begin{aligned}
-4\lambda\nu\sum_{i=2}^n(\psi_{x_i}a_{ii})_{x_1}\vartheta_{x_i}\vartheta_{x_1} &= -4\lambda\nu\sum_{i=2}^n(a_{ii})_{x_1}\psi_{x_i}\vartheta_{x_i}\vartheta_{x_1} \\
&\geq -4\lambda\nu\sum_{i=2}^n|(a_{ii})_{x_1}||\psi_{x_i}||\vartheta_{x_i}\vartheta_{x_1}| \\
&\geq -4\lambda\nu\sqrt{2\gamma}\sum_{i=2}^n|(a_{ii})_{x_1}||\vartheta_{x_i}\vartheta_{x_1}| \\
&= 2\lambda\nu\sqrt{2\gamma}\sum_{i=2}^n(-2|(a_{ii})_{x_1}||\vartheta_{x_i}\vartheta_{x_1}|) \\
&\geq -2\lambda\nu\sqrt{2\gamma}\sum_{i=2}^n((x_1+\delta_0)\vartheta_{x_i}^2 + \frac{1}{(x_1+\delta_0)}|(a_{ii})_{x_1}|^2\vartheta_{x_1}^2) \\
&\geq -2\lambda\nu\sqrt{2\gamma}(x_1+\delta_0)\sum_{i=2}^n\vartheta_{x_i}^2 \\
&\quad -2\lambda\nu\sqrt{2\gamma}\frac{M^2}{(x_1+\delta_0)}(n-1)\vartheta_{x_1}^2, \quad (2.21)
\end{aligned}$$

$$2\lambda\nu \sum_{i=2}^n (a_{ii})_{x_i} \psi_{x_i} \vartheta_{x_1}^2 \geq -2\lambda\nu M \sqrt{2\gamma} (n-1) \vartheta_{x_1}^2, \quad (2.22)$$

$$\begin{aligned} 4\lambda\nu \sum_{j=1}^m (\psi_{y_j} b_{jj})_{x_1} \vartheta_{y_j} \vartheta_{x_1} &= 4\lambda\nu \sum_{j=1}^m (b_{jj})_{x_1} \psi_{y_j} \vartheta_{y_j} \vartheta_{x_1} \\ &\geq -4\lambda\nu \sum_{j=1}^m |(b_{jj})_{x_1}| |\psi_{y_j}| |\vartheta_{y_j} \vartheta_{x_1}| \\ &\geq -4\lambda\nu \sqrt{2\gamma} \sum_{j=1}^m |(b_{jj})_{x_1}| |\vartheta_{y_j} \vartheta_{x_1}| \\ &= 2\lambda\nu \sqrt{2\gamma} \sum_{j=1}^m (-2|(b_{jj})_{x_1}| |\vartheta_{y_j} \vartheta_{x_1}|) \\ &\geq -2\lambda\nu \sqrt{2\gamma} \sum_{j=1}^m ((x_1 + \delta_0) \vartheta_{y_j}^2 + \frac{1}{(x_1 + \delta_0)} |(b_{jj})_{x_1}|^2 \vartheta_{x_1}^2) \\ &\geq -2\lambda\nu \sqrt{2\gamma} (x_1 + \delta_0) \sum_{j=1}^m \vartheta_{y_j}^2 - 2\lambda\nu \sqrt{2\gamma} \frac{M^2}{(x_1 + \delta_0)} m \vartheta_{x_1}^2, \quad (2.23) \end{aligned}$$

$$\begin{aligned} -2\lambda\nu \sum_{j=1}^m (b_{jj})_{y_j} \psi_{y_j} \vartheta_{x_1}^2 &\geq -2\lambda\nu \sum_{j=1}^m |(b_{jj})_{y_j}| |\psi_{y_j}| |\vartheta_{x_1}^2| \\ &\geq -2\lambda\nu M \sqrt{2\gamma} m \vartheta_{x_1}^2 \quad (2.24) \end{aligned}$$

eşitsizlikleri yazılabilir. Ayrıca  $I$ 'nın terimleri için

$$\begin{aligned} |(a_{ss} a_{ii} \psi_{x_i})_{x_s}| &= |(a_{ss})_{x_s} a_{ii} \psi_{x_i} + a_{ss} (a_{ii})_{x_s} \psi_{x_i} + a_{ss} a_{ii} \psi_{x_i x_s}| \\ &\leq M^2 \sqrt{2\gamma} + M^2 \sqrt{2\gamma} + M^2 = M^2 (2\sqrt{2\gamma} + 1), \end{aligned}$$

$$\begin{aligned} |(a_{ss} a_{ii} \psi_{x_i})_{x_i}| &= |(a_{ss})_{x_i} a_{ii} \psi_{x_i} + a_{ss} (a_{ii})_{x_i} \psi_{x_i} + a_{ss} a_{ii} \psi_{x_i x_i}| \\ &\leq M^2 \sqrt{2\gamma} + M^2 \sqrt{2\gamma} + M^2 = M^2 (2\sqrt{2\gamma} + 1), \end{aligned}$$

$$\begin{aligned} |(a_{ss} b_{jj} \psi_{y_j})_{x_s}| &= |(a_{ss})_{x_s} b_{jj} \psi_{y_j} + a_{ss} (b_{jj})_{x_s} \psi_{y_j} + a_{ss} b_{jj} \psi_{y_j x_s}| \\ &\leq M^2 \sqrt{2\gamma} + M^2 \sqrt{2\gamma} = 2M^2 \sqrt{2\gamma}, \end{aligned}$$

$$\begin{aligned} |(a_{ss} b_{jj} \psi_{y_j})_{y_j}| &= |(a_{ss})_{y_j} b_{jj} \psi_{y_j} + a_{ss} (b_{jj})_{y_j} \psi_{y_j} + a_{ss} b_{jj} \psi_{y_j y_j}| \\ &\leq M^2 \sqrt{2\gamma} + M^2 \sqrt{2\gamma} + M^2 = M^2 (2\sqrt{2\gamma} + 1), \end{aligned}$$

$$\begin{aligned} |(b_{kk} a_{ii} \psi_{x_i})_{y_k}| &= |(b_{kk})_{y_k} a_{ii} \psi_{x_i} + b_{kk} (a_{ii})_{y_k} \psi_{x_i} + b_{kk} a_{ii} \psi_{x_i y_k}| \\ &\leq M^2 \sqrt{2\gamma} + M^2 \sqrt{2\gamma} = 2M^2 \sqrt{2\gamma}, \end{aligned}$$

$$\begin{aligned}
|(b_{kk}a_{ii}\psi_{x_i})_{x_i}| &= |(b_{kk})_{x_i}a_{ii}\psi_{x_i} + b_{kk}(a_{ii})_{x_i}\psi_{x_i} + b_{kk}a_{ii}\psi_{x_ix_i}| \\
&\leq M^2\sqrt{2\gamma} + M^2\sqrt{2\gamma} + M^2 = M^2(2\sqrt{2\gamma} + 1),
\end{aligned}$$

$$\begin{aligned}
|(b_{kk}b_{jj}\psi_{y_j})_{y_k}| &= |(b_{kk})_{y_k}b_{jj}\psi_{y_j} + b_{kk}(b_{jj})_{y_k}\psi_{y_j} + b_{kk}b_{jj}\psi_{y_jy_k}| \\
&\leq M^2\sqrt{2\gamma} + M^2\sqrt{2\gamma} + M^2 = M^2(2\sqrt{2\gamma} + 1),
\end{aligned}$$

$$\begin{aligned}
|(b_{kk}b_{jj}\psi_{y_j})_{y_j}| &= |(b_{kk})_{y_j}b_{jj}\psi_{y_j} + b_{kk}(b_{jj})_{y_j}\psi_{y_j} + b_{kk}b_{jj}\psi_{y_jy_j}| \\
&\leq M^2\sqrt{2\gamma} + M^2\sqrt{2\gamma} + M^2 = M^2(2\sqrt{2\gamma} + 1)
\end{aligned}$$

eşitsizlikleri kullanarak

$$\begin{aligned}
&\sum_{s,i=2}^n (-2(a_{ss}a_{ii}\psi_{x_i})_{x_s}\vartheta_{x_i}\vartheta_{x_s}) \\
&\geq -2 \sum_{s,i=2}^n |(a_{ss}a_{ii}\psi_{x_i})_{x_s}||\vartheta_{x_i}\vartheta_{x_s}| \geq M^2(2\sqrt{2\gamma} + 1) \sum_{s,i=2}^n (-2|\vartheta_{x_i}\vartheta_{x_s}|) \\
&\geq -M^2(2\sqrt{2\gamma} + 1) \sum_{s,i=2}^n (\vartheta_{x_i}^2 + \vartheta_{x_s}^2) = -M^2(2\sqrt{2\gamma} + 1) \left( \sum_{s,i=2}^n \vartheta_{x_i}^2 + \sum_{s,i=2}^n \vartheta_{x_s}^2 \right) \\
&= -M^2(2\sqrt{2\gamma} + 1) \left( (n-1) \sum_{i=2}^n \vartheta_{x_i}^2 + (n-1) \sum_{s=2}^n \vartheta_{x_s}^2 \right) \\
&= -2M^2(2\sqrt{2\gamma} + 1)(n-1) \sum_{i=2}^n \vartheta_{x_i}^2,
\end{aligned}$$

$$\begin{aligned}
\sum_{s,i=2}^n (a_{ss}a_{ii}\psi_{x_i})_{x_i}\vartheta_{x_s}^2 &\geq - \sum_{s,i=2}^n |(a_{ss}a_{ii}\psi_{x_i})_{x_i}||\vartheta_{x_s}^2| \\
&\geq -M^2(2\sqrt{2\gamma} + 1) \sum_{s,i=2}^n \vartheta_{x_s}^2 \\
&= -M^2(2\sqrt{2\gamma} + 1)(n-1) \sum_{s=2}^n \vartheta_{x_s}^2,
\end{aligned}$$

$$\begin{aligned}
\sum_{s=2}^n \sum_{j=1}^m 2(a_{ss}b_{jj}\psi_{y_j})_{x_s}\vartheta_{y_j}\vartheta_{x_s} &\geq - \sum_{s=2}^n \sum_{j=1}^m 2|(a_{ss}b_{jj}\psi_{y_j})_{x_s}||\vartheta_{y_j}\vartheta_{x_s}| \\
&\geq 2M^2\sqrt{2\gamma} \sum_{s=2}^n \sum_{j=1}^m (-2|\vartheta_{y_j}\vartheta_{x_s}|) \\
&\geq -2M^2\sqrt{2\gamma} \sum_{s=2}^n \sum_{j=1}^m (\vartheta_{y_j}^2 + \vartheta_{x_s}^2) \\
&= -2M^2\sqrt{2\gamma} \left( (n-1) \sum_{j=1}^m \vartheta_{y_j}^2 + m \sum_{s=2}^n \vartheta_{x_s}^2 \right),
\end{aligned}$$

$$\begin{aligned}
-\sum_{s=2}^n \sum_{j=1}^m (a_{ss} b_{jj} \psi_{y_j})_{y_j} \vartheta_{x_s}^2 &\geq -\sum_{s=2}^n \sum_{j=1}^m |(a_{ss} b_{jj} \psi_{y_j})_{y_j}| |\vartheta_{x_s}^2| \\
&\geq -M^2(2\sqrt{2\gamma} + 1) \sum_{s=2}^n \sum_{j=1}^m \vartheta_{x_s}^2 \\
&= -M^2(2\sqrt{2\gamma} + 1) m \sum_{s=2}^n \vartheta_{x_s}^2,
\end{aligned}$$

$$\begin{aligned}
\sum_{k=1}^m \sum_{i=2}^n 2(b_{kk} a_{ii} \psi_{x_i})_{y_k} \vartheta_{x_i} \vartheta_{y_k} &\geq -\sum_{k=1}^m \sum_{i=2}^n 2|(b_{kk} a_{ii} \psi_{x_i})_{y_k}| |\vartheta_{x_i} \vartheta_{y_k}| \\
&\geq 2M^2 \sqrt{2\gamma} \sum_{k=1}^m \sum_{i=2}^n (-2|\vartheta_{x_i} \vartheta_{y_k}|) \\
&\geq -2M^2 \sqrt{2\gamma} \sum_{k=1}^m \sum_{i=2}^n (\vartheta_{x_i}^2 + \vartheta_{y_k}^2) \\
&= -2M^2 \sqrt{2\gamma} (m \sum_{i=2}^n \vartheta_{x_i}^2 + (n-1) \sum_{k=1}^m \vartheta_{y_k}^2),
\end{aligned}$$

$$\begin{aligned}
-\sum_{k=1}^m \sum_{i=2}^n (b_{kk} a_{ii} \psi_{x_i})_{x_i} \vartheta_{y_k}^2 &\geq -\sum_{k=1}^m \sum_{i=2}^n |(b_{kk} a_{ii} \psi_{x_i})_{x_i}| |\vartheta_{y_k}^2| \\
&\geq -M^2(2\sqrt{2\gamma} + 1) \sum_{k=1}^m \sum_{i=2}^n \vartheta_{y_k}^2 \\
&= -M^2(2\sqrt{2\gamma} + 1)(n-1) \sum_{k=1}^m \vartheta_{y_k}^2,
\end{aligned}$$

$$\begin{aligned}
\sum_{k,j=1}^m -2(b_{kk} b_{jj} \psi_{y_j})_{y_k} \vartheta_{y_j} \vartheta_{y_k} &\geq -2 \sum_{k,j=1}^m |(b_{kk} b_{jj} \psi_{y_j})_{y_k}| |\vartheta_{y_j} \vartheta_{y_k}| \\
&\geq M^2(2\sqrt{2\gamma} + 1) \sum_{k,j=1}^m (-2|\vartheta_{y_j} \vartheta_{y_k}|) \\
&\geq -M^2(2\sqrt{2\gamma} + 1) \sum_{k,j=1}^m (\vartheta_{y_j}^2 + \vartheta_{y_k}^2) \\
&= -2M^2(2\sqrt{2\gamma} + 1) m \sum_{j=1}^m \vartheta_{y_j}^2,
\end{aligned}$$

$$\begin{aligned}
\sum_{k,j=1}^m (b_{kk} b_{jj} \psi_{y_j})_{y_j} \vartheta_{y_k}^2 &\geq -\sum_{k,j=1}^m |(b_{kk} b_{jj} \psi_{y_j})_{y_j}| |\vartheta_{y_k}^2| \\
&\geq -M^2(2\sqrt{2\gamma} + 1) \sum_{k,j=1}^m \vartheta_{y_k}^2 \\
&= -M^2(2\sqrt{2\gamma} + 1) m \sum_{k=1}^m \vartheta_{y_k}^2
\end{aligned}$$

yazılabilir. Bunlara bağılı olarak

$$\begin{aligned}
I &\geq 2\lambda\nu(x_1 + \delta_0)^2(-2M^2(2\sqrt{2\gamma} + 1)(n - 1) \sum_{i=2}^n \vartheta_{x_i}^2 - M^2(2\sqrt{2\gamma} + 1)(n - 1) \sum_{s=2}^n \vartheta_{x_s}^2 \\
&\quad - 2M^2\sqrt{2\gamma}((n - 1) \sum_{j=1}^m \vartheta_{y_j}^2 + m \sum_{s=2}^n \vartheta_{x_s}^2) - M^2(2\sqrt{2\gamma} + 1)m \sum_{s=2}^n \vartheta_{x_s}^2 \\
&\quad - 2M^2\sqrt{2\gamma}(m \sum_{i=2}^n \vartheta_{x_i}^2 + (n - 1) \sum_{k=1}^m \vartheta_{y_k}^2) - M^2(2\sqrt{2\gamma} + 1)(n - 1) \sum_{k=1}^m \vartheta_{y_k}^2 \\
&\quad - 2M^2(2\sqrt{2\gamma} + 1)m \sum_{j=1}^m \vartheta_{y_j}^2 - M^2(2\sqrt{2\gamma} + 1)m \sum_{k=1}^m \vartheta_{y_k}^2) \\
&= 2\lambda\nu(x_1 + \delta_0)^2((-2M^2(2\sqrt{2\gamma} + 1)(n - 1) - M^2(2\sqrt{2\gamma} + 1)(n - 1) \\
&\quad - 2M^2\sqrt{2\gamma}m - M^2(2\sqrt{2\gamma} + 1)m - 2M^2\sqrt{2\gamma}m) \sum_{i=2}^n \vartheta_{x_i}^2 \\
&\quad + (-2M^2\sqrt{2\gamma}(n - 1) - 2M^2\sqrt{2\gamma}(n - 1) - M^2(2\sqrt{2\gamma} + 1)(n - 1) \\
&\quad - 2M^2(2\sqrt{2\gamma} + 1)m - M^2(2\sqrt{2\gamma} + 1)m) \sum_{j=1}^m \vartheta_{y_j}^2) \\
&= -2\lambda\nu(x_1 + \delta_0)^2 \\
&\quad \times ((3M^2(2\sqrt{2\gamma} + 1)(n - 1) + 4M^2\sqrt{2\gamma}m + M^2(2\sqrt{2\gamma} + 1)m) \sum_{i=2}^n \vartheta_{x_i}^2 \\
&\quad + (3M^2(2\sqrt{2\gamma} + 1)m + 4M^2\sqrt{2\gamma}(n - 1) + M^2(2\sqrt{2\gamma} + 1)(n - 1)) \sum_{j=1}^m \vartheta_{y_j}^2) \quad (2.25)
\end{aligned}$$

bulunur. (2.16) eşitsizliğindeki diğer terimler için aşağıdaki değerlendirmeler yapılabilir:

$$\begin{aligned}
& -(\sum_{i=2}^n (a_{ii})_{x_i} \vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j} \vartheta_{y_j})^2 \\
&= -(\sum_{i=2}^n (a_{ii})_{x_i} \vartheta_{x_i})^2 + 2 \sum_{i=2}^n \sum_{j=1}^m (a_{ii})_{x_i} (b_{jj})_{y_j} \vartheta_{x_i} \vartheta_{y_j} - (\sum_{j=1}^m (b_{jj})_{y_j} \vartheta_{y_j})^2.
\end{aligned}$$

Burada  $(\sum_{k=1}^m a_k)^2 \leq m \sum_{k=1}^m a_k^2$  olduğu dikkate alınarak

$$\begin{aligned}
-(\sum_{i=2}^n (a_{ii})_{x_i} \vartheta_{x_i})^2 &\geq -(M \sum_{i=2}^n \vartheta_{x_i})^2 \\
&\geq -M^2(n - 1) \sum_{i=2}^n \vartheta_{x_i}^2,
\end{aligned}$$

$$\begin{aligned}
2 \sum_{i=2}^n \sum_{j=1}^m (a_{ii})_{x_i} (b_{jj})_{y_j} \vartheta_{x_i} \vartheta_{y_j} &\geq -2 \sum_{i=2}^n \sum_{j=1}^m |(a_{ii})_{x_i}| |(b_{jj})_{y_j}| |\vartheta_{x_i} \vartheta_{y_j}| \\
&\geq M^2 \sum_{i=2}^n \sum_{j=1}^m (-2 |\vartheta_{x_i} \vartheta_{y_j}|) \\
&\geq -M^2 \sum_{i=2}^n \sum_{j=1}^m (\vartheta_{x_i}^2 + \vartheta_{y_j}^2) \\
&= -M^2 (m \sum_{i=2}^n \vartheta_{x_i}^2 + (n-1) \sum_{j=1}^m \vartheta_{y_j}^2),
\end{aligned}$$

$$\begin{aligned}
-(\sum_{j=1}^m (b_{jj})_{y_j} \vartheta_{y_j})^2 &\geq -(M \sum_{j=1}^m \vartheta_{y_j})^2 \\
&\geq -M^2 m \sum_{j=1}^m \vartheta_{y_j}^2
\end{aligned}$$

eşitsizlikleri elde edilir. Bu eşitsizlikler kullanılarak

$$\begin{aligned}
&-(\sum_{i=2}^n (a_{ii})_{x_i} \vartheta_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j} \vartheta_{y_j})^2 \\
&\geq -M^2 (n-1) \sum_{i=2}^n \vartheta_{x_i}^2 - M^2 (m \sum_{i=2}^n \vartheta_{x_i}^2 + (n-1) \sum_{j=1}^m \vartheta_{y_j}^2) \\
&\quad - M^2 m \sum_{j=1}^m \vartheta_{y_j}^2
\end{aligned} \tag{2.26}$$

bulunur. Ek olarak

$$2\lambda\nu \sum_{i=2}^n a_{ii} \psi_{x_i x_i} \vartheta_{x_1}^2 \geq -2\lambda\nu \vartheta_{x_1}^2 \sum_{i=2}^n |a_{ii}| = -2\lambda\nu M(n-1) \vartheta_{x_1}^2, \tag{2.27}$$

$$-2\lambda\nu \sum_{j=1}^m b_{jj} \psi_{y_j y_j} \vartheta_{x_1}^2 \geq -2\lambda\nu \vartheta_{x_1}^2 \sum_{j=1}^m |b_{jj}| = -2\lambda\nu Mm \vartheta_{x_1}^2 \tag{2.28}$$

elde edilir. Böylece (2.17)-(2.28) eşitsizlikleri kullanılarak

$$\psi^{\nu+1} (L_0 \varphi)^2 \chi^2 \geq E_1 \vartheta_{x_1}^2 + E_2 \sum_{i=2}^n \vartheta_{x_i}^2 + E_3 \sum_{j=1}^m \vartheta_{y_j}^2 + E_4 \vartheta^2 + d_1 + d_2 \tag{2.29}$$

yazılabilir. Burada

$$\begin{aligned}
E_1 &= 4\lambda\nu \delta (x_1 + \delta_0)^{-3} - 2\lambda\nu \delta \frac{M^2}{\varepsilon_0 (x_1 + \delta_0)} (n-1) - 2\lambda\nu \delta \frac{M^2}{\varepsilon_0 (x_1 + \delta_0)} m \\
&\quad - 2\lambda\nu \sqrt{2\gamma} \frac{M^2}{(x_1 + \delta_0)} (n-1) - 2\lambda\nu M \sqrt{2\gamma} (n-1) - 2\lambda\nu \sqrt{2\gamma} \frac{M^2}{(x_1 + \delta_0)} m \\
&\quad - 2\lambda\nu M \sqrt{2\gamma} m - 2\lambda\nu M (n-1) - 2\lambda\nu M m,
\end{aligned}$$

$$\begin{aligned}
E_2 &= 2\lambda\nu(x_1 + \delta_0)(\delta\alpha_1 - \delta\varepsilon_0 \\
&\quad - \sqrt{2\gamma} - (x_1 + \delta_0)(3M^2(2\sqrt{2\gamma} + 1)(n - 1) + 4M^2\sqrt{2\gamma}m + M^2(2\sqrt{2\gamma} + 1)m) \\
&\quad - (x_1 + \delta_0)\psi(M^2(n - 1) + M^2m)),
\end{aligned}$$

$$\begin{aligned}
E_3 &= 2\lambda\nu(x_1 + \delta_0)(\delta\alpha_2 - \delta\varepsilon_0 - \sqrt{2\gamma} \\
&\quad - (x_1 + \delta_0)(3M^2(2\sqrt{2\gamma} + 1)m + 4M^2\sqrt{2\gamma}(n - 1) + M^2(2\sqrt{2\gamma} + 1)(n - 1)) \\
&\quad - (x_1 + \delta_0)\psi(M^2(n - 1) + M^2m)).
\end{aligned}$$

$$\begin{aligned}
E_4 &= -2\lambda\nu(\delta((x_1 + \delta_0)^{-1}E_0)_{x_1} + (x_1 + \delta_0)(\sum_{i=2}^n (a_{ii}\psi_{x_i}E_0)_{x_i} - \sum_{j=1}^m (b_{jj}\psi_{y_j}E_0)_{y_j})) \\
&\quad - 2(x_1 + \delta_0)^2(k\psi^{\nu+1} - \lambda\nu\xi(\sum_{i=2}^n (a_{ii}\psi_{x_i})_{x_i} - \sum_{j=1}^m (b_{jj}\psi_{y_j})_{y_j}))
\end{aligned}$$

şeklinde tanımlıdır. Ayrıca  $\delta_0 \leq \frac{1}{2}\gamma$ ,  $\delta \geq 4$  ve  $0 < \delta x_1 < \gamma$  olduğundan

$$\begin{aligned}
x_1 &< \frac{\gamma}{\delta} \leq \frac{\gamma}{4}, \\
\delta_0 &\leq \frac{1}{2}\gamma
\end{aligned}$$

eşitsizlikleri taraf tarafa toplanırsa  $(x_1 + \delta_0) < \frac{3}{4}\gamma$  olarak bulunur.

Şimdi  $\vartheta_{x_1}^2$ 'nin katsayısı olan  $E_1$  toplamını düzenleyelim:

$$\begin{aligned}
E_1 &= 4\lambda\nu\delta(x_1 + \delta_0)^{-3} - 2\lambda\nu\delta\frac{M^2}{\varepsilon_0(x_1 + \delta_0)}(n + m - 1) \\
&\quad - 2\lambda\nu\sqrt{2\gamma}\frac{M^2}{(x_1 + \delta_0)}(n + m - 1) - 2\lambda\nu M\sqrt{2\gamma}(n + m - 1) \\
&\quad - 2\lambda\nu M(n + m - 1) \\
&= 2\lambda\nu(x_1 + \delta_0)^{-3}(2\delta - (x_1 + \delta_0)^2\delta\frac{M^2}{\varepsilon_0}(n + m - 1) \\
&\quad - (x_1 + \delta_0)^2\sqrt{2\gamma}M^2(n + m - 1) - (x_1 + \delta_0)^3M\sqrt{2\gamma}(n + m - 1) \\
&\quad - (x_1 + \delta_0)^3M(n + m - 1)) \\
&= 2\lambda\nu(x_1 + \delta_0)^{-3}(\delta + E_{11} - E_{12}).
\end{aligned}$$

Burada

$$\begin{aligned}
E_{11} &= \delta(1 - (x_1 + \delta_0)^2\frac{M^2}{\varepsilon_0}(n + m - 1)), \\
E_{12} &= (x_1 + \delta_0)^2\sqrt{2\gamma}M^2(n + m - 1) + (x_1 + \delta_0)^3M\sqrt{2\gamma}(n + m - 1) \\
&\quad + (x_1 + \delta_0)^3M(n + m - 1)
\end{aligned}$$

dir. Diğer taraftan, eğer  $0 < \gamma < \frac{2\sqrt{2\varepsilon_0}}{3M\sqrt{n+m-1}}$  alınırsa

$$\begin{aligned}
1 - (x_1 + \delta_0)^2 \frac{M^2}{\varepsilon_0} (n + m - 1) &> 1 - \left(\frac{3}{4}\gamma\right)^2 \frac{M^2}{\varepsilon_0} (n + m - 1) \\
&> 1 - \left(\frac{6\sqrt{2\varepsilon_0}}{12M\sqrt{n+m-1}}\right)^2 \frac{M^2}{\varepsilon_0} (n + m - 1) \\
&= 1 - \frac{72\varepsilon_0}{144M^2(n+m-1)} \frac{M^2}{\varepsilon_0} (n + m - 1) \\
&= \frac{1}{2}
\end{aligned}$$

yazılır. Böylece

$$E_{11} > \delta \left(1 - \left(\frac{3}{4}\gamma\right)^2 \frac{M^2}{\varepsilon_0} (n + m - 1)\right) > \frac{1}{2}\delta \quad (2.30)$$

bulunur.

Ayrıca  $0 < \gamma < 1$  olduğu göz önünde bulundurularak  $E_1$  içerisindeki diğer terimler için

$$\begin{aligned}
&(x_1 + \delta_0)^2 \sqrt{2\gamma} M^2 (n + m - 1) + (x_1 + \delta_0)^3 M \sqrt{2\gamma} (n + m - 1) \\
&+ (x_1 + \delta_0)^3 M (n + m - 1) \\
< &\left(\frac{3}{4}\gamma\right)^2 \sqrt{2\gamma} M^2 (n + m - 1) + \left(\frac{3}{4}\gamma\right)^3 M \sqrt{2\gamma} (n + m - 1) \\
&+ \left(\frac{3}{4}\gamma\right)^3 M (n + m - 1) \\
< &\sqrt{2} M^2 (n + m - 1) + M \sqrt{2} (n + m - 1) + M (n + m - 1) \\
= &(\sqrt{2}(M + 1) + 1) M (n + m - 1)
\end{aligned}$$

olur. Burada  $E'_{12} = (\sqrt{2}(M + 1) + 1) M (n + m - 1)$ ,  $\delta \geq \delta_1 = 2E'_{12} \implies \delta \geq 2E'_{12} \implies \frac{\delta}{2} \geq E'_{12} \implies \frac{\delta}{2} - E'_{12} \geq 0$  olarak alınırsa ve (2.30) eşitsizliği kullanılırsa

$$E_1 > 2\lambda\nu(x_1 + \delta_0)^{-3} \left(\delta + \frac{1}{2}\delta - E'_{12}\right) \geq 2\lambda\nu(x_1 + \delta_0)^{-3} \delta > 2\lambda\nu\gamma^{-3} \delta \quad (2.31)$$

bulunur.

Şimdi  $\vartheta_{x_i}^2$ 'nin katsayısı olan  $E_2$  toplamını düzenleyelim:

Bu amaçla bir  $\varepsilon_0$  sayısını  $0 < \varepsilon_0 < \frac{\alpha_1}{4}$  olacak şekilde seçelim. Bu durumda

$$\delta\alpha_1 - \delta\varepsilon_0 = \delta(\alpha_1 - \varepsilon_0) > \delta\left(\alpha_1 - \frac{\alpha_1}{4}\right) = \frac{3}{4}\delta\alpha_1 \quad (2.32)$$

yazılabilir. Eğer

$$\begin{aligned}
\delta_2 &= \frac{4}{\alpha_1} (\sqrt{2} + 3M^2(2\sqrt{2} + 1)(n - 1) + 4M^2\sqrt{2}m \\
&\quad + M^2(2\sqrt{2} + 1)m + M^2(n - 1) + M^2m)
\end{aligned}$$

olmak üzere  $\delta \geq \delta_2$  alınırsa

$$\begin{aligned}\frac{1}{4}\alpha_1\delta &\geq \sqrt{2} + 3M^2(2\sqrt{2} + 1)(n - 1) + 4M^2\sqrt{2}m \\ &\quad + M^2(2\sqrt{2} + 1)m + M^2(n - 1) + M^2m, \\ -\frac{1}{4}\alpha_1\delta &\leq -(\sqrt{2} + 3M^2(2\sqrt{2} + 1)(n - 1) + 4M^2\sqrt{2}m \\ &\quad + M^2(2\sqrt{2} + 1)m + M^2(n - 1) + M^2m)\end{aligned}$$

elde edilir. O halde

$$\begin{aligned}E_2 &= 2\lambda\nu(x_1 + \delta_0)(\delta\alpha_1 - \delta\varepsilon_0 \\ &\quad - \sqrt{2\gamma} - (x_1 + \delta_0)(3M^2(2\sqrt{2\gamma} + 1)(n - 1) + 4M^2\sqrt{2\gamma}m + M^2(2\sqrt{2\gamma} + 1)m) \\ &\quad - (x_1 + \delta_0)\psi(M^2(n - 1) + M^2m))\end{aligned}$$

olduğundan  $0 < \gamma < 1$  ve  $0 < \psi < 1$  olduğu dikkate alınırsa  $E_2$  içerisindeki terimler aşağıdaki şekilde değerlendirilebilir:

$$\begin{aligned}&\sqrt{2\gamma} + (x_1 + \delta_0)(3M^2(2\sqrt{2\gamma} + 1)(n - 1) + 4M^2\sqrt{2\gamma}m + M^2(2\sqrt{2\gamma} + 1)m) \\ &\quad + (x_1 + \delta_0)\psi(M^2(n - 1) + M^2m) \\ &< \sqrt{2\gamma} + \frac{3}{4}\gamma(3M^2(2\sqrt{2\gamma} + 1)(n - 1) + 4M^2\sqrt{2\gamma}m + M^2(2\sqrt{2\gamma} + 1)m) \\ &\quad + \frac{3}{4}\gamma\psi(M^2(n - 1) + M^2m) \\ &< \sqrt{2} + 3M^2(2\sqrt{2} + 1)(n - 1) + 4M^2\sqrt{2}m + M^2(2\sqrt{2} + 1)m + M^2(n - 1) + M^2m.\end{aligned}$$

Sonuç olarak

$$E_2 > 2\lambda\nu(x_1 + \delta_0)\left(\frac{3}{4}\delta\alpha_1 - \frac{1}{4}\delta\alpha_1\right) = \lambda\nu(x_1 + \delta_0)\delta\alpha_1 \quad (2.33)$$

bulunur.

Şimdi  $\vartheta_{y_j}^2$ 'nin katsayısı olan  $E_3$  toplamını düzenleyelim:

Bu amaçla bir  $\varepsilon_0$  sayısını  $0 < \varepsilon_0 < \frac{\alpha_2}{4}$  olacak şekilde seçelim. Bu durumda

$$\delta\alpha_2 - \delta\varepsilon_0 = \delta(\alpha_2 - \varepsilon_0) > \delta\left(\alpha_2 - \frac{\alpha_2}{4}\right) = \frac{3}{4}\delta\alpha_2 \quad (2.34)$$

yazılabilir. Eğer

$$\begin{aligned}\delta_3 &= \frac{4}{\alpha_2}(\sqrt{2} + 3M^2(2\sqrt{2} + 1)m + 4M^2\sqrt{2}(n - 1) \\ &\quad + M^2(2\sqrt{2} + 1)(n - 1) + M^2(n - 1) + M^2m)\end{aligned}$$

olmak üzere  $\delta \geq \delta_3$  alınırsa

$$\begin{aligned}\frac{1}{4}\alpha_2\delta &\geq \sqrt{2} + 3M^2(2\sqrt{2} + 1)m + 4M^2\sqrt{2}(n-1) \\ &\quad + M^2(2\sqrt{2} + 1)(n-1) + M^2(n-1) + M^2m, \\ -\frac{1}{4}\alpha_2\delta &\leq -(\sqrt{2} + 3M^2(2\sqrt{2} + 1)m + 4M^2\sqrt{2}(n-1) \\ &\quad + M^2(2\sqrt{2} + 1)(n-1) + M^2(n-1) + M^2m)\end{aligned}$$

elde edilir. O halde

$$\begin{aligned}E_3 &= 2\lambda\nu(x_1 + \delta_0)(\delta\alpha_2 - \delta\varepsilon_0 - \sqrt{2\gamma}) \\ &\quad - (x_1 + \delta_0)(3M^2(2\sqrt{2\gamma} + 1)m + 4M^2\sqrt{2\gamma}(n-1) + M^2(2\sqrt{2\gamma} + 1)(n-1)) \\ &\quad - (x_1 + \delta_0)\psi(M^2(n-1) + M^2m)\end{aligned}$$

olduğundan  $0 < \gamma < 1$  ve  $0 < \psi < 1$  olduğu göz önünde bulundurulursa  $E_3$  içerisindeki terimler aşağıdaki şekilde değerlendirilebilir:

$$\begin{aligned}&\sqrt{2\gamma} + (x_1 + \delta_0)(3M^2(2\sqrt{2\gamma} + 1)m + 4M^2\sqrt{2\gamma}(n-1) + M^2(2\sqrt{2\gamma} + 1)(n-1)) \\ &\quad + (x_1 + \delta_0)\psi(M^2(n-1) + M^2m) \\ &< \sqrt{2\gamma} + \frac{3}{4}\gamma(3M^2(2\sqrt{2\gamma} + 1)m + 4M^2\sqrt{2\gamma}(n-1) + M^2(2\sqrt{2\gamma} + 1)(n-1)) \\ &\quad + \frac{3}{4}\gamma\psi(M^2(n-1) + M^2m) \\ &< \sqrt{2} + 3M^2(2\sqrt{2} + 1)m + 4M^2\sqrt{2}(n-1) \\ &\quad + M^2(2\sqrt{2} + 1)(n-1) + M^2(n-1) + M^2m.\end{aligned}$$

Sonuç olarak

$$E_3 > 2\lambda\nu(x_1 + \delta_0)\left(\frac{3}{4}\delta\alpha_2 - \frac{1}{4}\delta\alpha_2\right) = \lambda\nu(x_1 + \delta_0)\delta\alpha_2 \quad (2.35)$$

bulunur.

Şimdi  $\vartheta^2$ 'nin  $E_4$  katsayısını ele alalım:

$$E_4 = E_{41} + E_{42} + 2(x_1 + \delta_0)^2\lambda\nu\xi\left(\sum_{i=2}^n(a_{ii}\psi_{x_i})_{x_i} - \sum_{j=1}^m(b_{jj}\psi_{y_j})_{y_j}\right).$$

Burada

$$\begin{aligned}
E_{41} = & -2\lambda^3\nu^3\delta^3((x_1 + \delta_0)^{-2}\psi^{-2\nu-2})_{x_1} - 2\lambda^3\nu^3\delta \sum_{i=2}^n (a_{ii}\psi^{-2\nu-2})_{x_1}\psi_{x_i}^2 \\
& + 2\lambda^3\nu^3\delta \sum_{j=1}^m (b_{jj}\psi^{-2\nu-2})_{x_1}\psi_{y_j}^2 - 2\lambda^3\nu^3\delta^2 \sum_{i=2}^n (a_{ii}\psi_{x_i}\psi^{-2\nu-2})_{x_i} \\
& - 2\lambda^3\nu^3(x_1 + \delta_0)^2 \sum_{i,s=2}^n (a_{ii}a_{ss}\psi_{x_i}\psi_{x_s}^2\psi^{-2\nu-2})_{x_i} \\
& + 2\lambda^3\nu^3(x_1 + \delta_0)^2 \sum_{i=2}^n \sum_{j=1}^m (a_{ii}b_{jj}\psi_{x_i}\psi_{y_j}^2\psi^{-2\nu-2})_{x_i}\psi_{y_j}^2 \\
& + 2\lambda^3\nu^3\delta^2 \sum_{j=1}^m (b_{jj}\psi_{y_j}\psi^{-2\nu-2})_{y_j} + 2\lambda^3\nu^3(x_1 + \delta_0)^2 \sum_{j=1}^m \sum_{i=2}^n (a_{ii}b_{jj}\psi_{y_j}\psi_{x_i}^2\psi^{-2\nu-2})_{y_j}\psi_{x_i}^2 \\
& - 2\lambda^3\nu^3(x_1 + \delta_0)^2 \sum_{j,k=1}^m (b_{jj}b_{kk}\psi_{y_j}\psi_{y_k}^2\psi^{-2\nu-2})_{y_j},
\end{aligned}$$

$$\begin{aligned}
E_{42} = & 2\lambda^2\nu^2(\delta((x_1 + \delta_0)^{-1}E_{01})_{x_1} + (x_1 + \delta_0)(\sum_{i=2}^n (a_{ii}\psi_{x_i}E_{01})_{x_i} - \sum_{j=1}^m (b_{jj}\psi_{y_j}E_{01})_{y_j})) \\
& - 2(x_1 + \delta_0)^2 k\psi^{\nu+1},
\end{aligned}$$

$$\begin{aligned}
E_{01} = & (x_1 + \delta_0)^{-1}(\nu + 1)\psi^{-\nu-2}\psi_{x_1}^2 \\
& + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}((\nu + 1)\psi^{-\nu-2}\psi_{x_i}^2 - \psi^{-\nu-1}\psi_{x_i x_i})) \\
& - \sum_{j=1}^m b_{jj}((\nu + 1)\psi^{-\nu-2}\psi_{y_j}^2 - \psi^{-\nu-1}\psi_{y_j y_j})
\end{aligned}$$

olarak tanımlıdır.

Şimdi  $E_{41}$  ifadesini değerlendirelim:

$\psi_{x_1} = \delta$  olduğu dikkate alınarak

$$\begin{aligned}
& -2\lambda^3\nu^3\delta^3((x_1 + \delta_0)^{-2}\psi^{-2\nu-2})_{x_1} \\
= & -2\lambda^3\nu^3\delta^3(-2(x_1 + \delta_0)^{-3}\psi^{-2\nu-2} - 2(\nu + 1)\psi^{-2\nu-3}\psi_{x_1}(x_1 + \delta_0)^{-2}) \\
= & 4\lambda^3\nu^3\delta^4(\nu + 1)(x_1 + \delta_0)^{-2}\psi^{-2\nu-3} + 4\lambda^3\nu^3\delta^3(x_1 + \delta_0)^{-3}\psi^{-2\nu-2}, \tag{2.36}
\end{aligned}$$

$$\begin{aligned}
& -2\lambda^3\nu^3\delta \sum_{i=2}^n (a_{ii}\psi^{-2\nu-2})_{x_1}\psi_{x_i}^2 \\
= & -2\lambda^3\nu^3\delta \sum_{i=2}^n ((a_{ii})_{x_1}\psi^{-2\nu-2} - 2(\nu + 1)\psi^{-2\nu-3}\psi_{x_1}a_{ii})\psi_{x_i}^2 \\
= & 4\lambda^3\nu^3\delta^2(\nu + 1)\psi^{-2\nu-3} \sum_{i=2}^n a_{ii}\psi_{x_i}^2 - 2\lambda^3\nu^3\delta\psi^{-2\nu-2} \sum_{i=2}^n (a_{ii})_{x_1}\psi_{x_i}^2, \tag{2.37}
\end{aligned}$$

$$\begin{aligned}
& 2\lambda^3\nu^3\delta\sum_{j=1}^m(b_{jj}\psi^{-2\nu-2})_{x_1}\psi_{y_j}^2 \\
= & 2\lambda^3\nu^3\delta\sum_{j=1}^m((b_{jj})_{x_1}\psi^{-2\nu-2} - 2(\nu+1)\psi^{-2\nu-3}\psi_{x_1}b_{jj})\psi_{y_j}^2 \\
= & -4\lambda^3\nu^3\delta^2(\nu+1)\psi^{-2\nu-3}\sum_{j=1}^mb_{jj}\psi_{y_j}^2 + 2\lambda^3\nu^3\delta\psi^{-2\nu-2}\sum_{j=1}^m(b_{jj})_{x_1}\psi_{y_j}^2, \tag{2.38}
\end{aligned}$$

$$\begin{aligned}
& -2\lambda^3\nu^3\delta^2\sum_{i=2}^n(a_{ii}\psi_{x_i}\psi^{-2\nu-2})_{x_i} \\
= & -2\lambda^3\nu^3\delta^2\sum_{i=2}^n((a_{ii})_{x_i}\psi_{x_i}\psi^{-2\nu-2} + a_{ii}\psi_{x_ix_i}\psi^{-2\nu-2} - 2(\nu+1)\psi^{-2\nu-3}\psi_{x_i}^2a_{ii}) \\
= & 4\lambda^3\nu^3\delta^2(\nu+1)\psi^{-2\nu-3}\sum_{i=2}^na_{ii}\psi_{x_i}^2 - 2\lambda^3\nu^3\delta^2\psi^{-2\nu-2}\sum_{i=2}^n((a_{ii})_{x_i}\psi_{x_i} + a_{ii}\psi_{x_ix_i}), \tag{2.39}
\end{aligned}$$

$$\begin{aligned}
& -2\lambda^3\nu^3(x_1+\delta_0)^2\sum_{i,s=2}^n(a_{ii}a_{ss}\psi_{x_i}\psi_{x_s}^2\psi^{-2\nu-2})_{x_i} \\
= & -2\lambda^3\nu^3(x_1+\delta_0)^2\sum_{i,s=2}^na_{ii}a_{ss}\psi_{x_i}\psi_{x_s}^2(-2(\nu+1)\psi^{-2\nu-3}\psi_{x_i}) \\
& -2\lambda^3\nu^3(x_1+\delta_0)^2\sum_{i,s=2}^n(a_{ii}a_{ss}\psi_{x_i}\psi_{x_s}^2)_{x_i}\psi^{-2\nu-2} \\
= & 4\lambda^3\nu^3(x_1+\delta_0)^2(\nu+1)\psi^{-2\nu-3}\sum_{i,s=2}^na_{ii}a_{ss}\psi_{x_i}^2\psi_{x_s}^2 \\
& -2\lambda^3\nu^3(x_1+\delta_0)^2\psi^{-2\nu-2}\sum_{i,s=2}^n(a_{ii}a_{ss}\psi_{x_i}\psi_{x_s}^2)_{x_i}, \tag{2.40}
\end{aligned}$$

$$\begin{aligned}
& 2\lambda^3\nu^3(x_1+\delta_0)^2\sum_{i=2}^n\sum_{j=1}^m(a_{ii}b_{jj}\psi_{x_i}\psi^{-2\nu-2})_{x_i}\psi_{y_j}^2 \\
= & 2\lambda^3\nu^3(x_1+\delta_0)^2\sum_{i=2}^n\sum_{j=1}^m((a_{ii})_{x_i}b_{jj}\psi_{x_i}\psi^{-2\nu-2} + a_{ii}(b_{jj})_{x_i}\psi_{x_i}\psi^{-2\nu-2} \\
& + a_{ii}b_{jj}\psi_{x_ix_i}\psi^{-2\nu-2} + a_{ii}b_{jj}\psi_{x_i}(-2(\nu+1)\psi^{-2\nu-3}\psi_{x_i}))\psi_{y_j}^2 \\
= & -4\lambda^3\nu^3(x_1+\delta_0)^2(\nu+1)\psi^{-2\nu-3}\sum_{i=2}^n\sum_{j=1}^ma_{ii}b_{jj}\psi_{x_i}^2\psi_{y_j}^2 \\
& + 2\lambda^3\nu^3(x_1+\delta_0)^2\psi^{-2\nu-2}\sum_{i=2}^n\sum_{j=1}^m((a_{ii})_{x_i}b_{jj}\psi_{x_i} + a_{ii}(b_{jj})_{x_i}\psi_{x_i} + a_{ii}b_{jj}\psi_{x_ix_i})\psi_{y_j}^2, \tag{2.41}
\end{aligned}$$

$$\begin{aligned}
& 2\lambda^3\nu^3\delta^2\sum_{j=1}^m(b_{jj}\psi_{y_j}\psi^{-2\nu-2})_{y_j} \\
= & 2\lambda^3\nu^3\delta^2\sum_{j=1}^m((b_{jj})_{y_j}\psi_{y_j}\psi^{-2\nu-2} + b_{jj}\psi_{y_j y_j}\psi^{-2\nu-2} - 2(\nu+1)\psi^{-2\nu-3}\psi_{y_j}^2 b_{jj}) \\
= & -4\lambda^3\nu^3\delta^2(\nu+1)\psi^{-2\nu-3}\sum_{j=1}^m b_{jj}\psi_{y_j}^2 + 2\lambda^3\nu^3\delta^2\psi^{-2\nu-2}\sum_{j=1}^m((b_{jj})_{y_j}\psi_{y_j} + b_{jj}\psi_{y_j y_j}), \quad (2.42)
\end{aligned}$$

$$\begin{aligned}
& 2\lambda^3\nu^3(x_1 + \delta_0)^2\sum_{j=1}^m\sum_{i=2}^n(a_{ii}b_{jj}\psi_{y_j}\psi^{-2\nu-2})_{y_j}\psi_{x_i}^2 \\
= & 2\lambda^3\nu^3(x_1 + \delta_0)^2\sum_{j=1}^m\sum_{i=2}^n((a_{ii})_{y_j}b_{jj}\psi_{y_j}\psi^{-2\nu-2} + a_{ii}(b_{jj})_{y_j}\psi_{y_j}\psi^{-2\nu-2} \\
& + a_{ii}b_{jj}\psi_{y_j y_j}\psi^{-2\nu-2} + a_{ii}b_{jj}\psi_{y_j}(-2(\nu+1)\psi^{-2\nu-3}\psi_{y_j}))\psi_{x_i}^2 \\
= & -4\lambda^3\nu^3(x_1 + \delta_0)^2(\nu+1)\psi^{-2\nu-3}\sum_{j=1}^m\sum_{i=2}^n a_{ii}b_{jj}\psi_{x_i}^2\psi_{y_j}^2 \\
& + 2\lambda^3\nu^3(x_1 + \delta_0)^2\psi^{-2\nu-2}\sum_{j=1}^m\sum_{i=2}^n((a_{ii})_{y_j}b_{jj}\psi_{y_j} + a_{ii}(b_{jj})_{y_j}\psi_{y_j} + a_{ii}b_{jj}\psi_{y_j y_j})\psi_{x_i}^2, \quad (2.43)
\end{aligned}$$

$$\begin{aligned}
& -2\lambda^3\nu^3(x_1 + \delta_0)^2\sum_{j,k=1}^m(b_{jj}b_{kk}\psi_{y_j}\psi_{y_k}^2\psi^{-2\nu-2})_{y_j} \\
= & -2\lambda^3\nu^3(x_1 + \delta_0)^2\sum_{j,k=1}^m b_{jj}b_{kk}\psi_{y_j}\psi_{y_k}^2(-2(\nu+1)\psi^{-2\nu-3}\psi_{y_j}) \\
& -2\lambda^3\nu^3(x_1 + \delta_0)^2\sum_{j,k=1}^m(b_{jj}b_{kk}\psi_{y_j}\psi_{y_k}^2)_{y_j}\psi^{-2\nu-2} \\
= & 4\lambda^3\nu^3(x_1 + \delta_0)^2(\nu+1)\psi^{-2\nu-3}\sum_{j,k=1}^m b_{jj}b_{kk}\psi_{y_j}^2\psi_{y_k}^2 \\
& -2\lambda^3\nu^3(x_1 + \delta_0)^2\psi^{-2\nu-2}\sum_{j,k=1}^m(b_{jj}b_{kk}\psi_{y_j}\psi_{y_k}^2)_{y_j} \quad (2.44)
\end{aligned}$$

elde edilir.

Bu durumda (2.36)-(2.44) eşitliklerinden  $E_{41}$  aşağıdaki şekilde yazılabilir:

$$\begin{aligned}
E_{41} = & 4\lambda^3\nu^3(\nu+1)\psi^{-2\nu-3}(\delta^4(x_1 + \delta_0)^{-2} + \delta^2\sum_{i=2}^n a_{ii}\psi_{x_i}^2 - \delta^2\sum_{j=1}^m b_{jj}\psi_{y_j}^2 + \delta^2\sum_{i=2}^n a_{ii}\psi_{x_i}^2 \\
& + (x_1 + \delta_0)^2\sum_{i,s=2}^n a_{ii}a_{ss}\psi_{x_i}^2\psi_{x_s}^2 - (x_1 + \delta_0)^2\sum_{i=2}^n\sum_{j=1}^m a_{ii}b_{jj}\psi_{x_i}^2\psi_{y_j}^2 - \delta^2\sum_{j=1}^m b_{jj}\psi_{y_j}^2 \\
& - (x_1 + \delta_0)^2\sum_{j=1}^m\sum_{i=2}^n a_{ii}b_{jj}\psi_{x_i}^2\psi_{y_j}^2 + (x_1 + \delta_0)^2\sum_{j,k=1}^m b_{jj}b_{kk}\psi_{y_j}^2\psi_{y_k}^2)
\end{aligned}$$

$$\begin{aligned}
& +4\lambda^3\nu^3\psi^{-2\nu-2}(\delta^3(x_1+\delta_0)^{-3}-\frac{\delta}{2}\sum_{i=2}^n(a_{ii})_{x_1}\psi_{x_i}^2+\frac{\delta}{2}\sum_{j=1}^m(b_{jj})_{x_1}\psi_{y_j}^2 \\
& -\frac{\delta^2}{2}\sum_{i=2}^n((a_{ii})_{x_i}\psi_{x_i}+a_{ii}\psi_{x_ix_i})-\frac{(x_1+\delta_0)^2}{2}\sum_{i,s=2}^n(a_{ii}a_{ss}\psi_{x_i}\psi_{x_s}^2)_{x_i} \\
& +\frac{(x_1+\delta_0)^2}{2}\sum_{i=2}^n\sum_{j=1}^m((a_{ii})_{x_i}b_{jj}\psi_{x_i}+a_{ii}(b_{jj})_{x_i}\psi_{x_i}+a_{ii}b_{jj}\psi_{x_ix_i})\psi_{y_j}^2 \\
& +\frac{\delta^2}{2}\sum_{j=1}^m((b_{jj})_{y_j}\psi_{y_j}+b_{jj}\psi_{y_jy_j})-\frac{(x_1+\delta_0)^2}{2}\sum_{j,k=1}^m(b_{jj}b_{kk}\psi_{y_j}\psi_{y_k}^2)_{y_j} \\
& +\frac{(x_1+\delta_0)^2}{2}\sum_{j=1}^m\sum_{i=2}^n((a_{ii})_{y_j}b_{jj}\psi_{y_j}+a_{ii}(b_{jj})_{y_j}\psi_{y_j}+a_{ii}b_{jj}\psi_{y_jy_j})\psi_{x_i}^2).
\end{aligned}$$

Yukarıdaki eşitlikte pozitif terimler atılırsa,  $E_4$  için aşağıdaki eşitsizlik elde edilir:

$$\begin{aligned}
E_4 \geq & 4\lambda^3\nu^3(\nu+1)\psi^{-2\nu-3}\delta^4(x_1+\delta_0)^{-2} \\
& +2\lambda^3\nu^3\psi^{-2\nu-2}\delta^2(2\delta(x_1+\delta_0)^{-3}-\frac{1}{\delta}\sum_{i=2}^n(a_{ii})_{x_1}\psi_{x_i}^2+\frac{1}{\delta}\sum_{j=1}^m(b_{jj})_{x_1}\psi_{y_j}^2 \\
& -\sum_{i=2}^n((a_{ii})_{x_i}\psi_{x_i}+a_{ii}\psi_{x_ix_i})-\frac{(x_1+\delta_0)^2}{\delta^2}\sum_{i,s=2}^n(a_{ii}a_{ss}\psi_{x_i}\psi_{x_s}^2)_{x_i} \\
& +\frac{(x_1+\delta_0)^2}{\delta^2}\sum_{i=2}^n\sum_{j=1}^m((a_{ii})_{x_i}b_{jj}\psi_{x_i}+a_{ii}(b_{jj})_{x_i}\psi_{x_i}+a_{ii}b_{jj}\psi_{x_ix_i})\psi_{y_j}^2 \\
& +\sum_{j=1}^m((b_{jj})_{y_j}\psi_{y_j}+b_{jj}\psi_{y_jy_j})-\frac{(x_1+\delta_0)^2}{\delta^2}\sum_{j,k=1}^m(b_{jj}b_{kk}\psi_{y_j}\psi_{y_k}^2)_{y_j} \\
& +\frac{(x_1+\delta_0)^2}{\delta^2}\sum_{j=1}^m\sum_{i=2}^n((a_{ii})_{y_j}b_{jj}\psi_{y_j}+a_{ii}(b_{jj})_{y_j}\psi_{y_j}+a_{ii}b_{jj}\psi_{y_jy_j})\psi_{x_i}^2) \\
& +E_{42}+2(x_1+\delta_0)^2\lambda\nu\xi(\sum_{i=2}^n(a_{ii}\psi_{x_i})_{x_i}-\sum_{j=1}^m(b_{jj}\psi_{y_j})_{y_j}). \tag{2.45}
\end{aligned}$$

Şimdi son eşitsizlikteki terimleri  $(x_1+\delta_0) < 1$ ,  $\delta \geq 4$ ,  $0 < \gamma < 1$ ,  $|\psi_{x_i}| \leq \sqrt{2\gamma}$ ,  $|\psi_{y_j}| \leq \sqrt{2\gamma}$ ,  $\|a_{ii}\|_{C^1(\overline{D \times G})} \leq M$ ,  $\|b_{jj}\|_{C^1(\overline{D \times G})} \leq M$ ,  $2 \leq i \leq n$ ,  $1 \leq j \leq m$  eşitsizliklerini ve  $\psi_{x_ix_s} = 1$  ( $i = s$ ;  $2 \leq i, s \leq n$ ),  $\psi_{y_jy_k} = 1$  ( $j = k$ ;  $1 \leq j, k \leq m$ ) olduğunu göz önünde bulundurarak

değerlendirelim:

$$\begin{aligned}
& 2\delta(x_1 + \delta_0)^{-3} - \frac{1}{\delta} \sum_{i=2}^n (a_{ii})_{x_1} \psi_{x_i}^2 + \frac{1}{\delta} \sum_{j=1}^m (b_{jj})_{x_1} \psi_{y_j}^2 \\
& - \sum_{i=2}^n ((a_{ii})_{x_i} \psi_{x_i} + a_{ii} \psi_{x_i x_i}) - \frac{(x_1 + \delta_0)^2}{\delta^2} \sum_{i,s=2}^n (a_{ii} a_{ss} \psi_{x_i} \psi_{x_s}^2)_{x_i} \\
& + \frac{(x_1 + \delta_0)^2}{\delta^2} \sum_{i=2}^n \sum_{j=1}^m ((a_{ii})_{x_i} b_{jj} \psi_{x_i} + a_{ii} (b_{jj})_{x_i} \psi_{x_i} + a_{ii} b_{jj} \psi_{x_i x_i}) \psi_{y_j}^2 \\
& + \sum_{j=1}^m ((b_{jj})_{y_j} \psi_{y_j} + b_{jj} \psi_{y_j y_j}) - \frac{(x_1 + \delta_0)^2}{\delta^2} \sum_{j,k=1}^m (b_{jj} b_{kk} \psi_{y_j} \psi_{y_k}^2)_{y_j} \\
& + \frac{(x_1 + \delta_0)^2}{\delta^2} \sum_{j=1}^m \sum_{i=2}^n ((a_{ii})_{y_j} b_{jj} \psi_{y_j} + a_{ii} (b_{jj})_{y_j} \psi_{y_j} + a_{ii} b_{jj} \psi_{y_j y_j}) \psi_{x_i}^2 \\
= & 2\delta(x_1 + \delta_0)^{-3} - \frac{1}{\delta} \sum_{i=2}^n (a_{ii})_{x_1} \psi_{x_i}^2 + \frac{1}{\delta} \sum_{j=1}^m (b_{jj})_{x_1} \psi_{y_j}^2 \\
& - \sum_{i=2}^n ((a_{ii})_{x_i} \psi_{x_i} + a_{ii} \psi_{x_i x_i}) + \sum_{j=1}^m ((b_{jj})_{y_j} \psi_{y_j} + b_{jj} \psi_{y_j y_j}) \\
& - \frac{(x_1 + \delta_0)^2}{\delta^2} \sum_{i,s=2}^n ((a_{ii})_{x_i} a_{ss} \psi_{x_i} \psi_{x_s}^2 + a_{ii} (a_{ss})_{x_i} \psi_{x_i} \psi_{x_s}^2 \\
& + a_{ii} a_{ss} \psi_{x_i x_i} \psi_{x_s}^2 + 2a_{ii} a_{ss} \psi_{x_i} \psi_{x_s} \psi_{x_s x_i}) \\
& + \frac{(x_1 + \delta_0)^2}{\delta^2} \sum_{i=2}^n \sum_{j=1}^m ((a_{ii})_{x_i} b_{jj} \psi_{x_i} + a_{ii} (b_{jj})_{x_i} \psi_{x_i} + a_{ii} b_{jj} \psi_{x_i x_i}) \psi_{y_j}^2 \\
& + \frac{(x_1 + \delta_0)^2}{\delta^2} \sum_{j=1}^m \sum_{i=2}^n ((a_{ii})_{y_j} b_{jj} \psi_{y_j} + a_{ii} (b_{jj})_{y_j} \psi_{y_j} + a_{ii} b_{jj} \psi_{y_j y_j}) \psi_{x_i}^2 \\
& - \frac{(x_1 + \delta_0)^2}{\delta^2} \sum_{j,k=1}^m ((b_{jj})_{y_j} b_{kk} \psi_{y_j} \psi_{y_k}^2 + b_{jj} (b_{kk})_{y_j} \psi_{y_j} \psi_{y_k}^2 \\
& + b_{jj} b_{kk} \psi_{y_j y_j} \psi_{y_k}^2 + 2b_{jj} b_{kk} \psi_{y_j} \psi_{y_k} \psi_{y_k y_j}) \\
\geq & 2\delta(x_1 + \delta_0)^{-3} - \frac{1}{\delta} \sum_{i=2}^n |(a_{ii})_{x_1} \psi_{x_i}^2| - \frac{1}{\delta} \sum_{j=1}^m |(b_{jj})_{x_1} \psi_{y_j}^2| \\
& - \sum_{i=2}^n (|(a_{ii})_{x_i} \psi_{x_i}| + |a_{ii} \psi_{x_i x_i}|) - \sum_{j=1}^m (|(b_{jj})_{y_j} \psi_{y_j}| + |b_{jj} \psi_{y_j y_j}|) \\
& - \frac{(x_1 + \delta_0)^2}{\delta^2} \sum_{i,s=2}^n (|(a_{ii})_{x_i} a_{ss} \psi_{x_i} \psi_{x_s}^2| + |a_{ii} (a_{ss})_{x_i} \psi_{x_i} \psi_{x_s}^2| \\
& + |a_{ii} a_{ss} \psi_{x_i x_i} \psi_{x_s}^2| + |2a_{ii} a_{ss} \psi_{x_i} \psi_{x_s} \psi_{x_s x_i}|)
\end{aligned}$$

$$\begin{aligned}
& -\frac{(x_1 + \delta_0)^2}{\delta^2} \sum_{i=2}^n \sum_{j=1}^m (|(a_{ii})_{x_i} b_{jj} \psi_{x_i} \psi_{y_j}^2| + |a_{ii}(b_{jj})_{x_i} \psi_{x_i} \psi_{y_j}^2| + |a_{ii} b_{jj} \psi_{x_i x_i} \psi_{y_j}^2|) \\
& -\frac{(x_1 + \delta_0)^2}{\delta^2} \sum_{j=1}^m \sum_{i=2}^n (|(a_{ii})_{y_j} b_{jj} \psi_{y_j} \psi_{x_i}^2| + |a_{ii}(b_{jj})_{y_j} \psi_{y_j} \psi_{x_i}^2| + |a_{ii} b_{jj} \psi_{y_j y_j} \psi_{x_i}^2|) \\
& -\frac{(x_1 + \delta_0)^2}{\delta^2} \sum_{j,k=1}^m (|(b_{jj})_{y_j} b_{kk} \psi_{y_j} \psi_{y_k}^2| + |b_{jj}(b_{kk})_{y_j} \psi_{y_j} \psi_{y_k}^2| \\
& + |b_{jj} b_{kk} \psi_{y_j y_j} \psi_{y_k}^2| + |2b_{jj} b_{kk} \psi_{y_j} \psi_{y_k} \psi_{y_k y_j}|) \\
> 2\delta - \sum_{i=2}^n |(a_{ii})_{x_1} \psi_{x_i}^2| - \sum_{j=1}^m |(b_{jj})_{x_1} \psi_{y_j}^2| \\
& - \sum_{i=2}^n (|(a_{ii})_{x_i} \psi_{x_i}| + |a_{ii} \psi_{x_i x_i}|) - \sum_{j=1}^m (|(b_{jj})_{y_j} \psi_{y_j}| + |b_{jj} \psi_{y_j y_j}|) \\
& - \sum_{i,s=2}^n (|(a_{ii})_{x_i} a_{ss} \psi_{x_i} \psi_{x_s}^2| + |a_{ii}(a_{ss})_{x_i} \psi_{x_i} \psi_{x_s}^2| \\
& + |a_{ii} a_{ss} \psi_{x_i x_i} \psi_{x_s}^2| + |2a_{ii} a_{ss} \psi_{x_i} \psi_{x_s} \psi_{x_s x_i}|) \\
& - \sum_{i=2}^n \sum_{j=1}^m (|(a_{ii})_{x_i} b_{jj} \psi_{x_i} \psi_{y_j}^2| + |a_{ii}(b_{jj})_{x_i} \psi_{x_i} \psi_{y_j}^2| + |a_{ii} b_{jj} \psi_{x_i x_i} \psi_{y_j}^2|) \\
& - \sum_{j=1}^m \sum_{i=2}^n (|(a_{ii})_{y_j} b_{jj} \psi_{y_j} \psi_{x_i}^2| + |a_{ii}(b_{jj})_{y_j} \psi_{y_j} \psi_{x_i}^2| + |a_{ii} b_{jj} \psi_{y_j y_j} \psi_{x_i}^2|) \\
& - \sum_{j,k=1}^m (|(b_{jj})_{y_j} b_{kk} \psi_{y_j} \psi_{y_k}^2| + |b_{jj}(b_{kk})_{y_j} \psi_{y_j} \psi_{y_k}^2| \\
& + |b_{jj} b_{kk} \psi_{y_j y_j} \psi_{y_k}^2| + |2b_{jj} b_{kk} \psi_{y_j} \psi_{y_k} \psi_{y_k y_j}|) \\
\geq 2\delta - M2\gamma(n-1) - M2\gamma m - M\sqrt{2\gamma}(n-1) - M(n-1) - M\sqrt{2\gamma}m - Mm \\
& - M^2 2\gamma \sqrt{2\gamma}(n-1)^2 - M^2 2\gamma \sqrt{2\gamma}(n-1)^2 - M^2 2\gamma(n-1)^2 - M^2 4\gamma(n-1) \\
& - M^2 2\gamma \sqrt{2\gamma}(n-1)m - M^2 2\gamma \sqrt{2\gamma}(n-1)m - M^2 2\gamma(n-1)m \\
& - M^2 2\gamma \sqrt{2\gamma}(n-1)m - M^2 2\gamma \sqrt{2\gamma}(n-1)m - M^2 2\gamma(n-1)m - M^2 2\gamma \sqrt{2\gamma}m^2 \\
& - M^2 2\gamma \sqrt{2\gamma}m^2 - M^2 2\gamma m^2 - M^2 4\gamma m \\
= 2\delta - (n-1)^2(M^2 2\gamma \sqrt{2\gamma} + M^2 2\gamma \sqrt{2\gamma} + M^2 2\gamma) \\
& - (n-1)(M2\gamma + M\sqrt{2\gamma} + M + M^2 4\gamma) - m^2(M^2 2\gamma \sqrt{2\gamma} + M^2 2\gamma \sqrt{2\gamma} + M^2 2\gamma) \\
& - m(M2\gamma + M\sqrt{2\gamma} + M + M^2 4\gamma) \\
& - (n-1)m(M^2 2\gamma \sqrt{2\gamma} + M^2 2\gamma \sqrt{2\gamma} + M^2 2\gamma + M^2 2\gamma \sqrt{2\gamma} + M^2 2\gamma \sqrt{2\gamma} + M^2 2\gamma)
\end{aligned}$$

$$\begin{aligned}
&= 2\delta - (n-1)^2(M^2 4\gamma \sqrt{2\gamma} + M^2 2\gamma) - (n-1)(M 2\gamma + M \sqrt{2\gamma} + M + M^2 4\gamma) \\
&\quad - m^2(M^2 4\gamma \sqrt{2\gamma} + M^2 2\gamma) - m(M 2\gamma + M \sqrt{2\gamma} + M + M^2 4\gamma) \\
&\quad - (n-1)m(M^2 8\gamma \sqrt{2\gamma} + M^2 4\gamma) \\
&> 2\delta - (n-1)^2(4\sqrt{2}M^2 + 2M^2) - (n-1)(2M + \sqrt{2}M + M + 4M^2) \\
&\quad - m^2(4\sqrt{2}M^2 + 2M^2) - m(2M + \sqrt{2}M + M + 4M^2) \\
&\quad - (n-1)m(8\sqrt{2}M^2 + 4M^2) \\
&= 2\delta - (4\sqrt{2}M^2 + 2M^2)((n-1)^2 + m^2 + 2(n-1)m) \\
&\quad - (2M + \sqrt{2}M + M + 4M^2)(n+m-1) \\
&= 2\delta - (4\sqrt{2} + 2)M^2(n+m-1)^2 - (\sqrt{2} + 3 + 4M)M(n+m-1).
\end{aligned}$$

Burada eğer

$$\delta \geq \delta_4 = \frac{1}{2}((4\sqrt{2} + 2)M^2(n+m-1)^2 + (\sqrt{2} + 3 + 4M)M(n+m-1))$$

alırsa

$$\begin{aligned}
&2\delta(x_1 + \delta_0)^{-3} - \frac{1}{\delta} \sum_{i=2}^n (a_{ii})_{x_1} \psi_{x_i}^2 + \frac{1}{\delta} \sum_{j=1}^m (b_{jj})_{x_1} \psi_{y_j}^2 \\
&- \sum_{i=2}^n ((a_{ii})_{x_i} \psi_{x_i} + a_{ii} \psi_{x_i x_i}) - \frac{(x_1 + \delta_0)^2}{\delta^2} \sum_{i,s=2}^n (a_{ii} a_{ss} \psi_{x_i} \psi_{x_s}^2)_{x_i} \\
&+ \frac{(x_1 + \delta_0)^2}{\delta^2} \sum_{i=2}^n \sum_{j=1}^m ((a_{ii})_{x_i} b_{jj} \psi_{x_i} + a_{ii} (b_{jj})_{x_i} \psi_{x_i} + a_{ii} b_{jj} \psi_{x_i x_i}) \psi_{y_j}^2 \\
&+ \sum_{j=1}^m ((b_{jj})_{y_j} \psi_{y_j} + b_{jj} \psi_{y_j y_j}) - \frac{(x_1 + \delta_0)^2}{\delta^2} \sum_{j,k=1}^m (b_{jj} b_{kk} \psi_{y_j} \psi_{y_k}^2)_{y_j} \\
&+ \frac{(x_1 + \delta_0)^2}{\delta^2} \sum_{j=1}^m \sum_{i=2}^n ((a_{ii})_{y_j} b_{jj} \psi_{y_j} + a_{ii} (b_{jj})_{y_j} \psi_{y_j} + a_{ii} b_{jj} \psi_{y_j y_j}) \psi_{x_i}^2 \\
&> 0
\end{aligned} \tag{2.46}$$

olur. Diğer taraftan

$$\delta^4(x_1 + \delta_0)^{-2} > \delta^4 \tag{2.47}$$

olduğundan

$$\begin{aligned}
& 2\lambda^3\nu^3\delta^4(\nu+1)(x_1+\delta_0)^{-2}\psi^{-2\nu-3} + E_{42} \\
> \lambda^2\nu^2(2\lambda\nu\delta^4(\nu+1)\psi^{-2\nu-3} \\
& + 2(\delta((x_1+\delta_0)^{-1}E_{01})_{x_1} + (x_1+\delta_0)(\sum_{i=2}^n(a_{ii}\psi_{x_i}E_{01})_{x_i} - \sum_{j=1}^m(b_{jj}\psi_{y_j}E_{01})_{y_j})) \\
& - 2(x_1+\delta_0)^2\frac{k}{\lambda^2\nu^2}\psi^{\nu+1}) \\
\geq \lambda^2\nu^2(2\lambda\nu\delta^4(\nu+1)\psi^{-2\nu-3} \\
& - 2|\delta((x_1+\delta_0)^{-1}E_{01})_{x_1} + (x_1+\delta_0)(\sum_{i=2}^n(a_{ii}\psi_{x_i}E_{01})_{x_i} - \sum_{j=1}^m(b_{jj}\psi_{y_j}E_{01})_{y_j})| \\
& - 2(x_1+\delta_0)^2\frac{k}{\lambda^2\nu^2}\psi^{\nu+1}) \\
> \lambda^2\nu^2(4\lambda\nu\delta^4 \\
& - 2|\delta((x_1+\delta_0)^{-1}E_{01})_{x_1} + (x_1+\delta_0)(\sum_{i=2}^n(a_{ii}\psi_{x_i}E_{01})_{x_i} - \sum_{j=1}^m(b_{jj}\psi_{y_j}E_{01})_{y_j})| \\
& - 2(x_1+\delta_0)^2\frac{k}{\lambda^2\nu^2}\psi^{\nu+1}) \\
\geq 0,
\end{aligned}$$

$$2\lambda^3\nu^3\delta^4(\nu+1)(x_1+\delta_0)^{-2}\psi^{-2\nu-3} + E_{42} > 0 \quad (2.48)$$

dır. Böylece (2.45) eşitsizliğinin, (2.46)-(2.48) eşitsizlikleri kullanılarak

$$\begin{aligned}
E_4 & > 2\lambda^3\nu^3\delta^4(\nu+1)\psi^{-2\nu-3} + 2\lambda\nu\xi(x_1+\delta_0)^2(\sum_{i=2}^n(a_{ii}\psi_{x_i})_{x_i} - \sum_{j=1}^m(b_{jj}\psi_{y_j})_{y_j}) \\
& > 2\lambda^3\nu^4\delta^4\psi^{-2\nu-3} + 2\lambda\nu\xi(x_1+\delta_0)^2(\sum_{i=2}^n(a_{ii}\psi_{x_i})_{x_i} - \sum_{j=1}^m(b_{jj}\psi_{y_j})_{y_j})
\end{aligned} \quad (2.49)$$

olduğu görülür. Sonuç olarak

$$\begin{aligned}
\psi^{\nu+1}(L_0\varphi)^2\chi^2 & \geq 2\lambda\nu\gamma^{-3}\delta\vartheta_{x_1}^2 + \lambda\nu(x_1+\delta_0)\delta\alpha_1\sum_{i=2}^n\vartheta_{x_i}^2 \\
& + \lambda\nu(x_1+\delta_0)\delta\alpha_2\sum_{j=1}^m\vartheta_{y_j}^2 + 2\lambda^3\nu^4\delta^4\psi^{-2\nu-3}\vartheta^2 \\
& + 2\lambda\nu\xi(x_1+\delta_0)^2(\sum_{i=2}^n(a_{ii}\psi_{x_i})_{x_i} - \sum_{j=1}^m(b_{jj}\psi_{y_j})_{y_j})\vartheta^2 \\
& + d_1(\vartheta) + d_2(\vartheta)
\end{aligned} \quad (2.50)$$

değerlendirmesi bulunur.

Diğer taraftan, aşağıdaki eşitlikler kullanılarak (2.50) değerlendirmesi  $\varphi$  fonksiyonu için elde edilir.

$$\begin{aligned}
\chi &= e^{\lambda\psi^{-\nu}}, \chi^2 = e^{2\lambda\psi^{-\nu}}, (\chi^2)_{x_i} = -2\lambda\nu\psi^{-\nu-1}\psi_{x_i}\chi^2, \\
\vartheta &= e^{\lambda\psi^{-\nu}}\varphi, \vartheta_{x_i} = -\lambda\nu\psi^{-\nu-1}\psi_{x_i}e^{\lambda\psi^{-\nu}}\varphi + \varphi_{x_i}e^{\lambda\psi^{-\nu}} = (\varphi_{x_i} - \lambda\nu\psi^{-\nu-1}\psi_{x_i}\varphi)\chi \\
\vartheta_{x_i}^2 &= (\varphi_{x_i} - \lambda\nu\psi^{-\nu-1}\psi_{x_i}\varphi)^2\chi^2 \\
&= (\varphi_{x_i}^2 - 2\lambda\nu\psi^{-\nu-1}\psi_{x_i}\varphi\varphi_{x_i} + \lambda^2\nu^2\psi^{-2\nu-2}\psi_{x_i}^2\varphi^2)\chi^2 \\
&= \varphi_{x_i}^2\chi^2 - 2\lambda\nu\psi^{-\nu-1}\psi_{x_i}\varphi\varphi_{x_i}\chi^2 + \lambda^2\nu^2\psi^{-2\nu-2}\psi_{x_i}^2\varphi^2\chi^2.
\end{aligned}$$

$\vartheta_{x_i}^2$ 'nin ifadesi için

$$\begin{aligned}
-2\lambda\nu\psi^{-\nu-1}\psi_{x_i}\varphi\varphi_{x_i}\chi^2 &= -\lambda\nu\psi^{-\nu-1}\psi_{x_i}(\varphi^2)_{x_i}\chi^2, \\
\lambda\nu(\psi^{-\nu-1}\psi_{x_i}\varphi^2\chi^2)_{x_i} &= \lambda\nu(-(\nu+1)\psi^{-\nu-2}\psi_{x_i}^2\varphi^2\chi^2 + \psi^{-\nu-1}\psi_{x_ix_i}\varphi^2\chi^2 \\
&\quad + \psi^{-\nu-1}\psi_{x_i}2\varphi\varphi_{x_i}\chi^2 + \psi^{-\nu-1}\psi_{x_i}\varphi^2(-2\lambda\nu\psi^{-\nu-1}\psi_{x_i}\chi^2)) \\
&= -\lambda\nu(\nu+1)\psi^{-\nu-2}\psi_{x_i}^2\varphi^2\chi^2 + \lambda\nu\psi^{-\nu-1}\psi_{x_ix_i}\varphi^2\chi^2 \\
&\quad + \lambda\nu\psi^{-\nu-1}\psi_{x_i}2\varphi\varphi_{x_i}\chi^2 - 2\lambda^2\nu^2\psi^{-2\nu-2}\psi_{x_i}^2\varphi^2\chi^2 \\
-\lambda\nu\psi^{-\nu-1}\psi_{x_i}(\varphi^2)_{x_i}\chi^2 &= -\lambda\nu(\nu+1)\psi^{-\nu-2}\psi_{x_i}^2\varphi^2\chi^2 + \lambda\nu\psi^{-\nu-1}\psi_{x_ix_i}\varphi^2\chi^2 \\
&\quad - 2\lambda^2\nu^2\psi^{-2\nu-2}\psi_{x_i}^2\varphi^2\chi^2 - \lambda\nu(\psi^{-\nu-1}\psi_{x_i}\varphi^2\chi^2)_{x_i}
\end{aligned}$$

eşitlikleri kullanılırsa  $\psi_{x_ix_i} = 1$  ( $2 \leq i \leq n$ ) olduğundan

$$\begin{aligned}
\vartheta_{x_i}^2 &= \varphi_{x_i}^2\chi^2 - \lambda^2\nu^2\psi^{-2\nu-2}\psi_{x_i}^2\varphi^2\chi^2 - \lambda\nu(\nu+1)\psi^{-\nu-2}\psi_{x_i}^2\varphi^2\chi^2 \\
&\quad + \lambda\nu\psi^{-\nu-1}\varphi^2\chi^2 - \lambda\nu(\psi^{-\nu-1}\psi_{x_i}\varphi^2\chi^2)_{x_i}
\end{aligned}$$

ve buna bağlı olarak  $\psi_{x_1} = \delta$ ,  $\psi_{x_1x_1} = 0$  olduğu göz önünde bulundurularak

$$\begin{aligned}
\vartheta_{x_1}^2 &= \varphi_{x_1}^2\chi^2 - \lambda^2\nu^2\delta^2\psi^{-2\nu-2}\varphi^2\chi^2 - \lambda\nu(\nu+1)\delta^2\psi^{-\nu-2}\varphi^2\chi^2 \\
&\quad - \lambda\nu(\delta\psi^{-\nu-1}\varphi^2\chi^2)_{x_1}
\end{aligned}$$

bulunur. Benzer şekilde,  $\psi_{y_jy_j} = 1$  ( $1 \leq j \leq m$ ) olduğu dikkate alınarak

$$\begin{aligned}
\vartheta_{y_j} &= (\varphi_{y_j} - \lambda\nu\psi^{-\nu-1}\psi_{y_j}\varphi)\chi, \\
\vartheta_{y_j}^2 &= \varphi_{y_j}^2\chi^2 - \lambda^2\nu^2\psi^{-2\nu-2}\psi_{y_j}^2\varphi^2\chi^2 - \lambda\nu(\nu+1)\psi^{-\nu-2}\psi_{y_j}^2\varphi^2\chi^2 \\
&\quad + \lambda\nu\psi^{-\nu-1}\varphi^2\chi^2 - \lambda\nu(\psi^{-\nu-1}\psi_{y_j}\varphi^2\chi^2)_{y_j}
\end{aligned}$$

elde edilir. Sonuç olarak,

$$\begin{aligned}
& 2\lambda\nu\gamma^{-3}\delta\vartheta_{x_1}^2 + \lambda\nu(x_1 + \delta_0)\delta\alpha_1 \sum_{i=2}^n \vartheta_{x_i}^2 + \lambda\nu(x_1 + \delta_0)\delta\alpha_2 \sum_{j=1}^m \vartheta_{y_j}^2 + 2\lambda^3\nu^4\delta^4\psi^{-2\nu-3}\vartheta^2 \\
= & 2\lambda\nu\gamma^{-3}\delta(\varphi_{x_1}^2\chi^2 - \lambda^2\nu^2\delta^2\psi^{-2\nu-2}\varphi^2\chi^2 \\
& - \lambda\nu(\nu + 1)\delta^2\psi^{-\nu-2}\varphi^2\chi^2 - \lambda\nu(\delta\psi^{-\nu-1}\varphi^2\chi^2)_{x_1}) \\
& + \lambda\nu(x_1 + \delta_0)\delta\alpha_1 \sum_{i=2}^n (\varphi_{x_i}^2\chi^2 - \lambda^2\nu^2\psi^{-2\nu-2}\psi_{x_i}^2\varphi^2\chi^2 - \lambda\nu(\nu + 1)\psi^{-\nu-2}\psi_{x_i}^2\varphi^2\chi^2 \\
& + \lambda\nu\psi^{-\nu-1}\varphi^2\chi^2 - \lambda\nu(\psi^{-\nu-1}\psi_{x_i}\varphi^2\chi^2)_{x_i}) \\
& + \lambda\nu(x_1 + \delta_0)\delta\alpha_2 \sum_{j=1}^m (\varphi_{y_j}^2\chi^2 - \lambda^2\nu^2\psi^{-2\nu-2}\psi_{y_j}^2\varphi^2\chi^2 - \lambda\nu(\nu + 1)\psi^{-\nu-2}\psi_{y_j}^2\varphi^2\chi^2 \\
& + \lambda\nu\psi^{-\nu-1}\varphi^2\chi^2 - \lambda\nu(\psi^{-\nu-1}\psi_{y_j}\varphi^2\chi^2)_{y_j}) + 2\lambda^3\nu^4\delta^4\psi^{-2\nu-3}\varphi^2\chi^2 \\
= & 2\lambda\nu\gamma^{-3}\delta\varphi_{x_1}^2\chi^2 + \lambda\nu(x_1 + \delta_0)\delta\alpha_1\chi^2 \sum_{i=2}^n \varphi_{x_i}^2 \\
& + \lambda\nu(x_1 + \delta_0)\delta\alpha_2\chi^2 \sum_{j=1}^m \varphi_{y_j}^2 + 2\lambda^3\nu^4\delta^4\psi^{-2\nu-3}\varphi^2\chi^2 \\
& - 2\lambda^2\nu^2\gamma^{-3}\delta(\delta\psi^{-\nu-1}\varphi^2\chi^2)_{x_1} - \lambda^2\nu^2(x_1 + \delta_0)\delta\alpha_1 \sum_{i=2}^n (\psi^{-\nu-1}\psi_{x_i}\varphi^2\chi^2)_{x_i} \\
& - \lambda^2\nu^2(x_1 + \delta_0)\delta\alpha_2 \sum_{j=1}^m (\psi^{-\nu-1}\psi_{y_j}\varphi^2\chi^2)_{y_j} \\
& - 2\lambda^3\nu^3\gamma^{-3}\delta^3\psi^{-2\nu-2}\varphi^2\chi^2 - \lambda^3\nu^3(x_1 + \delta_0)\delta\alpha_1\psi^{-2\nu-2}\varphi^2\chi^2 \sum_{i=2}^n \psi_{x_i}^2 \\
& - \lambda^3\nu^3(x_1 + \delta_0)\delta\alpha_2\psi^{-2\nu-2}\varphi^2\chi^2 \sum_{j=1}^m \psi_{y_j}^2 \\
& - \lambda^2\nu^2(2\gamma^{-3}\delta^3(\nu + 1)\psi^{-\nu-2} + (x_1 + \delta_0)\delta\alpha_1(\nu + 1)\psi^{-\nu-2} \sum_{i=2}^n \psi_{x_i}^2 \\
& + (x_1 + \delta_0)\delta\alpha_2(\nu + 1)\psi^{-\nu-2} \sum_{j=1}^m \psi_{y_j}^2 - (x_1 + \delta_0)\delta\alpha_1\psi^{-\nu-1}(n - 1) \\
& - (x_1 + \delta_0)\delta\alpha_2\psi^{-\nu-1}m)\varphi^2\chi^2
\end{aligned} \tag{2.51}$$

bulunur. Ayrıca  $(x_1 + \delta_0) < \frac{3}{4}\gamma$ ,  $\psi^{-1} > 1$ ,  $\sum_{i=2}^n \psi_{x_i}^2 \leq 2\gamma$ ,  $\sum_{j=1}^m \psi_{y_j}^2 \leq 2\gamma$  olduğundan  $\nu \geq \nu_1 = 2\gamma^{-3}\delta^{-1} + \frac{3}{2}\gamma^2\delta^{-3}(\alpha_1 + \alpha_2) + 1$  olmak üzere (2.51) eşitliğindeki dördüncü,

sekizinci, dokuzuncu ve onuncu terimler için

$$\begin{aligned}
& \lambda^3 \nu^4 \delta^4 \psi^{-2\nu-3} \varphi^2 \chi^2 - 2\lambda^3 \nu^3 \gamma^{-3} \delta^3 \psi^{-2\nu-2} \varphi^2 \chi^2 \\
& - \lambda^3 \nu^3 (x_1 + \delta_0) \delta \alpha_1 \psi^{-2\nu-2} \varphi^2 \chi^2 \sum_{i=2}^n \psi_{x_i}^2 - \lambda^3 \nu^3 (x_1 + \delta_0) \delta \alpha_2 \psi^{-2\nu-2} \varphi^2 \chi^2 \sum_{j=1}^m \psi_{y_j}^2 \\
= & \varphi^2 \chi^2 \psi^{-2\nu-2} (\lambda^3 \nu^4 \delta^4 \psi^{-1} - 2\lambda^3 \nu^3 \gamma^{-3} \delta^3 - \lambda^3 \nu^3 (x_1 + \delta_0) \delta \alpha_1 \sum_{i=2}^n \psi_{x_i}^2 \\
& - \lambda^3 \nu^3 (x_1 + \delta_0) \delta \alpha_2 \sum_{j=1}^m \psi_{y_j}^2) \\
> & \varphi^2 \chi^2 \psi^{-2\nu-2} (\lambda^3 \nu^4 \delta^4 - 2\lambda^3 \nu^3 \gamma^{-3} \delta^3 - \lambda^3 \nu^3 \frac{3}{4} \gamma \delta \alpha_1 2\gamma - \lambda^3 \nu^3 \frac{3}{4} \gamma \delta \alpha_2 2\gamma) \\
= & \varphi^2 \chi^2 \psi^{-2\nu-2} \lambda^3 \nu^3 \delta^4 (\nu - 2\gamma^{-3} \delta^{-1} - \frac{3}{2} \gamma^2 \delta^{-3} (\alpha_1 + \alpha_2)) \\
\geq & \lambda^3 \nu^3 \delta^4 \psi^{-2\nu-2} \varphi^2 \chi^2
\end{aligned} \tag{2.52}$$

yazılır. Bu sonuç ve onbirinci, onikinci ve onüçüncü terimler birlikte değerlendirilirse her

$\lambda \geq \lambda_1 = 2\gamma^{-3}(\nu + 1) + \frac{3}{2}\gamma^2(\nu + 1)(\alpha_1 + \alpha_2)$ ,  $0 < \psi < 1$  ve  $\nu \geq 1$  için

$$\begin{aligned}
& \lambda^3 \nu^3 \delta^4 \psi^{-2\nu-2} \varphi^2 \chi^2 - 2\lambda^2 \nu^2 \gamma^{-3} \delta^3 (\nu + 1) \psi^{-\nu-2} \varphi^2 \chi^2 \\
& - \lambda^2 \nu^2 (x_1 + \delta_0) \delta \alpha_1 (\nu + 1) \psi^{-\nu-2} \varphi^2 \chi^2 \sum_{i=2}^n \psi_{x_i}^2 \\
& - \lambda^2 \nu^2 (x_1 + \delta_0) \delta \alpha_2 (\nu + 1) \psi^{-\nu-2} \varphi^2 \chi^2 \sum_{j=1}^m \psi_{y_j}^2 \\
= & \lambda^2 \nu^3 \delta^4 \psi^{-2\nu-2} \varphi^2 \chi^2 (\lambda - 2\nu^{-1} \gamma^{-3} \delta^{-1} (\nu + 1) \psi^\nu \\
& - \nu^{-1} (x_1 + \delta_0) \delta^{-3} \alpha_1 (\nu + 1) \psi^\nu \sum_{i=2}^n \psi_{x_i}^2 - \nu^{-1} (x_1 + \delta_0) \delta^{-3} \alpha_2 (\nu + 1) \psi^\nu \sum_{j=1}^m \psi_{y_j}^2) \\
> & \lambda^2 \nu^3 \delta^4 \psi^{-2\nu-2} \varphi^2 \chi^2 (\lambda - 2\gamma^{-3}(\nu + 1) - \frac{3}{4} \gamma \alpha_1 (\nu + 1) 2\gamma - \frac{3}{4} \gamma \alpha_2 (\nu + 1) 2\gamma) \\
= & \lambda^2 \nu^3 \delta^4 \psi^{-2\nu-2} \varphi^2 \chi^2 (\lambda - 2\gamma^{-3}(\nu + 1) - \frac{3}{2} \gamma^2 (\nu + 1) (\alpha_1 + \alpha_2)) \\
\geq & 0
\end{aligned} \tag{2.53}$$

olur. O halde yukarıdaki eşitsizliklerden (2.51) eşitliğini  $\nu \geq \nu_1$  ve  $\lambda \geq \lambda_1$  için

$$\begin{aligned}
& 2\lambda \nu \gamma^{-3} \delta \vartheta_{x_1}^2 + \lambda \nu (x_1 + \delta_0) \delta \alpha_1 \sum_{i=2}^n \vartheta_{x_i}^2 \\
& + \lambda \nu (x_1 + \delta_0) \delta \alpha_2 \sum_{j=1}^m \vartheta_{y_j}^2 + 2\lambda^3 \nu^4 \delta^4 \psi^{-2\nu-3} \vartheta^2 \\
\geq & 2\lambda \nu \gamma^{-3} \delta \varphi_{x_1}^2 \chi^2 + \lambda \nu (x_1 + \delta_0) \delta \alpha_1 \chi^2 \sum_{i=2}^n \varphi_{x_i}^2 + \lambda \nu (x_1 + \delta_0) \delta \alpha_2 \chi^2 \sum_{j=1}^m \varphi_{y_j}^2 \\
& + \lambda^3 \nu^4 \delta^4 \psi^{-2\nu-3} \varphi^2 \chi^2 + d_3(\varphi)
\end{aligned} \tag{2.54}$$

olarak yazabiliriz. Burada

$$\begin{aligned} d_3(\varphi) = & -2\lambda^2\nu^2\gamma^{-3}\delta(\psi^{-\nu-1}\delta\varphi^2\chi^2)_{x_1} - \lambda^2\nu^2(x_1 + \delta_0)\delta\alpha_1 \sum_{i=2}^n (\psi^{-\nu-1}\psi_{x_i}\varphi^2\chi^2)_{x_i} \\ & - \lambda^2\nu^2(x_1 + \delta_0)\delta\alpha_2 \sum_{j=1}^m (\psi^{-\nu-1}\psi_{y_j}\varphi^2\chi^2)_{y_j} \end{aligned}$$

biçiminde tanımlıdır. Böylece

$$\begin{aligned} \psi^{\nu+1}(L_0\varphi)^2\chi^2 \geq & 2\lambda\nu\gamma^{-3}\delta\varphi_{x_1}^2\chi^2 + \lambda\nu(x_1 + \delta_0)\delta\alpha_1\chi^2 \sum_{i=2}^n \varphi_{x_i}^2 \\ & + \lambda\nu(x_1 + \delta_0)\delta\alpha_2\chi^2 \sum_{j=1}^m \varphi_{y_j}^2 + \lambda^3\nu^4\delta^4\psi^{-2\nu-3}\varphi^2\chi^2 \\ & + 2\lambda\nu\xi(x_1 + \delta_0)^2 \left( \sum_{i=2}^n (a_{ii}\psi_{x_i})_{x_i} - \sum_{j=1}^m (b_{jj}\psi_{y_j})_{y_j} \right) \varphi^2\chi^2 + \sum_{j=1}^3 d_j(\varphi) \end{aligned}$$

eşitsizliği bulunur ve ispat tamamlanır. ■

### 2.3 LEMMA 2.3

**Lemma 2.3** *Aşağıdaki eşitsizlik  $Q_\gamma$  da sağlanır:*

$$\begin{aligned} & 2\lambda\nu(x_1 + \delta_0)\chi^2\Lambda\varphi(L_0\varphi) \\ \geq & -2\lambda\nu\Lambda^0\chi^2(\varphi_{x_1}^2 + M(x_1 + \delta_0)^2(\sum_{i=2}^n \varphi_{x_i}^2 + \sum_{j=1}^m \varphi_{y_j}^2)) \\ & - 4\lambda^3\nu^3\chi^2\psi^{-2\nu-2}(\delta^2 + M2\gamma(n + m - 1))\Lambda^0\varphi^2 \\ & - C\lambda^2\nu^2\psi^{-\nu-1}\delta^2\chi^2\varphi^2 - 2\lambda\nu(x_1 + \delta_0)^2\chi^2\Lambda\xi\varphi^2 + d_4(\varphi). \end{aligned} \tag{2.55}$$

Burada  $C > 0$ ,  $a_{ii}$ ,  $b_{jj}$  katsayılarına ve  $n$ ,  $m$  ye bağlı olup

$$\begin{aligned} d_4(\varphi) = & 2\lambda\nu(\chi^2\Lambda\varphi\varphi_{x_1})_{x_1} - \lambda\nu((\chi^2\Lambda)_{x_1}\varphi^2)_{x_1} \\ & + 2\lambda\nu(x_1 + \delta_0)^2(\chi^2\Lambda \sum_{i=2}^n a_{ii}\varphi\varphi_{x_i})_{x_i} - \lambda\nu(x_1 + \delta_0)^2((\chi^2\Lambda \sum_{i=2}^n a_{ii})_{x_i}\varphi^2)_{x_i} \\ & - 2\lambda\nu(x_1 + \delta_0)^2(\chi^2\Lambda \sum_{j=1}^m b_{jj}\varphi\varphi_{y_j})_{y_j} + \lambda\nu(x_1 + \delta_0)^2((\chi^2\Lambda \sum_{j=1}^m b_{jj})_{y_j}\varphi^2)_{y_j} \\ & - (2\lambda\nu(x_1 + \delta_0)^2\chi^2\Lambda k\varphi^2)_\xi, \end{aligned}$$

$$\Lambda = \sum_{i=2}^n (a_{ii}\psi_{x_i})_{x_i} - \sum_{j=1}^m (b_{jj}\psi_{y_j})_{y_j},$$

$$\Lambda^0 = \max\{(\sqrt{2\gamma} + 1)M(n + m - 1) + M(n - 1), (\sqrt{2\gamma} + 1)M(n + m - 1) + Mm\}$$

şeklinde tanımlıdır.

**İspat.** İlk olarak aşağıdaki eşitliği ele alalım:

$$\begin{aligned}
& 2\lambda\nu(x_1 + \delta_0)\chi^2\Lambda\varphi(L_0\varphi) \\
= & 2\lambda\nu(x_1 + \delta_0)\chi^2\Lambda\varphi((x_1 + \delta_0)^{-1}\varphi_{x_1x_1} \\
& + (x_1 + \delta_0)(\sum_{i=2}^n a_{ii}\varphi_{x_ix_i} - \sum_{j=1}^m b_{jj}\varphi_{y_jy_j} - 2k\varphi_\xi - \xi\varphi)) \\
= & 2\lambda\nu\chi^2\Lambda\varphi\varphi_{x_1x_1} + 2\lambda\nu(x_1 + \delta_0)^2\chi^2\Lambda\varphi\sum_{i=2}^n a_{ii}\varphi_{x_ix_i} - 2\lambda\nu(x_1 + \delta_0)^2\chi^2\Lambda\varphi\sum_{j=1}^m b_{jj}\varphi_{y_jy_j} \\
& - 4\lambda\nu(x_1 + \delta_0)^2\chi^2\Lambda\varphi k\varphi_\xi - 2\lambda\nu(x_1 + \delta_0)^2\chi^2\Lambda\varphi\xi\varphi.
\end{aligned}$$

Bu eşitlikteki terimler  $a_{x_i}b_{x_i} = (a_{x_i}b)_{x_i} - a_{x_ix_i}b$  özdeşliğide göz önünde bulundurularak değerlendirilirse,

$$\begin{aligned}
2\lambda\nu\chi^2\Lambda\varphi\varphi_{x_1x_1} &= 2\lambda\nu((\chi^2\Lambda\varphi\varphi_{x_1})_{x_1} - (\chi^2\Lambda)_{x_1}\varphi\varphi_{x_1} - \chi^2\Lambda\varphi_{x_1}^2) \\
&= 2\lambda\nu(\chi^2\Lambda\varphi\varphi_{x_1})_{x_1} - 2\lambda\nu(\chi^2\Lambda)_{x_1}\varphi\varphi_{x_1} - 2\lambda\nu\chi^2\Lambda\varphi_{x_1}^2, \\
2\lambda\nu(\chi^2\Lambda)_{x_1}\varphi\varphi_{x_1} &= \lambda\nu(\chi^2\Lambda)_{x_1}(\varphi^2)_{x_1} \\
&= \lambda\nu(((\chi^2\Lambda)_{x_1}\varphi^2)_{x_1} - (\chi^2\Lambda)_{x_1x_1}\varphi^2), \\
2\lambda\nu\chi^2\Lambda\varphi\varphi_{x_1x_1} &= 2\lambda\nu(\chi^2\Lambda\varphi\varphi_{x_1})_{x_1} - \lambda\nu((\chi^2\Lambda)_{x_1}\varphi^2)_{x_1} \\
&\quad + \lambda\nu(\chi^2\Lambda)_{x_1x_1}\varphi^2 - 2\lambda\nu\chi^2\Lambda\varphi_{x_1}^2, \tag{2.56}
\end{aligned}$$

$$\begin{aligned}
2\lambda\nu(x_1 + \delta_0)^2\chi^2\Lambda\varphi\sum_{i=2}^n a_{ii}\varphi_{x_ix_i} &= 2\lambda\nu(x_1 + \delta_0)^2\chi^2\Lambda\sum_{i=2}^n a_{ii}\varphi\varphi_{x_ix_i} \\
&= 2\lambda\nu(x_1 + \delta_0)^2\chi^2\Lambda\sum_{i=2}^n a_{ii}((\varphi\varphi_{x_i})_{x_i} - \varphi_{x_i}^2) \\
&= 2\lambda\nu(x_1 + \delta_0)^2\chi^2\Lambda\sum_{i=2}^n a_{ii}(\varphi\varphi_{x_i})_{x_i} \\
&\quad - 2\lambda\nu(x_1 + \delta_0)^2\chi^2\Lambda\sum_{i=2}^n a_{ii}\varphi_{x_i}^2,
\end{aligned}$$

$$\begin{aligned}
& 2\lambda\nu(x_1 + \delta_0)^2\chi^2\Lambda\sum_{i=2}^n a_{ii}(\varphi\varphi_{x_i})_{x_i} \\
= & 2\lambda\nu(x_1 + \delta_0)^2(\chi^2\Lambda\sum_{i=2}^n a_{ii}\varphi\varphi_{x_i})_{x_i} - 2\lambda\nu(x_1 + \delta_0)^2(\chi^2\Lambda\sum_{i=2}^n a_{ii})_{x_i}\varphi\varphi_{x_i} \\
= & 2\lambda\nu(x_1 + \delta_0)^2(\chi^2\Lambda\sum_{i=2}^n a_{ii}\varphi\varphi_{x_i})_{x_i} - \lambda\nu(x_1 + \delta_0)^2(\chi^2\Lambda\sum_{i=2}^n a_{ii})_{x_i}(\varphi^2)_{x_i},
\end{aligned}$$

$$\begin{aligned}
& \lambda\nu(x_1 + \delta_0)^2 (\chi^2 \Lambda \sum_{i=2}^n a_{ii})_{x_i} (\varphi^2)_{x_i} \\
= & \lambda\nu(x_1 + \delta_0)^2 ((\chi^2 \Lambda \sum_{i=2}^n a_{ii})_{x_i} \varphi^2)_{x_i} - (\chi^2 \Lambda \sum_{i=2}^n a_{ii})_{x_i x_i} \varphi^2, \\
& 2\lambda\nu(x_1 + \delta_0)^2 \chi^2 \Lambda \varphi \sum_{i=2}^n a_{ii} \varphi_{x_i x_i} \\
= & 2\lambda\nu(x_1 + \delta_0)^2 (\chi^2 \Lambda \sum_{i=2}^n a_{ii} \varphi_{x_i})_{x_i} - \lambda\nu(x_1 + \delta_0)^2 ((\chi^2 \Lambda \sum_{i=2}^n a_{ii})_{x_i} \varphi^2)_{x_i} \\
& + \lambda\nu(x_1 + \delta_0)^2 (\chi^2 \Lambda \sum_{i=2}^n a_{ii})_{x_i x_i} \varphi^2 - 2\lambda\nu(x_1 + \delta_0)^2 \chi^2 \Lambda \sum_{i=2}^n a_{ii} \varphi_{x_i}^2, \tag{2.57} \\
-2\lambda\nu(x_1 + \delta_0)^2 \chi^2 \Lambda \varphi \sum_{j=1}^m b_{jj} \varphi_{y_j y_j} &= -2\lambda\nu(x_1 + \delta_0)^2 \chi^2 \Lambda \sum_{j=1}^m b_{jj} \varphi \varphi_{y_j y_j} \\
&= -2\lambda\nu(x_1 + \delta_0)^2 \chi^2 \Lambda \sum_{j=1}^m b_{jj} ((\varphi \varphi_{y_j})_{y_j} - \varphi_{y_j}^2) \\
&= -2\lambda\nu(x_1 + \delta_0)^2 \chi^2 \Lambda \sum_{j=1}^m b_{jj} (\varphi \varphi_{y_j})_{y_j} \\
&\quad + 2\lambda\nu(x_1 + \delta_0)^2 \chi^2 \Lambda \sum_{j=1}^m b_{jj} \varphi_{y_j}^2, \\
-2\lambda\nu(x_1 + \delta_0)^2 \chi^2 \Lambda \sum_{j=1}^m b_{jj} (\varphi \varphi_{y_j})_{y_j} \\
= & -2\lambda\nu(x_1 + \delta_0)^2 (\chi^2 \Lambda \sum_{j=1}^m b_{jj} \varphi \varphi_{y_j})_{y_j} + 2\lambda\nu(x_1 + \delta_0)^2 (\chi^2 \Lambda \sum_{j=1}^m b_{jj})_{y_j} \varphi \varphi_{y_j} \\
= & -2\lambda\nu(x_1 + \delta_0)^2 (\chi^2 \Lambda \sum_{j=1}^m b_{jj} \varphi \varphi_{y_j})_{y_j} + \lambda\nu(x_1 + \delta_0)^2 (\chi^2 \Lambda \sum_{j=1}^m b_{jj})_{y_j} (\varphi^2)_{y_j}, \\
& \lambda\nu(x_1 + \delta_0)^2 (\chi^2 \Lambda \sum_{j=1}^m b_{jj})_{y_j} (\varphi^2)_{y_j} \\
= & \lambda\nu(x_1 + \delta_0)^2 ((\chi^2 \Lambda \sum_{j=1}^m b_{jj})_{y_j} \varphi^2)_{y_j} - (\chi^2 \Lambda \sum_{j=1}^m b_{jj})_{y_j y_j} \varphi^2, \\
& -2\lambda\nu(x_1 + \delta_0)^2 \chi^2 \Lambda \varphi \sum_{j=1}^m b_{jj} \varphi_{y_j y_j} \\
= & -2\lambda\nu(x_1 + \delta_0)^2 (\chi^2 \Lambda \sum_{j=1}^m b_{jj} \varphi \varphi_{y_j})_{y_j} + \lambda\nu(x_1 + \delta_0)^2 ((\chi^2 \Lambda \sum_{j=1}^m b_{jj})_{y_j} \varphi^2)_{y_j} \\
& - \lambda\nu(x_1 + \delta_0)^2 (\chi^2 \Lambda \sum_{j=1}^m b_{jj})_{y_j y_j} \varphi^2 + 2\lambda\nu(x_1 + \delta_0)^2 \chi^2 \Lambda \sum_{j=1}^m b_{jj} \varphi_{y_j}^2, \tag{2.58}
\end{aligned}$$

$$-4\lambda\nu(x_1 + \delta_0)^2\chi^2\Lambda\varphi k\varphi_\xi = -(2\lambda\nu(x_1 + \delta_0)^2\chi^2\Lambda k\varphi^2)_\xi, \quad (2.59)$$

$$-2\lambda\nu(x_1 + \delta_0)^2\chi^2\Lambda\varphi\xi\varphi = -2\lambda\nu(x_1 + \delta_0)^2\chi^2\Lambda\xi\varphi^2 \quad (2.60)$$

elde edilir. (2.56)-(2.60) eşitlikleri kullanılarak

$$\begin{aligned} & 2\lambda\nu(x_1 + \delta_0)\chi^2\Lambda\varphi(L_0\varphi) \\ = & 2\lambda\nu(\chi^2\Lambda\varphi\varphi_{x_1})_{x_1} - \lambda\nu((\chi^2\Lambda)_{x_1}\varphi^2)_{x_1} + \lambda\nu(\chi^2\Lambda)_{x_1x_1}\varphi^2 - 2\lambda\nu\chi^2\Lambda\varphi_{x_1}^2 \\ & + 2\lambda\nu(x_1 + \delta_0)^2(\chi^2\Lambda\sum_{i=2}^n a_{ii}\varphi\varphi_{x_i})_{x_i} - \lambda\nu(x_1 + \delta_0)^2((\chi^2\Lambda\sum_{i=2}^n a_{ii})_{x_i}\varphi^2)_{x_i} \\ & + \lambda\nu(x_1 + \delta_0)^2(\chi^2\Lambda\sum_{i=2}^n a_{ii})_{x_ix_i}\varphi^2 - 2\lambda\nu(x_1 + \delta_0)^2\chi^2\Lambda\sum_{i=2}^n a_{ii}\varphi_{x_i}^2 \\ & - 2\lambda\nu(x_1 + \delta_0)^2(\chi^2\Lambda\sum_{j=1}^m b_{jj}\varphi\varphi_{y_j})_{y_j} + \lambda\nu(x_1 + \delta_0)^2((\chi^2\Lambda\sum_{j=1}^m b_{jj})_{y_j}\varphi^2)_{y_j} \\ & - \lambda\nu(x_1 + \delta_0)^2(\chi^2\Lambda\sum_{j=1}^m b_{jj})_{y_jy_j}\varphi^2 + 2\lambda\nu(x_1 + \delta_0)^2\chi^2\Lambda\sum_{j=1}^m b_{jj}\varphi_{y_j}^2 \\ & - (2\lambda\nu(x_1 + \delta_0)^2\chi^2\Lambda k\varphi^2)_\xi - 2\lambda\nu(x_1 + \delta_0)^2\chi^2\Lambda\xi\varphi^2 \\ = & \lambda\nu(\chi^2\Lambda)_{x_1x_1}\varphi^2 - 2\lambda\nu\chi^2\Lambda\varphi_{x_1}^2 + \lambda\nu(x_1 + \delta_0)^2(\chi^2\Lambda\sum_{i=2}^n a_{ii})_{x_ix_i}\varphi^2 \\ & - 2\lambda\nu(x_1 + \delta_0)^2\chi^2\Lambda\sum_{i=2}^n a_{ii}\varphi_{x_i}^2 - \lambda\nu(x_1 + \delta_0)^2(\chi^2\Lambda\sum_{j=1}^m b_{jj})_{y_jy_j}\varphi^2 \\ & + 2\lambda\nu(x_1 + \delta_0)^2\chi^2\Lambda\sum_{j=1}^m b_{jj}\varphi_{y_j}^2 - 2\lambda\nu(x_1 + \delta_0)^2\chi^2\Lambda\xi\varphi^2 + d_4(\varphi), \\ & 2\lambda\nu(x_1 + \delta_0)\chi^2\Lambda\varphi(L_0\varphi) \\ = & -2\lambda\nu\Lambda\chi^2(\varphi_{x_1}^2 + (x_1 + \delta_0)^2(\sum_{i=2}^n a_{ii}\varphi_{x_i}^2 - \sum_{j=1}^m b_{jj}\varphi_{y_j}^2)) \\ & + A\varphi^2 - 2\lambda\nu(x_1 + \delta_0)^2\chi^2\Lambda\xi\varphi^2 + d_4(\varphi) \end{aligned} \quad (2.61)$$

bulunur. Burada

$$A = \lambda\nu((\chi^2\Lambda)_{x_1x_1} + (x_1 + \delta_0)^2((\chi^2\Lambda\sum_{i=2}^n a_{ii})_{x_ix_i} - (\chi^2\Lambda\sum_{j=1}^m b_{jj})_{y_jy_j}))$$

dir. Diğer taraftan  $|\Lambda| \leq \Lambda^0$ ,  $|a_{ii}| \leq M$ ,  $2 \leq i \leq n$  ve  $|b_{jj}| \leq M$ ,  $1 \leq j \leq m$  olduğundan

$$\begin{aligned}
& -2\lambda\nu\Lambda\chi^2(\varphi_{x_1}^2 + (x_1 + \delta_0)^2(\sum_{i=2}^n a_{ii}\varphi_{x_i}^2 - \sum_{j=1}^m b_{jj}\varphi_{y_j}^2)) \\
\geq & -2\lambda\nu|\Lambda|\chi^2(\varphi_{x_1}^2 + (x_1 + \delta_0)^2(\sum_{i=2}^n |a_{ii}\varphi_{x_i}^2| + \sum_{j=1}^m |b_{jj}\varphi_{y_j}^2|)) \\
\geq & -2\lambda\nu\Lambda^0\chi^2(\varphi_{x_1}^2 + M(x_1 + \delta_0)^2(\sum_{i=2}^n \varphi_{x_i}^2 + \sum_{j=1}^m \varphi_{y_j}^2))
\end{aligned} \tag{2.62}$$

yazılabilir. Ayrıca  $A$  katsayısını değerlendirmek için

$$\begin{aligned}
(\chi^2\Lambda)_{x_1} &= (\chi^2)_{x_1}\Lambda + \chi^2\Lambda_{x_1}, \\
(\chi^2\Lambda)_{x_1x_1} &= ((\chi^2)_{x_1}\Lambda + \chi^2\Lambda_{x_1})_{x_1} \\
&= (\chi^2)_{x_1x_1}\Lambda + (\chi^2)_{x_1}\Lambda_{x_1} + (\chi^2)_{x_1}\Lambda_{x_1} + \chi^2\Lambda_{x_1x_1} \\
&= (\chi^2)_{x_1x_1}\Lambda + 2(\chi^2)_{x_1}\Lambda_{x_1} + \chi^2\Lambda_{x_1x_1}, \\
(\chi^2\Lambda a_{ii})_{x_i} &= (\chi^2)_{x_i}\Lambda a_{ii} + \chi^2\Lambda_{x_i}a_{ii} + \chi^2\Lambda(a_{ii})_{x_i}, \\
(\chi^2\Lambda a_{ii})_{x_ix_i} &= ((\chi^2)_{x_i}\Lambda a_{ii} + \chi^2\Lambda_{x_i}a_{ii} + \chi^2\Lambda(a_{ii})_{x_i})_{x_i} \\
&= (\chi^2)_{x_ix_i}\Lambda a_{ii} + (\chi^2)_{x_i}\Lambda_{x_i}a_{ii} + (\chi^2)_{x_i}\Lambda(a_{ii})_{x_i} \\
&\quad + (\chi^2)_{x_i}\Lambda_{x_i}a_{ii} + \chi^2\Lambda_{x_ix_i}a_{ii} + \chi^2\Lambda_{x_i}(a_{ii})_{x_i} \\
&\quad + (\chi^2)_{x_i}\Lambda(a_{ii})_{x_i} + \chi^2\Lambda_{x_i}(a_{ii})_{x_i} + \chi^2\Lambda(a_{ii})_{x_ix_i} \\
&= \chi^2(\Lambda_{x_ix_i}a_{ii} + \Lambda_{x_i}(a_{ii})_{x_i} + \Lambda_{x_i}(a_{ii})_{x_i} + \Lambda(a_{ii})_{x_ix_i}) \\
&\quad + (\chi^2)_{x_i}(\Lambda_{x_i}a_{ii} + \Lambda(a_{ii})_{x_i} + \Lambda_{x_i}a_{ii}) + (\chi^2)_{x_ix_i}\Lambda a_{ii} + (\chi^2)_{x_i}\Lambda(a_{ii})_{x_i}, \\
(\chi^2\Lambda a_{ii})_{x_ix_i} &= \Lambda((\chi^2)_{x_i}a_{ii})_{x_i} + (\chi^2)_{x_i}(2\Lambda_{x_i}a_{ii} + \Lambda(a_{ii})_{x_i}) \\
&\quad + \chi^2(\Lambda_{x_ix_i}a_{ii} + 2\Lambda_{x_i}(a_{ii})_{x_i} + \Lambda(a_{ii})_{x_ix_i}), \\
(\chi^2\Lambda b_{jj})_{y_j} &= (\chi^2)_{y_j}\Lambda b_{jj} + \chi^2\Lambda_{y_j}b_{jj} + \chi^2\Lambda(b_{jj})_{y_j}, \\
(\chi^2\Lambda b_{jj})_{y_jy_j} &= \Lambda((\chi^2)_{y_j}b_{jj})_{y_j} + (\chi^2)_{y_j}(2\Lambda_{y_j}b_{jj} + \Lambda(b_{jj})_{y_j}) \\
&\quad + \chi^2(\Lambda_{y_jy_j}b_{jj} + 2\Lambda_{y_j}(b_{jj})_{y_j} + \Lambda(b_{jj})_{y_jy_j})
\end{aligned}$$

bağıntıları yardımıyla

$$\begin{aligned}
A &= \lambda\nu((\chi^2)_{x_1x_1}\Lambda + 2(\chi^2)_{x_1}\Lambda_{x_1} + \chi^2\Lambda_{x_1x_1}) \\
&+ (x_1 + \delta_0)^2\left(\sum_{i=2}^n(\Lambda((\chi^2)_{x_i}a_{ii})_{x_i} + (\chi^2)_{x_i}(2\Lambda_{x_i}a_{ii} + \Lambda(a_{ii})_{x_i}))\right. \\
&+ \chi^2(\Lambda_{x_ix_i}a_{ii} + 2\Lambda_{x_i}(a_{ii})_{x_i} + \Lambda(a_{ii})_{x_ix_i})) \\
&- \sum_{j=1}^m(\Lambda((\chi^2)_{y_j}b_{jj})_{y_j} + (\chi^2)_{y_j}(2\Lambda_{y_j}b_{jj} + \Lambda(b_{jj})_{y_j})) \\
&+ \chi^2(\Lambda_{y_jy_j}b_{jj} + 2\Lambda_{y_j}(b_{jj})_{y_j} + \Lambda(b_{jj})_{y_jy_j}))
\end{aligned} \tag{2.63}$$

bulunur. Ayrıca  $\psi_{x_1} = \delta$  ve  $\psi_{x_1x_1} = 0$  olduğu göz önünde bulundurularak

$$\begin{aligned}
\chi^2 &= e^{2\lambda\psi^{-\nu}}, \\
(\chi^2)_{x_1} &= -2\lambda\nu\psi^{-\nu-1}\psi_{x_1}e^{2\lambda\psi^{-\nu}} = -2\lambda\nu\psi^{-\nu-1}\delta\chi^2, \\
(\chi^2)_{x_1x_1} &= -2\lambda\nu(-(\nu+1)\psi^{-\nu-2}\psi_{x_1}^2\chi^2 + \psi^{-\nu-1}\psi_{x_1x_1}\chi^2 \\
&\quad + \psi^{-\nu-1}\psi_{x_1}(-2\lambda\nu\psi^{-\nu-1}\psi_{x_1}\chi^2)) \\
&= 2\lambda\nu(\nu+1)\delta^2\psi^{-\nu-2}\chi^2 + 4\lambda^2\nu^2\delta^2\psi^{-2\nu-2}\chi^2 \\
&= 2\lambda\nu\chi^2(2\lambda\nu\delta^2\psi^{-2\nu-2} + (\nu+1)\delta^2\psi^{-\nu-2}), \\
(\chi^2)_{x_i} &= -2\lambda\nu\psi^{-\nu-1}\psi_{x_i}e^{2\lambda\psi^{-\nu}} = -2\lambda\nu\psi^{-\nu-1}\psi_{x_i}\chi^2, \\
(\chi^2)_{x_ix_i} &= -2\lambda\nu(-(\nu+1)\psi^{-\nu-2}\psi_{x_i}^2\chi^2 + \psi^{-\nu-1}\psi_{x_ix_i}\chi^2 + \psi^{-\nu-1}\psi_{x_i}(-2\lambda\nu\psi^{-\nu-1}\psi_{x_i}\chi^2)) \\
&= 2\lambda\nu\chi^2(2\lambda\nu\psi^{-2\nu-2}\psi_{x_i}^2 + (\nu+1)\psi^{-\nu-2}\psi_{x_i}^2 - \psi^{-\nu-1}\psi_{x_ix_i}), \\
(a_{ii}(\chi^2)_{x_i})_{x_i} &= (a_{ii})_{x_i}(\chi^2)_{x_i} + a_{ii}(\chi^2)_{x_ix_i} \\
&= (a_{ii})_{x_i}(-2\lambda\nu\psi^{-\nu-1}\psi_{x_i}\chi^2) \\
&\quad + a_{ii}(2\lambda\nu\chi^2(2\lambda\nu\psi^{-2\nu-2}\psi_{x_i}^2 + (\nu+1)\psi^{-\nu-2}\psi_{x_i}^2 - \psi^{-\nu-1}\psi_{x_ix_i})) \\
&= 2\lambda\nu\chi^2(a_{ii}(2\lambda\nu\psi^{-2\nu-2}\psi_{x_i}^2 + (\nu+1)\psi^{-\nu-2}\psi_{x_i}^2 - \psi^{-\nu-1}\psi_{x_ix_i}) \\
&\quad - (a_{ii})_{x_i}\psi^{-\nu-1}\psi_{x_i}), \\
(\chi^2)_{y_j} &= -2\lambda\nu\psi^{-\nu-1}\psi_{y_j}e^{2\lambda\psi^{-\nu}} = -2\lambda\nu\psi^{-\nu-1}\psi_{y_j}\chi^2, \\
(\chi^2)_{y_jy_j} &= -2\lambda\nu(-(\nu+1)\psi^{-\nu-2}\psi_{y_j}^2\chi^2 + \psi^{-\nu-1}\psi_{y_jy_j}\chi^2 + \psi^{-\nu-1}\psi_{y_j}(-2\lambda\nu\psi^{-\nu-1}\psi_{y_j}\chi^2)) \\
&= 2\lambda\nu\chi^2(2\lambda\nu\psi^{-2\nu-2}\psi_{y_j}^2 + (\nu+1)\psi^{-\nu-2}\psi_{y_j}^2 - \psi^{-\nu-1}\psi_{y_jy_j}),
\end{aligned}$$

$$\begin{aligned}
(b_{jj}(\chi^2)_{y_j})_{y_j} &= (b_{jj})_{y_j}(\chi^2)_{y_j} + b_{jj}(\chi^2)_{y_j y_j} \\
&= (b_{jj})_{y_j}(-2\lambda\nu\psi^{-\nu-1}\psi_{y_j}\chi^2) \\
&\quad + b_{jj}(2\lambda\nu\chi^2(2\lambda\nu\psi^{-2\nu-2}\psi_{y_j}^2 + (\nu+1)\psi^{-\nu-2}\psi_{y_j}^2 - \psi^{-\nu-1}\psi_{y_j y_j})) \\
&= 2\lambda\nu\chi^2(b_{jj}(2\lambda\nu\psi^{-2\nu-2}\psi_{y_j}^2 + (\nu+1)\psi^{-\nu-2}\psi_{y_j}^2 - \psi^{-\nu-1}\psi_{y_j y_j}) \\
&\quad - (b_{jj})_{y_j}\psi^{-\nu-1}\psi_{y_j})
\end{aligned}$$

eşitlikleri yazılabilir. Bu son eşitlikler kullanılırsa

$$\begin{aligned}
A &= \lambda\nu(2\lambda\nu\chi^2(2\lambda\nu\delta^2\psi^{-2\nu-2} + (\nu+1)\delta^2\psi^{-\nu-2})\Lambda \\
&\quad + 2(-2\lambda\nu\delta\psi^{-\nu-1}\chi^2)\Lambda_{x_1} + \chi^2\Lambda_{x_1 x_1} \\
&\quad + (x_1 + \delta_0)^2(\sum_{i=2}^n (\Lambda 2\lambda\nu\chi^2(a_{ii}(2\lambda\nu\psi^{-2\nu-2}\psi_{x_i}^2 + (\nu+1)\psi^{-\nu-2}\psi_{x_i}^2 - \psi^{-\nu-1}\psi_{x_i x_i}) \\
&\quad - (a_{ii})_{x_i}\psi^{-\nu-1}\psi_{x_i}) - 2\lambda\nu\psi^{-\nu-1}\psi_{x_i}\chi^2(2\Lambda_{x_i}a_{ii} + \Lambda(a_{ii})_{x_i}) \\
&\quad + \chi^2(\Lambda_{x_i x_i}a_{ii} + 2\Lambda_{x_i}(a_{ii})_{x_i} + \Lambda(a_{ii})_{x_i x_i})) \\
&\quad - \sum_{j=1}^m (\Lambda 2\lambda\nu\chi^2(b_{jj}(2\lambda\nu\psi^{-2\nu-2}\psi_{y_j}^2 + (\nu+1)\psi^{-\nu-2}\psi_{y_j}^2 - \psi^{-\nu-1}\psi_{y_j y_j}) \\
&\quad - (b_{jj})_{y_j}\psi^{-\nu-1}\psi_{y_j}) - 2\lambda\nu\psi^{-\nu-1}\psi_{y_j}\chi^2(2\Lambda_{y_j}b_{jj} + \Lambda(b_{jj})_{y_j}) \\
&\quad + \chi^2(\Lambda_{y_j y_j}b_{jj} + 2\Lambda_{y_j}(b_{jj})_{y_j} + \Lambda(b_{jj})_{y_j y_j})))
\end{aligned}$$

elde edilir. Diğer taraftan  $\psi_{x_i x_i} = 1$ ,  $\psi_{y_j y_j} = 1$  ve  $\psi_{x_i y_j} = 0$ ,  $2 \leq i \leq n$ ,  $1 \leq j \leq m$  olduğu dikkate alınarak

$$\begin{aligned}
\Lambda &= \sum_{i=2}^n (a_{ii}\psi_{x_i})_{x_i} - \sum_{j=1}^m (b_{jj}\psi_{y_j})_{y_j} \\
&= \sum_{i=2}^n ((a_{ii})_{x_i}\psi_{x_i} + a_{ii}\psi_{x_i x_i}) - \sum_{j=1}^m ((b_{jj})_{y_j}\psi_{y_j} + b_{jj}\psi_{y_j y_j}) \\
&= \sum_{i=2}^n (a_{ii})_{x_i}\psi_{x_i} + \sum_{i=2}^n a_{ii} - \sum_{j=1}^m (b_{jj})_{y_j}\psi_{y_j} - \sum_{j=1}^m b_{jj},
\end{aligned}$$

$$\begin{aligned}
\Lambda_{x_i} &= \sum_{i=2}^n ((a_{ii})_{x_i x_i}\psi_{x_i} + (a_{ii})_{x_i}\psi_{x_i x_i}) + \sum_{i=2}^n (a_{ii})_{x_i} \\
&\quad - \sum_{j=1}^m ((b_{jj})_{y_j x_i}\psi_{y_j} + (b_{jj})_{y_j}\psi_{y_j x_i}) - \sum_{j=1}^m (b_{jj})_{x_i} \\
&= \sum_{i=2}^n (a_{ii})_{x_i x_i}\psi_{x_i} + 2 \sum_{i=2}^n (a_{ii})_{x_i} - \sum_{j=1}^m (b_{jj})_{y_j x_i}\psi_{y_j} - \sum_{j=1}^m (b_{jj})_{x_i},
\end{aligned}$$

$$\begin{aligned}
\Lambda_{x_i x_i} &= \sum_{i=2}^n ((a_{ii})_{x_i x_i} \psi_{x_i} + (a_{ii})_{x_i x_i} \psi_{x_i x_i}) + 2 \sum_{i=2}^n (a_{ii})_{x_i x_i} \\
&\quad - \sum_{j=1}^m ((b_{jj})_{y_j x_i} \psi_{y_j} + (b_{jj})_{y_j x_i} \psi_{y_j x_i}) - \sum_{j=1}^m (b_{jj})_{x_i x_i} \\
&= \sum_{i=2}^n (a_{ii})_{x_i x_i} \psi_{x_i} + 3 \sum_{i=2}^n (a_{ii})_{x_i x_i} - \sum_{j=1}^m (b_{jj})_{y_j x_i} \psi_{y_j} - \sum_{j=1}^m (b_{jj})_{x_i x_i},
\end{aligned}$$

$$\begin{aligned}
\Lambda_{y_j} &= \sum_{i=2}^n ((a_{ii})_{x_i y_j} \psi_{x_i} + (a_{ii})_{x_i} \psi_{x_i y_j}) + \sum_{i=2}^n (a_{ii})_{y_j} \\
&\quad - \sum_{j=1}^m ((b_{jj})_{y_j y_j} \psi_{y_j} + (b_{jj})_{y_j} \psi_{y_j y_j}) - \sum_{j=1}^m (b_{jj})_{y_j} \\
&= \sum_{i=2}^n (a_{ii})_{x_i y_j} \psi_{x_i} + \sum_{i=2}^n (a_{ii})_{y_j} - \sum_{j=1}^m (b_{jj})_{y_j y_j} \psi_{y_j} - 2 \sum_{j=1}^m (b_{jj})_{y_j},
\end{aligned}$$

$$\begin{aligned}
\Lambda_{y_j y_j} &= \sum_{i=2}^n ((a_{ii})_{x_i y_j y_j} \psi_{x_i} + (a_{ii})_{x_i y_j} \psi_{x_i y_j}) + \sum_{i=2}^n (a_{ii})_{y_j y_j} \\
&\quad - \sum_{j=1}^m ((b_{jj})_{y_j y_j y_j} \psi_{y_j} + (b_{jj})_{y_j y_j} \psi_{y_j y_j}) - 2 \sum_{j=1}^m (b_{jj})_{y_j y_j} \\
&= \sum_{i=2}^n (a_{ii})_{x_i y_j y_j} \psi_{x_i} + \sum_{i=2}^n (a_{ii})_{y_j y_j} - \sum_{j=1}^m (b_{jj})_{y_j y_j y_j} \psi_{y_j} - 3 \sum_{j=1}^m (b_{jj})_{y_j y_j}
\end{aligned}$$

bulunur. Aşağıda  $|\psi_{x_i}| \leq \sqrt{2\gamma}$ ,  $|\psi_{y_j}| \leq \sqrt{2\gamma}$ ,  $\|a_{ii}\|_{C^3(\overline{D \times G})} \leq M$  ve  $\|b_{jj}\|_{C^3(\overline{D \times G})} \leq M$ ,  $2 \leq i \leq n$ ,  $1 \leq j \leq m$  eşitsizlikleri kullanılmıştır.

$$\begin{aligned}
|\Lambda| &\leq \sum_{i=2}^n |(a_{ii})_{x_i} \psi_{x_i}| + \sum_{i=2}^n |a_{ii}| + \sum_{j=1}^m |(b_{jj})_{y_j} \psi_{y_j}| + \sum_{j=1}^m |b_{jj}| \\
&\leq M\sqrt{2\gamma}(n-1) + M(n-1) + M\sqrt{2\gamma}m + Mm \\
&= M(\sqrt{2\gamma}+1)(n-1) + M(\sqrt{2\gamma}+1)m \\
&= (\sqrt{2\gamma}+1)M(n+m-1),
\end{aligned}$$

$$\begin{aligned}
|\Lambda_{x_i}| &\leq \sum_{i=2}^n |(a_{ii})_{x_i x_i} \psi_{x_i}| + 2 \sum_{i=2}^n |(a_{ii})_{x_i}| + \sum_{j=1}^m |(b_{jj})_{y_j x_i} \psi_{y_j}| + \sum_{j=1}^m |(b_{jj})_{x_i}| \\
&\leq M\sqrt{2\gamma}(n-1) + 2M(n-1) + M\sqrt{2\gamma}m + Mm \\
&= (\sqrt{2\gamma}+2)M(n-1) + (\sqrt{2\gamma}+1)Mm \\
&= (\sqrt{2\gamma}+1)M(n+m-1) + M(n-1),
\end{aligned}$$

$$\begin{aligned}
|\Lambda_{x_i x_i}| &\leq \sum_{i=2}^n |(a_{ii})_{x_i x_i x_i} \psi_{x_i}| + 3 \sum_{i=2}^n |(a_{ii})_{x_i x_i}| + \sum_{j=1}^m |(b_{jj})_{y_j x_i x_i} \psi_{y_j}| + \sum_{j=1}^m |(b_{jj})_{x_i x_i}| \\
&\leq M\sqrt{2\gamma}(n-1) + 3M(n-1) + M\sqrt{2\gamma}m + Mm \\
&= M\sqrt{2\gamma}(n+m-1) + M(3(n-1) + m),
\end{aligned}$$

$$\begin{aligned}
|\Lambda_{y_j}| &\leq \sum_{i=2}^n |(a_{ii})_{x_i y_j} \psi_{x_i}| + \sum_{i=2}^n |(a_{ii})_{y_j}| + \sum_{j=1}^m |(b_{jj})_{y_j y_j} \psi_{y_j}| + 2 \sum_{j=1}^m |(b_{jj})_{y_j}| \\
&\leq M\sqrt{2\gamma}(n-1) + M(n-1) + M\sqrt{2\gamma}m + 2Mm \\
&= M\sqrt{2\gamma}(n+m-1) + M(n+2m-1) \\
&= (\sqrt{2\gamma} + 1)M(n+m-1) + Mm,
\end{aligned}$$

$$\begin{aligned}
|\Lambda_{y_j y_j}| &\leq \sum_{i=2}^n |(a_{ii})_{x_i y_j y_j} \psi_{x_i}| + \sum_{i=2}^n |(a_{ii})_{y_j y_j}| + \sum_{j=1}^m |(b_{jj})_{y_j y_j y_j} \psi_{y_j}| + 3 \sum_{j=1}^m |(b_{jj})_{y_j y_j}| \\
&\leq M\sqrt{2\gamma}(n-1) + M(n-1) + M\sqrt{2\gamma}m + 3Mm \\
&= M\sqrt{2\gamma}(n+m-1) + M(n+3m-1)
\end{aligned}$$

dir. Diğer taraftan

$$\begin{aligned}
|\Lambda| &\leq \Lambda^0, \quad |\Lambda_{x_i}| \leq \Lambda^0, \quad |\Lambda_{y_j}| \leq \Lambda^0, \\
|\Lambda_{x_i x_i}| &\leq M\sqrt{2\gamma}(n+m-1) + M(3(n-1) + m), \\
|\Lambda_{y_j y_j}| &\leq M\sqrt{2\gamma}(n+m-1) + M(n+3m-1)
\end{aligned} \tag{2.64}$$

bağıntıları kullanılırsa

$$\begin{aligned}
|A| &\leq \lambda\nu\Lambda^0(|4\lambda^2\nu^2\delta^2\chi^2\psi^{-2\nu-2}| + |2\lambda\nu(\nu+1)\delta^2\chi^2\psi^{-\nu-2}| + |4\lambda\nu\delta\psi^{-\nu-1}\chi^2| \\
&\quad + (x_1 + \delta_0)^2(\sum_{i=2}^n (|4\lambda^2\nu^2\chi^2 a_{ii}\psi^{-2\nu-2}\psi_{x_i}^2| + |2\lambda\nu(\nu+1)\chi^2 a_{ii}\psi^{-\nu-2}\psi_{x_i}^2| \\
&\quad + |2\lambda\nu\chi^2 a_{ii}\psi^{-\nu-1}\psi_{x_i x_i}| + |2\lambda\nu\chi^2 (a_{ii})_{x_i}\psi^{-\nu-1}\psi_{x_i}| + |4\lambda\nu\psi^{-\nu-1}\psi_{x_i}\chi^2 a_{ii}| \\
&\quad + |2\lambda\nu\psi^{-\nu-1}\psi_{x_i}\chi^2 (a_{ii})_{x_i}| + 2|\chi^2 (a_{ii})_{x_i}| + |\chi^2 (a_{ii})_{x_i x_i}|) \\
&\quad + \sum_{j=1}^m (|4\lambda^2\nu^2\chi^2 b_{jj}\psi^{-2\nu-2}\psi_{y_j}^2| + |2\lambda\nu(\nu+1)\chi^2 b_{jj}\psi^{-\nu-2}\psi_{y_j}^2| + |2\lambda\nu\chi^2 b_{jj}\psi^{-\nu-1}\psi_{y_j y_j}| \\
&\quad + |2\lambda\nu\chi^2 (b_{jj})_{y_j}\psi^{-\nu-1}\psi_{y_j}| + |4\lambda\nu\psi^{-\nu-1}\psi_{y_j}\chi^2 b_{jj}| + |2\lambda\nu\psi^{-\nu-1}\psi_{y_j}\chi^2 (b_{jj})_{y_j}| \\
&\quad + 2|\chi^2 (b_{jj})_{y_j}| + |\chi^2 (b_{jj})_{y_j y_j}|)) \\
&\quad + \lambda\nu(\chi^2|\Lambda_{x_1 x_1}| + (x_1 + \delta_0)^2(\sum_{i=2}^n |\chi^2 a_{ii}||\Lambda_{x_i x_i}| + \sum_{j=1}^m |\chi^2 b_{jj}||\Lambda_{y_j y_j}|))
\end{aligned}$$

ve diğ er bir  ekilde

$$\begin{aligned}
|A| \leq & 4\lambda^3\nu^3\chi^2\psi^{-2\nu-2}\Lambda^0(\delta^2 + (x_1 + \delta_0)^2(\sum_{i=2}^n |a_{ii}\psi_{x_i}^2| + \sum_{j=1}^m |b_{jj}\psi_{y_j}^2|)) \\
& + \lambda^2\nu^2\psi^{-\nu-1}\delta^2\chi^2\Lambda^0(2(\nu + 1)\psi^{-1} + \frac{4}{\delta} \\
& + \frac{(x_1 + \delta_0)^2}{\delta^2}(\sum_{i=2}^n (|2a_{ii}(\nu + 1)\psi^{-1}\psi_{x_i}^2| + |2a_{ii}\psi_{x_i x_i}| \\
& + |2(a_{ii})_{x_i}\psi_{x_i}| + |4\psi_{x_i} a_{ii}| + |2\psi_{x_i}(a_{ii})_{x_i}|) \\
& + \sum_{j=1}^m (|2(\nu + 1)b_{jj}\psi^{-1}\psi_{y_j}^2| + |2b_{jj}\psi_{y_j y_j}| + |2(b_{jj})_{y_j}\psi_{y_j}| \\
& + |4\psi_{y_j} b_{jj}| + |2\psi_{y_j}(b_{jj})_{y_j}|))) \\
& + \lambda\nu\chi^2(\sum_{i=2}^n (|2(a_{ii})_{x_i}|\Lambda^0 + |(a_{ii})_{x_i x_i}|\Lambda^0) + \sum_{j=1}^m (|2(b_{jj})_{y_j}|\Lambda^0 + |(b_{jj})_{y_j y_j}|\Lambda^0) \\
& + |\Lambda_{x_1 x_1}| + (x_1 + \delta_0)^2(\sum_{i=2}^n |a_{ii}||\Lambda_{x_i x_i}| + \sum_{j=1}^m |b_{jj}||\Lambda_{y_j y_j}|))
\end{aligned}$$

elde edilir. Eđer

$$\begin{aligned}
\lambda \geq \lambda_2 = & \sum_{i=2}^n (|2(a_{ii})_{x_i}|\Lambda^0 + |(a_{ii})_{x_i x_i}|\Lambda^0) + \sum_{j=1}^m (|2(b_{jj})_{y_j}|\Lambda^0 + |(b_{jj})_{y_j y_j}|\Lambda^0) \\
& + |\Lambda_{x_1 x_1}| + (x_1 + \delta_0)^2(\sum_{i=2}^n |a_{ii}||\Lambda_{x_i x_i}| + \sum_{j=1}^m |b_{jj}||\Lambda_{y_j y_j}|)
\end{aligned}$$

olarak alınır ve  $\nu \geq 1$  olduđu g z  n nde bulundurulursa

$$\begin{aligned}
|A| \leq & 4\lambda^3\nu^3\chi^2\psi^{-2\nu-2}\Lambda^0(\delta^2 + (x_1 + \delta_0)^2(\sum_{i=2}^n |a_{ii}\psi_{x_i}^2| + \sum_{j=1}^m |b_{jj}\psi_{y_j}^2|)) \\
& + \lambda^2\nu^2\psi^{-\nu-1}\delta^2\chi^2\Lambda^0(2(\nu + 1)\psi^{-1} + \frac{4}{\delta} \\
& + \frac{(x_1 + \delta_0)^2}{\delta^2}(\sum_{i=2}^n (|2a_{ii}(\nu + 1)\psi^{-1}\psi_{x_i}^2| + |2a_{ii}\psi_{x_i x_i}| \\
& + |2(a_{ii})_{x_i}\psi_{x_i}| + |4\psi_{x_i} a_{ii}| + |2\psi_{x_i}(a_{ii})_{x_i}|) \\
& + \sum_{j=1}^m (|2(\nu + 1)b_{jj}\psi^{-1}\psi_{y_j}^2| + |2b_{jj}\psi_{y_j y_j}| + |2(b_{jj})_{y_j}\psi_{y_j}| \\
& + |4\psi_{y_j} b_{jj}| + |2\psi_{y_j}(b_{jj})_{y_j}|))) + \lambda^2\nu^2\chi^2
\end{aligned}$$

$$\begin{aligned}
&= 4\lambda^3\nu^3\chi^2\psi^{-2\nu-2}\Lambda^0(\delta^2 + (x_1 + \delta_0)^2(\sum_{i=2}^n |a_{ii}\psi_{x_i}^2| + \sum_{j=1}^m |b_{jj}\psi_{y_j}^2|)) \\
&\quad + \lambda^2\nu^2\psi^{-\nu-1}\delta^2\chi^2(\Lambda^0(2(\nu+1)\psi^{-1} + \frac{4}{\delta} \\
&\quad + \frac{(x_1 + \delta_0)^2}{\delta^2}(\sum_{i=2}^n (|2a_{ii}(\nu+1)\psi^{-1}\psi_{x_i}^2| + |2a_{ii}\psi_{x_i x_i}| \\
&\quad + |2(a_{ii})_{x_i}\psi_{x_i}| + |4\psi_{x_i}a_{ii}| + |2\psi_{x_i}(a_{ii})_{x_i}|) \\
&\quad + \sum_{j=1}^m (|2(\nu+1)b_{jj}\psi^{-1}\psi_{y_j}^2| + |2b_{jj}\psi_{y_j y_j}| + |2(b_{jj})_{y_j}\psi_{y_j}| \\
&\quad + |4\psi_{y_j}b_{jj}| + |2\psi_{y_j}(b_{jj})_{y_j}|))) + \frac{\psi^{\nu+1}}{\delta^2})
\end{aligned}$$

$$\begin{aligned}
&= 4\lambda^3\nu^3\chi^2\psi^{-2\nu-2}\Lambda^0(\delta^2 + (x_1 + \delta_0)^2(\sum_{i=2}^n |a_{ii}\psi_{x_i}^2| + \sum_{j=1}^m |b_{jj}\psi_{y_j}^2|)) \\
&\quad + C\lambda^2\nu^2\psi^{-\nu-1}\delta^2\chi^2
\end{aligned}$$

eşitsizliği elde edilir. Burada

$$\begin{aligned}
C &= \Lambda^0(2(\nu+1)\psi^{-1} + \frac{4}{\delta} + \frac{(x_1 + \delta_0)^2}{\delta^2}(\sum_{i=2}^n (|2a_{ii}(\nu+1)\psi^{-1}\psi_{x_i}^2| + |2a_{ii}\psi_{x_i x_i}| \\
&\quad + |2(a_{ii})_{x_i}\psi_{x_i}| + |4\psi_{x_i}a_{ii}| + |2\psi_{x_i}(a_{ii})_{x_i}|) + \sum_{j=1}^m (|2(\nu+1)b_{jj}\psi^{-1}\psi_{y_j}^2| \\
&\quad + |2b_{jj}\psi_{y_j y_j}| + |2(b_{jj})_{y_j}\psi_{y_j}| + |4\psi_{y_j}b_{jj}| + |2\psi_{y_j}(b_{jj})_{y_j}|))) + \frac{\psi^{\nu+1}}{\delta^2}
\end{aligned}$$

dir. O halde mutlak değer tanımından ve  $|a_{ii}| \leq M$ ,  $|b_{jj}| \leq M$ ,  $|\psi_{x_i}|^2 \leq 2\gamma$ ,  $|\psi_{y_j}|^2 \leq 2\gamma$ ,  $2 \leq i \leq n$ ,  $1 \leq j \leq m$  kabullerinden

$$\begin{aligned}
A &\geq -4\lambda^3\nu^3\chi^2\psi^{-2\nu-2}\Lambda^0(\delta^2 + (x_1 + \delta_0)^2(\sum_{i=2}^n |a_{ii}\psi_{x_i}^2| + \sum_{j=1}^m |b_{jj}\psi_{y_j}^2|)) \\
&\quad - C\lambda^2\nu^2\psi^{-\nu-1}\delta^2\chi^2 \\
&\geq -4\lambda^3\nu^3\chi^2\psi^{-2\nu-2}\Lambda^0(\delta^2 + M2\gamma(n-1) + M2\gamma m) - C\lambda^2\nu^2\psi^{-\nu-1}\delta^2\chi^2 \tag{2.65}
\end{aligned}$$

yazılabilir. (2.62) ve (2.65) eşitsizlikleri kullanılarak

$$\begin{aligned}
&2\lambda\nu(x_1 + \delta_0)\chi^2\Lambda\varphi(L_0\varphi) \\
&\geq -2\lambda\nu\Lambda^0\chi^2(\varphi_{x_1}^2 + (x_1 + \delta_0)^2M(\sum_{i=2}^n \varphi_{x_i}^2 + \sum_{j=1}^m \varphi_{y_j}^2)) \\
&\quad - 4\lambda^3\nu^3\chi^2\psi^{-2\nu-2}(\delta^2 + M2\gamma(n+m-1))\Lambda^0\varphi^2 \\
&\quad - C\lambda^2\nu^2\psi^{-\nu-1}\delta^2\chi^2\varphi^2 - 2\lambda\nu(x_1 + \delta_0)^2\chi^2\Lambda\xi\varphi^2 + d_4(\varphi)
\end{aligned}$$

ispat tamamlanır. ■

**Lemma 2.1'in ispatı.** (2.9) ve (2.55) eşitsizlikleri taraf tarafa toplanırsa,

$$\begin{aligned}
& \psi^{\nu+1}(L_0\varphi)^2\chi^2 + 2\lambda\nu(x_1 + \delta_0)\chi^2\Lambda\varphi(L_0\varphi) \\
\geq & 2\lambda\nu\gamma^{-3}\delta\varphi_{x_1}^2\chi^2 + \lambda\nu(x_1 + \delta_0)\delta\alpha_1\chi^2\sum_{i=2}^n\varphi_{x_i}^2 \\
& + \lambda\nu(x_1 + \delta_0)\delta\alpha_2\chi^2\sum_{j=1}^m\varphi_{y_j}^2 + \lambda^3\nu^4\delta^4\psi^{-2\nu-3}\varphi^2\chi^2 \\
& + 2\lambda\nu\xi(x_1 + \delta_0)^2\left(\sum_{i=2}^n(a_{ii}\psi_{x_i})_{x_i} - \sum_{j=1}^m(b_{jj}\psi_{y_j})_{y_j}\right)\varphi^2\chi^2 + \sum_{j=1}^3d_j(\varphi) \\
& - 2\lambda\nu\Lambda^0\chi^2(\varphi_{x_1}^2 + M(x_1 + \delta_0)^2(\sum_{i=2}^n\varphi_{x_i}^2 + \sum_{j=1}^m\varphi_{y_j}^2)) \\
& - 4\lambda^3\nu^3\chi^2\psi^{-2\nu-2}(\delta^2 + M2\gamma(n + m - 1))\Lambda^0\varphi^2 \\
& - C\lambda^2\nu^2\psi^{-\nu-1}\delta^2\chi^2\varphi^2 - 2\lambda\nu(x_1 + \delta_0)^2\chi^2\Lambda\xi\varphi^2 + d_4(\varphi) \\
= & \sum_{i=1}^4d_i(\varphi) + 2\lambda\nu\varphi_{x_1}^2\chi^2(\gamma^{-3}\delta - \Lambda^0) + \lambda\nu(x_1 + \delta_0)(\delta\alpha_1 - 2\Lambda^0M(x_1 + \delta_0))\chi^2\sum_{i=2}^n\varphi_{x_i}^2 \\
& + \lambda\nu(x_1 + \delta_0)(\delta\alpha_2 - 2\Lambda^0M(x_1 + \delta_0))\chi^2\sum_{j=1}^m\varphi_{y_j}^2 \\
& + 4\lambda^3\nu^3\psi^{-2\nu-3}\varphi^2\chi^2\left(\frac{\nu\delta^4}{4} - \psi\Lambda^0(\delta^2 + M2\gamma(n + m - 1)) - \frac{C}{\lambda\nu}\delta^2\psi^{\nu+2}\right) \\
> & \sum_{i=1}^4d_i(\varphi) + 2\lambda\nu\varphi_{x_1}^2\chi^2(\gamma^{-3}\delta - \Lambda^0) + \lambda\nu(x_1 + \delta_0)(\delta\alpha_1 - 2\Lambda^0M\gamma)\chi^2\sum_{i=2}^n\varphi_{x_i}^2 \\
& + \lambda\nu(x_1 + \delta_0)(\delta\alpha_2 - 2\Lambda^0M\gamma)\chi^2\sum_{j=1}^m\varphi_{y_j}^2 \\
& + 4\lambda^3\nu^3\psi^{-2\nu-3}\varphi^2\chi^2\left(\frac{\nu\delta^4}{4} - \psi\Lambda^0(\delta^2 + M2\gamma(n + m - 1)) - \frac{C}{\lambda\nu}\delta^2\psi^{\nu+2}\right) \tag{2.66}
\end{aligned}$$

bulunur. Burada

$$\gamma^{-3}\delta - \Lambda^0 \geq 1 \Rightarrow \delta \geq (1 + \Lambda^0)\gamma^3,$$

$$\delta\alpha_1 - 2\Lambda^0M\gamma \geq 1 \Rightarrow \delta \geq (1 + 2\Lambda^0M\gamma)/\alpha_1,$$

$$\delta\alpha_2 - 2\Lambda^0M\gamma \geq 1 \Rightarrow \delta \geq (1 + 2\Lambda^0M\gamma)/\alpha_2$$

dir. Diğer taraftan

$$0 < \psi < 1, \lambda \geq \lambda_3 = C, \nu \geq 1,$$

$$\nu \geq \nu_2 = \frac{4}{\delta^4}(1 + \Lambda^0(\delta^2 + M2\gamma(n + m - 1)) + \delta^2)$$

olduğundan

$$\begin{aligned} & \frac{\nu\delta^4}{4} - \psi\Lambda^0(\delta^2 + M2\gamma(n + m - 1)) - \frac{C}{\lambda\nu}\psi^{\nu+2}\delta^2 \\ & > \frac{\nu\delta^4}{4} - \Lambda^0(\delta^2 + M2\gamma(n + m - 1)) - \delta^2 \\ & \geq 1 \end{aligned}$$

olur. Burada  $\delta$ ,  $\lambda$  ve  $\nu$  parametrelerini  $\delta > \delta_*$ ,  $\lambda > \lambda_*$ ,  $\nu > \nu_*$ ,

$$\delta_* = \max\{4, \delta_1, \delta_2, \delta_3, \delta_4, \delta_5\},$$

$$\delta \geq \delta_5 = \max\{(1 + \Lambda^0)\gamma^3, (1 + 2\Lambda^0 M\gamma)/\alpha_1, (1 + 2\Lambda^0 M\gamma)/\alpha_2\},$$

$$\lambda_* = \max\{\lambda_1, \lambda_2, \lambda_3\}, \nu_* = \max\{\nu_1, \nu_2\}$$

olacak şekilde seçelim. Bu durumda (2.66) eşitsizliğinden

$$\begin{aligned} & \psi^{\nu+1}(L_0\varphi)^2\chi^2 + 2\lambda\nu(x_1 + \delta_0)\chi^2\Lambda\varphi(L_0\varphi) \\ & \geq 2\lambda\nu\varphi_{x_1}^2\chi^2 + \lambda\nu(x_1 + \delta_0)\chi^2 \sum_{i=2}^n \varphi_{x_i}^2 + \lambda\nu(x_1 + \delta_0)\chi^2 \sum_{j=1}^m \varphi_{y_j}^2 \\ & \quad + 4\lambda^3\nu^3\psi^{-2\nu-3}\varphi^2\chi^2 + \sum_{i=1}^4 d_i(\varphi) \end{aligned}$$

elde edilir. Böylece ispat tamamlanır.

## BÖLÜM 3

### ULTRAHİPERBOLİK SCHRÖDINGER DENKLEMİ İÇİN BİR TERS PROBLEM

**Problem 3.1** (1.1)-(1.2) bağıntılarından

$$u(x, y, 0) = a(x, y) \quad (3.1)$$

ek bilgisi yardımıyla  $(u, g)$  fonksiyonlar çiftinin bulunması problemini ele alalım.

Şimdi aşağıdaki kümeyi tanımlayalım:

$$\mathcal{U} = \{u(x, y, t); u \in C^3(Q), |\partial_t^\beta u|, |\partial_t^\beta u_{x_s}|, |\partial_t^\beta u_{x_s x_j}|, |\partial_t^\beta u_{y_i}|, |\partial_t^\beta u_{y_i y_l}| \leq K_u \exp(d_u t^2)\}.$$

Burada  $0 \leq \beta \leq 2$ ,  $\partial_t^\beta = \frac{\partial^\beta}{\partial t^\beta}$ ;  $s, j = 2, \dots, n$ ;  $i, l = 1, \dots, m$  ve  $K_u, d_u > 0$  sabitleri  $u(x, y, t)$  fonksiyonuna bağlıdır.

### 3.1 TERS PROBLEMİN ÇÖZÜMÜNÜN TEKLİĞİNİN ARAŞTIRILMASI

**Teorem 3.1** Kabul edelim ki

$$\begin{aligned} a_{ii} &\in C^3(\overline{D \times G}), \quad b_{jj} \in C^3(\overline{D \times G}), \\ \sum_{i=2}^n (a_{ii}(x, y))_{x_1} (\xi^i)^2 &\geq \alpha_1 |\xi|^2, \quad - \sum_{j=1}^m (b_{jj}(x, y))_{x_1} (\zeta^j)^2 \geq \alpha_2 |\zeta|^2, \end{aligned} \quad (3.2)$$

$$\sum_{i=2}^n a_{ii} \psi_{x_i}^2 \geq \alpha_3 \sum_{i=2}^n \psi_{x_i}^2, \quad - \sum_{j=1}^m b_{jj} \psi_{y_j}^2 \geq \alpha_4 \sum_{j=1}^m \psi_{y_j}^2 \quad (3.2')$$

olacak şekilde  $\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$  sabitleri mevcut olsun.

Burada  $\xi = (\xi_2, \dots, \xi_n) \in \mathbb{R}^{n-1}$  ve  $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_m) \in \mathbb{R}^m$  olarak tanımlıdır. Ayrıca

$$f \in C(Q), \quad |\partial_t^\beta f| \leq K_f \exp(d_f t^2), \quad f(x, y, 0) \neq 0$$

olsun. Problem 3.1 en çok bir  $(u, g)$  çözümüne sahiptir öyle ki  $u \in \mathcal{U}$ ,  $g \in C(D \times G)$ .

**İspat.** Kabul edelim ki  $a(x, y) = 0$  ve  $(u, g)$  Problem 3.1'in bir çözümü olsun. (2.5) denklemi, (2.6) koşulları ve (2.7) ifadesi kullanılarak

$$L_0\varphi_s = m_s, \quad s = 1, 2; \quad (3.3)$$

$$\varphi_s(0, 'x, y, \xi) = (\varphi_s)_{x_1}(0, 'x, y, \xi) = 0 \quad (3.4)$$

yazılabilir. Burada

$$L_0\varphi_s = (x_1 + \delta_0)^{-1}(\varphi_s)_{x_1x_1} + (x_1 + \delta_0)\left(\sum_{i=2}^n a_{ii}(\varphi_s)_{x_ix_i} - \sum_{j=1}^m b_{jj}(\varphi_s)_{y_jy_j} - 2k(\varphi_s)_\xi - \xi\varphi_s\right)$$

şeklinde olup  $m_1$  ve  $m_2$  ifadeleri için  $\widehat{F} = f_1 + if_2$ ,  $\widehat{w} = \varphi_1 + i\varphi_2$ ,  $\widehat{a_r w_{x_r}} = a_{1r} + ia_{2r}$ ,  $\widehat{a_0 w} = a_{10} + ia_{20}$ ,  $\widehat{b_l w_{y_l}} = b_{1l} + ib_{2l}$  tanımlamaları kullanılırsa

$$\begin{aligned} -\widehat{F} \frac{1}{f_0} \int_{-\infty}^{\infty} \xi \widehat{w} d\xi &= -(f_1 + if_2) \frac{1}{f_0} \int_{-\infty}^{\infty} \xi (\varphi_1 + i\varphi_2) d\xi \\ &= -(f_1 + if_2) \frac{1}{f_0} \left( \int_{-\infty}^{\infty} \xi \varphi_1 d\xi + i \int_{-\infty}^{\infty} \xi \varphi_2 d\xi \right) \\ &= -f_1 \frac{1}{f_0} \int_{-\infty}^{\infty} \xi \varphi_1 d\xi - if_1 \frac{1}{f_0} \int_{-\infty}^{\infty} \xi \varphi_2 d\xi \\ &\quad - if_2 \frac{1}{f_0} \int_{-\infty}^{\infty} \xi \varphi_1 d\xi + f_2 \frac{1}{f_0} \int_{-\infty}^{\infty} \xi \varphi_2 d\xi, \\ -\left(\sum_{r=0}^n \widehat{a_r w_{x_r}} + \sum_{l=1}^m \widehat{b_l w_{y_l}}\right) &= -\left(\sum_{r=0}^n (a_{1r} + ia_{2r}) + \sum_{l=1}^m (b_{1l} + ib_{2l})\right) \end{aligned}$$

eşitliklerinden

$$\begin{aligned} &(x_1 + \delta_0) \left( -\widehat{F} \frac{1}{f_0} \int_{-\infty}^{\infty} \xi \widehat{w} d\xi - \sum_{r=0}^n \widehat{a_r w_{x_r}} - \sum_{l=1}^m \widehat{b_l w_{y_l}} \right) \\ &= (x_1 + \delta_0) \left( -f_1 \frac{1}{f_0} \int_{-\infty}^{\infty} \xi \varphi_1 d\xi - if_1 \frac{1}{f_0} \int_{-\infty}^{\infty} \xi \varphi_2 d\xi \right. \\ &\quad \left. - if_2 \frac{1}{f_0} \int_{-\infty}^{\infty} \xi \varphi_1 d\xi + f_2 \frac{1}{f_0} \int_{-\infty}^{\infty} \xi \varphi_2 d\xi \right. \\ &\quad \left. - \sum_{r=0}^n (a_{1r} + ia_{2r}) - \sum_{l=1}^m (b_{1l} + ib_{2l}) \right) \\ &= m_1 + m_2 \end{aligned}$$

olduğu görülür. Burada  $w_{x_0} = w$  gösterimi kullanılmıştır. Böylece

$$\begin{aligned} m_1 &= (x_1 + \delta_0) \left( -f_1 \frac{1}{f_0} \int_{-\infty}^{\infty} \xi \varphi_1 d\xi + f_2 \frac{1}{f_0} \int_{-\infty}^{\infty} \xi \varphi_2 d\xi - \sum_{r=0}^n a_{1r} - \sum_{l=1}^m b_{1l} \right), \\ m_2 &= (x_1 + \delta_0) \left( -f_1 \frac{1}{f_0} \int_{-\infty}^{\infty} \xi \varphi_2 d\xi - f_2 \frac{1}{f_0} \int_{-\infty}^{\infty} \xi \varphi_1 d\xi - \sum_{r=0}^n a_{2r} - \sum_{l=1}^m b_{2l} \right) \end{aligned}$$

olarak bulunur. Diğer taraftan

$$\begin{aligned}
\left(\int_{-\infty}^{\infty} \xi \varphi_1 d\xi\right)^2 &= \left(\int_{-\infty}^{\infty} (1+\xi^2)^{-1/2} (1+\xi^2)^{1/2} \xi \varphi_1 d\xi\right)^2 \\
&\leq \int_{-\infty}^{\infty} (1+\xi^2)^{-1} d\xi \int_{-\infty}^{\infty} \xi^2 (1+\xi^2) \varphi_1^2 d\xi \\
&\leq C_1 \int_{-\infty}^{\infty} (1+\xi^2)^2 \varphi_1^2 d\xi,
\end{aligned}$$

$$\left(\int_{-\infty}^{\infty} \xi \varphi_2 d\xi\right)^2 \leq C_1 \int_{-\infty}^{\infty} (1+\xi^2)^2 \varphi_2^2 d\xi, \quad C_1 = \int_{-\infty}^{\infty} (1+\xi^2)^{-1} d\xi,$$

$$\left(\sum_{k=1}^m a_k\right)^2 \leq m \sum_{k=1}^m a_k^2, \quad (a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2)$$

eşitsizlikleri kullanılarak

$$\begin{aligned}
m_1^2 + m_2^2 &= (x_1 + \delta_0)^2 \left(-f_1 \frac{1}{f_0} \int_{-\infty}^{\infty} \xi \varphi_1 d\xi + f_2 \frac{1}{f_0} \int_{-\infty}^{\infty} \xi \varphi_2 d\xi - \sum_{r=0}^n a_{1r} - \sum_{l=1}^m b_{1l}\right)^2 \\
&\quad + (x_1 + \delta_0)^2 \left(-f_1 \frac{1}{f_0} \int_{-\infty}^{\infty} \xi \varphi_2 d\xi - f_2 \frac{1}{f_0} \int_{-\infty}^{\infty} \xi \varphi_1 d\xi - \sum_{r=0}^n a_{2r} - \sum_{l=1}^m b_{2l}\right)^2 \\
&\leq (x_1 + \delta_0)^2 \left(4f_1^2 \frac{1}{f_0^2} \left(\int_{-\infty}^{\infty} \xi \varphi_1 d\xi\right)^2 + 4f_2^2 \frac{1}{f_0^2} \left(\int_{-\infty}^{\infty} \xi \varphi_2 d\xi\right)^2 + 4\left(\sum_{r=0}^n a_{1r}\right)^2 + 4\left(\sum_{l=1}^m b_{1l}\right)^2\right. \\
&\quad \left.+ 4f_1^2 \frac{1}{f_0^2} \left(\int_{-\infty}^{\infty} \xi \varphi_2 d\xi\right)^2 + 4f_2^2 \frac{1}{f_0^2} \left(\int_{-\infty}^{\infty} \xi \varphi_1 d\xi\right)^2 + 4\left(\sum_{r=0}^n a_{2r}\right)^2 + 4\left(\sum_{l=1}^m b_{2l}\right)^2\right) \\
&\leq (x_1 + \delta_0)^2 \left(\left(4f_1^2 \frac{1}{f_0^2} + 4f_2^2 \frac{1}{f_0^2}\right) \left(\int_{-\infty}^{\infty} \xi \varphi_1 d\xi\right)^2 + \left(4f_2^2 \frac{1}{f_0^2} + 4f_1^2 \frac{1}{f_0^2}\right) \left(\int_{-\infty}^{\infty} \xi \varphi_2 d\xi\right)^2\right. \\
&\quad \left.+ 4(n+1) \sum_{r=0}^n (a_{1r}^2 + a_{2r}^2) + 4m \sum_{l=1}^m (b_{1l}^2 + b_{2l}^2)\right) \\
&= (x_1 + \delta_0)^2 \left(4 \frac{(f_1^2 + f_2^2)}{f_0^2} \left(\left(\int_{-\infty}^{\infty} \xi \varphi_1 d\xi\right)^2 + \left(\int_{-\infty}^{\infty} \xi \varphi_2 d\xi\right)^2\right)\right. \\
&\quad \left.+ 4(n+1) \sum_{r=0}^n (a_{1r}^2 + a_{2r}^2) + 4m \sum_{l=1}^m (b_{1l}^2 + b_{2l}^2)\right) \\
&\leq (x_1 + \delta_0)^2 \left(4 \frac{(f_1^2 + f_2^2)}{f_0^2} \left(C_1 \int_{-\infty}^{\infty} (1+\xi^2)^2 \varphi_1^2 d\xi + C_1 \int_{-\infty}^{\infty} (1+\xi^2)^2 \varphi_2^2 d\xi\right)\right. \\
&\quad \left.+ 4(n+1) \sum_{r=0}^n (a_{1r}^2 + a_{2r}^2) + 4m \sum_{l=1}^m (b_{1l}^2 + b_{2l}^2)\right) \\
&= (x_1 + \delta_0)^2 \left(4 \frac{C_1}{f_0^2} (f_1^2 + f_2^2) \int_{-\infty}^{\infty} (1+\xi^2)^2 (\varphi_1^2 + \varphi_2^2) d\xi\right. \\
&\quad \left.+ 4(n+1) \sum_{r=0}^n (a_{1r}^2 + a_{2r}^2) + 4m \sum_{l=1}^m (b_{1l}^2 + b_{2l}^2)\right) \tag{3.5}
\end{aligned}$$

elde edilir. Şimdi Lemma 2.1 de bulduğumuz

$$\begin{aligned} & \psi^{\nu+1}(L_0\varphi_s)^2\chi^2 + 2\lambda\nu(x_1 + \delta_0)\Lambda\varphi_s(L_0\varphi_s)\chi^2 \\ \leq & \psi^{\nu+1}(L_0\varphi_s)^2\chi^2 + (L_0\varphi_s)^2\chi^2 + \lambda^2\nu^2(x_1 + \delta_0)^2\Lambda^2\varphi_s^2\chi^2 \end{aligned}$$

eşitsizliğin sol tarafını değerlendirelim. Bu amaçla toplam sembolüde yukarıdaki eşitsizliğe dahil edilerek:

$$\begin{aligned} & \sum_{s=1}^2 (\psi^{\nu+1}(L_0\varphi_s)^2\chi^2 + 2\lambda\nu(x_1 + \delta_0)\Lambda\varphi_s(L_0\varphi_s)\chi^2) \\ \leq & \sum_{s=1}^2 (\psi^{\nu+1}(L_0\varphi_s)^2\chi^2 + (L_0\varphi_s)^2\chi^2 + \lambda^2\nu^2(x_1 + \delta_0)^2\Lambda^2\varphi_s^2\chi^2) \\ = & \sum_{s=1}^2 ((\psi^{\nu+1} + 1)(L_0\varphi_s)^2\chi^2 + \lambda^2\nu^2(x_1 + \delta_0)^2\Lambda^2\varphi_s^2\chi^2) \\ = & (\psi^{\nu+1} + 1)(m_1^2 + m_2^2)\chi^2 + \lambda^2\nu^2(x_1 + \delta_0)^2\Lambda^2(\varphi_1^2 + \varphi_2^2)\chi^2 \\ \leq & (\psi^{\nu+1} + 1)(x_1 + \delta_0)^2(4\frac{C_1}{f_0^2}(f_1^2 + f_2^2) \int_{-\infty}^{\infty} (1 + \xi^2)^2(\varphi_1^2 + \varphi_2^2)d\xi \\ & + 4(n+1) \sum_{r=0}^n (a_{1r}^2 + a_{2r}^2) + 4m \sum_{l=1}^m (b_{1l}^2 + b_{2l}^2))\chi^2 \\ & + \lambda^2\nu^2(x_1 + \delta_0)^2\Lambda^2(\varphi_1^2 + \varphi_2^2)\chi^2 \end{aligned} \tag{3.6}$$

yazılır. İkinci olarak Lemma 2.1 deki eşitsizliğin

$$\begin{aligned} & \psi^{\nu+1}(L_0\varphi_s)^2\chi^2 + 2\lambda\nu(x_1 + \delta_0)\Lambda\varphi_s(L_0\varphi_s)\chi^2 \\ \geq & 2\lambda\nu\chi^2(\varphi_s)_{x_1}^2 + \lambda\nu(x_1 + \delta_0)\chi^2 \sum_{i=2}^n (\varphi_s)_{x_i}^2 + \lambda\nu(x_1 + \delta_0)\chi^2 \sum_{j=1}^m (\varphi_s)_{y_j}^2 \\ & + 4\lambda^3\nu^3\psi^{-2\nu-3}\chi^2\varphi_s^2 + \sum_{i=1}^4 d_i(\varphi_s) \end{aligned}$$

sağ tarafını değerlendirelim:

$$\begin{aligned} & \sum_{s=1}^2 (2\lambda\nu\chi^2(\varphi_s)_{x_1}^2 + \lambda\nu(x_1 + \delta_0)\chi^2 \sum_{i=2}^n (\varphi_s)_{x_i}^2 + \lambda\nu(x_1 + \delta_0)\chi^2 \sum_{j=1}^m (\varphi_s)_{y_j}^2 \\ & + 4\lambda^3\nu^3\psi^{-2\nu-3}\chi^2\varphi_s^2 + \sum_{i=1}^4 d_i(\varphi_s)) \\ = & 2\lambda\nu\chi^2 \sum_{s=1}^2 (\varphi_s)_{x_1}^2 + \lambda\nu(x_1 + \delta_0)\chi^2 \sum_{s=1}^2 \sum_{i=2}^n (\varphi_s)_{x_i}^2 + \lambda\nu(x_1 + \delta_0)\chi^2 \sum_{s=1}^2 \sum_{j=1}^m (\varphi_s)_{y_j}^2 \\ & + 4\lambda^3\nu^3\psi^{-2\nu-3}\chi^2 \sum_{s=1}^2 \varphi_s^2 + \sum_{s=1}^2 \sum_{i=1}^4 d_i(\varphi_s). \end{aligned} \tag{3.7}$$

O halde (3.6) ve (3.7) birleştirilirse

$$\begin{aligned}
& 2\lambda\nu\chi^2 \sum_{s=1}^2 (\varphi_s)_{x_1}^2 + \lambda\nu(x_1 + \delta_0)\chi^2 \sum_{s=1}^2 \sum_{i=2}^n (\varphi_s)_{x_i}^2 + \lambda\nu(x_1 + \delta_0)\chi^2 \sum_{s=1}^2 \sum_{j=1}^m (\varphi_s)_{y_j}^2 \\
& + 4\lambda^3\nu^3\psi^{-2\nu-3}\chi^2 \sum_{s=1}^2 \varphi_s^2 + \sum_{s=1}^2 \sum_{i=1}^4 d_i(\varphi_s) \\
\leq & (\psi^{\nu+1} + 1)(x_1 + \delta_0)^2 \left( 4\frac{C_1}{f_0^2} (f_1^2 + f_2^2) \int_{-\infty}^{\infty} (1 + \xi^2)^2 (\varphi_1^2 + \varphi_2^2) d\xi \right. \\
& + 4(n+1) \sum_{r=0}^n (a_{1r}^2 + a_{2r}^2) + 4m \sum_{l=1}^m (b_{1l}^2 + b_{2l}^2) \Big) \chi^2 \\
& + \lambda^2\nu^2(x_1 + \delta_0)^2 \Lambda^2 (\varphi_1^2 + \varphi_2^2) \chi^2
\end{aligned} \tag{3.8}$$

elde edilir. (3.8) eşitsizliğini  $(1 + \xi^2)^2$  ile çarpıp  $(-\infty, \infty)$  aralığında integralini alalım:

$$\begin{aligned}
& \sum_{s=1}^2 \int_{-\infty}^{\infty} (2\lambda\nu(\varphi_s)_{x_1}^2 + \lambda\nu(x_1 + \delta_0) \sum_{i=2}^n (\varphi_s)_{x_i}^2 + \lambda\nu(x_1 + \delta_0) \sum_{j=1}^m (\varphi_s)_{y_j}^2 \\
& + 4\lambda^3\nu^3\psi^{-2\nu-3}\varphi_s^2) \chi^2 (1 + \xi^2)^2 d\xi + \sum_{s=1}^2 \sum_{i=1}^4 \int_{-\infty}^{\infty} d_i(\varphi_s) (1 + \xi^2)^2 d\xi \\
\leq & (\psi^{\nu+1} + 1)(x_1 + \delta_0)^2 \left( 4\frac{C_1}{f_0^2} \int_{-\infty}^{\infty} (f_1^2 + f_2^2) (1 + \xi^2)^2 d\xi \sum_{s=1}^2 \int_{-\infty}^{\infty} (1 + \xi^2)^2 \varphi_s^2 d\xi \right. \\
& + 4(n+1) \int_{-\infty}^{\infty} \sum_{r=0}^n (a_{1r}^2 + a_{2r}^2) (1 + \xi^2)^2 d\xi + 4m \int_{-\infty}^{\infty} \sum_{l=1}^m (b_{1l}^2 + b_{2l}^2) (1 + \xi^2)^2 d\xi \Big) \chi^2 \\
& + \lambda^2\nu^2(x_1 + \delta_0)^2 \Lambda^2 \chi^2 \sum_{s=1}^2 \int_{-\infty}^{\infty} (1 + \xi^2)^2 \varphi_s^2 d\xi.
\end{aligned} \tag{3.9}$$

Burada

$$\overline{f_0} = C_1 \max_{(x,y) \in D \times G} \left\{ \frac{1}{f_0^2} \right\}, \tag{3.10}$$

$$C_2 = \int_{-\infty}^{\infty} (f_1^2 + f_2^2) (1 + \xi^2)^2 d\xi < \infty \tag{3.11}$$

biçiminde tanımlıdır. (3.9) eşitsizliğinin sağ tarafındaki terimler için Parseval eşitliği kullanılır ve  $|\partial_t^\beta a_r| \leq C_0$  olduğu göz önünde bulundurulursa

$$\begin{aligned}
\int_{-\infty}^{\infty} (1 + \xi^2)^2 |\widehat{a_r w_{x_r}}|^2 d\xi &= \int_{-\infty}^{\infty} |\widehat{a_r w_{x_r}}|^2 d\xi + 2 \int_{-\infty}^{\infty} \xi^2 |\widehat{a_r w_{x_r}}|^2 d\xi + \int_{-\infty}^{\infty} \xi^4 |\widehat{a_r w_{x_r}}|^2 d\xi \\
&= \int_{-\infty}^{\infty} |a_r w_{x_r}|^2 dt + 2 \int_{-\infty}^{\infty} |\partial_t(a_r w_{x_r})|^2 dt + \int_{-\infty}^{\infty} |\partial_t^2(a_r w_{x_r})|^2 dt \\
&\leq 2 \int_{-\infty}^{\infty} \sum_{0 \leq \beta \leq 2} |\partial_t^\beta(a_r w_{x_r})|^2 dt \\
&\leq 2C_0' \int_{-\infty}^{\infty} \sum_{0 \leq \beta \leq 2} |\partial_t^\beta w_{x_r}|^2 dt
\end{aligned}$$

$$\begin{aligned}
&= 2C'_0 \left( \int_{-\infty}^{\infty} |w_{x_r}|^2 dt + \int_{-\infty}^{\infty} |\partial_t w_{x_r}|^2 dt + \int_{-\infty}^{\infty} |\partial_t^2 w_{x_r}|^2 dt \right) \\
&= 2C'_0 \left( \int_{-\infty}^{\infty} |\widehat{w_{x_r}}|^2 d\xi + \int_{-\infty}^{\infty} |\partial_t \widehat{w_{x_r}}|^2 d\xi + \int_{-\infty}^{\infty} |\partial_t^2 \widehat{w_{x_r}}|^2 d\xi \right) \\
&= 2C'_0 \left( \int_{-\infty}^{\infty} |\widehat{w_{x_r}}|^2 d\xi + \int_{-\infty}^{\infty} |i\xi \widehat{w_{x_r}}|^2 d\xi + \int_{-\infty}^{\infty} |(i\xi)^2 \widehat{w_{x_r}}|^2 d\xi \right) \\
&= 2C'_0 \int_{-\infty}^{\infty} (1 + \xi^2 + \xi^4) |\widehat{w_{x_r}}|^2 d\xi \\
&\leq 2C'_0 \int_{-\infty}^{\infty} (1 + \xi^2)^2 |\widehat{w_{x_r}}|^2 d\xi \\
&= 2C'_0 \int_{-\infty}^{\infty} (1 + \xi^2)^2 ((\varphi_1)_{x_r}^2 + (\varphi_2)_{x_r}^2) d\xi
\end{aligned}$$

elde edilir. Burada  $C'_0$ ,  $C_0$ 'a bağılı bir sabittir. Ayrıca  $|\widehat{a_r w_{x_r}}|^2 = a_{1r}^2 + a_{2r}^2$  olduğundan

$$\int_{-\infty}^{\infty} (a_{1r}^2 + a_{2r}^2) (1 + \xi^2)^2 d\xi \leq 2C'_0 \int_{-\infty}^{\infty} (1 + \xi^2)^2 ((\varphi_1)_{x_r}^2 + (\varphi_2)_{x_r}^2) d\xi$$

yazılabilir. Burada  $r = 0, \dots, n$  için toplam alınırsa

$$\begin{aligned}
&\int_{-\infty}^{\infty} \sum_{r=0}^n (a_{1r}^2 + a_{2r}^2) (1 + \xi^2)^2 d\xi \\
&\leq 2C'_0 \int_{-\infty}^{\infty} \left( \sum_{r=1}^n ((\varphi_1)_{x_r}^2 + (\varphi_2)_{x_r}^2) + \varphi_1^2 + \varphi_2^2 \right) (1 + \xi^2)^2 d\xi
\end{aligned} \tag{3.12}$$

elde edilir. Benzer şekilde  $|\partial_t^\beta b_l| \leq C_0$  olduğundan

$$\int_{-\infty}^{\infty} (1 + \xi^2)^2 |\widehat{b_l w_{y_l}}|^2 d\xi \leq 2C'_0 \int_{-\infty}^{\infty} (1 + \xi^2)^2 ((\varphi_1)_{y_l}^2 + (\varphi_2)_{y_l}^2) d\xi$$

yazılabilir. Burada  $|\widehat{b_l w_{y_l}}|^2 = b_{1l}^2 + b_{2l}^2$  yerine yazılırsa

$$\int_{-\infty}^{\infty} (b_{1l}^2 + b_{2l}^2) (1 + \xi^2)^2 d\xi \leq 2C'_0 \int_{-\infty}^{\infty} (1 + \xi^2)^2 ((\varphi_1)_{y_l}^2 + (\varphi_2)_{y_l}^2) d\xi$$

ve  $l = 1, \dots, m$  için toplam alınırsa

$$\int_{-\infty}^{\infty} \sum_{l=1}^m (b_{1l}^2 + b_{2l}^2) (1 + \xi^2)^2 d\xi \leq 2C'_0 \int_{-\infty}^{\infty} \sum_{l=1}^m ((\varphi_1)_{y_l}^2 + (\varphi_2)_{y_l}^2) (1 + \xi^2)^2 d\xi \tag{3.13}$$

bulunur. O halde (3.10)-(3.11) eşitlikleri ve (3.12)-(3.13) eşitsizlikleri (3.9) eşitsizliğinde yerine yazılırsa

$$\begin{aligned}
&\sum_{s=1}^2 \int_{-\infty}^{\infty} (2\lambda\nu(\varphi_s)_{x_1}^2 + \lambda\nu(x_1 + \delta_0) \sum_{i=2}^n (\varphi_s)_{x_i}^2 + \lambda\nu(x_1 + \delta_0) \sum_{j=1}^m (\varphi_s)_{y_j}^2 \\
&+ 4\lambda^3\nu^3\psi^{-2\nu-3}\varphi_s^2)\chi^2(1 + \xi^2)^2 d\xi + \sum_{s=1}^2 \sum_{i=1}^4 \int_{-\infty}^{\infty} d_i(\varphi_s)(1 + \xi^2)^2 d\xi
\end{aligned}$$

$$\begin{aligned}
&\leq (\psi^{\nu+1} + 1)(x_1 + \delta_0)^2(4\overline{f_0}C_2 \sum_{s=1}^2 \int_{-\infty}^{\infty} (1 + \xi^2)^2 \varphi_s^2 d\xi \\
&\quad + 8(n+1)C'_0 \int_{-\infty}^{\infty} (\sum_{r=1}^n ((\varphi_1)_{x_r}^2 + (\varphi_2)_{x_r}^2) + \varphi_1^2 + \varphi_2^2)(1 + \xi^2)^2 d\xi \\
&\quad + 8mC'_0 \int_{-\infty}^{\infty} \sum_{l=1}^m ((\varphi_1)_{y_l}^2 + (\varphi_2)_{y_l}^2)(1 + \xi^2)^2 d\xi \chi^2 \\
&\quad + \lambda^2 \nu^2 (x_1 + \delta_0)^2 \Lambda^2 \chi^2 \sum_{s=1}^2 \int_{-\infty}^{\infty} (1 + \xi^2)^2 \varphi_s^2 d\xi \\
&= (\psi^{\nu+1} + 1)(x_1 + \delta_0)^2(4\overline{f_0}C_2 \sum_{s=1}^2 \int_{-\infty}^{\infty} (1 + \xi^2)^2 \varphi_s^2 d\xi \\
&\quad + 8(n+1)C'_0 \sum_{s=1}^2 \int_{-\infty}^{\infty} (|\nabla_x \varphi_s|^2 + \varphi_s^2)(1 + \xi^2)^2 d\xi \\
&\quad + 8mC'_0 \sum_{s=1}^2 \int_{-\infty}^{\infty} |\nabla_y \varphi_s|^2 (1 + \xi^2)^2 d\xi \chi^2 \\
&\quad + \lambda^2 \nu^2 (x_1 + \delta_0)^2 \Lambda^2 \chi^2 \sum_{s=1}^2 \int_{-\infty}^{\infty} (1 + \xi^2)^2 \varphi_s^2 d\xi \tag{3.14}
\end{aligned}$$

elde edilir. Böylece (3.14) eşitsizliğinden

$$\begin{aligned}
&E_5 \chi^2 \sum_{s=1}^2 \int_{-\infty}^{\infty} (\varphi_s)_{x_1}^2 (1 + \xi^2)^2 d\xi + E_6 \chi^2 \sum_{s=1}^2 \int_{-\infty}^{\infty} \sum_{i=2}^n (\varphi_s)_{x_i}^2 (1 + \xi^2)^2 d\xi \\
&+ E_7 \chi^2 \sum_{s=1}^2 \int_{-\infty}^{\infty} \sum_{j=1}^m (\varphi_s)_{y_j}^2 (1 + \xi^2)^2 d\xi + E_8 \chi^2 \sum_{s=1}^2 \int_{-\infty}^{\infty} \varphi_s^2 (1 + \xi^2)^2 d\xi \\
&\leq - \sum_{s=1}^2 \sum_{i=1}^4 \int_{-\infty}^{\infty} d_i(\varphi_s)(1 + \xi^2)^2 d\xi \tag{3.15}
\end{aligned}$$

olduğu görülür. Burada

$$\begin{aligned}
E_5 &= 2\lambda\nu - 8(n+1)C'_0(\psi^{\nu+1} + 1)(x_1 + \delta_0)^2, \\
E_6 &= \lambda\nu(x_1 + \delta_0) - 8(n+1)C'_0(\psi^{\nu+1} + 1)(x_1 + \delta_0)^2, \\
E_7 &= \lambda\nu(x_1 + \delta_0) - 8mC'_0(\psi^{\nu+1} + 1)(x_1 + \delta_0)^2, \\
E_8 &= 4\lambda^3\nu^3\psi^{-2\nu-3} - 4(\psi^{\nu+1} + 1)(x_1 + \delta_0)^2\overline{f_0}C_2 \\
&\quad - 8(n+1)C'_0(\psi^{\nu+1} + 1)(x_1 + \delta_0)^2 - \lambda^2\nu^2(x_1 + \delta_0)^2\Lambda^2
\end{aligned}$$

olarak tanımlıdır.

Eğer  $\lambda > 1$ ,  $\nu \geq \nu_3 = \frac{1}{2} + 8(n+1)C'_0$ ,  $0 < \psi < 1$  ve  $(x_1 + \delta_0) < 1$  eşitsizlikleri dikkate

alınırsa

$$\begin{aligned}
E_5 &> 2\lambda\nu - 16(n+1)C'_0 \\
&\geq 2\lambda\left(\frac{1}{2} + 8(n+1)C'_0\right) - 16(n+1)C'_0 \\
&= \lambda + 16\lambda(n+1)C'_0 - 16(n+1)C'_0 \\
&> \lambda
\end{aligned} \tag{3.16}$$

olur.

İkinci olarak  $\lambda > 1$ ,  $\nu \geq \nu_4 = 1 + 16(n+1)C'_0$ ,  $0 < \psi < 1$ ,  $(x_1 + \delta_0) < 1$  eşitsizlikleri ve  $(x_1 + \delta_0) \neq 0$  olduğu göz önüne alınarak

$$\begin{aligned}
E_6 &> (x_1 + \delta_0)^2 \left( \lambda\nu \frac{1}{(x_1 + \delta_0)} - 16(n+1)C'_0 \right) \\
&> (x_1 + \delta_0)^2 (\lambda\nu - 16(n+1)C'_0) \\
&\geq (x_1 + \delta_0)^2 (\lambda(1 + 16(n+1)C'_0) - 16(n+1)C'_0) \\
&= (x_1 + \delta_0)^2 (\lambda + 16\lambda(n+1)C'_0 - 16(n+1)C'_0) \\
&> (x_1 + \delta_0)^2 \lambda
\end{aligned} \tag{3.17}$$

bulunur.

Benzer şekilde  $\lambda > 1$ ,  $\nu \geq \nu_5 = 1 + 16mC'_0$ ,  $0 < \psi < 1$ ,  $(x_1 + \delta_0) < 1$  seçilerek ve  $(x_1 + \delta_0) \neq 0$  olduğu dikkate alınarak

$$\begin{aligned}
E_7 &> (x_1 + \delta_0)^2 \left( \lambda\nu \frac{1}{(x_1 + \delta_0)} - 16mC'_0 \right) \\
&> (x_1 + \delta_0)^2 (\lambda\nu - 16mC'_0) \\
&\geq (x_1 + \delta_0)^2 (\lambda(1 + 16mC'_0) - 16mC'_0) \\
&= (x_1 + \delta_0)^2 (\lambda + 16\lambda mC'_0 - 16mC'_0) \\
&> (x_1 + \delta_0)^2 \lambda
\end{aligned} \tag{3.18}$$

elde edilir.

Son olarak  $\lambda \geq \max\{\lambda_4, \lambda_5\}$ ,  $\lambda_4 = (\Lambda^0)^2$ ,  $\lambda_5 = 8\overline{f_0}C_2 + 16(n+1)C'_0$ ,  $0 < \psi < 1$ ,  $(x_1 + \delta_0) < 1$ ,  $|\Lambda| \leq \Lambda^0$  ve  $\nu \geq 1$  alınırsa

$$E_8 > 4\lambda^3\nu^3\psi^{-2\nu-3} - 8\overline{f_0}C_2 - 16(n+1)C'_0 - \lambda^2\nu^2(\Lambda^0)^2$$

$$\begin{aligned}
\lambda^3\nu^3\psi^{-2\nu-3} - \lambda^2\nu^2(\Lambda^0)^2 &> \lambda^3\nu^3 - \lambda^2\nu^2\lambda_4 \\
&\geq \lambda^3\nu^3 - \lambda^3\nu^3 = 0,
\end{aligned}$$

$$\begin{aligned}
\lambda^3 \nu^3 \psi^{-2\nu-3} - 8\overline{f_0} C_2 - 16(n+1)C'_0 &> \lambda^3 \nu^3 - \lambda_5 \\
&\geq \lambda^3 \nu^3 - \lambda \\
&\geq \lambda^3 \nu^3 - \lambda^3 \nu^3 = 0,
\end{aligned}$$

$$E_8 > 2\lambda^3 \nu^3 \psi^{-2\nu-3} \quad (3.19)$$

bulunur. Yukarıdaki (3.16)-(3.19) eşitsizlikleri (3.15) eşitsizliğinde yerine yazılırsa ve  $\lambda > 1$ ,  $\nu \geq 1$  olduğu dikkate alınırsa

$$\begin{aligned}
&\sum_{s=1}^2 \int_{-\infty}^{\infty} (\lambda(\varphi_s)_{x_1}^2 + \lambda(x_1 + \delta_0)^2 \sum_{i=2}^n (\varphi_s)_{x_i}^2 + \lambda(x_1 + \delta_0)^2 \sum_{j=1}^m (\varphi_s)_{y_j}^2 \\
&+ 2\lambda^3 \nu^3 \psi^{-2\nu-3} \varphi_s^2) \chi^2 (1 + \xi^2)^2 d\xi \leq - \sum_{s=1}^2 \sum_{i=1}^4 \int_{-\infty}^{\infty} d_i(\varphi_s) (1 + \xi^2)^2 d\xi, \\
&\sum_{s=1}^2 \int_{-\infty}^{\infty} 2\lambda^3 \nu^3 \psi^{-2\nu-3} \varphi_s^2 \chi^2 (1 + \xi^2)^2 d\xi \leq - \sum_{s=1}^2 \sum_{i=1}^4 \int_{-\infty}^{\infty} d_i(\varphi_s) (1 + \xi^2)^2 d\xi, \\
&\sum_{s=1}^2 \int_{-\infty}^{\infty} 2\psi^{-2\nu-3} \varphi_s^2 \chi^2 (1 + \xi^2)^2 d\xi \leq - \frac{1}{\lambda^3 \nu^3} \sum_{s=1}^2 \sum_{i=1}^4 \int_{-\infty}^{\infty} d_i(\varphi_s) (1 + \xi^2)^2 d\xi
\end{aligned}$$

elde edilir. Son eşitsizlikte  $2(1 + \xi^2)^2 > 1$ ,  $\chi^2 > 1$  alınır ve  $0 < \psi < 1$  olduğu göz önünde bulundurulursa

$$\sum_{s=1}^2 \int_{-\infty}^{\infty} \varphi_s^2 d\xi \leq - \frac{1}{\lambda^3 \nu^3} \sum_{s=1}^2 \sum_{i=1}^4 \int_{-\infty}^{\infty} d_i(\varphi_s) (1 + \xi^2)^2 d\xi$$

olur. Eğer  $\lambda \rightarrow \infty$  için limite geçilirse  $\varphi_1 = \varphi_2 = 0$  olur. Böylece  $g = 0$  bulunur ve ispat tamamlanır. ■



## BÖLÜM 4

### SONUÇ

Bu tezde, ultrahiperbolik Schrödinger denklemi için bir ters problem ele alınmış ve başlangıç anında verilen bir ek bilgi yardımıyla çözümün tekliği araştırılmıştır. Burada kullanılan temel araç Lavrenti'ev vd. (1986) da verilmiş olan Carleman tipi bir eşitsizliktir. Bu yöntem, Klivanov ve Timonov (2004) ve Amirov ve Yamamoto (2008) tarafından parabolik, hiperbolik ve Schrödinger denklemlerine uygulanmıştır. Ayrıca tezde ele alınan problem çalışılırken Fourier analizinin kavram ve araçlarından faydalanılmıştır. Ultrahiperbolik Schrödinger denklemleri ek boyutlar içerdiğinden başta Sicim teorisi olmak üzere modern fizik kuramları açısından önem taşımaktadır.



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