GENERATING TREE METHOD AND APPLICATIONS TO PATTERN-AVOIDING INVERSION SEQUENCES

A THESIS SUBMITTED TO

THE GRADUATE SCHOOL OF ENGINEERING AND SCIENCE

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR

THE DEGREE OF

MASTER OF SCIENCE

IN

MATHEMATICS

By

Melis Gezer

May 2024

GENERATING TREE METHOD AND APPLICATIONS TO PATTERN-AVOIDING INVERSION SEQUENCES

By Melis Gezer

May 2024

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Gökhan Yıldırım(Advisor)

Alexander Goncharov

Emrah Sercan Yılmaz

Approved for the Graduate School of Engineering and Science:

Orhan Arıkan
Director of the Graduate School

ABSTRACT

GENERATING TREE METHOD AND APPLICATIONS TO PATTERN-AVOIDING INVERSION SEQUENCES

Melis Gezer
M.S. in Mathematics
Advisor: Gökhan Yıldırım
May 2024

An inversion sequence of length n is an integer sequence $e = e_1 \cdots e_n$ such that $0 \leq e_i < i$ for each $0 \leq i \leq n$. We use I_n to denote the set of inversion sequences of length n. Let $[k] := \{0, 1, \cdots, k-1\}$ denote the alphabet and τ be a word of length k over this alphabet. A pattern of length k is simply a word over the alphabet $[k]$. We say an inversion sequence $e \in I_n$ contains the pattern τ of length k if it contains a sub-sequence of length k that is order isomorphic to τ; otherwise, e avoids the pattern τ. For a given pattern τ, we use $I_n(\tau)$ to denote the set of all τ-avoiding inversion sequences of length n. Firstly, we review the enumeration of inversion sequences that avoid patterns of length three. We then study an enumeration method based on generating trees and the kernel method to enumerate pattern-avoiding inversion sequences for general patterns. Then, we provide sampling algorithms for pattern-avoiding inversion sequences and apply them to some specific patterns. Based on extensive simulations, we study some statistics such as the number of zeros, the number of distinct elements, the number of repeated elements, and the maximum elements. Finally, we present a bijection between $I_n(0312)$ and $I_n(0321)$ that preserves these statistics.

Keywords: inversion sequences, pattern avoidance, generating functions, generating trees, kernel method, random sampling.

ÖZET

ÜRETEÇ AĞAÇLAR YÖNTEMİ VE MOTİF İÇERMEYEN TERS-ÇEVRİM DİZİLERİNE UYGULAMALARI

Melis Gezer
Mathematics, Yüksek Lisans
Tez Danışmanı: Gökhan Yıldırım
Mayıs 2024

Uzunluğu n olan bir ters-çevrim dizisi, her $0 \leq i \leq n$ için $0 \leq e_i < i$ şartını sağlayan $e = e_1 \cdots e_n$ şeklinde verilmiş bir tamsayı dizisidir. Uzunluğu n olan ters-çevrim dizileri kümesi I_n ile gösterilir. Uzunluğu k olan motifler, $\{0, 1, 2, \ldots, k-1\}$ alfabesi üzerinde verilmiş kelimeler ile temsil edilir. Verilen k uzunluğunda bir τ motifi için, uzunluğu n olan bir ters-çevrim dizisi $e \in I_n$, τ ile aynı göreceli sıralamaya sahip k uzunluğunda bir alt-dizi içeriyorsa, motifi içeriyor deriz. Aksi takdirde, bu ters-çevrim dizisi bu motifi içermez. Verilen bir τ motifi için, τ'yu içermeyen n uzunluğundaki ters-çevrim dizileri kümesi $I_n(\tau)$ ile gösterilir. Öncelikle uzunluğu 3 olan motifleri içermeyen ters-çevrim dizilerinin nasıl sayıldığını inceliyoruz. Sonrasında, verilen daha genel motifler için, motif içermeyen ters-çevrim dizilerini saymak için üreteç ağaçlar ve çekirdek yöntemlerini çalışıyoruz. Sonra, motif içermeyen ters-çevrim dizileri için örnekleme algoritması veriyoruz ve bunu bazı motiflere uyguluyoruz. Bu algoritma ile yaptığımız çalışmalara dayanarak, belirli motifleri içermeyen ters-çevrim dizilerinin bazı istatistikleri üzerine çalışıyoruz. Bu istatistikler: sıfır olan elemanların sayısı, tekrar eden elemanların sayısı, tekrar etmeyen elemanların sayısı, soldan sağa en büyük olan elemanların sayısı ve dizinin en büyük elemanıdır. Son olarak $I_n(0312)$ ve $I_n(0321)$ kümeleri arasında birebir, örten ve bu istatistikleri koruyan bir fonksiyon olduğunu gösteriyoruz.

Anahtar sözcükler: ters-çevrim dizisi, motif içermeme, üreteç fonksiyonlar, üreteç ağaçlar, rastgele örneklem.

Acknowledgement

Foremost, I would like to express my gratitude to my advisor, Gökhan Yıldırım, for his invaluable feedback, guidance, and unwavering support throughout the completion of my thesis.

I also would like to thank my thesis committee members, Alexandre Goncharov and Emrah Sercan Yılmaz, for their time and valuable feedback.

I would also like to extend my heartfelt appreciation to my friends Semra, Zilan, Lütfiye, Zeynep, and Esat, whose presence inspires me to keep going for their unwavering friendship and invaluable discussions. Furthermore, I am grateful to many other friends who listened and motivated me during this period; their support is immeasurable.

Special gratitude is extended to my beloved mother, whose unwavering love, support, and sacrifices have been the cornerstone of my journey.

I would like to acknowledge that this thesis was partially supported by Tübitak-Bideb-no: 120F352. The numerical calculations were partially performed at ULAKBIM-TÜBİTAK, High Performance and Grid Computing Center (TRUBA resources).

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Pattern avoidance has intriguing connections to various areas of mathematics, including algebra, geometry, and computer science, and it has applications in fields such as sorting algorithms, data structures, biological sequence analysis, and quantum chemistry; for further details, see [1, 2, 3, 4]. The study of pattern-avoiding permutations and inversion sequences reveals rich combinatorial structures and often involves sophisticated enumerative techniques to count the number of structures avoiding a given pattern. Pattern-restricted permutations are of particular interest in combinatorics because they generalize classical sorting problems and connect to various mathematical structures, including tableaux, trees, and graphs. They also appear in statistical mechanics and the study of sorting algorithms. Inversion sequences provide a way to encode permutations and study their properties related to inversions. These sequences are useful in various areas of combinatorics and have applications in algorithms and the analysis of sorting processes.

1.1 Permutations and inversion sequences

Let $[n] := \{1, 2, \dots, n\}$. A permutation of length n over the set $[n]$ is a word $\sigma_1\sigma_2 \dots \sigma_n$ with no repeated letters and $\sigma_i \in [n]$ for all $i = 1, \dots, n$. For example, $\sigma = 32618475$ is a permutation of length 8.

We use S_n to denote the set of all permutations of $[n]$. For instance, $S_1 = \{1\}$, $S_2 = \{12, 21\}$, $S_3 = \{123, 132, 213, 231, 312, 321\}$.

Definition 1.1.1. *An inversion sequence of length n is any integer sequence $e = (e_1, e_2, \dots, e_n)$ where $0 \leq e_i < i$ for all $i \in \{1, \dots, n\}$.*

For simplicity, we use $e = e_1 \dots e_n$ instead of $(e_1, e_2, \dots, e_n)$. For example, $e = 0010113641$ is an inversion sequence of length 10. Note that e_1 is always zero because of the condition $0 \leq e_1 < 1$.

Let I_n denote the set of all inversion sequences of length n. For instance, $I_1 = \{0\}$, $I_2 = \{00, 01\}$, $I_3 = \{000, 001, 002, 010, 011, 012\}$.

There is a bijection between I_n and S_n defined as follows: $\phi : S_n \mapsto I_n$ such that $\phi(\sigma_1 \dots \sigma_n) = e_1 \dots e_n$ where $e_i = |\{j : \sigma_j > \sigma_i \text{ and } j < i\}|$.

For example, let $\sigma = 13524$, then $\phi(13524) = 00021 \in I_5$.

Conversely, to obtain the permutation $\sigma \in S_n$ from a given inversion sequence $e \in I_n$, we read the inversion sequences from the end, and we write an element from $[n]$ in such a way that there are exactly e_i numbers greater than i positioned to the left of i. For example, $\phi^{-1}(010) = 213$.

1.1.1 Pattern-avoiding permutations and inversion sequences

The study of pattern-avoiding permutations has a long history. During the 1880s, MacMahon studied generating functions of the permutations that avoid some monotone patterns, for further details, see [5]. From historical origins to the present day, the investigation of patterns in permutations has found applications in computational biology and theoretical physics; for more details, see [2].

Definition 1.1.2. *A pattern of length k is simply a permutation in S_k, $\tau = \tau_1\tau_2\ldots\tau_k \in S_k$. A permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in S_n$ contains the pattern $\tau = \tau_1\tau_2\ldots\tau_k \in S_k$ if there exist $i_1 < i_2 < \ldots < i_k$ such that $\sigma_{ia} <_{ib}$ if and only if $\tau_a < \tau_b$ for all $1 \leq a, b \leq k$. Otherwise, σ avoids pattern τ.*

As an example, the permutation $\sigma = 87651234 \in S_8$ contains the pattern $\tau = 123 \in S_3$ because of the sub-sequences $\sigma_5\sigma_6\sigma_7 = 123$, $\sigma_5\sigma_6\sigma_8 = 124$ and $\sigma_6\sigma_7\sigma_8 = 234$. On the other hand, σ avoids the pattern $\tau' = 132$ since there is no sub-sequence of length three of σ that corresponds to τ'.

Let $S_n(\tau) = \{\sigma \in S_n \mid \sigma \text{ avoids } \tau\}$. For example, $S_4(132) = \{1234, 2134, 2314, 2341, 3124, 3241, 3214, 3412, 3421, 4123, 4231, 4213, 4312, 4321\}$. The avoidance sequence for a given pattern τ is the counting sequence $\{|S_1(\tau)|, |S_2(\tau)|, |S_3(\tau)|\ldots\}$.

Similarly, we define $S_n(B) = \bigcap_{\tau\in B} S_n(\tau)$. The avoidance sequence of B is the counting sequence $\{|S_1(B)|, |S_2(B)|, |S_3(B)|\ldots\}$.

From a combinatorial perspective, pattern-avoiding permutations establish connections between many combinatorial structures as their avoidance sequences count well-known combinatorial structures, for further details see [3].

Definition 1.1.3. *For permutations, two patterns $\tau, \tau' \in S_k$ are said to be Wilf-equivalent, denoted by $\tau \sim_{\mathrm{W}} \tau'$, if $|S_n(\tau)| = |S_n(\tau')|$ for all $n \geq 1$.*

Note that $\sim_{\mathrm{W}}$ defines an equivalence relation on S_k. The equivalence classes of this relation are called Wilf classes. If two patterns τ and τ' are Wilf-equivalent, then we say that they are in the same Wilf class. Let $(S_k, \sim_{\mathrm{W}})$ denotes Wilf classes in S_k.

For $(S_2, \sim_{\mathrm{W}})$, there is only one permutation of length n that avoids 12, which is the permutation $n(n-1)\ldots21$. Therefore $|S_n(12)| = 1$ for all $n \geq 1$. Similarly, we have $|S_n(21)| = 1$ for all $n \geq 1$. Therefore, there is only one Wilf class for this case.

In the study of pattern-avoiding permutations, it is known that there exists only one Wilf class in $(S_3, \sim_{\rm w})$, and the avoidance sequence for each $\tau \in S_3$ corresponds to Catalan numbers; for further details see [6, 7, 8].

The first comprehensive study on pattern-avoiding permutations was presented in a paper by Simion and Schmidt [8] in 1985. In addition to other results, they established a direct bijection between $S_n(123)$ and $S_n(132)$.

In analogy to pattern-avoiding permutations, the pattern avoidance concept is studied for inversion sequences. The extensive study on pattern-avoiding inversion sequences began with two independent papers published by Mansour and Shattuck [9] and Corteel, Martinez, Savage, and Weselcouch [10] in 2015 and 2016, respectively. In these papers, the enumeration of pattern-avoiding inversion sequences for the patterns of length 3 is solved, except for the patterns 010 and 100. Then, Lin and Yan [11] studied the Wilf-equivalence classes for the pairs of patterns of length 3. They showed there are 48 Wilf classes among 78 pairs. Following that, Kotsireas, Mansour, and Yıldırım [12] developed an algorithmic approach based on generating tree method to enumerate pattern-avoiding inversion sequences.

Note that there are repeating elements in inversion sequences; therefore, we use a modified definition for patterns for the inversion sequences.

Definition 1.1.4. *A pattern of length k is a word $\tau = \tau_1\tau_2\ldots\tau_k$ with $\tau_i \in \{0, 1, \ldots, k-1\}$ for all i with the constraint that τ_i can be j only if $j-1$ is already present in τ.*

For a given pattern $\tau = \tau_1\tau_2\ldots\tau_k$, we obtain patterns that are order isomorphic to τ by replacing all the occurrences of the i^{th} smallest entry of τ with $i-1$ for all i. For instance, the pattern 022 is order isomorphic to 011.

Let $\mathcal{P}_n$ be the set of patterns of length n up to order isomorphism. For instance, $\mathcal{P}_3 = \{000, 001, 010, 011, 012,\ 021, 100, 102, 120, 101, 110, 201, 210\}$.

Definition 1.1.5. *An inversion sequence $e \in I_n$ contains the pattern τ of length k if there is a sub-sequence of length k in e that is order isomorphic to τ ; otherwise, e avoids the pattern τ.*

Observe that the inversion sequence $e = 00213011$ contains pattern 000 since the sub-sequences $e_1e_2e_6 = 000$ and $e_4e_7e_8 = 111$ of e is an occurrence of 000. It also contains 102 because the sub-sequence $e_3e_4e_5 = 213$ is an occurrence of 102. However, e avoids 110.

Let $I_n(\tau)$ to denote the set of all τ-avoiding inversion sequences of length n. The counting sequence of $I_n(\tau)$ is the integer sequence $\{|I_1(\tau)|,\ |I_2(\tau)|,\ |I_3(\tau)|, \ldots\}$.

Definition 1.1.6. *Two patterns, τ and τ', are Wilf-equivalent if $|I_n(\tau)| = |I_n(\tau')|$ for all $n \geq 1$.*

While there are 13 patterns in $\mathcal{P}_3$, there exist 11 Wilf classes in $\mathcal{P}_3$. Because $I_n(101) = I_n(110)$ and $I_n(201) = I_n(210)$. For more details, see [10].

Definition of pattern avoidance can be extended from a single pattern to a set of patterns as $I_n(B) = \bigcap_{\tau \in B} I_n(\tau)$. The counting sequence of $I_n(B)$ is the integer sequence $\{|I_1(B)|,\ |I_2(B)|,\ |I_3(B)|, \ldots\}$.

1.2 Generating tree method

The generating tree method is a powerful combinatorial technique used to systematically construct and enumerate combinatorial objects by representing them as nodes in a tree. In this method, each node corresponds to a specific combinatorial object, and edges represent the generation of new objects from existing ones by applying certain rules or operations. Starting from a root node, which represents the simplest form of the object (often the empty object or a base case), the tree is generated by recursively applying the rules to create child nodes. This approach not only provides a clear visual and structural representation of the combinatorial class but also facilitates the derivation of recursive formulas and generating functions. The generating tree method is particularly useful in studying pattern-avoiding permutations, tilings, lattice paths, and other combinatorial structures, as it allows for the systematic exploration of their properties and the enumeration of specific subclasses by tracking the growth and connections within the tree. When combined with the kernel method, it leads to solutions to many enumeration problems in a systematic way. The kernel method is a sophisticated technique used in enumerative combinatorics to solve functional equations that arise in the enumeration of combinatorial structures, see [13]. It is particularly effective for handling generating functions that involve multiple variables. The core idea of the kernel method is to manipulate the functional equation to eliminate certain variables, thereby simplifying the problem. This is typically achieved by identifying a "kernel" term in the equation that can be set to zero, allowing the remaining terms to be solved more easily. The method often involves clever substitutions and algebraic manipulations to isolate the desired generating function. The kernel method has been successfully applied to a variety of combinatorial problems, such as counting lattice paths, tilings, and pattern-avoiding permutations.

A combinatorial class $\mathcal{C}$ is a set of discrete objects with an associated size function such that for each n, there is a finitely many objects of size n. For a class $\mathcal{C}$, we define $\mathcal{C}_n$ to be the objects in $\mathcal{C}$ of size n. A generating tree for $\mathcal{C}$ is a rooted, labeled tree where the nodes represent the objects of $\mathcal{C}$ and have the following properties:

1. Every object in $\mathcal{C}$ is represented exactly once in the tree.
2. Every element of $\mathcal{C}_n$ are located at level n in the tree.
3. The children of objects are obtained by a set of succession rules that determines the number of children and their labels.

The generating tree method, in most cases, is an efficient tool to find the enumeration sequence of the related combinatorial class.

In 1995, Julian West used the generating tree method to enumerate permutations with forbidden sub-sequences in [14] and [15]. Afterward, Barcucci, Del Lungo, Pergola, and Pinzani made significant contributions to this field by describing a large number of combinatorial classes with generating trees, see [16, 17].

We present some examples from [15] and [18].

Example 1.2.1. *Fibonacci Tree*

The generating tree has the following succession rules:

Root : (1)

Rules : $(1) \rightsquigarrow (2)$

$(2) \rightsquigarrow (1), (2).$

According to succession rules, the node with label 1 *has one child with label* 2 *and the node with label* 2 *has two children with labels* 1 *and* 2*, respectively. The node initializes the tree with the label* 1*. See figure 1.1.*

The main question is: how many nodes are at the level n*? Let* g_n *be the number of nodes at level* n*, and the root is at level* 0*. Our goal is to compute the following generating function:*

$$G(x) = \sum_{n=0}^{\infty} g_n x^n = g_0 + g_1 x + g_2 x^2 + g_3 x^3 + \dots$$

Here, $[x^n]G(x) = g_n$ *gives the total number of children at level* n*.*

Let $G_i(x)$ *denote the generating function corresponding to the sub-tree with root label* i*. In this example, the labels of the nodes are* 1 *or* 2*, and we have two succession rules. We write two functional equations by using* $G_i(x)$ *according to the given succession rules. These are,*

$$G_1(x) = 1 + xG_2(x)$$

and

$$G_2(x) = 1 + xG_1(x) + xG_2(x).$$

These two functional equations are solved by substitution, then we get the following equation for $G_1(x)$*.*

$$G_1(x) = \frac{1}{1 - x - x^2}. \tag{1.1}$$

Note that $\sum_{n=0}^{\infty} F_{n+1} x^n = \dfrac{1}{1 - x - x^2}$ *where* F_n *is the* n^{th} *Fibonacci Number.*

Hence, the generating function corresponding to the Fibonacci generating tree is given by (1.1)*. Figure 1.1 shows the first 7 levels of the Fibonacci generating tree.*

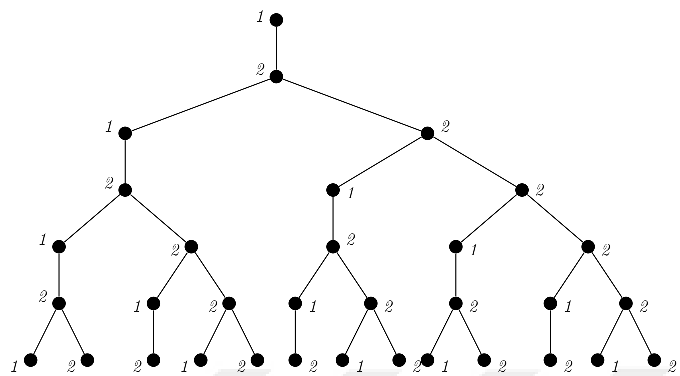


Figure 1.1: Fibonacci generating tree

Example 1.2.2. *Catalan Tree*

The generating tree has the following succession rules:

Root : (1)

Rules : $(1) \rightsquigarrow (2)$

$(m) \rightsquigarrow (2), (3), \ldots, (m+1)$ *where* $m > 1$.

We assign labels to the nodes according to their number of children. Following the succession rules, the node initializes the tree with the label 1*. The node with label* 1 *has one child with label* 2*, and the node with label* m *has* m *children with labels* $2, 3, \ldots (m+1)$*, respectively. Let* g_n *be the number of nodes at level* n*, and the root is at level* 0*. See figure 1.2. Our goal is to compute the generating function* $G(x) = \sum_{n=0}^{\infty} g_n x^n$*. Let* $G_i(x)$ *denote the generating function corresponding to the sub-tree with root label* i*. We write functional equations using* $G_i(x)$ *according to the given succession rules. These are,*

$$G_m(x) = 1 + x\left[G_2(x) + G_3(x) + \cdots + G_{m+1}(x)\right] \quad \textit{where } m \geq 1.$$

As $m \geq 1$*, we have infinite number of* $G_i(x)$*. Therefore, we introduce a multi-variable generating function to deal with only one function instead of an infinite number of functions.*

Let $G(x,u) = \sum_{m=1}^{\infty} G_m(x) u^{m-1} = G_1(x) + G_2(x)u + G_3(x)u^2 + \ldots$

Note that, $G(x,0) = G_1(x)$ *and* $G_{m+1}(x) - G_m(x) = xG_{m+2}(x)$*. By multiplying* *by u^{m-1} and summing over $m \geq 1$, we obtain*

$$\sum_{m=1}^{\infty} G_{m+1}(x)u^{m-1} - \sum_{m=1}^{\infty} G_m(x)u^{m-1}.$$

By changing the limits of the summation, we have

$$\frac{1}{u}\sum_{m=2}^{\infty} G_m(x)u^{m-1} - G(x,u) = \frac{x}{u^2}\sum_{m=3}^{\infty} G_m(x)u^{m-1}.$$

Then we arrange the equation and get

$$\frac{1}{u}G(x,u) - \frac{1}{u}G_1(x) - G(x,u) = \frac{x}{u^2}G(x,u) - \frac{x}{u^2}G_1(x) - \frac{x}{u}G_2(x).$$

Finally, we obtain the following functional equation:

$$\left(\frac{1}{u} - 1 - \frac{x}{u^2}\right)G(x,u) = -\frac{x}{u^2}G_1(x) + \frac{1}{u}.$$

This type of functional equation can be systematically solved using the kernel method; for further details, see [13]. Here the kernel equation is $\frac{1}{u} - 1 - \frac{x}{u^2} = 0$ *and its roots are:*

$$u = \frac{1 \pm \sqrt{1-4x}}{2}.$$

Power series expansion of the root $u = \frac{1+\sqrt{1-4x}}{2}$ *gives negative coefficients. As we consider only positive coefficients in enumeration problems, we choose the root* $u = \frac{1-\sqrt{1-4x}}{2}$.

Notice that, $u = \frac{1-\sqrt{1-4x}}{2} = xC(x)$ *where $C(x)$ is the generating function of Catalan Numbers, then we obtain*

$$G_1(x) = C(x).$$

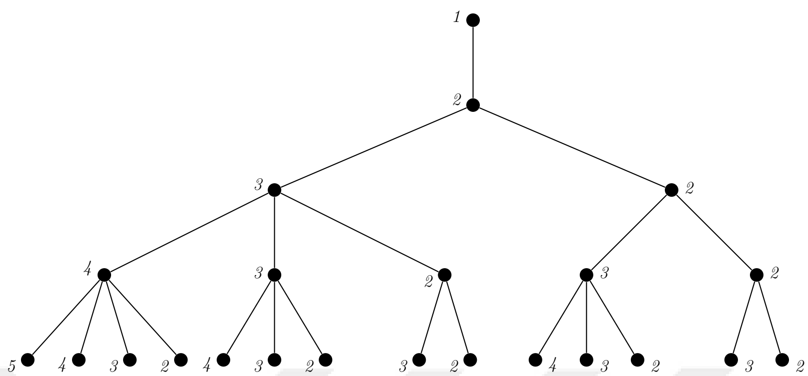


Figure 1.2: Catalan generating tree

Example 1.2.3. *Schröder Tree*
The generating tree has the following succession rules:
Root : (2),
Rules : $(m) \rightsquigarrow (3)(4)\dots(m)(m+1)(m+1)$ *where* $m \geq 2$
Let g_n *be the number of nodes at level* n*, and the root is at level 0. See figure 1.3. We want to compute the generating function* $G(x) = \sum_{n=0}^{\infty} g_n x^n$*. Let* $G_i(x)$ *denote the generating function corresponding to the sub-tree with root label* i*. The succession rules yield the following equations:*

$$G_m(x) = 1 + x\left[G_3(x) + G_4(x) + \cdots + G_m(x) + G_{m+1}(x) + G_{m+1}(x)\right]$$

where $m \geq 2$.

Let $G(x,u) = \sum_{m=2}^{\infty} G_m(x)u^{m-2} = G_2(x) + G_3(x)u + G_4(x)u^2 + \dots$

Notice that, $G(x,0) = G_2(x)$.

We also have,

$$G_{m+1}(x) - G_m(x) = 2xG_{m+2}(x) - xG_{m+1}(x).$$

By multiplying u^{m-2} *and summing over* $m \geq 2$,

$$\sum_{m=2}^{\infty} G_{m+1}(x)u^{m-2} - \sum_{m=2}^{\infty} G_m(x)u^{m-2} = 2x\sum_{m=2}^{\infty} G_{m+2}(x)u^{m-2} - x\sum_{m=2}^{\infty} G_{m+1}(x)u^{m-2}.$$

By changing the limits of the summation,

$$\frac{1}{u}\sum_{m=3}^{\infty} G_m(x)u^{m-2} - G(x,u) = \frac{2x}{u^2}\sum_{m=4}^{\infty} G_m(x)u^{m-2} - \frac{x}{u}\sum_{m=3}^{\infty} G_m(x)u^{m-2}.$$

Then, we arrange and simplify this equation and obtain the following functional equation.

$$\left(u - u^2 - 2x + ux\right) G(x,u) = x(u-2)G_2(x) + u. \tag{1.2}$$

By using kernel method with $u = \dfrac{x+1-\sqrt{1-6x+x^2}}{2}$, *we obtain,*

$$G_2(x) = \frac{1-x-\sqrt{1-6x+x^2}}{2x}.$$

Notice that this is the generating function of Schröder numbers; for further details, see Appendix B.

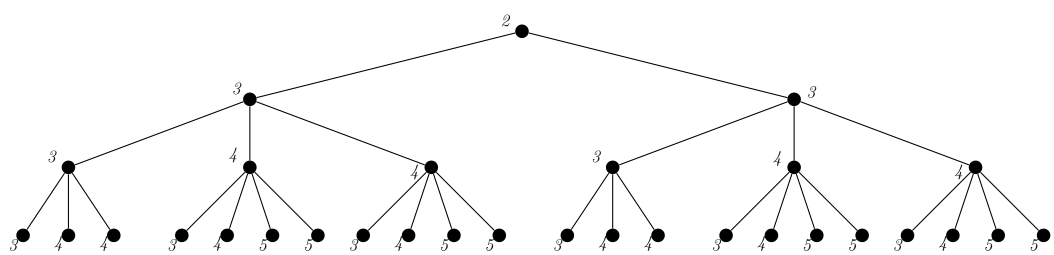


Figure 1.3: Schröder generating tree

Example 1.2.4. *Motzkin Tree*

The generating tree has the following succession rules:

Root : (1),

Rules : $(m) \rightsquigarrow (1)(2)\dots(m-1)(m+1)$ *where* $m \geq 1$.

Let g_n *be the number of nodes at level* n *and the root is at level 0. See figure 1.4. We want to compute the generating function* $G(x) = \sum_{n=0}^{\infty} g_n x^n$. *Let* $G_i(x)$ *denote the generating function corresponding to the sub-tree with root label* i. *The rules yield the following equations:*

$$G_m(x) = 1 + x\left[G_1(x) + G_2(x) + \cdots + G_{m-1}(x) + G_{m+1}(x)\right]$$

where $m \geq 1$.

Let $G(x,u) = \sum_{m=1}^{\infty} G_m(x)u^{m-1} = G_1(x) + G_1(x)u + G_3(x)u^2 + \dots$

Notice that, $G(x,0) = G_1(x)$.

We also have,

$$G_{m+1}(x) - G_m(x) = xG_m(x) - xG_{m+1}(x) + xG_{m+2}(x). \tag{1.3}$$

By simplifying,

$$G_{m+1}(x) - G_m(x) = \frac{x}{1+x} G_{m+2}(x).$$

By multiplying u^{m-1} *and summing over* $m \geq 1$

$$\sum_{m=1}^{\infty} G_{m+1}(x)u^{m-1} - \sum_{m=1}^{\infty} G_m(x)u^{m-1} = \frac{x}{1+x} \sum_{m=1}^{\infty} G_{m+2}(x)u^{m-1}. \tag{1.4}$$

By changing the limits of the summation,

$$\sum_{m=2}^{\infty} G_m(x)u^{m-2} - G(x,u) = \frac{x}{1+x} \sum_{m=3}^{\infty} G_m(x)u^{m-3}. \tag{1.5}$$

Then, we arrange the equation and obtain the following functional equation.

$$\big((1+x)u - (1+x)u^2 - x\big)\, G(x,u) = ((1+x)u - x - u)\, G_1(x) + u.$$

By using kernel method with $u = \dfrac{1+x-\sqrt{1-2x-3x^2}}{2(1+x)}$, *we obtain*

$$G_2(x) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}.$$

This is the generating function of Motzkin numbers.

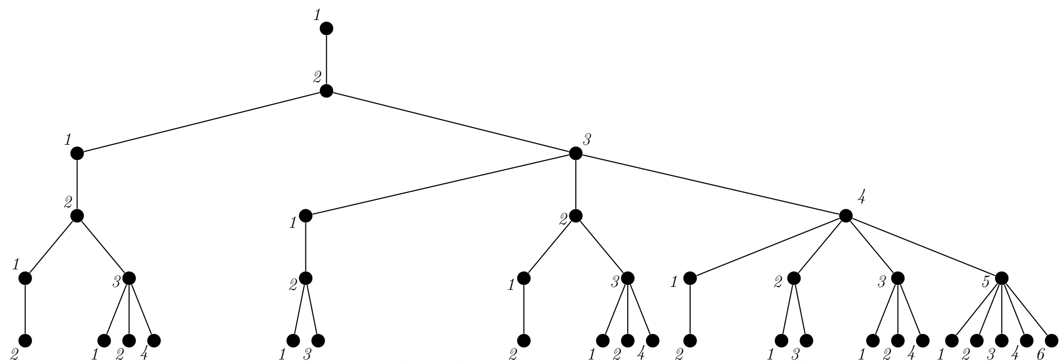


Figure 1.4: Motzkin generating tree

In the rest of this thesis, we study various pattern-avoiding inversion sequence classes. In the next chapter, we review the enumeration of inversion sequences avoiding a pattern of length three, primarily following [10]. In the third chapter, we study applications of generating tree method to pattern-avoiding inversion sequences.

Chapter 2

Inversion Sequences Avoiding Patterns of Length Three

In this chapter, we review the enumeration of inversion sequences, avoiding the set of patterns of length three. Additionally, we will study their correspondence with well-known combinatorial structures. We review the proofs from [10, 9, 19] and [18].

Pattern τ	$\|I_n(\tau)\|$ for $n = 1, 2, \dots 7$	OEIS
000	$1, 2, 5, 16, 61, 272, 1385$	A000111
001	$1, 2, 4, 8, 16, 32, 64$	A000079
011	$1, 2, 5, 15, 52, 203, 877$	A000110
012	$1, 2, 5, 13, 34, 89, 233$	A001519
021	$1, 2, 6, 22, 90, 394, 1806$	A006318
101 or 110	$1, 2, 6, 23, 105, 549, 3207$	A113227
102	$1, 2, 6, 22, 89, 381, 1694$	A200753
120	$1, 2, 6, 23, 103, 515, 2803$	A263778
201 or 210	$1, 2, 6, 24, 118, 674, 4306$	A263777
100	$1, 2, 6, 23, 106, 565, 3399$	A263780
010	$1, 2, 5, 15, 53, 215, 979$	A263779

Table 2.1: List of length-three patterns and the counting sequence of inversion sequences avoiding the corresponding pattern.

The detailed research on pattern-avoiding inversion sequences began with two independent papers published by Mansour and Shattuck [9] and Corteel et al. [10] in 2015 and 2016, respectively. In these studies, Mansour and Shattuck investigated inversion sequences that avoid patterns of length three with no repetition, while Corteel et al. delved into avoiding such patterns with repetitions. Both of these studies showed that some pattern-avoiding inversion sequences are enumerated by well-known counting sequences such as Fibonacci numbers, Schröder numbers, and Euler up/down numbers. We refer to Appendix B for more details about these counting sequences. Specifically, both [9] and [10] demonstrated that $I_n(012)$ is enumerated by odd-indexed Fibonacci numbers, and $I_n(021)$ is enumerated by large Schröder numbers. Additionally, [10] revealed that $I_n(000)$ is counted by Euler up/down numbers, $I_n(001)$ is counted by 2^{n-1}, and $I_n(011)$ is counted by Bell numbers. Mansour and Shattuck find an algebraic generating function for the counting sequence of $I_n(102)$ and a functional equation for the counting sequence of $I_n(120)$. Moreover, these studies showed that $I_n(101)$ and $I_n(110)$, as well as $I_n(201)$ and $I_n(210)$, have the same cardinality. In 2022, Testart derived a recurrence relation for the enumeration of $I_n(010)$. Subsequently, in 2023, Kotsireas, Mansour, and Yıldırım presented the functional equation for the enumeration of $I_n(100)$ using the generating tree method. Thanks to all of these studies the counting sequences for avoiding each 13 patterns of length three have appeared in the Online Encyclopedia of Integer Sequences (OEIS).

2.1 Class $I_n(012)$

We will see that pattern-avoidance restriction induces a structure on inversion sequences. Understanding such structures sometimes plays a fundamental role in enumerating pattern-avoiding inversion sequences. Recall that a sequence of integers $e_1e_2\ldots e_n$ is called weakly decreasing if $e_i \geq e_j$ for all $i < j$. Observe that the positive elements of the inversion sequences that avoid 012 must form a weakly decreasing sequence. Because, if there exist two positive elements e_i and e_j of $e \in I_n(012)$ with $e_i < e_j$ where $i < j$, then the sub-sequence $e_1e_ie_j$ would be isomorphic to 012.

We study the enumeration of $I_n(012)$ from [10]. First of all, we define the map α_k from the set of integer sequences to the set of integer sequences such that $\alpha_k(e_1e_2\dots e_n) = f_1f_2\dots f_n$ where $f_i = 0$ if $e_i = 0$, otherwise $f_i = e_i + k$ for any integer k.

For example $\alpha_{-1}(01214) = 00103$ and $\alpha_3(01214) = 04547$.

Theorem 2.1.1. *$|I_n(012)| = F_{2n-1}$ where F_n is the n^{th} Fibonacci number.*

Proof. Let $n \geq 3$, we set $a_n := |I_n(012)|$. If $e = e_1e_2\dots e_{n-1} \in I_{n-1}(012)$, then $e_1e_2\dots e_{n-1}0 \in I_n(012)$ and $e_1e_2\dots e_{n-1}1 \in I_n(012)$. This is because adding 0 or 1 at the end of an inversion sequence avoiding 012 does not create a 021 pattern. Therefore, there are a_{n-1} elements of $I_n(012)$ that end in zero. Similarly, there are a_{n-1} elements of $I_n(012)$ that end in one. Let $A_n := \{e \in I_n(012) \mid e_n > 1\}$ and $B_n := \{e \in I_n(012) \mid e_n > 0\}$. Here, the set A_n consists of inversion sequences of length n that avoid 012, excluding those that end in 0 or 1. So, $|A_n| = a_n - a_{n-1} - a_{n-1}$. Similarly, the set B_n consists of inversion sequences of length n that avoid 012, excluding those that end in 0. So, $|B_n| = a_n - a_{n-1}$.

Let $f : B_{n-1} \mapsto A_n$ be the function defined by $f(e) = 0\alpha(e)$.

For example, if $002101 \in B_6$, then $f(002101) = 0003202 \in A_7$.

Note that f has an inverse as follows. $f^{-1} : A_n \mapsto B_{n-1}$ is defined by removing the first zero element and decreasing each nonzero element by 1. For instance, $000302 \in A_6$, then $f^{-1}(000302) = 00201 \in B_5$.

Thanks to this bijection, we have $|B_{n-1}| = |A_n|$ and this equation gives $a_n = 3a_{n-1} - a_{n-2}$. The recurrence relation of odd-indexed Fibonacci numbers is given by the same equation, see Appendix A.2.1. Therefore $|I_n(012)| = F_{2n-1}$. □

For example, $I_3(012) = \{000, 001, 002, 010, 011\}$, $I_4(012) = \{0000, 0001, 0002, 0003, 0010, 0011, 0020, 0021, 0022, 0100, 0101, 0110, 0111\}$.

2.2 Class $I_n(000)$

$I_n(000)$ is enumerated by Euler up/down numbers. We know that 0-1-2 increasing trees are enumerated by Euler up/down numbers; for further details, see Appendix B. We will study the bijection between $I_n(000)$ and 0-1-2 increasing trees presented in [10].

Recall that 0-1-2 increasing trees are rooted unordered trees with n vertices, each having at most two children. Each node is assigned a unique label from the set $[n]$, which ascends along every path from the root to a leaf.

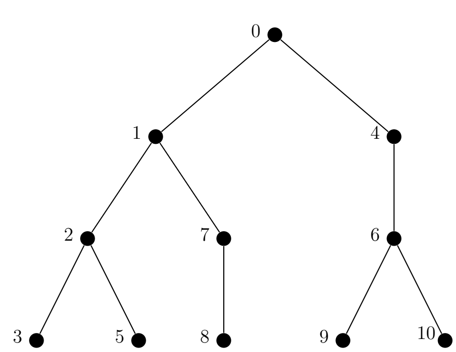
Notice that an inversion sequence avoids 000 if and only if each element appears at most twice. For example, $0011234327 \in I_{10}(000)$.

Theorem 2.2.1. *$|I_n(000)| = E_{n+1}$ where E_n is the n^{th} Euler up/down number.*

Proof. Let T_n be the set of 0-1-2 increasing trees with n vertices. Define the function $f : I_n(000) \mapsto T_{n+1}$ as e_i is the parent of i in T_{n+1}.

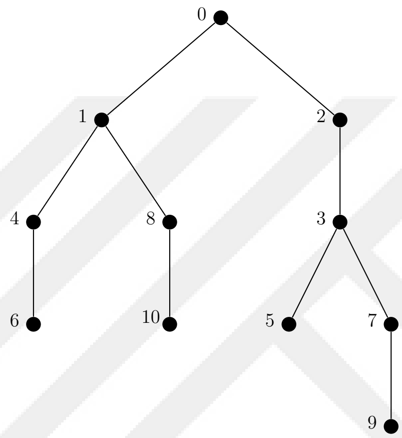
If $e \in I_n(000)$, then e_i can be the parent of at most two children, otherwise it contains 000. Moreover, recall that an inversion sequence $e = e_1e_2 \dots e_n$ has the property $0 \leq e_i < i$ for all $i = 1, 2, \dots n$. If e_i is the parent of i, then $e_i < i$ implies that the label of each parent is less than the label of its children in the corresponding tree under the function f. Thus, the corresponding tree is an 0-1-2 increasing tree.

To illustrate, if $e = 0120241766 \in I_{10}(000)$, then the corresponding 0-1-2 increasing tree under the function f is as follows.



Note that f has an inverse as follows. $f^{-1} : T_{n+1} \mapsto I_n(000)$ is defined by finding e_i as the label of the parent of the child with the label i.

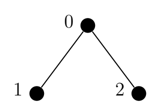
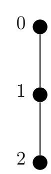
For example, the inversion sequence e that corresponds to the 0-1-2 tree given below, is $e = 0021343178 \in I_{10}(000)$.

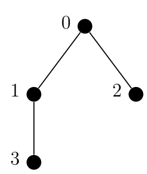
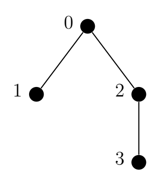
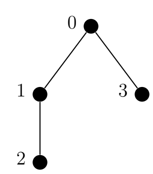


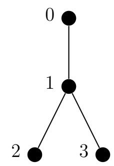
It follows from this bijection that $f : I_n(000) \mapsto T_{n+1}$ implies $|I_n(000)| = E_{n+1}$.

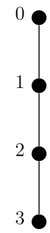
$\square$

The elements of the set $I_n(000)$ and the corresponding 0-1-2 trees are given below for $n = 2$ and $n = 3$.

$f(00) =$  $f(01) =$ 

$f(001) =$  $f(002) =$  $f(010) =$ 

$f(011) =$ 

$f(012) =$ 

2.3 Class $I_n(021)$

$I_n(021)$ is enumerated by large Schröder numbers S_{n-1}. Recall that S_n gives the number of Schröder paths of semi-length n. For further details about large Schröder numbers, see Appendix A. In this section, first, we study enumeration of $I_n(021)$ and then, we review the bijection between $I_n(021)$ and Schröder paths of semi-length $n-1$ from [10].

Observe that an inversion sequence avoids the pattern 021 if and only if its positive elements are weakly increasing. Because if there are two positive elements e_i and e_j of $e \in I_n(021)$ with $e_i > e_j$ such that $i < j$, then the sub-sequence $e_1e_ie_j$ is isomorphic to the pattern 021. The element of an inversion sequence that satisfies the condition $e_{j+1} = j$ is called a maximal element.

Theorem 2.3.1. *$|I_n(021)| = S_{n-1}$ where S_n is the n^{th} large Schröder number.*

Proof. Let $e \in I_n(021)$. Assume that $e_{j+1} = j$ is the right-most maximal element.

If $j = 0$, then $e_1 = 0$. Thus we have two cases for e:

1. e is an inversion sequence of length one, that is $e = 0$
2. e is an inversion sequence of length n in the form of $e = 0e'$ where $e' \in I_{n-1}(021)$. For example, if $e = 001220 \in I_6(021)$, then right-most maximal element of e is $e_1 = 0$. We observe $e' = 01220 \in I_5(021)$.

If $j > 0$, then the third case for e is as follows:

3. e is in the form of

$$\underbrace{e_1e_2\dots e_j}_{\in I_j(021)}\underbrace{e_{j+1}}_{=j}e_{j+2}e_{j+3}\dots e_n$$

The sub-sequence of e given as $e_1e_2\dots e_j$ is an inversion sequence avoiding the pattern 021 of length j. For the sub-sequence $e_{j+2}e_{j+3}\dots e_n$, either $e_{j+i}=0$ or $j\leq e_{j+i}<j+i-1$ where $i=2,\dots,n-j$. Note that if $e_{j+i}<j$, then the sub-sequence $e_1e_je_{j+i}$ forms 021. By subtracting $j-1$ from the positive elements of $e_{j+2}\dots e_n$ and adding a 0 to the beginning of the sub-sequence, we obtain an element of $I_{n-j}(021)$.

To illustrate, let $e=0001355067\in I_{10}(021)$. The right-most maximal element of e is $e_6=5$. The sub-sequence $00013\in I_5(021)$ and we obtain $01023\in I_5(021)$ by subtracting 4 from positive elements of the sub-sequence 5067 and adding a zero to the beginning.

Let $a_n:=|I_n(021)|$. We know that $a_1=1$ and we have the following recurrence relation from cases 2 and 3.

$$a_n=a_{n-1}+\sum_{j=1}^{n-1}a_ja_{n-j}\ \text{ where }\ n\geq 2. \tag{2.1}$$

Let $A(x)=\sum\limits_{n\geq1}a_nx^n=x+\sum\limits_{n\geq2}a_nx^n$. By substituting (2.1),

$$A(x)=x+\sum_{n\geq2}a_{n-1}x^n+\sum_{n\geq2}\sum_{j=1}^{n-1}a_ja_{n-j}x^n. \tag{2.2}$$

In the equation (2.2), the first, second, and third terms correspond to case 1, case 2, and case 3, respectively. The third term yields $A^2(x)$ from the multiplication property of the power series. For more details, see Appendix B. Then we obtain:

$$A(x)=x+xA(x)+A^2(x). \tag{2.3}$$

The two solutions of the equation $A^2(x) + (x-1)A(x) + x = 0$ are

$$\frac{1-x+\sqrt{x^2-6x+1}}{2} \quad \text{and} \quad \frac{1-x-\sqrt{x^2-6x+1}}{2}.$$

The first solution does not have non-negative coefficients. Therefore, we choose the second one.

$$A(x) = \frac{1-x-\sqrt{x^2-6x+1}}{2} = xs(x).$$

where $s(x)$ is the generating function of large Schröder numbers.

$$A(x) = x\sum_{n\geq 0} S_n x^n = \sum_{n\geq 0} S_n x^{n+1} = \sum_{n\geq 1} S_{n-1} x^n.$$

where S_n is the n^{th} large Schröder number. Therefore, $|I_n(021)| = S_{n-1}$. □

2.3.1 The bijection between $I_n(021)$ and Schröder paths of semi-length $n-1$

We study the following bijection from [10]. Let $\mathcal{S}_n$ be the set of Schröder paths of semi-length n. We denote the steps $(1,1)$, $(1,-1)$, and $(2,0)$ by the letters U, D, and F, respectively.

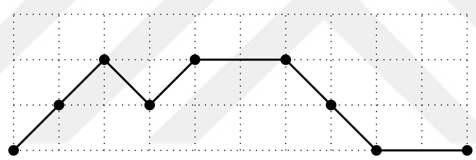
Next, define a map $\rho_n : I_n(021) \mapsto \mathcal{S}_{n-1}$ as follows. Let $e_{j+1} = j$ be the right-most maximal element of e. Set $\rho(0)$ as the empty path. Recall the function α_k defined previously.

$$\rho_n(e) = \begin{cases} F\rho_{n-1}(e_2 \dots e_n) & \text{if } j = 0 \\ U\rho_j(e_1 \dots e_j) D \rho_{n-j}(\alpha_{1-j}(0e_{j+2} \dots e_n)) & \text{otherwise} \end{cases}$$

Example 2.3.1. *Let $e = 010240 \in I_6(021)$. Then, we find $\rho_6(e)$ as follows.*

$$\begin{aligned}
\rho_6(e) &= \rho_6(0102\mathbf{4}0) \\
&= U\rho_4(0\mathbf{1}02)D\rho_2(\mathbf{0}0) \\
&= UU\rho_1(0)D\rho_3(00\mathbf{2})DF\rho_1(0) \\
&= UUDU\rho_2(\mathbf{0}0)DDF \\
&= UUDUF\rho_1(0)DDF \\
&= UUDUFDDF
\end{aligned}$$

Notice that bold elements represent maximal elements. Then, the corresponding Schröder path is as follows:



Note that for a given Schröder path, the first return to the x-axis gives the maximal element of the corresponding inversion sequence.

Remark: If the Schröder path of length n returns to the x-axis at k where $k \in (1, n)$, then we call it as decomposable, otherwise indecomposable.

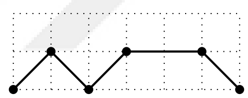
The function ρ_n has an inverse as follows. We define $\rho_n^{-1} : \mathcal{S}_{n-1} \mapsto I_n(021)$ by following procedure:

1. If the given Schröder path of semi-length $n-1$ is decomposable, then find the coordinate of the first return to the x-axis $(2k, 0)$ and set $e_{k+1} = k$. Next, decompose the given Schröder path into two Schröder paths by cutting it at the coordinate $(2k, 0)$ where the path returns to the x-axis for the first time. Find corresponding inversion sequences for both of them. Append the inversion sequence that corresponds to the left sub-path to the beginning of $e_{k+1} = k$ and append the inversion sequence that corresponds to the right sub-path to the end of $e_{k+1} = k$ by removing its initial zero-term and increasing each positive element by $k-1$.

2. If the given Schröder path of semi-length $n-1$ is indecomposable, then set $e_n = n-1$. Next, modify the path by removing the first step U and the last step D from the sub-path and find the corresponding inversion sequences. Append the inversion sequence corresponding to the modified path to the beginning of $e_{k+1} = k$.

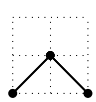
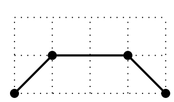
We obtain corresponding inversion sequences $e \in I_n(021)$ for a given Schröder path of semi-length $n-1$ by applying this procedure until each element of e is determined.

Example 2.3.2. *The Schröder path $UDUFD$ corresponds the inversion sequences* 0102. *We find it as follows.*



Notice that the given Schröder path is the semi-length of three. Therefore, the length of e is four. The first return to the x-axis occurs at $(2, 0)$. *This gives* $e_2 = 1$. *We decompose the path by cutting it at* $(2, 0)$.

Now, we have two sub-paths:

 *and* 

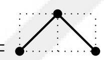
The left sub-path is semi-length of one, and because of that, the corresponding inversion sequence is the length of two. The first return to the x-axis occurs at $(2, 0)$, *and it gives* $e_2 = 1$ *for the first sub-path. Then, we remove the first U step and last D step and obtain an empty path which corresponds to* $e_1 = 0$. *Hence, the left sub-path gives the inversion sequence* 01.

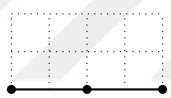
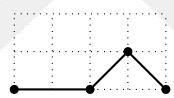
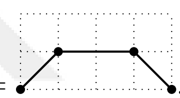
The right sub-path is a semi-length of two, and the corresponding inversion sequence is the length of two; therefore, the corresponding inversion sequence is the length of three. The first return to the x-axis occurs at $(4, 0)$, *and it gives* $e_3 = 2$.

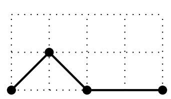
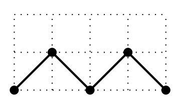
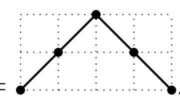
Then, we remove the first U step and last D step and obtain an F step which corresponds $e_1e_2 = 00$. Hence, the right sub-path gives the inversion sequence 002.

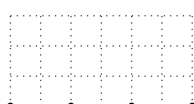
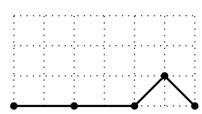
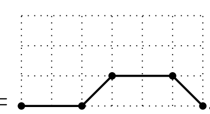
Finally, we concatenated these two inversion sequences 01 and 002 by removing the initial zero-term of the right sub-path, and we found $e = 0102$.

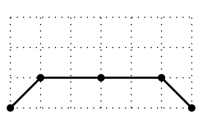
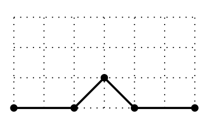
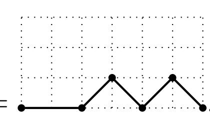
The elements of the set $I_n(021)$ and the corresponding Schröder paths are given below for $n = 2, 3, 4$.

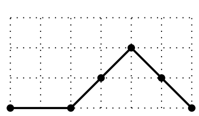
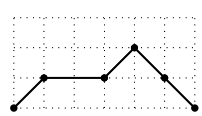
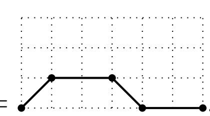
$\rho_2(00) =$  , $\rho_2(01) =$ 

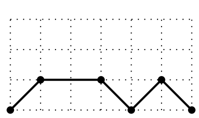
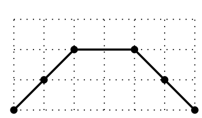
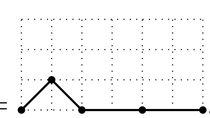
$\rho_3(000) =$  , $\rho_3(001) =$  , $\rho_3(002) =$  ,

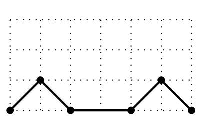
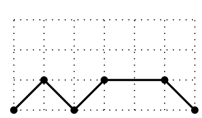
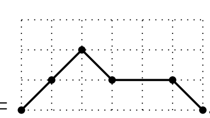
$\rho_3(010) =$  , $\rho_3(011) =$  , $\rho_3(012) =$  ,

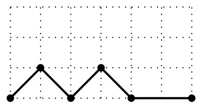
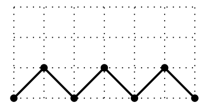
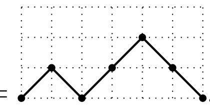
$\rho_4(0000) =$  , $\rho_4(0001) =$  , $\rho_4(0002) =$  ,

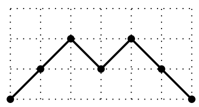
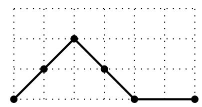
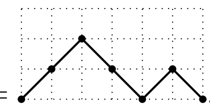
$\rho_4(0003) =$  , $\rho_4(0010) =$  , $\rho_4(0011) =$  ,

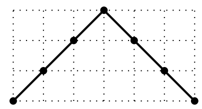
$\rho_4(0012) =$  , $\rho_4(0013) =$  , $\rho_4(0020) =$  ,

$\rho_4(0022) =$  , $\rho_4(0023) =$  , $\rho_4(0100) =$  ,

$\rho_4(0101) =$  , $\rho_4(0102) =$  , $\rho_4(0103) =$  ,

$\rho_4(0110) =$  , $\rho_4(0111) =$  , $\rho_4(0112) =$  ,

$\rho_4(0113) =$  , $\rho_4(0120) =$  , $\rho_4(0122) =$  ,

$\rho_4(0123) =$ 

2.4 Class $I_n(010)$

We follow [19] for the enumeration of $I_n(010)$. Let $\begin{bmatrix} n \\ k \end{bmatrix}$ be the unsigned Stirling number of the first kind which counts the number of permutations of n elements with k disjoint cycles; for further details see Appendix B.

Lemma 2.4.1. *Let $U_{n,k}$ be the set of $010-$avoiding words $\omega = \omega_1\omega_2\ldots\omega_n$ of length n on the alphabet $\{1,\ldots,k\}$ such that ω contains all letters $\{1,\ldots,k\}$ and $\omega_1 = k$. Let $a_{n,k} = |U_{n,k}|$. For all $n, k \geq 1$,*

$$a_{n,k} = \begin{bmatrix} n \\ n+1-k \end{bmatrix}.$$

Proof. Let $n \geq k \geq 2$ and $\omega \in U_{n,k}$. Note that all letters 1 in ω are consecutive. Because 1 is the smallest letter of the alphabet, and if there is another letter between 1's, then the pattern 010 is contained. We will examine ω in two cases:

1. If ω contains several letters 1, then we obtain a word $\omega' \in U_{n-1,k}$ by removing one of 1's.

2. If ω contains a single letter 1, then we obtain a word $\omega' \in U_{n-1,k-1}$ by removing it and subtracting 1 from all other letters. Note that $n-1$ words $\omega \in U_{n,k}$ gives the same $\omega' \in U_{n-1,k-1}$, because $\omega_1 = k > 1$, there are $n-1$ possible positions for the letter 1 in ω.

Then, we have the following recurrence relation:

$$a_{n,k} = a_{n-1,k} + (n-1) \cdot a_{n-1,k-1}. \tag{2.4}$$

Notice that this recurrence relation is also satisfied by the unsigned Stirling numbers of the first kind $\begin{bmatrix} n \\ n+1-k \end{bmatrix}$. For more details, see Appendix A. $\square$

Example 2.4.1. *For $n = 4$ and $k = 3$, we have $\omega \in U_{4,3}$ on the alphabet $\{1, 2, 3\}$ and $\omega_1 = 3$. Recall ω contains all letters of the alphabet $1, 2, 3$, and all letters 1 are consecutive. The list of all possible ω words is below.*

$$\{3112, 3122, 3123, 3132, 3211, 3212, 3213, 3221, 3231, 3312, 3321\}.$$

We obtain $312, 321 \in U_{3,3}$ by removing one of the 1's from the words $3112, 3211$. Let us look at the words with a single letter 1 and apply the operations given in Lemma 2.4.1.

- $3122, 3212, 3221$ *yields* $211 \in U_{3,2}$,
- $3123, 3213, 3231$ *yields* $212 \in U_{3,2}$, *and*
- $3132, 3312, 3321$ *yields* $221 \in U_{3,2}$.

Therefore, the recurrence relation (2.4) is satisfied for $n = 4$, $k = 3$ with $a_{4,3} = 11$, $a_{3,3} = 2$ and $a_{3,2} = 3$.

Theorem 2.4.1. *Let $B_{n,m,d}$ be the set of $010-$avoiding inversion sequences that have length n, maximum m, and contain exactly d distinct values. Let $b_{n,m,d} = |B_{n,m,d}|$, so that we have*

$$|I_n(010)| = \sum_{m=0}^{n-1} \sum_{d=0}^{n} b_{n,m,d}.$$

The numbers $b_{n,m,d}$ satisfy the following recurrence relation:

$$b_{n,m,d} = \sum_{i=0}^{d-1} \binom{m-i}{d-i-1} \sum_{p=m+1}^{n} \begin{bmatrix} n-p+1 \\ n-p-d+i+2 \end{bmatrix} \sum_{j=0}^{m-1} b_{p-1,j,i}. \tag{2.5}$$

Proof. Let $e = e_1 \dots e_n \in B_{n,m,d}$ and p be the index of the left-most position of the value m in e. We split e into two sub-sequences such that $e = \beta\gamma$ where $\beta = e_1 \dots e_{p-1}$ and $\gamma = e_p \dots e_n$.

Note that β and γ have no values in common. Because if they share common values, these common values with the element $e_p = m$ cause the occurrence of the pattern 010.

Note that, $\beta \in B_{p-1,j,i}$ where $j < m$ and $i < d$. Assume that n, m, d, p, j, i and sub-sequence β is already chosen, then we count how many sequences γ are there for a fixed β.

We know that γ is a 010−avoiding word of length $n - p + 1$ with $d - i$ distinct values. The elements of γ can be from the set $\{0, \dots, m\} - \bigcup\limits_{k=1}^{i} d_k$ where d_k is the distinct values of β. Note that this set has $m + 1 - i$ elements. Next, we choose an alphabet for the word γ from this set. As we know γ starts with $e_p = m$, then it remains to choose $d - i - 1$ values among $m - i$ possible values. Therefore. there are $\binom{m-i}{d-i-1}$ choices.

After we determine the letter set for the word γ, we arrange them into a word 010−avoiding by using lemma 2.4.1. As the length of γ is $n - p + 1$ and the alphabet consists of $d - i$ distinct letters, we have $\begin{bmatrix} n-p+1 \\ n-p+1+1-(d-i) \end{bmatrix}$ number of choices for γ by using lemma 2.4.1. Then, we have equation 2.5.

$\square$

2.5 Class $I_n(011)$

First of all, we observe that if $e \in I_n(011)$, then positive entries of e are distinct. Otherwise, two same elements cause the occurrence of the pattern 011 with $e_1 = 0$. We follow [10] for the following theorem.

Theorem 2.5.1. *Let $I_{n,k}(011)$ be the set of* 011−*avoiding inversion sequences of* *length n with k zeros. Then,*

$$|I_{n,k}(011)| = S_{n,k}.$$

where $S_{n,k}$ is the Stirling numbers of the second kind.

Proof. We obtain $e_1 \ldots e_n \in I_{n,k}(011)$ by appending e_n to $e_1 \ldots e_{n-1} \in I_{n-1}(011)$ in two ways:

1. If $e_1 \ldots e_{n-1} \in I_{n-1,k-1}(011)$, then append $e_n = 0$.
2. If $e_1 \ldots e_{n-1} \in I_{n-1,k}(011)$, then it contains $n-1-k$ distinct positive elements. Here, e_n should be a new positive distinct entry; otherwise, the pattern 011 occurs. By eliminating the elements that are used before, we have $(n-1)-(n-1-k) = k$ choices for e_n.

Let $a_{n,k} := |I_{n,k}(011)|$. Then, we have the following recurrence relation.

$$a_{n,k} = ka_{n-1,k} + a_{n-1,k-1}.$$

Notice that this recurrence relation is the same as the recurrence relation of Stirling numbers of the second kind, given in Appendix A. $\square$

Chapter 3

Enumerating Pattern-Avoiding Inversion Sequences Using Generating Tree Method

In 2022, Kotsireas, Mansour, and Yıldırım presented an algorithmic approach that utilizes generating trees to find counting sequences of pattern-avoiding inversion sequence classes; for further details, see [12, 20]. For the remainder of this thesis, we will refer to it as the KMY algorithm.

Let B be a set of patterns. The class of inversion sequences which avoids B is given by $\mathcal{I}_B = \bigcup_{n=1}^{\infty} I_n(B)$. We use $\mathcal{T}(B)$ to denote the representing tree for the class $\mathcal{I}_B$. The representing tree $\mathcal{T}(B)$ has the following properties.

1. The root is 0, which is an inversion sequence of length one, and it stays at level 1.
2. Objects from $I_n(B)$ are located at n^{th} level of the tree
3. The parent of inversion sequences $e = e_1e_2 \dots e_n \in I_n(B)$ is the unique inversion sequence $e = e_1e_2 \dots e_{n-1} \in I_{n-1}(B)$.
4. The children of $e = e_1e_2 \dots e_{n-1} \in I_{n-1}(B)$ are obtained from the set

$\{e_1e_2\ldots e_n \mid e_n = 0,\ldots,n-1\}$, ensuring they avoid the patterns in B.

5. The arrangement of nodes is from left to right. Let $e = e_1e_2\ldots e_{n-1}i$ and $e' = e_1e_2\ldots e_{n-1}j$ be the children of the same parent $e_1e_2\ldots e_{n-1}$. In this case, e appears to the left of e' if $i < j$.

See Figure 3.1 for the first five levels of $\mathcal{T}(000)$. Observe that the label of each child is obtained by appending possible entries at the end of the label of its parent by obeying the pattern-avoiding restriction. For example, the children of 00 are only 001 and 002. Because 000 does not obey the pattern-avoiding restriction.

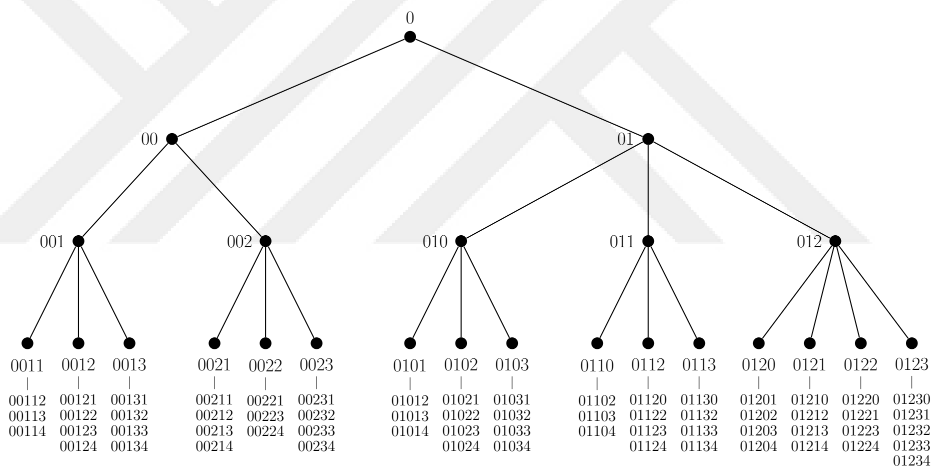


Figure 3.1: First five levels of $\mathcal{T}(000)$

Let $\mathcal{T}(B;e)$ denote the sub-tree consisting of the inversion sequences e as the root and its descendants in $\mathcal{T}(B)$. The main question is finding isomorphic sub-trees within the context of plane tree isomorphism, which we denote by $\mathcal{T}(B;e) \cong \mathcal{T}(B;e')$ where $e, e' \in \mathcal{T}(B)$ are labels of two distinct nodes. We write $e \sim e'$ if $\mathcal{T}(B;e) \cong \mathcal{T}(B;e')$. We represent each equivalence class by the inversion sequence, which is located at the left-most node at the lowest level of $\mathcal{T}(B)$. Hence, we relabel the nodes with the representative inversion sequences of its equivalence class. We denote this relabelled tree by $T[B]$.

In order to find isomorphic sub-trees, it is crucial to understand that we can convert any pattern $\tau = \tau_1\ldots\tau_\ell \in B$ over the alphabet $\{0,1,\ldots,k\}$ to an inversion sequence.

Definition 3.0.1. *Let* τ *be a pattern. It can be extended to an inversion sequence in the form* $\alpha^{(1)}\tau_1\alpha^{(2)}\tau_2\ldots\alpha^{(\ell)}\tau_\ell$ *where* $\alpha^{(j)}$ *is a word over the alphabet* $\{0,1,\ldots,j-1\}$ *such that the length of the inversion sequence* $\alpha^{(1)}\tau_1\alpha^{(2)}\tau_2\ldots\alpha^{(j)}\tau_j$ *is minimal for each* $j=1,2,\ldots,\ell$. *Note that, some words* $\alpha^{(j)}$*s might be empty.*

Let L_τ denote the set of all inversion sequences that are derived from the pattern τ. Note that, if the pattern τ is already an inversion sequence, then $L_\tau=\{\tau\}$.

Example 3.0.1. *Let* $\mathcal{P}_3$ *be the set of patterns of length three.*

$$\mathcal{P}_3=\{000,001,010,011,012,021,100,101,110,102,120,201,210\}.$$

The patterns $000,001,010,011,012$ *are inversion sequences. The set* L_τ *is given below for the other patterns.*

$$L_{021}=\{0021,0121\},\ L_{100}=\{0100\},\ L_{101}=\{0101\},\ L_{110}=\{0110\},$$
$$L_{102}=\{0102\},\ L_{120}=\{0120\},\ L_{201}=\{00201,01201\},\ L_{210}=\{00210,01210\}.$$

Remark: The minimality condition in Definition 3.0.1 implies that the maximum length of any pattern in L_τ is $k+\ell$.

Note that,any inversion sequence e avoids the pattern set B if and only if e avoids $L_B=\bigcup\limits_{\{\tau\in B\}} L_\tau$.

Next, we discuss an important lemma and its proof from [12] that is based on the extension of patterns to inversion sequences to determine isomorphic sub-trees of $\mathcal{T}(B)$.

Lemma 3.0.1. *Let* t *be the length of the longest pattern in* B. *We have that* $\mathcal{T}(B;e)\cong\mathcal{T}(B;e')$ *for two inversion sequences* $e,e'\in\mathcal{T}$ *if and only if* $\mathcal{T}^{2t}(B;e)\cong\mathcal{T}^{2t}(B;e')$ *where* $\mathcal{T}^m(B;e)$ *denotes the finite tree corresponding to the first* m *level of* $\mathcal{T}(B;e)$.

Proof. Suppose that every pattern in B is an inversion sequence and τ is the longest pattern having a length of t. For $e, e' \in \mathcal{T}(B)$, it is clear that $\mathcal{T}(B;e) \cong \mathcal{T}(B;e') \implies \mathcal{T}^{2t}(B;e) \cong \mathcal{T}^{2t}(B;e')$.

To show $\mathcal{T}^{2t}(B;e) \cong \mathcal{T}^{2t}(B;e') \implies \mathcal{T}(B;e) \cong \mathcal{T}(B;e')$, we give a proof of $\mathcal{T}(B;e) \ncong \mathcal{T}(B;e') \implies \mathcal{T}^{2t}(B;e) \ncong \mathcal{T}^{2t}(B;e')$.

Assume that $\mathcal{T}(B;e) \ncong \mathcal{T}(B;e')$. We read the nodes of $\mathcal{T}(B;e)$ and $\mathcal{T}(B;e')$ from top to bottom and from left to right, and denote them as e_j and e'_j with $e_0 = e$ and $e'_0 = e'$. Since $\mathcal{T}(B;e) \ncong \mathcal{T}(B;e')$, there exists $s \geq 0$ minimal such that the number of children of e_j is equal to the number of children of e'_j for $j = 0, 1, \ldots, s-1$ and the number of children of e_s is not equal to the number of children of e'_s.

Let i_j be the number of children of e_j and e'_j for $j = 1, \ldots, s-1$. The set of children of e_j is $\{e_jp_1, \ldots, e_jp_{i_j}\}$. Similarly the set of e'_j is $\{e'_jq_1, \ldots, e'_jq_{i_j}\}$. Then, there exists a bijection α such that $\alpha(p_i) = q_i$ where $1 \leq i \leq i_j$.

Let a_s be the number of children of e_s and b_s be the number of children of e'_s. Assume that $a_s < b_s$. Set $i_s = b_s$ and observe that the set $\{e_sp_1, \ldots, e_sp_{i_s}\}$ contains at least one inversion sequence that contains a pattern from the pattern set B while all of the elements of the set $\{e'_sp_1, \ldots, e'_sp_{i_s}\}$ avoids B.

Let e_sp_k be the first inversion sequences which does not avoid B where $k \leq i_s$. As e_sp_k is descendant of e, we can denote e_sp_k by ef_s. Similarly e'_sq_k is an inversion sequence which avoids B and it is denoted by $e'f'_s$.

Any occurrence of τ in ef_s can use at most $t-1$ letters of f_s. Because we assumed that any pattern in B is an inversion sequence, that means τ starts with 0 and this 0 matches the first element of e. Therefore, at least one element of τ corresponds to an element from e and this means that the remaining $t-1$ elements correspond to the elements of f_s. Therefore, there is a sub-sequence $g = f_{s_{r_1}} \ldots f_{s_{r_m}}$ of minimal length m such that the word eg contains τ and $m \leq t-1$. However, the word $e'g'$ avoids B where $g' = f'_{s_{r_1}} \ldots f'_{s_{r_m}}$ is a sub-sequence of f'_s.

Since ef_s is an inversion sequence, then there exists an inversion sequence $e\tilde{g} \in L_{eg}$. The maximum length of $\tilde{g}$ is $t-1+t-1$. Note that $e\tilde{g}$ contains τ while

$e'\tilde{g'}$ avoids τ. Therefore, if $\mathcal{T}(B;e) \ncong \mathcal{T}(B;e')$, then $\mathcal{T}^{2t-1}(B;e) \ncong \mathcal{T}^{2t-1}(B;e')$. (Because the root is level 1). In other words, if the first $2t$ levels of the trees are isomorphic, then we say these trees are isomorphic, that is $\mathcal{T}^{2t}(B;e) \cong \mathcal{T}^{2t}(B;e') \implies \mathcal{T}(B;e) \cong \mathcal{T}(B;e')$, $\square$

The KMY algorithm aims to find $T[B]$ for the set of patterns B with the convention that $0 \notin B$.

Let the steps of the KMY algorithm be $i = 1, \ldots, M$ for any positive integer M. We define the following sets:

- A_i is the set of all children obtained at step i, which is the all labels at level i of $\mathcal{T}(B)$.
- B_i is the set of all new equivalence classes obtained at step i.
- R is the set of succession rules.

The algorithm is as follows:

1. Initialize $T[B]$ by the root with label $e = 0 \in I_1(B)$ and set $A_1 = \{0\}$, $B_1 = \{0\}$, and $R = \emptyset$.
2. For all $i = 2, \ldots, M$,
   (a) Let N_e be the set of all children of e for any $e \in B_{i-1}$. Therefore, $A_i = \bigcup\limits_{e \in B_{i-1}} N_e$.
   (b) For each child $e \in A_i$, find e' such that $e \sim e'$ where $e' \in \bigcup\limits_{j=1}^{i-1} Q_j$ using the Lemma 3.0.1, if it is possible. Otherwise, add e to the set B_i.
   (c) Write a succession rule for each $e \in B_i$ such that $e \rightsquigarrow e^{(1)}e^{(2)}\ldots e^{(s)}$ where e^j is the label of the j^{th} child of e, from left to right, in $\mathcal{T}[B]$ and add it to the set R.

Note that, if we obtain $A_i = \emptyset$ at step $(2.a)$, then we have a finite $\mathcal{T}[B]$. In conclusion, the KMY algorithm gives the set of succession rules, R, for $\mathcal{T}[B]$. Then, we write generating functions for each succession rule. The coefficients of the generating function that corresponds to the root give the counting sequence of $I_n(B)$.

Example 3.0.2. *Let $B = \{000, 010, 012\}$. We construct $\mathcal{T}(B)$ for the class $\mathcal{I}_B$. Figure 3.2 shows first 5 level of $\mathcal{T}(B)$. We initialize $\mathcal{T}(B)$ by the root 0 and we construct the rest of the $\mathcal{T}(B)$ by adding a last element to each inversion sequences by obeying the pattern avoidance restrictions of the pattern set B, in a recursive manner. See the figure 3.2. Observe that we can not append one more entry to the inversion sequences at the last level: otherwise, they do not avoid B. Thus, $\mathcal{T}(B)$ is a finite tree.*

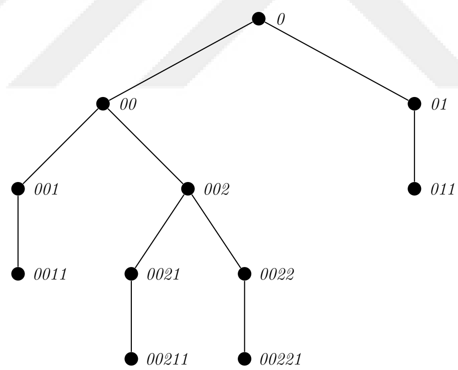


Figure 3.2: $\mathcal{T}(B)$ where $B = \{000, 010, 012\}$.

Moreover, inversion sequences at leaves yield isomorphic sub-trees such that $\mathcal{T}(B; 011) \cong \mathcal{T}(B; 0011) \cong \mathcal{T}(B; 00211) \cong \mathcal{T}(B; 00221)$. Therefore, we write $\mathcal{T}(B; 011)$ instead of $\mathcal{T}(B; 0011)$, $\mathcal{T}(B; 00211)$ and $\mathcal{T}(B; 00221)$ in $\mathcal{T}[B]$. Similarly, we observe that $\mathcal{T}(B; 01) \cong \mathcal{T}(B, 001) \cong \mathcal{T}(B, 0021) \cong \mathcal{T}(B, 0022)$. Then we write $\mathcal{T}(B; 01)$ instead of $\mathcal{T}(B; 001)$, $\mathcal{T}(B; 0021)$ and $\mathcal{T}(B; 0022)$ in $\mathcal{T}[B]$. See $\mathcal{T}[B]$ in the figure 3.3.

Next, we apply the KMY algorithm.

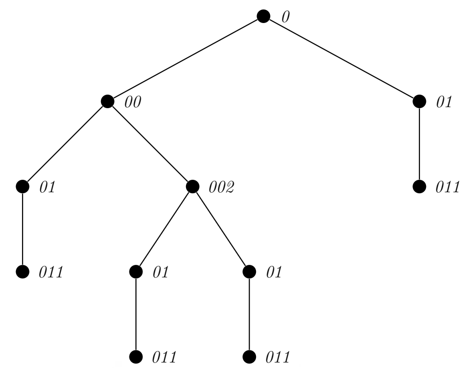


Figure 3.3: $\mathcal{T}[B]$ where $B = \{000, 010, 012\}$

.

i	A_i	*Comments*	B_i
1	$\{0\}$		$\{0\}$
2	$\{00, 01\}$	$00 \nsim 0;\ 01 \nsim 0;\ 00 \nsim 01$	$\{00, 01\}$
3	$\{001, 002, 011\}$	$001 \sim 01;\ 002, 011 \nsim v \in B_1 \cup B_2$	$\{002, 011\}$
4	$\{0011, 0021, 0022\}$	$0011 \sim 011;\ 0021 \sim 01; 0022 \sim 01$	$\emptyset$
5	$\{00211, 00221\}$	$00211 \sim 011;\ 00221 \sim 011$	$\emptyset$

The generating tree $\mathcal{T}[B]$ *has the following succession rules:*

Root : 0,

Rules : $0 \rightsquigarrow 00, 01,$

$00 \rightsquigarrow 01, 002,$

$01 \rightsquigarrow 011,$

$002 \rightsquigarrow 01, 01.$

We want to find the generating function $R(x) = \sum_{n=1}^{\infty} |I_n(B)| x^n$. *Let* $A_w(x)$ *denote the generating function for the number of nodes in the sub-tree* $\mathcal{T}[B; w]$. *Then, we have*

$$R(x) = x + xA_{00}(x) + xA_{01}(x) \tag{3.1}$$

and

$$\begin{aligned} A_{00}(x) &= x + xA_{01}(x) + xA_{002}(x) \\ A_{01}(x) &= x + xA_{011}(x) \\ A_{002}(x) &= x + 2xA_{01}(x) \\ A_{011}(x) &= x \end{aligned} \tag{3.2}$$

By substituting equations in (3.2) *into* (3.1)*, we obtain the following result.*

$$R(x) = x + 2x^2 + 3x^3 + 3x^4 + 2x^5. \tag{3.3}$$

Notice that $[x^n]R(x)$ *gives the number of vertices in* n^{th} *level in the tree* $\mathcal{T}(B)$*.*

3.1 Class $I_n(000, 021)$

In this section, we study the enumeration of $I_n(000, 021)$ by following [12]. The figure 3.4 shows the first 5 levels of $\mathcal{T}(B)$ where $B = \{000, 021\}$. We observe that the generating tree of $I_n(000, 021)$ is infinite. Therefore the succession rules depend on a variable m such that m goes to infinite.

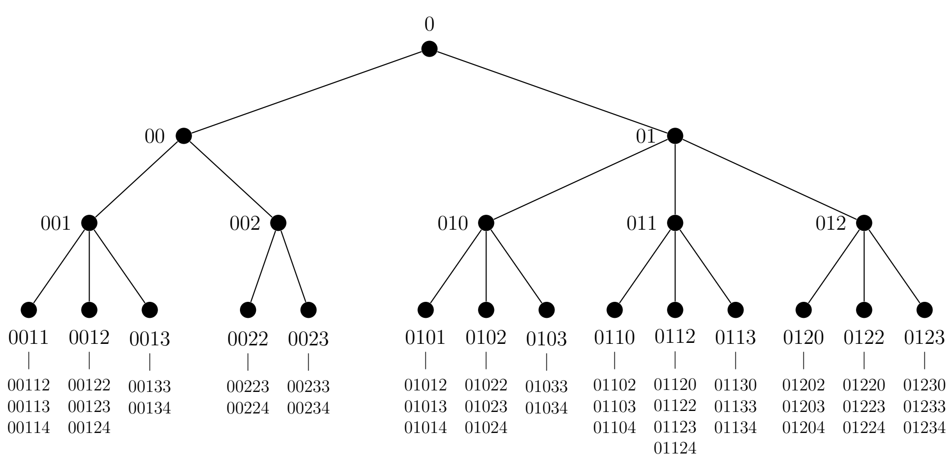


Figure 3.4: The first 5 levels of $\mathcal{T}(B)$ where $B = \{000, 021\}$.

Let $B = \{000, 021\}$. Then we introduce some notation to represent corresponding inversion sequences.

We define $r_0 = 0$, $a_0 = 00$, $b_0 = 002$, $c_1 = 011$, $d_1 = 01$, $a_m = 0011\dots mm$, $b_m = 0011\dots(m-1)(m-1)m$, with $m \geq 1$, $c_m = 01122\dots mm$, and $d_m = 01122\dots(m-1)(m-1)m$ with $m \geq 2$. The generating tree $\mathcal{T}[B]$ has the following succession rules:

$$
\begin{aligned}
&\textit{Root} \; : r_0, \\
&\textit{Rules} : r_0 \rightsquigarrow a_0 d_1, \\
&\qquad a_m \rightsquigarrow b_{m+1} b_m \dots b_0, \quad m \geq 0 \\
&\qquad b_m \rightsquigarrow a_m b_m b_{m-1} \dots b_0, \quad m \geq 0 \\
&\qquad c_m \rightsquigarrow a_m d_{m+1} d_m \dots d_1, \quad m \geq 1 \\
&\qquad d_m \rightsquigarrow b_m c_m d_m d_{m-1} \dots d_1, \quad m \geq 1.
\end{aligned}
$$

Figure 3.5 shows the first 5 levels of generating tree for $I_n(000, 021)$ with respect to the succession rules.

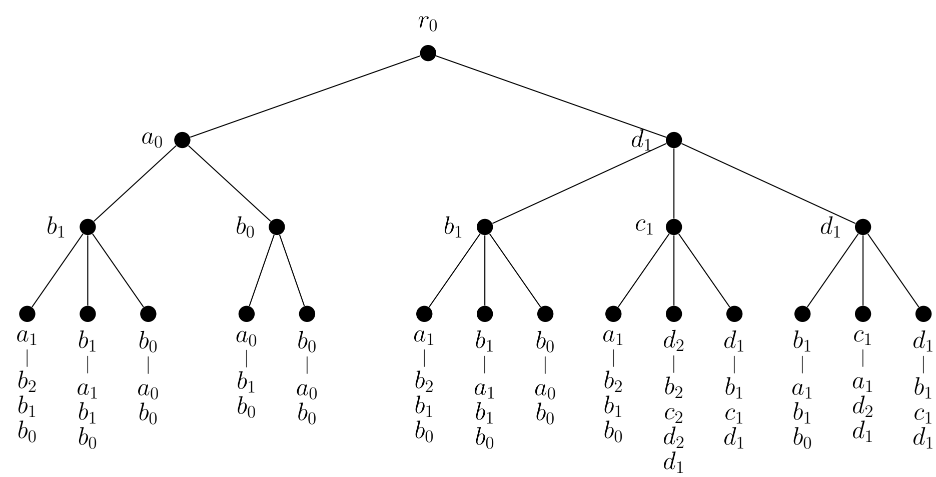
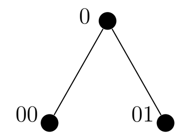


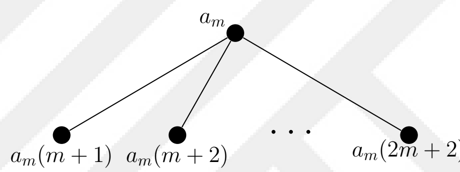
Figure 3.5: The first 5 levels of $\mathcal{T}[B]$ with respect to rules where $B = \{000, 021\}$.

Verification for the given succession rules is as follows:

1. $r_0 \rightsquigarrow a_0 d_1$:
   We initialize the tree $\mathcal{T}[B]$ by the root 0.

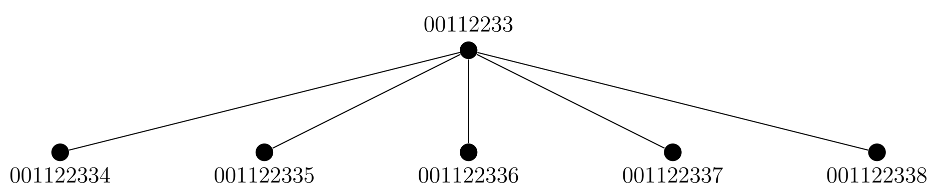


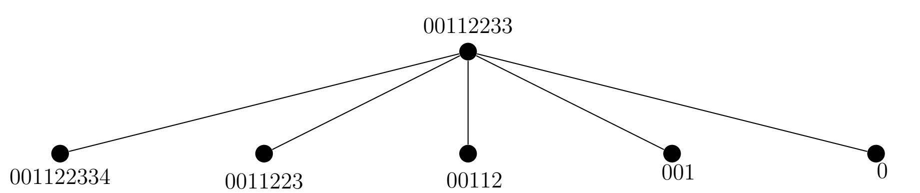
2. $a_m \rightsquigarrow b_{m+1}b_m \dots b_0,\ m \geq 0$:
   The children of a_m is in the form of $a_m j = 0011mmj$ where $j > m$. This is because if $j \leq m$, then the sequence does not avoid the pattern set B. For example, if $m = 3$, then $001122330, 001122331, 001122332, 001122333 \notin I_n(B)$ and $001122334, 001122335, 001122336 \in I_n(B)$.

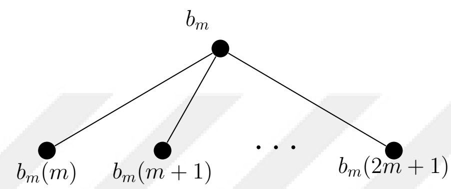
   Notice that, the first child of a_m is $a_m(m+1) = b_{m+1}$. The sub-tree $\mathcal{T}(B; a_m(m+k))$ is isomorphic to the sub-tree $\mathcal{T}(B; b_{m+2-k})$ by removing the letters $m+2-k, m+3-k, \dots, m$ and decrease each letter greater than m by $2k-2$. Thus, lemma 3.0.1 gives that the children of the node with label a_m are exactly the nodes labelled by $b_{m+1}, b_m, \dots, b_0$ that is, $a_m \rightsquigarrow b_{m+1}b_m \dots b_0$, with $m \geq 0$.

   To illustrate this, $\mathcal{T}(B; 00112233)$ and $\mathcal{T}[B; 00112233]$ are as follows,



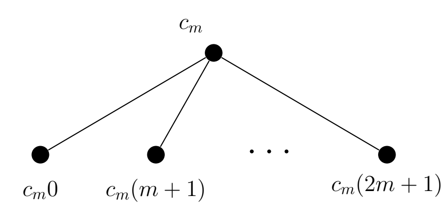


3. $b_m \rightsquigarrow a_m b_m b_{m-1} \dots b_0,\ m \geq 0$
The children of b_m is in the form of $b_m j = 0011 \dots (m-1)(m-1)mj$ where $j \geq m$. Because if $j < m$, then the sequence contains 000 or 021. For example, if $m = 3$, then $00112230, 00112231, 00112232 \notin I_n(B)$ and $00112233, 00112234, 00112235, 00112236, 00112237 \in I_n(B)$.



Observe that, the first child of b_m is $b_m m = a_m$ The sub-tree $\mathcal{T}(B; b_m(m+k+1))$ is isomorphic to the sub-tree $\mathcal{T}(B; b_{m-k})$ by removing the letters $m-k, \dots, m$ and decrease each letter greater than m by $2k-2$. Thus, lemma 3.0.1 gives that the children of the node with label b_m are exactly the nodes labelled by $a_m, b_m, \dots, b_0$ that is, $b_m \rightsquigarrow a_m b_m b_{m-1} \dots b_0$, with $m \geq 0$.

4. $c_m \rightsquigarrow a_m d_{m+1} d_m \dots d_1,\ m \geq 1$:
The children of c_m is in the form of $c_m j = 01122 \dots mmj$ where $j \in \{0, m+1, \dots, 2m+1\}$; otherwise, the sequence contains the patterns 000 or 021. For example if m=3, then the children of c_3 are: $01122330, 01122334, 01122335, 01122336, 01122337$.



The sub-tree $\mathcal{T}(B; c_m 0)$ is isomorphic to the sub-tree $\mathcal{T}(B; a_m)$ when the last zero is deleted, and a zero is appended at the beginning. Also the sub-tree $\mathcal{T}(B; c_m(m+k+1))$ is isomorphic to the sub-tree $\mathcal{T}(B; d_m(m-k+1))$ by removing the letters $m-k+1, \dots, m$ and decrease each letter greater than m by $2k$.

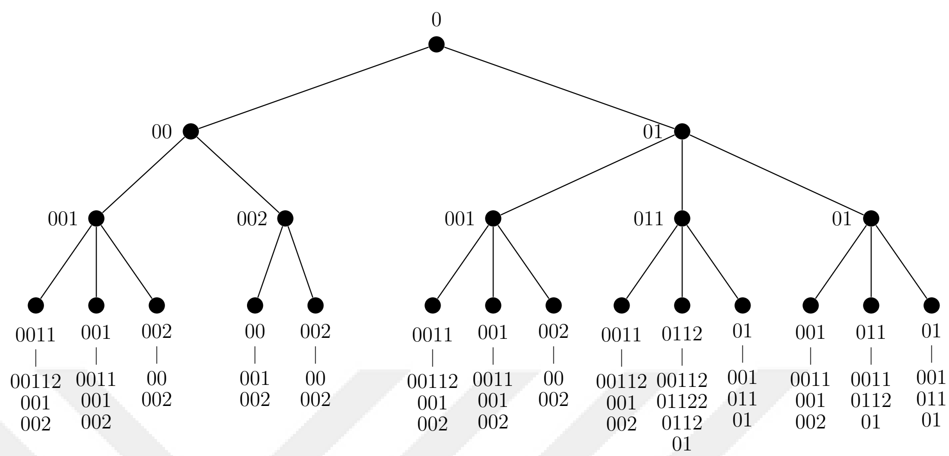
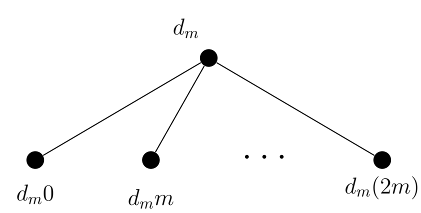


Figure 3.6: The first 5 levels of $\mathcal{T}[B]$ where $B = \{000, 021\}$.

5. $d_m \rightsquigarrow b_m c_m d_m d_{m-1} \ldots d_1,\ m \geq 1$:
The children of d_m is in the form of $d_m j = 01122\ldots(m-1)(m-1)mj$ where $j \in \{0, m, \ldots, 2m\}$; otherwise, the sequence contains the patterns 000 or 021. For example if m=3, then the children of d_3 are: $0112230, 0112233, 0112234, 0112235, 0112236$.



The sub-tree $\mathcal{T}(B; d_m 0)$ is isomorphic to the sub-tree $\mathcal{T}(B; b_m)$ when the last zero is deleted, and a zero is appended at the beginning. Also the sub-tree $\mathcal{T}(B; d_m m)$ is isomorphic to the sub-tree $\mathcal{T}(B; c_m)$. In addition, the sub-tree $\mathcal{T}(B; d_m(m+k+1))$ is isomorphic to the sub-tree $\mathcal{T}(B; d_{m-k})$ by removing the letters $m-k, \ldots, m$ and decrease each letter greater than m by $2k+1$.

We introduce generating functions for each of the succession rules in order to obtain an explicit formula for the generating function of $I_n(B)$.

Generating functions for the succession rules are as follows:

Nodes	Generating Functions	Corresponding Sub-tree
r_0	$R(x)$	$\mathcal{T}(B;0)$
a_m	$A_m(x)$	$\mathcal{T}(B;a_m)$
b_m	$B_m(x)$	$\mathcal{T}(B;b_m)$
c_m	$C_m(x)$	$\mathcal{T}(B;c_m)$
d_m	$D_m(x)$	$\mathcal{T}(B;d_m)$

Then we have the following functional equations:

$$R(x) = x + xA_0(x) + xD_1(x) \tag{3.4}$$

$$A_m(x) = x + x\sum_{j=0}^{m+1} B_j(x), \quad m \geq 0 \tag{3.5}$$

$$B_m(x) = x + xA_m(x) + x\sum_{j=0}^{m} B_j(x), \quad m \geq 0 \tag{3.6}$$

$$C_m(x) = x + xA_m(x) + x\sum_{j=1}^{m+1} D_j(x), \quad m \geq 1 \tag{3.7}$$

$$D_m(x) = x + xB_m(x) + xC_m(x) + x\sum_{j=1}^{m} D_j(x), \quad m \geq 1 \tag{3.8}$$

Notice that the generating functions which correspond to the succession rules are infinite. Thus, we define multi-variable generating functions to obtain finite functional equations.

$$A(x,u) = \sum_{m\geq 0} A_m(x)u^m = A_0(x) + A_1(x)u + A_2(x)u^2 + \dots \tag{3.9}$$

$$B(x,u) = \sum_{m\geq 0} B_m(x)u^m = B_0(x) + B_1(x)u + B_2(x)u^2 + \dots \tag{3.10}$$

$$C(x,u) = \sum_{m\geq 1} C_m(x)u^{m-1} = C_1(x) + C_2(x)u + C_3(x)u^2 + \dots \tag{3.11}$$

$$D(x,u) = \sum_{m\geq 1} D_m(x)u^{m-1} = D_1(x) + D_2(x)u + D_3(x)u^2 + \dots \tag{3.12}$$

Notice that, $A(x,0) = A_0(x)$, $B(x,0) = B_0(x)$, $C(x,0) = C_1(x)$ and $D(x,0) = D_1(x)$. Next, we seek simpler expressions for these multi-variable generating functions.

$$A(x,u) = A_0(x) + A_1(x)u + A_2(x)u^2 + A_3(x)u^3 + \dots$$

We substitute (3.5) into (3.9).

$$\begin{aligned} A(x,u) = {} & (x + xB_0(x) + xB_1(x)) \\ & + (x + xB_0(x) + xB_1(x) + xB_2(x))u \\ & + (x + xB_0(x) + xB_1(x) + xB_2(x) + xB_3(x))u^2 \\ & + (x + xB_0(x) + xB_1(x) + xB_2(x) + xB_3(x) + xB_4(x))u^3 + \dots \end{aligned} \tag{3.13}$$

We arrange (3.13) as follows.

$$\begin{aligned} A(x,u) = {} & x(1 + u + u^2 + \dots) + xB_0(x)(1 + u + u^2 + \dots) \\ & + xB_1(x)(1 + u + \dots) + xB_2(x)(u + u^2 + \dots) \\ & + xB_3(x)(u^2 + u^3 + \dots) + xB_4(x)(u^3 + \dots) + \dots \end{aligned} \tag{3.14}$$

Then we write the closed form of the power series in (3.14).

$$\begin{aligned} A(x,u) &= \frac{x}{1-u} + \frac{x}{1-u} \cdot \frac{1}{u}(B_0(x)u + B_1(x)u + B_2(x)u^2 + B_3(x)u^3 + \dots) \\ A(x,u) &= \frac{x}{1-u} + \frac{x}{1-u} \cdot \frac{1}{u}(B_0(x)u - B_0(x) + B_0(x) + B_1(x)u + B_2(x)u^2 \\ & \quad + B_3(x)u^3 + B_4(x)u^4 + \dots) \\ A(x,u) &= \frac{x}{1-u} + \frac{x}{1-u} \cdot \frac{1}{u}(B_0(x)u - B_0(x)) + \frac{x}{1-u} \cdot \frac{1}{u}(B_0(x) + B_1(x)u \\ & \quad + B_2(x)u^2 + B_3(x)u^3 + B_4(x)u^4 + \dots) \\ A(x,u) &= \frac{x}{1-u} + \frac{x}{1-u} \cdot \frac{1}{u} \cdot (u-1)B_0(x) + \frac{x}{1-u} \cdot \frac{1}{u}B(x,u) \end{aligned} \tag{3.15}$$

Using partial fraction, $\dfrac{1}{(1-u)u} = \dfrac{1}{1-u} + \dfrac{1}{u}$, we obtain:

$$A(x,u) = \frac{x}{1-u} + \frac{x}{u}[B(x,u) - B(x,0)] + \frac{x}{1-u}B(x,u). \tag{3.16}$$

We obtain a simpler expression for $B(x,u)$ as follows.

$$B(x,u) = B_0(x) + B_1(x)u + B_2(x)u^2 + B_3(x)u^3 + \dots$$

We substitute (3.6) into (3.10).

$$\begin{aligned} B(x,u) = &(x + xA_0(x) + xB_0(x)) \\ &+ (x + xA_1(x) + xB_0(x) + xB_1(x))u \\ &+ (x + xA_2(x) + xB_0(x) + xB_1(x) + xB_2(x))u^2 \\ &+ (x + xA_3(x) + xB_0(x) + xB_1(x) + xB_2(x) + xB_3(x))u^3 + \dots \end{aligned} \tag{3.17}$$

We arrange (3.17) as follows.

$$\begin{aligned} B(x,u) = &x(1 + u + u^2 + \dots) + x(A_0(x) + A_1(x)u + A_2(x)u^2 + \dots) \\ &+ xB_0(x)(1 + u + \dots) + xB_1(x)(u + u^2 + \dots) \\ &+ xB_2(x)(u^2 + u^3 + \dots) + xB_3(x)(u^3 + u^4 + \dots) + \dots \end{aligned} \tag{3.18}$$

Next, we write the closed form of the power series in (3.18).

$$\begin{aligned} B(x,u) &= \frac{x}{1-u} + xA(x,u) + \frac{x}{1-u}B_0(x) + \frac{xu}{1-u}B_1(x) + \frac{xu^2}{1-u}B_2(x) + \dots \\ B(x,u) &= \frac{x}{1-u} + xA(x,u) + \frac{x}{1-u}(B_0(x) + B_1(x)u + B_2(x)u^2 + \dots) \end{aligned}$$

Then, we obtain the following functional equation:

$$B(x,u) = \frac{x}{1-u} + xA(x,u) + \frac{x}{1-u}B(x,u) \tag{3.19}$$

We obtain a simpler expression for $C(x,u)$ as follows.

$$C(x,u) = C_1(x) + C_2(x)u + C_3(x)u^2 + C_4(x)u^3 \dots$$

We substitute (3.7) into (3.11).

$$\begin{aligned} C(x,u) &= (x + xA_1(x) + xD_1(x) + xD_2(x)) \\ &\quad + (x + xA_2(x) + xD_1(x) + xD_2(x) + xD_3(x))u \\ &\quad + (x + xA_3(x) + xD_1(x) + \ldots + xD_4(x))u^2 \\ &\quad + (x + xA_4(x) + xD_1(x) + \ldots + xD_5(x))u^3 + \ldots \end{aligned} \tag{3.20}$$

We arrange (3.20) as follows.

$$\begin{aligned} C(x,u) &= x(1 + u + u^2 + \ldots) + x(A_1(x) + A_2(x)u + \ldots) \\ &\quad + xD_1(x)(1 + u + \ldots) + xD_2(x)(1 + u + \ldots) \\ &\quad + xD_3(x)(u + u^2 + \ldots) + xD_4(x)(u^2 + u^3 + \ldots) \\ &\quad + xD_5(x)(u^3 + u^4 + \ldots) + \ldots \end{aligned} \tag{3.21}$$

Then, we write the closed form of the power series in (3.21).

$$\begin{aligned} C(x,u) &= \frac{x}{1-u} + \frac{x}{u}(A_1(x)u + A_2(x)u^2 + A_3(x)u^3 + A_4(x)u^4 + \ldots) \\ &\quad + \frac{x}{1-u}(D_1(x) + D_2(x) + D_3(x)u + D_4(x)u^2 + D_5(x)u^3 + \ldots) \\ C(x,u) &= \frac{x}{1-u} + \frac{x}{u}(-A_0(x) + A_0(x) + A_1(x)u + A_2(x)u^2 + A_3(x)u^3 + \ldots) \\ &\quad + \frac{x}{1-u} \cdot \frac{1}{u}(D_1(x)u + D_2(x)u + D_3(x)u^2 + D_4(x)u^3 + D_5(x)u^4 + \ldots) \\ C(x,u) &= \frac{x}{1-u} - \frac{x}{u}A_0(x) + \frac{x}{u}A(x,u) + \frac{x}{1-u} \cdot \frac{1}{u}D_1(x)u \\ &\quad + \frac{x}{1-u} \cdot \frac{1}{u}(-D_1(x) + D_1(x) + D_2(x)u + D_3(x)u^2 + D_4(x)u^3 + \ldots) \\ C(x,u) &= \frac{x}{1-u} - \frac{x}{u}A_0(x) + \frac{x}{u}A(x,u) + \frac{x}{1-u}D_1(x) - \frac{x}{1-u} \cdot \frac{1}{u}D_1(x) \\ &\quad + \frac{x}{1-u} \cdot \frac{1}{u}D(x,u) \end{aligned}$$

Hence, we have the following functional equation:

$$C(x,u) = \frac{x}{1-u} + \frac{x}{u}(A(x,u) + D(x,u) - A(x,0) - D(x,0)) + \frac{x}{1-u}D(x,u). \tag{3.22}$$

We obtain a simpler expression for $D(x,u)$ as follows.

$$D(x,u) = D_1(x) + D_2(x)u + D_3(x)u^2 + D_4(x)u^3 \ldots$$

We substitute (3.8) into (3.12).

$$\begin{aligned} D(x,u) = &(x + xB_1(x) + xC_1(x) + xD_1(x)) \\ &+ (x + xB_2(x) + xC_2(x) + xD_1(x) + xD_2(x))u \\ &+ (x + xB_3(x) + xC_3(x) + xD_1(x) + xD_2(x) + xD_3(x))u^2 + \ldots \end{aligned} \tag{3.23}$$

We arrange (3.23) as follows.

$$\begin{aligned} D(x,u) = &(x + xB_1(x) + xC_1(x) + xD_1(x)) \\ &+ (x + xB_2(x) + xC_2(x) + xD_1(x) + xD_2(x))u \\ &+ (x + xB_3(x) + xC_3(x) + xD_1(x) + xD_2(x) + xD_3(x))u^2 \\ &+ (x + xB_4(x) + xC_4(x) + xD_1(x) + xD_2(x) + xD_3(x) + \ldots \\ D(x,u) = &x(1 + u + u^2 + \ldots) + x(B_1(x) + B_2(x)u + B_3(x)u^2 + B_4(x)u^3 + \ldots) \\ &+ x(C_1(x) + C_2(x)u + C_3(x)u^2 + C_4(x)u^3 + \ldots) + xD_1(x)(1 + u + \ldots) \\ &+ xD_2(x)(u + u^2 + \ldots) + xD_3(x)(u^2 + u^3 + \ldots) + xD_4(x)(u^3 + \ldots) + \ldots \end{aligned}$$

Then, we write the closed form of the power series.

$$\begin{aligned} D(x,u) = &\frac{x}{1-u} + \frac{x}{u}(-B_0(x) + B_0(x) + B_1(x)u + B_2(x)u^2 + \ldots) \\ &+ xC(x) + \frac{x}{1-u}(D_1(x) + D_2(x)u + D_3(x)u^2 + \ldots). \end{aligned}$$

Then we obtain,

$$D(x,u) = \frac{x}{1-u} + \frac{x}{u}(B(x,u) - B(x,0)) + xC(x) + \frac{x}{1-u}D(x,u) \tag{3.24}$$

Below we see all of the simpler forms of multi-variable generating functions that correspond to the succession rules.

$$
\begin{aligned}
A(x,u) &= \frac{x}{1-u} + \frac{x}{u}[B(x,u) - B(x,0)] + \frac{x}{1-u}B(x,u) \\
B(x,u) &= \frac{x}{1-u} + xA(x,u) + \frac{x}{1-u}B(x,u) \\
C(x,u) &= \frac{x}{1-u} + \frac{x}{u}(A(x,u) + D(x,u) - A(x,0) - D(x,0)) + \frac{x}{1-u}D(x,u) \\
D(x,u) &= \frac{x}{1-u} + \frac{x}{u}(B(x,u) - B(x,0)) + xC(x) + \frac{x}{1-u}D(x,u).
\end{aligned}
$$

Using (3.16) and (3.19),

$$B(x,u) - A(x,u) = xA(x,u) - \frac{x}{u}(B(x,u) - B(x,0)). \tag{3.25}$$

From (3.19), we have

$$B(x,u) = \frac{1-u}{1-u-x}\left(\frac{x}{1-u} + xA(x,u)\right) \tag{3.26}$$

and

$$B(x,0) = \frac{1-u}{1-x}\left(x + xA(x,0)\right). \tag{3.27}$$

By substituting these equations into (3.25), we get the following functional equation.

$$\frac{(1-x)u - x^2 - u^2}{u(1-u-x)}A(x,u) = \frac{-x^2}{u(1-x)}A(x,0) + \frac{x}{(1-u-x)(1-x)}. \tag{3.28}$$

By using the kernel method,

$$u = \frac{1 - x - \sqrt{1-2x-3x^2}}{2} = x^2 m(x).$$

where $m(x)$ is the generating function of Motzkin numbers, we obtain:

$$A(x,0) = xm^2(x). \tag{3.29}$$

We substitute (3.29) into (3.28).

$$A(x,u) = \frac{xm(x)(x^2m(x)-u)}{u^2+(x-1)u+x^2}. \tag{3.30}$$

Next, we substitute (3.30) into (3.19).

$$B(x,u) = \frac{xm(x)((u+x)x^2m(x)-u+x^2)}{u^2+(x-1)u+x^2}. \tag{3.31}$$

By substituting (3.22) into (3.24) with using (3.30) and (3.31), we obtain

$$\begin{aligned} \frac{((1-x)u-x^2-u^2)^2}{u(1-u)}D(x,u) + \frac{x^2((1-x)u-x^2-u^2)}{u}D(x,0) = \\ \frac{xm(x)((x^2-x-1)u^2+(x^3-3x^2+1)u+x^4+x^2)}{1-u} + \\ \frac{x^3m^2(x)((x-1)u^2+(x^2-x+3)u+x^3+x^2-x-2)}{1-u}. \end{aligned} \tag{3.32}$$

By differentiating (3.32) respect to u and taking $u = x^2m(x)$, after some algebraic simplifications, we get

$$D(x,0) = \frac{x((x^2+2x-2)m(x)-x+1)((1-x)m(x)-2)}{(1+x)(1-3x)}. \tag{3.33}$$

Therefore, the following result is obtained in [12] based on the equation (3.4).

Theorem 3.1.1. *The generating function* $R(x) = \sum_{n\geq 1} |I_n(000,021)|x^n$ *is given by*

$$\frac{3x^3+x^2-3x+1}{2x^2\sqrt{1-2x-3x^2}} - \frac{(1-x)^2}{2x^2} = x+2x^2+5x^3+14x^4+39x^5+111x^6+317x^7+\dots$$

Moreover, by sequence A002426 in OEIS, we get for all $n \geq 1$

$$|I_n(000,021)| = \frac{1}{2}(3a_{n-1}+a_n-3a_{n+1}+a_{n+2})$$

where $a_n = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{2k}{k}$.

3.2 Class $I_n(100,021)$ and $I_n(110,021)$

In this section, we study the enumeration of $I_n(100,021)$ and $I_n(110,021)$ by following [12]. First of all, we introduce some notation to represent corresponding inversion sequences. The generating tree $\mathcal{T}[B]$ where $B = \{100,021\}$ has the following succession rules.

$$
\begin{aligned}
\textit{Root} &: a_1,\\
\textit{Rules} &: a_m \rightsquigarrow a_{m+1}b_m b_{m-1}\dots b_1, \quad m \geq 1\\
& \quad b_m \rightsquigarrow c_m b_{m+1} b_m \dots b_1, \quad m \geq 1\\
& \quad c_m \rightsquigarrow c_{m+1} d_m d_{m-1} \dots d_1 e, \quad m \geq 1\\
& \quad d_m \rightsquigarrow d_{m+1} d_m \dots d_1 e, \quad m \geq 1.\\
& \quad e \rightsquigarrow d_1 e
\end{aligned}
$$

where $a_m = 0^m$, $b_m = 0^m1$, $c_m = 0^m10$, $d_m = 0^m102$ with $m \geq 1$ and $e = 0103$.

The generating tree $\mathcal{T}[B]$ where $B = \{110,021\}$ has the following succession rules:

$$
\begin{aligned}
\textit{Root} &: a_1,\\
\textit{Rules} &: a_m \rightsquigarrow a_{m+1}b_m b_{m-1}\dots b_1, \quad m \geq 1\\
& \quad b_m \rightsquigarrow c_m b_{m+1} b_m \dots b_1, \quad m \geq 1\\
& \quad c_m \rightsquigarrow c_{m+1} d_m d_{m-1} \dots d_1 e, \quad m \geq 1\\
& \quad d_m \rightsquigarrow d_{m+1} d_m \dots d_1 e, \quad m \geq 1.\\
& \quad e \rightsquigarrow d_1 e
\end{aligned}
$$

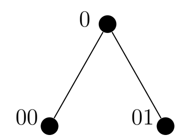
where $a_m = 0^m$, $b_m = 0^m1$, $c_m = 0^m11$, $d_m = 0^m112$ with $m \geq 1$ and $e = 0113$.

Note that these generating trees have the same succession rules. This yields the following corollary.

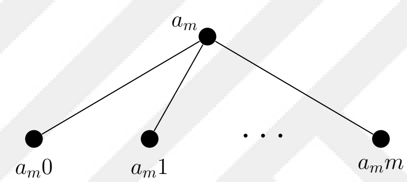
Corollary 3.2.1. *For all $n \geq 1$, $|I_n(100,021)| = |I_n(110,021)|$.*

Verification for the succession rules of $\mathcal{T}[B]$ where $B = \{110,021\}$ is as follows.

1. $a_m \rightsquigarrow a_{m+1}b_m b_{m-1}\dots b_1, \ \ m \geq 1$:
   Note that, $a_1 = 0$. The initialization of the tree verifies the rule for $m = 1$.

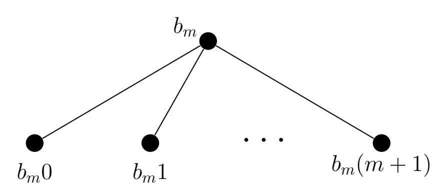
   

   The children of a_m is in the form of $a_m j = 0^m j$ where $j \in \{0, 1, \dots, m\}$.

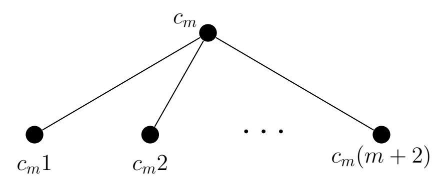
   Notice that, the first child is $a_m 0 = a_{m+1}$ and the second child is $a_m 1 = b_m$. The sub-tree $\mathcal{T}(\{110, 021\}; a_m j)$ is isomorphic to the sub-tree $\mathcal{T}(\{110, 021\}; b_{m-j+1})$ for $j \geq 0$ when we remove $j-1$ leading zeros from $a_m j$ and decrease the last element of $a_m j$ by $j - 1$.

2. $b_m \rightsquigarrow c_m b_{m+1} b_m \dots b_1, \ \ m \geq 1$:
   The children of b_m is in the form of $b_m j = 0^m 1 j$ where $j \in \{0, 1, \dots, m, m+1\}$.

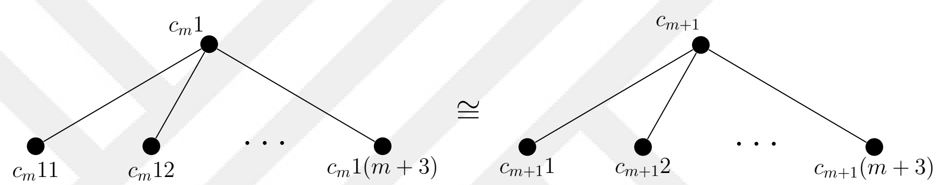
   

   Obviously, the first child is c_m. The sub-tree $\mathcal{T}(\{110, 021\}; b_m j)$ is isomorphic to the sub-tree $\mathcal{T}(\{110, 021\}; b_{m-j+2})$ when we remove $j - 1$ leading zeros from $a_m j$, setting first non-zero element as zero and decrease the last element of $a_m j$ by $j - 1$.

3. $c_m \rightsquigarrow c_{m+1} d_m d_{m-1} \dots d_1 e, \ \ m \geq 1$:
   The children of c_m is in the form of $c_m j = 0^m 10 j$ where $j \in \{1, \dots, m+2\}$.

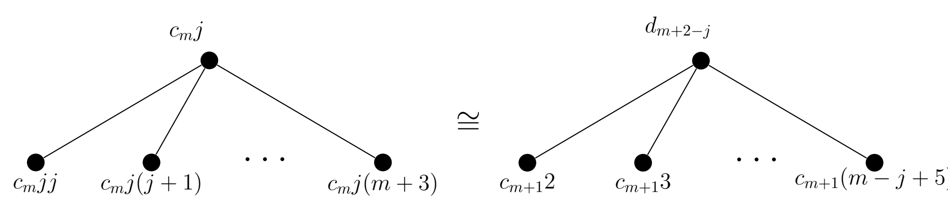


The sub-tree $\mathcal{T}(\{110, 021\}; c_m1)$ is isomorphic to the sub-tree $\mathcal{T}(\{110, 021\}; c_{m+1})$ by adding zero at the beginning and deleting the last element 1.



Note that, the children of c_m1 is in the form of $c_m1j = 0^m101j$ where $j \in \{1, \ldots m+3\}$ and the children of c_{m+1} is in the form of $c_{m+1}j = 0^{m+1}10j$ where $j \in \{1, \ldots m+3\}$. Otherwise, they do not avoid the pattern set.

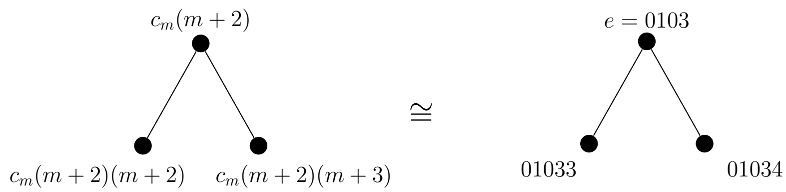
The sub-tree $\mathcal{T}(\{110, 021\}; c_mj)$ is isomorphic to the sub-tree $\mathcal{T}(\{110, 021\}; d_{m+2-j})$ where $j \geq 2$ by removing $j-2$ zeros from the beginning and adding $j-2$ to the last two elements.



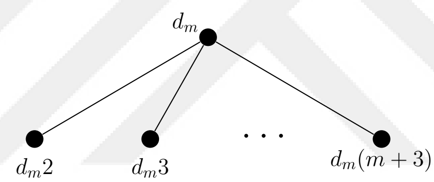
Note that, the children of c_mj is in the form of $c_mjk = 0^m10jk$ where $k \in \{j, \ldots m+3\}$ and the children of d_{m+2-j} is in the form of $d_{m+2-j}k = 0^{m+2-j}102k$ where $k \in \{2, \ldots, m-j+5\}$. Otherwise, they do not avoid the pattern set.

The sub-tree $\mathcal{T}(\{110, 021\}; c_m(m+2))$ is isomorphic to the sub-tree $\mathcal{T}(\{110, 021\}; e)$ by removing $m-1$ zeros from the beginning and adding

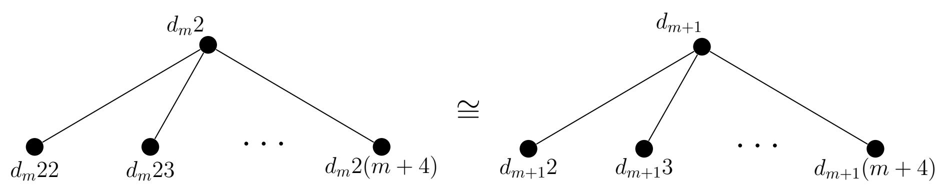
$1 - m$ to the last two elements.



4. $d_m \rightsquigarrow d_{m+1} d_m \ldots d_1 e, \;\; m \geq 1$:
   The children of d_m is in the form of $d_m j = 0^m 102j$ where $j \in \{2, \ldots, m+3\}$.

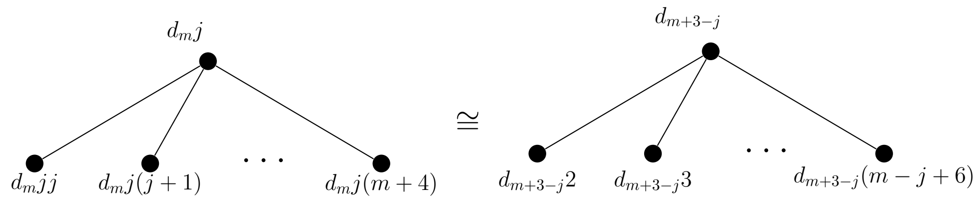


The sub-tree $\mathcal{T}(\{110, 021\}; d_m2)$ is isomorphic to the sub-tree $\mathcal{T}(\{110, 021\}; d_{m+1})$ by adding zero at the beginning and removing the second to last element 2.



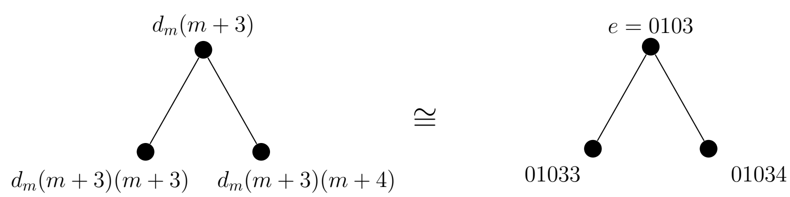
Note that, the children of d_m2 is in the form of $d_m2j = 0^m1022j$ where $j \in \{2, \ldots, m+4\}$ and the children of d_{m+1} is in the form of $d_{m+1}j = 0^{m+1}102j$ where $j \in \{2, \ldots, m+4\}$. Otherwise, they do not avoid the pattern set.

The sub-tree $\mathcal{T}(\{110, 021\}; d_m j)$ is isomorphic to the sub-tree $\mathcal{T}(\{110, 021\}; d_{m+3-j})$ where $j \geq 3$ by removing $j - 3$ zeros from the beginning, removing second to the last element and adding $2 - j$ to the last two elements.

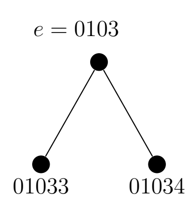


Note that, the children of $d_m j$ is in the form of $d_m jk = 0^m 102jk$ where $k \in \{j, \ldots, m+4\}$ and the children of d_{m+3-j} is in the form of $d_{m+3-j}k = 0^{m+3-j}102k$ where $k \in \{2, \ldots, m-j+5\}$. Otherwise, they do not avoid the pattern set.

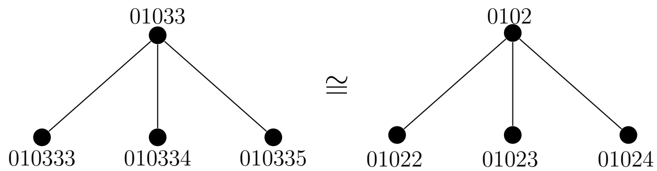
The sub-tree $\mathcal{T}(\{110, 021\}; d_m(m+3))$ is isomorphic to the sub-tree $\mathcal{T}(\{110, 021\}; e)$ by removing $m-1$ zeros from the beginning, removing third to the last element 2, and subtracting m to the last two elements.



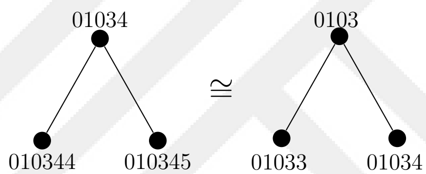
5. $e \rightsquigarrow d_1 e$:
   The children of $e = 0103$ are 01033 and 01034.



The sub-tree $\mathcal{T}(\{110, 021\}; 01033)$ is isomorphic to the sub-tree $\mathcal{T}(\{110, 021\}; d_1)$ by removing the second to the last element, and after that, decreasing last two elements by 1.



The sub-tree $\mathcal{T}(\{110, 021\}; 01034)$ is isomorphic to the sub-tree $\mathcal{T}(\{110, 021\}; e)$ by removing the second to the last element, and after that, decreasing last element by 1.



Generating functions for the succession rules are as follows:

Nodes	Generating Functions	Corresponding Sub-tree
a_m	$A_m(x)$	$\mathcal{T}(B; a_m)$
b_m	$B_m(x)$	$\mathcal{T}(B; b_m)$
c_m	$C_m(x)$	$\mathcal{T}(B; c_m)$
d_m	$D_m(x)$	$\mathcal{T}(B; d_m)$
e	$E(x)$	$\mathcal{T}(B; e)$

Then we have the following functional equations.

$$A_m(x) = x + xA_{m+1}(x) + x\sum_{j=1}^{m} B_j(x), \quad m \geq 1 \tag{3.34}$$

$$B_m(x) = x + xC_m(x) + x\sum_{j=1}^{m+1} B_j(x), \quad m \geq 1 \tag{3.35}$$

$$C_m(x) = x + xC_{m+1}(x) + x\sum_{j=1}^{m} D_j(x) + xE(x), \quad m \geq 1 \tag{3.36}$$

$$D_m(x) = x + x\sum_{j=1}^{m+1} D_j(x) + xE(x), \quad m \geq 1 \tag{3.37}$$

$$E(x) = x + xD_1(x) + xE(x) \tag{3.38}$$

Next, we define the following multivariable generating functions.

$$A(x,u) = \sum_{m\geq 1} A_m(x)u^{m-1} = A_1(x) + A_2(x)u + A_3(x)u^2 + \dots \tag{3.39}$$

$$B(x,u) = \sum_{m\geq 1} B_m(x)u^{m-1} = B_1(x) + B_2(x)u + B_3(x)u^2 + \dots \tag{3.40}$$

$$C(x,u) = \sum_{m\geq 1} C_m(x)u^{m-1} = C_1(x) + C_2(x)u + C_3(x)u^2 + \dots \tag{3.41}$$

$$D(x,u) = \sum_{m\geq 1} D_m(x)u^{m-1} = D_1(x) + D_2(x)u + D_3(x)u^2 + \dots \tag{3.42}$$

Note that, $A(x,0) = A_1(x)$, $B(x,0) = B_1(x)$, $C(x,0) = C_1(x)$ and $D(x,0) = D_1(x)$.

Next, we seek simpler expressions for these multi-variable generating functions.

$$A(x,u) = A_1(x) + A_2(x)u + A_3(x)u^2 + \dots \tag{3.43}$$

We substitute (3.34) into (3.43).

$$\begin{aligned} A(x,u) &= (x + xA_2(x) + xB_1(x)) \\ &\quad + (x + xA_3(x) + xB_1(x) + xB_2(x))u \\ &\quad + (x + xA_4(x) + xB_1(x) + xB_2(x) + xB_3(x))u^2 + \dots \end{aligned} \tag{3.44}$$

We arrange (3.44) as follows.

$$\begin{aligned} A(x,u) = \frac{x}{1-u} &+ \frac{x}{u}\left(-A_1(x) + A_1(x) + A_2(x)u + A_3(x)u^2 + A_4(x)u^3 + \dots\right) \\ &+ \frac{x}{1-u}\left(B_1(x) + B_2(x)u + B_3(x)u^2 + \dots\right). \end{aligned}$$

Then we get,

$$A(x,u) = \frac{x}{1-u} + \frac{x}{u}(A(x,u) - A(x,0)) + \frac{x}{1-u}B(x,u). \tag{3.45}$$

We obtain a simpler expression for $B(x,u)$ as follows.

$$B(x,u) = B_1(x) + B_2(x)u + B_3(x)u^2 + \dots \tag{3.46}$$

We substitute (3.40) into (3.46)

$$\begin{aligned} B(x,u) &= (x + xC_1(x) + xB_1(x) + xB_2(x)) \\ &\quad + (x + xC_2(x) + xB_1(x) + xB_2(x) + xB_3(x))u \\ &\quad + (x + xC_3(x) + xB_1(x) + xB_2(x) + xB_3(x) + xB_4(x))u^2 + \dots \end{aligned} \tag{3.47}$$

We arrange (3.47) as follows.

$$\begin{aligned} B(x,u) &= x(1 + u + \dots) + x(C_1(x) + C_2(x)u + \dots) \\ &\quad + xB_1(x)(1 + u + \dots) + xB_2(x)(1 + u + \dots) \\ &\quad + xB_3(x)(u + u^2 + \dots) + xB_4(x)(u^2 + u^3 \dots) + \dots \end{aligned} \tag{3.48}$$

Then we write the closed form of the power series in (3.48).

$$B(x,u) = \frac{x}{1-u} + xC(x,u) + \frac{x}{1-u} \cdot \frac{1}{u}\left(B_1(x)u + B_2(x)u + B_3(x)u^2 + \dots\right)$$

$$\begin{aligned} B(x,u) &= \frac{x}{1-u} + xC(x,u) + \frac{x}{1-u} \cdot \frac{1}{u}B_1(x)u \\ &\quad + \frac{x}{1-u} \cdot \frac{1}{u}\left(-B_1(x) + B_1(x) + B_2(x)u + B_3(x)u^2 + B_4(x)u^3 + \dots\right) \end{aligned}$$

$$B(x,u) = \frac{x}{1-u} + xC(x,u) + \frac{x}{1-u}B_1(x) - \frac{x}{1-u} \cdot \frac{1}{u}B_1(x) + \frac{x}{1-u} \cdot \frac{1}{u}B(x,u)$$

Using partial fraction, $\frac{x}{(1-u)u} = \frac{x}{1-u} + \frac{x}{u}$, we obtain

$$B(x,u) = \frac{x}{1-u} + xC(x,u) + \frac{x}{u}\left(B(x,u) - B(x,0)\right) + \frac{x}{1-u}B(x,u). \tag{3.49}$$

We obtain a simpler expression for $C(x,u)$ as follows.

$$C(x,u) = C_1(x) + C_2(x)u + C_3(x)u^2 + \dots \tag{3.50}$$

We substitute (3.41) into (3.50)

$$\begin{aligned} C(x,u) &= (x + xC_2(x) + xD_1(x) + xE(x)) \\ &\quad + (x + xC_3(x) + xD_1(x) + xD_2(x) + xE(x))u \\ &\quad + (x + xC_4(x) + xD_1(x) + xD_2(x) + xD_3(x) + xE(x))u^2 \\ &\quad + \dots \end{aligned} \tag{3.51}$$

We arrange (3.51).

$$\begin{aligned} C(x,u) &= x(1 + u + \dots) + x(C_2(x) + C_3(x)u + \dots) \\ &\quad + xD_1(x)(1 + u + \dots) + xD_2(x)(u + u^2 + \dots) \\ &\quad + xD_3(x)(u^2 + u^3 + \dots) + \dots \\ &\quad + xE(x)(1 + u + \dots) \end{aligned} \tag{3.52}$$

Then we write the closed form of the power series in (3.52).

$$\begin{aligned} C(x,u) &= \frac{x}{1-u} + \frac{x}{u}\left(-C_1(x) + C_1(x) + C_2(x)u + C_3(x)u^2 + \dots\right) \\ &\quad + \frac{x}{1-u}\left(D_1(x) + D_2(x)u + D_3(x)u^2 + \dots\right) + \frac{x}{1-u}E(x) \\ C(x,u) &= \frac{x}{1-u} - \frac{x}{u}C_1(x) + \frac{x}{u}C(x,u) + \frac{x}{1-u}D(x,u) + \frac{x}{1-u}E(x) \end{aligned}$$

Then, we obtain

$$C(x,u) = \frac{x}{1-u} + \frac{x}{u}\left(C(x,u) - C(x,0)\right) + \frac{x}{1-u}D(x,u) + \frac{x}{1-u}E(x,u). \tag{3.53}$$

We obtain a simpler expression for $D(x,u)$ as follows.

$$D(x,u) = D_1(x) + D_2(x)u + D_3(x)u^2 + \dots \tag{3.54}$$

We substitute (3.42) into (3.54).

$$\begin{aligned} D(x,u) = &(x + xD_1(x) + xD_2(x) + xE(x)) \\ &+ (x + xD_1(x) + xD_2(x) + xD_3(x) + xE(x))u \\ &+ (x + xD_1(x) + xD_2(x) + xD_3(x) + xD_4(x) + xE(x))u^2 \\ &+ \ldots \end{aligned} \tag{3.55}$$

We arrange (3.55).

$$\begin{aligned} D(x,u) = &x(1 + u + \ldots) + xD_1(1 + u + \ldots) + xD_2(1 + u + u^2 + \ldots) \\ &+ xD_3(u + u^2 + \ldots) + xD_4(u^2 + u^3 \ldots) + \ldots + xE(x)(1 + u + \ldots) \end{aligned} \tag{3.56}$$

Next, we write the closed form of the power series in (3.56).

$$\begin{aligned} D(x,u) &= \frac{x}{1-u} + \frac{x}{1-u}\left(D_1(x) + D_2(x) + D_3(x)u + \ldots\right) + \frac{x}{1-u}E(x) \\ D(x,u) &= \frac{x}{1-u} + \frac{x}{u(1-u)}\left(D_1(x)u - D_1(x) + D_1(x) + D_2(x)u + \ldots\right) \\ &\quad + \frac{x}{1-u}E(x) \\ D(x,u) &= \frac{x}{1-u} + \frac{x}{u(1-u)}\left(D_1(x)u - D_1(x)\right) + \frac{x}{u(1-u)}D(x,u) + \frac{x}{1-u}E(x) \end{aligned}$$

Using partial fraction, $\dfrac{x}{(1-u)u} = \dfrac{x}{1-u} + \dfrac{x}{u}$, we obtain

$$D(x,u) = \frac{x}{1-u} + \frac{x}{u}\left(D(x,u) - D(x,0)\right) + \frac{x}{1-u}D(x,u) + \frac{x}{1-u}E(x,u). \tag{3.57}$$

Also, we have the following simplified functional equation for $E(x)$.

$$E(x) = \frac{x}{1-x} + \frac{x}{1-x}D(x,0). \tag{3.58}$$

Using (3.57) and (3.58),

$$\begin{aligned} D(x,u) = &\frac{x}{(1-x)(1-u)} + \frac{x}{u}\left(D(x,u) - D(x,0)\right) + \frac{x}{1-u}D(x,u) \\ &+ \frac{x^2}{(1-x)(1-u)}D(x,0). \end{aligned} \tag{3.59}$$

We rearrange the equation (3.59)

$$\left(1 - \frac{x}{u} - \frac{x}{1-u}\right) D(x,u) = \frac{x}{(1-x)(1-u)} + \left(\frac{x^2}{(1-x)(1-u)} - \frac{x}{u}\right) D(x,0).$$

We apply the kernel method with $u = xc(x)$ where $c(x) = \dfrac{1-\sqrt{1-4x}}{2x}$ is the generating function of Catalan numbers. Hence, we have

$$D(x,0) = \frac{xc^2(x)}{1 - x - x^2c^2(x)} \tag{3.60}$$

and

$$E(x) = xc^2(x). \tag{3.61}$$

By substituting (3.60) and (3.61) in (3.57), we obtain

$$D(x,u) = \frac{xc^2(x)(xc(x) - u)}{u^2 - u + x}. \tag{3.62}$$

Next, we substitute (3.61) and (3.62) in (3.53).

$$C(x,u) = \frac{x}{1-u} + \frac{x}{u}(C(x,u) - C(x,0)) + \frac{x^2c^2(x)(xc(x) - u)}{(1-u)(u^2 - u + x)} + \frac{x^2c^2(x)}{1-u}. \tag{3.63}$$

By applying kernel method with $u = x$, we get

$$C(x,0) = xc^3(x) \tag{3.64}$$

and

$$C(x,u) = \frac{(1 - x - u)c(x) + u - 1}{u^2 - u + x}. \tag{3.65}$$

Similarly, we obtain the following equations.

$$B(x,0) = \frac{xc(x)}{1 - 2c(x)}$$

and

$$A(x,0) = \frac{x}{1-x} + \frac{x}{1-x}B(x,x).$$

Finally, the following theorem is presented in [12].

Theorem 3.2.1. *The generating function* $R(x) = \sum_{n\geq 1} |I_n(110, 021)|x^n$ *is given by*

$$\frac{(1-3x)^2}{2x^2\sqrt{1-4x}} - \frac{(1-3x)(1-x)}{2x^2}.$$

Moreover, $|I_n(110, 021)| = \frac{n^2+n+6}{2(n+3)(n+2)}\binom{2n+2}{n+1}$.

Chapter 4

Inversion Sequences Avoiding Patterns of Length Four

In 2022, Letong Hong and Rupert Li studied all Wilf equivalence classes of inversion sequences avoiding patterns of length four, except one unresolved pattern 3012; for details, see [21]. They also stated that the pattern 3012 possibly belongs to the same equivalence class with 3201 and 3210. Nontrivial equivalence classes are as follows.

$$\begin{gathered}1011 \equiv 1101 \equiv 1110 \\ 2110 \equiv 2101 \equiv 2011 \\ 0221 \equiv 0212 \\ 0312 \equiv 0321 \\ 1102 \equiv 1012 \\ 2201 \equiv 2210 \\ 2301 \equiv 2310 \\ 3201 \equiv 3210 \stackrel{?}{\equiv} 3012\end{gathered} \tag{4.1}$$

In this thesis, we studied patterns 0321, 0312, 0221, and 0212. First, we designed sampling algorithms to generate random objects from the classes $I_n(0221)$, $I_n(0212)$, $I_n(0321)$, and $I_n(0312)$ to observe their structure. Based on our random samples, we study some statistics such as the number of zeros, the number of distinct elements, the number of repeated elements, the maximum elements, and the number of left-to-right maximum elements. Also, we present an explicit bijection between $I_n(0321)$ and $I_n(0312)$ that preserves these statistics.

4.1 Sampling algorithms

Suppose we are given a pattern τ. For a large n, how can we obtain random samples from $I_n(\tau)$ uniformly randomly? This is computationally a challenging task because determining whether a given sequence contains a specific pattern or not is difficult. The Markov Chain Monte Carlo (MCMC) method is a powerful approach to sample objects from complex discrete structures, see [22]. In this section, we briefly describe the method without further details. The MCMC method involves running a random walk on the set $I_n(\tau)$ for a sufficiently long time and then obtaining some samples. The MCMC algorithm we utilized to generate pattern-avoiding inversion sequences is given below. We follow similar ideas first studied in the context of pattern-avoiding permutations by Madras and Liu; for details, see [23].

Algorithm: Generating an inversion sequence of length n avoiding pattern τ.

```
Set e = e_1 e_2 ... e_n from I_n(τ).
for iteration = 1, 2, ..., M do
    Choose i uniformly from {1, 2, ..., n}
    Choose j uniformly from {0, ..., i − 1}
    Let ẽ = e_1 e_2 ... e_{i−1} j e_{i+1} ... e_n
    if ẽ avoids τ then
        e ← ẽ
    else
        e ← e
    end if
end for
```

First, we fix a pattern τ and n. Then, we start our random walk on $I_n(\tau)$ with a specific element $e = e_1e_2 \dots e_n \in I_n(\tau)$ (for instance, we can use $e = 00\dots0$ if it avoids τ). At each step we randomly modify e at only one entry by considering the pattern condition. In the algorithm, we need to check whether $\tilde{e}$ contains the pattern τ or not. The efficiency of the algorithm is mostly determined by this step. We do not need to search τ in the whole modified sequence $\tilde{e}$. We know e does not contain τ and $\tilde{e}$ is obtained from e only by changing one entry. So, we pose the following question and analyze the possible cases: What are the possibilities for this single-entry-change to generate the pattern in $\tilde{e}$, even though e does not contain the pattern?

4.1.1 A sampling algorithm for $I_n(0321)$

We set $e = 00\ldots00$ to initialize the algorithm because it is easy to see that $e = 00\ldots00 \in I_n(0321)$. Then, we use the following facts:

- If $\tilde{e}$ contains the pattern 0321, then there exist non-zero entries e_k, e_l, e_m such that $e_k > e_l > e_m$ and $k < l < m$. Hence, $e_1e_ke_le_m$ yields the pattern 0321 and j would be one of e_k, e_l, and e_m.
- Note that j can not play the role of 0 in the pattern 0321. Because if it does, this means that there are elements e_k, e_l, e_m such that $e_k > e_l > e_m$ and $k < l < m$ after the position of j and these e_k, e_l, e_m would create the pattern 0321 with e_1 before single-entry-change.

Example 4.1.1. *Generating an* 0321*-avoiding inversion sequence of length* 12 *by using the algorithm.*
Suppose at some step of this procedure we have $e = 001232052602 \in I_{12}(0321)$. *Choose* i *from* $\{1,\ldots,12\}$ *uniformly randomly. Suppose* $i = 9$. *Now, we choose* j *from* $\{0,\ldots,8\}$ *uniformly randomly.*

$$\begin{aligned}
&\textit{If } j = 1, \textit{ then } \quad \tilde{e} = 001232051602 \notin I_{12}(0321) &&\longrightarrow \quad \textit{j plays the role of 1}\\
&\textit{If } j = 3, \textit{ then } \quad \tilde{e} = 001232053602 \notin I_{12}(0321) &&\longrightarrow \quad \textit{j plays the role of 2}\\
&\textit{If } j = 7, \textit{ then } \quad \tilde{e} = 001232057602 \notin I_{12}(0321) &&\longrightarrow \quad \textit{j plays the role of 3}
\end{aligned}$$

In summary, the possibilities for this single-entry-change to generate the pattern in $\tilde{e}$, even though e does not contain the pattern 0321 are as follows:

1. If j plays the role of 3 in the pattern 0321, then there exist at least two non-zero elements smaller than j in the sub-sequence $e_{i+1} \dots e_n$ so that 0321 occurs.

2. If j plays the role of 2 in the pattern 0321, then there exists at least one non-zero element larger than j in the sub-sequence $e_1 \dots e_{i-1}$ and there exists at least one element smaller than j in the sub-sequence $e_{i+1} \dots e_n$.

3. If j plays the role of 1 in the pattern 0321, then there exist at least two elements larger than j in the sub-sequence $e_1 \dots e_{i-1}$ that forms the pattern 10.

Note that we look at sub-sequences of $\tilde{e}$ to check whether they contain pattern 10 instead of searching the pattern 0321 in the whole sequence. This makes the algorithm more efficient.

4.1.2 A sampling algorithm for $I_n(0312)$

We again set $e = 00 \dots 00$ to initialize the algorithm. Then, we use the following facts:

- If $\tilde{e}$ contains the pattern 0312, then there exists non-zero entries e_k, e_l, e_m such that $e_k > e_m > e_l$ and $k < l < m$. Hence, $e_1 e_k e_l e_m$ yields the pattern 0312 and j would be one of e_k ,e_l, and e_m.

- Note that j can not play the role of 0 in the pattern 0312. Because if it does, this means that there are elements e_k, e_l, e_m such that $e_k > e_m > e_l$ and $k < l < m$ after the position of j and these e_k, e_l, e_m would create the pattern 0312 with e_1 before single-entry-change.

Example 4.1.2. *Generating an* 0312*-avoiding inversion sequence of length* 12 *by using the algorithm.*

Suppose at some step of this procedure we have $e = 002310008448 \in I_{12}(0312)$*. Choose* i *from* $\{1,\ldots,12\}$ *uniformly randomly. Suppose* $i = 10$*. Now, we choose* j *from* $\{0,\ldots,9\}$*.*

$$\begin{array}{llll} \textit{If } j = 3, \textit{ then} & \tilde{e} = 002310008348 \notin I_{12}(0312) & \longrightarrow & j \textit{ plays the role of 1.} \\ \textit{If } j = 2, \textit{ then} & \tilde{e} = 002310008248 \notin I_{12}(0312) & \longrightarrow & j \textit{ plays the role of 2.} \\ \textit{If } j = 9, \textit{ then} & \tilde{e} = 002310008948 \notin I_{12}(0312) & \longrightarrow & j \textit{ plays the role of 3.} \end{array}$$

In summary, the possibilities for this single-entry-change to generate the pattern in $\tilde{e}$, even though e does not contain the pattern 0312 are as follows:

1. If j plays the role of 3 in the pattern 0312, then there exist at least two non-zero elements smaller than j in the sub-sequence $e_{i+1}\ldots e_n$ that forms the pattern 01.
2. If j plays the role of 2 in the pattern 0312, then there exists at least one non-zero element larger than j and one element smaller than j in the sub-sequence $e_1\ldots e_{i-1}$ that forms the pattern 10.
3. If j plays the role of 1 in the pattern 0312, then there exists at least one element larger than j in the sub-sequence $e_1\ldots e_{i-1}$ and there exists at least one element larger than j in the sub-sequence $e_{i+1}\ldots e_n$ that forms the pattern 10.

4.2 Numerical results

The plot of an inversion sequence $e = e_1\ldots e_n$ is the set of points (i, e_i) for $i = 1, 2, \ldots, n$. Figure 4.1 shows the plot of $e = 0110345609 \in I_{10}$.

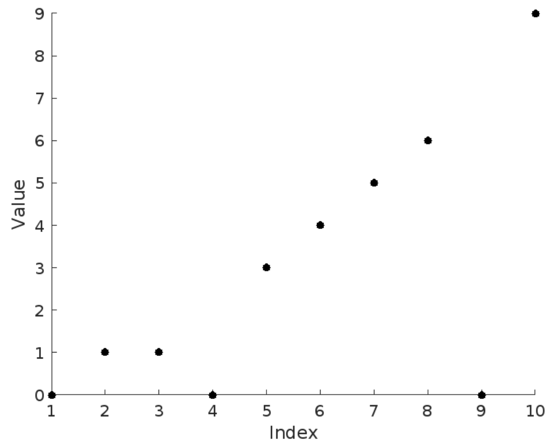


Figure 4.1: The plot of $e = 0110345609 \in I_{10}$.

Let $e = e_1 \ldots e_n$ be an inversion sequence. If $e_i \geq e_j$ where $\forall j < i$, then e_i is called a left-to-right maximum entry.

We provide the plot of an inversion sequence that avoids certain patterns by representing zero entries in red and left-to-right maximum entries in blue. This characterization helps analyze inversion sequences based on the placement of these specific elements; see Figure 4.2. From these plots, we observe that inversion sequences avoiding different patterns exhibit variations in the placement of entries; see Figure 4.3.

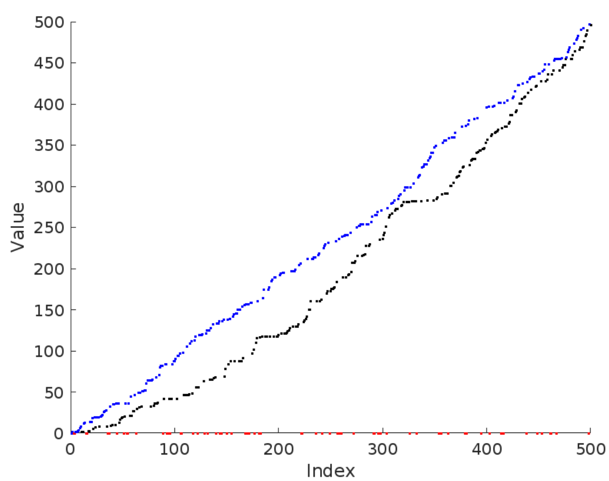
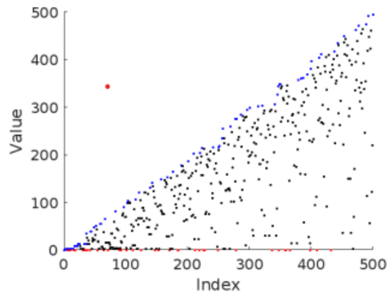
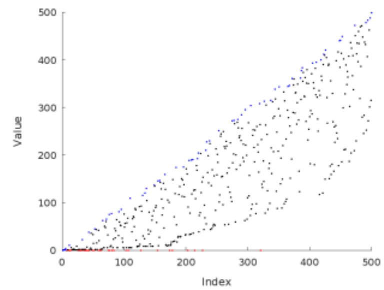


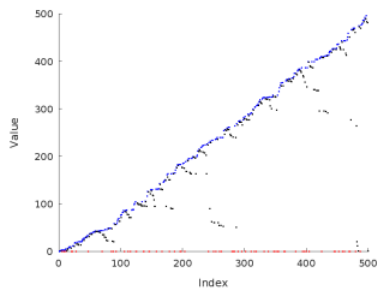
Figure 4.2: The plot of a randomly generated 0321-avoiding inversion sequence of length 500. The red dots correspond to zero entries. The blue dots correspond to the left-to-right maximum entries.



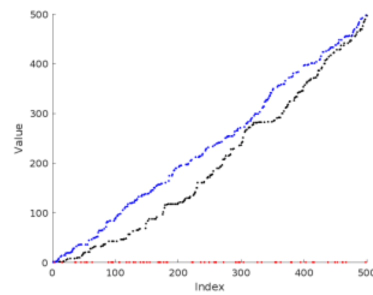
A random 0212-avoiding inversion sequence



A random 0221-avoiding inversion sequence



A random 0312-avoiding inversion sequence



A random 0321-avoiding inversion sequence

Figure 4.3: The plots of inversion sequences of length 500 that avoids different patterns. The red dots correspond to zero entries. The blue dots correspond to left-to-right maximum entries.

We also created heatmaps for inversion sequences that avoid certain patterns to understand where entries are more likely to occur. Figure 4.4 shows four different heatmaps of four different pattern-avoiding inversion sequences of length 500; these patterns are 0212, 0221, 0312, and 0321. Each heatmap includes 100 inversion sequences generated uniformly randomly using our algorithms. Lighter areas represent fewer entries, while darker areas indicate more.

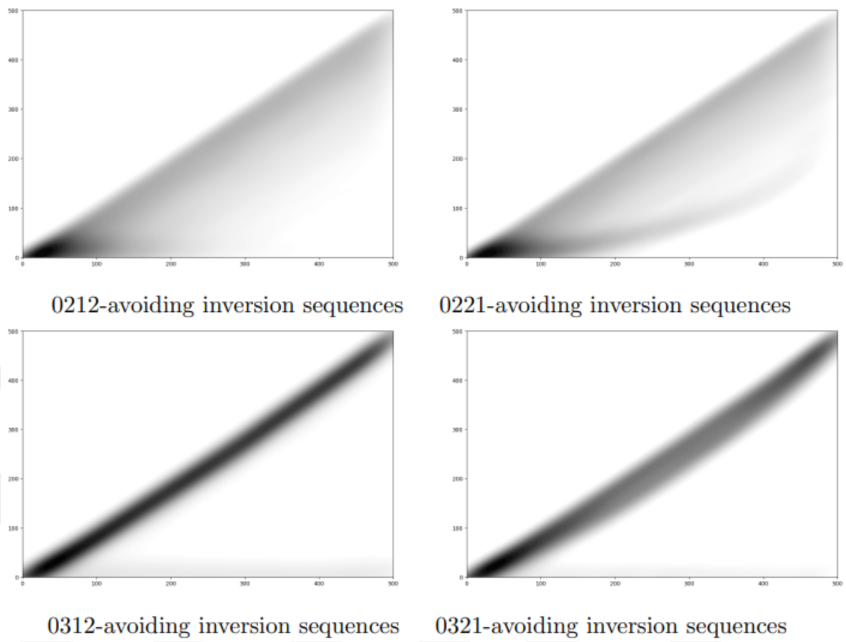


Figure 4.4: Heatmaps of pattern-avoiding inversion sequences of length 500; these patterns are 0212, 0221, 0312, and 0321. Each heatmap includes 100 inversion sequences generated randomly using our algorithms. Lighter areas represent fewer entries, while darker pixels indicate more.

Our numerical calculations lead us to conjecture that the bijection between $I_n(0312)$ and $I_n(0321)$ preserves the following statistics:

For any $e = e_1e_2 \dots e_n \in I_n$, we define

$$\begin{aligned} zeros(e) &= |\{1 \leq i \leq n \mid e_i = 0\}| \\ dist(e) &= |\{e_1, e_2, \dots, e_n\}\ | \\ repeats(e) &= |\{1 \leq i \leq n-1 \mid \{e_i \in e_{i+1}, e_{i+2}, \dots, e_n\}| \\ max(e) &= max(e_1, e_2, \dots, e_n) \\ lrmax(x) &= |\{1 \leq i \leq n \mid e_i \geq e_j \text{ where } \forall j < i\}|. \end{aligned} \tag{4.2}$$

We define the generating functions for classes of inversion sequences that avoid these specific patterns with the mentioned statistics as follows.

$$H_B(x, y, z, t, u, v) = \sum_{n \geq 1} \sum_{e \in I_n} x^n y^{zeros(e)} z^{dist(e)} t^{repeats(e)} u^{max(e)} v^{lrmax(e)} \tag{4.3}$$

In Section 4.3, we present a bijection between $I_n(0321)$ and $I_n(0312)$. This bijection implies that

$$H_{0312}(x, y, z, t, u, v) = H_{0321}(x, y, z, t, u, v).$$

4.3 A bijection between $I_n(0321)$ and $I_n(0312)$

We define a bijection

$$\phi : I_n(0321) \mapsto I_n(0312)$$

satisfying

$$\begin{aligned} zeros(e) &= zeros(\phi(e)) \\ dist(e) &= dist(\phi(e)) \\ repeats(e) &= repeats(\phi(e)) \\ max(e) &= max(\phi(e)) \\ lrmax(e) &= lrmax(\phi(e)) \end{aligned}$$

First, we observe the structure of any inversion sequence $e \in I_n(0321)$. For example, the plot of $e = (0, 0, 0, 2, 1, 3, 0, 7, 0, 1, 1, 3, 9, 12, 3, 14, 11, 15, 15, 1) \in I_{20}(0321)$ is given in Figure 4.5. Note that any 0321-avoiding inversion sequence has either zero elements or weakly-increasing sub-sequence between each left-to-right maxima.

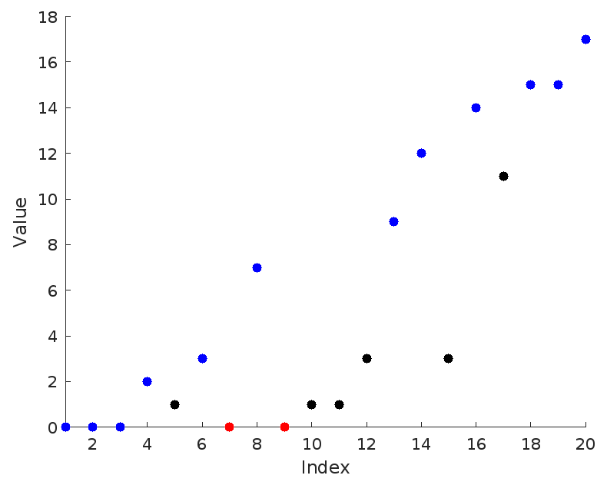


Figure 4.5: Plot of $e = (0, 0, 0, 2, 1, 3, 0, 7, 0, 1, 1, 3, 9, 12, 3, 14, 11, 15, 15, 1) \in I_{20}(0321)$. The red dots correspond to zero entries. The blue dots correspond to left-to-right maximum entries.

Then we observe the structure of any inversion sequence $e' \in I_n(0312)$. For example, the plot of $e' = (0, 1, 2, 3, 0, 0, 6, 0, 5, 8, 6, 5, 0, 4, 2, 0, 8, 14, 10, 2) \in I_{20}(0312)$ is given in Figure 4.6. Note that any 0312-avoiding inversion sequence has either zero elements or weakly-decreasing sub-sequence between each left-to-right maxima.

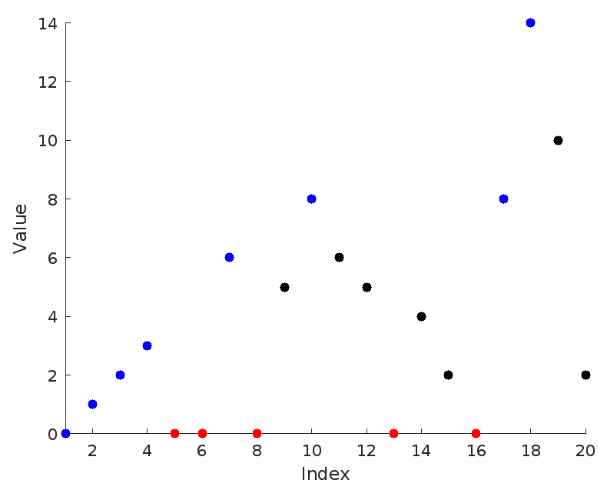


Figure 4.6: Plot of $e' = (0, 1, 2, 3, 0, 0, 6, 0, 5, 8, 6, 5, 0, 4, 2, 0, 8, 14, 10, 2) \in I_{20}(0312)$. The red dots correspond to zero entries. The blue dots correspond to left-to-right maximum entries

We define $\phi : I_n(0321) \mapsto I_n(0312)$ as follows:

Given $e \in I_n(0321)$, we define $e' = \phi(e) \in I_n(0312)$ as follows. For each $i = 1, 2, \ldots, n$, in that order,

- If e_i is a zero or left-to-right maximum entry, then $e'_i = e_i$.
- Let M be the multi-set of non-zero and non-left-to-right maximum entries of e. If $e_i \in M$, then e'_i is assigned the largest possible entry from M, ensuring that e'_i does not become a left-to-right maximum entry in e'. Then, we subsequently remove this entry from the set M.

Next we define $\beta : I_n(0312) \mapsto I_n(0321)$ as follows:

Given $e \in I_n(0312)$, we define $e' = \beta(e) \in I_n(0321)$ as follows. For each $i = 1, 2, \ldots, n$, in that order,

- If e_i is a zero or left-to-right maximum entry, then $e'_i = e_i$.
- If $e_i \in M$, then e'_i is assigned the smallest possible entry from M, and subsequently remove this entry from the set M.

Example 4.3.1. *Let $e = 0110412750 \in I_{10}(0321)$. To find $\phi(e)$, we find zero and left-to-right maximum entries and leave them unchanged as follows.*

$$0\,1\,1\,0\,4\,\_\,\_\,7\,\_\,0$$

Then, $M = \{1, 2, 5\}$.

For the first blank, we choose 2 as the largest possible entry from the set M, which does not create a new left-to-right maximum element.

$$0\,1\,1\,0\,4\,2\,\_\,7\,\_\,0$$

Then, $M = \{1, 5\}$ and we choose 1 for the second blank. We obtain

$$0\,1\,1\,0\,4\,2\,1\,7\,\_\,0$$

Then, $M = \{5\}$*. Finally, we choose* 5 *for the last blank.*

$$0\,1\,1\,0\,4\,2\,1\,7\,5\,0$$

Therefore, $\phi(0110412750) = 0110421750 \in I_{10}(0312)$*.*

Conversely, let $e = 0110421750 \in I_{10}(0312)$*. To find* $\beta(e)$*, we leave zero and left-to-right maximum entries unchanged.*

$$0\,1\,1\,0\,4\,\_\,\_\,7\,\_\,0$$

Then, $M = \{2, 1, 5\}$*. For the remain entries, we choose* 1*,* 2*, and* 5 *respectively. Then we have*

$$0\,1\,1\,0\,4\,1\,2\,7\,5\,0$$

Therefore, $\beta(0110421750) = 0110412750 \in I_{10}(0321)$*.*

Note that $\beta(\phi(e)) = e$ for all $e \in I_n(0321)$. Because we fix zero and left-to-right maximum elements when applying maps ϕ and β, their positions do not change. On the other hand, when we apply the map ϕ, we arrange the non-zero elements in the sub-sequences between two left-to-right maximum elements in decreasing order. Afterward, we apply the map β by arranging the non-zero elements in the sub-sequences between two left-to-right maximum elements in increasing order. Therefore, we obtain the same $e \in I_n(0321)$. Conversely, we have $\phi(\beta(e)) = e$ for all $e \in I_n(0312)$. We conclude that $\beta = \phi^{-1}$. Hence we have a bijection between $I_n(0321)$ and $I_n(0312)$.

Appendix A

Generating Functions

The fundamental objective in enumerative combinatorics is counting the elements of a finite set. Typically, we have an infinite collection of finite sets C_i indexed by a set I (such as non-negative integers $\mathbb{N}$), and aim to simultaneously count the number $f(i)$ of elements in each C_i.

A generating function is an object used to represent a counting function $f(i)$, generally expressed as a formal power series.

There are two types of generating functions: ordinary generating functions and exponential generating functions. When $I = \mathbb{N}$, the ordinary generating function $F(x)$ is defined as the formal power series:

$$F(x) = \sum_{n \geq 0} f_n x^n$$

while the exponential generating function $F(x)$ is the formal power series defined as,

$$F(x) = \sum_{n \geq 0} f_n \frac{x^n}{n!}.$$

In formal power series, we don't focus on assigning specific values to x or addressing questions of convergence and divergence.

In this thesis, we only focus on ordinary generating functions.

We represent the coefficient of x^n in $F(x)$ using the notation $[x^n]F(x)$, thus

$$f_n = [x^n]F(x).$$

Addition and multiplication operations can be applied to generating functions as follows.

$$\sum_{n\geq 0} a_n x^n + \sum_{n\geq 0} b_n x^n = \sum_{n\geq 0}(a_n + b_n)x^n$$

$$\left(\sum_{n\geq 0} a_n x^n\right) \cdot \left(\sum_{n\geq 0} b_n x^n\right) = \sum_{k\geq 0} c_k x^k$$

where

$$c_k = \left(\sum_{\ell=0}^{k} a_\ell b_{k-\ell}\right)$$

Proposition A.0.1. *A power series $f(x) = a_0 + a_1x + a_2x^2 + \ldots$ has a multiplicative inverse if and only if $a_0 \neq 0$.*

Proof. Let $g(x) = b_0 + b_1x + b_2x^2 + \ldots$ be a power series. Assume $g(x)$ satisfies $f(x)g(x) = 1$

$$(a_0 + a_1x + a_2x^2 + \ldots)(b_0 + b_1x + b_2x^2 + \ldots) = 1$$
$$a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \ldots = 1$$

By equating coefficients,

$$\begin{aligned} a_0b_0 &= 1 \\ a_0b_1 + a_1b_0 &= 0 \\ a_0b_2 + a_1b_1 + a_2b_0 &= 0 \\ &\vdots \end{aligned}$$

We find a unique solution for b_0 in the first equation only if $a_0 \neq 0$. □

Proposition A.0.2. *A power series $f(x) = a_1x + a_2x^2 + a_3x^3 + ...$ has a compositional inverse, denoted by $f^{<-1>}(x)$, if and only if $a_1 \neq 0$, in which case $f^{<-1>}(x)$ is unique. Moreover, if $g(x)$ satisfies either $f(g(x)) = x$ or $g(f(x)) = x$*

then $f^{<-1>}(x) = g(x)$.

Proof. Assume $g(x)$ satisfies $f(g(x)) = x$.

$$a_1(b_1x + b_2x^2 + \dots) + a_2(b_1x + b_2x^2 + \dots)^2 + \dots = x$$

By equating coefficients,

$$\begin{aligned} a_1b_1 &= 1 \\ a_1b_2 + a_2b_1^2 &= 0 \\ a_1b_3 + 2a_2b_1b_2 + a_3b_1^3 &= 0 \\ &\vdots \end{aligned}$$

We find a unique solution for b_1 in the first equation only if $a_1 \neq 0$. $\square$

Appendix B

Combinatorics Background

B.1 Fibonacci numbers

The Fibonacci numbers, denoted by F_n, are defined by the following recurrence relation:

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2 \tag{B.1}$$

with initial conditions $F_0 = 0$, $F_1 = 1$.

Fibonacci sequence appears as A000045 in the OEIS: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34 \ldots$

The generating function can be computed from the recurrence relation as follows.

Let $f(x) = \sum_{n=0}^{\infty} F_n x^n$. Then we have,

$$f(x) = x + \sum_{n=2}^{\infty} (F_{n-1} + F_{n-2}) x^n$$

$$f(x) = x + x \sum_{n=2}^{\infty} F_{n-1} x^{n-1} + x^2 \sum_{n=2}^{\infty} F_{n-2} x^{n-2}$$

$$f(x) = x + x f(x) + x^2 f(x).$$

Hence, we have

$$f(x) = \frac{x}{1 - x - x^2}. \tag{B.2}$$

The roots of $1 - x - x^2 = 0$ are

$$\frac{-1 \pm \sqrt{5}}{2}.$$

Let

$$\varphi = \frac{1 + \sqrt{5}}{2}, \quad \psi = \frac{1 - \sqrt{5}}{2}.$$

Note that, $\varphi \cdot \psi = -1$. Hence $\varphi = \dfrac{-1}{\psi}$

$$\begin{aligned} 1 - x - x^2 &= -(x + \varphi)(x + \psi) \\ &= -\varphi(\varphi^{-1}x + 1)\psi(\psi^{-1}x + 1) \\ &= (\varphi^{-1}x + 1)(\psi^{-1}x + 1) \\ &= (1 - \psi x)(1 - \varphi x). \end{aligned}$$

By using partial fractions,

$$\begin{aligned} \frac{x}{1 - x - x^2} &= \frac{x}{(1 - \varphi x)(1 - \psi x)} \\ &= \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \varphi x} - \frac{1}{1 - \psi x} \right). \end{aligned}$$

Now we have,

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \varphi x} - \frac{1}{1 - \psi x} \right) \\ &= \frac{1}{\sqrt{5}} \left(\sum_{n=0}^{\infty} \varphi^n x^n - \sum_{n=0}^{\infty} \psi^n x^n \right) \\ &= \frac{1}{\sqrt{5}} \left(\sum_{n=0}^{\infty} (\varphi^n - \psi^n) x^n \right). \end{aligned}$$

Therefore, coefficients are given by,

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

B.1.1 Odd and even indexed Fibonacci numbers

The generating function for the Fibonacci sequence is decomposed into its odd and even parts as,

$$\begin{aligned} f(x) &= f_{odd}(x) + f_{even}(x) \\ \sum_{n=0}^{\infty} F_n x^n &= \sum_{n=0}^{\infty} F_{2n+1} x^{2n+1} + \sum_{n=0}^{\infty} F_{2n} x^{2n} \end{aligned}$$

where $f_{odd}(x) = \dfrac{f(x) - f(-x)}{2}$ and $f_{even}(x) = \dfrac{f(x) + f(-x)}{2}$.

Recall that $f(x) = \dfrac{x}{1 - x - x^2}$. Then we have,

$$\begin{aligned} f_{odd} &= \frac{1}{2}\left[\frac{x}{1-x-x^2} + \frac{x}{1+x-x^2}\right] \\ &= \frac{x - x^3}{1 - 3x^2 + x^4}. \end{aligned}$$

Note that $\sum_{n=0}^{\infty} F_{2n+1} x^{2n+1} = \dfrac{x - x^3}{1 - 3x^2 + x^4}$. Then,

$$\sum_{n=0}^{\infty} F_{2n+1} x^{2n} = \frac{1 - x^2}{1 - 3x^2 + x^4}$$

and hence

$$\sum_{n=0}^{\infty} F_{2n+1} x^{n} = \frac{1 - x}{1 - 3x + x^2}.$$

Let $a_n = F_{2n+1}$. Since the generating function of odd-indexed Fibonacci numbers is

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1-x}{1-3x+x^2},$$

by multiplying both sides by $1-3x+x^2$,

$$\sum_{n=0}^{\infty} a_n x^n - 3x\sum_{n=0}^{\infty} a_n x^n + x^2\sum_{n=0}^{\infty} a_n x^n = 1-x.$$

Next, we change the limits of the summation.

$$a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n = 3a_0 x + 3\sum_{n=2}^{\infty} a_{n-1} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n + 1 - x.$$

Using $a_0 = 1$ and $a_1 = 2$, we obtain the recurrence relation for odd-index Fibonacci numbers as follows.

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} (3a_{n-1} - a_{n-2}) x^n.$$

Finally, we have

$$a_n = 3a_{n-1} - a_{n-2}.$$

Similarly, the generating function for the even-indexed Fibonacci sequence is given as follows.

$$\begin{aligned} f_{even}(x) &= \frac{1}{2}\left[\frac{x}{1-x-x^2} - \frac{x}{1+x-x^2}\right] \\ &= \frac{x^2}{1-3x^2+x^4}. \end{aligned}$$

Note that $\sum\limits_{n=0}^{\infty} F_{2n} x^{2n} = \dfrac{x^2}{1-3x^2+x^4}$. Then,

$$\sum_{n=0}^{\infty} F_{2n} x^n = \frac{x}{1-3x+x^2}.$$

Let $b_n = F_{2n}$. Then, the generating function of odd-indexed Fibonacci numbers is as follows.

$$\sum_{n=0}^{\infty} b_n x^n = \frac{x}{1-3x+x^2}.$$

B.2 Alternating permutations

Let S_n denote the set of all permutations of $[n]$. A permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in S_n$ is called alternating if $\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \ldots$. In other words, if i is even, then $\sigma_i < \sigma_{i+1}$ and if i is odd, then $\sigma_i > \sigma_{i+1}$. Similarly σ is reverse alternating if $\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \ldots$. For example, $7452613 \in S_7$ is an alternating permutation and $2716354 \in S_7$ is a reverse alternating permutation.

Let Alt_n denote the set of alternating permutations in S_n. For instance $Alt_1 = \{1\}$, $Alt_2 = \{21\}$, $Alt_3 = \{213, 312\}$, and $Alt_4 = \{2143, 3142, 3241, 4132, 4231\}$.

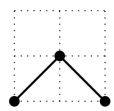
Let $E_n = |Alt_n|$. The number E_n is called the Euler or up/down number, and its sequence appears as A000111 in the OEIS: $1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, ...$

In 1994, Kuznetsov and I. M. Pak presented a bijection between alternating permutations and 0-1-2 increasing trees in [24].

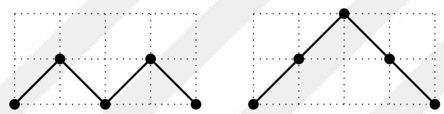
B.3 Dyck paths and Catalan numbers

Dyck paths are closely linked to several combinatorial structures, such as Catalan numbers, balanced parentheses, binary trees, and non-crossing partitions. They are widely used in various mathematical disciplines due to their elegant properties and rich combinatorial significance. A Dyck path of semi-length n is a lattice path in $\mathbb{Z}^2$ from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$; ensuring that the path remains above the x-axis.

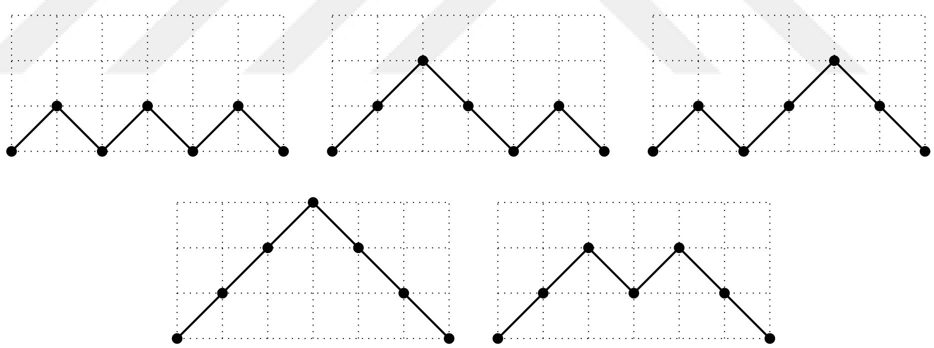
The Dyck path of semi-length one:



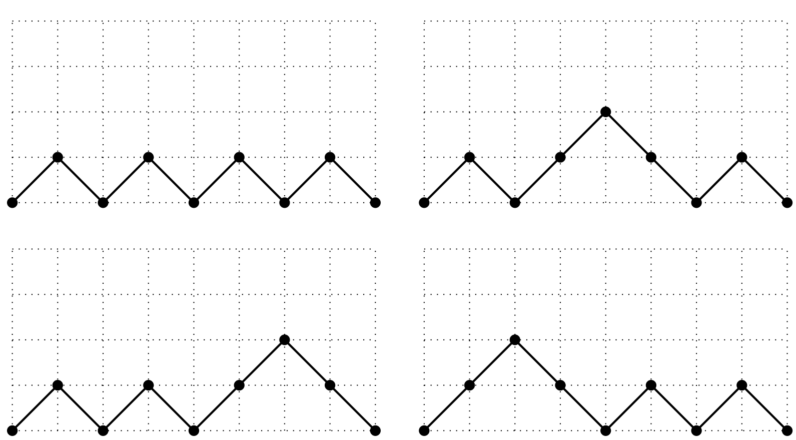
The two Dyck paths of semi-length two:

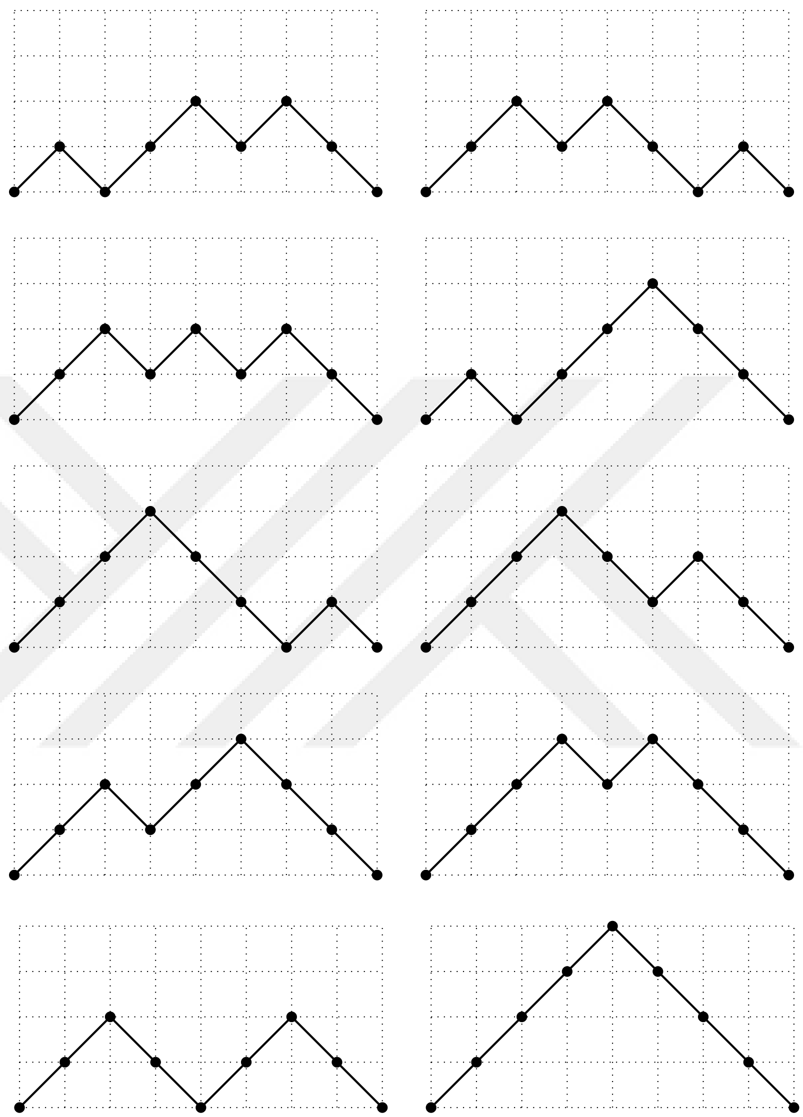


The five Dyck paths of semi-length three:



The fourteen Dyck paths of semi-length four:





The Catalan numbers C_n are a sequence of natural numbers that satisfy the recurrence relation:

$$C_n = \sum_{k=1}^{n} C_{k-1}C_{n-k} \text{ and } C_0 = 1.$$

The Catalan numbers are commonly found in various counting problems, often related to recursively defined objects like Dyck paths, Ballot sequences, binary trees, plane trees, convex polygon triangulations, and similar objects.

The sequence of Catalan numbers is given in the OEIS as A000108:

$1, 1, 2, 5, 14, 42, 132, 429, \ldots$

Let's compute the generating function, $c(x) = \sum_{n=0}^{\infty} C_n x^n$, for the Catalan numbers. By using the recurrence relation in its definition, we get

$$c(x) = C_0 + \sum_{n \geq 1} C_n x^n$$

$$c(x) = 1 + \sum_{n \geq 1} \sum_{k=1}^{n} C_{k-1} C_{n-k} x^n$$

$$c(x) = 1 + x(\sum_{n \geq 0} C_n x^n)^2$$

$$c(x) = 1 + xc(x)^2.$$

By using the quadratic formula, we find the roots of $c(x) = 1 + xc(x)^2$ as follows.

$$\frac{1 \pm \sqrt{1-4x}}{2x}.$$

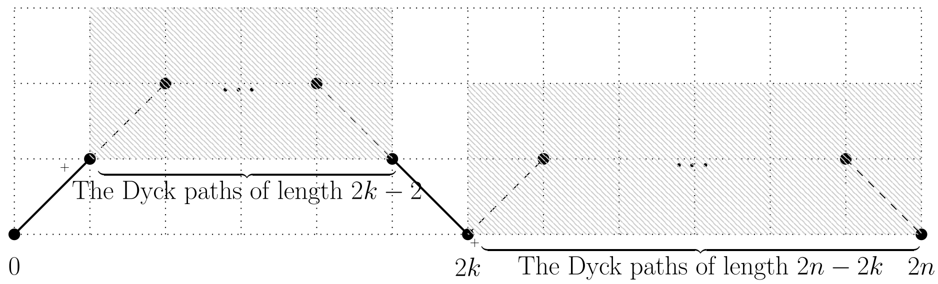
$\frac{1+\sqrt{1-4x}}{2x}$ yields negative coefficients. Thus, we choose $\frac{1-\sqrt{1-4x}}{2x}$. By using generalized binomial theorem, we obtain the following formula.

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{B.3}$$

For this section, we use [25].

Proposition B.3.1. *The Dyck paths are enumerated by Catalan numbers.*

Proof. Let D_n denote the number of Dyck paths of semi-length n. Let a Dyck path of semi-length n be given. Assume that $(2k, 0)$ is the coordinate that the path touches the x-axis for the first time.



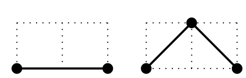
Notice that a Dyck path starts with the step $(1,1)$ and the path arrives at $(2k,0)$ by the step $(1,-1)$. In other words, the first step and the last step are fixed as $(1,1)$ and $(1,-1)$ in the interval $[0,2k]$. Therefore, there are D_{k-1} Dyck paths in $[0,2k]$ and D_{n-k} Dyck paths in $[2k,2n]$. The number of Dyck paths of semi-length n is given by the following recurrence relation:

$$D_n = \sum_{k=1}^{n} D_{k-1}D_{n-k} \ .$$

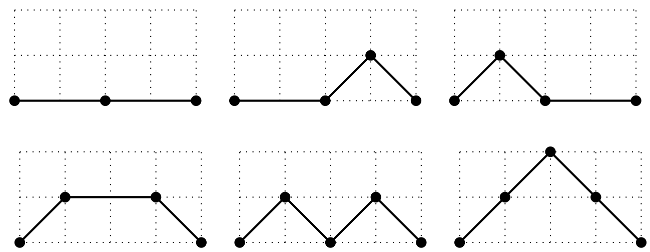
This recursion is the same as the Catalan number's recursion. Therefore $D_n = C_n$. $\square$

B.4 Schröder paths and Large Schröder numbers

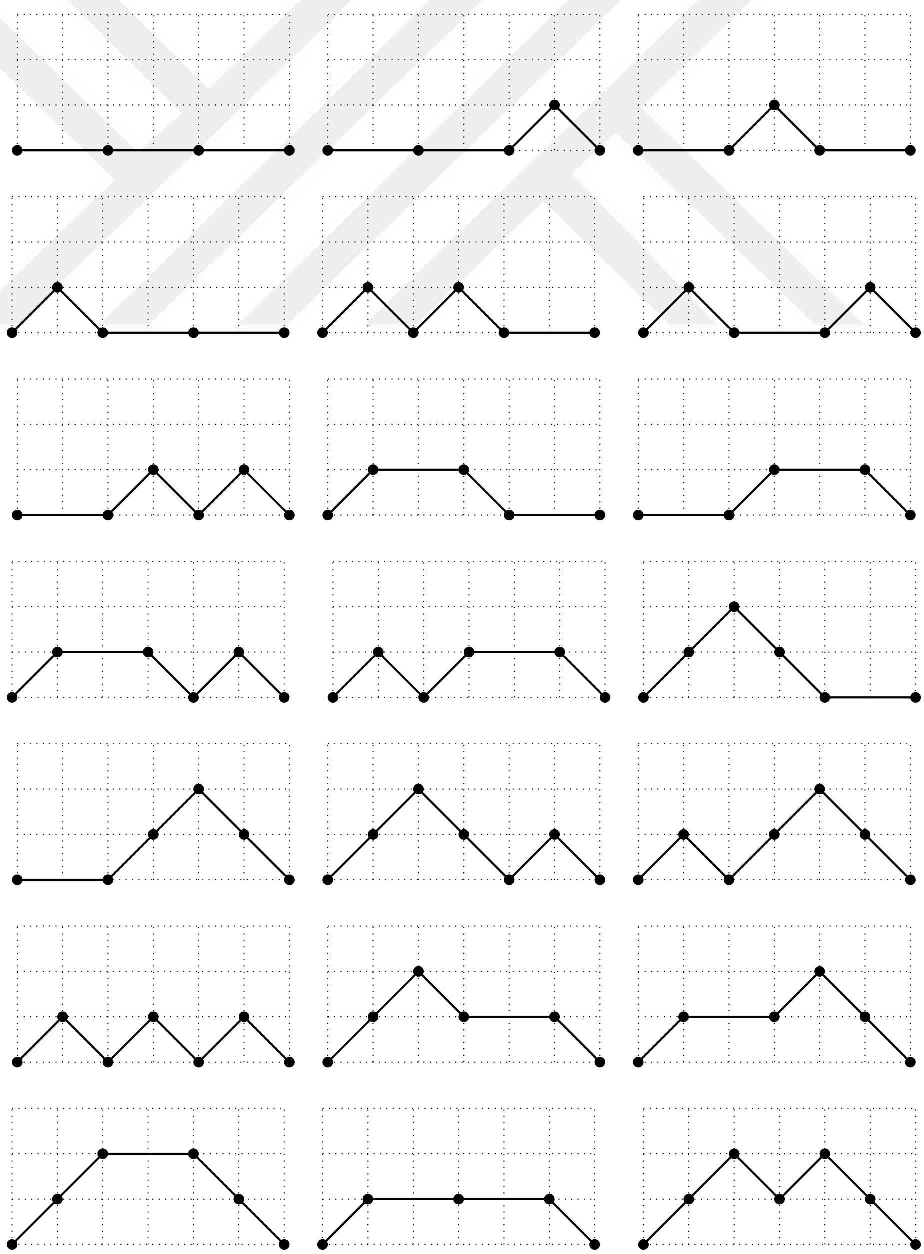
A Schröder path of semi-length n is a lattice path in $\mathbb{Z}^2$ from $(0,0)$ to $(2n,0)$ with steps $(1,1)$, $(1,-1)$ and $(2,0)$; ensuring that the path remains above the x-axis. The two Schröder paths of semi-length one:

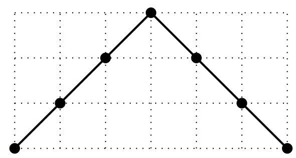


The six Schröder paths of semi-length two:



The twenty-two Schröder paths of semi-length three:



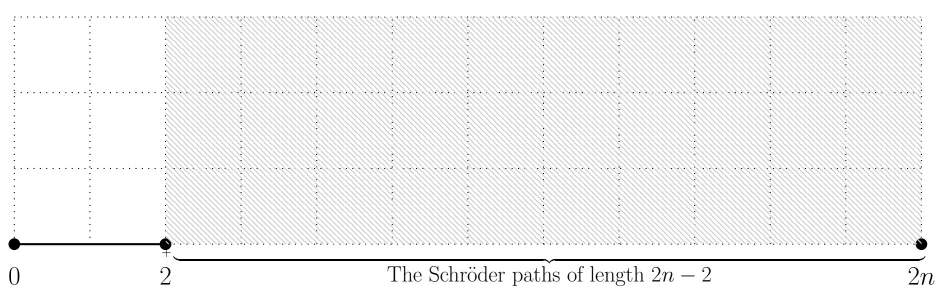


Let S_n be the number of Schröder paths of semi-length n. The sequence of S_n is called large Schröder numbers, and it appears in OEIS as A006318. A few terms of large Schröder numbers $1, 2, 6, 22, 90, 394, \ldots$

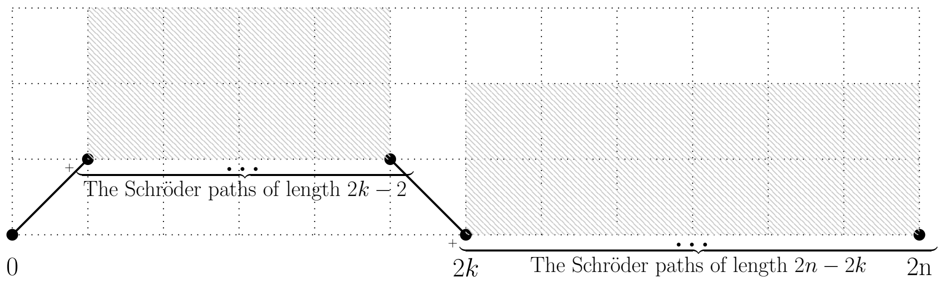
Proposition B.4.1. *Large Schröder numbers satisfy the following recurrence relation.*

$$S_n = S_{n-1} + \sum_{k=1}^{n} S_{k-1}S_{n-k} \text{ where } S_0 = 1,\ S_1 = 2.$$

Proof. Observe that a Schröder path starts with either step $(1,1)$ or step $(2,0)$. If the initial step is $(2,0)$, then the remaining part will be a Schröder path of length $2n-2$. This means, there are S_{n-1} Schröder paths in the interval $[2, 2n]$.



Assume that $(2k, 0)$ is the coordinate that the path touches the x-axis for the first time. The path arrives at $(2k, 0)$ by the step $(1,-1)$. If the initial step is $(1,-1)$, then we see Schröder paths of length $2k-2$ in the interval $[1, 2k-1]$ and Schröder paths of length $2n-2k$ in the interval $[2k, 2n]$. Hence, there are S_{k-1} choices for the paths in $[1, 2k-1]$ and S_{n-k} choices for the paths in $[2k, 2n]$.



Summarizing, we get the recurrence relation:

$$S_n = S_{n-1} + \sum_{k=1}^{n} S_{k-1} S_{n-k} \text{ where } S_0 = 1,\ S_1 = 2.$$

□

Consider generating function for the large Schröder numbers $s(x) = \sum_{n=0}^{\infty} S_n x^n$ where S_n is the n^{th} large Schröder number. Using the recurrence relation, we obtain the following functional equation.

$$\begin{aligned}
s(x) &= S_0 + \sum_{n\geq 1} S_n x^n \\
s(x) &= 1 + \sum_{n\geq 1} S_{n-1} x^n + \sum_{n\geq 1} \sum_{k=1}^{n} S_{k-1} S_{n-k} x^n \\
s(x) &= 1 + x \sum_{n\geq 1} S_{n-1} x^{n-1} + x \sum_{n\geq 1} \sum_{k=1}^{n} S_{k-1} S_{n-k} x^{n-1} \\
s(x) &= 1 + xs(x) + xs(x)^2.
\end{aligned}$$

The two solutions of the equation $xs(x)^2 + (x-1)s(x) + 1 = 0$ are

$$s_1(x) = \frac{-x + 1 + \sqrt{x^2 - 6x + 1}}{2x}$$

and

$$s_2(x) = \frac{-x + 1 - \sqrt{x^2 - 6x + 1}}{2x}.$$

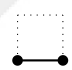
The first solution $s_1(x)$ cannot be the generating function of the large Schröder numbers as it does not give nonnegative integers. Therefore the generating function of the Schröder numbers is given by:

$$s(x) = \frac{-x + 1 - \sqrt{x^2 - 6x + 1}}{2x}.$$

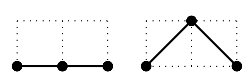
B.5 Motzkin paths and Motzkin numbers

A Motzkin path of length n is a lattice path in $\mathbb{Z}^2$ from $(0, 0)$ to $(n, 0)$ with steps $(1, 1)$, $(1, -1)$ and $(1, 0)$, ensuring that the path remains above the x-axis.

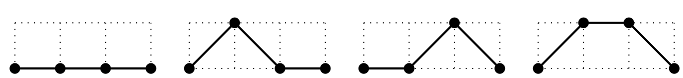
The Motzkin path of length one:



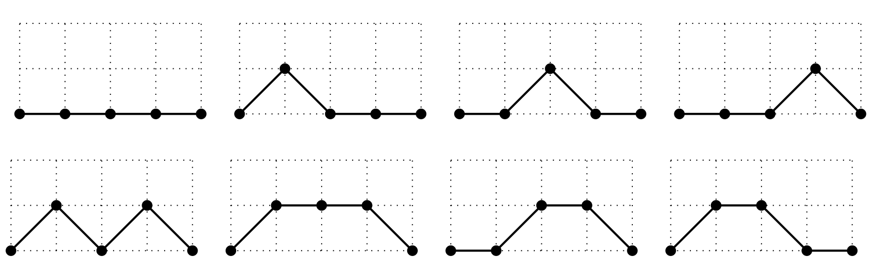
The two Motzkin paths of length two:

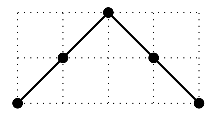


The four Motzkin paths of length three:



The nine Motzkin paths of length four:



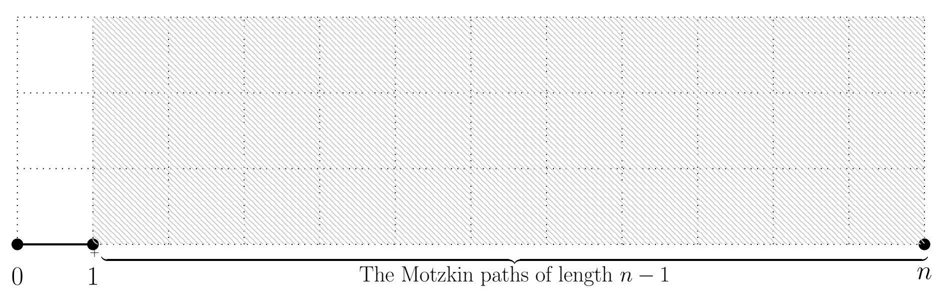


Let M_n be the number of Motzkin paths of length n. The sequence of M_n is called Motzkin numbers, and it is given in OEIS as A001006. A few terms of Motzkin numbers are $1, 1, 2, 4, 9, 21, 51, 127, \ldots$

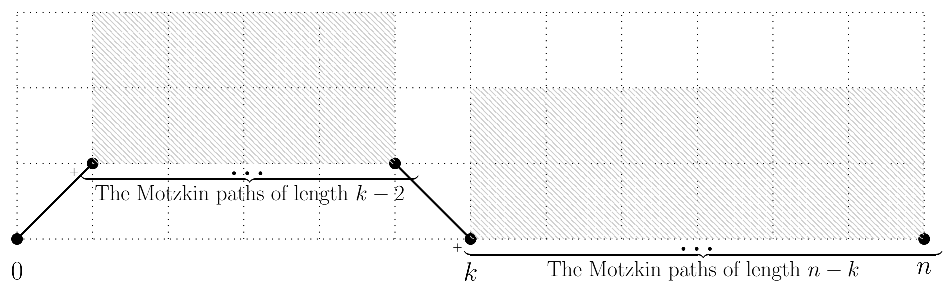
Proposition B.5.1. *Motzkin numbers satisfy the following recurrence relation.*

$$M_n = M_{n-1} + \sum_{k=2}^{n} M_{k-2}M_{n-k} \text{ where } M_0 = 1,\ M_1 = 1.$$

Proof. Note that a Motzkin path starts with either step $(1, 1)$ or step $(1, 0)$. If the path starts with step $(1, 0)$, then the remaining part will be a Motzkin path of length $n - 1$. Hence, there are M_{n-1} Motzkin paths in the interval $[1, n]$.



Suppose that $(k, 0)$ is the coordinate that the path touches the x-axis for the first time. The path arrives at $(k, 0)$ by the step $(1, -1)$. If the first step is $(1, -1)$, then there exist Motzkin paths of length $k - 2$ in the interval $[1, k - 1]$ and Motzkin paths of length $n - k$ in the interval $[k, n]$. Therefore, there are M_{k-2} choices for the paths in $[1, k - 1]$ and M_{n-k} choices for the paths in $[k, 2]$.



Then, we get the recurrence relation:

$$M_n = M_{n-1} + \sum_{k=2}^{n} M_{k-2} M_{n-k} \text{ where } M_0 = 1,\ M_1 = 1.$$

$\square$

Consider the generating function for the Motzkin numbers $m(x) = \sum_{n=0}^{\infty} M_n x^n$ where M_n is the n^{th} Motzkin number. Using the recurrence relation, we obtain the following functional equation:

$$m(x) = M_0 + \sum_{n\geq 1} M_n x^n$$

$$m(x) = 1 + \sum_{n\geq 1} M_{n-1} x^n + \sum_{n\geq 1} \sum_{k=2}^{n} M_{k-2} M_{n-k} x^n$$

$$m(x) = 1 + \sum_{n\geq 1} M_{n-1} x^n + x^2 \sum_{n\geq 1} \sum_{k=1}^{n-1} M_{k-1} M_{n-k-1} x^{n-2}$$

$$m(x) = 1 + x m(x) + x^2 m(x)^2.$$

The two solutions of the equation $x^2 m(x)^2 + (x-1)m(x) + 1 = 0$ are

$$m_1(x) = \frac{-x + 1 + \sqrt{1 - 2x - 3x^2}}{2x^2}$$

and

$$m_2(x) = \frac{-x + 1 - \sqrt{1 - 2x - 3x^2}}{2x^2}.$$

As the solution $m_1(x)$ does not give nonnegative integers, the generating function of the Motzkin numbers is given by the second solution:

$$m(x) = \frac{-x + 1 - \sqrt{1 - 2x - 3x^2}}{2x^2}.$$

B.6 Stirling numbers

Another representation of permutations is cycle notation. In cycle notation, each element is mapped to the one on its right, except for the last element within each cycle, which loops back to the first element within the same cycle.

Example B.6.1. *n-permutations with k cycles are given below for $n = 1, 2, 3$.*

n	*2-line notation*	*Cycle Notation*	k
1	$\binom{1}{1}$	(1)	1
2	$\binom{1\ 2}{1\ 2}$	(1)(2)	2
2	$\binom{1\ 2}{2\ 1}$	(12)	1
3	$\binom{1\ 2\ 3}{1\ 2\ 3}$	(1)(2)(3)	3
3	$\binom{1\ 2\ 3}{1\ 3\ 2}$	(1)(23)	2
3	$\binom{1\ 2\ 3}{2\ 1\ 3}$	(12)(3)	2
3	$\binom{1\ 2\ 3}{2\ 3\ 1}$	(123)	1
3	$\binom{1\ 2\ 3}{3\ 1\ 2}$	(132)	1
3	$\binom{1\ 2\ 3}{3\ 2\ 1}$	(13)(2)	2

Definition B.6.1. *The number of n-permutations with k cycles is called the unsigned Stirling numbers of the first kind and is denoted by $c_{n,k}$.*

The unsigned Stirling numbers of the first kind satisfy the following recurrence relation.

$$c_{n+1,k} = nc_{n,k} + c_{n,k-1}.$$

Definition B.6.2. *The number $s_{n,k} = (-1)^{n-k} c_{n,k}$ is called the Stirling number of the first kind.*

Definition B.6.3. *A partition of the set* $[n]$ *into* r *blocks is a distribution of the elements of* $[n]$ *into* r *disjoint non-empty sets, called blocks so that each element is placed into exactly one block.*

Definition B.6.4. *The number of partitions of* $[n]$ *into* k *nonempty parts is denoted by* $S_{n,k}$. *The numbers* $S_{n,k}$ *are called the Stirling numbers of the second kind.*

We set $S_{0,0} = 1$ by convention. The parts of a partition of $[n]$ are also called the blocks of that partition.

Example B.6.2. *The set* $[4]$ *has seven partitions into two nonempty parts, namely* $\{\{1,2,3\}\{4\}\}, \{\{1,2,4\}\{3\}\}, \{\{1,3,4\}\{2\}\}, \{\{2,3,4\}\{1\}\}, \{\{1,2\}\{3,4\}\}, \{\{1,3\}\{2,4\}\}, \{\{1,4\}\{2,3\}\}$. *Therefore,* $S_{4,2} = 7$.

The following table provides numerical values for $S_{n,k}$ where $0 \leq k \leq n$ and $0 \leq n \leq 7$.

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$n=0$	1							
$n=1$	0	1						
$n=2$	0	1	1					
$n=3$	0	1	3	1				
$n=4$	0	1	7	6	1			
$n=5$	0	1	15	25	10	1		
$n=6$	0	1	31	90	65	15	1	
$n=7$	0	1	63	301	350	140	21	1

This triangular array is given in OEIS as A008277.

Theorem B.6.1. *For all positive integers* $k < n$,

$$S_{n,k} = S_{n-1,k-1} + kS_{n-1,k}. \tag{B.4}$$

Proof. To obtain a partition of $[n]$ into k blocks, we can partition $[n-1]$ into k blocks and place n into any of these blocks in $kS_{n-1,k}$ ways, or we can put n in a block by itself and partition $[n-1]$ into $k-1$ blocks in $S_{n-1,k-1}$ ways. Hence, (B.4) follows. □

Definition B.6.5. *The number of all set partitions of $[n]$ into nonempty parts is denoted by B_n, and is called the n^{th} Bell number with $B_0 = 1$. Therefore, $B_n = \sum_{k=0}^{n} S_{n,k}$.*

The sequence B_n is given in OEIS as A000110. A few terms of this sequence are as follows,

$$1, 1, 2, 5, 15, 52, 203, 877 \ldots$$

B.7 Trees

We follow the definitions from [26].

Definition B.7.1. *A graph $G = (V, E)$ consists of a set V of elements called vertices and a set E of elements called edges where an edge consists of an unordered pair of vertices. We will write $V(G)$ and $E(G)$ for the vertex and edge set of G, respectively.*

Geometrically, we think of the vertices as nodes and the edges as line segments or curves joining them. An edge e connecting vertices v and w is written $e = vw$. In this case we say that edge e contains v and w, or that e has endpoints v and w. Figure B.1 shows a graph $G(v, e)$ with the vertex set $V(G) = \{v_0, v_1, v_2\}$ and the edge set $E(G) = \{e_0, e_1, e_2\}$. The edges of this graph are $e_0 = v_0v_1$, $e_1 = v_1v_2$ and $e_2 = v_2v_0$.

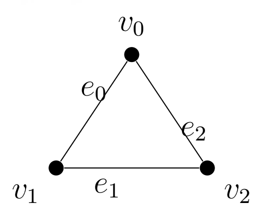


Figure B.1: A graph with 3 vertices and 3 edges.

Definition B.7.2. *A walk of length ℓ in G is a sequence of vertices $W : v_0, v_1, \ldots, v_\ell$ such that $v_{i-1}v_i \in E$ for $1 \leq i \leq \ell$. We say that the walk is from v_0 to v_ℓ, or is a $v_0 - v_\ell$ walk, or that v_0, v_ℓ are the endpoints of W.*

We call a walk W a path if all the vertices are distinct, and we usually use letters like P for paths. In particular, we will use W_n or P_n to denote a walk or a path having n vertices, respectively.

A cycle of length ℓ in G is a sequence of distinct vertices $C : v_1, v_2 \ldots v_\ell$ such that we have distinct edges $v_{i-1}v_i$ for $1 \leq i \leq \ell$, and subscripts are taken modulo ℓ so that $v_0 = v_\ell$.

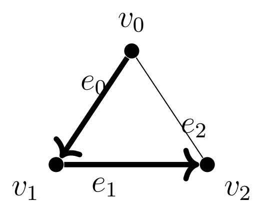


Figure B.2: A path $P_2 : v_0, v_1, v_2$.

In a cycle, the length is both the number of vertices and the number of edges. The notation C_n will be used for a cycle with n vertices and we will call this an n-cycle. We call G acyclic if it contains no cycles.

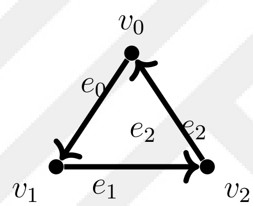


Figure B.3: A cycle $C_3 : v_0, v_1, v_2$.

Definition B.7.3. *A graph G is connected if, for every pair of vertices $v, w \in V$, there is a walk in G from v to w.*

Definition B.7.4. *A graph G is a tree if it is both connected and acyclic. We denote a tree as $T(E, V)$.*

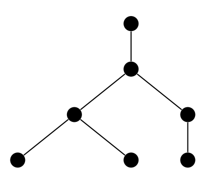


Figure B.4: A tree with 7 vertices.

A rooted tree is a tree with selected root $r \in V$. We denote a rooted tree as $T(E, V, r)$.

Definition B.7.5. *Two rooted trees $T_1(E_1, V_1, r_1)$ and $T_2(E_2, V_2, r_2)$ are isomorphic if there is a bijection between the vertex sets $\phi : V_1 \mapsto V_2$ such that*

$$(u, v) \in E_1 \iff (\phi(u), \phi(v)) \in E_2 \quad \forall u, v \in V_1$$

and $\phi(r_1) = r_2$.

An ordered tree is a tree in which the relative order of the children is fixed for each node. Otherwise, it is called an unordered tree.

An ordered tree is called a plane tree. Let T_n be the set of plane trees with n vertices. Note that, $T_1 = 1$, $T_2 = 1$, $T_3 = 2$, $T_4 = 5 \ldots$ See the figure B.5.

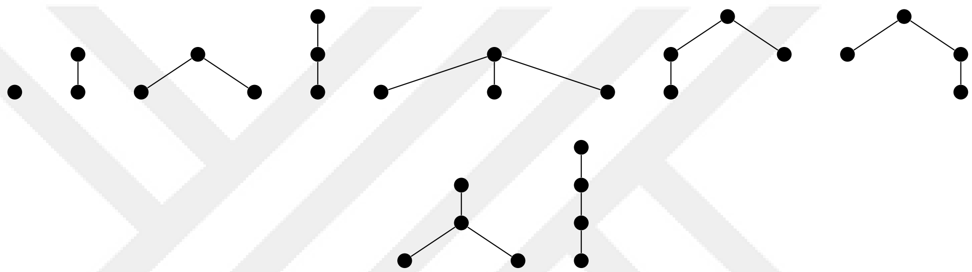


Figure B.5: The plane trees T_n for $n = 1, 2, 3, 4$.

A labeled tree is a tree in which each vertex is given a unique label.

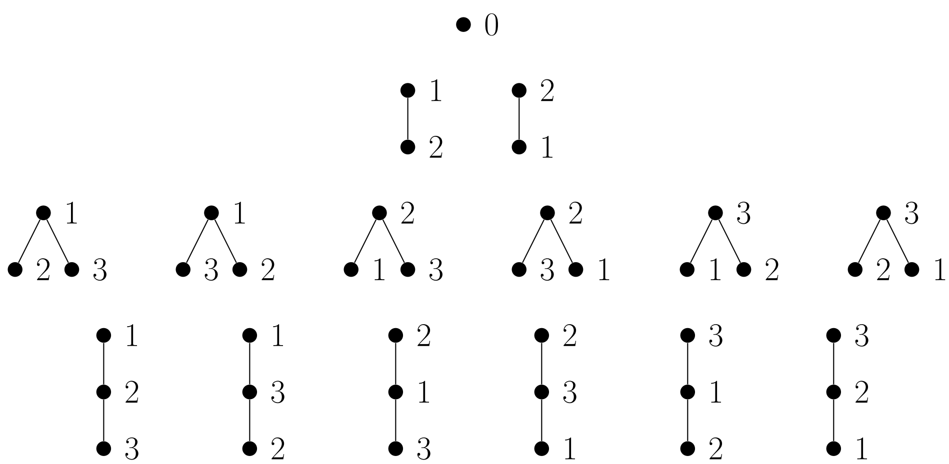


Figure B.6: The labeled plane trees with n vertices where $n = 1, 2, 3$

It is important to observe that there exist 12 labeled plane trees with 3 vertices, whereas there are 9 labeled unordered trees with the same number of vertices. The distinction between labeled plane trees and labeled unordered trees lies in the consideration of the relative order of children. In a labeled plane tree, the relative order of children is significant. This means that if we change the order

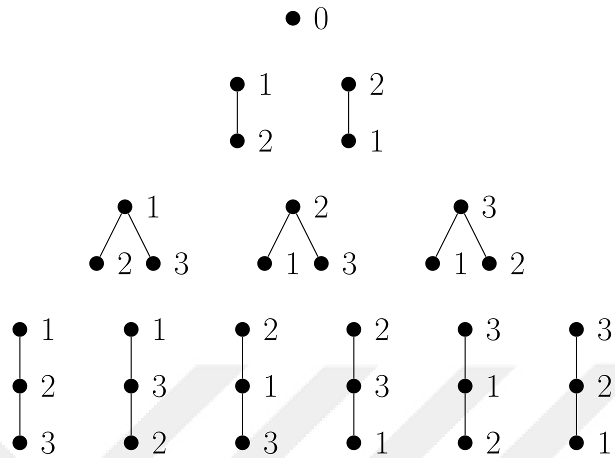
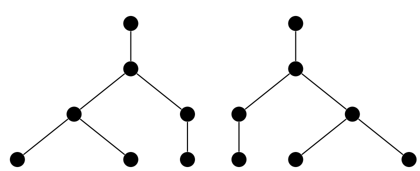


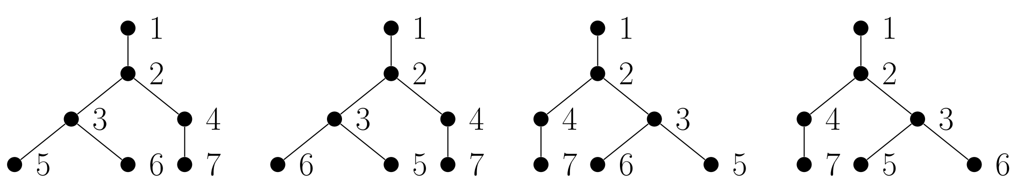
Figure B.7: The labeled unordered trees with n vertices where $n = 1, 2, 3$

of the children, we create a different plane tree. As a result, we have more possibilities for labeled plane trees with the same set of vertices. On the other hand, labeled unordered trees do not consider the relative order of children. Different arrangements of children in an unordered tree are considered equivalent and represent the same tree. Therefore, there are fewer possibilities for labeled unordered trees with the same set of vertices.

Example B.7.1. *The following unordered trees with* 7 *vertices are isomorphic.*



Example B.7.2. *The following labeled unordered trees with* 7 *vertices are isomorphic.*



Bibliography

[1] L. Pudwell, "From permutation patterns to the periodic table," *Notices Amer. Math. Soc.*, vol. 67, no. 7, pp. 994–1001, 2020.

[2] M. Albert, D. Bevan, M. Bóna, and I. Miklós, "Genomics, Pattern Avoidance, and Statistical Mechanics (Dagstuhl Seminar 18451)," *Dagstuhl Reports*, vol. 8, no. 11, pp. 1–20, 2019.

[3] S. Kitaev, *Patterns in Permutations and Words*. Springer Science & Business Media, 2011.

[4] M. Bóna, *Introduction to Enumerative Combinatorics*. Walter Rudin student series in advanced mathematics, McGraw-Hill Education, 2007.

[5] P. A. MacMahon, *Combinatory Analysis*. Cambridge University Press, 1915/1916.

[6] D. E. Knuth, *The Art of Computer Programming*. Addison-Wesley, 1969.

[7] D. Rotem, "On a correspondence between binary trees and a certain type of permutation," *Information Processing Letters*, vol. 4, no. 3, pp. 58–61, 1975.

[8] R. Simion and F. W. Schmidt, "Restricted permutations," *European Journal of Combinatorics*, vol. 6, no. 4, pp. 383–406, 1985.

[9] T. Mansour and M. Shattuck, "Pattern avoidance in inversion sequences," *Pure Mathematics and Applications*, vol. 25, no. 2, pp. 157 – 176, 2016.

[10] S. Corteel, M. A. Martinez, C. D. Savage, and M. Weselcouch, “Patterns in inversion sequences I,” *Discrete Mathematics & Theoretical Computer Science*, vol. 18, no. 2, 2015.

[11] C. Yan and Z. Lin, “Inversion sequences avoiding pairs of patterns,” *Discrete Mathematics & Theoretical Computer Science*, vol. 22, no. 1, 2020.

[12] I. Kotsireas, T. Mansour, and G. Yıldırım, “An algorithmic approach based on generating trees for enumerating pattern-avoiding inversion sequences,” *Journal of Symbolic Computation*, vol. 120, 2024.

[13] H. Prodinger, “The kernel method: A collection of examples,” *Séminaire Lotharingien de Combinatoire [electronic only]*, vol. 50, 2004.

[14] J. West, “Generating trees and the catalan and schröder numbers,” *Discrete Mathematics*, vol. 146, no. 1-3, pp. 247–262, 1995.

[15] J. West, “Generating trees and forbidden subsequences,” *Discrete Mathematics*, vol. 157, no. 1-3, pp. 363–374, 1996.

[16] E. Barcucci, A. D. Lungo, E. Pergola, and R. Pinzani, “A methodology for plane tree enumeration,” *Discrete Mathematics*, vol. 180, no. 1-3, pp. 45–64, 1998.

[17] E. P. Elena Barcucci, Alberto Del Lungo and R. Pinzani, “Eco:a methodology for the enumeration of combinatorial objects,” *Journal of Difference Equations and Applications*, vol. 5, no. 4-5, pp. 435–490, 1999.

[18] C. Banderier, P. Flajolet, D. Gardy, M. Bousquet-Melou, A. Denise, and D. Gouyou-Beauchamps, “Generating functions for generating trees,” *Discrete Mathematics*, vol. 246, no. 1-3, pp. 29–55, 2002.

[19] B. Testart, “Inversion sequences avoiding the pattern 010.” `https://arxiv.org/abs/2212.07222`, 2023.

[20] T. Mansour and G. Yıldırım, “Inversion sequences avoiding 021 and another pattern of length four,” *Discrete Mathematics & Theoretical Computer Science*, vol. vol. 25:2, Nov. 2023.

[21] L. Hong and R. Li, "Length-four pattern avoidance in inversion sequences," *The Electronic Journal of Combinatorics*, vol. 29, Dec. 2022.

[22] D. Levin and Y. Peres, *Markov Chains and Mixing Times*. MBK, American Mathematical Society, 2017.

[23] N. Madras and H. Liu, "Random pattern-avoiding permutations," in *Algorithmic probability and combinatorics*, vol. 520 of *Contemp. Math.*, pp. 173–194, Amer. Math. Soc., Providence, RI, 2010.

[24] A. Kuznetsov, I. Pak, and A. Postnikov, "Increasing trees and alternating permutations," *Russian Mathematical Surveys*, vol. 49, pp. 79–114, 1994.

[25] R. P. Stanley, *Enumerative Combinatorics*. Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2 ed., 2011.

[26] B. Sagan, *Combinatorics: The Art of Counting*. Graduate Studies in Mathematics, American Mathematical Society, 2020.