PRESERVATION OF IMPLEMENTABILITY UNDER ALGEBRAIC OPERATIONS

A Master's Thesis

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PRESERVATION OF IMPLEMENTABILITY UNDER ALGEBRAIC OPERATIONS

Graduate School of Economics and Social Sciences
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by

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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

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ABSTRACT

PRESERVATION OF IMPLEMENTABILITY UNDER ALGEBRAIC OPERATIONS

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In this thesis, we investigate whether union and intersection preserve Nash and subgame perfect implementability. Nash implementability is known to be preserved under union. Here we first show that, under some reasonably mild assumptions, Nash implementability is also preserved under intersection. The conjunction of these two results yields an almost lattice-like structure for Nash implementable social choice rules. Next, we carry over these results to subgame perfect implementability by employing similar arguments. Finally, based on the fact that Nash implementable social choice rules are closed under union, we provide a new characterization of Nash implementability, which also exemplifies the potential use of our findings for further research.

Keywords: Social Choice Theory, Nash Implementability, Subgame Perfect Equilibrium Implementability, Characterization of Nash Implementability.

ÖZET

UYGULANABİLİRLİĞİN CEBİRSEL İŞLEMLER ALTINDA KORUNMASI

DOĞAN, SERHAT

Yüksek Lisans, Ekonomi Bölümü

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Bu tezde Nash ve alt oyun yetkin uygulanabilirliğin birleşim ve kesişim altında korunup korunmadığını inceliyoruz. Nash uygulanabilirliğin birleşim altında korunduğu bilinmektedir. Bu çalışmada önce Nash uygulanabilirliğin bazı makul varsayımlarla kesişim altında da korunduğunu gösteriyoruz. Bu iki sonucu birlikte kullanarak, Nash uygulanabilir sosyal seçme kurallarının kafesimsi bir yapıya sahip olduğunu belirliyoruz. Daha sonra benzer yöntemlerle bu sonuçların alt oyun yetkin uygulanabilirlik için de geçerli olduğunu gösteriyoruz. Son olarak, aynı zamanda bu çalışmadaki bulgularımızın baska ne tür araştırmalarda kullanılabileceğini de örnekleyecek biçimde, Nash uygulanabilir sosyal seçme kurallarının birleşim altında kapalı olmalarını kullanarak, Nash uygulanabilirliğin yeni bir karakterizasyonunu elde ediyoruz.

Anahtar Kelimeler: Sosyal Seçim Kuramı, Nash Uygulanabilirlik, Alt Oyun Yetkin Uygulanabilirlik, Nash Uygulanabilirliğin Karakterizasyonu.

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CHAPTER 1

INTRODUCTION

Although the foundations of implementation theory were laid roughly forty years ago, the study of algebraic structures pertaining to sets of implementable rules seems not to have attracted much attention so far. It is only rather recently that the preservation of implementability according to certain well known solution concepts under union and intersection of social choice rules has been looked into. Given a solution concept s, in case sets of s-implementable social choice rules turn out to exhibit a nice algebraic structure, this would shed some further light on s-implementability from a different angle, possibly leading to new characterizations. A full treatment of this problem would thus also require that one looks into what binary operations on sets of social choice rules are more relevant in the context of s- implementability for a given solution concept s.

Among the few studies concerning the implementability of the union or intersection of implementable social choice rules, Benoit et al. (2007) are the first one that we are aware of. They show that the union of two Nash-implementable social choice correspondences is also Nash-implementable, while the same need not be true for intersection. Benoit et al. (2007), however, leave the question concerning intersection partially unsolved for the case, where agents are not allowed to have indifferences. Kutlu (2008) fills in this gap by giving an example showing that , even when indifferences are not allowed, the intersection of two Nash-implementable social choice correspondences need not be Nash-implementable.

As for dominant strategy implementability, counter examples provided by Benoit et al. (2007) illustrate that it is not preserved under either union or intersection. There are some other studies concerned with preservation of properties other than implementability as Maskin monotonicity under union or intersection of social choice rules. Kara and Snmez (1997), on the other hand, consider matching rules from a similar angle and show that the intersection of Pareto optimal and individually rational rules need not be Nash implementable, which may be regarded as a forerunner of the non-preservation of Nash implementability under intersection of social choice rules.

This thesis focuses on positive results concerning preservation of implementability under intersection and union, which also are shown to lead to a new characterization of Nash implementability. Chapter 2 specifies certain conditions which suffice for preservation of Nash implementability under intersection. More specifically, it is shown that the intersection of two nonempty-valued, Pareto optimal, Nash implementable social choice rules with Pareto-rich domains is again Nash implementabie. In chapter 3, subgame perfect implementability is shown to be always preserved under union, while preservation under intersection is obtained when one confines himself to nonempty-valued Pareto optimal social choice rules. In chapter 4, we provide a constructive characterization of Nash implementability. The yardstick this characterization introduces for Nash implementability is especially easy to use for social choice rules whose critical profiles can easily be found. Chapter 5 closes the thesis with some concluding remarks.

CHAPTER 2

NASH IMPLEMENTABILITY

2.1 Preliminaries

Throughout the thesis, A will denote the finite set of alternatives and N will denote the finite set of individuals. A linear order is a transitive, anti-symmetric, and complete binary relation. $\mathcal{L}(A)$ is the set of all linear orders on A. An element of $\mathcal{L}(A)^N$ will be called a preference profile. A social choice rule (SCR) is a function $F : \mathcal{L}(A)^N \to 2^A$. For any $a \in A$ and $P \in \mathcal{L}(A)$ the lower contour set of a at P is the set of elements which are at most as good as a, that is $L(P, a) := \{b \in A : aPb\}$.

An SCR F is Maskin-monotonic if for any $R, R' \in \mathcal{L}(A)^N$ and for any $a \in A$: $[a \in F(R)$, and for all $i \in N$ $L(R_i, a) \subset L(R'_i, a)$ imply $a \in F(R')]$.

An SCR F is Pareto optimal if F chooses none of the Pareto dominated alternatives in R.

Consider any abstract set M_i for each $i \in N$, called the message space of the agent i. $M = \prod\limits_{i \in N} M_i$ is called the message space. Take a function $g : M \to A$, called the outcome function. Then the pair (M, g) is called a mechanism. Given a mechanism $\Gamma = (M, g)$ and a preference profile $R \in \mathcal{L}(A)^N$. Then (M, g, R) defines a normal form game, where each agent have strategy space M_i and for any joint strategy $m \in M$ preferences are induced by g and R.

A solution concept σ is defined as $\sigma(M, g) : \mathcal{L}(A)^N \to 2^M$. Given an SCR F, a solution concept σ and a mechanism $\Gamma = (M, g)$ is said to $\sigma-$implement F if for any $R \in \mathcal{L}(A)^N$, one has $g(\sigma((M, g, R)) = F(R)$. An SCR F is said to be $\sigma-$ implementable if there exists a mechanism which $\sigma-$implements F. And finally let σ_{NE} denote the Nash Equilibria.

2.2 Intersection For Nash-Implementability

Benoit et al. (2007) have shown Nash-Implementability is closed under union when alternative set has more than two elements. In this section we will prove that intersection is also preserved under some reasonable assumptions. First we need to define a richness condition for our domain to have this result.

Definition. Let $D \subset \mathcal{L}(A)^N$, D satisfies Pareto Richness condition if $R \in D$ and aR_ib for each $i \in N$, then $\exists R' \in D$ where $L(R_i, a) \subset L(R'_i, a)$ and $\forall i \in N$, $L(R'_i, a) = L(R'_i, b) \cup \{a\}$.

Proposition 1. *Let $|N| \geq 3$ and F and G be two non-empty valued Pareto optimal Nash-Implementable SCRs, which are defined on a Pareto rich domain, then $F \cap G$ is Nash-Implementable as well.*

Proof. First let's introduce the mechanism which supposedly implements $F \cap G$ then prove it. Since F and G are Nash-Implementable then there exist (M^F, g^F) and (M^G, g^G) mechanisms which Nash-Implements F and G respectively. For each $i \in N$ let's define $M_i = M_i^F \times M_i^G \times \mathbb{R}$ and $M = \prod_{i \in N} M_i$. So for any $m \in M$, $m = ((m_1^F, m_1^G, r_1), (m_2^F, m_2^G, r_2)...(m_n^F, m_n^G, r_n))$. Let's define g as follows,

$$g(m) = \begin{cases} g^F(m^F), & \text{if } \Sigma_{i \in N} r_i \geq 0 \\ g^G(m^G), & \text{if } \Sigma_{i \in N} r_i < 0 \end{cases}$$

Now let's prove (M, g) Nash-Implements $F \cap G$.

Let $m \in M$ be a Nash equilibrium then either $g(m) = g^F(m^F)$ or $g(m) = g^G(m^G)$. Let $g(m) = g^F(m^F)$, if m^F is not a Nash equilibrium in (M^F, g^F) then one of the agents can deviate and get advantageous, since both F and G are non-empty valued there must exist a Nash equilibrium, therefore m^F must be a Nash equilibrium. Moreover is one agent prefers $g^G(m^G)$ over $g^F(m^F)$ then just by changing r_i one can make $g^G(m^G)$ chosen. Thus for each agent $g^F(m^F)$ more preferable than $g^G(m^G)$. Besides any single agent deviation from m^G must result some outcome which is not better than $g^F(m^F)$ too. Let R be the the original preferences of agents. If $g^F(m^F) \neq g^G(m^G)$ then clearly $\forall i \in A$, $g^F(m^F) R_i g^G(m^G)$ besides we know that if $\tilde{m}^G = (\tilde{m}^G_i, m^G_{-i})$ then $g^F(m^F) R_i g^G(\tilde{m}^G)$ since m is a Nash solution. We know D is a Pareto rich domain therefore there exist R' where $L(R_i, g^F(m^F)) \subset L(R'_i, g^F(m^F))$ and $\forall i \in N$, $L(R'_i, g^F(m^F)) = L(R'_i, g^G(m^G)) \cup \{g^F(m^F)\}$. If the original preferences was R' then obviously m would become a Nash solution still, and $g^G(m^G) \in G(R)$ as well, which contradicts the fact that G is Pareto optimal since $g^G(m^G)$ is Pareto dominated by $g^F(m^F)$. Thus $g^F(m^F) = g^G(m^G)$ then clearly m^G is a Nash solution as well which means $g^G(m^G) \in G(R)$. Since we already have $g^F(m^F) \in F(R)$, $g^F(m^F) = g^G(m^G) \in F(R) \cap G(R)$. Therefore we have proven $g(\sigma_{NE}(M, g, R)) \subset F \cap G(R)$.

For any $a \in F \cap G(R)$ there exist $\bar{m}^F$,$\bar{m}^G$ Nash solutions which both have outcome a. If $m_i = (\bar{m}^F, \bar{m}^G, .)$ for each $i \in N$, m is clearly a Nash solution which gives outcome a regardless of r_is. Thus obviously $F \cap G(R) \subset g(\sigma_{NE}(M, g, R))$. Finally we have reached $F \cap G(R) = g(\sigma_{NE}(M, g, R))$ which concludes the proof. □

We already have a favorable result regarding the Preservation of Nash-Implementability under union. By supplementing the above result we can deduce the set of all Pareto optimal Nash-Implementable SCRs defined on a Pareto rich domain almost forms a lattice structure. We just have nonemptyness condition in the way. Besides, that set is not very scarce, for instance on the full domain the rule which selects the top choices of a particular agent set which has more than one element satisfies these properties. Such rules are Maskin-monotonic and satisfies No-Veto power, thus Nash-Implementable, they are Pareto optimal too.

Apart from above SCRs we can also find another family of such SCRs. In order to define a rule in that family let us first define the mechanism then show the corresponding SCR satisfy our conditions. Let R be a preference profile where a is not Pareto dominated. So for $i \in N$ let $M_i = (L(R_i, a), \mathbb{R})$, $M = \prod_{i \in N} M_i$ and let the outcome function be the choice of the agent with highest real number, with the tie breaker as lowest indiced agent. Now let's prove that the function, which that mechanism Nash-Implements, satisfies our conditions.

That mechanism basically gives any agent the ability to choose any alternative among $L(R_i, a)$, by choosing sufficiently large real numbers. Observe that $\bigcup_{i \in N} L(R_i, a) = A$ since a is not Pareto dominated. Now let's prove $F(R) = g(\sigma_{NE}(M, g, R))$, is Pareto optimal. If for some R, there exist $b, c \in A$ which b Pareto dominates c and $c \in F(R)$ then there exist a Nash solution $m \in M$ where $g(m) = c$, but by above observation we know that for some $i \in N$, $b \in L(R_i, a)$ so agent i can make b chosen, where himself and all others benefit, thus there cannot exist a Nash solutions which has outcome c. Therefore the induced function F is Pareto optimal and Nash-Implementable.

There are a few remarks about these rules, worths to be denoted.

Remark 1. Let F belongs to the family defined above, then for any R, $|F(R)| \leq 1$.

Remark 2. Let F be the rule which is induced by R and a then, $\forall b \in A$, $b \in F(R')$ if and only if for every $i \in N$, $L(R_i, a) \subset L(R'_i, b)$.

Clearly that family is pretty large, and also any function which result from the any union of these rules is also a Pareto optimal and Nash-Implementable.

Moreover any intersection of these SCRs is also Nash-Implementable even though they are not non-empty valued for the most of the cases.That makes the set of all Pareto optimal and Nash-Implementable SCRs is not scarce at all. One can ask whether any Pareto optimal Maskin-monotonic SCR is Nash-Implementable or not. The answer to that question is not necessarily. The following SCR is both Pareto optimal and Maskin-monotonic but not Nash-Implementable.

Example 1. Let F^a be the SCR which is defined on full domain $\mathcal{L}(A)^N$ and satisfies the followings,

$$
\begin{array}{ll}
(i) & a \in F^a(R) \text{ iff a is top choice for every agent in } R \\
(ii) & b \neq a, b \in F^a(R) \text{ iff b is Pareto optimal in } R
\end{array}
$$

That SCR is not empty valued at any profile. Now let's show it cannot be Nash-Implementable via any mechanism.

Proof. Assume F^a can be Nash-Implementable via mechanism (M, g), and let the original preferences of individuals be R which is represented below.

R_1	R_2	$\dots$	R_n
$\vdots$	a	a	a
$\vdots$	b	b	b
b	$\vdots$	$\dots$	$\vdots$
a	$\vdots$	$\dots$	$\vdots$

Now, since in that case b is Pareto optimal there should exist $m \in M$ where $g(m) = b$ and m is a Nash-equilibrium. Now in this case *agent* 1 cannot deviate anything except a which would prevent m to be a Nash-equilibrium. And there should exist $m'_1 \in M_1$ which satisfies $g(m'_1, m_{-1}) = a$. Assume not, then if *agent* 1 changes the positions of a and b, b still would be a Nash-equilibrium even though it is not Pareto optimal. That contradicts the fact that (M, g) Nash-Implements F^a. But now in this case (m'_1, m_{-1}) become a

Nash-Equilibrium when *agent* 1 changes the positions of a and b. But F^a chooses a only unanimity case. Therefore we got contradiction in each case, which leads F^a cannot be Nash-Implementable via any mechanism. $\square$

So we have acquired the set of all Pareto optimal Nash-Implementable SCRs is strictly smaller than set of all Pareto optimal Maskin-monotonic SCRs.

2.3 Union For Two Agents

In this section we will prove that for two agents Nash-Implementability is not preserved under union, on the contrary of above cases, by giving a counter example. This result prevents us from searching a basis for the Nash-Implementable choice rules for two agents.

Example 2. Let $N = \{1, 2\}$, $A = \{a, b\}$, Let

$$R^1 = \begin{pmatrix} a & a \\ b & b \end{pmatrix}, R^2 = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, R^3 = \begin{pmatrix} b & a \\ a & b \end{pmatrix}, R^4 = \begin{pmatrix} b & b \\ a & a \end{pmatrix}.$$

Now Let's define $F_1(R^1) = a, F_1(R^2) = a, F_1(R^3) = \emptyset, F_1(R^4) = b$ and $F_2(R^1) = a, F_2(R^2) = \emptyset, F_2(R^3) = a, F_1(R^4) = b$. We are going to prove that F_1 and F_2 are Nash-Implementable while $F_1 \cup F_2$ is not. When $M_1 = \{m_1^1, m_1^2, m_1^3\}$ and $M_2 = \{m_2^1, m_2^2, m_2^3\}$, the following form game Nash-Implements F_1, It can be verified easily. Since the roles of agent 1 and agent 2 are reversed in F_2, we can use a similar game for implementing F_2 just by interchanging their roles.

	m_1^1	m_1^2	m_1^3
m_2^1	a	a	b
m_2^2	a	b	b
m_2^3	a	b	a

Thus F_1 and F_2 are both Nash-Implementable but we can deduce that

$F = F_1 \cup F_2$ is not Nash-Implementable, since it is a well known 2 agent SCR which is not Nash-Implementable. Therefore union does not preserve Nash-Implementability when there is only two agents.

CHAPTER 3

SUBGAME PERFECT EQUILIBRIUM IMPLEMENTABILITY

Another solution concept which is widely investigated is Subgame Perfect Equilibrium (SPE). In the literature several characterizations of SPE implementability is given. But we will consider whether it is preserved under union and intersection or not. We will be able to find out positive results for both operations. It turns out union preserves SPE implementability always, while intersection needs some assumptions as in previous chapters.

3.1 Preliminaries

We will confine our attention to mechanisms which are representable by *extensive form games*. A mechanism is a triplet $\Gamma = (Y, M, g)$ where Y denotes the set of histories, M is message space and g is the outcome function. $M = \Pi_{i=1}^{n} M_i$ and for any $i \in N$, $M_i = \Pi_{y \in Y} M_i^y$ where for any $y \in Y$ if y is non-terminal history, $\exists i \in N$ such that $M_i^y \neq \{\emptyset\}$ and if y is a terminal history, $\forall i \in N$, $M_i^y = \{\emptyset\}$. Let $\overline{Y}$ denote the set of all terminal histories and $m^y = (m_1^y, \ldots, m_n^y)$ denote the message vector at history y. There is an initial history $\emptyset \in Y$, moreover histories and messages are tied by some property that for any non-terminal history y and a message vector at history y, m^y, we have $(y, m^y) = y'$ where $y' \in Y$. And by that property each history can be represented by a finite sequence of message vectors, i.e. $(\emptyset, m^1, m^2, \ldots m^k) = y$. As Γ conatins finitely many stages, for any $m \in M$, these message vectors lead to a unique terminal history from inital history. Sometimes we call the terminal history a path.

The outcome function $g : M \to A$ specifies an outcome for each terminal history, hence for each strategy profile. Given a preference profile R, the pair (Γ, R) constitutes an extensive form game.

By construction of Γ for any $y \in Y$ and $m \in M$, there exist a unique path which leads y to a terminal history by using message vectors of m. Let $m : y$ denote that unique terminal history. The usual terminal history induced by m is $m : \emptyset$. We call m a SPE if for all $i \in N$, $y \in Y$ and $m'_i \in M_i$, we have $g(m : y) R_i g((m'_i, m_{-i}) : y)$. Let $\sigma_{SPE}(\Gamma, R)$ denote the set of all subgame perfect equilibria of (Γ, R).

An SCR F is SPE implementable if there exist a mechanism Γ, such that $F(R) = g(\sigma_{SPE}(\Gamma, R))$ for any R on the domain of F.

3.2 Results

Proposition 2. *If F and G are two non-empty valued Pareto optimal SPE implementable SCRs then $F \cap G$ is also SPE implementable.*

Proof. In order to prove that proposition we will construct a mechanism which implements $F \cap G$. Let $\Gamma^F = (Y^F, M^F, g^F)$ and $\Gamma^G = (Y^G, M^G, g^G)$ be the mechanisms which implement F and G respectively. Now define $Y = \{\emptyset\} \sqcup Y^F \sqcup Y^G$, where $\sqcup$ denotes disjoint union. Thus initial history of Y^F and Y^G are included in Y as $\emptyset^F$ and $\emptyset^G$. This basically introduces a new initial history and from that node game will continue either on Y^F or on Y^G. The message set for every history except $\emptyset$ is inherited from previous mechanisms. Let $M_i = \mathbb{R} \cup M_i^F \cup M_i^G$ and $M = \Pi_{i=1}^n M_i$. For initial history each agent will send a real number simultaneously as message and if the sum of these numbers is positive game will continue from $\emptyset^F$, otherwise $\emptyset^G$ will be the following history. After that movement everything occurs as in original mechanisms, and outcome function g attains what g^F attains to any terminal point which belongs to Y^F, similarly g^G to terminal points of Y^G. And finally $\Gamma = (Y, M, g)$.

Now Let's prove that this mechanism SPE implements $F \cap G$. For any R, if $m \in M$ is a SPE of game (Γ, R), then clearly m^F and m^G both should be SPE of Γ^F and Γ^G respectively, by definition of SPE and construction of Γ. Thus $g(m : \emptyset^F) \in F(R)$ and $g(m : \emptyset^G) \in G(R)$, if these two outcomes are not same then clearly there there should exist two agents who have opposite orders for these two outcomes since both F and G does not pick Pareto dominated alternatives, which will lead a contradiction with m being a SPE since these two agents can deviate from one two another just by changing their messages for the initial history. Thus $g(m : \emptyset^F) = g(m : \emptyset^G) \in (F \cap G)(R)$ which results $g(\sigma_{SPE}(\Gamma, R)) \subset (F \cap G)(R)$. And for any $a \in (F \cap G)(R)$ let m^F and m^G be SPEs which leads a, thus for any $r_i \in \mathbb{R}$, $m = (r_i, m_i^F, M_i^G)$ is trivially a SPE which leads a. Therefore we got $(F \cap G)(R) = g(\sigma_{SPE}(\Gamma, R))$, which concludes the proof. □

Proposition 3. *If F and G are two SPE implementable SCRs then $F \cup G$ is also SPE implementable.*

Proof. Again as in previous cases we will construct a mechanism which SPE implements $F \cup G$. Let $\Gamma^F, \Gamma^G, Y, M, g$ be defined as above. But we will define the movement at $\emptyset$ different from before. If at most one of the agents sends a positive number as message then continue from $\emptyset^F$, otherwise go to $\emptyset^G$.

Now Let's prove that mechanism SPE implements $F \cup G$. Similar as above if m is a SPE then we should have $g(m : \emptyset^F) \in F(R)$ and $g(m : \emptyset^G) \in G(R)$, and since $g(m)$ is either $g(m : \emptyset^F)$ or $g(m : \emptyset^G)$ that means $g(m) \in (F \cup G)(R)$ therefore, $g(\sigma_{SPE}(\Gamma, R)) \subset (F \cup G)(R)$. And for any $a \in (F \cup G)(R)$ WLOG assume $a \in F(R)$, let m^F and m^G be SPEs of Γ^F and Γ^G, where $g^F(m^F) = a$.

If all of the agents send a negative real number as a message for initial history, that is $m_i = (r_i, m_i^F, m_i^G)$ where $r_i < 0$, then obviously $g(m) = a$ and m will satisfy all necessary conditions of being a SPE. Thus we can deduce $(F \cup G)(R) = g(\sigma_{SPE}(\Gamma, R))$, which means $F \cup G$ is SPE implementable. $\square$

CHAPTER 4

A CHARACTERIZATION OF NASH IMPLEMENTABILITY

As mentioned before, by Benoit et al. (2007), it is known that if $|N| \geq 3$, union of Nash-Implementable SCRs is also Nash-Implementable. In this chapter we will construct a basis for all of the Nash-Implementable choice rules which will also help us to construct a characterization for Nash-Implementability. In order to accomplish this we will define a family of SCRs which will be called "*Critical-Profile Rules Family*"

Definition. For any $R \in \mathcal{L}(A)^N$, $a \in A$, let $B = \cup_{i=1}^{n} L(R_i, a)$ and $B \subset C \subset A$, then $F_{(a,R,C)}$ is defined as follows.

(i) $a \in F_{(a,R,C)}(R')$ iff R' is a Maskin-monotonic improvement of R w.r.t a

(ii) $b \in F_{(a,R,C)}(R')$ iff $\exists i \in N$ s.t. $b \in L(R_i, a) \subset L(R'_i, b)$ and $\forall j \neq i, C \subset L(R_j, b)$

(iii) $c \in F_{(a,R,C)}(R')$ iff $c \in (C \setminus B)$ and $\forall i \in N, C \subset L(R_i, c)$

This choice rule can attain two elements only just in the case when first and third conditions satisfied at the same time. Otherwise it can attain at most one value for all preference profiles.

Proposition 4. *If $|N| \geq 3$, $F_{(a,R,C)}$ is Nash-Implementable.*

Proof. We can check Nash-Implementability of that SCR by Moore-Repullo's characterization theorem, but we can also prove it by constructing a mechanism. Let $M_i = \{a, 0\} \times SL(R_i, a) \times C \times \mathbb{R}$ where $SL(R_i, a)$ denote the strict lower contour set (if empty let it be $\{\emptyset\}$), with generic element $m_i = (a_i, b_i, c_i, r_i)$. $M = \Pi_{i=1}^n M_i$ and $g : M \to A$ as follows,

$$g(m) = \begin{cases} a & \text{if } \forall i \in N, a_i = a \\ b_i & \text{if } a_i = 0 \text{ and } \forall j \in N \setminus \{i\}, a_j = a \\ c_i & \text{o.w. and } \forall j \in N \setminus \{i\}, r_i > r_j \end{cases}$$

Now let's find out all Nash Equilibria of that mechanism under any R. Let m be a Nash Equilibrium, if at m at least two agents pick 0 as a_i instead of a then all agents should have the same maximal element c_i over B, otherwise at least one would deviate. If exactly one agent pick 0 as a_i and b_i is chosen, then all other agents should have b_i as the maximal element over B and agent i should prefer b_i over $L(R_i, a)$ otherwise one would deviate, and such profiles are exactly the preferences where b_i could be chosen. And finally if none of the agents pick 0 and none wants to deviate, that means each agent prefer a among $L(R_i, a)$ which is again exactly which the rule chooses. Thus we have proved the outcomes of Nash Equilibria in each case coincides with the outcomes of our choice rule. Thus given (M, g) mechanism, Nash-Implements $F_{(a,R,C)}$. $\square$

Let F be a Nash-Implementable SCR with range C and R be an a-critical profile, with $b \in L(R_i, a)$. Since R is a critical profile then for any m Nash equilibrium of (M, g, R), agent i must be able to deviate to any alternative which belongs to $L(R_i, a)$. In case of a non-reachable alternative, one can rearrange R by taking that alternative above a and that message would still be a Nash equilibrium which contradicts by the fact that R is a Critical Profile. Thus there exist a m_i' such that $g(m_i', m_{-i}) = b$, so if we replace the places of a and b at the preference of agent i, and take b to the top for all other agents (m_i', m_{-i}) is a Nash equilibrium for that preference profile, thus

F must pick b for this case, which is also the same thing as our family do. Thus the Critical-Profile Rule $F_{(a,R,C)}$ includes only implicitly induced choices by the existence of a critical profile. It has no redundant information, thus has no binding or contradicting case. Which means $F_{(a,R,C)}(R') \subset F(R')$

Before giving out our characterization for the Nash-Implementable choice rules we need to define critical profile for a given choice rule. Let R be a choice rule which satisfies $a \in F(R)$. If a is not chosen for any other choice profile where all agents have the same or narrower lower contour sets as in R for a and at least one of the agents have a strictly narrower lower contour set we will call R as an a-critical profile of F. Now for any Maskin-monotonic SCR F with range C let $C_a(F) \subset L(A)^N$ denote the set of all a-critical profiles.

Proposition 5. *For any Maskin-monotonic SCR F with range C,*

$$F(R) = \bigcup_{a \in A} \bigcup_{R' \in C_a(F)} F_{(a,R',C)}(R)$$

if and only if F is Nash-Implementable.

Proof. Let the range of F be C. If F is Nash-Implementable pick and critical profile R' of F with respect to a, so at any profile where $F_{(a,R',C)}$ chooses an alternative, F should choose the same alternative as well. So $F_{(a,R',C)}(R) \subset F(R)$ for every preference profile R. Thus we can deduce $\cup_{a \in A} \cup_{R' \in C_a(F)} F_{(a,R',C)}(R) \subset F(R)$. And since all critical profiles is included in the union we can also claim $F(R) \subset \cup_{a \in A} \cup_{R' \in C_a(F)} F_{(a,R',C)}(R)$, which concludes the proof of if part. For the reverse implication since we have proved any Critical-Profile Rule is Nash-Implementable, and Nash-Implementability is preserved under union we can claim F is Nash-Implementable as well. □

This characterization theorem turns out to be pretty easy to verify when the set of critical profiles is easy to find. Critical-Profile Rules Family is a basis for all Nash-Implementable SCRs, one can argue its minimality. If R

and R' have the exact same lower contour sets for all agents then obviously $F_{(a,R',C)} = F_{(a,R,C)}$, If we ignore these repetitions, we can claim a kind of minimality for this family. The minimality of basis in linear algebra is defined by linearly independency, which is basically states none of the elements of the basis is in the span of other elements. We can define a similar minimality definition for this situation as well. The proper minimality definition here should be congruent to necessity of each element, which means any element in the basis cannot be expressed as the union of some other the elements of the basis. Thus under that minimality definiton we have the following result.

Proposition 6. *Critical-Profile Rules Family is a minimal basis of Nash-Implementable choice rules under union.*

Proof. $F_{(a,R',C)}$ chooses a in any maskin-monotonic improvement of R' and any alternative b except a in cases where at least $|N|-1$ agents top b, so if in R' there are at least 2 agents who don't place a at the top, If we want to express $F_{(a,R',C)}$ as a union, we should use a SCR which chooses a at R', and this can occur only if we use $F_{(a,R,D)}$ where R' is a maskin monotonic improvement of R, if R' is a strict improvement then $F_{(a,R,D)}$ chooses a more than necessary parts which leads a contradiction, thus both R and R' should have the same lower contour sets. Moreover $D \subset C$ should be satisfied because $F_{(a,R,D)}$ has image D while $F_{(a,R',C)}$ have image C and $F_{(a,R,D)} \subset F_{(a,R',C)}$ should be satisfied. Finally if $D \neq C$ then all the alternatives except a will be chosen in $F_{(a,R,D)}$ when they are top alternative among D for all agents, but that is not true for $F_{(a,R',C)}$ which again will give a contradiction, thus $D = C$ should be satisfied. Thus $F_{(a,R,D)} = F_{(a,R',C)}$ which gives a contradiction, means if at least 2 agents doesn't top a at R' then $F_{(a,R',C)}$ cannot be expressed as the union of other rules, thus it cannot be excluded from the basis. Therefore nearly all of the elements of the Critical-Profile Rules Family are necessary for spanning the space. A problem occurs when a is not the top choice of at most one agent. For instance let R and R' be two different profiles, where for every $j \neq i$, $L(R_j, a) = L(R'_j, b) = C$ moreover let $L(R_i, a) = L(R'_i, b)$, one can easily verify that $F_{a,R,C} = F_{b,R',C}$, but in this case we can also ignore repetitions since they all result the same SCR. $\square$

Therefore we have shown the minimality of our basis.

CHAPTER 5

CONCLUSION

In this thesis we have attempted to discover some important properties pertaining to the algebraic structure of social choice rules that are implementable according to a given solution concept, with an aim to possibly obtain new characterizations of the implementability in question. The two implementabilities that have been studied here are those according to Nash and subgame perfect Nash equilibrium notions.

The main result of chapter 2 was that Nash implementability is also preserved under intersection when we impose some further conditions upon the social choice rules considered. It is important, however, to note that it is the observations made and the examples given in chapter 2 that actually comprise the conceptual core behind the construction of the Critical-Profile-Rules Family introduced in chapter 4, which in turn is used in our characterization theorem. It was preservation of Nash implementability under union rather than intersection that was used in our characterization of Nash implementability. One may, of course, try to use the theorem of chapter 2 concerning intersection to learn more about the set of Nash implementable social choice rules as well, without hoping to obtain a full characterization though, due to the restrictive assumption of that theorem.

It is only natural to expect that similar constructive characterizations may also be obtained for subgame perfect implementability along with implementability according to some other well-known solution concepts. Given a solution concept s, families of social choice rules, which act as a kind of basis in the sense that they span the entire set of s-implementable social choice rules with respect to certain algebraic operations and are minimal in that regard, are surely to reflect certain intrinsic properties of s-implementability, and thus will be useful in understanding implementability in general from a different angle.

Therefore, further research topics also include doing the same exercise, which we have gone through here in the context of Nash and subgame perfect implementability, for other solution concepts as strong Nash, undominated Nash or Bayesian Nash. The results obtained in this study make such extensions to look promising.

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