CRITICAL EXPONENT FOR THE SEMILINEAR WAVE EQUATION WITH SPACE-DEPENDENT DAMPING

by

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ABSTRACT

CRITICAL EXPONENT FOR THE SEMILINEAR WAVE EQUATION WITH SPACE-DEPENDENT DAMPING

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The thesis is a survey concerning with the global existence and blow-up of solutions of the Cauchy problem for the semilinear damped wave equations

$$
\begin{aligned}
&u_{tt} - \Delta u + V(x)u_t = |u|^p, && x \in \mathbf{R}^n, \quad t > 0,\\
&u(x,0) = \varepsilon u_0(x), \quad u_t(x,0) = \varepsilon u_1(x), && x \in \mathbf{R}^n,
\end{aligned}
$$

where $1 < p < \frac{n+2}{n-2}$ is a real number, $\varepsilon > 0$ is a small number and $V(x)$ is the potential or the friction term like $V(x) \sim V_0(1 + |x|)^{-\alpha}$ for large $|x|$, where $V_0 > 0$ and $\alpha \in [0, 1)$ are constants. The number $p_c = 1 + \frac{2}{n-\alpha}$ is the critical value of p in the sense that if $1 < p \leq p_c$, then solutions blow-up in finite time; if $p_c < p < \frac{n+2}{n-2}$, then there exist small data global solutions.

Keywords: semilinear damped wave equation, space dependent damping, global solution, blow up of the solution, decay rate of the energy.

ÖZET

UZAY DEĞİŞKENİNE BAĞLI SÖNÜM TERİMLİ YARILİNEER DALGA DENKLEMİ İÇİN KRİTİK ÜS

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Bu tezde, $1 < p < \frac{n+2}{n-2}$ bir sayı ve $\varepsilon > 0$ küçük bir sayı olmak üzere uzay değişkenine bağlı sönüm terimli yarı lineer dalga denklemi için verilen

$$
\begin{aligned}
&u_{tt} - \Delta u + V(x)u_t = |u|^p, && x \in \mathbf{R}^n, \quad t > 0, \\
&u(x,0) = \varepsilon u_0(x), \quad u_t(x,0) = \varepsilon u_1(x), && x \in \mathbf{R}^n
\end{aligned}
$$

Cauchy probleminin global çözümlerinin varlığı ve çözümlerin patlaması incelenmiştir. Burada $V_0 > 0$ ve $\alpha \in [0, 1)$ olmak üzere büyük $|x|$ değerleri için, $V(x) \sim V_0(1 + |x|)^{-\alpha}$ şeklinde bir sürtünme terimi olup, $p_c = 1 + \frac{2}{n-\alpha}$ sayısı aşağıdaki anlamda p sayısının kritik değeridir: $1 < p \leq p_c$ iken problemin çözümleri patlar; $p_c < p < \frac{n+2}{n-2}$ ise problemin küçük veri global çözumleri vardır.

Anahtar Kelimeler: Sönüm terimli yarı lineer dalga denklemi, uzay bağımlı sönüm terimi, global çözüm, çözümün patlaması, enerji azalma oranı

To my family and my supervisor

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CHAPTER 1

INTRODUCTION AND BASIC FACTS

1.1 Introduction

Let $n \geq 1$ be an integer and $x = (x_1, x_2, \cdots, x_n)$ be a point in $\mathbf{R}^n$, and $\Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$ be the Laplace operator. We consider the Cauchy problem for the semilinear damped wave equation with space-dependent potential

$$u_{tt} - \Delta u + V(x)u_t = |u|^p, \qquad x \in \mathbf{R}^n, \quad t > 0, \tag{1.1}$$

$$u(x,0) = \varepsilon u_0(x), \quad u_t(x,0) = \varepsilon u_1(x), \quad x \in \mathbf{R}^n, \tag{1.2}$$

where $u = u(x,t)$ is scalar-valued unknown function, $\varepsilon > 0$ is a small parameter which measures the smallness of the amplitude of solutions, p is the power of nonlinearity such that

$$1 < p < \frac{n+2}{n-2} \quad (n \geq 3), \qquad 1 < p < +\infty \quad (n = 1,2),$$

u_0 and u_1 are compactly supported initial data from the energy space:

$$u_0 \in H^1(\mathbf{R}^n), \qquad u_1 \in L^2(\mathbf{R}^n),$$
$$\operatorname{supp}(u_i) \subset B(K) = \{x \in \mathbf{R}^n : |x| < K\}, K > 0, \; i = 0,1$$

and V is a radial positive C^1-function of x, referred as a friction coefficient or potential (see Ikawa [9]), satisfying the condition

$$V(x) \sim V_0(1 + |x|)^{-\alpha}, \quad |x| \to \infty \tag{1.3}$$

where $V_0 > 0$ is a constant and $\alpha \in [0, 1)$. Condition (1.3) and Definition 1.15 (see Section 1.2) imply that condition (1.3) is equivalent to the following two-sided inequalities

$$V_1(1 + |x|)^{-\alpha} \leq V(x) \leq V_2(1 + |x|)^{-\alpha}, \quad \alpha \in [0, 1) \tag{1.4}$$

for some positive constants V_1 and V_2.

Equations of the form (1.1) appear in models for traveling waves in a nonhomogeneous gas with damping that changes with the position. The term $V(x)u_t$ in (1.1) is the space-dependent damping term (also called a dissipative term). The nonlinear term $|u|^p$ in (1.1) is the focusing type of nonlinearity, which is sometimes called "blowing up" term. We should say that if the nonlinear term $-|u|^p$ is replaced by $|u|^p$, it is well known that there exists global solutions without any smallness condition on the data (u_0, u_1) (see Grillakis [6] and Struwe [35]). The energy $E_u(t)$ for solutions of problem (1.1)-(1.2) is defined by

$$E_u(t) := \frac{1}{2} \int_{\mathbf{R^n}} (|u_t|^2 + |\nabla u|^2)\, dx, \tag{1.5}$$

which will be often used in the sequel.

We study questions of global existence, blow up and asymptotic behavior as $t \to \infty$ for solutions of (1.1)-(1.2) by deriving sharp decay estimates. Our interest is focused on the so-called critical exponent p_c, which is the number defined by the following property:

- If $p_c < p$, then all small data solutions of (1.1)-(1.2) are global.

- If $1 < p \leq p_c$, all solutions of (1.1)-(1.2) with data positive on average blow-up in finite time regardless of the smallness of the data.

It is well known that if the damping is missing, that is, if $V(x) = 0$, the critical exponent $p_w(n)$ for the wave equation

$$u_{tt} - \Delta u = |u|^p \tag{1.6}$$

is the positive root $p_w(n)$ of the quadratic equation

$$(n-1)p^2 - (n+1)p - 2 = 0, \tag{1.7}$$

where $n \geq 2$ is the space dimension ($p_w(1) = \infty$, see Sideris [31], Rammaha [30]). The proof of this fact, known as Strauss' conjecture [33], took almost 20 years and effort of many mathematicians, beginning with John [17], Glassey [4], Sideris [32], Strauss [34], Zhou [46] and ending with Lindblad and Sogge [21], Georgiev, Lindblad and Sogge [3] and Tataru [36].

In [37] and [38], Todorova and Yordanov solved the critical exponent problem for the wave equation (1.1) when the potential $V(x) = 1$. The main result in [37] and [38] is that the critical exponent p_c of the damped equation

$$u_t - \Delta u + u_t = |u|^p. \tag{1.8}$$

is exactly $p_c = 1 + \frac{2}{n}$, which is the same as famous Fujita's critical exponent (see [2]) for the semilinear heat equation

$$u_t - \Delta u = |u|^p. \tag{1.9}$$

We should clarify this fact since it has been conjectured previously that the damped wave equation (1.8) has the diffusive structure for large times (as $t \to \infty$) (see Bellout [1]) and this suggests that (1.8) should have a similar critical exponent as one can observe for the semilinear heat equation (1.9).

More precisely, in [37] and [38], Todorova and Yordanov proved small data global existence for the damped wave equation (1.1) with $V(x) = 1$ if $p > 1+\frac{2}{n}$; however if $1 < p < 1+\frac{2}{n}$, they prove blow up for all solutions of (1.1) with data positive on average. In [38] the authors proved other results, which also indicate parabolic asymptotic profile for solutions of equation (1.1) with constant potential $V(x)$ for large exponent $p > p_c$. Later on Zhang [43] proved that the critical exponent $p_c = 1 + \frac{2}{n}$ belongs to the blow up region. Ikehata and Tanizawa [15] considered the global existence part of (1.8) for noncompactly supported data. There are many related results to the so-called diffusion phenomenon, and we quote some of them below: Han and Milani [5], Ikehata [10], Ikehata et al [11], Ikehata and Nishihara [12], Ikehata and Ohta [14], Hayashi et al [7], Hosono and Ogawa [8], Marcati and Nishihara [22], Nishihara [28, 29], Li and Zhou [20], Narazaki [27], Zhang [43], and the references therein.

The main goal in this thesis is discuss the critical exponent problem for (1.1) under the condition (1.3) for the potential $V(x)$.

We observe that the critical exponent p_c varies according to the slow and fast and critically decaying potential $V(x)$.

In the case of slow decaying potential, that is, if V satisfies condition (1.3) when $\alpha \in [0, 1)$, then the critical exponent for problem (1.1)-(1.2) is

$$p_c = 1 + \frac{2}{n - \alpha}.$$

In the case of fast decaying potential, namely if $\alpha > 1$ in (1.3), then the energy

$E_u(t)$ of the linear part of equation (1.1)

$$u_{tt} - \Delta u + V(x)u_t = 0 \tag{1.10}$$

does not decay. The result of Mochizuki [24] shows that $E_u(t)$ of (1.10) approaches a non-zero constant as $t \to \infty$ if $V(x) = O(|x|^{-1-\delta})$ with $\delta > 0$. In this case we expect that equation (1.1) loses its "parabolicity" asymptotic effects and turns back to the regime of pure wave equation. Respectively, we expect that the critical exponent p_c of the damped wave equation in the case of fast decaying potential $\alpha > 1$ jumps to the critical exponent of the wave equation Strauss' number $p_w(n)$, that is, $p_c = p_w(n)$ for any $\alpha > 1$.

The case of critically decaying potential $\alpha = 1$ is very delicate. This case is a transition from asymptotic parabolicity to the pure hyperbolic regime. The energy decay of equation (1.10) depends in a very interesting way on the coefficient V_0 of the potential $V(x)$: The critical exponent $p_c = 1 + \frac{1}{n-1}$ for large $V_0 \geq n - 1$, while for $0 < V_0 < n - 1$ the critical exponent p_c increases when $V_0 \to 0$.

In the thesis we show further interesting phenomena due to the presence of the damping term. We also derive the exact decay rate for the energy $E_u(t)$ and L^2-and L^{p+1}-norms of solutions in the global existence part for exponents of nonlinearity $p > p_c$.

In order to get a sharp critical exponent result we will need sharp decay estimates for the corresponding linear equation (1.10). This problem is quite delicate in the case of space-dependent potential. Todorova and Yordanov [39, 40] already derived almost an optimal decay for solutions of (1.10). The key idea is to derive asymptotically a very good approximation for the fundamental solution of equation (1.10).

The thesis is organized as follows. In the next section, definitions and basic facts are given. Chapter 2 involves the main part of the thesis. In this chapter we deal with a particular problem (see problem (1.1)-(1.2)) due to Ikehata, Todorova and Yordanov [16] about the critical exponent problem, for which it is shown that if $p > p_c$, all solutions blow-up regardless of the smallness of the initial data; and if $1 < p < p_c$, depending on the smallness of the initial data, problem has a global solution. Furthermore, in the case of existence of small data global solutions, sharp decay estimates for solutions and the energy $E_u(t)$ of (1.1)-(1.2), are derived. In Chapter 3 the results are discussed by giving some remarks.

1.2 Basic facts and definitions

In this section we give definitions, notations and general facts which will be used in the thesis.

Definition 1.1. Let Ω be an open, connected set (domain) in $\mathbf{R}^n$ and $1 \leq p$ be a real number. We denote by $L^p(\Omega)$ the class of all measurable functions u, defined on Ω for which

$$\int_\Omega |u(x)|^p dx < \infty.$$

$L^p(\Omega)$ is a Banach space with the norm

$$||u||_p := \int_\Omega |u(x)|^p dx < \infty.$$

In particular, for $p = 2$, $L^2(\Omega)$ is a Hilbert space with the inner product

$$(u, v) := \int_\Omega u(x)v(x)dx$$

and the norm

$$||u|| := \int_\Omega |u(x)|^2 dx$$

for $u, v \in L^2(\Omega)$.

For $p = \infty$, $L^\infty(\Omega)$ is a Banach space with the norm

$$||u||_\infty = ess \sup_{x\in\Omega}|u(x)|.$$

Definition 1.2. Let $\Omega \subset \mathbf{R^n}$ be an open, connected set and let $\Phi : \Omega \to \mathbf{R}$ be a function. Support of Φ can be defined as

$$\text{supp}(\Phi) = \overline{\{x \in \Omega : \Phi(x) \neq 0\}}.$$

Definition 1.3. Let $\Omega \subset \mathbf{R^n}$ be an open, connected set. A function $u : \Omega \to \mathbf{R}$ is said to be locally integrable if for every compact set $K \subset \Omega$,

$$\int_K |u(x)|dx < \infty.$$

We denote by $L^1_{loc}(\Omega)$ the space of locally integrable functions defined on Ω.

Definition 1.4. Let $C_0^\infty(\Omega)$ is the space of infinitely differentiable functions with compact support. This space is also known as the space of all test functions defined on Ω.

Definition 1.5. Let $x \in \mathbf{R^n}$ with coordinates $x = (x_1, ..., x_n)$. A multi-index is an n-tuple $\alpha = (\alpha_1, ..., \alpha_n)$ $(\alpha_i \in N)$.
If we set

$$|\alpha| = \sum_{i=1}^{n} \alpha_i,$$

then

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... x_n^{\alpha_n}}$$

represents the α-th order partial differentiation operator.

Definition 1.6. Let Ω be a domain. Suppose $u, v \in L^1_{loc}(\Omega)$, and α is a multi-index. We say that v is the α^{th}-weak partial derivative of u, written

$$D^\alpha u = v,$$

provided

$$\int_\Omega u D^\alpha \phi dx = (-1)^{|\alpha|} \int_\Omega v \phi dx$$

for all test functions $\phi \in C_0^\infty(\Omega)$.

Definition 1.7. A sequence of functions $\{\phi_m\}$ in $C_0^\infty(\Omega)$ is said to converge to 0 if there exists a fixed compact set $K \subset \Omega$ such that $\text{supp}(\phi_m) \subset K$ for all m and $\{\phi_m\}$ and all its derivatives converge uniformly to zero on K.

Definition 1.8. A linear functional T from $C_0^\infty(\Omega)$ to $\mathbf{R}$ is said to be a distribution, or generalized function if whenever $\phi_m \to 0$ in C_0^∞, we have $T(\phi_m) \to 0$.

Definition 1.9. The space $C([0,T], X)$ consists of all continuous functions $u : [0,T] \to X$ such that

$$||u||_{C([0,T],X)} \equiv \max_{0 \le t \le T} ||u(t,.)||_X < \infty.$$

Definition 1.10. Let k be a non-negative integer and let $1 \le p \le \infty$. Then we define $W^{k,p}(\Omega)$ to be set of all distributions $u \in L^p(\Omega)$ such that $D^\alpha u \in L^p(\Omega)$ for $|\alpha| \le k$. In $W^{k,p}(\Omega)$, we define a norm by

$$||u||_{k,p} := \left(\sum_{|\alpha| \le k} ||D^\alpha u||_p^p \right)^{1/p} \quad if \quad 1 \le p < \infty$$

and

$$||u||_{k,\infty} := \max_{0 \le |\alpha| \le k} ||D^\alpha u||_\infty \quad if \quad p = \infty.$$

For $p = 2$ we define an inner product by

$$(u, v)_k := \sum_{|\alpha| \leq k} \int_{\Omega} D^{\alpha} u(x) D^{\alpha} v(x) dx.$$

We also use the notation $H^k(\Omega)$ for $W^{k,2}(\Omega)$ and $L^2(\Omega)$ for $W^{0,2}(\Omega)$.

Definition 1.11. By $W_0^{k,p}(\Omega)$ we denote the closure of $C_0^{\infty}(\Omega)$ in $W^{k,p}(\Omega)$. This means that $u \in W_0^{k,p}(\Omega)$ if and only if there exist functions $u_m \in C_0^{\infty}(\Omega)$ such that $u_m \to u \in W_0^{k,p}(\Omega)$.

Definition 1.12. (The divergence theorem) Let Ω be bounded open subset of $\mathbf{R^n}$ with smooth (or piecewise smooth) boundary $\partial\Omega$. Then

$$\int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} ds = \int_{\Omega} \nabla \cdot \mathbf{F} dx.$$

Definition 1.13. (**Big** O **notation**) We say that a function $f(x)$ is less than order $g(x)$ as x approaches a, denoted by $f(x) = o(g(x))$ as $x \to a$, if the ratio $\dfrac{f(x)}{g(x)}$ approaches 0 as x approaches a.

Definition 1.14. (**Little** o **notation**) We say that a function $f(x)$ is order of $g(x)$ as x approaches a, denoted by $f(x) = O(g(x))$ as $x \to a$, if the ratio $\dfrac{f(x)}{g(x)}$ is bounded for x near a.

Definition 1.15. (**Asymptotically equal**) Given two functions, $f(x)$ and $g(x)$, we say that $f(x)$ is similar to (or asymptotically equal to) $g(x)$ as $x \to \infty$, written $f(x) \sim g(x)$ as $x \to \infty$; if $\lim_{x \to \infty} \dfrac{f(x)}{g(x)} = 1$.

Inequalities

Cauchy's inequality with ε:

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}, \quad \forall a, b \in \mathbf{R}, \quad \forall \varepsilon > 0.$$

Young's inequality: Let $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad (a, b > 0).$$

Hölder's inequality. Assume $1 \leq p,\ q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\int_\Omega |u(x)v(x)| dx \leq ||u||_p ||v||_q$$

for $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$.

Gagliardo-Nirenberg's inequality. Let $1 \leq r < p \leq \infty,\ 1 \leq q \leq p$ and $m \geq 0$. Then, the inequality

$$||u||_p \leq C||D^m u||_q^\theta ||u||_r^{1-\theta} \quad \forall u \in W^{m,q}(\mathbf{R^n}) \cap L^r(\mathbf{R^n})$$

holds with some $C = C(n, \mu, j, q, r, \theta) > 0$ and

$$\theta = \left(\frac{1}{r} - \frac{1}{p}\right)\left(\frac{m}{n} + \frac{1}{r} - \frac{1}{q}\right)^{-1},$$

provided that $0 < \theta \leq 1$ ($0 < \theta < 1$ if $1 < q < \infty$ and $m - \frac{n}{q}$ is a nonnegative integer).

Poincaré's inequality. Let Ω be a bounded open set in $\mathbf{R^n}$ and $1 < p < \infty$. Then there exists a positive constant $C = C(\Omega, p)$ such that

$$||u||_p \leq C||\nabla u||_p$$

for every $u \in W_0^{1,p}(\Omega)$. In particular, we have

$$||u|| \leq C||\nabla u||$$

for every $u \in H_0^1(\Omega)$.

CHAPTER 2

SEMILINEAR WAVE EQUATION WITH SPACE DEPENDENT DAMPING TERM

2.1 Introduction

This chapter contains the main part of the thesis. We study small data global existence of solutions along with deriving decay estimates, and blow-up of local solutions of problem (1.1)-(1.2):

$$u_{tt} - \Delta u + V(x)u_t = |u|^p, \qquad x \in \mathbf{R}^n, \quad t > 0,$$
$$u(x,0) = \varepsilon u_0(x), \quad u_t(x,0) = \varepsilon u_1(x), \quad x \in \mathbf{R}^n,$$

where $\varepsilon > 0$ is a small number to be chosen later at our disposal, and

- The space dependent potential V satisfies condition (1.3).
- The exponent p satisfies: $1 < p < \frac{n+2}{n-2}$ if $n \geq 3$, and $1 < p < \infty$ if $n = 1, 2$.
- u_0 and u_1 are compactly supported initial data such that $u_0 \in H^1(\mathbf{R}^n)$ and $u_1 \in L^2(\mathbf{R}^n)$.

The main reference of this chapter is [16].

2.2 Small data global existence of solutions

In this section it is shown that if $p > p_c$, then problem (1.1)-(1.2) has small data global solutions. In this case, the exact decay rates for the energy (see (2.1)) and L^2

and L^{p+1}-norms of solutions are also studied.

We start with a well known classical local existence result for (1.1)-(1.2). First, for any $T > 0$, let us define the space $X(0,T)$ as

$$X(0,T) := C([0,T); H^1(\mathbf{R}^n)) \cap C^1([0,T); L^2(\mathbf{R}^n))$$

and denote by $T(\varepsilon) > 0$ the life span of solutions (1.1)-(1.2): the largest value that solutions exist for $x \in \mathbf{R^n}$, $0 < t < T(\varepsilon)$. When $T(\varepsilon) = \infty$, problem (1.1)-(1.2) admits a global in time solution, while we have only a local in time solution on $t \in [0, T(\varepsilon))$ when $T(\varepsilon) < \infty$.

Then, the following local existence of solutions for (1.1)-(1.2), which can be proved by a simple modification of the result in Strauss [34] or [26], is as follows.

Proposition 2.1. *Let $V(x)$ satisfy condition (1.3) and $p_c < p < \frac{n+2}{n-2}$ if $n \geq 3$, and $1 < p < \infty$ if $n = 1, 2$. Then, for each $u_0 \in H^1(\mathbf{R^n})$ and $u_1 \in L^2(\mathbf{R^n})$, there exists a maximal time $T_m = T_m(\varepsilon) > 0$ such that problem (1.1)-(1.2) has a unique solution $u \in X(0, T_m)$. If $T_m < +\infty$, then*

$$\limsup_{t \to T_m^+} (\|u_t(t,\cdot)\| + \|\nabla u(t,\cdot)\|) = +\infty.$$

Moreover, the finite propagation speed property holds:

$$\text{If } \ u_0(x) = u_1(x) = 0 \ \text{ for } \ |x| > R, \ \text{ then } \ u(t,x) = 0 \ \text{ for } \ |x| > t + R, \ \ t \in [0, T_m)$$

for some $R > 0$.

The small data global existence result is as the following.

Theorem 2.1. *Let the potential $V(x)$ satisfy (1.3) and $p_c < p < \frac{n+2}{n-2}$ for $n \geq 3$, and $p_c < p < \infty$ for $n = 1, 2$. Then there exists a number $\varepsilon_0 > 0$ such that for any*

$0 < \varepsilon < \varepsilon_0$, problem (1.1)-(1.2) has a unique solution u belongs to $X(0, +\infty)$ and satisfies the estimates

$$\begin{aligned}
&\int_{\mathbf{R^n}} e^{\frac{(\kappa-\delta)|x|^{2-\alpha}}{t}} u(t,x)^2 dx \le C t^{-m+\frac{\alpha}{2-\alpha}},\\
&\int_{\mathbf{R^n}} e^{\frac{\kappa|x|^{2-\alpha}}{t}} (u_t(t,x)^2 + |\nabla u(t,x)|^2) dx \le C t^{-m-1},\\
&\int_{\mathbf{R^n}} e^{\frac{\kappa|x|^{2-\alpha}}{t}} |u(t,x)|^{p+1} dx \le C t^{-m-1-\gamma},
\end{aligned} \tag{2.1}$$

for large $t \gg 1$, where

$$\kappa = \frac{(m-\delta)V_0}{(2-\alpha)(n-\alpha)}, \quad m = \frac{n-\alpha}{2-\alpha} - 2\delta,$$

$$\gamma = \frac{(p-1)(n-\alpha)-2}{2-\alpha} - (p-1)\delta > 0$$

and $\delta > 0$ is a small number.

The proof of Theorem 2.1 is quite involved and can be proved by deriving the standard and weighted energy estimates for the solution u. We mainly split the proof Theorem 2.1 into two parts.

The first part that we begin with the observation that $T_m \to \infty$ as $\varepsilon \to 0^+$, is relatively easy and relies on the standard energy estimate, that is, it depends on an estimate for $E_u(t)$.

2.2.1 Lower bound for the life span of solutions

Let $[0, T_m)$ denote the maximal interval of existence of the local solution u. It follows from Proposition 2.1 that $T_m = +\infty$ if the energy $E_u(t)$, defined by (1.5), is bounded on every compact subset of $[0, T_m)$. Hence we have to show that $E_u(t)$, or another equivalent norm, does not blow up in finite time. Here, we assume that $\varepsilon > 0$

is sufficiently small number.

Deriving a lower bound of $T_m \to \infty$ as $\varepsilon \to 0^+$ is based on the standard energy inequality, associated with problem (1.1)-(1.2):

$$\begin{aligned} E_u(t) &\leq E_u(0) + \frac{1}{p+1}\|u(0,\cdot)\|_{p+1}^{p+1} + \frac{1}{p+1}\|u(t,\cdot)\|_{p+1}^{p+1} \\ &= \frac{1}{2}(\|u_1\|^2 + \|\nabla u_0\|^2)\varepsilon^2 + \frac{\varepsilon^{p+1}}{p+1}\|u_0\|_{p+1}^{p+1} + \frac{1}{p+1}\|u(t,\cdot)\|_{p+1}^{p+1}. \end{aligned} \tag{2.2}$$

To obtain inequality (2.2) we have taken the inner product of equation (1.1) by u_t in $[0, T) \times \mathbf{R^n}$, applying the divergence theorem and using the finite speed of propagation property.

The following lemma gives a lower bound for T_m.

Lemma 2.1. *Under the assumptions of Theorem 2.1, the life span of T_m and energy satisfy*

$$T_m > C_2\varepsilon^{-(p-1)/(1-\theta)(p+1)} - R,$$

$$E_u(t) \leq 2C_1\varepsilon^2, \quad t \in [0, C_2\varepsilon^{-(p-1)/(1-\theta)(p+1)} - R],$$

where $\theta = \frac{n(p-1)}{2(p+1)}$ and C_1, $C_2 > 0$ are constants depending on n, p, and the data (u_0, u_1). Thus, $T_m \to \infty$ as $\varepsilon \to 0^+$.

Proof. Since $\varepsilon > 0$ is small and $p > 1$, $\varepsilon^{p+1} < \varepsilon^2$ and hence from (2.2) we have

$$E_u(t) \leq C_1\varepsilon^2 + \frac{1}{p+1}\|u(t,\cdot)\|_{p+1}^{p+1}, \tag{2.3}$$

where $C_1 = \frac{1}{2}(\|u_1\|^2 + \|\nabla u_0\|^2) + \frac{1}{p+1}\|u_0\|_{p+1}^{p+1}$. Now applying the Gagliardo-Nirenberg and Poincaré inequalities together with finite propagation we can bound the last term in (2.3) as follows:

$$\begin{aligned}
\|u(t,\cdot)\|_{p+1}^{p+1} &\leq C\|\nabla u(t,\cdot)\|^{\theta(p+1)}\|u(t,\cdot)\|^{(1-\theta)(p+1)} \\
&\leq C(1+t+R)^{(1-\theta)(p+1)}\|\nabla u(t,\cdot)\|^{p+1}.
\end{aligned}$$

Now using this bound and recalling the definition of $E_u(t)$ we have

$$E_u(t) \leq C_1\varepsilon^2 + C(t+R)^{(1-\theta)(p+1)}E_u(t)^{\frac{p+1}{2}}, \tag{2.4}$$

where $C > 0$ depends on n and p. Let us denote by T the first time such that $E_u(T) = 2C_1\varepsilon^2$. Since $E_u(t)$ is continuous and $E_u(0) < 2C_1\varepsilon^2$, we conclude that $E_u(t) < 2C_1\varepsilon^2$ for all $t \in [0,T)$. From estimate (2.4) with $t = T$ we have

$$2C_1\varepsilon^2 \leq C_1\varepsilon^2 + C(T+R)^{(1-\theta)(p+1)}(2C_1\varepsilon^2)^{\frac{p+1}{2}}.$$

Solving this inequality for T yields the following lower bound:

$$T \geq C_2\varepsilon^{-(p-1)/(1-\theta)(p+1)} - R,$$

where $C_2 > 0$ depends on n, p, and the data (u_0, u_1). Since $E_u(t) \leq 2C_1\varepsilon^2$ for all $t \in [0,T]$, the life span of u satisfies $T_m > T$. It is clear that $T \to \infty$ as $\varepsilon \to 0^+$. □

We can assume that $C_2\varepsilon^{-(p-1)/(1-\theta)(p+1)} - R < t < T_m$, so we deal with arbitrarily large t by choosing sufficiently small ε.

It remains to obtain estimates (2.1) to complete the proof of Theorem (2.1). In this case the standard energy $E_u(t)$ is not very convenient for precise estimates of u, so we use a weighted energy instead by constructing suitable multipliers (positive weight functions). A starting point fully depends on the work due to Todorova and Yordanov [39]. Indeed, for solution u on $[0, T_m(\varepsilon))$ of problem (1.1)-(1.2) we set $u = wv$, or

$v = uw^{-1}$, where w is an approximate solution of the linear part of (1.1)-(1.2)

$$u_{tt} - \Delta u + V(x)u_t = 0$$

and can be defined by

$$w(t,x) := t^{-m} e^{-m_1 \frac{\phi(x)}{t}}, \tag{2.5}$$

where m and m_1 are suitable parameters that can be determined explicitly in some certain cases. Here $\phi \in C^2(\mathbf{R^n})$ is a positive solution of the Poisson equation:

$$\Delta\phi(x) = V(x), \quad x \in \mathbf{R^n}. \tag{2.6}$$

To get the main decay estimates (2.1) we need a special solution of the Poisson equation, which is important to construct the approximate solution.

For the sake of completeness we give radially symmetric positive solutions of the Poisson equation (2.6) in the case when V is a radially symmetric (see [39]). For non-radial coefficient $V(x)$ the solution $\phi(x)$ of the Poisson equation (2.6) can be constructed by the method in [42].

2.2.2 Poisson equation in $\mathbf{R^n}$

Proposition 2.2. *Let V be a radially symmetric function in $C^1(\mathbf{R}^n)$, $n \geq 1$, which satisfies condition (1.4). Then equation (2.6) admits a solution $\phi \in C^2(\mathbf{R}^n)$, such that*

$$\begin{aligned} &(A1) \qquad \phi_0(1+|x|)^{2-\alpha} \leq \phi(x) \leq \phi_1(1+|x|)^{2-\alpha}, \\ &(A2) \qquad m(V) := \liminf_{|x|\to+\infty} \frac{V(x)\phi(x)}{|\nabla V(x)|^2} > 0, \end{aligned}$$

where ϕ_0 and ϕ_1 are positive constants depending on V_0, n and α.

Proof. We first solve equation (2.6) for a radial function $V(x)$. We will find three different representations of $V(x)$ depending on whether $n = 1$, $n = 2$, or $n \geq 3$.

When $n = 1$, we consider even functions $V(x)$ and $\phi(x)$. Integrating the equation twice on $[0, r]$ with $r > 0$, we have

$$\phi(r) = 1 + \int_0^r (r - \tau)\, V(\tau)\, d\tau, \quad r \in \mathbf{R}.$$

However, we can consider only non-negative r since $\phi(r)$ is an even function.

When $n \geq 2$, we rely on radial symmetry to simplify the Poisson equation:

$$\frac{d^2\phi}{dr^2} + \frac{n-1}{r}\frac{d\phi}{dr} = V(r),$$

where $r = |x|$. Notice that we can multiply both sides of the equation by r^{n-1} and rewrite the left side to obtain

$$\frac{d}{dr}\left(r^{n-1}\frac{d\phi}{dr}\right) = r^{n-1}V(r).$$

Integrating this on $[0, r]$, $r > 0$ and using the condition $\phi'(0) = 0$ we get

$$\frac{d\phi(r)}{dr} = r^{1-n}\int_0^r \tau^{n-1}V(\tau)\, d\tau. \tag{2.7}$$

If we repeat the integration and choose $\phi(0) = 1$, the resulting solution is

$$\phi(r) = 1 + \int_0^r \left(\ln\frac{r}{\tau}\right)\tau V(\tau)\, d\tau, \quad n = 2, \tag{2.8}$$

and

$$\phi(r) = 1 + \frac{1}{n-2}\int_0^r \left(1 - \frac{\tau^{n-2}}{r^{n-2}}\right)\tau V(\tau)\, d\tau, \quad n \geq 3. \tag{2.9}$$

We can write the above integrals for $\phi(r)$ in a common formula as

$$\phi(r) = 1 + r^2 \int_0^1 K_n(\tau)V(r\tau)\, d\tau, \quad r > 0, \tag{2.10}$$

where

$$K_n(\tau) = \begin{cases} 1-\tau & \text{if } n = 1, \\ \tau \ln \frac{1}{\tau} & \text{if } n = 2, \\ \frac{1}{n-2}\tau(1-\tau^{n-2}) & \text{if } n \geq 3. \end{cases}$$

It is easy to see that there are positive constants $L_n > \ell_n > 0$, such that

$$K_n(\tau) \leq L_n, \qquad \tau \in [0,1], \quad \text{and} \quad K_n(\tau) \geq \ell_n, \qquad \tau \in [1/4, 1/2].$$

To estimate ϕ, we substitute the lower and upper bounds of $V(r)$ into formula (2.10). Thus we obtain

$$1 + V_1 \ell_n I_0(r) r^2 \leq \phi(r) \leq 1 + V_2 L_n I_1(r) r^2, \quad r > 0, \tag{2.11}$$

with

$$I_k(r) = \int_{(1-k)/4}^{1/(2-k)} (1 + r\tau)^{-\alpha}\ d\tau, \quad k = 0, 1.$$

Now the trivial estimates

$$I_0(r) \geq \int_{1/4}^{1/2} (1+r)^{-\alpha}\, d\tau = (1+r)^{-\alpha}/4,$$

$$I_1(r) \leq \int_1^0 (r\tau)^{-\alpha}\, d\tau = r^{-\alpha}/(1-\alpha),$$

and (2.11) yield the two-sided bounds on $\phi(r)$, which is equivalent to claim $(A1)$.

To show the second claim ($A2$), we use

$$\frac{d\phi(r)}{dr} = r \int_0^1 \tau^{n-1} V(r\tau)\, d\tau, \quad r > 0,$$

which is valid in any dimension $n \geq 1$. Clearly the assumptions on V imply

$$1 + V_1 J_0(r) r \leq \frac{d\phi(r)}{dr} \leq 1 + V_2 J_1(r) r, \quad r > 0, \tag{2.12}$$

with

$$J_k(r) = \int_{(1-k)/2}^1 \tau^{n-1}(1 + r\tau)^{-\alpha}\, d\tau, \quad k = 0, 1.$$

Employing the simple estimates

$$J_0(r) \geq (1 + r)^{-\alpha}/2,$$
$$J_1(r) \leq r^{-\alpha}/(1 - \alpha),$$

in (2.12), we can get the following two-sided inequality

$$\phi_2(1 + |x|)^{2-\alpha} \leq |\nabla\phi(x)| \leq \phi_3(1 + |x|)^{2-\alpha}, \quad |x| \geq 1,$$

for some positive constants ϕ_2 and ϕ_3. Thus it follows from the above two-sided bounds for $\phi(x)$ and $|\nabla\phi(x)|$ that $m(V) > 0$, which verifies claim ($A2$). □

The following example shows that in some certain special cases $m(V)$ can be calculated explicitly.

Example 2.1. If $V(x) > 0$ is a radially symmetric C^1-function and

$$V(x) \sim V_0 |x|^{-\alpha}, \quad |x| \to \infty, \tag{2.13}$$

with $V_0 > 0$, then condition (1.4) is trivially satisfied and the Poisson equation (2.6) has a solution with the following properties:

$$\phi(x) \sim \frac{V_0}{(2-\alpha)(n-\alpha)}|x|^{2-\alpha}, \quad |\nabla\phi(x)| \sim \frac{V_0}{n-\alpha}|x|^{1-\alpha}, \quad |x| \to \infty.$$

In this case, we can compute the limit $m(V)$ explicitly by using (2.7) and the asymptotics

$$V(r) = V_0 r^{-\alpha} + o(r^{-\alpha}).$$

Thus we have

$$\frac{d\phi(r)}{dr} = r^{1-n}\int_0^r \tau^{n-1}[V_0\tau^{-\alpha} + o(\tau^{-\alpha})]d\tau, \quad r \to \infty.$$

Hence

$$\begin{aligned} \frac{d\phi(r)}{dr} &= \frac{V_0}{n-\alpha} r^{1-\alpha} + o(r^{1-\alpha}), \\ \phi(r) &= \frac{V_0}{(n-\alpha)(2-\alpha)} r^{2-\alpha} + o(r^{2-\alpha}), \end{aligned}$$

as $r \to \infty$. Now the definition of $m(V)$ gives

$$m(V) = \frac{n-\alpha}{2-\alpha}.$$

Remark 2.1. *For non-radial coefficient $V(x)$ the solution $\phi(x)$ of the Poisson equation (2.6) can be constructed by the method in [42].*

2.2.3 Weighted energy identity

Now we derive a weighted energy identity for the solution u of problem (1.1)-(1.2) by using the change of variables $u = wv$,or $v = uw^{-1}$, where w is a positive C^2 function.

A simple computation implies the following equation on $[0, T_m(\varepsilon))$:

$$v_{tt} - \Delta v + A(t,x)v_t + B(t,x).\nabla v + Q(t,x)v = w^{p-1}|v|^p, \tag{2.14}$$

where

$$\begin{aligned} A(t,x) &= V + 2w^{-1}w_t, \\ B(t,x) &= -2w^{-1}\nabla w, \\ Q(t,x) &= w^{-1}(w_{tt} - \Delta w + Vw_t), \end{aligned} \tag{2.15}$$

and the dot product is $x \cdot y = x_1y_1 + \cdots + x_ny_n$.

For a given P and w in $C^2((0,\infty)\times \mathbf{R}^n)$ we form the multiplier $Pv_t + wv$, where the first coefficient P and the approximate solution w will be determined later. We multiply (2.14) with $Pv_t + wv$:

$$(v_{tt} - \Delta v + A(t,x)v_t + B(t,x).\nabla v + Q(t,x)v)(Pv_t + wv) = w^{p-1}|v|^p(Pv_t + wv).$$

We will rewrite the left-hand side in divergence form and apply the divergence theorem. There are many terms, unfortunately, so we split the work into 5 steps.

Step 1. Multiply (2.14) by v_t and rearrange the terms on the left-hand side:

$$\begin{aligned} &\left(\frac{v_t^2 + |\nabla v|^2}{2}\right)_t - \nabla\cdot(v_t\nabla v) + Av_t^2 + v_tB\cdot\nabla v + Q\left(\frac{v^2}{2}\right)_t \\ &\qquad = \frac{1}{p+1}(w^{p-1}|v|^pv)_t - \frac{1}{p+1}(w^{p-1})_t|v|^pv. \end{aligned}$$

Step 2. Multiply (2.14) by v and rearrange the terms on the left side:

$$(v_tv)_t - \nabla\cdot(v\nabla v) - v_t^2 + |\nabla v|^2 + A\left(\frac{v^2}{2}\right)_t + B\cdot\nabla\frac{v^2}{2} + Cv^2 = w^{p-1}|v|^pv.$$

Step 3. Multiply the identity from Step 1 by P and integrate by parts over $\mathbf{R}^n$.

$$
\begin{aligned}
&\frac{d}{dt}\int P\frac{v_t^2+|\nabla v|^2}{2}\,dx-\int P_t\frac{v_t^2+|\nabla v|^2}{2}\,dx+\int \nabla P\cdot v_t\nabla v\,dx\\
&+\int (PAv_t^2+v_tPB\cdot\nabla v)\,dx+\frac{d}{dt}\int PQ\frac{v^2}{2}\,dx-\int (PQ)_t\frac{v^2}{2}\,dx\\
&=\frac{1}{p+1}\frac{d}{dt}\int Pw^{p-1}|v|^p v\,dx-\frac{1}{p+1}\int (Pw^{p-1})_t|v|^p v\,dx.
\end{aligned}
$$

Notice that boundary terms vanish since the support of $v(t,\cdot)$ is compact.

Step 4. Multiply the identity from Step 2 by w and integrate by parts over $\mathbf{R}^n$.

$$
\begin{aligned}
&\frac{d}{dt}\int wv_tv\,dx-\int w_tv_tv\,dx+\int \nabla w\cdot v\nabla v\,dx\\
&+\int (w|\nabla v|^2-wv_t^2)\,dx+\frac{d}{dt}\int wA\frac{v^2}{2}\,dx-\int (wA)_t\frac{v^2}{2}\,dx\\
&-\int \nabla\cdot(wB)\frac{v^2}{2}\,dx+\int wv^2\,dx\\
&=\frac{1}{p+1}\int Pw^{p-1}|v|^p v\,dx.
\end{aligned}
$$

In the final form we have used

$$
w_tv_tv=\frac{1}{2}(w_tv^2)_t-\frac{1}{2}w_{tt}v^2,\quad \nabla w\cdot v\nabla v=\frac{1}{2}\nabla\cdot(v^2\nabla w)-\frac{1}{2}v^2\Delta w,
$$

so the above identity becomes

$$
\begin{aligned}
&\frac{d}{dt}\int (wv_tv-\frac{1}{2}wtv^2)\,dx+\int \frac{1}{2}w_{tt}v^2\,dx-\int \frac{1}{2}\Delta wv^2\,dx\\
&+\int (w|\nabla v|^2-wv_t^2)\,dx+\frac{d}{dt}\int wA\frac{v^2}{2}\,dx-\int (wA)_t\frac{v^2}{2}\,dx\\
&-\int \nabla\cdot(wB)\frac{v^2}{2}\,dx+\int wQv^2\,dx\\
&=\frac{1}{p+1}\int Pw^{p-1}|v|^p v\,dx.
\end{aligned}
$$

Step 5. Add the final identities from Step 3 and Step 4 and combine terms with the same order of derivatives. This yields the following result.

Proposition 2.3. *Let v be the solution of (2.14). If P and w are C^2-functions, then*

$$\begin{aligned} \frac{d}{dt}E(v_t, \nabla v)(t) \quad &+ \quad F(t) + G(t) \\ &= \quad H(t) + \frac{d}{dt}\Big\{\frac{1}{p+1}\int_{\mathbf{R^n}} Pw^{p-1}|v|^p v\, dx\Big\}, \quad t \in [0, T_m(\varepsilon)) \end{aligned} \tag{2.16}$$

where

$$E(v_t, \nabla v)(t) := \frac{1}{2}\int_{\mathbf{R^n}}\{P(v_t^2 + |\nabla v|^2) + 2wv_tv + (QP + w_t + Vw)v^2\}dx, \tag{2.17}$$

is the weighted energy and

$$\begin{aligned} F(t) \quad &= \quad \frac{1}{2}\int_{\mathbf{R^n}}(-P_t + 2VP + 4P(\log w)_t - 2w)v_t^2 dx \\ &+ \quad \int_{\mathbf{R^n}}(\nabla P - 2P\nabla \log w).v_t\nabla v dx + \frac{1}{2}\int_{\mathbf{R^n}}(-P_t + 2w)|\nabla v|^2 dx \end{aligned} \tag{2.18}$$

$$G(t) = \tfrac{1}{2}\int_{\mathbf{R^n}}[Qw - (QP)_t]v^2 dx,$$

$$H(t) = \int_{\mathbf{R^n}}\Big\{w^p - \tfrac{1}{p+1}(Pw^{p-1})_t\Big\}|v|^p v dx,$$

where the values of $A(t,x)$ and $B(t,x)$ from (2.15) have been used in the final form.

2.2.4 Construction of weight functions

In this section we construct the weights P and w in Proposition 2.3 so that the weighted energy $E(v, \nabla v)(t)$ becomes positive definite, that is, $E(v, \nabla v)(t) \geq 0$.

Now assume that ϕ and $m(V)$ have properties $(A1)$-$(A2)$ in Proposition 2.2. For a given small $\delta \in (0, \frac{1}{2}m(V))$ and a large $S_0 > 0$ we set

$$m := m(V) - 2\delta; \quad m_1 := m(V) - \delta, \quad S(x) = m_1\phi(x) + S_0, \tag{2.19}$$

and define

$$P(t, x) := \frac{3}{4}\left(\frac{6}{t} + \frac{S(x)}{t^2}\right)^{-1} w(t, x), \tag{2.20}$$

where w is the approximate solution defined in (2.5). Thus the two weights w and P are defined by (2.5) and (2.20), respectively. First of all we note that such P and w are defined independently of the solution itself, and satisfy the following properties.

Lemma 2.2. *Let $\alpha \in [0, 1)$ given as in condition (1.3). There exists a large number $t_0 > 0$ such that for $t \geq t_0$ the following inequalities hold*

(i) $\quad C_1 P(t, x) \geq t^\alpha w(t, x),$

(ii) $\quad |w_t(t, x)| \leq C_1(t^{-\alpha} w(t, x),$

(iii) $\quad -P_t + w \geq 0,$

(iv) $\quad Q \geq 0, \quad Q_t \leq 0, \quad Qw - (QP)_t = Q(w - P_t) - Q_t P \geq 0,$

(v) $\quad (-P_t + 2w)(-P_t + 2VP + 4P(\ln w)_t - 2w) \geq |\nabla P - 2P\nabla \ln w|^2$

with some constant $C_1 > 0$.

Proof. By a simple calculation we first obtain

$$\begin{aligned} Pt^{-\alpha}w^{-1} &= \frac{3}{4}\frac{t^{2-\alpha}}{6t + (m(V) - \delta)\phi(x)} \\ &\geq \frac{3}{4}\frac{t^{2-\alpha}}{6t + C_0(1 + |x|)^{2-\alpha}} \\ &\geq \frac{3}{4}\frac{t^{2-\alpha}}{6t + C_0(1 + t + R)^{2-\alpha}} \end{aligned}$$

for some constant $C_0 > 0$. Since

$$\lim_{t\to+\infty} \frac{t^{2-\alpha}}{6t + C_0(1+t+R)^{2-\alpha}} = \frac{1}{C_1} > 0,$$

claim *(i)* is verified. By a similar way we can show claim*(ii)*.

Proof of (*iii*). We first show that

$$\frac{\partial_t P(t,x)}{P(t,x)} \le \frac{-m+1}{t} + \frac{4}{3}\frac{S(x)}{t^2}.$$

We calculate $\partial_t P/P = \partial_t \ln P$ using definition (2.5):

$$\frac{\partial_t P(t,x)}{P(t,x)} = -\frac{m}{t} + \frac{S(x)}{t^2} + \Big(\frac{6}{t} + \frac{2S(x)}{t^2}\Big)\Big(6 + \frac{S(x)}{t}\Big)^{-1}.$$

Since

$$\Big(6 + \frac{S(x)}{t}\Big)^{-1} \le \frac{1}{6},$$

we have

$$\frac{\partial_t P(t,x)}{P(t,x)} \le -\frac{m}{t} + \frac{S(x)}{t^2} + \frac{1}{6}\Big(\frac{6}{t} + \frac{2S(x)}{t^2}\Big) = \frac{-m+1}{t} + \frac{4}{3}\frac{S(x)}{t^2}.$$

Thus using definitions (2.5)and (2.20) we find that

$$\begin{aligned} -\partial_t P + w &= \Big(-\frac{\partial_t P}{P} + \frac{w}{P}\Big)P \\ &\ge \Big(\frac{m-1}{t} - \frac{4}{3}\frac{S(x)}{t^2} + \frac{4}{3}\frac{S(x)}{t^2}\Big)P \\ &\ge 0. \end{aligned}$$

Proof of (*iv*). It is enough to show that $Q \ge 0$ and $Q_t \le 0$ since the third inequality follows from (*iii*) and these inequalities. Let us calculate the first-order and second-order derivatives of P:

$$w_t = \Big(-\frac{m}{t} + \frac{S(x)}{t^2}\Big)w, \quad w_{tt} = \Big(-\frac{m}{t} + \frac{S(x)}{t^2}\Big)^2 P + \Big(\frac{m}{t^2} - \frac{2S(x)}{t^3}\Big)w$$
$$\nabla w = -\frac{\nabla S(x)}{t}w, \quad \Delta w = \Big(-\frac{\Delta S(x)}{t} + \frac{|\nabla S(x)|^2}{t^2}\Big)w.$$

Substituting these derivatives into (2.15) we get

$$Q = \frac{\Delta S - mV}{t} + \frac{VS - |\nabla S(x)|^2}{t^2} + \Big(-\frac{m}{t} + \frac{S(x)}{t^2}\Big)^2 + \frac{m}{t^2} - \frac{2S(x)}{t^3}.$$

Also, by the definition, we can list a few useful properties of $S(x)$ (see [39] for details).

(S1) $\Delta S(x) = (m+\delta)V(x)$ for all x,

(S2) $S(x) = O(|x|^{2-\alpha})$ for large x,

(S3) $(1 - \frac{\delta}{2m(V)})V(x)S(x) - |\nabla S(x)|^2 \geq 0$ for all x.

Now employing these properties in the formula for Q we see that

$$Q \geq \frac{\delta V(x)}{t} + \frac{\delta_1 V(x)S(x)}{t^2} - \frac{2S(x)}{t^3} = \frac{\delta V(x)}{t} + \frac{S(x)(\delta_1 V(x) - 2/t)}{t^2},$$

where $\delta_1 = \delta/(2m(V))$. Clearly we can assume that $|x| \leq t + R$ in these estimates. From $V(x) \geq V_1(1+|x|)^{-\alpha}$, with $\alpha \in [0,1)$, we conclude that $\delta_1 V(x) - 2/t > 0$ for sufficiently large time t. Hence $Q > 0$.

The proof of $Q_t \leq 0$ is similar, since $-tQ_t$ and Q have identical leading coefficients t^{-1} and proportional coefficients t^{-2}.

Proof of (v). We can estimate the first factor on the left-hand side by (iii):

$$-P_t + 2w = (-P_t + w) + w \geq w.$$

The second factor needs more work. We rewrite

$$-P_t + 2VP + 4P(\ln w)_t - 2w = \Big(-\frac{P_t}{P} + 2V\Big)P + \Big(4\frac{w_t}{w} - 2\frac{w}{P}\Big)P$$

and use the estimate for $-P_t/P$ in the proof of (iii) :

$$\begin{aligned}
&- P_t + 2VP + 4P(\ln w)_t - 2w \\
&\geq \Big(\frac{m-1}{t} - \frac{4}{3}\frac{S(x)}{t^2} + 2V(x)\Big)P + \Big(-\frac{4m}{t} + \frac{4S(x)}{t^2} - \frac{8}{3}\frac{S(x)}{t^2}\Big)P.
\end{aligned}$$

Regrouping terms on the right-hand side, we obtain

$$-P_t + 2VP + 4P(\ln w)_t - 2w \geq \Big(\frac{-3m+7}{t} + 2V(x)\Big)P \geq V(x)P,$$

for sufficiently large t. The latter inequality follows from

$$V(x) \geq V_1(1 + t + R)^{-\alpha} \geq (3m + 17)t^{-1},$$

whenever $|x| \leq t + R$ and t is sufficiently large. Thus we multiply the above lower bounds of the two factors in (v):

$$\begin{aligned}
(-P_t + 2w)(-P_t + 2VP + 4P(\ln w)_t - 2w) &\geq wVP \\
&= \frac{4V(x)(6t + S(x))}{3t^2}P^2 \\
&\geq \frac{4}{4}\frac{V(x)S(x)}{t^2}P^2, \qquad (2.21)
\end{aligned}$$

where we have used $w/P = 4(6t + S(x))/(3t^2)$ from the definitions of w and P. It remains to bound the right-hand side of inequality (iii). Since $\nabla P/P = \nabla \ln P$, we calculate

$$\begin{aligned}\nabla P - 2P \ln w &= (\nabla \ln P - 2\nabla \ln w)P \\ &= (-\frac{\nabla S(x)}{6t + S(x)} + \frac{\nabla S(x)}{t})P \\ &= \frac{(5t + S(x))(\nabla S(x)}{t(6t + S(x))}P\end{aligned}$$

and from its square

$$(\nabla P - 2P \ln w)^2 = \frac{(5t + S(x))^2 |\nabla S(x)|^2}{t^2(6t + S(x))^2} P^2.$$

Condition $(S3)$ implies that $|\nabla S(x)|^2 \leq V(x)S(x)$, so the above expression can be compared with the lower bound (2.21):

$$\frac{(5t + S(x))^2 |\nabla S(x)|^2}{t^2(6t + S(x))^2} P^2 \leq \frac{V(x)S(x)(5t + S(x))^2}{t^2(6t + S(x))^2} P^2 \leq \frac{V(x)S(x)}{t^2} P^2.$$

Hence condition (*iii*) holds. □

Remark 2.2. *Based on Lemma 2.1 by taking* $\varepsilon > 0$ *sufficiently small we can make the life span* $T_m = T_m(\varepsilon)$ *of the solution of (1.1)-(1.2) large enough such that* $T_m(\varepsilon) > C_2\varepsilon^{-(p-1)/(1-\theta)(p+1)} - R > t_0$ *, where* $t_0 > 0$ *is the time defined in Lemma 2.2 and* $C_2\varepsilon^{-(p-1)/(1-\theta)(p+1)} - R$ *is the lower bound for the life span from Lemma 2.2. This gives us an important bound for* $E_u(t_0)$*:*

$$E_u(t_0) \leq 2C_1\varepsilon^2, \tag{2.22}$$

which will be used in the sequel.

2.2.5 Weighted energy estimates

In this section we show that the weighted energy $E(v_t, \nabla v)(t)$ is a bounded function of time for the weights P and w constructed in Lemma 2.2. Another problem is to show that the weighted energy $E(v_t, \nabla v)(t)$ is positive definite, so it can be used for decay estimates of v and the original solution $u = wv$.

By using conditions (iii) and (v) in Lemma 2.2 we see that $F(t)$ in (2.18) is a positive definite quadratic form, therefore $F(t) \geq 0$. Also $G(t) \geq 0$ because of condition (iv). Therefore, after integrating (2.16) over $[t_0, t]$, where $t_0 < t < T_m$, we have

$$
\begin{aligned}
E(v_t, \nabla v)(t) \leq E(v_t, \nabla v)(t_0) &+ \frac{1}{p+1} \int_{\mathbf{R^n}} P(t_0, x) w(t_0, x)^{p-1} |v(t_0, x)|^{p+1} dx \\
&+ \frac{1}{p+1} \int_{\mathbf{R^n}} P w^{p-1} |v|^{p+1} dx + \int_{t_0}^{t} H(s) ds. \qquad (2.23)
\end{aligned}
$$

We need estimates for the third and fourth terms of the right hand side of the inequality(2.23). Now we introduce a new function:

$$
W(t) := \int_{\mathbf{R^n}} P(t, x)(v_t^2 + |\nabla v|^2) dx + \int_{\mathbf{R^n}} V(x) w(t, x) v(t, x)^2 dx.
$$

For the third term of the right hand side of (2.23) we have the following crucial estimate. In order to proceed, however, we will show that $W(t)$ is bounded by $E_u(t)$.

Lemma 2.3. *For each $t \in [0, T_m)$ it is true that*

$$
\int_{\mathbf{R^n}} w(t, x) v(t, x)^2 dx \leq C_R(t) ||\nabla u(t, \cdot)||^2,
$$

$$
E(v_t, \nabla v)(t) \leq C_R(t) E_u(t),
$$

with some t-dependent constant $C_R(t)$ satisfying

$$
\lim_{t \to +\infty} C_R(t) = +\infty.
$$

We omit the proof since it elementarily follows from the fact that $v = u/w$, $w = t^{-m}e^{-m_1\phi(x)/t}$ and the compact support of the data.

Thus, $W(t)$ is well defined and continuous for $t \in [t_0, T_m)$. For the fourth term of the right-hand side of (2.23) we have the following crucial estimate.

Lemma 2.4. *Let $V(x)$ satisfy condition (1.3) and $\alpha \in [0,1)$. If $p_c < p$, then there is a number $\gamma > 0$, which depends on p, n, α and δ such that*

$$\int_{\mathbf{R^n}} (1 + \frac{S(x)}{t}) P w^{p-1}|v|^{p+1} dx \leq C t^{-\gamma} W(t)^{(p+1)/2}, \quad t \in [t_0, T_m].$$

Proof. From definitions of $w(t,x)$ and $P(t,x)$ we have

$$\begin{aligned} P w^{p-1}|v|^{p+1} &= \frac{3}{4}(\frac{6}{t} + \frac{S}{t^2})^{-1} w^p |v|^{p+1} \\ &\leq C t w^p |v|^{p+1}, \end{aligned}$$

and

$$\begin{aligned} \frac{S(x)}{t} P w^{p-1}|v|^{p+1} &= \frac{3}{4} t \frac{S(x)}{t}(6 + \frac{S(x)}{t})^{-1} w^p |v|^{p+1} \\ &\leq \frac{3}{4} t w^p |v|^{p+1}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\mathbf{R^n}} (1 + \frac{S(x)}{t}) P w^{p-1}|v|^{p+1} dx &\leq C t \int_{\mathbf{R^n}} w^p |v|^{p+1} dx \\ &= C t^{-(pm-1)} \int_{\mathbf{R^n}} e^{(pS(x))/t} |v|^{p+1} dx. \end{aligned} \tag{2.24}$$

By setting

$$\psi(t,x) = \frac{1}{2}\frac{S(x)}{t}, \quad \eta = \frac{2p}{p+1},$$

we can rewrite (2.24) in the form

$$\int_{\mathbf{R^n}} (1 + \frac{S(x)}{t}) P w^{p-1} |v|^{p+1} dx \le C t^{-(pm-1)} \|e^{-\eta\psi(t,\cdot)} v\|_{p+1}^{p+1}. \tag{2.25}$$

To estimate the weighted norm $\|e^{-\eta\psi(t,\cdot)} v\|_{p+1}$ we use Gagliardo-Nirenberg inequality and get

$$\|e^{-\eta\psi(t,\cdot)} v\|_{p+1} \le \|e^{-\eta\psi(t,\cdot)} v\|^{\theta} \|\nabla (e^{-\eta\psi(t,\cdot)} v)\|^{1-\theta}, \tag{2.26}$$

where

$$\theta = 1 - n(\frac{1}{2} - \frac{1}{p+1}).$$

Since

$$|\nabla (e^{-\eta\psi} v)|^2 = \eta^2 e^{-2\eta\psi} |\nabla\psi|^2 v^2 - 2\eta e^{-2\eta\psi} v \nabla\psi . \nabla v + e^{-2\eta\psi} |\nabla v|^2,$$

integrating by parts we obtain

$$\begin{aligned} 2\eta \int_{\mathbf{R^n}} e^{-2\eta\psi} v \nabla v \cdot \nabla\psi dx &= \eta \int_{\mathbf{R^n}} e^{-2\eta\psi} \nabla v^2 \cdot \nabla\psi dx \\ &= \eta \int_{\mathbf{R^n}} (e^{-2\eta\psi} \nabla\psi) \cdot \nabla v^2 dx \\ &= -\eta \int_{\mathbf{R^n}} v^2 (e^{-2\eta\psi} \Delta\psi - 2\eta e^{-2\eta\psi} |\nabla\psi|^2) dx \\ &= 2\eta^2 \int_{\mathbf{R^n}} e^{-2\eta\psi} v^2 |\nabla\psi|^2 dx - \eta \int_{\mathbf{R^n}} v^2 e^{-2\eta\psi} \Delta\psi dx. \end{aligned}$$

Thus we find

$$\begin{aligned} \int_{\mathbf{R^n}} |\nabla (e^{-\eta\psi} v)|^2 dx &= \int_{\mathbf{R^n}} e^{-2\eta\psi} |\nabla v|^2 dx + \int_{\mathbf{R^n}} e^{-2\eta\psi} (\eta\Delta\psi - \eta^2 |\nabla\psi|^2) v^2 dx \\ &\le \int_{\mathbf{R^n}} e^{-2\eta\psi} |\nabla v|^2 dx + \eta \int_{\mathbf{R^n}} e^{-2\eta\psi} (\Delta\psi) v^2 dx. \end{aligned}$$

Therefore, we can rewrite (2.26), as follows

$$\|e^{-\eta\psi(t)}v\|_{p+1} \leq \|e^{-\eta\psi(t)}v\|^{\theta}\Big(\int_{\mathbf{R^n}} e^{-2\eta\psi}|\nabla v|^2 dx$$

$$+\eta\int_{\mathbf{R^n}} e^{-2\eta\psi}(\Delta\psi)v^2 dx\Big)^{(1-\theta)/2}. \tag{2.27}$$

Now we estimate the right-hand side of (2.2.5). The definitions of $\psi(t,x)$ and $\phi(x)$ imply that

$$\Delta\psi(t,x) = \frac{1}{2t}\Delta S(x) = \frac{m(V)-\delta}{2t}\Delta\phi(x) = \frac{m(V)-\delta}{2t}V(x)$$

and hence we have

$$\int_{\mathbf{R^n}} e^{-2\eta\psi}(\Delta\psi)v(t,x)^2 dx = \frac{m(V)-\delta}{2t}\int_{\mathbf{R^n}} V(x)e^{-2\eta\psi(t,x)}v(t,x)^2 dx.$$

Also since

$$e^{-2\eta\psi(t,x)} = e^{-\frac{S(x)}{t}}t^{-m}e^{-\frac{(p-1)S(x)}{(p+1)t}}t^m \leq t^m w(t,x), \tag{2.28}$$

we get

$$\int_{\mathbf{R^n}} e^{-2\eta\psi}(\Delta\psi)v(t,x)^2 dx \leq Ct^{m-1}\int_{\mathbf{R^n}} V(x)w(t,x)v(t,x)^2 dx. \tag{2.29}$$

To estimate the weighted norm $\|e^{-\eta\psi(t)}v\|$ in (2.2.5) we use the following decomposition:

$$e^{-2\eta\psi(t,x)}v(t,x)^2 = e^{-S(x)/t}t^{-m}t^m e^{-(S(x)(p-1))/((p+1)t)}t^{-\alpha/(2-\alpha)}t^{\alpha/(2-\alpha)}v^2$$

$$= t^m w(t,x)v(t,x)^2 e^{-(S(x)(p-1))/((p+1)t)}t^{-\alpha/(2-\alpha)}t^{\alpha/(2-\alpha)}. \tag{2.30}$$

Furthermore, there exists a constant $C > 0$ such that for any $x > 0$ it is true that

$$x^{\alpha/(2-\alpha)} \leq Ce^{x(p-1)/(p+1)}, \quad \alpha \in [0,1), \quad \frac{\alpha}{2-\alpha} < 1.$$

So, with a generous constant $C > 0$ we have

$$
\begin{aligned}
Ce^{-(p-1)S(x)/(p+1)t}t^{-\alpha/(2-\alpha)} &\leq Ct^{-\alpha/(2-\alpha)}(\frac{S(x)}{t})^{-\alpha/(2-\alpha)} \\
&= CS(x)^{-\alpha/(2-\alpha)} \\
&= C(m(V)-\delta)^{-\alpha/(2-\alpha)}\phi(x)^{-\alpha/(2-\alpha)} \\
&\leq CV(x).
\end{aligned}
$$

This inequality combined with (2.30) implies

$$
e^{-2\eta\psi(t,x)}v(t,x)^2 \leq Ct^m w(t,x)t^{\alpha/(2-\alpha)+m}v(t,x)^2 V(x),
$$

and we get

$$
e^{-2\eta\psi(t,x)}v(t,x)^2 \leq Ct^{\alpha/(2-\alpha)+m}w(t,x)v(t,x)^2 V(x),
$$

which shows the desired estimate

$$
\|e^{-\eta\psi(t)}v\|^2 \leq Ct^{m+(\alpha/(2-\alpha))}\int_{\mathbf{R^n}} V(x)w(t,x)v(t,x)^2 dx. \tag{2.31}
$$

On the other hand, in order to estimate the quantity $\|e^{-\eta\psi(t)}\nabla v\|$ of (2.2.5) we use (2.28). In fact, from the definition of P and (2.28) we see that

$$
\begin{aligned}
e^{-2\eta\psi(t,x)} &= t^m w(t,x)e^{-\frac{p-1}{p+1}\frac{S(x)}{t}} \\
&= t^m\frac{4}{3}(\frac{6}{t}+\frac{S(x)}{t^2})P(t,x)e^{-\frac{p-1}{p+1}\frac{S(x)}{t}} \\
&= \frac{4}{3}t^{m-1}P(t,x)\{(6+\frac{S(x)}{t})e^{-\frac{p-1}{p+1}\frac{S(x)}{t}}\} \\
&\leq Ct^{m-1}P(t,x),
\end{aligned}
$$

where $C > 0$ is a constant determined by the fact that the function

$$x \to (6 + x)e^{-Kx} \quad (K > 0)$$

is bounded above, that is, for all $x \geq 0$, $(6 + x)e^{-Kx} \leq C$. This implies

$$\int_{\mathbf{R^n}} e^{-2\eta\psi} |\nabla v|^2 dx \leq Ct^{m-1} \int_{\mathbf{R^n}} P(t, x)|\nabla v|^2 dx. \tag{2.32}$$

Therefore, by using (2.2.5), (2.29), (2.31) and (2.32) we have

$$||e^{-\eta\psi(t)} v||_{p+1}^{p+1} \leq C\Big(t^{m+(\alpha/(2-\alpha))} \int_{\mathbf{R^n}} V(x)w(t, x)v(t, x)^2 dx\Big)^{(p+1)\theta/2}$$

$$\times\Big(t^{(m-1)} \int_{\mathbf{R^n}} V(x)w(t, x)v(t, x)^2 dx + t^{(m-1)} \int_{\mathbf{R^n}} P(t, x)|\nabla v(t, x)|^2 dx\Big)^{(p+1)(1-\theta)/2}$$

$$\begin{aligned} &\leq C\Big(t^{m+(\alpha/(2-\alpha))} W(t)\Big)^{(p+1)\theta/2} \Big(t^{m-1)} W(t)\Big)^{(p+1)(1-\theta)/2} \\ &= Ct^{\{(m+\frac{\alpha}{2-\alpha})\frac{\theta}{2}+(m-1)\frac{1-\theta}{2}\}(p+1)} W(t)^{\frac{p+1}{2}} \quad (t > t_0). \end{aligned}$$

This inequality together with (2.25) gives

$$\int_{\mathbf{R^n}} (1 + \frac{S(x)}{t}) P w^{p-1} |v|^{p+1} dx \quad \leq \quad Ct^{-\gamma} W(t)^{\frac{p+1}{2}}, \quad t > t_0,$$

where

$$\begin{aligned} \gamma &= (pm - 1) - \Big\{(m + \frac{\alpha}{2-\alpha})\frac{\theta}{2} + (m-1)\frac{1-\theta}{2}\Big\}(p + 1), \\ &= \frac{(p-1)(n-\alpha) - 2}{2 - \alpha} - (p - 1)\delta. \end{aligned}$$

Finally, from the assumption $p_c < p$ we find $\gamma > 0$, which implies the desired estimate.

□

Now by using the result in Lemma 2.4 we are able to estimate the fourth term of the right hand side of (2.23) as follows.

Lemma 2.5. *Under the assumptions in Lemma 2.4 we have*

$$\int_{t_0}^{t} H(s)ds \leq C \int_{t_0}^{t} s^{-1-\gamma} W(s)^{(p+1)/2} ds, \quad t \in [t_0, T_m)$$

with some constant $C > 0$ and $\gamma > 0$ is the constant determined in Lemma 2.4.

Proof. It follows from the definitions of P and w that

$$\begin{aligned} K(t,x) : &= w^p - \frac{1}{p+1}(Pw^{p-1})t \\ &= \frac{1}{p+1} Pw^{p-1}\{(p+1)\frac{w}{P} - \frac{P_t}{P} - (p-1)\frac{w_t}{w}\}. \end{aligned}$$

Since

$$\frac{P_t}{P} = (\frac{6}{t} + \frac{S(x)}{t^2})^{-1}(\frac{6}{t^2} + \frac{2S(x)}{t^3}) + \frac{w_t}{w}$$

and

$$\frac{w_t}{w} = -\frac{m}{t} + \frac{S(x)}{t^2}\frac{w}{P} = \frac{4}{3}(\frac{6}{t} + \frac{S(x)}{t^2}),$$

we have

$$
\begin{aligned}
K(t,x) = Pw^{p-1}\Big\{&\frac{4}{3}(\frac{6}{t}+\frac{S(x)}{t^2}) - \frac{p}{p+1}(\frac{S(x)}{t^2}-\frac{m}{t}) \\
&- \frac{1}{p+1}(\frac{6}{t}+\frac{S(x)}{t^2})^{-1}(\frac{6}{t^2}+\frac{2S(x)}{t^3})\Big\} \\
&\le Pw^{p-1}\{\frac{4}{3}(\frac{6}{t}+\frac{S(x)}{t^2}) + \frac{pm}{(p+1)t}\} \\
&= t^{-1}Pw^{p-1}\{(8+\frac{pm}{p+1}) + \frac{4}{3}\frac{S(x)}{t}\} \\
&\le Ct^{-1}Pw^{p-1}(1+\frac{S(x)}{t}),
\end{aligned}
$$

which implies

$$
w(t,x)^p - \frac{1}{p+1}(P(t,x)w(t,x)^{p-1})_t \le Ct^{-1}(1+\frac{S(x)}{t})P(t,x)w(t,x)^{p-1}.
$$

This shows

$$
\begin{aligned}
\int_{t_0}^{t} H(s)ds &\le \int_{t_0}^{t}\int_{\mathbf{R^n}}\Big(w(s,x)^p - \frac{1}{p+1}(P(s,x)w(s,x)^{p-1})_t\Big)|v(s,x)|^{p+1}dxds \\
&\le C\int_{t_0}^{t} s^{-1}\int_{\mathbf{R^n}}(1+\frac{S(x)}{s})P(s,x)w(s,x)^{p-1}|v(s,x)|^{p+1}dxds. \qquad (2.33)
\end{aligned}
$$

Thus, by using Lemma 2.4 and estimate (2.33) we can derive the estimate in Lemma 2.5.

From Lemma 2.4, Lemma 2.5, and the weighted energy inequality (2.23) we get the following estimate

$$
\begin{aligned}
&E(v_t,\nabla v) = \frac{1}{2}\int_{\mathbf{R^n}} P(t,x)(v_t^2+|\nabla v|^2)dx + \frac{1}{2}\int_{\mathbf{R^n}}[2wv_tv + (QP+w_t+Vw)v^2]dx \\
&\le E(v_t,\nabla v)(t_0) + CW(t)^{(p+1)/2} + C\int_{t_0}^{t} s^{-1-\gamma}W(s)^{(p+1)/2}ds, \quad t\ge t_0. \qquad (2.34)
\end{aligned}
$$

Since $w > 0$ satisfies

$$w_{tt} - \Delta w + V w_t \geq 0,$$

and $P > 0$, then $QP \geq 0$. Since $\frac{d}{dt}(wv^2) - w_t v^2 = 2wvv_t$, we can rewrite (2.34) as follows:

$$E(v_t, \nabla v) = \frac{1}{2}\int_{\mathbf{R^n}} P(t,x)(v_t^2 + |\nabla v|^2)dx + \frac{1}{2}\int_{\mathbf{R^n}} Vwv^2 dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbf{R^n}} wv^2 dx$$

$$\leq E(v_t, \nabla v)(t_0) + CW(t)^{(p+1)/2} + C\int_{t_0}^{t} s^{-1-\gamma} W(s)^{(p+1)/2} ds, \quad t \geq t_0. \tag{2.35}$$

□

We need one more preparation.

Lemma 2.6. *Let $\alpha \in [0,1), c_0 > 0, a > 0, R > 0$ and $E_0 > 0$ be given real numbers, and let $f \in C([t_0, T_m))$ be a monotone increasing function. If a function $h \in C^1([t_0, T_m))$ satisfies the inequality*

$$h'(t) + \frac{c_0}{(a+t+R)^\alpha} h(t) \leq E_0 + f(t), \tag{2.36}$$

then the following estimate holds

$$h(t) \leq h(t_0) + C(E_0 + f(t))(a+t+R)^\alpha, \quad t \geq t_0,$$

with some constant $C > 0$.

Proof. Since

$$\frac{d}{dt}\{e^{\frac{c_0}{1-\alpha}(a+t+R)^{1-\alpha}} h(t)\} \quad = \quad e^{\frac{c_0}{1-\alpha}(a+t+R)^{1-\alpha}}(h'(t) + \frac{c_0}{(a+t+R)^\alpha} h(t)),$$

we multiply both sides of inequality (2.36) by $e^{\frac{c_0}{1-\alpha}(a+t+R)^{1-\alpha}}$ we get

$$\frac{d}{dt}\{e^{\frac{c_0}{1-\alpha}(a+t+R)^{1-\alpha}}h(t)\} \leq e^{\frac{c_0}{1-\alpha}(a+t+R)^{1-\alpha}}(E_0 + f(t))$$

and integrating over $[t_0, t]$ $(t < T_m)$ we have

$$\begin{aligned} e^{\frac{c_0}{1-\alpha}(a+t+R)^{1-\alpha}}h(t) &\leq e^{\frac{c_0}{1-\alpha}(a+t_0+R)^{1-\alpha}}h(t_0) + E_0 \int_{t_0}^{t} e^{\frac{c_0}{1-\alpha}(a+s+R)^{1-\alpha}}ds \\ &\quad + \int_{t_0}^{t} e^{\frac{c_0}{1-\alpha}(a+s+R)^{1-\alpha}}f(s)ds \\ &\leq e^{\frac{c_0}{1-\alpha}(a+t+R)^{1-\alpha}}h(t_0) + (E_0 + f(t)) \int_{t_0}^{t} e^{\frac{c_0}{1-\alpha}(a+s+R)^{1-\alpha}}ds, \end{aligned}$$

where the monotone increasing property of the function $f(t)$ has been used. Since

$$\begin{aligned} \int_{t_0}^{t} e^{\frac{c_0}{1-\alpha}(a+s+R)^{1-\alpha}}ds &= \int_{a+t_0+R}^{a+t+R} e^{C_1\tau^{1-\alpha}}d\tau \\ &= \frac{1}{1-\alpha}\int_{(a+t_0+R)^{1-\alpha}}^{(a+t+R)^{1-\alpha}} e^{C_1 z} z^{\frac{\alpha}{1-\alpha}}dz \\ &\leq e^{C_1(a+t+R)^{1-\alpha}}\frac{(a+t+R)^{\alpha}}{C_1(1-\alpha)}, \end{aligned}$$

where $C_1 = c_0/(1-\alpha)$, we obtain the desired estimate. □

Let

$$M(t) := \max_{0\leq s\leq t} W(s). \tag{2.37}$$

Note that the function $t \to M(t)$ is monotone increasing and continuous in $[t_0, T_m)$. To verify the latter property, let $t_0 \leq t_1 < t_2 \leq t_3 < T_m$ and use

$$\begin{aligned} &0 \leq M(t_2) - M(t_1) \\ &\quad \leq \sup\{|W(s_2) - W(s_1)| : \ t_0 \leq s_1 < s_2 \leq t_3 \text{ and } s_2 - s_1 \leq t_2 - t_1\}. \end{aligned}$$

The upper bound goes to 0 as $t_2 - t_1 \to 0$, since W is uniformly continuous on $[t_0, t_3]$.

After the above results we can prove the following lemma.

Lemma 2.7. *Let* $\alpha \in [0, 1)$. *Then the following bound holds*

$$\int_{\mathbf{R^n}} wv^2 dx \leq \int_{\mathbf{R^n}} wv^2 dx \,|_{t=t_0}$$

$$+C(E(v_t, \nabla v)(t_0) + M(t)^{(p+1)/2}$$

$$+ \int_{t_0}^{t} s^{-1-\gamma} W(s)^{(p+1)/2} ds) t^{\alpha},$$

for $t \in [t_0, T_m)$ *with large* $t_0 > 0$.

Proof. It follows from (2.2.5) that

$$\begin{aligned}
&\frac{1}{4} \int_{\mathbf{R^n}} V(x) wv^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbf{R^n}} wv^2 dx \\
&\leq E(v_t, \nabla v)(t_0) + CW(t)^{(p+1)/2} + C \int_{t_0}^{t} s^{-1-\gamma} W(s)^{(p+1)/2} ds, \quad t \in [t_0, T_m).
\end{aligned}$$

Now by using condition (1.3) and (2.37) we get

$$\begin{aligned}
&\frac{1}{2} \frac{V_0}{(1 + t + R)^{\alpha}} \int_{\mathbf{R^n}} wv^2 dx + \frac{d}{dt} \int_{\mathbf{R^n}} wv^2 dx \\
&\leq 2E(v_t, \nabla v)(t_0) + CM(t)^{(p+1)/2} + C \int_{t_0}^{t} s^{-1-\gamma} W(s)^{\frac{p+1}{2}} ds, \quad t \in [t_0, T_m).
\end{aligned}$$

Since the function

$$t \to CM(t)^{\frac{p+1}{2}} + C \int_{t_0}^{t} s^{-1-\gamma} W(s)^{\frac{p+1}{2}} ds$$

is monotone increasing, we can apply Lemma 2.6 with

$$h(t) = \int_{\mathbf{R^n}} wv^2 dx, \quad E_0 = 2E(v_t, \nabla v)(t_0), \quad a = 1, \quad c_0 = \frac{V_0}{2},$$

$$f(t) = C(M(t)^{\frac{p+1}{2}} + \int_{t_0}^{t} s^{-1-\gamma} W(s)^{\frac{p+1}{2}} ds),$$

and obtain the desired estimate. □

Employing these results, we go to the final stage of the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $t_0 < T < T_m$. The Cauchy-Schwarz inequality implies that

$$|2wv_t v| \leq \sigma t^{\alpha} w v_t^2 + \sigma^{-1} t^{-\alpha} w v^2$$

for any $\sigma > 0$. So, we have

$$2wv_t v \geq -\sigma t^{\alpha} w v_t^2 - \sigma^{-1} t^{-\alpha} w v^2.$$

Thus from (2.34) we get

$$\int_{\mathbf{R^n}} P(v_t^2 + |\nabla v|^2) dx - \int_{\mathbf{R^n}} \sigma t^{\alpha} w v_t^2 dx - \sigma^{-1} t^{-\alpha} \int_{\mathbf{R^n}} w v^2 dx + \int_{\mathbf{R^n}} (w_t + Vw) v^2 dx$$

$$\leq 2E(v_t, \nabla v)(t_0) + CW(t)^{(p+1)/2} + C \int_{t_0}^{t} s^{-1-\gamma} W(s)^{(p+1)/2} ds. \qquad (2.38)$$

This implies

$$\int_{\mathbf{R^n}} (P - \sigma t^{\alpha} w)(v_t^2 + |\nabla v|^2) dx + \int_{\mathbf{R^n}} (w_t + Vw - \sigma^{-1} t^{-\alpha} w) v^2 dx$$

$$\leq 2E(v_t, \nabla v)(t_0) + CW(t)^{(p+1)/2} + C \int_{t_0}^{t} s^{-1-\gamma} W(s)^{(p+1)/2} ds, \quad t \in [t_0, T_m).$$

Because of Lemma 2.2 we have

$$2E(v_t, \nabla v)(t_0) + CW(t)^{(p+1)/2} + C \int_{t_0}^{t} s^{-1-\gamma} W(s)^{(p+1)/2} ds$$

$$\geq \int_{\mathbf{R^n}} P(1 - \sigma C_1)(v_t^2 + |\nabla v|^2) dx + \int_{\mathbf{R^n}} (Vw + (w_t - \sigma^{-1} t^{-\alpha} w)) v^2 dx$$

$$\geq (1-\sigma C_1)\int_{\mathbf{R^n}} P(v_t^2+|\nabla v|^2)dx+\int_{\mathbf{R^n}} Vwv^2dx-(C_1+\sigma^{-1})t^{-\alpha}\int_{\mathbf{R^n}} wv^2dx.$$

for $t\in[t_0,T_m)$. This together with Lemma 2.7 yields

$$\begin{aligned}
&(1-\sigma C_1)\int_{\mathbf{R^n}} P(v_t^2+|\nabla v|^2)dx+\int_{\mathbf{R^n}} Vwv^2dx\\
&\leq 2E(v_t,\nabla v)(t_0)+CW(t)^{(p+1)/2}+C\int_{t_0}^t s^{-1-\gamma}W(s)^{(p+1)/2}ds\\
&\quad+C(C_1+\sigma^{-1})t^{-\alpha}\Big\{\int_{\mathbf{R^n}} wv^2dx|_{t=t_0}+t^{\alpha}(E(v_t,\nabla v)(t_0)\\
&\quad+M(t)^{(p+1)/2}+\int_{t_0}^t s^{-1-\gamma}W(s)^{(p+1)/2}ds\Big\}\\
&\leq 2E(v_t,\nabla v)(t_0)+Ct^{-\alpha}\int_{\mathbf{R^n}} wv^2dx|_{t=t_0}+CE(v_t,\nabla v)(t_0)\\
&\quad+CM(t)^{(p+1)/2}+C\int_{t_0}^t s^{-1-\gamma}W(s)^{(p+1)/2}ds,\quad t\in[t_0,T_m),
\end{aligned}$$

where $C=C_\sigma>0$ is a constant (recall (2.37) for the definition of M(t)). By taking $\sigma>0$ sufficiently small we have

$$\begin{aligned}
&\int_{\mathbf{R^n}} P(v_t^2+|\nabla v|^2)dx+\int_{\mathbf{R^n}} Vwv^2dx\\
&\qquad\leq C\Big(E(v_t,\nabla v)(t_0)+\int_{\mathbf{R^n}} wv^2dx|_{t=t_0}\Big)+CM(t)^{(p+1)/2}\\
&\qquad+C\int_{t_0}^t s^{-1-\gamma}W(s)^{(p+1)/2}ds,
\end{aligned}$$

for all $t\in[t_0,T_m)$. This estimate together with Lemma 2.3 gives the following crucial bound:

$$W(t)\leq C_{t_0}E_u(t_0)+CM(t)^{(p+1)/2}+\frac{t_0^{-\gamma}}{\gamma}\Big(\max_{0\leq s\leq t}W(s)\Big)^{(p+1)/2},\quad t\in[t_0,T_m).$$

We know from the bound (2.22) that $E_u(t_0) \leq 2C_1\varepsilon^2$, so we have the inequality

$$M(t) \leq C_{t_0}\varepsilon^2 + CM(t)^{(p+1)/2}, \quad t \in [t_0, T_m)$$

with another constant C_{t_0}, which also depends on n, p and the initial data (u_0, u_1). By Lemma 2.3, it is obvious that $M(t_0) \leq C^*\varepsilon^2$, where C^* is a constant. By using the above estimate and standard arguments as in [41], upon possible additional decreasing of $\varepsilon > 0$, we get

$$M(t) \leq C_{t_0}\varepsilon^2, \quad t \in [t_0, T_m). \tag{2.39}$$

This implies $T_m = +\infty$.

Finally, we derive the decay estimates in Theorem 2.1. Indeed from (2.39) we have

$$\int_{\mathbf{R^n}} P(t,x)(v_t(t,x)^2 + |\nabla v(t,x)|^2)dx + \int_{\mathbf{R^n}} V(x)w(t,x)v(t,x)^2 dx \leq C\varepsilon^2 \tag{2.40}$$

for $t \geq t_0$. It follows from the definition of $v = uw^{-1}$ that the second term of the left-hand side of (2.40) satisfies the estimate

$$\int_{\mathbf{R^n}} V(x)w(t,x)^{-1}u(t,x)^2 dx \leq t^m \int e^{\frac{(m-\delta)\phi(x)}{t}} V(x)u^2 dx \leq C\varepsilon^2. \tag{2.41}$$

Further, by using the bounds for $\phi(x)$, namely

$$\phi_0(1 + |x|)^{2-\alpha} \leq \phi(x) \leq \phi_1(1 + |x|)^{2-\alpha}$$

for $x \in \mathbf{R^n}$ and (1.3) we get the estimate

$$\begin{aligned} V(x) &\geq C(\phi(x))^{-\frac{\alpha}{2-\alpha}} \\ &= Ct^{-\frac{\alpha}{2-\alpha}}\left(\frac{\phi(x)}{t}\right)^{-\frac{\alpha}{2-\alpha}} \\ &\geq Ct^{-\frac{\alpha}{2-\alpha}}e^{-\delta\frac{\phi(x)}{t}}, \end{aligned} \tag{2.42}$$

where $C > 0$ and $t \geq t_0$ is sufficiently large. We complete the decay estimate for L^2-norm of solution u by substituting this lower bound for $V(x)$ into inequality (2.41). Next, we derive the energy decay estimate for the solution u of problem (1.1)-(1.2). Notice that

$$v_t^2 = (-w^{-2}w_t u + w^{-1}u_t)^2 \geq \frac{1}{2}w^{-2}u_t^2 - 3w^{-4}w_t^2u^2,$$

and

$$|\nabla v|^2 = |-w^{-2}u\nabla w + w^{-1}\nabla u|^2 \geq \frac{1}{2}w^{-2}|\nabla u|^2 - 3w^{-4}|\nabla w|^2u^2.$$

Thus we have

$$\frac{1}{2}Pw^{-2}(u_t^2 + |\nabla u|^2) \leq P(v_t^2 + |\nabla v|^2) + 3Pw^{-4}(w_t^2 + |\nabla w|^2)u^2.$$

Integrating this inequality over $\mathbf{R^n}$, and applying (2.40) we obtain

$$\frac{1}{2}\int_{\mathbf{R^n}} Pw^{-2}(u_t^2 + |\nabla u|^2)dx \leq C\varepsilon^2 + 3\int_{\mathbf{R^n}} Pw^{-4}(w_t^2 + |\nabla w|^2)u^2dx. \tag{2.43}$$

It is easy to check that P and w satisfy

$$Pw^{-3}(w_t^2 + |\nabla w|^2) \leq CV(x).$$

By using the above inequality and estimates (2.43) and (2.41) we get

$$\frac{1}{2}\int_{\mathbf{R^n}} Pw^{-2}(u_t^2+|\nabla u|^2)dx \le C\varepsilon^2 + C\int_{\mathbf{R^n}} w^{-1}V(x)u^2dx \le C\varepsilon^2.$$

This implies the energy decay estimate in Theorem 2.1.

Finally, from (2.39) and Lemma 2.4 we find that

$$\int_{\mathbf{R^n}} (\frac{6}{t}+\frac{S(x)}{t^2})^{-1}w^{-1}|u|^{p+1}dx \le Ct^{-\gamma},$$

so that the decay estimate for the L^{p+1} norm as written in Theorem 2.1 holds.

Corollary 2.1. *Under the same assumptions as in Theorem 2.1 the following estimates hold*

$$\begin{aligned}
\int_{\mathbf{R^n}} u(t,x)^2dx &\le Ct^{-(\frac{n-\alpha}{2-\alpha}-\frac{\alpha}{2-\alpha}-2\delta)},\\
\int_{\mathbf{R^n}} (u_t(t,x)^2+|\nabla u(t,x)|^2)dx &\le Ct^{-(\frac{n-\alpha}{2-\alpha}+1-2\delta)},\\
\int_{\mathbf{R^n}} |u(t,x)|^{p+1}dx &\le Ct^{-(p\frac{n-\alpha}{2-\alpha}-\frac{\alpha}{2-\alpha}-(p+1)\delta)},
\end{aligned}$$

for large $t \gg 1$, *and small* $\delta > 0$.

Another important consequence of the main theorem is the following.

Corollary 2.2. *Under the same assumptions as in Theorem 2.1, the solution of (1.1)-(1.2) satisfies*

$$\int_{|x|^{2-\alpha}\ge t^{1+p}} (u_t(t,x)^2+|\nabla u(t,x)|^2)dx \le C_Rt^{-m}e^{-\kappa t^p}$$

for an arbitrary fixed $p > 0$.

Namely, the decay rate of the energy under consideration in the region $|x|^{2-\alpha} \ge t^{1+p}\quad (p>0)$, is exponential. This shows parabolic asymptotic profile of solutions of the problem (1.1)-(1.2).

2.3 Blow-up of solutions

In this section we prove that solutions of problem (1.1)-(1.2) blow-up in the case when the power p of the nonlinearity $|u|^p$ is subcritical or critical, that is, $1 < p \leq p_c$. We apply the method of test functions developed by Zhang [43, 45].

The blow-up result in the case when $1 < p \leq p_c$ is as follows.

Theorem 2.2. *Let the potential $V(x)$ satisfy condition (1.3) and let $1 < p \leq p_c$. If the initial data (u_0, u_1) satisfies*

$$\int_{\mathbf{R^n}} (u_1(x) + V(x)u_0(x))\, dx > 0,$$

then the solution of problem (1.1)-(1.2) does not exist globally for any $\varepsilon > 0$.

Proof. First we find a non-negative $\phi \in C_0^\infty(\mathbf{R} \times \mathbf{R^n})$, such that

$$\phi(t,x) = \begin{cases} 1, & \text{if } (t,x) \in [-1,1] \times B(R), \\ 0, & \text{if } (t,x) \in (\mathbf{R} \times \mathbf{R^n}) \setminus ([-2,2] \times B(2R)), \end{cases}$$

satisfying the condition

$$|D^2\phi(t,x)|^4 + |D\phi(t,x)|^2 \leq C\phi(t,x), \quad (t,x) \in \mathbf{R} \times \mathbf{R^n},$$

where $R > 0$ is a constant, $D = (\partial_t, \nabla)$ and $C > 0$ is some constant. We can see the reference [43] for the existence of such functions. Then the test function ϕ_T is defined by

$$\phi_T(t,x) = \phi\Big(\frac{t}{T^{2-\alpha}}, \frac{x}{T}\Big), \quad (t,x) \in \mathbf{R} \times \mathbf{R^n},$$

with some large parameter T. Let P_T be the subset of $\mathbf{R} \times \mathbf{R^n}$ where $\phi_T = 1$ and

$$Q_T = (\mathrm{supp}(D^2\phi_T) \cup \mathrm{supp}(D\phi_T)) \cap ([0,+\infty) \times \mathbf{R^n}),$$

that is, Q_T is the support of derivatives restricted to $t \geq 0$. It is easy to see that

$$P_T \supset \{(t,x) : t \leq T^{2-\alpha} \text{and} \quad |x| \leq RT\},$$

$$Q_T \subset \{(t,x) : t > T^{2-\alpha} \text{or} \quad |x| > RT\}. \tag{2.44}$$

We will prove the theorem by making use of the method of contradiction. To do so, we assume that a global solution u exists. This means that $T_m = +\infty$ when

$$1 < p \leq p_c = 1 + 2/(n-\alpha),$$

and

$$\int_{\mathbf{R^n}} (u_1(x) + V(x)u_0(x))\, dx > 0.$$

To arrive at a contradiction, we multiply equation (1.1) by ϕ_T^q, with $q = \frac{2p}{p-1}$, and integrate by parts over $[0,+\infty) \times \mathbf{R^n}$:

$$\begin{aligned}\int_0^\infty \int_{\mathbf{R^n}} u(\partial_t^2 \phi_T^q - \Delta \phi_T^q - V\partial_t \phi_T^q) dxdt = &\int_0^\infty \int_{\mathbf{R^n}} |u|^p \phi_T^q dxdt \\ &+ \int_{\mathbf{R^n}} (Vu_0 + u_1) dx.\end{aligned} \tag{2.45}$$

Here we use $\phi_T(0,x) = 1$, $\partial_t \phi_T(0,x) = 0$, and the initial conditions on u to evaluate boundary integrals at $t = 0$. Next, we estimate the integral on the left-hand side by Hölder's inequality and compare it with the integral on the right-hand side. A straightforward calculation yields

$$\begin{aligned}|\partial_t^2 \phi_T^q - \Delta \phi_T^q - V\partial_t \phi_T^q| &\leq C(\phi_T^{q-1}|D^2\phi_T| + \phi_T^{q-2}|D\phi_T|^2 + V\phi_T^{q-1}|\partial_t \phi_T|) \\ &\leq C(T^{2\alpha-4}\phi_T^{q-3/4} + T^{-2}\phi_T^{q-3/4}) \\ &+ C(T^{-2}\phi_T^{q-1} + T^{2\alpha-4}\phi_T^{q-1} + T^{\alpha-2}V\phi_T^{q-1/2})\end{aligned}$$

for $(t,x) \in Q_T$, while $\partial_t^2 \phi_T^q - \Delta \phi_T^q - V\partial_t \phi_T^q$ is identically 0 for $(t,x) \notin Q_T$. Since $\alpha \in [0,1)$ and $\phi_T \le C$, the expression is bounded by $C(T^{-2}\phi_T^{q-1} + T^{\alpha-2}V\phi_T^{q-1/2})$. Thus, the left side of identity (2.45) satisfies

$$\begin{aligned}
&\Big| \int_0^\infty \int_{\mathbf{R^n}} u(\partial_t^2 \phi_T^q - \Delta \phi_T^q - V\partial_t \phi_T^q) dx dt \Big| \\
&\le C \int_0^\infty \int_{Q_T} |u|(T^{-2}\phi_T^{q-1} + T^{\alpha-2}V\phi_T^{q-1/2}) dx dt \\
&\le C \Big(\int_0^\infty \int_{Q_T} |u|^p \phi_T^q dx dt \Big)^{1/p} (I(T))^{(p-1)/p},
\end{aligned} \tag{2.46}$$

with

$$\begin{aligned}
I(T) = T^{-2p/(p-1)} &\int_0^\infty \int_{\mathbf{R^n}} \phi_T^{q-p/(p-1)} dx dt \\
&+ T^{(\alpha-2)p/(p-1)} \int_0^\infty \int_{\mathbf{R^n}} V^{p/(p-1)} \phi_T^{q-p/(2(p-1))} dx dt.
\end{aligned}$$

Notice that in the above identity all exponent of ϕ_T are positive since

$$V(x) = V_0(1+|x|)^{-\alpha}$$

satisfies the condition (1.3). Hence

$$I(T) \le CT^{-2p/(p-1)} \int_0^{2T^{2-\alpha}} \int_{|x| \le 2RT} dx dt$$

$$+CT^{(\alpha-2)p/(p-1)} \int_0^{2T^{2-\alpha}} \int_{|x|\le 2RT} (1+|x|)^{-\alpha p/(p-1)} dx dt,$$

which simplifies to

$$I(T) \leq C \begin{cases} T^{-2p/(p-1)+2-\alpha+n} + T^{(\alpha-2)p/(p-1)+2-\alpha}, & \text{if } \alpha p/(p-1) > n, \\ T^{-2p/(p-1)+2-\alpha+n} + T^{(\alpha-2)p/(p-1)+2-\alpha} \log T, & \text{if } \alpha p/(p-1) = n, \\ T^{-2p/(p-1)+2-\alpha+n} + T^{-2p/(p-1)+2-\alpha+n}, & \text{if } \ \alpha p/(p-1) < n. \end{cases}$$

An upper bound for $I(T)$ is

$$I(T) \leq CT^{-2p/(p-1)+2-\alpha+n}.$$

We can return to (2.46) and write

$$\Big| \int_0^\infty \int_{\mathbf{R^n}} u(\partial_t^2 \phi_T^q - \Delta \phi_T^q - V \partial_T \phi_T^q) dxdt \Big|$$

$$\leq C \Big(\int_0^\infty \int_{Q_T} |u|^p \phi_T^q dxdt \Big)^{1/p} T^{-2+(2-\alpha+n)(p-1)/p},$$

where C is independent of T. Substituting this estimate into (2.45) we have

$$\int_0^\infty \int_{\mathbf{R^n}} |u|^p \phi_T^q dxdt$$

$$\leq C \Big(\int_0^\infty \int_{Q_T} |u|^p \phi_T^q dxdt \Big)^{1/p} T^{-2+(2-\alpha+n)(p-1)/p}. \tag{2.47}$$

Finally, we show that the above inequality can not hold as $T \to \infty$. If $p \leq p_c$, then

$$-2 + (2 - \alpha + n)(p-1)/p \leq 0,$$

and (2.3) shows that

$$\Big(\int_0^\infty \int_{P_T} |u|^p dxdt \Big)^{(p-1)/p} \leq C.$$

Letting $T \to \infty$ and using (2.44), we conclude that $u \in L^p([0, +\infty) \times \mathbf{R^n})$. Hence

(2.44) also implies that $\| u \|_{L^p(Q_T)} \to 0$ as $T \to \infty$. Passing to the limit in (2.3), we see that $\|u\|_{L^p([0,+\infty)\times\mathbf{R^n})} \leq 0$ for any $1 < p \leq p_c$. This is impossible, since u is non-trivial solution. $\square$

CHAPTER 3

CONCLUSION

In the thesis we have investigated global existence and long time behavior of solutions as $t \to \infty$ of a Cauchy problem for the semilinear damped wave equations (1.1) having the power type of nonlinearity:

$$u_{tt} - \Delta u + V(x)u_t = |u|^p$$

in $[0, \infty) \times \mathbf{R}^n$, where $u = u(t, x)$ is a real-valued unknown function, p is the exponent of the nonlinear term with $1 < p < \frac{n+2}{n-2}$, and V is a positive space-dependent potential behaving like $|x|^{-\alpha}$ with $\alpha \in [0, 1)$, which decays to zero for large x (see (1.3)). The initial data (u_0, u_1) (see (1.2)) are compactly supported and taken from the energy space $H^1 \times L^2$. and satisfy the smallness condition.

We have observed that existence of solutions as a global or local depends on a value of p, called a critical value and denoted by p_c, which separates the values of p such that for the values p on the left side of p_c, that is, $1 < p < p_c$, we have a global solution provided that the initial data is small; while there is no global solution when $p \le p_c$ whatever the initial data is. It is shown that $p_c = 1 + \frac{2}{n-\alpha}$, which depends not only p and the space dimensions n ($n \ge 1$), but also α, the decay rate of the potential V.

In the case of existence of small data global solutions, this occurs when $1 < p < p_c$, the long-time behavior of solutions have been studied by deriving decay rates of the energy $E_u(t)$ (see (1.5) for the definition) and L^{p+1}-norms of solutions as $t \to \infty$. It is shown that the decay rates are polynomial and depend on the parameters p, α and n (see Theorem 2.1). Here the method for obtaining these decay rates, the strengthened multiplier method developed in Todorova and Yordanov [39] for the corresponding linear equation of (1.1), has been used, and it has relied on the positive solutions of the Poisson equation (see (2.6)).

We have also seen that in deriving precise decay rates for solutions weighted energy method is more appropriate if we can construct suitable weight functions. The construction of these weights is based on diffusive structure of the semilinear damped wave equations, meaning that solutions of the damped wave equations have same asymptotic profile with solutions of semilinear heat equation for large times. This fact can be.

Finally, in the case when $p > p_c$ it is proved that there is no longer a global solution by constructing suitable test functions developed by Zhang [43, 45] and using the method by contradiction.

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