COMPARISON OF TDOA BASED SOURCE LOCALIZATION METHODS AND IMPROVEMENT IN SOURCE LOCALIZATION ACCURACY WITH SENSOR POSITION UNCERTAINTIES

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ABSTRACT

Comparison of TDOA Based Source Localization Methods and Improvement in Source Localization Accuracy with Sensor Position Uncertainties

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The accurate and precise localization of an emitting source has applications in a number of fields including wireless communications, military, surveillance, geophysics and navigation systems. This thesis investigates various existing source localization methods based on time difference of arrival (TDOA) measurements for the case (*i*) when the knowledge of the sensor positions is assumed to be perfect and exact without any uncertainties in them and for the case (*ii*) when sensor position errors are assumed to be present and taken into consideration when formulating the source localization algorithms. The relative localization performance of each method is studied by comparing the Root Mean Square Errors (RMSEs) and Cramer-Rao Lower Bounds (CRLBs) of the corresponding resulting estimates for various sensor distribution scenarios and simulation conditions. Moreover, the theoretical analysis predicting the significant improvement in localization performance when sensor position errors are considered in contrast to the case when they are ignored, is corroborated through simulation.

Keywords: Source Localization, Time Difference of Arrival (TDOA), Sensor Position Errors, Least-Squares, Nonconvex Problem, Constrained Optimization

ÖZET

Varış Zaman Farkı Temelli Kaynak Konumlandırma Yöntemlerinin Karşılaştırılması ve Algılayıcı Konumu Hataları Dikkate Alındığında Kaynak Konumlandırma Doğruluğundaki İyileşme

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Yayın yapan bir kaynağın doğru ve hassas konumlanması, kablosuz iletişim, askeri, gözetim, jeofizik ve navigasyon sistemleri gibi çeşitli alanlarda uygulamalara sahiptir. Bu tezde, varış zaman farkı (VZF) ölçümlerine dayalı mevcut çeşitli kaynak konumlandırma yöntemleri, (*i*) algılayıcı konum bilgisi herhangi bir hata olmaksızın tam ve doğru varsayıldığı ve (*ii*) kaynak konumlandırma algoritması formülize edilirken algılayıcı konum hataları mevcut varsayıldığı dikkate alınarak incelenmektedir. Her yöntemin nispi konumlandırma performansı, farklı kaynak ve algılayıcı dağılımı ve benzetim senaryoları ve koşulları için, elde edilen kestirimlerin kök ortalama kare hatası (RMSE)'yi ve Cramér Rao alt sınır (CRLB)'yi karşılaştırmasıyla incelenmiştir. Ayrıca, algılayıcı konum hataları göz ardı edildikleri durumun aksine algılayıcı konum hataları dikkate alındığında konumlandırma performansındaki önemli iyileşmeyi öngören teorik analizi pekiştirilmiştir.

Anahtar Sözcükler: Kaynak Konumlandırma, Varış Zaman Farkı (VZF), Algılayıcı Konum Hataları, En Küçük Kareler, Sınırlı Optimumlaştırma

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Aslam Shaikh

July, 2020

28/07/2020

STATEMENT OF COMPLIANCE WITH ETHICAL PRINCIPLES AND RULES

I hereby truthfully declare that this thesis is an original work prepared by me; that I have behaved in accordance with the scientific ethical principles and rules throughout the stages of preparation, data collection, analysis and presentation of my work; that I have cited the sources of all the data and information that could be obtained within the scope of this study, and included these sources in the references section; and that this study has been scanned for plagiarism with "scientific plagiarism detection program" used by Anadolu University, and that "it does not have any plagiarism" whatsoever. I also declare that, if a case contrary to my declaration is detected in my work at any time, I hereby express my consent to all the ethical and legal consequences that are involved.

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GLOSSARY OF SYMBOLS AND ABBREVIATIONS

AOA	Angle of Arrival
BLUE	Best Linear Unbiased Estimator
BS	Base Station
CLS	Constrained Least Squares
CRLB	Cramér Rao Lower Bound
CWLS	Constrained Weighted Least Squares
Convex CWLS	Convex Constrained Weighted Least Squares
DOA	Direction of Arrival
EVD	Eigenvalue Decomposition
FDOA	Frequency Difference of Arrival
GPS	Global Positioning System
LLS	Linear Least Squares
LOS	Line-of-sight
ML	Maximum Likelihood
MSE	Mean Square Error
MS	Mobile Station
MVUE	Minimum Variance Unbiased Estimator
NCC	Normalized Cross-correlation
NLS	Non-Linear Least Squares
NLOS	Non Line-of-sight
PDF	Probability Density Function
PSAP	Public Safety Answering Point
QCQP	Quadratically Constrained Quadratic Optimization
RDOA	Range Difference of Arrival
RMSE	Root Mean Square Error
RSS	Received Signal Strength

SDP	Semi-definite Programming
SNR	Signal-to-Noise Ratio
SI	Spherical Interpolation
SM	Subspace Minimization
TOA	Time of Arrival
TDOA	Time Difference of Arrival
2sWLS	Two Step Weighted Least Squares
UAV	Unmmanned Aerial Vehicle
WLS	Weighted Least Squares
2D	Two-Dimensional
3D	Three-Dimensional

1. INTRODUCTION

The accurate and precise localization of a signal source has been a problem of significant interest for the signal processing community, thanks to the number of applications in various areas such as wireless communications, surveillance, target tracking, radar, sonar, acoustics, navigation, geophysics and other sensor networks [1, 2]. Moreover, new federal requirements in terms of localization accuracy in various operating environments has been a motivation for focused research efforts aimed at accurate and efficient source localization. For example, in the US, Federal Communications Commission (FCC) mandates that the wireless service providers must report the location of call initiating mobile station (MS) to an Emergency 911 (E-911) at the public safety answering points (PSAPs) with an accuracy of 50 meters for 67% of all wireless E-911 calls [3]. Nonetheless, a higher accuracy is still desirable and research done over the recent past has been improving the accuracy performance further.

The techniques of wireless source localization, with respect to the existing infrastructure and technologies, can be categorized into two broad categories. (1) Mobile-based (*forward-link*) localization systems and (2) Network-based (or *reverse link*) localization systems [4]. In the first scenario, the MS which serves as a receiver determines its own location by measuring the signal parameters of an external system, for example, cellular network or the Global Positioning System (GPS) it operates on. In the second scenario, the parent system determines the position of the MS (as a signal source) by measuring its signal parameters at the base stations which act as receiving sensors. These sensors capture the received signals and extract the relevant data and send them to a fusion processor for the estimation of the source location. The fusion processor is the site of implementation of the localization algorithms.

The advantages in terms of lower-cost, size, weight and battery consumption at the MS handset have rendered network-based systems preferable over mobile-based systems. Moreover, in the GPS-based localization system, the mobile receiver needs signals from at least four satellites out of the global network of 24 GPS satellites which can be challenging in urban settings with limited satellite visibility, increased multipath problems and frequent signal blockages, although a hybrid implementation based on both GPS system and cellular network can also be used for improved localization performance. In general, the GPS-based approach has a relatively higher accuracy but the performance tends to degrade significantly in urban settings due to signal loss or attenuation from the GPS satellites. These aforementioned considerations and difficulties have led the researchers to rather improve and implement the network-based techniques for source localization.

As mentioned earlier, network-based source localization method makes use of some signal parameters of the emitter source signal to localize it. These signal parameters usually are Time of Arrival (TOA), Time Difference of Arrival (TDOA), Frequency Difference of Arrival (FDOA), Direction of Arrival (DOA) (or the Angle of Arrival (AOA)) and the Received Signal Strength (RSS). Usually one of these parameters is enough to localize the signal source (or target) but techniques making use of two or more parameters result in more precise localization measurements [5]. An array of spatially-separated receiver nodes collaborate together to localize the source (or target) via trilateration (or multilateration) for range-based methods and triangulation for angle-based ones. It is usually assumed that the position of all the receiver nodes are known. In a scenario when these nodes are dynamic or moving, a positioning technique such as GPS is employed to first localize the position of the receiver nodes. In other circumstances, multiple receiver nodes may cooperate to find their own positions before any attempt to localize the target is made [1].

Time of Arrival (TOA) is the one-way propagation time of the signal travelling from a source to a receiver. Time Difference of Arrival (TDOA) is the difference in the arrival times of the source signal to a pair of receivers. Direction of Arrival (DOA) is the angle of arrival of the source signal observed from the receiver's viewpoint. Received Signal Strength (RSS) is the average signal power received at the receiver. The techniques based upon these parameters have their respective advantages and shortcomings. TOA based techniques are very intuitive but it necessitates the precise synchronization of the target and receivers's clock for the accurate localization. In the case of unsynchronized clocks, large positioning errors can be seen which is highly undesirable in many applications, especially in the military where the target is inherently noncooperative and hence time synchronization is not possible. TDOA based techniques essentially solves the time synchronization problem as synchronization is only required across the receivers. RSS based techniques are cheap and quite simple to implement because the synchronization among the source and receivers is not required and doesn't have LOS (line of sight) requirements unlike other techniques but the signal decay model it employs is subject to a number of parameters which are volatile and hence the localization information it generates is usually not very precise and hence cannot meet the precision requirements of many applications. The DOA based techniques require antenna arrays at each sensor which makes it expensive.

The study carried out in this thesis work is centered around the TDOA based technique because, as mentioned earlier, time synchronization is not an issue in this case. Hence the localization scheme independent of the nature of the source or target is possible with TDOA. The source doesn't have to cooperate in any manner with the receivers. This attribute of TDOA is especially attractive for military applications where the source (enemy warplanes, for instance) is inherently uncooperative and sensors have to be passive to remain unnoticed by these enemy systems.

TDOA based localization techniques are implemented in two steps. In the first step, TDOA values are estimated using the output measurements of the receiving sensors. In the next step, the target position is estimated from the TDOA values obtained in the first step along with the information about the sensor positions. The accuracy of the estimated target position depends primarily on the accuracy of these TDOA values and also on the geometry of the receiver sensor network [6].

1.1. Related works and our motivation

There are numerous studies [7, 8, 9] on TDOA based source localization and researchers have developed a number of novel algorithms for various localization scenarios. Some of them are more robust and efficient than others. However, some might be more suitable than others depending upon the application need. Maximum Likelihood (ML) method is regarded as an optimal technique for parameter estimation. However, it has been demonstrated that a closed-form maximum likelihood estimator (MLE) does not exist owing to the non-linearity of the problem [10]. An iterative method which makes use of the Taylor series expansion has been introduced to obtain the ML estimate [11, 10]. Since the formulation of the ML target localization with TDOA results in a non-convex optimization problem, semidefinite relaxation is proposed in [12] to convert the non-convex ML optimization to a convex SDP problem, but the performance is shown to be poor when the measurement noise level is small as is presented in its simulation results.

Iterative methods, however, tends to be computationally expensive and needs an initial guess close to the true value (to avoid the local minima) which is not feasible for many practical applications. Closed-form solution, hence, is attractive because it does not require an initial guess, unlike iterative solutions, and doesn't suffer from the local convergence problem. Thus, suboptimal closed-form methods have received significant attention in the research community with regards to TDOA based source localization. It can be considered as a trade-off between the disadvantages of iterative methods and the simplicity of closed-form ones. These methods, despite their suboptimality, are useful for real-time systems with low computational capacity. Furthermore, these methods can be used to obtain an initial estimate for the iterative methods including iterative ML, mentioned earlier. In a number of works over the years, these suboptimal methods are formulated and explored, including spherical interpolation [13], spherical intersection [14], subspace minimization [15], two-stage weighted least squares [8].

Apart from methods that only make use of TDOA measurements, there are also hybrid methods that use other signal parameters such as FDOA [16, 17, 18], DOA [19], RSS [20] and TOA [21] along with TDOA. Hybrid methods tend to result in more accurate localization.

A large body of work devoted to developing algorithms for source localization have assumed the receiver locations to be exact and perfectly known. Sensor position uncertainties, which might occur as a result of inherent measurement errors in measuring any quantity, were not considered. In real-world scenario, accurate information on sensor location may not be available for various reasons. For example, sensors could be attached on-board on unmanned aerial vehicles (UAVs) or aeroplanes whose positions cannot be known exactly (and precisely) even with advanced GPS technology [22]. Moreover, in a sensor network, sensors are randomly placed in a surveillance area so that it is difficult to know their locations accurately. Hence, in recent years, researchers have tackled the source localization problem by also taking sensor position errors into account [6, 23, 24, 25]. As demonstrated in [26], even a small sensor position error can result in a considerable degradation in the localization accuracy.

1.2. Thesis Scope and Objective

In this thesis, source localization problem based upon TDOA measurements was investigated assuming that (*i*) sensor position noise is absent or (*ii*) sensor position noise is present. Various source localization methods, found in the literature, were studied separately taking these assumptions into consideration. Their relative performance was compared under different simulation conditions and scenarios and some interesting observations under these conditions were noted and remarked upon. For some methods, a few ideas from other similar works found in the literature, were incorporated to slightly improve their estimation performance or reduce the computational complexity. Theoretical results showing the significant degradation in localization performance when the sensor position errors are ignored was confirmed using simulation results.

1.3. Overview

The remainder of the thesis is organized as follows. In Chapter 2, Section 2.1 briefly discusses the important methods for the estimation of TDOA values. It is followed by the theoretical discussion on TDOA based localization. Then, Section 2.3 introduces the problem statement and finally, in the subsequent sections, TDOA based source localization methods are mathematically formulated starting briefly with the non-linear based techniques (Section 2.4.1) such as non-linear least squares (NLS) and iterative ML and then proceeding towards linear techniques (Section 2.4.2) such as linear least squares (LLS), subspace minimization (SM) to name just a few. The discussion on the localization methods based on linear techniques covers the major part of the thesis. In Chapter 3, corresponding TDOA based source localization methods taking sensor position uncertainties into account are formulated. In Chapter 4, simulation results for various cases and scenarios are presented. Localization performance of various methods given in the previous chapters (Chapters 2 and 3) are compared under various simulation conditions for a number of distribution scenarios. Chapter 5 summarizes the work and gives conclusions and recommendations for future studies. In Appendix A, Cramer Rao lower bound (CRLB) for the exact sensor position case (A.1) and for the noisy sensor position cases (A.2) is obtained. In Appendix B, proof of the lemma and theorem used in formulating the algorithm given in Section 3.7 is given.

2. TDOA SOURCE LOCALIZATION: EXACT SENSOR POSITIONS

2.1. Estimation of TDOA values

As the name implies, TDOA estimation is based upon the measurement of the difference in time between the signals arriving at a pair of receivers. A pair of receivers gives one TDOA measurement. For the localization performance to be satisfactory, the obtained TDOA values must also be accurate. For that reason, efforts are made to reduce the error in the estimation of TDOA values as much as possible. Consequently, depending upon the ambient physical conditions, sensor properties, signal characteristics of the received signal and other influencing factors, most appropriate TDOA estimation technique is employed. Since a detailed explanation of these techniques are not within the scope of this thesis work, only a short introduction to these estimation techniques are mentioned in the following paragraphs.

In general, there are two methods to obtain the TDOA measurements. In one of these methods, namely leading edge detection [27], the time instant at which the sensor output surpasses a predefined threshold value is assumed to be the signal arrival time. Then, this *arrival time* of the signal to the reference sensor is subtracted from that of the others which gives the TDOA values. In the other method, namely correlation method [28], the output of the reference sensor and that of the other sensors are cross-correlated as shown below. The maximum of this correlation function over an unknown delay t' (which needs to be estimated) gives the estimate of the true TDOA value between the two received signals $x_1(t)$ and $x_2(t)$, denoted by $\hat{\tau}$ as follows:

$$\hat{\tau} = \arg\max_{t'} \sum_{t=t_1}^{t_2} x_1(t)x_2(t-t') \tag{2.1}$$

Furthermore, the normalized cross-correlation (NCC), given below, is more useful and offers robustness with respect to the gain difference between the two signals $x_1(t)$ and $x_2(t)$ as follows:

$$\hat{\tau} = \arg\max_{t'} \frac{\sum_{t=t_1}^{t_2} x_1(t)x_2(t-t')}{\sqrt{\sum_{t=t_1}^{t_2} {x_1}^2(t)}\sqrt{\sum_{t=t_1}^{t_2} {x_2}^2(t)}} \tag{2.2}$$

Usually the cross-correlation technique is preferred because it allows the asynchronous operation between the source and the receivers. Even though, TDOA based localization allows the clock offset between the source and the receivers, synchronization between the receivers is nonetheless necessary for the accurate TDOA measurement.

2.2. TDOA Localization as Hyperbolic Positioning

TDOA localization is also known as hyperbolic positioning since each TDOA measurement results in a hyperbola passing between two receivers used in getting the TDOA measurement. A hyperbola is the locus of a point in a plane such that the difference of distances from two fixed points (known as foci) is a constant. As shown

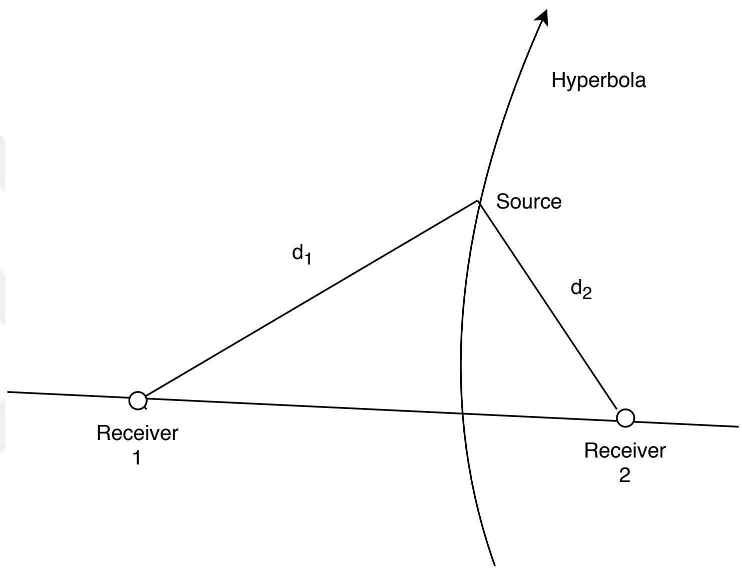


Figure 2.1: The hyperbola of a TDOA measurement

in Figure 2.1, the range difference

$$d_1 - d_2$$

calculated from the TDOA measurement implies a hyperbola along which the source will lie. With more receivers, there will be more hyperbolas for each sensor pair. The intersection point of these hyperbolas is the possible location of the source.

2.3. Problem Statement

TDOA based localization system is based on finding the source location using TDOA values measured at the receivers. Usually one of the receivers is taken as a reference sensor relative to which TDOA values are taken. In case of MS positioning, the base station which first receives the target or source signal is considered as the reference station. As is obvious, N receivers will result in $(N - 1)$ TDOA values. In a 2D scenario, two TDOA values from three receivers are enough to localize the target assuming that the positions of the receivers are known. In a 3D scenario, three

TDOA values from 4 receivers are enough to localize the target. As mentioned in the previous chapter, TDOA localization consists of two steps. In the first step, TDOA values are estimated from the receivers' outputs and in the second step, these TDOA values are processed using suitable algorithms to estimate the source position. The following section deals with this second step assuming that we have obtained TDOA values through the first step. For the simplicity of illustration, the problem is formulated for 2D case. 3D formulation would merely result in an increase in matrix and vector dimensions and degrees of polynomial.

2.3.1. Generalized Measurement Model

In general, TDOA based localization can be modeled in the following manner assuming the sensor positions are exactly known. *The case of uncertainties in the sensor positions will be explored later in the next chapter.* In fact this model is representative of all the localization models irrespective of the parameter used, not only for TDOA.

$$\mathbf{r} = \mathbf{f}(\mathbf{x}) + \mathbf{n} \tag{2.3}$$

where $\mathbf{r}$ is the measurement vector, $\mathbf{x}$ is the source position to be determined, $\mathbf{f}(\mathbf{x})$ is a known nonlinear function and $\mathbf{n}$ is an additive zero-mean noise vector. The source localization problem, in general, is to estimate $\mathbf{x}$ given $\mathbf{r}$.

2.3.2. Formulation of TDOA Based Source Localization Problem

Let us consider an array of N passive, stationary sensors. Sensors are placed in a 2D plane as shown in the Figure 2.2. Let

$$\mathbf{x} = [\ x \quad y\ ]^T$$

be the unknown source position (to be estimated) and

$$\mathbf{x}_i = [\ x_i \quad y_i\ ]^T$$

be the known positions of the i^{th} sensor, $i = 1, 2, ..., N$, where $N \geq 3$ is the number of receivers. The true distance between the source and the i^{th} sensor, denoted by d_i, is given by the norm of $\mathbf{x}$ and $\mathbf{x}_i$ as follows:

$$d_i = \|\mathbf{x} - \mathbf{x}_i\|_2 = \sqrt{(x - x_i)^2 + (y - y_i)^2} \qquad i = 1, 2, ..., N \tag{2.4}$$

Let us assume, without loss of generality that the source emits a signal at time t_0 and the i^{th} sensor receives it at time $t_i; i = 1, 2, ..., N$ with $N \geq 3$; that is, t_i's

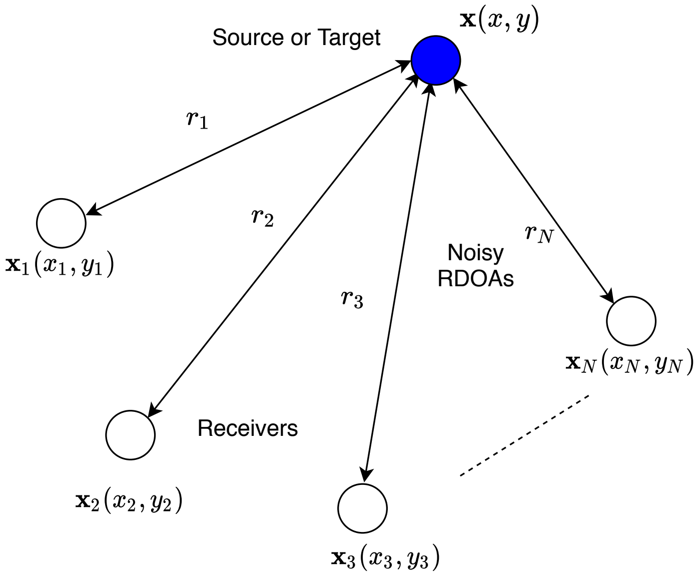


Figure 2.2: Sensor distribution

are the available TOAs (which are noisy). Note that there are $N(N-1)/2$ distinct TDOAs from all the paired combination of sensors, denoted by $t_{k,i} = (t_k - t_0) - (t_i - t_0) = t_k - t_i$; $k, i = 1, 2, ..., N$ with $k > i$. However, there are only $(N-1)$ non-redundant TDOAs. For example, in the case of $N = 3$ sensors, the distinct TDOAs are $t_{2,1}, t_{3,1}$ and $t_{3,2}$ and we can clearly see that $t_{3,2} = (t_{3,1} - t_{2,1})$ is redundant. So, to decrease complexity without compromising the localization performance, we should calculate all $N(N-1)/2$ TDOAs and reduce them to $(N-1)$ nonredundant TDOAs as measurement data for source localization [1].

Without loss of generality, if the first sensor is chosen as the reference sensor, available noisy TDOA between 1^{st} and i^{th} sensor is given as follows:

$$t_{i,1} = t_i - t_1 \tag{2.5}$$

For simplicity purposes, we convert all TDOAs to RDOAs and TOAs to their corresponding distance values using the simple relation as $ROA = c * TOA$ and $RDOA = c * TDOA$ where c is the speed of propagation of the signal and ROA is the distance between the source and the receiver. So the above Equation 2.5 can be rewritten as

$$r_{i,1} = r_i - r_1 \tag{2.6}$$

where $r_{i,1} = c * t_{i,1}$ and $r_i = c * t_i$.

Now, taking hint from Generalized Measurement Model 2.3 in the above section, Noisy RDOAs $r_{i,1}$ can be written as

$$\begin{aligned} r_{i,1} &= f_{i,1}(\mathbf{x}) + n_{i,1} \\ r_{i,1} &= d_{i,1} + n_{i,1} \end{aligned} \qquad i = 2, 3, ..., N \tag{2.7}$$

where

$$d_{i,1} = f_{i,1}(\mathbf{x}) = d_i - d_1 = \sqrt{(x - x_i)^2 + (y - y_i)^2} - \sqrt{(x - x_1)^2 + (y - y_1)^2} \tag{2.8}$$

are the noise-free RDOA values and $n_{i,1}$ represent zero-mean additive Gaussian noises.
Hence, the TDOA signal model in matrix form is

$$\begin{aligned} \mathbf{r} &= \mathbf{f}(\mathbf{x}) + \mathbf{n} \\ \mathbf{r} &= \mathbf{d} + \mathbf{n} \end{aligned} \tag{2.9}$$

where

$$\begin{aligned}
\mathbf{f}(\mathbf{x}) &= \mathbf{d} = \begin{bmatrix} \|\mathbf{x} - \mathbf{x}_2\| - \|\mathbf{x} - \mathbf{x}_1\| \\ \|\mathbf{x} - \mathbf{x}_3\| - \|\mathbf{x} - \mathbf{x}_1\| \\ \vdots \\ \|\mathbf{x} - \mathbf{x}_N\| - \|\mathbf{x} - \mathbf{x}_1\| \end{bmatrix} \\
\mathbf{f}(\mathbf{x}) &= \mathbf{d} = \begin{bmatrix} \sqrt{(x - x_2)^2 + (y - y_2)^2} - \sqrt{(x - x_1)^2 + (y - y_1)^2} \\ \sqrt{(x - x_3)^2 + (y - y_3)^2} - \sqrt{(x - x_1)^2 + (y - y_1)^2} \\ \vdots \\ \sqrt{(x - x_N)^2 + (y - y_N)^2} - \sqrt{(x - x_1)^2 + (y - y_1)^2} \end{bmatrix} \\
\mathbf{r} &= \begin{bmatrix} r_{2,1} & r_{3,1} & \cdots & r_{N,1} \end{bmatrix}^T \\
\mathbf{n} &= \begin{bmatrix} n_{2,1} & n_{3,1} & \cdots & n_{N,1} \end{bmatrix}^T
\end{aligned} \tag{2.10}$$

If the noisy RDOA vector is $\mathbf{r} = \begin{bmatrix} r_{2,1} & r_{3,1} & \cdots & r_{N,1} \end{bmatrix}^T$, true RDOA vector is $\mathbf{d} = \begin{bmatrix} d_{2,1} & d_{3,1} & \cdots & d_{N,1} \end{bmatrix}^T$, noise vector is $\mathbf{n} = \begin{bmatrix} n_{2,1} & n_{3,1} & \cdots & n_{N,1} \end{bmatrix}^T$ and the covariance matrix of $\mathbf{r}$ is $\mathbf{Q}$, then the PDF of $\mathbf{r}$ (paramaterized by $\mathbf{x}$) can be written as follows:

$$p(\mathbf{r}; \mathbf{x}) = \frac{1}{(2\pi)^{(N-1)/2} |\mathbf{Q}|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{r} - \mathbf{d})^T \mathbf{Q}^{-1} (\mathbf{r} - \mathbf{d})\right\} \tag{2.11}$$

Since $\mathbf{d}$ is a non-linear function of $\mathbf{x}$, the PDF of $\mathbf{r}$ is related to the source position

$\mathbf{x}$ and hence the PDF of $\mathbf{r}$ is paramaterized by $\mathbf{x}$ which is denoted by the notation $p(\mathbf{r};\mathbf{x})$ in Equation 2.11.

Minimum mean square error (MMSE) criteria usually produces estimators which cannot be realized. That is due to the fact that the resulting formulation contains a bias term which depends on the unknown parameter to be estimated. Hence, it would be a good idea to limit the estimator to be unbiased. Moreover, it would be useful to find the one which attains the Cramer-Rao Lower Bound (CRLB) [29]. The CRLB gives a lower bound on variance achievable by any unbiased estimator using the same data, and thus it can be used as a criterion to compare with the mean square error (MSE) of various positioning algorithms and evaluate their estimation performance. The estimator which attains the CRLB is termed as efficient. Efficient estimators might not always exist and hence the following theorem presented in [29] allows us to quickly determine if such an estimator exist.

Theorem 1 Assume $\boldsymbol{\theta}$ to be the unknown parameter vector to be estimated, $\mathbf{x}$ be the observation vector and $p(\mathbf{x};\boldsymbol{\theta})$ represent the probability density function (PDF) of $\mathbf{x}$. Let's also assume that the PDF $p(\mathbf{x};\boldsymbol{\theta})$ satisfies the "regularity" condition 2.12 given below as

$$\mathrm{E}\left[\frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right] = 0 \quad \textit{for all } \boldsymbol{\theta} \tag{2.12}$$

where we take expectation with respect to $p(\mathbf{x};\boldsymbol{\theta})$. Then the variance of any unbiased estimator must satisfy the following

$$\mathrm{var}(\hat{\boldsymbol{\theta}}) \geq \frac{1}{-\mathrm{E}\left[\frac{\partial^2 \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^2}\right]} \tag{2.13}$$

where the derivative is calculated at the true value of $\boldsymbol{\theta}$ and the expectation is taken with respect to $p(\mathbf{x};\boldsymbol{\theta})$.
Moreover, an unbiased estimator that attains the bound for all $\boldsymbol{\theta}$ can only be found if and only if

$$\frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \mathbf{I}(\boldsymbol{\theta})(\mathbf{g}(\mathbf{x}) - \boldsymbol{\theta}) \tag{2.14}$$

for some functions $\mathbf{g}$ and $\mathbf{I}$. $\mathbf{g}(\mathbf{x})$ is the minimum variance estimator (MVU) in Equation 2.14 and $[\mathbf{I}^{-1}(\boldsymbol{\theta})]_{ii}$ is the variance of the i^{th} element of the estimator vector where $\mathbf{I}^{-1}(\boldsymbol{\theta})$ is the inverse of $\mathbf{I}(\boldsymbol{\theta})$. The quantity $\mathbf{I}(\boldsymbol{\theta})$ is known as the Fisher Information Matrix (FIM).

In Appendix A, we have derived the CRLB for the source localization problem based on TDOA measurements. We have also demonstrated that the regularity condition (Equation 2.12) is satisfied. Also, for this problem, we found $\partial \ln p(\mathbf{x};\boldsymbol{\theta})/\partial \boldsymbol{\theta}$

given in Equation 2.14 to be

$$\frac{\partial \ln p(\mathbf{r};\mathbf{x})}{\partial \mathbf{x}} = \frac{\partial \mathbf{d}^T}{\partial \mathbf{x}} \mathbf{Q}^{-1}(\mathbf{r} - \mathbf{d}) \tag{2.15}$$

Since the noise-free RDOA vector $\mathbf{d}$ is a nonlinear function of the source location $\mathbf{x}$, it is not possible to put Equation 2.15 in the form of Equation 2.14, which indicates that an unbiased estimator attaining the CRLB does not exist for the given localization problem.

2.4. Source Localization Algorithms

Source localization algorithms could be divided into two main categories based on whether linear or non-linear methodology was applied [1]. In general, non-linear technique directly uses Equation 2.3 to solve for $\mathbf{x}$ by minimizing the least square (LS) or the weighted least squares (WLS) cost function built from the following error function:

$$\mathbf{e}_{nonlinear} = \mathbf{r} - \mathbf{f}(\tilde{\mathbf{x}}) \tag{2.16}$$

where $\tilde{\mathbf{x}} = \begin{bmatrix} \tilde{x} & \tilde{y} \end{bmatrix}^T$ is the optimization variable for $\mathbf{x}$. The former results in Nonlinear Least Squares (NLS) estimator and the latter (weighted version) results in Maximum Likelihood (ML) estimator.

Alternatively, the linear approach converts Equation 2.3 into a set of linear equations in $\mathbf{x}$ as follows:

$$\mathbf{b} = \mathbf{A}\mathbf{x} + \mathbf{q} \tag{2.17}$$

where $\mathbf{b}$ and $\mathbf{A}$ are known and $\mathbf{q}$ is the new transformed noise vector. Using the Equation 2.17, the following error function is constructed as

$$\mathbf{e}_{linear} = \mathbf{b} - \mathbf{A}\tilde{\mathbf{x}} \tag{2.18}$$

Using the LS or WLS techniques on Equation 2.18 results in LLS (Linear Least Squares), Weighted Least Squares (WLS) and subspace estimator, which will be explained in detail in the subsequent sections.

2.4.1. Nonlinear Techniques

As mentioned earlier, the nonlinear approach seeks to find the source location directly from Equations 2.7 and 2.9 which results in the NLS and ML estimators.

Accuracy of these schemes is generally high and they are simple to implement. Especially, when the noise statistics is unavailable, NLS method is a practical choice. However, global convergence may not be guaranteed owing to their multimodal cost function, which makes them unalluring for some applications. [1]

2.4.1.1 Nonlinear Least Squares (NLS)

The NLS method minimizes the LS cost function directly obtained from Equation 2.7. With the use of Equations 2.7 and 2.9, the NLS cost function for TDOA-based positioning, denoted by $J_{NLS}(\tilde{\mathbf{x}})$, is

$$J_{NLS}(\tilde{\mathbf{x}}) = \sum_{i=2}^{N} \left(r_{i,1} - \sqrt{(\tilde{x} - x_i)^2 + (\tilde{y} - y_i)^2} + \sqrt{(\tilde{x} - x_1)^2 + (\tilde{y} - y_1)^2} \right)^2 \tag{2.19}$$

Or in the matrix form

$$J_{NLS}(\tilde{\mathbf{x}}) = (\mathbf{r} - \mathbf{f}(\tilde{\mathbf{x}}))^T (\mathbf{r} - \mathbf{f}(\tilde{\mathbf{x}})) \tag{2.20}$$

The NLS position estimate is then given by

$$\hat{\mathbf{x}}_{NLS} = \arg\min_{\tilde{\mathbf{x}}} J_{NLS}(\tilde{\mathbf{x}}) \tag{2.21}$$

The optimization problem (Equation 2.21) is non-convex and thus it is a little bit challenging to find $\hat{\mathbf{x}}$ because there are local minima along with the global minimum in the two-dimensional surface of the cost function $J_{NLS}(\tilde{\mathbf{x}})$. In general, there are two schemes to solve Equation 2.21. In the first scheme, global exploration is performed by applying grid search or random search techniques like genetic algorithm and particle swarm optimization. The second scheme, on the contrary, deals with local search, in which $\hat{\mathbf{x}}$ is found iteratively based on an initial guess $\hat{\mathbf{x}}^0$, where 0 referes to the 0th iteration. If the initial position estimate $\hat{\mathbf{x}}^0$ is close enough to $\mathbf{x}$, $\hat{\mathbf{x}}$ can be obtained in a certain number of iterations. Newton-Rhapson, Gauss-Newton and steepest descent methods are three regularly used local search techniques. [1]

2.4.1.2 Maximum Likelihood Estimation (MLE)

Maximum likelihood (ML) estimation, which is based on the maximum likelihood principle, is the most useful approach for finding the practical estimator. It can be considered as an alternative to the Minimum Variance Unbiased (MVU) estimator which is preferable when the MVU estimator does not exist or cannot be realized even if does exist. It is a simple procedure which allows it to be implemented for complicated estimation problems. In addition to that, for most practical cases, its performance is optimal for large enough data. It can be thought of as an approximate MVU estimator due to its approximate efficiency. [29] Moreover, ML estimator can be seen as a weighted version of the NLS technique by making use of the known noise covariance.

Given that the noise distribution is known, the ML approach seeks to maximize the PDF $p(\mathbf{r};\mathbf{x})$ of the available noisy data $\mathbf{r}$. The problem is to find $\mathbf{x}$ which gives the maximum value of $p(\mathbf{r};\mathbf{x})$ given in Equation 2.11. Since the natural logarithm is a monotonically increasing function, taking the natural logarithm of Equation 2.11 and finding $\mathbf{r}$ which maximizes this resulting log PDF function would give the same result as

$$\hat{\mathbf{x}}_{ML} = \arg\max_{\mathbf{x}}\{p(\mathbf{r};\mathbf{x})\} = \arg\max_{\mathbf{x}}\{\ln(p(\mathbf{r};\mathbf{x}))\} \tag{2.22}$$

The natural logarithm of Equation 2.11 gives

$$\ln(p(\mathbf{r};\mathbf{x})) = -\ln((2\pi)^{(N-1)/2}|\mathbf{Q}|^{1/2}) - \frac{1}{2}(\mathbf{r}-\mathbf{d})^T\mathbf{Q}^{-1}(\mathbf{r}-\mathbf{d})\} \tag{2.23}$$

As the first term on the right hand side of Equation 2.23 is independent of $\mathbf{x}$, maximizing Equation 2.23 is equivalent to minimizing the second term, hence the ML estimate can be written as follows:

$$\hat{\mathbf{x}}_{ML} = \arg\min_{\mathbf{x}} (\mathbf{r}-\mathbf{d})^T\mathbf{Q}^{-1}(\mathbf{r}-\mathbf{d}) \tag{2.24}$$

Using Equation 2.7 we can write the above Equation 2.24 as :

$$\begin{aligned}\hat{\mathbf{x}}_{ML} &= \arg\min_{\tilde{\mathbf{x}}} (\mathbf{r}-\mathbf{f}(\tilde{\mathbf{x}}))^T\mathbf{Q}^{-1}(\mathbf{r}-\mathbf{f}(\tilde{\mathbf{x}})) \\ \hat{\mathbf{x}}_{ML} &= \arg\min_{\tilde{\mathbf{x}}} J_{ML}(\tilde{\mathbf{x}})\end{aligned} \tag{2.25}$$

where

$$\begin{aligned}J_{ML}(\tilde{\mathbf{x}}) &= (\mathbf{r}-\mathbf{f}(\tilde{\mathbf{x}}))^T\mathbf{Q}^{-1}(\mathbf{r}-\mathbf{f}(\tilde{\mathbf{x}})) \\ &= \sum_{i=2}^{N}\frac{\left(r_{i,1}-\sqrt{(\tilde{x}-x_i)^2+(\tilde{y}-y_i)^2}+\sqrt{(\tilde{x}-x_1)^2+(\tilde{y}-y_1)^2}\right)^2}{\sigma_i^2}\end{aligned} \tag{2.26}$$

Comparing Equations 2.20 and 2.21 with Equation 2.25, we observe that ML estimator is a weighted version of NLS estimator. So it can be said that in the presence of zero-mean Gaussian noise, the former generalizes the latter. When $\mathbf{Q}^{-1}$ is proportional to the identity matrix or σ_i^2 are identical, the ML estimator take the same form as NLS method. Even if the additive noise is assumed not to be Gaussian in Equation 2.7, the estimator in Equation 2.24 is still useful. In this scenario, the estimator is considered as the (weighted) least squares estimator where $\mathbf{Q}^{-1}$ becomes the weighting matrix [10]. The detailed discussion on weighted least squares is given in Section 2.4.2.2.

In order to find the ML estimate given in Equation 2.24, we simply take the partial derivative of Equation 2.23 with respect to $\mathbf{x}$ and equate it to zero as follows:

$$\frac{\partial \ln(p(\mathbf{r};\mathbf{x}))}{\partial \mathbf{x}} = \frac{\partial \mathbf{d}^T}{\partial \mathbf{x}} \mathbf{Q}^{-1}(\mathbf{r}-\mathbf{d}) = \mathbf{0}_{2\times 1} \tag{2.27}$$

In Equation 2.10, it can be clearly seen that $\mathbf{d}$ is a nonlinear function of $\mathbf{x}$ which means that there is no closed-form solution for MLE. In such cases, we invoke numerical methods to find the estimate. In the case of MLE, one of such methods is called Iterative ML which is proposed in [10, 11]. The following section examines it in detail.

2.4.1.3 Iterative ML Method

Let's assume $\hat{\mathbf{x}}^0$ to be the initial guess. Taylor series expansion around this initial estimate can be written up to the first order as follows:

$$\mathbf{d}(\mathbf{x}) \simeq \mathbf{d}(\hat{\mathbf{x}}^0) + \mathbf{J}_{\hat{\mathbf{x}}^0}(\mathbf{x}-\hat{\mathbf{x}}^0) \tag{2.28}$$

where

$$\mathbf{J}_{\hat{\mathbf{x}}^0} \triangleq \left.\frac{\partial \mathbf{d}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\hat{\mathbf{x}}^0} = \begin{bmatrix} \left.\frac{\partial d_{2,1}}{\partial x}\right|_{\mathbf{x}=\hat{\mathbf{x}}^0} & \left.\frac{\partial d_{2,1}}{\partial y}\right|_{\mathbf{x}=\hat{\mathbf{x}}^0} \\ \left.\frac{\partial d_{3,1}}{\partial x}\right|_{\mathbf{x}=\hat{\mathbf{x}}^0} & \left.\frac{\partial d_{3,1}}{\partial y}\right|_{\mathbf{x}=\hat{\mathbf{x}}^0} \\ \vdots & \vdots \\ \left.\frac{\partial d_{N,1}}{\partial x}\right|_{\mathbf{x}=\hat{\mathbf{x}}^0} & \left.\frac{\partial d_{N,1}}{\partial y}\right|_{\mathbf{x}=\hat{\mathbf{x}}^0} \end{bmatrix} \tag{2.29}$$

Here, $\mathbf{J}_{\hat{\mathbf{x}}^0}$ is the Jacobian matrix of $\mathbf{d}(\mathbf{x^0}) = \mathbf{f}(\mathbf{x^0})$ computed at $\hat{\mathbf{x}}^0$. Substituting Equation 2.28 into the ML cost function (Equation 2.24), we get

$$J_{ML}(\mathbf{x}) = (\mathbf{r}'_{\hat{\mathbf{x}}^0} - \mathbf{J}_{\hat{\mathbf{x}}^0}\mathbf{x})^T \mathbf{Q}^{-1}(\mathbf{r}'_{\hat{\mathbf{x}}^0} - \mathbf{J}_{\hat{\mathbf{x}}^0}\mathbf{x}) \tag{2.30}$$

where

$$\mathbf{r}'_{\hat{\mathbf{x}}^0} = \mathbf{r} - \mathbf{d}(\hat{\mathbf{x}}^0) + \mathbf{J}_{\hat{\mathbf{x}}^0}\hat{\mathbf{x}}^0$$

Taking the gradient of Equation 2.30 and equating to zero, we get

$$\nabla_{\mathbf{x}} J_{ML}(\mathbf{x})\big|_{\mathbf{x}=\hat{\mathbf{x}}^1} = 2\mathbf{J}^T_{\hat{\mathbf{x}}^0}\mathbf{Q}^{-1}\mathbf{J}_{\hat{\mathbf{x}}^0}\hat{\mathbf{x}}^1 - 2\mathbf{J}^T_{\hat{\mathbf{x}}^0}\mathbf{Q}^{-1}\mathbf{r}_{\hat{\mathbf{x}}^0} = \mathbf{0}_{2\times 1} \tag{2.31}$$

Solving Equation 2.31 results in

$$\hat{\mathbf{x}}^1 = (\mathbf{J}^T_{\hat{\mathbf{x}}^0}\mathbf{Q}^{-1}\mathbf{J}_{\hat{\mathbf{x}}^0})^{-1}\mathbf{J}^T_{\hat{\mathbf{x}}^0}\mathbf{Q}^{-1}\mathbf{r}_{\hat{\mathbf{x}}^0} \tag{2.32}$$

Equation 2.32 can be extended to the general form as

$$\hat{\mathbf{x}}^{k+1} = (\mathbf{J}^T_{\hat{\mathbf{x}}^k}\mathbf{Q}^{-1}\mathbf{J}_{\hat{\mathbf{x}}^k})^{-1}\mathbf{J}^T_{\hat{\mathbf{x}}^k}\mathbf{Q}^{-1}\mathbf{r}_{\hat{\mathbf{x}}^k} \tag{2.33}$$

where

$$\mathbf{r}_{\hat{\mathbf{x}}^k} = \mathbf{r} - \mathbf{d}(\hat{\mathbf{x}}^k) + \mathbf{J}_{\hat{\mathbf{x}}^k}\hat{\mathbf{x}}^k \tag{2.34}$$

and

$$\mathbf{J}_{\hat{\mathbf{x}}^k} \triangleq \frac{\partial \mathbf{d}(\mathbf{x})}{\partial \mathbf{x}}\bigg|_{\mathbf{x}=\hat{\mathbf{x}}^k} \tag{2.35}$$

Source location estimate is found by iterating Equations 2.33, 2.34 and 2.35 until $\hat{\mathbf{x}}$ converges. Convergence is usually decided by some stopping criteria which includes a number of iterations and in some cases by a very small positive constant ε where $\|\hat{\mathbf{x}}^{k+1} - \hat{\mathbf{x}}^k\|_2 < \varepsilon$.

As is the case for all iterative methods, the iterative ML method is also computationally expensive compared to the closed form methods and needs an initial guess to start the algorithm. Estimation performance, hence, is dependent on the accuracy of this initial guess. Moreover, global convergence is not guaranteed.

2.4.2. Linear Techniques

The main idea of linear techniques is to transform the original nonlinear equation set 2.7 into a set of linear equations with zero-mean errors, given that the measurement noise is sufficiently small. This linearization would result in the corresponding optimization cost function to be unimodal which means that the global solution is always guaranteed [1]. Hence, the closed-form estimator could be obtained which is highly desirable for real-time systems or for applications having low computational capability. Moreover, closed form estimators can also be used to obtain an initial estimate for the iterative methods like nonlinear least squares and iterative ML mentioned in the previous Section 2.4.1. This section deals with the closed-form estimators for TDOA based source localization problem.

2.4.2.1 Linear Least Squares (LLS)

In this method, equation set 2.7 is firstly converted into a set of linear equations in $\mathbf{x}$ which then allows for the application of the ordinary LS technique to find the source location. During this conversion, an intermediate variable has to be introduced which is a function of the source position.
Let's rewrite the Equation 2.7 as follows:

$$r_{i,1} + \sqrt{(x - x_1)^2 + (y - y_1)^2} = \sqrt{(x - x_i)^2 + (y - y_i)^2} + n_{i,1} \tag{2.36}$$

Squaring both sides of Equation 2.36, we get

$$\begin{aligned} 2(x_1 - x_i)(x - x_1) + 2(y_1 - y_i)(y - y_1) - 2r_{i,1}\sqrt{(x - x_1)^2 + (y - y_1)^2} \\ + {n_{i,1}}^2 + 2n_{i,1}\sqrt{(x - x_i)^2 + (y - y_i)^2} \\ = {r_{i,1}}^2 - (x_1 - x_i)^2 - (y_1 - y_i)^2 \end{aligned} \tag{2.37}$$

Let

$$m_{i,1} = {n_{i,1}}^2 + 2n_{i,1}\sqrt{(x - x_i)^2 + (y - y_i)^2} \tag{2.38}$$

be the transformed noise component and

$$d_1 = \sqrt{(x - x_1)^2 + (y - y_1)^2} \tag{2.39}$$

be the dummy (intermediate) variable. Now, using Equations 2.38 and 2.39, Equation 2.37 can be written as

$$\begin{aligned} 2(x_1 - x_i)(x - x_1) + 2(y_1 - y_i)(y - y_1) - 2r_{i,1}d_1 + m_{i,1} \\ = {r_{i,1}}^2 - (x_1 - x_i)^2 - (y_1 - y_i)^2 \end{aligned} \tag{2.40}$$

As can be clearly seen, the above Equation 2.40 is linear in $\mathbf{x}$, if the intermediate variable d_1 is considered as an independent variable (which we will assume to be the case). It can be written in the matrix form as

$$\mathbf{A}\boldsymbol{\theta} + \mathbf{q} = \mathbf{b} \tag{2.41}$$

where

$$\mathbf{A} = 2\begin{bmatrix} x_1 - x_2 & y_1 - y_2 & -r_{2,1} \\ x_1 - x_3 & y_1 - y_3 & -r_{3,1} \\ \vdots & \vdots & \vdots \\ x_1 - x_N & y_1 - y_N & -r_{N,1} \end{bmatrix} \tag{2.42a}$$

$$\boldsymbol{\theta} = \begin{bmatrix} x - x_1 & y - y_1 & d_1 \end{bmatrix}^T \tag{2.42b}$$

$$\mathbf{q} = \begin{bmatrix} m_{2,1} & m_{3,1} & \cdots & m_{N,1} \end{bmatrix}^T \tag{2.42c}$$

$$\mathbf{b} = \begin{bmatrix} {r_{2,1}}^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2 \\ {r_{3,1}}^2 - (x_1 - x_3)^2 - (y_1 - y_3)^2 \\ \vdots \\ {r_{N,1}}^2 - (x_1 - x_N)^2 - (y_1 - y_N)^2 \end{bmatrix} \tag{2.42d}$$

Here, $\mathbf{b}$ is the transformed form of available RDOAs $\{r_{i,1}\}$, $i = 2, 3, ..., N$, $\mathbf{A}$ is the linear matrix form of the nonlinear function $\mathbf{f}(\mathbf{x})$ and $\boldsymbol{\theta}$ is the unknown vector to be determined to obtain the source location $\mathbf{x}$. For sufficiently small measurement noise or high signal-to-noise (SNR) conditions, the squared component of noise i.e. ${n_{i,1}}^2$ in Equation 2.38 can be ignored so that

$$m_{i,1} \approx 2n_{i,1}\sqrt{(x - x_i)^2 + (y - y_i)^2}$$

and hence $\mathbf{q}$ (Equation 2.42c) becomes

$$\mathbf{q} \approx \begin{bmatrix} 2n_{2,1}\sqrt{(x - x_2)^2 + (y - y_2)^2} \\ 2n_{3,1}\sqrt{(x - x_3)^2 + (y - y_3)^2} \\ \vdots \\ 2n_{N,1}\sqrt{(x - x_N)^2 + (y - y_N)^2} \end{bmatrix} \tag{2.43}$$

In such a scenario, $\mathbf{q}$ can be considered a zero-mean vector i.e $\mathrm{E}(\mathbf{q}) \approx \mathbf{0}$ and hence we can approximate Equation 2.41 as

$$\mathbf{A}\boldsymbol{\theta} \approx \mathbf{b} \tag{2.44}$$

A least squares cost criteria based on the above Equation 2.44 can be used to obtain the estimate of $\boldsymbol{\theta}$ as

$$\hat{\boldsymbol{\theta}}_{LLS} = \arg\min_{\tilde{\boldsymbol{\theta}}} J_{LLS}(\tilde{\boldsymbol{\theta}}) \tag{2.45}$$

where

$$J_{LLS}(\tilde{\boldsymbol{\theta}}) = (\mathbf{A}\tilde{\boldsymbol{\theta}} - \mathbf{b})^T(\mathbf{A}\tilde{\boldsymbol{\theta}} - \mathbf{b}) \tag{2.46}$$
$$= \tilde{\boldsymbol{\theta}}^T\mathbf{A}^T\mathbf{A}\tilde{\boldsymbol{\theta}} - 2\tilde{\boldsymbol{\theta}}^T\mathbf{A}^T\mathbf{b} + \mathbf{b}^T\mathbf{b}$$

is the LS cost function based on Equation 2.44 which is quadratic in $\tilde{\boldsymbol{\theta}}$ which means that there is a unique minimum in $J_{LLS}(\tilde{\boldsymbol{\theta}})$. The $\tilde{\boldsymbol{\theta}}$ which corresponds to this unique minimum is the source location estimate $\hat{\boldsymbol{\theta}}$ which is given by differentiating Equation 2.46 with respect to $\tilde{\boldsymbol{\theta}}$ and equating the result to zero as

$$\left.\frac{\partial J_{LLS}(\tilde{\boldsymbol{\theta}})}{\partial \tilde{\boldsymbol{\theta}}}\right|_{\tilde{\boldsymbol{\theta}}=\boldsymbol{\theta}} = \mathbf{0} \tag{2.47}$$
$$\Rightarrow 2\mathbf{A}^T\mathbf{A}\boldsymbol{\theta} - 2\mathbf{A}^T\mathbf{b} = \mathbf{0}$$
$$\Rightarrow \mathbf{A}^T\mathbf{A}\boldsymbol{\theta} = \mathbf{A}^T\mathbf{b}$$
$$\Rightarrow \hat{\boldsymbol{\theta}}_{LLS} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$$

The source location estimate $\hat{\mathbf{x}}$ is obtained from the first and second element of $\hat{\boldsymbol{\theta}}_{LLS}$ as

$$\hat{\mathbf{x}} = \begin{bmatrix} \hat{\theta}_1 + x_1 & \hat{\theta}_2 + y_1 \end{bmatrix} \tag{2.48}$$

In order to have the matrix inverse of $\mathbf{A}^T\mathbf{A}$ in Equation 2.47, the minimum number of sensors required is $N = 4$ because of the introduction of the additional (intermediate) variable of d_1 in Equation 2.40.

2.4.2.2 Weighted Least Square (WLS)

As was studied in the previous section, the LLS method is quite straightforward and simple to implement but its estimation performance is only optimal as long as the noise in the linear equations are independent and identically distributed which is rarely seen in the practical scenarios. WLS, as the weighted version of LLS, overcomes this limitation with only a slight increase in computational complexity while improving the localization accuracy. The improvement is due to the use of correlation between the TDOA values. Since all the TDOAs are determined with respect to the reference sensor, there is a nonzero correlation between the TDOA values which is reflected in the form of a symmetric weighting matrix $\mathbf{W}$ included in the WLS cost function denoted by $J_{WLS}(\tilde{\boldsymbol{\theta}})$ which is given as

$$J_{WLS}(\tilde{\boldsymbol{\theta}}) = (\mathbf{A}\tilde{\boldsymbol{\theta}} - \mathbf{b})^T\mathbf{W}(\mathbf{A}\tilde{\boldsymbol{\theta}} - \mathbf{b}) \tag{2.49}$$
$$= \tilde{\boldsymbol{\theta}}^T\mathbf{A}^T\mathbf{W}\mathbf{A}\tilde{\boldsymbol{\theta}} - 2\tilde{\boldsymbol{\theta}}^T\mathbf{A}^T\mathbf{W}\mathbf{b} + \mathbf{b}^T\mathbf{W}\mathbf{b}$$

Since $\mathbf{q}$ is a zero-mean vector (refer Equation 2.43), based on Equation 2.41, we get $\mathrm{E}\{\mathbf{b}\} = \mathbf{A}\boldsymbol{\theta}$, which relates to the linear unbiased data model [30, 29]. Therefore, we can invoke the best linear unbiased estimator (BLUE) approach to calculate the optimum $\mathbf{W}$ which is equal to the inverse of the nondiagonal covariance matrix $\boldsymbol{\Psi}$ of $\mathbf{q}$ given as

$$\mathbf{W} = \boldsymbol{\Psi}^{-1}$$

where

$$\boldsymbol{\Psi} = \mathrm{cov}(\mathbf{q}) = \mathrm{E}\left\{\mathbf{q}\mathbf{q}^T\right\} \approx 4diag(d_2, d_3, ..., d_N)\mathbf{Q}diag(d_2, d_3, ..., d_N) \tag{2.50}$$

Using Equation 2.8, we rewrite the above expression as

$$\begin{aligned} &\approx 4diag(d_{2,1} + d_1, d_{3,1} + d_1, ..., d_{N,1} + d_1)\mathbf{Q} \\ &\ast diag(d_{2,1} + d_1, d_{3,1} + d_1, ..., d_{N,1} + d_1) \end{aligned} \tag{2.51}$$

Since $d_{i,1}$'s are not available, we replace it with $r_{i,1}$ which are available. So we get

$$\begin{aligned} &\approx 4diag(r_{2,1} + d_1, r_{3,1} + d_1, ..., r_{N,1} + d_1)\mathbf{Q} \\ &\ast diag(r_{2,1} + d_1, r_{3,1} + d_1, ..., r_{N,1} + d_1) \end{aligned} \tag{2.52}$$

So in matrix form, $\mathbf{W} = \Psi^{-1}$ can be written as

$$\mathbf{W} = \Psi^{-1} = (4\mathbf{RQR})^{-1} \tag{2.53}$$

where $\mathbf{R} = diag(r_{2,1} + d_1, r_{3,1} + d_1, ..., r_{N,1} + d_1)$ Moreover, since d_1 given in Equation 2.42 is not available, we first estimate it using the LLS estimator (Equation 2.47). So the Equation 2.52 can be rewritten as

$$\begin{aligned} &\approx 4diag(r_{2,1} + [\hat{\boldsymbol{\theta}}]_3, r_{3,1} + [\hat{\boldsymbol{\theta}}]_3, ..., r_{N,1} + [\hat{\boldsymbol{\theta}}]_3)\mathbf{Q} \\ &\ast diag(r_{2,1} + [\hat{\boldsymbol{\theta}}]_3, r_{3,1} + [\hat{\boldsymbol{\theta}}]_3, ..., r_{N,1} + [\hat{\boldsymbol{\theta}}]_3) \end{aligned}$$

where $[\hat{\boldsymbol{\theta}}]_3$ is the LLS estimate for d_1.

Finally, using the similar approach as Equations 2.46 and 2.47 we compute the WLS estimate of $\boldsymbol{\theta}$ by minimizing the cost function (Equation 2.49) as

$$\begin{aligned} \hat{\boldsymbol{\theta}}_{WLS} &= \arg\min_{\tilde{\boldsymbol{\theta}}} J_{WLS}(\tilde{\boldsymbol{\theta}}) \\ &= (\mathbf{A}^T\mathbf{W}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{W}\mathbf{b} \end{aligned} \tag{2.54}$$

The WLS source location estimate is given by Equation 2.48.

Now we proceed on to find the expected value and covariance of $\hat{\boldsymbol{\theta}}_{WLS}$. We know that the estimated $\hat{\boldsymbol{\theta}}_{WLS}$ can be written as the sum of true $\boldsymbol{\theta}$ and the error term $\mathbf{e}_{WLS}$ given as

$$\hat{\boldsymbol{\theta}}_{WLS} = \boldsymbol{\theta} + \mathbf{e}_{WLS} \tag{2.55}$$

Multiplying both sides of Equation 2.54 with $\mathbf{A}^T\mathbf{W}\mathbf{A}$ and substituting Equation 2.55, we get

$$\mathbf{A}^T\mathbf{W}\mathbf{A}(\boldsymbol{\theta} + \mathbf{e}_{WLS}) = \mathbf{A}^T\mathbf{W}\mathbf{b} \tag{2.56}$$

Simplifying for $\mathbf{e}_{WLS}$ gives

$$\begin{aligned} \mathbf{e}_{WLS} &= (\mathbf{A}^T\mathbf{W}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{W}(\mathbf{b} - \mathbf{A}\boldsymbol{\theta}) \\ &= (\mathbf{A}^T\mathbf{W}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{W}\mathbf{q} \end{aligned} \tag{2.57}$$

Note that, since second-order noise terms are ignored, $\mathbf{q}$ approximately becomes a zero-mean vector so that $\mathrm{E}(\mathbf{q}) \approx \mathbf{0}$. Hence the expected value and covariance of $\hat{\boldsymbol{\theta}}_{WLS}$ can be written as

$$\mathrm{E}[\hat{\boldsymbol{\theta}}_{WLS}] = \boldsymbol{\theta} + \mathrm{E}[\mathbf{e}_{WLS}] \approx \boldsymbol{\theta} \tag{2.58}$$

$$\begin{aligned} &\mathrm{cov}(\hat{\boldsymbol{\theta}}_{WLS}) \approx \mathrm{E}[\mathbf{e}_{WLS}\mathbf{e}_{WLS}^T] \\ &\approx (\mathbf{A}^T\mathbf{W}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{W}\mathbf{W}^{-1}\mathbf{W}\mathbf{A}(\mathbf{A}^T\mathbf{W}\mathbf{A})^{-1} \\ &\approx (\mathbf{A}^T\mathbf{W}\mathbf{A})^{-1} \end{aligned} \tag{2.59}$$

The results obtained in 2.58 and 2.59 will be used in the following section 2.4.2.3.

2.4.2.3 Two Step Weighted Least Squares (2sWLS)

WLS method can be thought of as improvement over the LLS method, especially when the disturbances in the linear equations are not independent and identically distributed. However, localization performance can be further improved by making use of the relationship between unknown source position and the intermediate variable represented by Equation 2.39. Equation 2.58 shows that $\hat{\boldsymbol{\theta}}_{WLS}$ can be treated as a random vector whose expected value is equal to the true value so that $\hat{\boldsymbol{\theta}}_{WLS}$ can be written as

$$\hat{\boldsymbol{\theta}}_{WLS} \triangleq \begin{bmatrix} \hat{\theta}_{WLS,1} \\ \hat{\theta}_{WLS,2} \\ \hat{\theta}_{WLS,3} \end{bmatrix} = \begin{bmatrix} \theta_1 + e_{\theta_1} \\ \theta_2 + e_{\theta_2} \\ \theta_3 + e_{\theta_3} \end{bmatrix} \tag{2.60}$$

given

$$\boldsymbol{\theta} \triangleq \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \tag{2.61}$$

is the true parameter vector referred to in Equation 2.42b. Here, $e_{\theta_j}, j = 1, 2, 3$ are the estimation errors.

Assuming that estimated $\hat{\boldsymbol{\theta}}$ is sufficiently close to true $\boldsymbol{\theta}$, we can write

$$\begin{aligned} \hat{\theta}^2_{WLS,1} - \theta_1^2 &= (\theta_1 + e_{\theta_1})^2 - \theta_1^2 \approx 2\theta_1 e_{\theta_1} \\ \hat{\theta}^2_{WLS,2} - \theta_2^2 &= (\theta_2 + e_{\theta_2})^2 - \theta_2^2 \approx 2\theta_2 e_{\theta_2} \\ \hat{\theta}^2_{WLS,3} - \theta_3^2 &= (\theta_3 + e_{\theta_3})^2 - \theta_3^2 \approx 2\theta_3 e_{\theta_3} \end{aligned} \tag{2.62}$$

Using Equations 2.62 and based on Equation 2.39 we construct [1]

$$\mathbf{h} = \mathbf{G}\mathbf{z} + \mathbf{w} \tag{2.63}$$

$$\mathbf{w} = \mathbf{h} - \mathbf{G}\mathbf{z} \tag{2.64}$$

where

$$\mathbf{h} = \begin{bmatrix} \hat{\theta}^2_{WLS,1} \\ \hat{\theta}^2_{WLS,2} \\ \hat{\theta}^2_{WLS,3} \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{z} = \begin{bmatrix} \theta_1^2 \\ \theta_2^2 \end{bmatrix} \tag{2.65}$$

and from Equation 2.62, we can write

$$\mathbf{w} \approx \begin{bmatrix} 2\theta_1 e_{\theta_1} \\ 2\theta_2 e_{\theta_2} \\ 2\theta_3 e_{\theta_3} \end{bmatrix} \tag{2.66}$$

The covariance of $\mathbf{w}$, denoted by $\boldsymbol{\Phi}$ is given as

$$\boldsymbol{\Phi} = \mathrm{cov}(\mathbf{w}) = 4\mathbf{P}\mathrm{cov}(\hat{\boldsymbol{\theta}}_{WLS})\mathbf{P} \tag{2.67}$$

where $\mathbf{P} = diag(\theta_1, \theta_2, \theta_3)$ and $\mathrm{cov}(\hat{\boldsymbol{\theta}}_{WLS})$ is given in Equation 2.59. The WLS estimate of $\mathbf{z}$ is then given by

$$\begin{aligned} \hat{\mathbf{z}} &= \arg\min_{\tilde{\mathbf{z}}} (\mathbf{G}\tilde{\mathbf{z}} - \mathbf{h})^T \boldsymbol{\Phi}^{-1} (\mathbf{G}\tilde{\mathbf{z}} - \mathbf{h}) \\ &= (\mathbf{G}^T \boldsymbol{\Phi}^{-1} \mathbf{G})^{-1} \mathbf{G}^T \boldsymbol{\Phi}^{-1} \mathbf{h} \end{aligned} \tag{2.68}$$

$\mathbf{P}$ could be estimated using $\hat{\boldsymbol{\theta}}_{WLS}$ and an approximate $\boldsymbol{\Phi}$ can be easily obtained.

Finally the improved two-step WLS (2sWLS) estimate is obtained as

$$\hat{\boldsymbol{\theta}}_{2sWLS} = \begin{bmatrix} \pm\sqrt{[\hat{\mathbf{z}}]_1} \\ \pm\sqrt{[\hat{\mathbf{z}}]_2} \end{bmatrix} \tag{2.69}$$

where $[\hat{\mathbf{z}}]_1$ and $[\hat{\mathbf{z}}]_1$ are first and the second elements of $\hat{\mathbf{z}}$ respectively. As there is no sign infomation for θ in $\hat{\mathbf{z}}$ there are four different θ candidates. To resolve this issue, we simply take the sign information from the elements of the WLS estimate $\hat{\boldsymbol{\theta}}_{WLS}$ [1]. So we can write

$$\hat{\boldsymbol{\theta}}_{2sWLS} = \begin{bmatrix} \operatorname{sgn}\left([\hat{\boldsymbol{\theta}}_{WLS}]_1\right)\left|\sqrt{[\hat{\mathbf{z}}]_1}\right| \\ \operatorname{sgn}\left([\hat{\boldsymbol{\theta}}_{WLS}]_2\right)\left|\sqrt{[\hat{\mathbf{z}}]_2}\right| \end{bmatrix} \tag{2.70}$$

Finally, the improved two-step WLS (2sWLS) source location estimate $\hat{\mathbf{x}}_{2sWLS}$ is obtained from the first and second element of $\hat{\boldsymbol{\theta}}$ as

$$\hat{\mathbf{x}}_{2sWLS} = \begin{bmatrix} [\hat{\boldsymbol{\theta}}_{2sWLS}]_1 + x_1 & [\hat{\boldsymbol{\theta}}_{2sWLS}]_2 + y_1 \end{bmatrix} \tag{2.71}$$

where $[\hat{\boldsymbol{\theta}}_{2sWLS}]_1$ and $[\hat{\boldsymbol{\theta}}_{2sWLS}]_2$ are the two elements of $\hat{\boldsymbol{\theta}}_{2sWLS}$ given in Equation 2.70.

The 2sWLS source localization algorithm can be summarized as follows :

1. A and b are determined using the RDOA measurements and sensor positions.
2. Initial source position estimate $\hat{\boldsymbol{\theta}}_{LLS}$ is obtained using Equation 2.47.
3. Using the estimated d_1 from step 2 and RDOA values, approximate $\mathbf{R}$ is calculated.
4. Using $\mathbf{R}$ and $\mathbf{Q}$, $\mathbf{W}$ is calculated using Equation 2.53.
5. Using Equation 2.54 initial source location is again updated.
6. $\mathbf{P}$, $\mathbf{h}$ is constructed using the obtained source position in step 5.
7. Using $\mathbf{P}$ and $cov(\mathbf{w})$ in Equation 2.59, $\mathbf{\Phi}$ is calculated.
8. $\hat{\mathbf{z}}$ is calculated from Equation 2.68.
9. Using Equation 2.70, 2sWLS source position estimate is obtained.

2.4.2.4 Spherical Interpolation (SI)

Spherical Interpolation (SI) is a closed-form method to obtain the source location estimate [13]. In this method, a new equation error linear in d_1 is obtained which is minimized with respect to d_1 to find its estimate $\hat{d}_1$. Given $\hat{d}_1$, the source location $\mathbf{x} = \begin{bmatrix} x & y \end{bmatrix}^T$ is obtained in the following manner: We rewrite Equation 2.44 as

$$\begin{aligned} \mathbf{A}\boldsymbol{\theta} &= \mathbf{b} \\ \begin{bmatrix} \mathbf{B} & \mathbf{r} \end{bmatrix}\begin{bmatrix} \boldsymbol{\theta}_1^T & d_1 \end{bmatrix} &= \mathbf{b} \\ \mathbf{B}\boldsymbol{\theta}_1 + d_1\mathbf{r} &= \mathbf{b} \end{aligned} \tag{2.72}$$

where $\mathbf{B}$ is a submatrix of $\mathbf{A}$ built from the first two columns of $\mathbf{A}$, $\mathbf{r}$ is the remaining third column of $\mathbf{A}$ and $\boldsymbol{\theta}_1$ contains the first two elements of $\boldsymbol{\theta}$. Equation 2.72 can be rewritten as

$$\mathbf{B}\boldsymbol{\theta}_1 = (\mathbf{b} - d_1\mathbf{r}) \tag{2.73}$$

Assuming d_1 is known, least squares estimate of the $\boldsymbol{\theta}$ is given as

$$\hat{\boldsymbol{\theta}}_{1-SI} = (\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T(\mathbf{b} - d_1\mathbf{r}) \tag{2.74}$$

If we substitute $\hat{\boldsymbol{\theta}}_{1-SI}$ given in Equation 2.74 to the Equation 2.72, we get

$$\begin{aligned} \mathbf{B}(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T(\mathbf{b} - d_1\mathbf{r}) + d_1\mathbf{r} &= \mathbf{b} \\ d_1(\mathbf{I} - \mathbf{B}(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T)\mathbf{r} &= (\mathbf{I} - \mathbf{B}(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T)\mathbf{b} \end{aligned} \tag{2.75}$$

Let

$$\mathbf{S} \triangleq \mathbf{I} - \mathbf{B}(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T \tag{2.76}$$

Then least squares estimate of d_1 can be written as

$$\hat{d}_1 = \frac{\mathbf{r}^T\mathbf{S}^T\mathbf{S}\mathbf{b}}{\mathbf{r}^T\mathbf{S}^T\mathbf{S}\mathbf{r}} \tag{2.77}$$

Finally substituting $\hat{d}_1$ from Equation 2.77 into Equation 2.74, we get

$$\hat{\boldsymbol{\theta}}_{1-SI} = (\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\left(\mathbf{I} - \frac{\mathbf{r}\mathbf{r}^T\mathbf{S}^T\mathbf{S}}{\mathbf{r}^T\mathbf{S}^T\mathbf{S}\mathbf{r}}\right)\mathbf{b} \tag{2.78}$$

Source location $\mathbf{x}$ could be found as

$$\hat{\mathbf{x}} = \begin{bmatrix} [\boldsymbol{\theta}_{1-SI}]_1 + x_1 & [\boldsymbol{\theta}_{1-SI}]_2 + y_1 \end{bmatrix} \tag{2.79}$$

It should be pointed out that since SI method solves the same linear set of equations (without taking constraints into account) it is equivalent to the LLS method given in the previous section 2.4.2.1. In [31], it has been asserted that SI method is identical to the LLS method.

2.4.2.5 Subspace Minimization (SM)

Subspace Minimization (SM) is the closed form method for TDOA based source localization presented in [15]. This method is based on the idea of eliminating $\mathbf{r}$ in equation 2.72 by multiplying Equation 2.72 by a projection matrix $\mathbf{P}$ which has $\mathbf{r}$ in its null space, i.e.

$$\mathbf{P}\mathbf{r} = \mathbf{0} \tag{2.80}$$

This matrix $\mathbf{P}$ can be obtained as

$$\mathbf{P}_k = (\mathbf{I} - \mathbf{Z}^k)\mathbf{R}, k = 1, 2, 3, ... \tag{2.81}$$

where

$$\mathbf{Z} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 & \vdots \\ \vdots & \cdots & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix} \tag{2.82}$$

$$\mathbf{R} = \begin{bmatrix} r_{2,1} & 0 & 0 & 0 \\ 0 & r_{3,1} & 0 & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & r_{N,1} \end{bmatrix}^{-1} \tag{2.83}$$

$\mathbf{Z}$ is known as the circular shift matrix. The matrix $\mathbf{P_k}$ has this property for any value of k since

$$\mathbf{R}\mathbf{r} = \mathbf{1} \triangleq [1, 1, ..., 1]^T \tag{2.84}$$

and

$$(\mathbf{I} - \mathbf{Z}^k)\mathbf{1} = \mathbf{1} - \mathbf{Z}^k\mathbf{1} = \mathbf{1} - \mathbf{1} = \mathbf{0} \tag{2.85}$$

In order to simplify the following discussion we assume $k = 1$ so that $\mathbf{P_1}$ is the projection matrix. Now, multiplying Equation 2.72 with $\mathbf{P_1}$ gives

$$\begin{aligned} \mathbf{P}_1(\mathbf{B}\boldsymbol{\theta}_1 + d_1\mathbf{r}) &= \mathbf{P}_1\mathbf{b} \\ \mathbf{P}_1\mathbf{B}\boldsymbol{\theta}_1 &= \mathbf{P}_1\mathbf{b} \end{aligned} \tag{2.86}$$

Least Squares (LS) estimate of Equation 2.86 can be obtained as

$$\hat{\boldsymbol{\theta}}_{1-SM} = (\mathbf{B}^T\mathbf{P}_1^T\mathbf{P}_1\mathbf{B})^{-1}\mathbf{B}^T\mathbf{P}_1^T\mathbf{P}_1\mathbf{b} \tag{2.87}$$

In general, for all k we can write

$$\hat{\boldsymbol{\theta}}_{1-SM} = (\mathbf{B}^T\mathbf{P}_k^T\mathbf{P}_k\mathbf{B})^{-1}\mathbf{B}^T\mathbf{P}_k^T\mathbf{P}_k\mathbf{b} \tag{2.88}$$

It should be noted that the projection matrix $\mathbf{P_k}$ is singular with rank $N-2$. To have a nonsingular $\mathbf{P}_k\mathbf{B}$, it is necessary (although not sufficient) that the number of rows of $\mathbf{B}$ be larger than that of its columns i.e. $N-1>n$ where n is the dimension of the space or length of the source location vector $\mathbf{x}$. Thus for a unique solution of Equation 2.86 to exist we must have $N>n+1$ sensors which means that we must at least have 4 sensors for our 2-Dimension case to implement this method. It can be seen from Equation 2.78 the estimate $\hat{\boldsymbol{\theta}}_{1-SI}$ is the LS estimate of

$$\mathbf{B}\boldsymbol{\theta}_1 = \left(\mathbf{I} - \frac{\mathbf{r}\mathbf{r}^T\mathbf{S}^T\mathbf{S}}{\mathbf{r}^T\mathbf{S}^T\mathbf{S}\mathbf{r}}\right)\mathbf{b} \tag{2.89}$$

Multiplying the above Equation Equation 2.89 by $\mathbf{P_k}$ results in

$$\begin{aligned}\mathbf{P}_k\mathbf{B}\boldsymbol{\theta}_1 &= \mathbf{P}_k\mathbf{b} - \mathbf{P}_k\mathbf{r}\frac{\mathbf{r}^T\mathbf{S}^T\mathbf{S}}{\mathbf{r}^T\mathbf{S}^T\mathbf{S}\mathbf{r}}\mathbf{b} \\ &= \mathbf{P}_k\mathbf{b}\end{aligned} \tag{2.90}$$

which is the same as Equation 2.86 for $k=1$. This clearly demonstrates that the SM approach is equivalent to the SI approach discussed in the previous section 2.4.2.4.

2.4.2.6 Constrained Least Squares (CLS)

In section 2.4.2.1, the nonlinear Equation set 2.7 was linearized by introducing a new variable d_1 given in Equation 2.39 and assuming it to be independent of the source location $\mathbf{x}$ to make the resulting equation set linear in $\mathbf{x}$. However, d_1 is a function of the true source location $\mathbf{x}$ and the LLS estimate $\hat{\boldsymbol{\theta}}_{LLS}$, given in Equation 2.47 for source location, does not take this relationship into account. In section 2.4.2.3, this ignored relationship is exploited in an implicit manner to improve the WLS estimate given in section 2.4.2.2 even further. In [32], the authors propose to incorporate the relationship represented by Equation 2.39 explicitly on the minimization problem (Equation2.45) through the use of Lagrange multiplier technique. The minimization problem, hence, can be written formally as

$$\min \ (\mathbf{A}\boldsymbol{\theta} - \mathbf{b})^T(\mathbf{A}\boldsymbol{\theta} - \mathbf{b})^T \tag{2.91}$$

$$s.t. \ \boldsymbol{\theta}^T\boldsymbol{\Sigma}\boldsymbol{\theta} = 0, \ \theta_3 \geq 0 \tag{2.92}$$

where $\boldsymbol{\Sigma} \triangleq diag(1,1,-1)$. The equality constraint in Equation 2.92 is the matrix representation of the equation Equation 2.39 while the inequality constraint in Equation 2.92 arises from the fact that θ_3 is a non-negative quantity. Therefore, ignoring the inequality constraint, the corresponding Lagrangian can be written as

$$\mathcal{L}(\boldsymbol{\theta},\eta) = (\mathbf{A}\boldsymbol{\theta} - \mathbf{b})^T(\mathbf{A}\boldsymbol{\theta} - \mathbf{b}) + \eta\boldsymbol{\theta}^T\boldsymbol{\Sigma}\boldsymbol{\theta} \tag{2.93}$$

where η is the Lagrange multiplier to be determined. Constrained least squares (CLS) estimate is obtained by minimizing the Lagrange function (Equation 2.93) as

$$\hat{\boldsymbol{\theta}}_{CLS} \triangleq \arg\min_{\boldsymbol{\theta}}\{L(\boldsymbol{\theta},\eta)\} \tag{2.94}$$

Expanding the above Lagrange function (Equation 2.93) gives

$$\begin{aligned} \mathcal{L}(\boldsymbol{\theta},\eta) &= (\boldsymbol{\theta}^T\mathbf{A}^T - \mathbf{b}^T)(\mathbf{A}\boldsymbol{\theta} - \mathbf{b}) + \eta\boldsymbol{\theta}^T\boldsymbol{\Sigma}\boldsymbol{\theta} \\ &= \boldsymbol{\theta}^T\mathbf{A}^T\mathbf{A}\boldsymbol{\theta} - 2\mathbf{b}^T\mathbf{A}\boldsymbol{\theta} + \mathbf{b}^T\mathbf{b} + \eta\boldsymbol{\theta}^T\boldsymbol{\Sigma}\boldsymbol{\theta} \\ &= \boldsymbol{\theta}^T(\mathbf{A}^T\mathbf{A} + \eta\boldsymbol{\Sigma})\boldsymbol{\theta} - 2\mathbf{b}^T\mathbf{A}\boldsymbol{\theta} + \mathbf{b}^T\mathbf{b} \end{aligned} \tag{2.95}$$

The estimate of $\boldsymbol{\theta}$ is obtained by differentiating $\mathcal{L}(\boldsymbol{\theta},\eta)$ with respect to $\boldsymbol{\theta}$ and then equating the results to zero as follows:

$$\frac{\partial\mathcal{L}(\boldsymbol{\theta},\eta)}{\partial\boldsymbol{\theta}} = 2(\mathbf{A}^T\mathbf{A} + \eta\boldsymbol{\Sigma})\boldsymbol{\theta} - 2\mathbf{A}^T\mathbf{b} = 0 \tag{2.96}$$

Solving Equation 2.96 for $\boldsymbol{\theta}$ gives the CLS estimate as

$$\hat{\boldsymbol{\theta}}_{CLS} = (\mathbf{A}^T\mathbf{A} + \eta\boldsymbol{\Sigma})^{-1}\mathbf{A}^T\mathbf{b} \tag{2.97}$$

where η is yet to be determined. Once we obtain η, the CLS estimate of the source location can be calculated from Equation 2.97. To find η, we can directly impose the quadratic constraint given in Equation 2.92 by substituting Equation 2.97 into Equation 2.92 which results in

$$\mathbf{b}^T\mathbf{A}(\mathbf{A}^T\mathbf{A} + \eta\boldsymbol{\Sigma})^{-1}\boldsymbol{\Sigma}(\mathbf{A}^T\mathbf{A} + \eta\boldsymbol{\Sigma})^{-1}\mathbf{A}^T\mathbf{b} = 0 \tag{2.98}$$

Considering $\mathbf{\Sigma}^{-1} = \mathbf{\Sigma}$ we can write

$$\begin{aligned}(\mathbf{A}^T\mathbf{A} + \eta\mathbf{\Sigma})^{-1} &= \mathbf{\Sigma}(\mathbf{A}^T\mathbf{A}\mathbf{\Sigma} + \eta\mathbf{\Sigma}\mathbf{\Sigma})^{-1} \\ &= \mathbf{\Sigma}(\mathbf{A}^T\mathbf{A}\mathbf{\Sigma} + \eta\mathbf{I})^{-1} \end{aligned} \tag{2.99}$$

Using Equation 2.99, Equation 2.98 can be rewritten as

$$\mathbf{b}^T\mathbf{A}\mathbf{\Sigma}(\mathbf{A}^T\mathbf{A}\mathbf{\Sigma} + \eta\mathbf{I})^{-1}(\mathbf{A}^T\mathbf{A}\mathbf{\Sigma} + \eta\mathbf{I})^{-1}\mathbf{A}^T\mathbf{b} = 0 \tag{2.100}$$

By using eigenvalue decomposition, $\mathbf{A}^T\mathbf{A}\mathbf{\Sigma}$ can be factorized as

$$\mathbf{A}^T\mathbf{A}\mathbf{\Sigma} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{-1} \tag{2.101}$$

where $\mathbf{\Lambda} \triangleq diag(\zeta_{1,}\zeta_2, \zeta_3)$ represents a 3×3 diagonal matrix whose diagonal elements contain the eigenvalues of $\mathbf{A}^T\mathbf{A}\mathbf{\Sigma}$ ordered in a decreasing manner and $\mathbf{U}$ represents a 3×3 matrix whose columns contain the corresponding eigenvectors. Substituting Equation 2.101 in Equation 2.99 results in

$$\mathbf{p}^T(\mathbf{\Lambda} + \eta\mathbf{I})^{-2}\mathbf{q} = 0 \tag{2.102}$$

where

$$\begin{aligned}\mathbf{p} &= \mathbf{U}^T\Sigma\mathbf{A}^T\mathbf{b} = [\ p_1 \quad p_2 \quad p_3\ ]^T \\ \mathbf{q} &= \mathbf{U}^{-1}\mathbf{A}^T\mathbf{b} = [\ q_1 \quad q_2 \quad q_3\ ]^T\end{aligned} \tag{2.103}$$

Equation 2.102 can be rewritten in terms of the elements of the given vectors given in Equation 2.103 as

$$f(\eta) \triangleq \sum_{i=1}^{3} \frac{p_i q_i}{(\eta + \zeta_i)^2} = 0 \tag{2.104}$$

which is a polynomial of degree 4 which indicates that there are four roots of Equation 2.104 each related to a local extremum. Out of these four η values, we want to find the optimum η corresponding to the global minimum. The authors have proposed in [33] and [34] to choose the root which minimizes the cost function (Equation 2.91). Even though the inequality constraint $\theta_3 \geq 0$ is ignored in the algorithm, it is a good idea to choose the root which results in $\hat{\theta}_3$ being positive (since it's a positive quantity) thus satisfying this constraint too so that both the constraints will have been considered in obtaining the CLS estimate which, for the most part, should correspond to the global minimum. Once the optimal η is identified, Equation 2.97 gives the optimal CLS estimate. As in [18], we solve for η iteratively by using the improved Newton's method with step size $1 - a^k$, where $a \in [0, 1]$ is an attenuation factor and k is the iteration number as shown below:

$$\eta_{k+1} = \eta_k - (1 - \alpha^k)\frac{f(\eta_k)}{f'(\eta_k)}, f'(\eta_k) \neq 0, a \in [0, 1] \tag{2.105}$$

It should be noted that the CLS estimate given in Equation 2.97 gets reduced to the LLS estimate given in 2.47 when the term $\eta\boldsymbol{\Sigma} = 0$. This happens under low noise conditions when the expected value of η is zero [34]. To demonstrate this, we can rewrite Equation 2.97 as

$$[(\bar{\mathbf{A}} + \Delta\mathbf{A})^T(\bar{\mathbf{A}} + \Delta\mathbf{A}) + \eta\boldsymbol{\Sigma}](\boldsymbol{\theta} + \Delta\boldsymbol{\theta}) = (\bar{\mathbf{A}} + \Delta\mathbf{A})^T(\bar{\mathbf{b}} + \Delta\mathbf{b}) \tag{2.106}$$

where $\mathbf{A} = \bar{\mathbf{A}} + \Delta\mathbf{A}$, $\mathbf{b} = \bar{\mathbf{b}} + \Delta\mathbf{b}$ and $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta} + \Delta\boldsymbol{\theta}$. The barred quantity represents the noise free term and delta represents the corresponding noise. Assuming sufficiently low noise conditions, second-order error terms can be ignored to obtain

$$\eta\boldsymbol{\Sigma}(\boldsymbol{\theta} + \Delta\boldsymbol{\theta}) \approx \bar{\mathbf{A}}^T\Delta\mathbf{b} + \Delta\mathbf{A}^T\bar{\mathbf{b}} - \bar{\mathbf{A}}^T\Delta\mathbf{A}\boldsymbol{\theta} - \Delta\mathbf{A}^T\bar{\mathbf{A}}\boldsymbol{\theta} - \bar{\mathbf{A}}^T\bar{\mathbf{A}}\Delta\boldsymbol{\theta} \tag{2.107}$$

Since $\mathrm{E}[\Delta\mathbf{A}] = \mathbf{0}_{(N-1)\times 3}$, $\mathrm{E}[\Delta\mathbf{b}] = \mathbf{0}_{(N-1)\times 1}$ and $\mathrm{E}[\Delta\boldsymbol{\theta}] = \mathbf{0}_{3\times 1}$, taking the expectation of Equation 2.107 results in

$$\mathrm{E}[\eta] = 0 \tag{2.108}$$

So under high signal-to-noise (SNR) conditions, the LLS method can be considered as an approximation of the CLS method.

2.4.2.7 Constrained Weighted Least Squares (CWLS)

In the previous section 2.4.2.6, we studied that the CLS method explicitly incorporates the constraints in the minimization problem, unlike the 2sWLS method studied in section 2.4.2.3 making CLS more robust and elegant than 2sWLS. However, in CLS, the correlation among the TDOA values is not taken into account which was considered in WLS and 2sWLS. In order to enhance the estimation performance of CLS, this correlation could be used in the form of a weighting matrix similarly as in WLS and 2sWLS method which gives rise to CWLS method. In other words, CWLS method can be considered as a weighted version of the CLS. From section 2.4.2.2 we recall the WLS cost function is given as

$$J_{WLS}(\boldsymbol{\theta}) = (\mathbf{A}\boldsymbol{\theta} - \mathbf{b})^T\mathbf{W}(\mathbf{A}\boldsymbol{\theta} - \mathbf{b}) \tag{2.109}$$

Using Equation 2.92 the minimization problem can be written as

$$\min \ (\mathbf{A}\boldsymbol{\theta} - \mathbf{b})^T \mathbf{W}(\mathbf{A}\boldsymbol{\theta} - \mathbf{b})^{\mathbf{T}} \tag{2.110}$$

$$s.t. \ \boldsymbol{\theta}^T \boldsymbol{\Sigma} \boldsymbol{\theta} = 0, \ \theta_3 \geq 0 \tag{2.111}$$

Therefore, ignoring the inequality constraint, the corresponding Lagrangian can be written as

$$\mathcal{L}'(\boldsymbol{\theta},\eta') = (\mathbf{A}\boldsymbol{\theta} - \mathbf{b})^T \mathbf{W}(\mathbf{A}\boldsymbol{\theta} - \mathbf{b}) + \eta' \boldsymbol{\theta}^T \boldsymbol{\Sigma} \boldsymbol{\theta} \tag{2.112}$$

where η' is the Lagrange multiplier to be determined. Constrained weighted least squares (CWLS) estimate is obtained by minimizing the Lagrange function (Equation 2.112) as

$$\hat{\boldsymbol{\theta}}_{CWLS} \triangleq \arg\min_{\boldsymbol{\theta}} \{\mathcal{L}'(\boldsymbol{\theta},\eta')\} \tag{2.113}$$

Expanding the above Lagrange function gives

$$\begin{aligned} \mathcal{L}'(\boldsymbol{\theta},\eta') &= (\boldsymbol{\theta}^T \mathbf{A}^T - \mathbf{b}^T)\mathbf{W}(\mathbf{A}\boldsymbol{\theta} - \mathbf{b}) + \eta' \boldsymbol{\theta}^T \boldsymbol{\Sigma} \boldsymbol{\theta} \\ &= \boldsymbol{\theta}^T \mathbf{A}^T \mathbf{W} \mathbf{A} \boldsymbol{\theta} - 2\mathbf{b}^T \mathbf{W} \mathbf{A} \boldsymbol{\theta} + \mathbf{b}^T \mathbf{W} \mathbf{b} + \eta' \boldsymbol{\theta}^T \boldsymbol{\Sigma} \boldsymbol{\theta} \\ &= \boldsymbol{\theta}^T (\mathbf{A}^T \mathbf{W} \mathbf{A} + \eta' \boldsymbol{\Sigma})\boldsymbol{\theta} - 2\mathbf{b}^T \mathbf{W} \mathbf{A} \boldsymbol{\theta} + \mathbf{b}^T \mathbf{W} \mathbf{b} \end{aligned} \tag{2.114}$$

The estimate of $\boldsymbol{\theta}$ is obtained by differentiating $\mathcal{L}'(\boldsymbol{\theta},\eta)$ with respect to $\boldsymbol{\theta}$ and then equating the results to zero as follows:

$$\frac{\partial \mathcal{L}'(\boldsymbol{\theta},\eta')}{\partial \boldsymbol{\theta}} = 2(\mathbf{A}^T \mathbf{W} \mathbf{A} + \eta' \boldsymbol{\Sigma})\boldsymbol{\theta} - 2\mathbf{A}^T \mathbf{W} \mathbf{b} = 0 \tag{2.115}$$

Solving Equation 2.115 for $\boldsymbol{\theta}$ gives the CWLS estimate as

$$\hat{\boldsymbol{\theta}}_{CWLS} = (\mathbf{A}^T \mathbf{W} \mathbf{A} + \eta' \boldsymbol{\Sigma})^{-1} \mathbf{A}^T \mathbf{W} \mathbf{b} \tag{2.116}$$

where η' is yet to be determined. Once we obtain η', the CWLS estimate of the source location can be calculated from Equation 2.116. To find η', we can directly impose the quadratic constraint given in 2.111 by substituting Equation 2.116 into Equation 2.111 which results in

$$\mathbf{b}^T \mathbf{W} \mathbf{A} (\mathbf{A}^T \mathbf{W} \mathbf{A} + \eta \boldsymbol{\Sigma})^{-1} \boldsymbol{\Sigma} (\mathbf{A}^T \mathbf{W} \mathbf{A} + \eta \boldsymbol{\Sigma})^{-1} \mathbf{A}^T \mathbf{W} \mathbf{b} = 0 \tag{2.117}$$

Considering $\boldsymbol{\Sigma}^{-1} = \boldsymbol{\Sigma}$ we can write

$$\begin{aligned} (\mathbf{A}^T \mathbf{W} \mathbf{A} + \eta' \boldsymbol{\Sigma})^{-1} &= \boldsymbol{\Sigma} (\mathbf{A}^T \mathbf{W} \mathbf{A} \boldsymbol{\Sigma} + \eta' \boldsymbol{\Sigma}\boldsymbol{\Sigma})^{-1} \\ &= \boldsymbol{\Sigma} (\mathbf{A}^T \mathbf{W} \mathbf{A} \boldsymbol{\Sigma} + \eta' \mathbf{I})^{-1} \end{aligned} \tag{2.118}$$

Using Equation 2.118, Equation 2.117 can be written as

$$\mathbf{b}^T\mathbf{W}\mathbf{A}\boldsymbol{\Sigma}(\mathbf{A}^T\mathbf{W}\mathbf{A}\boldsymbol{\Sigma}+\eta\mathbf{I})^{-1}(\mathbf{A}^T\mathbf{W}\mathbf{A}\boldsymbol{\Sigma}+\eta\mathbf{I})^{-1}\mathbf{A}^T\mathbf{W}\mathbf{b}=0 \tag{2.119}$$

By using eigenvalue decomposition, $\mathbf{A}^T\mathbf{W}\mathbf{A}\boldsymbol{\Sigma}$ can be factorized as

$$\mathbf{A}^T\mathbf{W}\mathbf{A}\boldsymbol{\Sigma}=\mathbf{U}'\boldsymbol{\Lambda}'\mathbf{U}'^{-1} \tag{2.120}$$

where $\boldsymbol{\Lambda}' \triangleq diag(\zeta_1', \zeta_2', \zeta_3')$ represents a 3×3 diagonal matrix whose diagonal elements contain the eigenvalues of $\mathbf{A}^T\mathbf{W}\mathbf{A}\boldsymbol{\Sigma}$ ordered in a decreasing manner and $\mathbf{U}'$ represents a 3×3 matrix whose columns contain the corresponding eigenvectors. Substituting Equation 2.120 in Equation 2.118 results in

$$\mathbf{p}'^T(\boldsymbol{\Lambda}'+\eta'\mathbf{I})^{-2}\mathbf{q}'=0 \tag{2.121}$$

where

$$\begin{aligned} \mathbf{p}' &= \mathbf{U}'^T\boldsymbol{\Sigma}\mathbf{A}^T\mathbf{b} = [\ p_1' \quad p_2' \quad p_3'\ ] \\ \mathbf{q}' &= \mathbf{U}'^{-1}\mathbf{A}^T\mathbf{b} = [\ q_1' \quad q_2' \quad q_3'\ ] \end{aligned} \tag{2.122}$$

Equation 2.121 can be rewritten in terms of the elements of the given vectors given in Equation 2.122 as

$$f(\eta') \triangleq \sum_{i=1}^{3} \frac{p_i' q_i'}{(\eta'+\zeta_i')^2} = 0 \tag{2.123}$$

which is a polynomial of degree 4 which indicates that there are four roots of Equation 2.123 each related to a local extremum. Out of these four η' values, we want to find the optimum η' corresponding to the global minimum. It is proposed in [33] and [34] to choose the root which minimizes the cost function 2.109. Even though the inequality constraint $\theta_3 \geq 0$ is ignored in the algorithm, it is a good idea to choose the root which results in $\hat{\theta}_3$ being positive thus satisfying this constraint too. In this way, both the constraints will have been considered in obtaining the CWLS estimate which, for the most part, should correspond to the global minimum. Once the optimal η' is identified, Equation 2.97 gives the optimal CWLS estimate. In a similar manner to CLS method, we solve for η' by using the improved Newton's method (refer Equation 2.105) with step size $1-a^k$, where $a\in[0,1]$ is an attenuation factor and k is the iteration number.

The algorithm given in [33] with little addition from us in step 3 is illustrated as follows :

1. Set $\mathbf{W}=\mathbf{I}_{(N-1)}$, where $\mathbf{I}_{(N-1)}$ represents the identity matrix of order $(N-1)$.

2. Find all the roots of Equation 2.123 by using a standard root finding algorithm (improved Newton's method in our case).

3. Take only the real roots of Equation 2.123 (since the Lagrange multiplier is always real for a real optimization problem) and substitute them into Equation 2.116 to obtain the corresponding $\hat{\boldsymbol{\theta}}$ candidates. Then choose the solution $\hat{\boldsymbol{\theta}}_{CWLS}$ from those candidates which makes the expression (Equation 2.109) minimum and satisfies the inequality constraint $\theta_3 \geq 0$ at the same time.

4. Calculate $\mathbf{W}$ according to Equation 2.53 using the obtained d_1 in step 3 (which is the third element of the chosen candidate solution).

5. Repeat steps 2, 3 and 4 until $\hat{\boldsymbol{\theta}}$ converges.

3. TDOA SOURCE LOCALIZATION : NOISY SENSOR POSITIONS

3.1. Introduction

In the previous chapter 2, a number of TDOA based source localization algorithms were studied. However, these algorithms were developed based on the crucial assumption that the sensor positions are exactly known. Sensor position uncertainties, which might occur as a result of measurement errors (as in measuring any physical quantity) were not taken into consideration. Moreover, in real-world settings, accurate information on sensor position may not be available for various reasons. For example, sensors could be placed on-board on unmanned aerial vehicles (UAVs) or aeroplanes whose positions cannot be known precisely even with advanced GPS technology. Furthermore, in a sensor network, sensors are randomly placed in a surveillance area so that it is difficult to know their locations accurately. Ignoring the error in sensor position can cause significant degradation in the localization accuracy as is demonstrated in [26]. Hence, in recent years, researchers have tackled the source localization problem by also taking sensor position uncertainties into account [6, 23, 24].

3.2. Problem Formulation

Let us consider a network of N passive, stationary sensors. Sensors are placed in a 2D plane as shown in the Figure 2.2 and the TDOA information they collect is used to estimate the unknown source position $\mathbf{x} = [\ x \quad y\ ]^T$. The true positions $\mathbf{x}_i^0 = [\ x_i^0 \quad y_i^0\ ]^T$ of these sensors are unknown in practice and only their approximate (noisy) values $\mathbf{x}_i = [\ x_i \quad y_i\ ]^T$, $i = 1, 2, ..., N$ and $N \geq 3$ are available to the source localization algorithms. In precise terms, the relationship between $\mathbf{x}_i$ and $\mathbf{x}_i^0$ can be written as follows:

$$\mathbf{x}_i = \mathbf{x}_i^0 + \Delta\mathbf{x}_i \tag{3.1}$$

where $\Delta\mathbf{x}_i$ is the noise in sensor position $\mathbf{x}_i$ and the corresponding noise vector $\Delta\mathbf{s} = [\ \Delta\mathbf{x}_1^T \quad \Delta\mathbf{x}_2^T \quad \cdots \quad \Delta\mathbf{x}_N^T\ ]^T$ is assumed to be zero-mean Gaussian distributed with covariance matrix $\mathbf{Q}_x$. In more compact form, we can write

$$\mathbf{s} = \mathbf{s}^0 + \Delta\mathbf{s} \tag{3.2}$$

where

$$\begin{aligned} \mathbf{s} &= [\ \mathbf{x}_1^T \quad \mathbf{x}_2^T \quad \cdots \quad \mathbf{x}_N^T\ ]^T && (3.3) \\ \mathbf{s}^0 &= [\ \mathbf{x}_1^{0^T} \quad \mathbf{x}_2^{0^T} \quad \cdots \quad \mathbf{x}_N^{0^T}\ ]^T && (3.4) \end{aligned}$$

In order to get rid of measurements with large errors, we assume that a reliable non-line-of-sight detection algorithm such as [35] has been applied prior to the localization attempt so that all the measurements we utilize for source localization are limited to line-of-sight (LOS) propagation. Without loss of generality, if the first sensor is chosen as the reference sensor, available noisy TDOA between 1^{st} and i^{th} sensor is given as follows:

$$t_{i,1} = t_i - t_1, \qquad i = 2, 3, ..., N. \tag{3.5}$$

For simplicity purposes, we convert all TDOAs to RDOAs and TOAs to their corresponding distance values using the simple relation as $ROA = c * TOA$ and $RDOA = c * TDOA$ where c is the speed of propagation of the signal and ROA is the distance between the source and the receiver. So the above equation 3.5 can be rewritten as

$$r_{i,1} = r_i - r_1, \qquad i = 2, 3, ..., N. \tag{3.6}$$

where $r_{i,1} = c * t_{i,1}$ and $r_i = c * t_i$.

Now, taking hint from Generalized Measurement Model 2.3 in the previous chapter, Noisy RDOAs $r_{i,1}$ can be written as

$$\begin{aligned} r_{i,1} &= f_{i,1}(\mathbf{x}) + \Delta r_{i,1} \\ r_{i,1} &= r_{i,1}^0 + \Delta r_{i,1} \end{aligned} \qquad i = 2, 3, ..., N \tag{3.7}$$

where

$$r_{i,1}^0 = r_i^0 - r_1^0 \tag{3.8}$$

and

$$r_i^0 = \left\| \mathbf{x} - \mathbf{x}_i^0 \right\| = \sqrt{(x - x_i^0)^2 + (y - y_i^0)^2} \tag{3.9}$$

In the above Equation 3.7, $r_{i,1}^0$ represents the noise-free RDOA values and $\Delta r_{i,1}$ is the additive noise corrupting $r_{i,1}^0$. The corresponding noise vector

$$\Delta \mathbf{r} = [\ \Delta r_{2,1} \quad \Delta r_{3,1} \quad \cdots \quad \Delta r_{N,1}\ ]^T$$

is assumed to be zero-mean Gaussian with covariance $\mathbf{Q}_r$. r_i^0 is the true distance from the source $\mathbf{x}$ to the i^{th} sensor $\mathbf{x}_i$. In order to simplify the notation, equation 3.7 can be rewritten in the vector form as

$$\mathbf{r} = \mathbf{r}^0 + \Delta\mathbf{r} \tag{3.10}$$

where

$$\begin{aligned} \mathbf{r} &= \begin{bmatrix} r_{2,1} & r_{3,1} & \cdots & r_{N,1} \end{bmatrix}^T & (3.11) \\ \mathbf{r}^0 &= \begin{bmatrix} r_{2,1}^0 & r_{3,1}^0 & \cdots & r_{N,1}^0 \end{bmatrix}^T & (3.12) \end{aligned}$$

For the ease of exposition, the sensor position noise vector $\Delta\mathbf{s}$ is assumed to be independent of the RDOA error vector $\Delta\mathbf{r}$. This assumption is reasonable if $\Delta\mathbf{r}$ and $\Delta\mathbf{s}$ are obtained from two different kinds of sources. In situations when they are correlated, the solution can be easily obtained by making slight modifications to that of the original uncorrelated case. Now, Equation 3.8 can be rewritten as

$$r_{i,1}^0 + r_1^0 = r_i^0 \tag{3.13}$$

Squaring both sides of Equation 3.13 and simplifying it results in

$$\begin{aligned} r_{i,1}^0 + r_1^0 &= r_i^0 \\ r_{i,1}^{0^2} + 2r_{i,1}^0 r_1^0 + r_1^{0^2} &= r_i^{0^2} \\ r_{i,1}^{0^2} + 2r_{i,1}^0 r_1^0 &= (x - x_i^0)^2 + (y - y_i^0)^2 - (x - x_1^0)^2 - (y - y_1^0)^2 \\ r_{i,1}^{0^2} + 2r_{i,1}^0 r_1^0 &= (x_i^{0^2} + y_i^{0^2}) - (x_1^{0^2} + y_1^{0^2}) - 2x(x_i^0 - x_1^0) - 2y(y_i^0 - y_1^0) \\ r_{i,1}^{0^2} - R_i^{0^2} + R_1^{0^2} &= -2x(x_i^0 - x_1^0) - 2y(y_i^0 - y_1^0) - 2r_{i,1}^0 r_1^0 \\ r_{i,1}^{0^2} - R_i^{0^2} + R_1^{0^2} &= -2x(\mathbf{x}_i^0 - \mathbf{x}_1^0)^T\mathbf{x} - 2r_{i,1}^0 r_1^0 & (3.14) \end{aligned}$$

where

$$R_i^0 = \sqrt{x_i^{0^2} + y_i^{0^2}} \qquad i = 2, 3, ..., N \tag{3.15}$$

We can see that Equation 3.14 is linear in terms of $\mathbf{x}$ and r_1^0. The linearization is achieved by introducing an intermediate variable r_1^0 (which is related to the true source location $\mathbf{x}$) and assuming it to be independent of $\mathbf{x}$. However, r_1^0 is the true distance between the unknown source $\mathbf{x}$ and the true position of the first (reference) sensor $\mathbf{x}_1^0$ which is not available. So, we expand r_1^0 around the noisy sensor position

$\mathbf{x}_1$ using Taylor-series expansion up to the linear error term as follows:

$$\begin{aligned} r_1^0 &= \left\|\mathbf{x}-\mathbf{x}_1^0\right\| \\ &= \left\|\mathbf{x}-\mathbf{x}_1+\Delta\mathbf{x}_1\right\| \approx r_1+\mathbf{g}_{\mathbf{x},\mathbf{x}_1}^T\Delta\mathbf{x}_1 \end{aligned} \tag{3.16}$$

where

$$r_1 = \|\mathbf{x}-\mathbf{x}_1\| \tag{3.17}$$

and

$$\mathbf{g}_{\mathbf{x},\mathbf{x}_1} = \frac{\mathbf{x}-\mathbf{x}_1}{\|\mathbf{x}-\mathbf{x}_1\|} \tag{3.18}$$

By substituting $r_{i,1}^0 = r_{i,1} - \Delta r_{i,1}$ and $\mathbf{x}_i^0 = \mathbf{x}_i - \Delta\mathbf{x}_i$ (Equations 3.7 and 3.1), the set of linear equations in 3.14 can be rewritten as

$$\mathbf{q} = \mathbf{b} - \mathbf{A}\boldsymbol{\theta} \tag{3.19}$$

where

$$\mathbf{b} = \begin{bmatrix} r_{2,1}^2 - R_2^2 + R_1^2 \\ \vdots \\ r_{N,1}^2 - R_N^2 + R_1^2 \end{bmatrix} \tag{3.20}$$

$$\mathbf{A} = -2\begin{bmatrix} (\mathbf{x}_2-\mathbf{x}_1)^T & r_{2,1} \\ \vdots & \vdots \\ (\mathbf{x}_N-\mathbf{x}_1)^T & r_{N,1} \end{bmatrix} \tag{3.21}$$

$$\boldsymbol{\theta} = \begin{bmatrix} x & y & r_1 \end{bmatrix}^T \tag{3.22}$$

and $R_i = \sqrt{x_i^2 + y_i^2}$, $\quad r_1 = \|\mathbf{x}-\mathbf{x}_1\|$.
Rearranging Equation 3.19 results in

$$\mathbf{b} = \mathbf{A}\boldsymbol{\theta} + \mathbf{q} \tag{3.23}$$

which shows that $\mathbf{b}$ is linear with respect to the parameter vector $\boldsymbol{\theta}$ (assuming that $\mathbf{x}$ and r_1 are independent) and corrupted by the error vector $\mathbf{q}$ which can be written in terms of $\Delta\mathbf{r}$ and $\Delta\mathbf{s}$ as follows:

$$\mathbf{q} = \mathbf{B}\Delta\mathbf{r} + \Delta\mathbf{r}\otimes\Delta\mathbf{r} + \mathbf{D}\Delta\mathbf{s} + \Delta\mathbf{s}\otimes\Delta\mathbf{s} \tag{3.24}$$

where $\otimes$ represents the element wise multiplication. Ignoring the second order terms in Equation 3.24, the error vector $\mathbf{q}$ can be approximated as follows:

$$\mathbf{q} \approx \mathbf{B}\Delta\mathbf{r} + \mathbf{D}\Delta\mathbf{s} \tag{3.25}$$

where

$$\mathbf{B} = 2\begin{bmatrix} r_2^0 & 0 & \cdots & 0 \\ 0 & r_3^0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_N^0 \end{bmatrix} \tag{3.26}$$

$$\mathbf{D} = 2\begin{bmatrix} -r_{2,1}^0\mathbf{g}_{\mathbf{x},\mathbf{x}_1}^T - (\mathbf{x}-\mathbf{x}_1)^T & (\mathbf{x}-\mathbf{x}_2)^T & \mathbf{0}^T & \cdots & \mathbf{0}^T \\ -r_{3,1}^0\mathbf{g}_{\mathbf{x},\mathbf{x}_1}^T - (\mathbf{x}-\mathbf{x}_1)^T & \mathbf{0}^T & (\mathbf{x}-\mathbf{x}_3)^T & \cdots & \mathbf{0}^T \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -r_{N,1}^0\mathbf{g}_{\mathbf{x},\mathbf{x}_1}^T - (\mathbf{x}-\mathbf{x}_1)^T & \mathbf{0}^T & \mathbf{0}^T & \cdots & (\mathbf{x}-\mathbf{x}_N)^T \end{bmatrix} \tag{3.27}$$

and $\mathbf{g}_{\mathbf{x},\mathbf{x}_1}$ is defined in Equation 3.18.

The likelihood function of the unknown parameter $\boldsymbol{\theta}$ given the available measurement $\mathbf{b}$ is given as

$$L(\boldsymbol{\theta};\mathbf{b}) = \frac{1}{(2\pi)^{(N-1)/2}|\mathbf{Q}_0|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{b}-\mathbf{A}\boldsymbol{\theta})^T{\mathbf{Q}_0}^{-1}(\mathbf{b}-\mathbf{A}\boldsymbol{\theta})\right\} \tag{3.28}$$

where $\mathbf{Q}_0 = \mathrm{E}[\mathbf{q}\mathbf{q}^T]$. Taking the logarithm of Equation 3.28 results in

$$\ln L(\boldsymbol{\theta};\mathbf{b}) = C - \frac{1}{2}(\mathbf{b}-\mathbf{A}\boldsymbol{\theta})^T{\mathbf{Q}_0}^{-1}(\mathbf{b}-\mathbf{A}\boldsymbol{\theta}) \tag{3.29}$$

where C is a constant because it does not depend on the unknown parameter $\boldsymbol{\theta}$. Considering the approximate error vector in Equation 3.25 obtained by ignoring the second order terms, we have $\mathbf{Q}_0 \approx \mathbf{Q}$ where

$$\mathbf{Q} \triangleq \mathbf{B}\mathbf{Q}_r\mathbf{B}^T + \mathbf{D}\mathbf{Q}_x\mathbf{D}^T \tag{3.30}$$

Since $\mathbf{q}$ given in Equation 3.25 is approximated to be a linear combination of independent Gaussian noises $\Delta\mathbf{r}$ and $\Delta\mathbf{s}$ with the known covariance matrices $\mathbf{Q}_r$ and $\mathbf{Q}_x$, the minimum variance unbiased (MVU) estimator of the rearranged linear model given in Equation 3.23 can be approximated by the Maximum Likelihood (ML) estimator as follows:

$$\arg\min_{\boldsymbol{\theta}} (\mathbf{b}-\mathbf{A}\boldsymbol{\theta})^T\mathbf{Q}^{-1}(\mathbf{b}-\mathbf{A}\boldsymbol{\theta}) \tag{3.31}$$

The ML estimator given above is approximate because of the following reasons :

1. It is not directly based upon the RDOA measurements but on the linearized model given in Equation 3.23 which assumes the independence between θ and r_1.

2. It does not use the actual error vector to calculate $\mathbf{Q}$ but its approximated form given in Equation 3.25.

3.3. Weighted Least Squares (WLS)

The ML estimator (Equation 3.31) is also equivalent to the weighted least squares (WLS) estimator in which case $\mathbf{Q}^{-1}$ can be considered as the weighting matrix so the WLS solution of $\boldsymbol{\theta}$ which minimizes $\mathbf{q}^T\mathbf{W}\mathbf{q}$ is given as

$$\hat{\boldsymbol{\theta}}_{WLS} = (\mathbf{A}^T\mathbf{W}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{W}\mathbf{b} \tag{3.32}$$

where $\mathbf{W} = \mathbf{Q}^{-1} = \mathrm{E}[\mathbf{q}\mathbf{q}^T]^{-1}\big|_{\boldsymbol{\theta}=\boldsymbol{\theta}^0}$ and $\boldsymbol{\theta}^0$ represents the true solution of $\boldsymbol{\theta}$.

The weighting matrix $\mathbf{W}$ is not known in practice since $\mathbf{B}$ in Equation 3.26 and $\mathbf{D}$ in Equation 3.27 contain the true source position and true sensor positions which are not available. Similarly as in [26], we first take $\mathbf{W} = \mathbf{Q}_r{}^{-1}$ and then use Equation 3.32 to obtain an initial estimate $\hat{\boldsymbol{\theta}}_{WLS}$ to calculate $\mathbf{B}$ and $\mathbf{D}$ which are subsequently used to generate the improved weighting matrix using Equation 3.30 (and taking its inverse). This improved $\mathbf{W}$ can then be used again in Equation 3.32 to obtained a more refined solution. We can repeat this process for a few times to obtain a satisfactory $\mathbf{W}$ and a more accurate solution.

Now we proceed on to find the expected value and covariance of $\hat{\boldsymbol{\theta}}_{WLS}$. We know that the estimated $\hat{\boldsymbol{\theta}}_{WLS}$ can be written as the sum of true $\boldsymbol{\theta}$ and the estimation noise $\mathbf{e}_{WLS}$ given as

$$\hat{\boldsymbol{\theta}}_{WLS} = \boldsymbol{\theta} + \mathbf{e}_{WLS} \tag{3.33}$$

Multiplying both sides of Equation 3.32 with $\mathbf{A}^T\mathbf{W}\mathbf{A}$ and substituting Equation 3.33, we get

$$\mathbf{A}^T\mathbf{W}\mathbf{A}(\boldsymbol{\theta} + \mathbf{e}_{WLS}) = \mathbf{A}^T\mathbf{W}\mathbf{b} \tag{3.34}$$

Simplifying for $\mathbf{e}_{WLS}$ gives

$$\begin{aligned}\mathbf{e}_{WLS} &= (\mathbf{A}^T\mathbf{W}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{W}(\mathbf{b} - \mathbf{A}\boldsymbol{\theta}) \\ &= (\mathbf{A}^T\mathbf{W}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{W}\mathbf{q}\end{aligned} \tag{3.35}$$

Note that, since second-order noise terms are ignored, $\mathbf{q}$ approximately becomes a zero-mean vector so that $\mathrm{E}(\mathbf{q}) \approx \mathbf{0}$. Hence the expected value and covariance of $\hat{\boldsymbol{\theta}}_{WLS}$ can be written as

$$\mathrm{E}[\hat{\boldsymbol{\theta}}_{WLS}] = \boldsymbol{\theta} + \mathrm{E}[\mathbf{e}_{WLS}] \approx \boldsymbol{\theta} \tag{3.36}$$

$$\begin{aligned}\text{cov}(\hat{\boldsymbol{\theta}}_{WLS}) &\approx \text{E}[\mathbf{e}_{WLS}\mathbf{e}_{WLS}^T] \\ &\approx (\mathbf{A}^T\mathbf{W}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{W}\mathbf{W}^{-1}\mathbf{W}\mathbf{A}(\mathbf{A}^T\mathbf{W}\mathbf{A})^{-1} \\ &\approx (\mathbf{A}^T\mathbf{W}\mathbf{A})^{-1}\end{aligned} \tag{3.37}$$

The results obtained in Equations 3.36 and 3.37 will be used in the following section 3.4.

3.4. Two Step Weighted Least Squares (2sWLS)

WLS method obtains the source location estimate ignoring the relationship between the unknown source position $\mathbf{x}$ and the intermediate variable r_1 represented by equation 3.17. However, localization performance can be further improved by making use of this relationship which can also be expanded as

$${r_1}^2 = (x - x_1)^2 + (y - y_1)^2 \tag{3.38}$$

Based on the above Equation 3.38, and assuming that $\hat{\boldsymbol{\theta}}_{WLS}$ is sufficiently close to θ, we construct another set of equations as follows:

$$\mathbf{w} = \mathbf{h} - \mathbf{G}\mathbf{z} \tag{3.39}$$

where

$$\mathbf{h} = \begin{bmatrix} ([\hat{\theta}_{WLS}]_1 - x_1)^2 \\ ([\hat{\theta}_{WLS}]_2 - y_1)^2 \\ [\hat{\theta}_{WLS}]_3^2 \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{z} = \begin{bmatrix} (x - x_1)^2 \\ (y - y_1)^2 \end{bmatrix} \tag{3.40}$$

The WLS minimization of $\mathbf{w}^T\boldsymbol{\Phi}\mathbf{w}$ results in

$$\begin{aligned}\hat{\mathbf{z}} &= \arg\min_{\tilde{\mathbf{z}}} (\mathbf{h} - \mathbf{G}\tilde{\mathbf{z}})^T\boldsymbol{\Phi}(\mathbf{h} - \mathbf{G}\tilde{\mathbf{z}}) \\ &= (\mathbf{G}^T\boldsymbol{\Phi}\mathbf{G})^{-1}\mathbf{G}^T\boldsymbol{\Phi}\mathbf{h}\end{aligned} \tag{3.41}$$

where $\boldsymbol{\Phi} = \text{E}[\mathbf{w}\mathbf{w}^T]^{-1}\big|_{\mathbf{z}=\mathbf{z}^0}$ is the weighting matrix and $\mathbf{z^0}$ is the true solution of $\mathbf{z}$. Using Equation 3.33 and substituting $\mathbf{x}_s = \mathbf{x}_s^0 + \Delta\mathbf{x}$ (Equation 3.2), we can easily approximate the estimation error term $\mathbf{w}$ from Equation 3.39 by ignoring the second order term as follows [36]:

$$\mathbf{w}\big|_{\mathbf{z}=\mathbf{z}^0} \approx \mathbf{B}_1\mathbf{e}_{WLS} + \mathbf{D}_1\Delta\mathbf{x} \tag{3.42}$$

$$\mathbf{B}_1 = 2\begin{bmatrix} x - x_1 & 0 & 0 \\ 0 & y - y_1 & 0 \\ 0 & 0 & r_1^0 \end{bmatrix} \tag{3.43}$$

$$\mathbf{D}_1 = 2\begin{bmatrix} \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T \\ r_{2,1}^0 \mathbf{g}_{\mathbf{x},\mathbf{x}_1}^T + (\mathbf{x} - \mathbf{x}_1)^T & \mathbf{0}^T & \mathbf{0}^T & \mathbf{0}^T \end{bmatrix} \tag{3.44}$$

where $\mathbf{g}_{\mathbf{x},\mathbf{x}_1}$ is defined in Equation 3.18. Hence we obtain the weighting matrix as

$$\mathbf{\Phi} = [\mathbf{B}_1 \text{cov}(\hat{\boldsymbol{\theta}}_{WLS}) \mathbf{B}_1^T + \mathbf{D}_1 \mathbf{Q}_x \mathbf{D}_1^T]^{-1} \tag{3.45}$$

$\mathbf{z}$ contains the source position $\mathbf{x}$ within it as can be seen from Equation 3.40. As can be seen from Equation 3.40, the elements of $\mathbf{z}$ are in square form. To retrieve the source position $\mathbf{x}$, we take the square root of $\mathbf{z}$ which would introduce the sign ambiguity as can be seen below:

$$\begin{bmatrix} \pm\sqrt{[\hat{\mathbf{z}}]_1} \\ \pm\sqrt{[\hat{\mathbf{z}}]_2} \end{bmatrix} \tag{3.46}$$

where $[\hat{\mathbf{z}}]_1$ and $[\hat{\mathbf{z}}]_1$ are first and the second elements of $\hat{\mathbf{z}}$ respectively. As there is no sign infomation for $\mathbf{x}$ in $\hat{\mathbf{z}}$ there are four different $\mathbf{x}$ candidates. To resolve this issue, we simply take the sign information from the elements of the WLS estimate $\hat{\boldsymbol{\theta}}_{WLS}$. So, finally, we get

$$\hat{\mathbf{x}}_{(2sWLS)} = \mathbf{S}\begin{bmatrix} \sqrt{[\hat{\mathbf{z}}]_1} \\ \sqrt{[\hat{\mathbf{z}}]_2} \end{bmatrix} + \mathbf{x}_1 \tag{3.47}$$

where $\mathbf{S} = diag\{\text{sgn}([\hat{\boldsymbol{\theta}}_{WLS}]_1 - x_1), \text{sgn}([\hat{\boldsymbol{\theta}}_{WLS}]_2 - y_1)\}$.

Comments : The two step WLS method given in this section has higher computational complexity than that of the corresponding method given in the previous chapter which does not take into account the sensor position uncertainties. That is because two step WLS method given in this section involves the computation of $\mathbf{D}\mathbf{Q}_x\mathbf{D}^T$ and $\mathbf{D}_1\mathbf{Q}_x\mathbf{D}_1^T$ in Equation 3.30 and Equation 3.45 respectively. However, the size of the matrices to be inverted remains the same so that the increase in computational complexity is not very significant. So the two step WLS method which takes sensor position errors into account is superior to the one given in the previous chapter (which assumes the sensor positions to be exactly known) for a slight increase in computational complexity.

3.5. Constrained Least Squares (CLS)

In the previous section 3.2, the nonlinear Equation set given in 3.7 was linearized by introducing a new variable r_1 (Equation 3.17) and assuming it to be independent of the source location $\mathbf{x}$ to make the resulting equation set linear in $\mathbf{x}$. However, r_1 is a function of the true source location $\mathbf{x}$ and the WLS estimate $\hat{\boldsymbol{\theta}}_{WLS}$ given in Equation 3.32 for source location does not take this relationship into account. In section Equation 3.4, this ignored relationship is exploited in an implicit manner to improve the WLS estimate given in section Equation 3.3 even further. In [18], the authors propose to incorporate the relationship Equation 3.17 explicitly on the minimization problem Equation 3.31 through the use of Lagrange multiplier technique. The minimization problem, hence, can be written formally as

$$\begin{aligned} &\min_{\boldsymbol{\theta}} (\mathbf{b} - \mathbf{A}\boldsymbol{\theta})^T(\mathbf{b} - \mathbf{A}\boldsymbol{\theta}) \\ &\quad s.t. (\boldsymbol{\theta} - \tilde{\mathbf{x}}_1)^T \boldsymbol{\Sigma}(\boldsymbol{\theta} - \tilde{\mathbf{x}}_1) = 0,\ \boldsymbol{\theta}_3 \geq 0 \end{aligned} \tag{3.48}$$

where $\boldsymbol{\Sigma} \triangleq diag(1, 1, -1)$. The equality constraint in Equation 3.48 is the matrix representation of the equation Equation 3.17 while the inequality constraint in Equation 3.48 arises from the fact that $\boldsymbol{\theta}_3 = r_1$ is a non-negative quantity. Therefore, ignoring the inequality constraint, the corresponding Lagrangian can be written as

$$\mathcal{L}(\boldsymbol{\theta}, \eta) = (\mathbf{b} - \mathbf{A}\boldsymbol{\theta})^T(\mathbf{b} - \mathbf{A}\boldsymbol{\theta}) + \eta(\boldsymbol{\theta} - \tilde{\mathbf{x}}_1)^T \boldsymbol{\Sigma}(\boldsymbol{\theta} - \tilde{\mathbf{x}}_1) \tag{3.49}$$

where η is the Lagrange multiplier to be determined. Constrained least squares (CLS) estimate is obtained by minimizing the Lagrange function (Equation 3.49) as

$$\hat{\boldsymbol{\theta}}_{CLS} \triangleq \arg\min_{\boldsymbol{\theta}} \{L(\boldsymbol{\theta},\eta)\} \tag{3.50}$$

Expanding the above Lagrange function Equation 3.50 gives

$$\begin{aligned} \mathcal{L}(\boldsymbol{\theta}, \eta) &= (\mathbf{b}^T - \boldsymbol{\theta}^T\mathbf{A}^T)(\mathbf{b} - \mathbf{A}\boldsymbol{\theta}) + \eta(\boldsymbol{\theta}^T - \tilde{\mathbf{x}}_1^T)(\boldsymbol{\Sigma}\boldsymbol{\theta} - \boldsymbol{\Sigma}\tilde{\mathbf{x}}_1) \\ &= \mathbf{b}^T\mathbf{b} - 2\mathbf{b}^T\mathbf{A}\boldsymbol{\theta} + \boldsymbol{\theta}^T\mathbf{A}^T\mathbf{A}\boldsymbol{\theta} + \eta(\boldsymbol{\theta}^T\boldsymbol{\Sigma}\boldsymbol{\theta} - \boldsymbol{\theta}^T\boldsymbol{\Sigma}\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_1^T\boldsymbol{\Sigma}\boldsymbol{\theta} + \tilde{\mathbf{x}}_1^T\boldsymbol{\Sigma}\tilde{\mathbf{x}}_1) \end{aligned} \tag{3.51}$$

The estimate of $\boldsymbol{\theta}$ is obtained by differentiating $\mathcal{L}(\boldsymbol{\theta},\eta)$ with respect to $\boldsymbol{\theta}$ and then equating the results to zero as follows:

$$\frac{\partial \mathcal{L}(\boldsymbol{\theta}, \eta)}{\partial \boldsymbol{\theta}} = 2(-\mathbf{b}^T\mathbf{A} + \boldsymbol{\theta}^T\mathbf{A}^T\mathbf{A} + \eta\boldsymbol{\theta}^T\boldsymbol{\Sigma} - \eta\tilde{\mathbf{x}}_1^T\boldsymbol{\Sigma}) = 0 \tag{3.52}$$

or

$$\mathbf{A}^T\mathbf{A}\boldsymbol{\theta} + \eta\boldsymbol{\Sigma}\boldsymbol{\theta} = \mathbf{A}^T\mathbf{b} + \eta\boldsymbol{\Sigma}\tilde{\mathbf{x}}_1 \tag{3.53}$$

Solving Equation 3.53 for $\boldsymbol{\theta}$ gives the CLS estimate as

$$\hat{\boldsymbol{\theta}}_{CLS}(\eta) = (\mathbf{A}^T\mathbf{A} + \eta\boldsymbol{\Sigma})^{-1}(\mathbf{A}^T\mathbf{b} + \eta\boldsymbol{\Sigma}\tilde{\mathbf{x}}_1) \tag{3.54}$$

where η is yet to be determined. Once we obtain η, the CLS estimate of the source location can be calculated from Equation 3.54. To find η, we can directly impose the quadratic constraint given in Equation 3.48 by substituting Equation 3.54 into Equation 3.48 which results in

$$(\hat{\boldsymbol{\theta}}_{CLS}(\eta) - \tilde{\mathbf{x}}_1)^T\boldsymbol{\Sigma}(\hat{\boldsymbol{\theta}}_{CLS}(\eta) - \tilde{\mathbf{x}}_1) = 0 \tag{3.55}$$

Substituting $\hat{\boldsymbol{\theta}}_{CLS}(\eta)$ from Equation 3.54 back to Equation 3.55 results in a high order polynomial which is the function of η. As in [18], we solve for η by using the improved Newton's method (refer Equation 2.105) with step size $1 - a^k$, where $a \in [0, 1]$ is an attenuation factor and k is the iteration number.

3.6. Constrained Weighted Least Squares (CWLS)

In the previous section 3.5, we studied that the CLS method explicitly incorporates the constraints in the minimization problem, unlike the 2sWLS method studied in section 3.4 making CLS more robust and elegant than 2sWLS. However, in CLS, the correlation among the TDOA values is not taken into account which was considered in WLS and 2sWLS methods. In order to enhance the estimation performance of CLS, this correlation could be used in the form of a weighting matrix similarly as in WLS and 2sWLS method which gives rise to CWLS method. In other words, CWLS method can be considered as a weighted version of the CLS. From 3.31 we recall the unconstrained WLS estimator is given as

$$\arg\min_{\boldsymbol{\theta}} (\mathbf{b} - \mathbf{A}\boldsymbol{\theta})^T\mathbf{W}(\mathbf{b} - \mathbf{A}\boldsymbol{\theta})^T \tag{3.56}$$

Using Equation 3.48 the minimization problem can be written as

$$\begin{aligned} &\min_{\boldsymbol{\theta}} (\mathbf{b} - \mathbf{A}\boldsymbol{\theta})^T\mathbf{W}(\mathbf{b} - \mathbf{A}\boldsymbol{\theta}) \\ &\quad s.t. (\boldsymbol{\theta} - \tilde{\mathbf{x}}_1)^T\boldsymbol{\Sigma}(\boldsymbol{\theta} - \tilde{\mathbf{x}}_1) = 0,\ \boldsymbol{\theta}_3 \geq 0 \end{aligned} \tag{3.57}$$

Therefore, ignoring the inequality constraint, the corresponding Lagrangian can be written as

$$\mathcal{L}'(\boldsymbol{\theta}, \eta') = (\mathbf{b} - \mathbf{A}\boldsymbol{\theta})^T\mathbf{W}(\mathbf{b} - \mathbf{A}\boldsymbol{\theta}) + \eta'(\boldsymbol{\theta} - \tilde{\mathbf{x}}_1)^T\boldsymbol{\Sigma}(\boldsymbol{\theta} - \tilde{\mathbf{x}}_1) \tag{3.58}$$

where η' is the Lagrange multiplier to be determined. Constrained weighted least squares (CWLS) estimate is obtained by minimizing the Lagrange function (Equation 3.58) as

$$\hat{\boldsymbol{\theta}}_{CWLS} \triangleq \arg\min_{\boldsymbol{\theta}}\{L'(\boldsymbol{\theta},\eta')\} \tag{3.59}$$

Expanding the above Lagrange function gives

$$\begin{aligned}\mathcal{L}'(\boldsymbol{\theta},\eta') &= (\mathbf{b}^T - \boldsymbol{\theta}^T\mathbf{A}^T)\mathbf{W}(\mathbf{b} - \mathbf{A}\boldsymbol{\theta}) + \eta'(\boldsymbol{\theta}^T - \tilde{\mathbf{x}}_1^T)(\boldsymbol{\Sigma}\boldsymbol{\theta} - \boldsymbol{\Sigma}\tilde{\mathbf{x}}_1)\\ &= \mathbf{b}^T\mathbf{W}\mathbf{b} - 2\mathbf{b}^T\mathbf{W}\mathbf{A}\boldsymbol{\theta} + \boldsymbol{\theta}^T\mathbf{A}^T\mathbf{W}\mathbf{A}\boldsymbol{\theta} + \eta'(\boldsymbol{\theta}^T\boldsymbol{\Sigma}\boldsymbol{\theta} - \boldsymbol{\theta}^T\boldsymbol{\Sigma}\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_1^T\boldsymbol{\Sigma}\boldsymbol{\theta} + \tilde{\mathbf{x}}_1^T\boldsymbol{\Sigma}\tilde{\mathbf{x}}_1)\end{aligned} \tag{3.60}$$

The estimate of $\boldsymbol{\theta}$ is obtained by differentiating $\mathcal{L}'(\boldsymbol{\theta},\eta')$ with respect to $\boldsymbol{\theta}$ and then equating the results to zero as follows:

$$\frac{\partial\mathcal{L}'(\boldsymbol{\theta},\eta')}{\partial\boldsymbol{\theta}} = 2(-\mathbf{b}^T\mathbf{W}\mathbf{A} + \boldsymbol{\theta}^T\mathbf{A}^T\mathbf{W}\mathbf{A} + \eta'\boldsymbol{\theta}^T\boldsymbol{\Sigma} - \eta'\tilde{\mathbf{x}}_1^T\boldsymbol{\Sigma}) = 0 \tag{3.61}$$

or

$$\mathbf{A}^T\mathbf{W}\mathbf{A}\boldsymbol{\theta} + \eta'\boldsymbol{\Sigma}\boldsymbol{\theta} = \mathbf{A}^T\mathbf{W}\mathbf{b} + \eta'\boldsymbol{\Sigma}\tilde{\mathbf{x}}_1 \tag{3.62}$$

Solving Equation 3.62 for $\boldsymbol{\theta}$ gives the CWLS estimate as

$$\hat{\boldsymbol{\theta}}_{CWLS}(\eta') = (\mathbf{A}^T\mathbf{W}\mathbf{A} + \eta'\boldsymbol{\Sigma})^{-1}(\mathbf{A}^T\mathbf{W}\mathbf{b} + \eta'\boldsymbol{\Sigma}\tilde{\mathbf{x}}_1) \tag{3.63}$$

where η' is yet to be determined. Once we obtain η', the CWLS estimate of the source location can be calculated from Equation 3.63. To find η', we can directly impose the quadratic constraint given in Equation 3.57 by substituting Equation 3.63 into Equation 3.57 which results in

$$(\hat{\boldsymbol{\theta}}_{CWLS}(\eta') - \tilde{\mathbf{x}}_1)^T\boldsymbol{\Sigma}(\hat{\boldsymbol{\theta}}_{CWLS}(\eta') - \tilde{\mathbf{x}}_1) = 0 \tag{3.64}$$

Substituting $\hat{\boldsymbol{\theta}}_{CWLS}(\eta')$ from Equation 3.63 back to Equation 3.64 results in a high order polynomial which is the function of η'. In a similar manner to CLS method, we solve for η' by using the improved Newton's method (refer Equation 2.105) with step size $1-a^k$, where $a \in [0,1]$ is an attenuation factor and k is the iteration number.

3.7. Convex Constrained Weighted Least Squares (Convex CWLS)

In the previous section 3.6, the minimization problem given in Equation 3.57 is solved by exploiting the Lagrange multiplier technique. In this section, we will explore an alternative method to solve the problem. In [23] the authors reformulated the non-convex minimization problem in Equation 3.57 to a convex optimization problem by utilizing the hidden convexity of the problem. It should be noted that the original problem, Equation 3.57, which is an indefinite quadratic optimization problem subject to a single quadratic equality constraint is non-convex in nature which entails that the global solution is not guaranteed. However, since the problem is reformulated to be convex, the new method can obtain the global optimal source location estimate. We proceed by reiterating that original non-convex problem has some hidden convexity in it which will be exploited to reformulate it to a convex problem. In [37], the authors discovered that under a suitable simultaneous diagonalization assumption the indefinite quadratically constrained quadratic optimization problem (QCQP) is equivalent to a convex minimization problem with simple linear constraints. We reformulate Equation 3.57 to match with the model given in [37] as follows:

$$\begin{aligned} \min \ & \boldsymbol{\theta}_1^T \mathbf{A}_1 \theta_1 - 2\mathbf{b}_1^T \boldsymbol{\theta}_1 \\ s.t. \ & \boldsymbol{\theta}_1^T \boldsymbol{\Sigma} \boldsymbol{\theta}_1 = 0 \end{aligned} \tag{3.65}$$

where

$$\boldsymbol{\theta}_1 = \boldsymbol{\theta} - \tilde{\mathbf{x}}_1 \tag{3.66}$$

$$\mathbf{A}_1 = \mathbf{A}^T \mathbf{W} \mathbf{A} \tag{3.67}$$

$$\mathbf{b}_1 = \mathbf{A}^T \mathbf{W}(\mathbf{b} - \mathbf{A}\tilde{\mathbf{x}}_1) \tag{3.68}$$

Lemma 1. Consider the problem given in Equation 3.65 where $\mathbf{A}_1$ and $\boldsymbol{\Sigma}$ are the symmetric matrices. There exists a nonsingular matrix $\mathbf{S}$ such that

$$\mathbf{S}^T \mathbf{A}_1 \mathbf{S} = \mathbf{M} := diag(m_1, m_2, m_3) \tag{3.69}$$

$$\mathbf{S}^T \boldsymbol{\Sigma} \mathbf{S} = \mathbf{N} := diag(n_1, n_2, n_3) \tag{3.70}$$

The proof for Lemma 1 is given in Appendix B. The algorithm below illustrates the steps for simulataneous diagonalization of the matrices $\mathbf{A_1}$ and $\boldsymbol{\Sigma}$ in Equation 3.65.
Algorithm 1. <u>Simultaneous Diagonalization</u> [6]

1. Do eigenvalue decomposition (EVD) on $\mathbf{A_1}$, which gives a vector $\mathbf{d}$ of eigenvalues and a matrix $\mathbf{V}$ whose columns are the corresponding eigenvectors.
2. Let $\mathbf{V}_1 = \mathbf{V} diag(\mathbf{e})$, where $\mathbf{e}(i) = 1/\sqrt{\mathbf{d}(i)}$.
3. Perform EVD on ${\mathbf{V}_1}^T \boldsymbol{\Sigma} \mathbf{V}_1$ and get the matrix $\mathbf{V}_2$ with the corresponding eigenvectors.
4. Let $\mathbf{S} = \mathbf{V}_1 \mathbf{V}_2$. Then $\mathbf{S}^T \mathbf{A}_1 \mathbf{S}$ and $\mathbf{S}^T \boldsymbol{\Sigma} \mathbf{S}$ are diagonal.

Using the property of simultaneous diagonalization the non-convex optimization problem (Equation 3.65) can be converted into a convex minimization problem as follows :

Theorem 2. The CWLS source localization problem (Equation 3.65) is equivalent to the following:

$$\begin{aligned} \min \ & \sum_{j=1}^{3} m_j v_j - 2\,|a_j|\sqrt{v_j} \\ s.t. \ & \sum_{j=1}^{3} n_j v_j = 0, \mathbf{v} \in \mathbb{R}_+^3 \end{aligned} \tag{3.71}$$

which is a convex minimization problem where

$$\mathbf{S}^T \mathbf{b}_1 = \mathbf{a} \coloneqq [a_1, a_2, a_3]^T$$

An optimal solution $\mathbf{v}^* = [v_1, v_2, v_3]$ exists for the convex problem 3.71, so that the corresponding solution of Equation 3.65 is given by

$$\boldsymbol{\theta}_1^* = \mathbf{Sc}, \ \ c_j = \operatorname{sgn}(a_j)\sqrt{v_j^*}, \ \ j = 1, ...3 \tag{3.72}$$

where $\operatorname{sgn}(a_j) = 1$ if $a_j > 0$, $\operatorname{sgn}(a_j) = 0$ if $a_j = 0$ and $\operatorname{sgn}(a_j) = -1$ if $a_j < 0$. The proof for Theorem 2 is given in Appendix B.

The convex CWLS method can be summarized as follows :

1. Set $\mathbf{W} = \mathbf{Q}_r^{-1}$ and use 3.32 to find the WLS estimate of Equation 3.57 (while ignoring the constraint). The obtained estimate is used to construct an improved weighting matrix $\mathbf{W} = \mathbf{Q}^{-1}$ from Equation 3.30. Repeat the process for three iterations to obtain the satisfactory $\mathbf{W}$.

2. As described in Algorithm 1, simultaneously diagonalize the matrices $\mathbf{A}_1$ and $\mathbf{\Sigma}$.

3. Solve the corresponding convex optimization problem (Equation 3.71) using CVX toolbox in MATLAB and obtain a source location estimate from Equations 3.72 and 3.66.

4. NUMERICAL SIMULATIONS

In this chapter, we present the simulation results which illustrates the relative performance of the closed-form source location estimators given in Chapter 2 (2.4.2.1) and Chapter 3. We apply these estimators to various localization scenario (fixed versus randomly generated source and sensor positions, for instance) and compare the results through Monte Carlo simulations.

4.1. Simulation Results for Exact Sensor Positions Scenario

In this section, we present the simulation results for closed-form estimators (section 2.4.2.1) when the sensor positions are assumed to be exactly known. We simulate under various assumptions and conditions which are as follows :

1. All the measurements we utilize for source localization results from line-of-sight (LOS) propagation. NLOS (non-line-of-sight) problem is assumed to be solved prior to the localization step.
2. The localization accuracy is evaluated by the RMSE (root mean square error) which is defined as

$$RMSE(\mathbf{x}) = \sqrt{\frac{\sum_{l=1}^{L} \|\hat{\mathbf{x}}_l - \mathbf{x}\|^2}{L}}$$

   where $\hat{\mathbf{x}}_l$ denotes the source estimate at lth iteration and L is the total number of Monte Carlo iterations.
3. Noisy RDOA measurement vector $\mathbf{r}$ is constructed from the true RDOA vector $\mathbf{d}$ by adding zero-mean Gaussian noises with covariance matrix $\mathbf{Q} = \sigma_n^2 \mathbf{T}$ where

$$\mathbf{T} = \begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 2 \end{bmatrix}$$

4.1.1. Scenario 1 : Randomly Distributed Source and Sensor Positions

In this simulation scenario, we consider a random geometry in which the source and sensor positions are randomly selected from a uniform distribution defined over some interval. The goal of this simulation is to evaluate the relative performance of the closed-form estimators. We have set the simulation conditions as follows:

1. The number of sensors $N = 5$.

2. For each iteration, the x and y coordinates of the source and sensor positions are randomly selected from $[\ 0 \quad 15\ ]$ range so that they always fall within an area of 15×15 square units.

3. σ_n increases in 10 dB steps from -30 dB to 30 dB where dB values are related to σ_n as follows :

$$20\log_{10}(\sigma_n)dB$$

4. Number of Monte Carlo runs for each dB level : 1500

Figure 4.1 shows the outcome of the simulation. The solid lines (except for CLS) represents the estimators which make use of weighting matrix $\mathbf{W}$ to obtain the source location estimates. These estimators, in general, perform better compared to the estimators not making use of the weighting matrix. It is obvious from the figure that WLS and CWLS method performed better than LLS and CLS methods respectively. Moreover, it can be clearly seen in the figure that the effect of the weighting matrix is pronounced when the noise level is moderate. But as the noise level becomes too high, the effect gradually tends to diminish. It is because the $\mathbf{R}$ matrix (Equation 2.53), which is used to construct $\mathbf{W}$, becomes increasingly noisy.

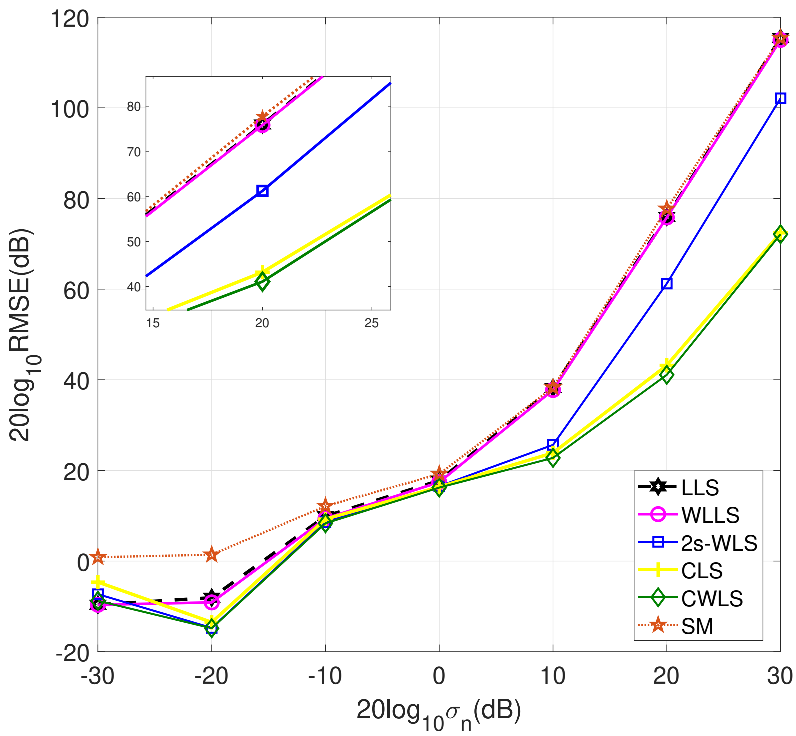


Figure 4.1: Comparison of the localization performance for random distribution scenario.

Aside from that, Figure 4.1 clearly shows the dramatic performance improvement when the relationship between the auxillary variable d_1 (Equation 2.39) and source location $\mathbf{x}$ is taken into account as in the second stage of 2sWLS method or in the form of constraints to the optimization problem as in the case of CLS and CWLS method. Furthermore, CLS and CWLS methods performed better than 2sWLS method. This goes on to show that applying constraints directly into the optimization problem results in better estimation performance than applying the constraints implicitly as is done in 2sWLS method.

Remark: We have seen through the use of simulations that the Spherical Interpolation (SI) method resulted in exactly the same source location estimates as is given by the LLS method which confirms the claim made in [31]. For this reason, we decided to omit the results of SI in the plots included in this chapter. It was also claimed in the same paper that Subspace Minimization (SM) results in the identical estimates to that of LLS method. However, the particular formulation of this method we used gave a slighly varying estimates to that of LLS method. Hence, the results of SM are included in the given plots.

4.1.2. Scenario 2 : Fixed Source and Sensor Positions

In this simulation scenario, we consider a fixed geometry in which the source and sensor positions are fixed, unlike the previous random distribution scenario 4.1.1. The goal of this simulation is to evaluate the performance of the estimators relative to the CRLB. The source and sensor distribution is illustrated in the Figure 4.2.

We have set the simulation conditions as follows:

1. Number of sensors $N = 5$
2. σ_n increases in 5 dB steps from -30 dB to 30 dB.
3. Number of Monte Carlo runs for each dB level : 1500 runs.
4. Sensor position coordinates are : $\mathbf{x}_1 = [\ 5 \quad 5\ ]^T$, $\mathbf{x}_2 = [\ 20 \quad 5\ ]^T$, $\mathbf{x}_3 = [\ 35 \quad 20\ ]^T$, $\mathbf{x}_4 = [\ 20 \quad 35\ ]^T$, $\mathbf{x}_5 = [\ 5 \quad 35\ ]^T$ and
   the source position coordinate is $\mathbf{x} = [\ 13 \quad 15\ ]^T$.

Figure 4.3 shows the outcome of the simulation. CRLB is also plotted for comparison. Similar to the random distribution scenario given in the previous section 4.1.1, fixed distribution scenario also maintains the order of the closed-form estimators in terms of RMSE performance. As usual CWLS method performs the

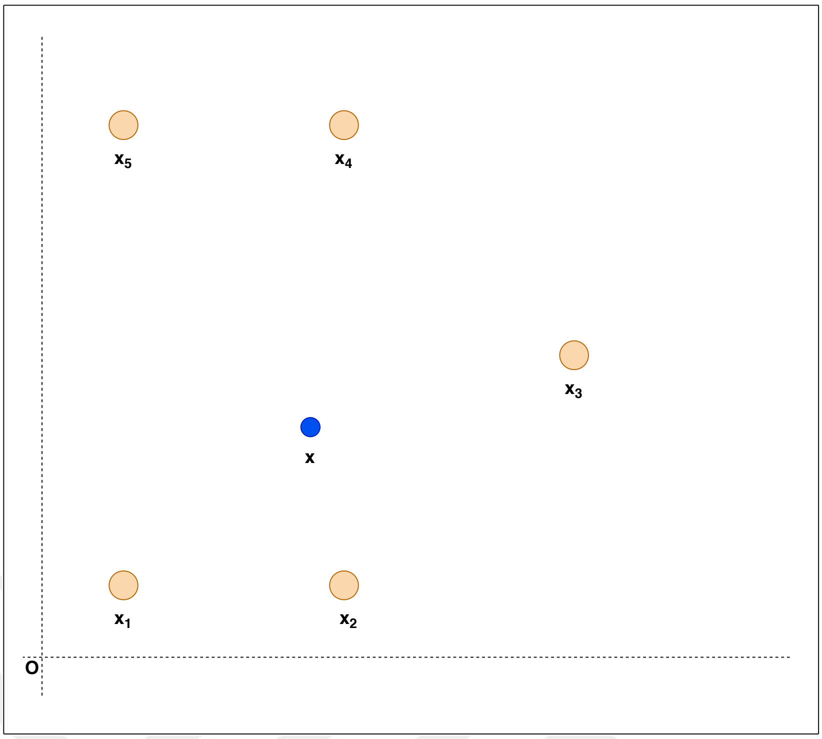


Figure 4.2: Source and sensor distribution for fixed distribution scenario.

best while LLS performs the worst. 2sWLS method shows similar RMSE performance as CLS method up to the 10 dB noise level after which the performance of 2sWLS degrades, culminating in approximately 50 dB degradation compared to CLS method at 30 dB noise. One attribute that is very noticeable in the Figure 4.3 is the robustness of CLS and CWLS methods. The corresponding RMSEs increase very gradually with the increase in noise, unlike LLS, WLS, 2sWLS and SM methods where we observe a dramatic spike in RMSE in high noise levels.

4.1.3. Effect of Number of Sensors on Localization Performance

In this section we discuss the simulation result on the effect of increase in the number of sensors N on the localization performance. As can clearly be seen in the Figure 4.4, all the source location estimation methods displayed a marked decrease in RMSE as N increased from 5 to 10. Usually, the localization algorithms are designed and optimized to be used with the minimum number of sensors possible; however, when an application needs an enhanced localization precision, the number of sensors used for localization can be increased. The result shows that LLS, WLS and SM methods performed better than 2sWLS, CLS and WLS methods when $N > 7$. However, cost considerations usually limits the number of sensors to the minimum in which case CLS, CWLS and 2sWLS would be preferred.

The details of the simulation are as follows:

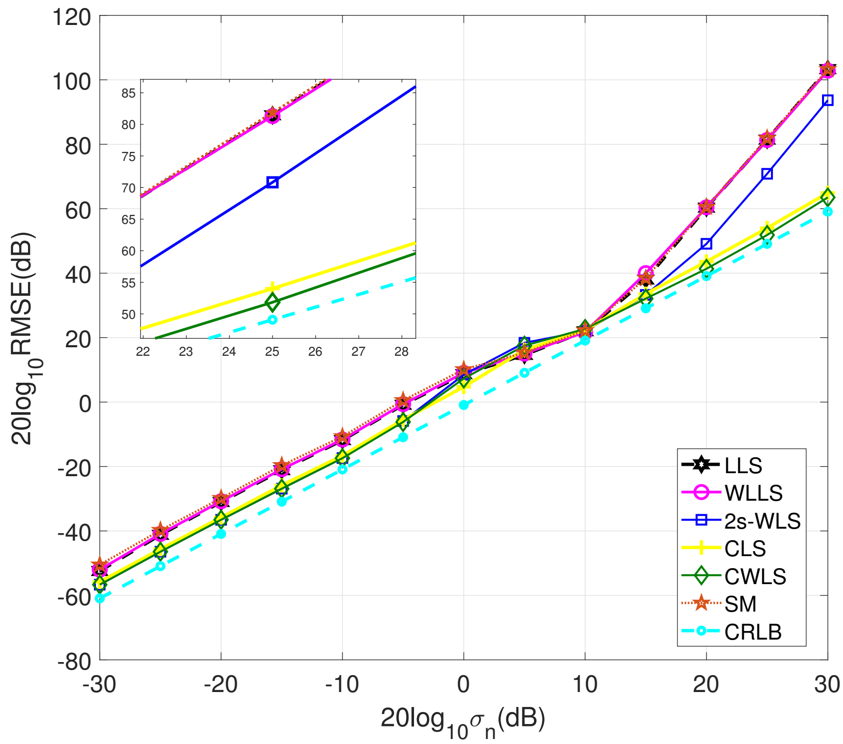


Figure 4.3: Comparison of the localization performance for fixed source and sensor distribution.

1. For each iteration, the x and y coordinates of the source and sensor positions are randomly selected from $[\ 0 \quad 15\ ]$ range so that they always fall within an area of 15×15 square units and the source position is fixed to $[\ 4 \quad 4\ ]^T$.
2. σ_n^2 is set to 0.1
3. Number of Monte Carlo trials for each N: 1500

4.2. Simulation Results for Noisy Sensor Positions Scenario

In this section, we present the simulation results for closed-form estimators (Chapter 3) when the sensor positions are assumed to be noisy. For the simulations, we consider a fixed geometry in which the source and sensor positions are fixed. The source and sensor distribution is illustrated in the Figure 4.2. The goal of this simulation is to evaluate the performance of the estimators relative to the CRLB. In addition to the simulation details given in the exact sensor position scenario 4.1, the following details were used in the simulation process:

1. Noisy RDOA measurement vector $\mathbf{r}$ (Equation 3.10) is constructed from the true RDOA vector $\mathbf{r^0}$ by adding zero-mean gaussian noises $\Delta\mathbf{r}$ with covariance

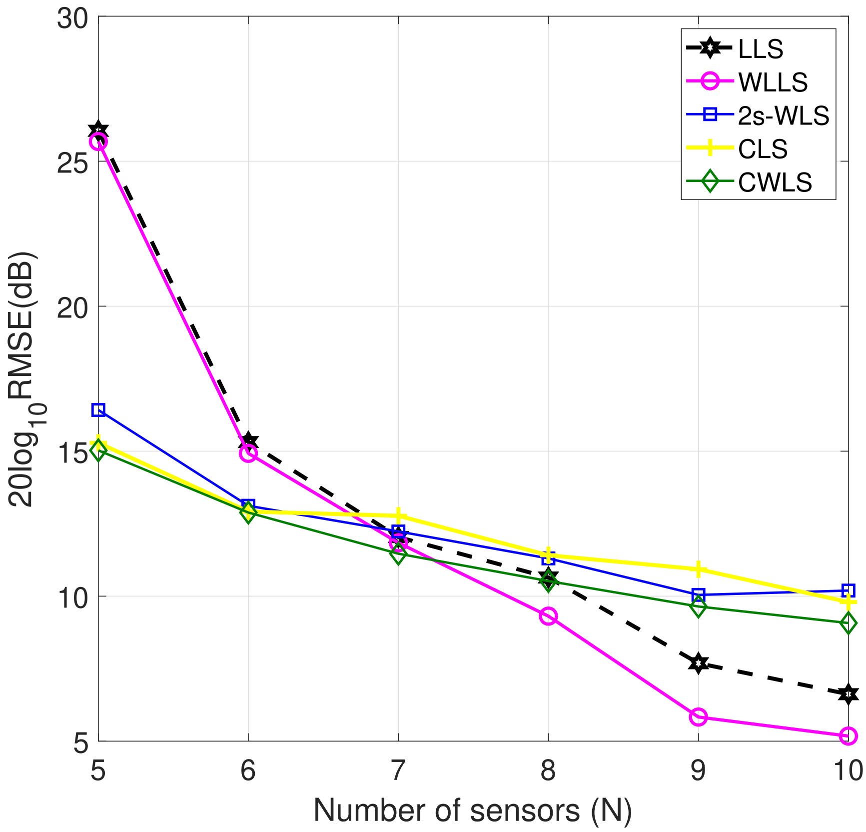


Figure 4.4: RMSE (dB) vs the No. of Sensors (N)

matrix $\mathbf{Q_r} = \sigma_r^2 \mathbf{T_r}$ where

$$\mathbf{T_r} = \begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 2 \end{bmatrix}$$

2. Noisy sensor position vector $\mathbf{s}$ is constructed from the true sensor position vector $\mathbf{s^0}$ by adding zero-mean gaussian noises $\Delta\mathbf{r}$ with covariance matrix $\mathbf{Q_x} = \sigma_x^2 \mathbf{T_x}$ where

$$\mathbf{T}_x = diag\{1, 1, 2, 2, 3, 3, 4, 4, 5, 5\}$$

3. Sensor position coordinates are : $\mathbf{x}_1 = [\ 5 \quad 5\ ]^T$, $\mathbf{x}_2 = [\ 20 \quad 5\ ]^T$, $\mathbf{x}_3 = [\ 35 \quad 20\ ]^T$, $\mathbf{x}_4 = [\ 20 \quad 35\ ]^T$, $\mathbf{x}_5 = [\ 5 \quad 35\ ]^T$
and the source position coordinate is $\mathbf{x} = [\ 13 \quad 15\ ]^T$.

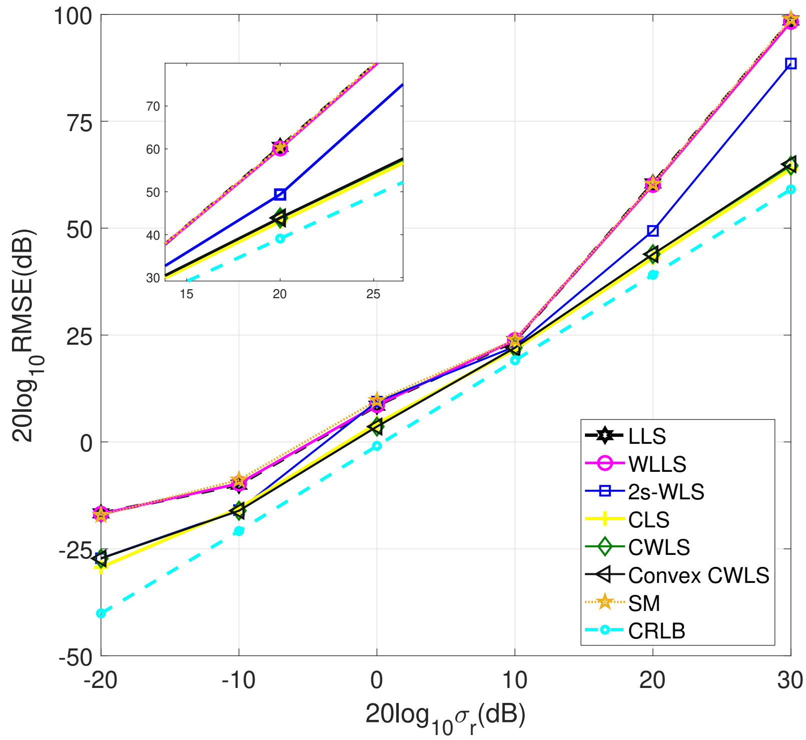


Figure 4.5: Comparison of the localization performance when σ_x is fixed

4.2.1. Scenario 1: Fixed Sensor Position Noise Variance σ_x^2

In this simulation scenario, we aim to compare the performance of various localization methods at various RDOA noise levels while the sensor position noise is fixed to a certain level. We have set the simulation conditions as follows:

1. Number of sensors $N = 5$
2. σ_r increases in 10 dB steps from −30 dB to 30 dB while σ_x is fixed to −40 dB.
3. Number of Monte Carlo runs for each dB level : 1500

Figure 4.5 shows the outcome of the simulation. CRLB is also plotted for comparison. Similar to the results obtained in the exact sensor position scenario given in the previous section 4.1, noisy sensor position case also maintains the order of the closed-form estimators in terms of RMSE performance except for the mild discrepancies seen in the former exact scenario. The CLS method seems to perform a little better than CWLS method beyond 20 dB which can be attributed to the fact that $\mathbf{B}$ matrix which is used to construct $\mathbf{W}$ becomes increasingly noisy as the noise level becomes too high. Besides, convex CWLS method clearly performs better than all of the given methods in low noise conditions which is because this method is formulated to reach the global minimum while the same cannot be guaranteed for CLS and CWLS method.

4.2.2. Scenario 2: Fixed RDOA Noise Variance σ_r^2

In this simulation scenario, we aim to compare the performance of various localization methods at various sensor position noise levels while the RDOA noise is fixed to a certain level. We have set the simulation conditions as follows:

1. Number of sensors $N = 5$
2. σ_x increases in 10 dB steps from −30 dB to 30 dB while σ_r is fixed to −40 dB.
3. Number of Monte Carlo runs for each dB level : 1500

Figure 4.6 shows the outcome of the simulation. CRLB is also plotted for comparison. It can be seen that the localization performance is robust until 0 dB beyond which the performance tends to degrade. Nevertheless, methods taking relationship exhibited by d_1 into account performs consistently better than that of those ignoring it.

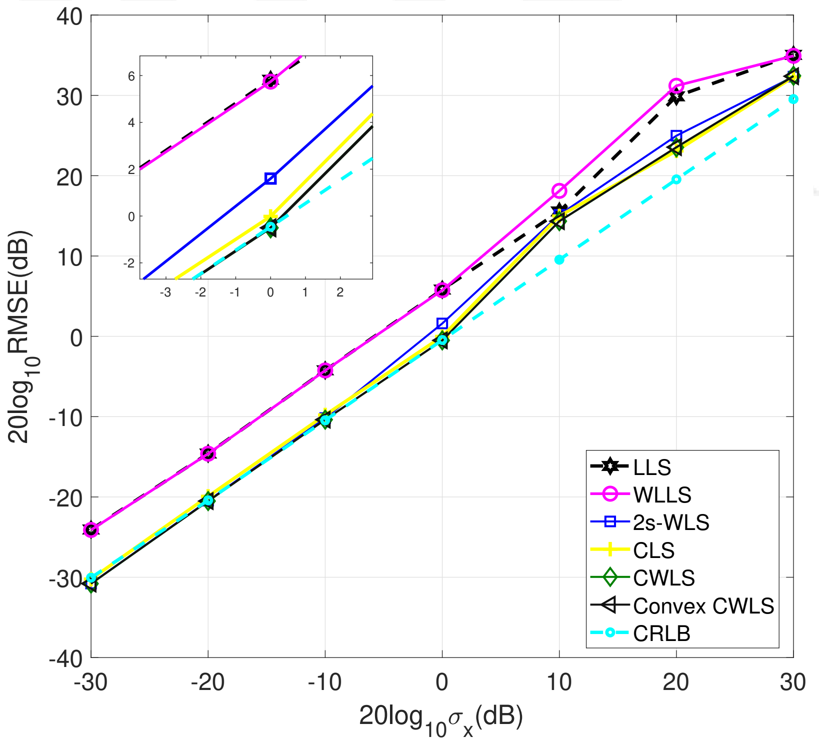


Figure 4.6: Comparison of the localization performance when σ_r is fixed.

4.2.3. Scenario 3: Both σ_x^2 and σ_r^2 vary simulataneously

In this simulation scenario, we aim to compare the localization performance of various localization methods for various values of p which is related to σ_x^2 and σ_r^2 as follows:

$$\sigma_x = p \times 10\sqrt{3}$$
$$\sigma_r = p \times 0.1$$

The same details given in Section 4.2 were used for the simulation process. The number of sensors were taken to be 5 and for each value of p, 1000 Monte Carlo trials were run. Figure 4.7 shows the outcome of the simulation. It can be clearly seen in Figure 4.7 that convex CWLS performed consistently better than other methods. It is because convex CWLS method is designed to perform better in low noise conditions. Moreover CWLS method showed an improved performance compared to CLS and 2sWLS method which is also expected in low noise conditions where the effect of weighting matrix is influential.

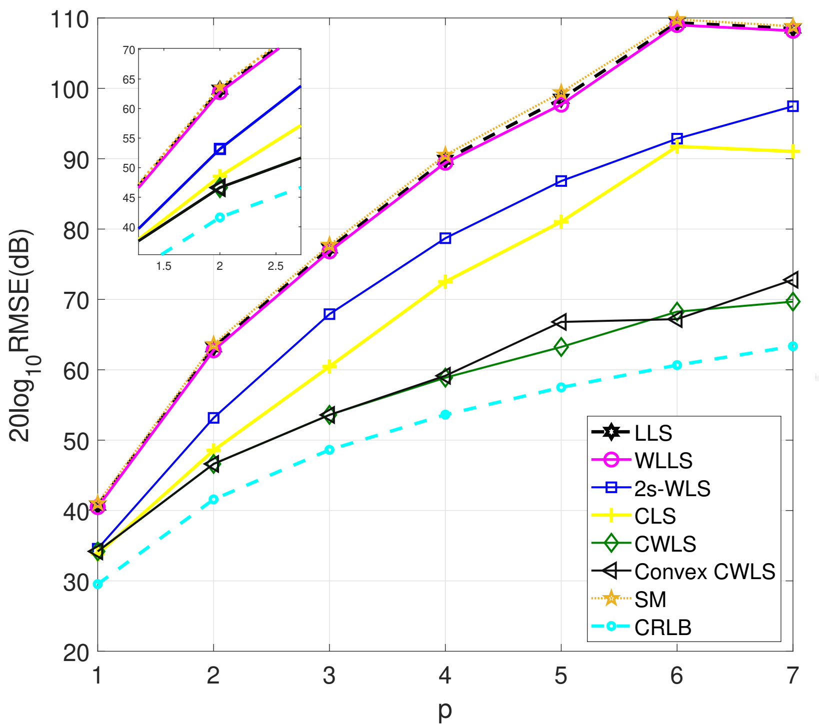
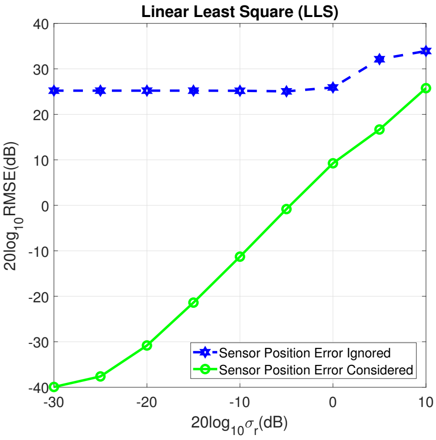


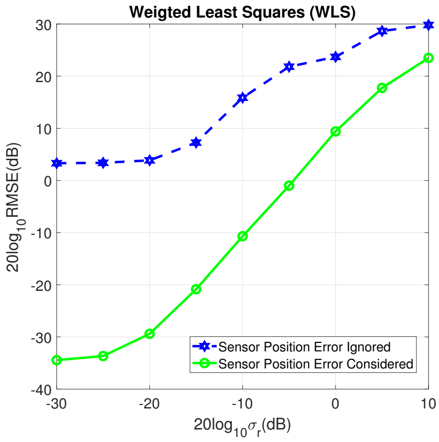
Figure 4.7: RMSE vs p. Comparison of localization performance as both σ_r and σ_r varies as function of p.

4.3. Effect of Ignoring Sensor Position Errors on Localization Performance

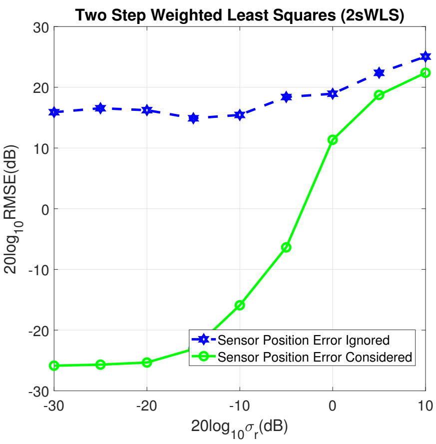
In this section, we present the simulation results on the effect of ignoring sensor position errors on the localization accuracy. [26] has discussed this issue in detail. Here, we only compare the RMSE of the source location estimates obtained from the algorithms assuming perfect knowledge of the source positions (Chapter 2) with that of the source location estimates obtained from the algorithms taking sensor



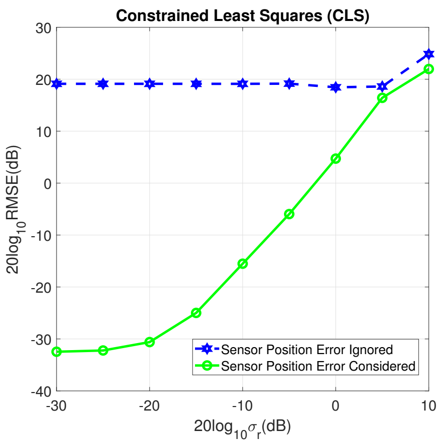
(a) fig 1



(b) fig 2



(c) fig 3



(d) fig 4

(e) fig 5

Figure 4.8: Variation of RMSEs when the sensor error position is ignored versus when they are taken into consideration

position errors into consideration (Chapter 3). In addition to the details given in 4.2, we have set the following simulation conditions:

1. Number of sensors $N = 5$
2. σ_r increases in 5 dB steps from −30 dB to 10 dB while σ_x is fixed to −30 dB.
3. Number of Monte Carlo runs for each dB level : 1500

Figure 4.8 shows the results of the simulation. As can clearly be seen in the simulation results, there is a significant degradation in localization performance when the sensor locations are assumed to be exactly known while they, in reality, are erroneous. For $\sigma_r = 30$ dB, there is a 40 dB increase in RMSE. The difference in RMSE seems to decrease as σ_r increases because RDOA noise becomes relatively dominant so the effect of sensor position noise is diminished.

5. CONCLUSION AND FUTURE WORKS

The accurate and precise localization of a radiating (emitting) radio source has numerous applications in fields such as wireless communications, military, geophysics, surveillance and navigation systems. In this thesis, various source localization methods based on time difference of arrival (TDOA) measurements were investigated. We briefly discussed the non-linear approaches (2.4.1) to source localization and non-linear least squares (NLS) and iterative ML methods were formulated as its instances. Even though the accuracy of these schemes are generally found to be high, they have a very high computational requirements which is not desirable for many applications. So, as a trade-off between estimation accuracy and computational requirements, sub-optimal closed-form estimators (2.4.2) were investigated subsequently which are obtained by converting the set of non-linear TDOA equations to linear equations. These closed-form source location estimators are derived separately assuming (i) the knowledge of the sensor positions to be exactly known and (ii) that the true sensor positions are not available and only the erroneous, noisy versions of them are available to the localization algorithms.

Regardless of the simulations conditions and localization scenarios, we have consistently noticed a pattern in all the simulation results. That the algorithm using weighting matrix $\mathbf{W}$ performed better, in mild noise conditions, than their corresponding counterparts not making use of $\mathbf{W}$. Moreover, the algorithms which made use of the relationship between source location and auxiliary variable (d_1 in exact sensor position case and r_1^0 in noisy sensor position case) performed significantly better than those ignoring it. Even among those which made use of this relationship, the ones explicitly integrating this relationship as a constraint in the optimization problem, for example, in CLS, CWLS and convex CWLS methods, performed significantly better than the ones using the relationship in the second stage to obtain a refined LS solution, for example, in 2sWLS method.

Even though the estimators were inherently sub-optimal due to the conditions in which they were derived, the performance was remarkably close to CRLB for mild noise conditions.

The significant performance improvement in terms of localization accuracy when the sensor position uncertainties are taken into account was also demonstrated through simulation in Figure 4.8.

Although, the TDOA measurement data were generated artificially in accordance with the assumptions made in the formulation of source localization algorithms, and hence reliably worked together with these algorithms to show their good performance, as a future work, actual real-world data could be used to test the reliability of these algorithms. Morever, source localization algorithms resistant to the rank-deficiency or ill-conditioning of **A** matrix (which arises in some source and sensor placement scnearios) could also be investigated.

In this thesis, closed-form localization algorithms were studied for static sensor case (assuming that sensor position errors are present). Nevertheless, the ideas could be extended to the moving sensor case and new algorithms could be explored which has better localization performance and computational efficiency compared to the existing algorithms mentioned in the literature. Following a similar approach to [22], where the author proposed an efficient mechanism for path planning (or trajectory optimization) of the moving sensors (as when they are attached to UAVs) in the case of range-only localization, we can also attempt to include these ideas, while also exploring new ones, to the TDOA case.

Appendix A.

Cramer-Rao Lower Bound for TDOA Based Source Location Estimator

A.1. CRLB For Exact (Noiseless) Sensor Position Scenario

Theorem 3 Assume $\boldsymbol{\theta}$ to be the unknown parameter vector to be estimated, $\mathbf{x}$ be the observation vector and $p(\mathbf{x};\boldsymbol{\theta})$ represent the probability density function (PDF) of $\mathbf{x}$. Let's also assume that the PDF $p(\mathbf{x};\boldsymbol{\theta})$ satisfies the regularity condition A.1 given below as

$$\mathrm{E}\left[\frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right] = 0 \quad for\ all\ \boldsymbol{\theta} \tag{A.1}$$

Then the variance of any unbiased estimator must satisfy the following

$$\mathrm{var}(\hat{\boldsymbol{\theta}}_i) \geq \left[\mathbf{I}^{-1}(\boldsymbol{\theta})\right]_{ii} \tag{A.2}$$

where $\hat{\boldsymbol{\theta}}_i$ is the i^{th} element of the parameter vector $\boldsymbol{\theta}$ and $[\mathbf{I}^{-1}(\boldsymbol{\theta})]_{ii}$ is its variance where $\mathbf{I}^{-1}(\boldsymbol{\theta})$ is the inverse of $\mathbf{I}(\boldsymbol{\theta})$. The quantity $\mathbf{I}(\boldsymbol{\theta})$ is known as the Fisher Information Matrix (FIM) which can be calculated as

$$\mathbf{I}(\boldsymbol{\theta}) = -\mathrm{E}\left[\frac{\partial^2 \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^2}\right] = \mathrm{E}\left[\left(\frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right)^2\right] \tag{A.3}$$

where we take expectations with respect to $p(\mathbf{x};\boldsymbol{\theta})$.

In Equation 2.11 the PDF of the available noisy RDOA vector $\mathbf{r}$ is given as

$$p(\mathbf{r};\mathbf{x}) = \frac{1}{(2\pi)^{(N-1)/2}|\mathbf{Q}|^{1/2}} \exp\left\{\frac{-1}{2}(\mathbf{r}-\mathbf{d})^T\mathbf{Q}^{-1}(\mathbf{r}-\mathbf{d})\right\} \tag{A.4}$$

where $\mathbf{x}$ is the source position to be estimated, $\mathbf{d}$ is the true RDOA vector, $\mathbf{Q}$ is the covariance matrix of the noise vector $\mathbf{n}$.

Taking the natural logarithm of the above PDF (Equation A.4) gives

$$\ln p(\mathbf{r};\mathbf{x}) = \frac{1}{(2\pi)^{(N-1)/2}|\mathbf{Q}|^{1/2}} - \frac{1}{2}(\mathbf{r}-\mathbf{d})^T\mathbf{Q}^{-1}(\mathbf{r}-\mathbf{d}) \tag{A.5}$$

Now, taking the partial derivative of Equation A.5 with respect to $\mathbf{x}$ and then taking the expectation of the resulting expression gives

$$\mathrm{E}\left[\frac{\partial \ln p(\mathbf{r};\mathbf{x})}{\partial \mathbf{x}}\right] = \mathrm{E}\left[\frac{\partial \mathbf{d}^T}{\partial \mathbf{x}}\mathbf{Q}^{-1}(\mathbf{r}-\mathbf{d})\right] = \mathbf{0}_{2\times 1} \tag{A.6}$$

The above Equation A.6 shows that the PDF given in Equation A.4 satisfies the regularity condition given in Equation A.1. Consequently, as illustrated in Theorem 3, a lower bound of the variance of unbiased estimator for the TDOA based source localization problem can be obtained. Firstly, we calculate $\mathbf{I}(\mathbf{x})$ given in Equation A.3 as

$$\begin{aligned}
\mathbf{I}(\mathbf{x}) &= \mathrm{E}\left[\left(\frac{\partial \ln p(\mathbf{r};\mathbf{x})}{\partial \mathbf{x}}\right)^2\right] \\
&= \frac{\partial \mathbf{d}^T}{\partial \mathbf{x}}\mathbf{Q}^{-1}\mathrm{E}\left[(\mathbf{r}-\mathbf{d})(\mathbf{r}-\mathbf{d})^T\right]\mathbf{Q}^{-1}\frac{\partial \mathbf{d}}{\partial \mathbf{x}}
\end{aligned} \tag{A.7}$$

We know that

$$\mathrm{E}\left[(\mathbf{r}-\mathbf{d})(\mathbf{r}-\mathbf{d})^T\right] = \mathrm{E}[\mathbf{n}\mathbf{n}^T] = \mathbf{Q}$$

So Equation A.7 gets reduced to

$$\mathbf{I}(\mathbf{x}) = \left.\left(\frac{\partial \mathbf{d}^T}{\partial \mathbf{x}}\mathbf{Q}^{-1}\frac{\partial \mathbf{d}}{\partial \mathbf{x}}\right)\right|_{\mathbf{x}=\mathbf{x}_{true}} \tag{A.8}$$

where $\mathbf{x}_{true}$ is the true source location. The result (Equation A.8) is also given in [1] which is the expression for FIM when the noise is zero-mean Gaussian distributed.

To calculate $\partial\mathbf{d}/\partial\mathbf{x}$ we firstly calculate individual $\partial d_{i,1}/\partial\mathbf{x}$ as follows:

$$\begin{aligned}
\frac{\partial d_{i,1}}{\partial \mathbf{x}} &= \begin{bmatrix} \frac{\partial\left(\sqrt{(x-x_i)^2+(y-y_i)^2}-\sqrt{(x-x_1)^2+(y-y_1)^2}\right)}{\partial x} \\ \frac{\partial\left(\sqrt{(x-x_i)^2+(y-y_i)^2}-\sqrt{(x-x_1)^2+(y-y_1)^2}\right)}{\partial y} \end{bmatrix}^T \\
&= \begin{bmatrix} \left(\frac{x-x_i}{d_i}-\frac{x-x_1}{d_1}\right) \\ \left(\frac{y-y_i}{d_i}-\frac{y-y_1}{d_1}\right) \end{bmatrix}^T
\end{aligned} \tag{A.9}$$

Now stacking $\partial d_{i,1}/\partial\mathbf{x}$'s vertically, $i = 2, 3, ..., N$ we get $\partial\mathbf{d}/\partial\mathbf{x}$ as

$$\frac{\partial \mathbf{d}}{\partial \mathbf{x}} = \begin{bmatrix} \left(\frac{x-x_2}{d_2}-\frac{x-x_1}{d_1}\right) & \left(\frac{y-y_2}{d_2}-\frac{y-y_1}{d_1}\right) \\ \left(\frac{x-x_3}{d_3}-\frac{x-x_1}{d_1}\right) & \left(\frac{y-y_3}{d_3}-\frac{y-y_1}{d_1}\right) \\ \vdots & \vdots \\ \left(\frac{x-x_N}{d_N}-\frac{x-x_1}{d_1}\right) & \left(\frac{y-y_N}{d_N}-\frac{y-y_1}{d_1}\right) \end{bmatrix}_{(N-1)\times 2} \tag{A.10}$$

A.2. CRLB For Noisy Sensor Position Scenario

In the previous section A.1, CRLB for TDOA based source location estimator for the exact (noiseless) sensor position scenario was derived. In this section, we derive the CRLB for the scenario in which we do not have information about the true sensor positions rather we only have noisy sensor position information available to us. Similary as in A.3, we can write the FIM $\mathbf{I}(\boldsymbol{\theta})$ as follows:

$$\mathbf{I}(\boldsymbol{\theta}) = -\mathrm{E}\left[\frac{\partial^2 \ln p(\mathbf{v};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^2}\right] = \mathrm{E}\left[\left(\frac{\partial \ln p(\mathbf{v};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right)^T \left(\frac{\partial \ln p(\mathbf{v};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right)\right] \tag{A.11}$$

where $\mathbf{v} = [\mathbf{r}^T, \mathbf{s}^T]^T$ is the measurement vector that contains the noisy TDOA values $\mathbf{r}$ given in 3.11 and noisy sensor positions $\mathbf{s}$ given in 3.2. $p(\mathbf{v};\boldsymbol{\theta})$ is probability density function (pdf) of $\mathbf{v}$ parameterized by the parameter vector $\boldsymbol{\theta}$. Since the exact sensor positions are unknown, sensor positions are also included in the parameter vector so that $\boldsymbol{\theta}$ becomes $\boldsymbol{\theta} = [\mathbf{x}^T, \mathbf{s}^{o^T}]^T$. We assume that the noisy $\mathbf{r}$ and $\mathbf{s}$ was corrupted by mutually independent zero-mean Gaussain noises $\Delta\mathbf{r}$ and $\Delta\mathbf{x}$ with covariance matrices $\mathbf{Q}_r$ and $\mathbf{Q}_x$ so that the mean is $\mathbf{r^0}$ and $\mathbf{s^0}$ respectively. Therefore, we can write $p(\mathbf{v}) = p(\mathbf{r}).p(\mathbf{s})$. The logarithm of the pdf of the available data vector $\mathbf{v} = [\mathbf{r}^T, \mathbf{s}^T]^T$ conditioned on the unknowns $\boldsymbol{\theta} = [\mathbf{x}^T, \mathbf{s}^{o^T}]^T$ is

$$\begin{aligned}
\ln p(\mathbf{v};\boldsymbol{\theta}) &= \ln p(\mathbf{r};\mathbf{x},\mathbf{s}^0) + \ln p(\mathbf{s};\mathbf{x},\mathbf{s}^0) \\
&= C - \frac{1}{2}(\mathbf{r}-\mathbf{r}^0)^T \mathbf{Q}_r^{-1}(\mathbf{r}-\mathbf{r}^0) - \frac{1}{2}(\mathbf{s}-\mathbf{s}^0)^T \mathbf{Q}_x^{-1}(\mathbf{s}-\mathbf{s}^0)
\end{aligned} \tag{A.12}$$

where C is a constant that does not depend on the unknowns $\boldsymbol{\theta} = [\mathbf{x}^T, \mathbf{s}^{o^T}]^T$. Applying partial derivatives with respect to the unknowns $\boldsymbol{\theta} = [\mathbf{x}^T, \mathbf{s}^{o^T}]^T$ twice and then taking expectation yields

$$\mathbf{I}(\boldsymbol{\theta}) = \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Y}^T & \mathbf{Z} \end{bmatrix} \tag{A.13}$$

where

$$\begin{aligned}
\mathbf{X} &= -\mathrm{E}\left(\frac{\partial^2 \ln p(\mathbf{v};\boldsymbol{\theta})}{\partial \mathbf{x} \partial \mathbf{x}^T}\right) = \left(\frac{\partial \mathbf{r}^0}{\partial \mathbf{x}}\right)^T \mathbf{Q}_r^{-1} \left(\frac{\partial \mathbf{r}^0}{\partial \mathbf{x}}\right) \\
\mathbf{Y} &= -\mathrm{E}\left(\frac{\partial^2 \ln p(\mathbf{v};\boldsymbol{\theta})}{\partial \mathbf{x} \partial \mathbf{s}^{0^T}}\right) = \left(\frac{\partial \mathbf{r}^0}{\partial \mathbf{x}}\right)^T \mathbf{Q}_r^{-1} \left(\frac{\partial \mathbf{r}^0}{\partial \mathbf{s}^0}\right) \\
\mathbf{Z} &= -\mathrm{E}\left(\frac{\partial^2 \ln p(\mathbf{v};\boldsymbol{\theta})}{\partial \mathbf{s}^0 \partial \mathbf{s}^{0^T}}\right) = \left(\frac{\partial \mathbf{r}^0}{\partial \mathbf{s}^0}\right)^T \mathbf{Q}_r^{-1} \left(\frac{\partial \mathbf{r}^0}{\partial \mathbf{s}^0}\right) + \left(\frac{\partial \mathbf{s}^0}{\partial \mathbf{s}^0}\right)^T \mathbf{Q}_x^{-1} \left(\frac{\partial \mathbf{s}^0}{\partial \mathbf{s}^0}\right) \\
&= \left(\frac{\partial \mathbf{r}^0}{\partial \mathbf{s}^0}\right)^T \mathbf{Q}_r^{-1} \left(\frac{\partial \mathbf{r}^0}{\partial \mathbf{s}^0}\right) + \mathbf{Q}_x^{-1}
\end{aligned}$$

The partial derivatives $\partial \mathbf{r}^0/\partial \mathbf{x}$ and $\partial \mathbf{r}^0/\partial \mathbf{s}^0$ are given as

$$\frac{\partial \mathbf{r}^0}{\partial \mathbf{x}} = \begin{bmatrix} \mathbf{g}^T_{\mathbf{x},\mathbf{x}^0_2} - \mathbf{g}^T_{\mathbf{x},\mathbf{x}^0_1} \\ \vdots \\ \mathbf{g}^T_{\mathbf{x},\mathbf{x}^0_N} - \mathbf{g}^T_{\mathbf{x},\mathbf{x}^0_1} \end{bmatrix} \tag{A.14}$$

$$\frac{\partial \mathbf{r}^0}{\partial \mathbf{s}^0} = \begin{pmatrix} \mathbf{g}^T_{\mathbf{x},\mathbf{x}^0_1} & -\mathbf{g}^T_{\mathbf{x},\mathbf{x}^0_2} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{g}^T_{\mathbf{x},\mathbf{x}^0_1} & \mathbf{0} & -\mathbf{g}^T_{\mathbf{x},\mathbf{x}^0_3} & \vdots & \mathbf{0} \\ \vdots & \mathbf{0} & \vdots & \ddots & \mathbf{0} \\ \mathbf{g}^T_{\mathbf{x},\mathbf{x}^0_1} & \mathbf{0} & \cdots & \mathbf{0} & -\mathbf{g}^T_{\mathbf{x},\mathbf{x}^0_N} \end{pmatrix} \tag{A.15}$$

where

$$\mathbf{g}_{\mathbf{x},\mathbf{x}^0_i} = \frac{\mathbf{x} - \mathbf{x}^0_i}{\|\mathbf{x} - \mathbf{x}^0_i\|} \tag{A.16}$$

Taking inverse of Equation A.13 gives the CRLB. The minimum variance of the source location estimates is given by the sum of the first two diagonal elements in $\mathbf{I}(\boldsymbol{\theta})^{-1}$.

Using the partitioned matrix inversion formula given in [38], the CRLB of the unknown source location $\mathbf{x}$ is given as

$$\begin{aligned} CRLB(\mathbf{x}) &= (\mathbf{X} - \mathbf{Y}\mathbf{Z}^{-1}\mathbf{Y}^T)^{-1} \\ &= \mathbf{X}^{-1} + \mathbf{X}^{-1}\mathbf{Y}(\mathbf{Z} - \mathbf{Y}^T\mathbf{X}^{-1}\mathbf{Y})^{-1}\mathbf{Y}^T\mathbf{X}^{-1} \end{aligned} \tag{A.17}$$

In the previous section, for non-noisy sensor position scenario Equation A.8 gives the FIM which is equal to $\mathbf{X}$. Thus $\mathbf{X}^{-1}$ gives the CRLB in this case. Hence, the second term in Equation A.17 represents the increase in CRLB when sensor position noise $\Delta\mathbf{s}$ is present.

Appendix B.

Proofs

B.1. Proof of Lemma 1

Lemma 1 facilitates an efficient mechanism to solve the CWLS source localization problem (Equation 3.65), with the condition that the simultaneous diagonalization assumption is satisfied. Lemma 2 given below provides a sufficient condition for simultaneous diagonalization [39].

Lemma 2 (Horn and Johnson [39]): Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$ be two symmetric matrices and suppose that there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \mathbf{A} + \beta \mathbf{B} \succ \mathbf{0},$$

then there exists a nonsingular matrix $\mathbf{C} \in \mathbb{R}^{n \times n}$ such that both $\mathbf{C}^T \mathbf{A} \mathbf{C}$ and $\mathbf{C}^T \mathbf{B} \mathbf{C}$ are diagonal.

Proof of Lemma 1. The sufficient condition in Lemma 2 is satisfied when either $\mathbf{A}$ or $\mathbf{B}$ is positive definite. Since $\mathbf{\Sigma}$ in Equation 3.65 is indefinite, it remains to be established that $\mathbf{A}_1 \in \mathbb{R}^{3 \times 3}$ is positive definite. The weighting matrix $\mathbf{W}$ is positive definite with rank $N-1$, so $\mathbf{A_1}$ is positive semi-definite with $rank(\mathbf{A}_1) \leq N-1$. The rank of $\mathbf{A}$ depends on the available positions of the sensors $\mathbf{x}_i$. Moreover, the last column and the first two columns of $\mathbf{A}$ has nonlinear relationship, so in general, the columns of $\mathbf{A}$ can be linearly independent (if the sensors are not aligned linearly) and hence it can have a full column rank of 3. Then, the rank of $\mathbf{A_1}$ can be 3 if and only if $N \geq 4$, which is satisfied since at least 4 sensors are needed for WLS/CWLS source localization in 2-D cases. Therefore, $\mathbf{A_1}$ is actually positive definite. Thus, the matrices $\mathbf{A_1}$ and $\mathbf{\Sigma}$ can be simultaneously diagonalized [23].

B.2. Proof of Theorem 2

The simultaneous diagonalization of $\mathbf{A}_1$ and $\mathbf{\Sigma}$ in Lemma 1 (Equations 3.69 and 3.70) results in the following: [6]

$$\mathbf{A}_1 = (\mathbf{S}^T)^{-1} \mathbf{M} \mathbf{S}^{-1} \tag{B.1}$$

$$\mathbf{\Sigma} = (\mathbf{S}^T)^{-1} \mathbf{N} \mathbf{S}^{-1} \tag{B.2}$$

Substituting Equations B.1 and B.2 into the non-convex problem (Equation 3.65), we have

$$\begin{aligned} \min_{\boldsymbol{\theta}_1} \;\; & \boldsymbol{\theta}_1^T(\mathbf{S}^T)^{-1}\mathbf{M}\mathbf{S}^{-1}\boldsymbol{\theta}_1 - 2\mathbf{b}_1^T\mathbf{S}\mathbf{S}^{-1}\boldsymbol{\theta}_1 \\ s.t. \;\; & \boldsymbol{\theta}_1(\mathbf{S}^T)^{-1}\mathbf{N}\mathbf{S}^{-1}\boldsymbol{\theta}_1 = 0 \end{aligned} \tag{B.3}$$

Since $\mathbf{S}^{-1}\boldsymbol{\theta}_1 \in R^3$ we can denote $\mathbf{S}^{-1}\boldsymbol{\theta}_1 = \mathbf{w} := [w_1, w_2, w_3]$. Hence, problem represented by Equation 3.65 is equivalent to the following

$$\begin{aligned} \min \;\; & \sum_{j=1}^{3} m_j w_j^2 - 2a_j w_j \\ s.t. \;\; & \sum_{j=1}^{3} n_j w_j^2 = 0 \end{aligned} \tag{B.4}$$

It can be clearly seen that the constraint of B.4 is still non-convex. However, if we set $v_j = w_j^2$, it can be transformed into a linear constraint. Moreover, in order to make the objective of minimization problem (Equation B.4) as small as possible, $a_j w_j$ should be non-negative which is guaranteed when w_j is chosen to have the same sign as a_j. Therefore, problem B.4 is equivalent to problem 3.71.

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