VARIATION SEMINORM AND CONVERGENCE IN VARIATION SEMINORM FOR OPERATORS

by

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THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
THE ABANT İZZET BAYSAL UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
IN
THE DEPARTMENT OF MATHEMATICS

NOVEMBER 2013

Approval of the Graduate School of Natural and Applied Sciences

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ABSTRACT

VARIATION SEMINORM AND CONVERGENCE IN VARIATION SEMINORM FOR OPERATORS

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November 2013, 68 pages

This thesis is a survey on the property of variation seminorm and on some approximation properties of operators not only in normed spaces but also in variation seminorm. It is indicated in this thesis that the variation detracting property satisfies the approximation.One of the fundamental problems of analysis is to approximate a given function f in some sense or other by functions having certain properties, and generally, by functions which have better properties than f.

This thesis consists of five chapters. In the first chapter, we have an introduction and give some necessary definitions and concepts which is used in this thesis.In the second chapter, we give the fundamental definitions and theorems providing the base of approximation theory including Bernstein operators and Szász-Mirakjan operators. We introduce Bernstein operators and Szász-Mirakjan operators and we give their convergence. We show the rate of convergence of these operators. In the third chapter, we give the definition of finite variation, some important theorems about it. We introduce the Stieltjes integral and BV[a,b]-space and its norm property and TV[a,b]-space and its seminorm property. In chapter four, we investigate the variation detracting property

of Bernstein Polynomials and the convergence of the linear positive operator, Bernstein polynomials, in variation seminorm and the rate of its convergence in variation seminorm. In chapter five, we investigate the variation detracting property of Szász-Mirakjan operator.We show the convergence of Szász-Mirakjan operators in variation seminorm and its rate of convergence in variation seminorm.

Keywords: Variation seminorm; Finite variation; Stieltjes integral; TV[a,b]-space; BV[a,b]-space; Variation detracting property; Bernstein polynomials; Szász-Mirakjan operators.

ÖZET

VARYASYON YARI NORMU VE OPERATÖRLERİN VARYASYON YARI NORMUNDA YAKINSAKLIKLARI

ONAR, Gizem

Yüksek lisans, Matematik Bölümü

Tez Yöneticisi: Doç. Dr. Harun Karslı

Kasım 2013, 68 sayfa

Bu tez, varyasyon yarı normu özelliğini ve yakınsaklık teorisindeki operatörlerin sadece normlu uzaylarda değil,varyasyon yarı normunda da yakınsaklık özellikleri üzerine yapılan bir çalışmadır. Tezde, yaklaşım sağlayan özelliğin varyasyon korumalı özelliği olduğu belirtilmiştir. Analizin temel problemlerinden biri f gibi bazı kötü özelliklere sahip bir fonksiyona, daha iyi özellikleri olan başka bir fonksiyonla yaklaşmaktır.

Bu tez beş bölüm içerir. Birinci ünitede giriş yapılmış ve tezde kullanılan bazı önemli tanımlar ve yapılar verilmiştir. İkinci ünitede tezde açıklanan Bernstein polinomlarının ve Szász-Mirakjan operatörlerinin ortaya çıkışını sağlayan yaklaşım teorisinin temelini oluşturan tanımlar ve teoremler verilmiştir. Bernstein polinomları ve Szász-Mirakjan operatörleri tanıtılmış ve yakınsaklıkları incelenmiştir. Ayrıca bu operatörlerin yakınsaklık hızları gösterilmiştir. Üçüncü ünitede sonlu varyasyon tanımı ve bununla ilgili bazı önemli teoremler verilmiştir. Stieltjes integrali, BV-[a,b] uzayı tanıtılmış ve norm özellikleri incelenmiştir, TV-[a,b] uzayı tanıtılmış ve yarınorm özellikleri incelenmiştir. Dördüncü ünitede Bernstein operatörünün varyasyon korumalı özelliği,

varyasyon yarı normunda lineer pozitif operatör Bernstein operatörünün yakınsaklığı ve yakınsaklık hızı incelenmiştir. Beşinci ünitede Szász-Mirakjan operatörlerinin varyasyon korumalı özelliği incelenmiştir. Szász-Mirakjan operatörlerinin varyasyon yarı normunda yakınsaklığı ve yakınsaklık hızı incelenmiştir.

Anahtar Kelimeler: Varyasyon yarı normu; Sonlu varyasyon; Stieltjes integrali; TV[a,b]-uzayı; BV[a,b]-uzayı; Varyasyon Korumalı özelliği; Bernstein polinomları; Szász-Mirakjan operatörleri.

To my family

ACKNOWLEDGEMENTS

I would like to express sincere appreciation to my supervisor, Assoc. Prof. Dr. Harun Karslı for his motivation, helpful discussions, encouragement, patience and constant guidance during this work. I would also like to thank other examining committee members for their suggestions.

I am grateful to express my appreciation to my family, relatives and friends for their continuous helps.

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NOTATIONS

$f_n(x) \rightrightarrows f(x)$	*Uniform convergence of* $\{f_n\}$ *to* f
$\mathbb{N}$	$\mathbb{N} = \{1, 2, ...\}$ *the set of natural numbers*
$AC[a, b]$	*The set of absolutely continuous functions on the closed interval* $[a, b]$
$TV[a, b]$	*The set of functions of total variation on the closed interval* $[a, b]$
$BV[a, b]$	*The set of functions of bounded variation on the closed interval* $[a, b]$
$C_{\rho,K_f}[0, \infty)$	*The weighted space for the function* f
$\omega(f; \delta)$	*The modulus of continuity of* f
$V_{[a,b]}[f]$	*Variation of the function* f
$(B_n f)(x)$	*Bernstein polynomials*
$(S_n f)(x)$	*Szasz − Mirakjan operators*
$\|.\|_{BV}$	*The norm defined by* $\|.\|_{BV} = V_{[a,b]}[f] + \|f(c)\|, a \leq c \leq b,$ *on the* $BV[a, b]$ *space*
$\|.\|_{TV}$	*The norm defined by* $\|.\|_{TV} = V_{[a,b]}[f]$ *on the* $TV[a, b]$ *space*
$O - o$	*Landau symbols*

CHAPTER 1

INTRODUCTION AND PRELIMINARIES

In this chapter, we give some necessary definitions and concepts, which is used in this thesis.

1.1 Introduction

In the approximation theory, we concern with the ability to approximate functions by simpler and more easily calculated functions. In the theory of Fourier series, we find some of the first results of approximation theory.

The first significant density result were those of Weierstrass, who proved in 1885 the density of algebraic polynomials in the class of continuous real-valued functions on the compact interval and the density of trigonometric polynomials in the class of $2\pi - periodic\ continuous\ real - valued\ functions$. In 1905, Borel proposed determination of an interpolation process, that allows finding polynomials $P(x)$, which converge uniformly to the function $f \in C[a,b]$.

In 1912, Bernstein gave an another proof of the Weierstrass approximation theorem on $C[0,1]$ instead of $C[a,b]$ by using the probability theory. For these, Bernstein introduced a polynomial, that is;

Let f be a function defined on the interval $[0,1]$ and let $\mathbb{N} = \{1,2,3,...\}$. The classical Bernstein operators $B_n f$ applied to f are defined as;

$$(B_n f)(x) = \sum_{k=0}^{n} f\Big(\frac{k}{n}\Big) p_{k,n}(x) \ , \quad 0 \le x \le 1 \ , \quad n \in \mathbb{N},$$

where $p_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ is the Bernstein basis.

At first, Lorentz showed the operators B_n have the property

$$V_{[0,1]}[B_n f] \leq V_{[0,1]}[f] \qquad (n \in \mathbb{N}).$$

and it is called "Variation Detracting property".

Then,the study of Bardaro, Butzer, Stens and Vinti made the variation detracting property more important. They pointed out that in order to obtain a convergence result in the variation seminorm, it is necessary and important to state the variation detracting property.

For the convergence in variation of $B_n f$ to f, if $f \in TV[0,1]$, then it is easy to see that

$$lim_{n\to\infty} V_I[B_n f - f] = 0$$

does not hold.

However, since $B_n f$ is absolutely continuous for all $n \in \mathbb{N}$ and the set $AC[0,1]$, consisting of all absolutely continuous functions on $[0,1]$, is a closed subspace as $TV[0,1]$ with respect to the convergence induced by the seminorm $\|f\|_{TV(I)}$, then it implies that $f \in AC[0,1]$. In addition, it is well-known that convergence in variation of a sequence $(f_n) \subset AC(I)$ towards f exactly means the convergence of the derivatives f_n' to f' in the $L_1(I) - norm$. That is;

If $lim_{w\to\infty} V_I[T_w f - f] = 0$ holds for a sequence $(f_w)_{w\in J}$ in $AC(I)$, then also $f \in AC(I)$ and $V_I[f_w - f] = \int_I |f_w'(u) - f'(u)| du$.

1.2 Some Basic Concepts

Definition 1.1. [6] Let M be a subset of a metric space X. Then a point x_0 of X (which may or may not be a point of M) is called an accumulation point of M (or limit point of M) if every neighborhood of x_0 contains at least one point $y \in M$ distinct from x_0.

Example 1.1. $X = \mathbb{R}$, $d(x,y) = |x - y|$ and $M = \left\{1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...\right\}$. $0 \notin M$, let $x_0 = 0$. If we take any neighborhood of 0,then we have at least one element in that neighborhood.

$x_0 \notin M$, but x_0 is an accumulation point of M.

Definition 1.2. [6] The set consisting of all points of M and the accumulation points of M is called the closure of M and is denoted by $\overline{M}$.

Example 1.2. Let $M = \left\{1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...\right\}$, then $\overline{M} = \left\{1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...\right\} \bigcup \{0\}$ since the accumulation point of M is 0.

Note that $\overline{M}$ is the smallest closed set containing M.

Definition 1.3. [6] A subset M of a metric space X is said to be dense in X if $\overline{M} = X$.

Definition 1.4. *i*) A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* at a point x_0 if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|x - x_0| < \delta \ \ implies \ \ |f(x) - f(x_0)| < \epsilon.$$

The function f is a *continuous function* if it is continuous at every point in its domain.

ii) Every open image in the range has an open set in the domain, then the function is continuous.

iii) Let f be a function. If $x_n \to x \Rightarrow f(x_n) \to f(x)$, then f is continuous.

Definition 1.5. A function f defined on an interval I is said to be uniformly continuous on I if to each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x_2) - f(x_1)| < \varepsilon$ for every points x_1, x_2 of I for which $|x_1 - x_2| < \delta$.

Example 1.3. Let $X = \mathbb{R}$ and $f(x) = 3x + 7$. Then f is uniformly continuous on X.

Proof. Choose $\varepsilon > 0$. Let $\delta = \frac{\varepsilon}{3}$. Choose $x_0 \in \mathbb{R}$ and $x \in \mathbb{R}$. Let $|x - x_0| < \delta$. Then $|f(x) - f(x_0)| = |(3x + 7) - (3x_0 + 7)| = 3|x - x_0| < 3\delta = \varepsilon$ □

Note that a function which is continuous and bounded on a closed interval is also uniformly continuous at that interval. Indeed, Heine-Borel theorem proves it.

Definition 1.6. [2] Let $I \subset \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is said to be absolutely continuous on I if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| \leq \varepsilon$$

for every finite number of overlapping intervals $(a_k, b_k), k = 1, 2, 3, ..., n$ with $[a_k, b_k] \subset I$ and

$$\sum_{k=1}^{n} (b_k - a_k) \leq \delta.$$

The space of all absolutely continuous functions $f : I \to \mathbb{R}$ is denoted by AC(I).

$$AC(I) = \{f : I \to \mathbb{R} \mid f \in AC(I)\}$$

Note that an absolutely continuous function is uniformly continuous.

Example 1.4. The function $f(x) = \sqrt{x}$ is absolutely continuous on $[0, 1]$.

Definition 1.7. A sequence of functions f_n is said to converge pointwisely on an interval $[a, b]$ to a function f if for each $\varepsilon > 0$ and for each x in $[a, b]$, there corresponds an integer N(depending on x and ε both) such that

$$|f_n(x) - f(x)| < \varepsilon \;\; for \;\; all \;\; n \geq N.$$

Example 1.5. Let $X = [0, 1]$. For every $n \in N$ and every $x \in [0, 1]$, let $f_n = x^n$. Then for every $x \in [0, 1]$, $f_n(x) \to f(x)$ as $n \to \infty$, where

$$f(x) = \begin{cases} 0, & \text{if} \quad 0 \leq x < 1, \\ 1, & \text{if} \quad x = 1. \end{cases}$$

Definition 1.8. A sequence of functions f_n is said to converge uniformly on an interval [a,b] to a function f if for any $\varepsilon > 0$ and for all $x \in [a, b]$, there exists an integer N (independent of x but dependent on ε) such that for all $x \in [a, b]$,

$$|f_n(x) - f(x)| < \varepsilon \;\; for \;\; all \;\; n \geq N.$$

Example 1.6. The sequence $\{f_n\} = \left\{\frac{nx^2+1}{nx+1}\right\}$ converges uniformly on the interval $[1, 2]$.

Note that if $f_n \rightrightarrows f$ on X, then f is continuous on X.

CHAPTER 2

FUNDAMENTAL THEOREMS

In this chapter, we give the fundamental definitions and theorems providing the base of approximation theory including Bernstein polynomials and Szász-Mirakjan operators. We introduce the Bernstein polynomials and Szász-Mirakjan operators and we give their convergence and their convergence rate.

2.1 Chebyshev System

$C[a,b]$ denotes the space of all continuous real-valued functions and $C^* = C^*[-\pi,\pi]$ is the space of $2\pi - periodic$ and continuous real-valued functions.
Both of these spaces are normed spaces with the norm

$$f \in C[a,b], \quad then \quad \|f\| = sup_{x\in[a,b]}|f(x)|.$$

[10] A set of functions $\{u_0, u_1, u_2, ..., u_n\} \subseteq C[a,b]$ is called a Chebyshev system on $[a,b]$ if each function $u \in span\{u_0, u_1, ..., u_n\}$ has at least n zero unless it is identically zero on $[a,b]$.

If $\{u_0, u_1, ..., u_n\} = \{x^0, x^1, ..., x^n\}$, then we get that every polynomial is in Chebyshev system.
For the periodic case, $\{u_0, u_1, u_2, ..., u_n\} \subseteq C^*[a,b]$ and any $u \in span\{u_0, u_1, ..., u_n\}$ has at most n-zeros unless it is identically zero.
$U_n = span\{u_0, u_1, ..., u_n\}$ is called a Chebyshev space. A chebyshev space can be characterized as follows;
If $a \leq x_0 < x_1 < x_2 < ... < x_n \leq b$, and y_i^n are real numbers, then there is a unique $u \in span\{u_0, u_1, ..., u_n\}$ such that

$$u(x_i) = y_i, \qquad i = 0, 1, 2, ..., n.$$

Let $f \in C[a, b]$ $(or \ \ C^*)$ and U_n be a Chebyshev subspace of $C[a, b]$ $(or \ \ C^*)$. A function $u^* \in U_n$ is called a best approximation to f from U_n if

$$\|f - u^*\| = inf_{u \in U_n} \|f - u\|.$$

The following theorem of Chebyshev gives the existence and uniqueness of u^*.

Theorem 2.1. *Let $f \in C[a, b]$ and $\{u_0, u_1, ..., u_n\}$ be a Chebyshev system on $[a, b]$. Then the best approximation u^* to f exists and is unique.*

If u is any function in $sp\{u_0, u_1, ..., u_n\}$, then u is the best approximation to f $\Leftrightarrow$ there exists points $a \leq x_0 < x_1 < x_2 < ... < x_n \leq b$ such that $f - u$ takes the values $\|f - u\|$ at the points $x_i, i = 0, 1, 2, ..., n$.

The following theorem is due to Weierstrass.

Theorem 2.2. *(Weierstrass, 1885) [11] Let f be a continuous function defined on $[a, b]$. To each $\varepsilon > 0$ there corresponds a polynomial P such that $\|f(x) - P(x)\| < \varepsilon$ for all $x \in [a, b]$. That is, each continuous real-valued function f on $[a, b]$ is approximated by algebraic polynomials.*

Proof. The original proof of Weierstrass:

Consider the following solution of the partial differential equation;

$$u(x, t) = \frac{n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-n^2(t-x)^2} f(x) dx.$$

In general, we write

$$W_n f = \frac{n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-n^2(t-x)^2} f(x) dx.$$

Note that $(W_n f)(x) = \frac{n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-n^2(t-x)^2} f(x) dx$ is called Gauss-Weierstrass integral operators with the kernel $G - W \ \ kernel, w_n f = \frac{n}{\sqrt{2\pi}} e^{-n^2(t-x)^2}$.

We say

$$w_n(z) = \frac{n}{\sqrt{2\pi}} e^{-n^2z^2} \quad (z = t - x), n \in \mathbb{N}.$$

To prove the Weierstrass theorem, firstly we show the properties of $w_n(z)$.

Some properties;

i) For $\forall n \in \mathbb{N}$ and $\forall z \in \mathbb{R}$, $w_n(z) > 0$.

ii) $w_n(z)$ is an even function as a function of z.

iii) Recall that $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$. Then, $\int_{-\infty}^{+\infty} w_n(t)dt = 1$.

iv)

$$\begin{aligned} lim_{n\to+\infty} w_n(z) &= 0, \ \ whenever \quad z \neq 0, \\ lim_{n\to+\infty} w_n(z) &= +\infty, \ \ whenever \quad z = 0. \end{aligned}$$

v) We set any $\delta > 0$ and $z \geq \delta$. Then $w_n(z)$ takes its supremum on $[\delta, +\infty)$ at the point δ. That is, $w_n(z) = \frac{n}{\sqrt{2\pi}} e^{-n^2z^2}$ is the supremum on $(-\infty, -\delta] \cup [\delta, +\infty)$.Hence,

$$lim_{n\to+\infty} w_n(\delta) = 0.$$

In other words,

$$lim_{n\to+\infty}(sup_{|z|\geq\delta} w_n(z)) = 0.$$

Now, we can prove the theorem.Let us consider the expansion of $e^{-n^2(t-x)^2}$.

$$\begin{aligned} e^{-n^2(t-x)^2} &= \sum_{k=0}^{\infty} \frac{(-n^2(t-x)^2)^k}{k!} \\ &= lim_{m\to+\infty} \sum_{k=0}^{m} \frac{(-n^2(t-x)^2)^k}{k!} = lim_{m\to+\infty} S_m. \end{aligned}$$

Now, we consider

$$P_m(f;x) = \frac{n}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} S_m(t-x) f(t)dt$$

Firstly, we calculate $|W_n - P_m|$.

$$
\begin{aligned}
|W_n - P_m| &= \frac{n}{\sqrt{2\pi}}\Big|\int_{-\infty}^{\infty} e^{-n^2(t-x)^2} f(t)dt - \int_{-\infty}^{\infty} S_m(t-x)f(t)dt\Big| \\
&\leq \frac{n}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Big|e^{-n^2(t-x)^2} - S_m(t-x)\Big|f(t)dt.
\end{aligned}
$$

Note that in the classical case, f is continuous and bounded,

$$
|W_n - P_m| \leq \frac{n}{\sqrt{2\pi}}\varepsilon\int_{-\infty}^{\infty}|f(t)|dt.
$$

In this case, we have some additional conditions:

i) if $f \in L_1$, this solves the problem.

ii) if we consider $[a, b]$ instead of $(-\infty, +\infty)$.

We know that f is not in L_1. Therefore, we will use the second condition. This yields

$$
|W_n - P_m| \leq \frac{n}{\sqrt{2\pi}}\varepsilon M(b-a)\varepsilon = \tilde{\epsilon}.
$$

$$
\begin{aligned}
P_m(f;x) &= \frac{n}{\sqrt{2\pi}}\int_a^b \sum_{k=0}^{m}\frac{(-1)^k n^{2k}(t-x)^{2k}}{k!}f(t)dt \\
&= \frac{n}{\sqrt{2\pi}}\sum_{k=0}^{m}\frac{(-1)^k n^{2k}}{k!}\int_a^b (t-x)^{2k}f(t)dt
\end{aligned}
$$

where $(t-x)^{2k} = \sum_{p=0}^{2k} c_k^p(-1)^p x^p t^{2k-p}$.

$$
\begin{aligned}
P_m(f;x) &= \frac{n}{\sqrt{2\pi}}\sum_{k=0}^{m}\frac{(-1)^k n^{2k}}{k!}\sum_{p=0}^{2k} c_k^p(-1)^p x^p\int_a^b t^{2k-p}f(t)dt \\
&= \sum_{\nu=0}^{2m} A_\nu x^\nu
\end{aligned}
$$

where A_ν are the coefficients.

This means that $P_m(f;x)$ is a polynomial with degree $2m$.

By the triangle inequality, we know that $|f - P_m| \leq |f - W_n| + |W_n - P_m|$. Now, we will need to show that $|f - W_n| \leq \varepsilon$.

$$
\begin{aligned}
|W_n(f;x) - f(x)| &= \Big|\frac{n}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(t)w_n(t-x)dt - f(x)\Big| \\
&= \Big|\frac{n}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(t)w_n(t-x)dt - f(x)\frac{n}{\sqrt{\pi}}\int_{-\infty}^{\infty} w_n(t-x)dt\Big| \\
&= \Big|\frac{n}{\sqrt{\pi}}\int_{-\infty}^{\infty}[f(t) - f(x)]w_n(t-x)dt\Big| \\
&\leq \frac{n}{\sqrt{\pi}}\int_{-\infty}^{\infty}|f(t) - f(x)|w_n(t-x)dt \\
&= \frac{n}{\sqrt{\pi}}\Big\{\int_{-\infty}^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^{+\infty}\Big\}|f(t) - f(x)|w_n(t-x)dt
\end{aligned}
$$

Here, we say

$$
\begin{aligned}
I_1 &= \int_{-\infty}^{x-\delta} |f(t) - f(x)|w_n(t-x)dt \\
I_2 &= \int_{x-\delta}^{x+\delta} |f(t) - f(x)|w_n(t-x)dt \\
I_3 &= \int_{x+\delta}^{+\infty} |f(t) - f(x)|w_n(t-x)dt
\end{aligned}
$$

We start with I_1.

From the property (v), since $w_n(t-x)$ takes its maximum value in $x-\delta$, we write $t = x - \delta$.

$$
\begin{aligned}
I_1 &= \int_{-\infty}^{x-\delta} |f(t) - f(x)|w_n(t-x)dt \\
&\leq w_n(-\delta)\int_{-\infty}^{x-\delta} |f(t) - f(x)|dt \\
&\leq w_n(-\delta)\int_{-\infty}^{+\infty} |f(t) - f(x)|dt
\end{aligned}
$$

By the triangle inequality,we find

$$
\begin{aligned}
I_1 &\leq w_n(-\delta)\int_{-\infty}^{+\infty} |f(t)|dt + w_n(-\delta)\int_{-\infty}^{+\infty} |f(x)|dt \\
&\leq w_n(-\delta)2M.
\end{aligned}
$$

Since $\int_{-\infty}^{+\infty} w_n(t-x)dt = \varepsilon$, then $\int_{x-\delta}^{x+\delta} w_n(t-x)dt < \varepsilon$. So we have

$$\begin{aligned} I_2 &= \int_{x-\delta}^{x+\delta} |f(t)-f(x)|w_n(t-x)dt \\ &< \varepsilon \end{aligned}$$

Similarly, for I_3, from the property (v), since $w_n(t-x)$ takes its maximum value in $x+\delta$, we write $t = x+\delta$.

$$\begin{aligned} I_3 &= \int_{x+\delta}^{\infty} |f(t)-f(x)|w_n(t-x)dt \\ &\leq w_n(+\delta)\int_{x+\delta}^{\infty} |f(t)-f(x)|dt \\ &\leq w_n(+\delta)\int_{-\infty}^{+\infty} |f(t)-f(x)|dt \\ (by\ \ the\ \ triangle\ \ inequality,) &\leq w_n(+\delta)\int_{-\infty}^{+\infty} |f(t)|dt + w_n(+\delta)\int_{-\infty}^{+\infty} |f(x)|dt \\ &\leq w_n(+\delta)2M. \end{aligned}$$

In conclusion,

$$|W_n(f;x)-f(x)| \leq 2Mw_n(-\delta) + \varepsilon + 2Mw_n(\delta).$$

Taking its limit, we have

$$|W_n(f;x)-f(x)| \leq \tilde{\varepsilon} \qquad (\forall \tilde{\varepsilon} > 0).$$

□

Definition 2.1. Let V be a real linear vector space. A function which assigns to each vector $v \in V$ the real number $||v||$ is a *norm* on V if and only if it satisfies, for all $v, w \in V$ and $k \in \mathbb{R}$, the following axioms:

i) $||v|| \geq 0$,

ii) $||v|| = 0$ if and only if $v = 0$, *iii*) $||v + w|| \leq ||v|| + ||w||$,

iv) $||kv|| = |k|||v||$.

A linear space V together with a norm is called a *normed space*. The real number $||v||$ is called the *norm* of the vector v.

Note that seminorm satisfies the properties (i),(iii) and (iv) of the norm, but the condition that $||v|| = 0$ implies that $v = 0$ is not required.When it holds, the seminorm is a norm.

Definition 2.2. [2] The element f of the space L_2 is called a limit of the sequence $f_1, f_2, f_3, ..., f_n, ...$ of elements of the same space, if for every $\varepsilon > 0$, there exists a natural number N such that

$$||f_n - f||_{L_2} < \varepsilon, \quad \forall n > N.$$

We say that the sequence (f_n) converges to the element in L_2 norm and we write

$$lim_{n\to\infty} f_n(x) = f(x) \ \ or \ \ f_n \to f$$

or in the professional books;

$$l.i.m f_n(x) = f(x).$$

The relation $f_n \to f$ means; $||f_n - f||_{L_2} \to 0$ or $lim_{n\to\infty} \int_a^b [f_n(x) - f(x)]^2 dx = 0$.

Definition 2.3. Let X be any nonempty set. A collection $\mathcal{A}$ of subsets of X is called an algebra on X if it satisfies,

i) if $A \in \mathcal{A}$, $A^c \in \mathcal{A}$

ii) if $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$.

Definition 2.4. Let A be an algebra. An algebra norm A is a map $||.|| : A \to [0, +\infty)$ such that $(A, ||.||)$ is called a normed space.

If $||a.b|| \leq ||a||.||b|| \ \ (a, b \in A)$, then the normed algebra $(A, ||.||)$ is a Banach algebra.

Definition 2.5. (Unitral subalgebra, Seperating points) Let K be a compact metric space. Consider the Banach algebra

$$C(K, \mathbb{R}) = \{f : \quad K \to \mathbb{R} \ | \ f \ \ is \ \ continuous\}$$

with the norm $\|f\|_\infty = sup_{t \in K} |f(t)|$. Then,

i) $A \subset C(K, \mathbb{R})$ is a unital subalgebra if $1 \in A$ and $f, g \in A$, $\alpha, \beta \in \mathbb{R}$ implies that $\alpha f + \beta g \in A$ and $fg \in A$.

ii) $A \subset C(K, \mathbb{R})$ seperates points of K if for all $s, t \in K$ with $s \neq t$, there exists $f \in A$ such that $f(s) \neq f(t)$.

Theorem 2.3. *(Stone-Weierstrass Theorem) Let X be a compact set and let $C(X)$ denote the space of continuous real-valued functions defined on X. Assume A is a subalgebra of C(X). Then A is dense in C(X) in the uniform norm if and only if A separates points and for each $x \in X$ there exists an $f \in A$ satisfying $f(x) \neq 0$.*

Another proof of Weierstrass Theorem:
This proof is due to Bernstein(1912).
Before we give the proof of this theorem, we will give some necessary definitions.

Definition 2.6. [6] A mapping from a normed space to a normed space is called an *operator.*

Definition 2.7. (Linear operator)[6] A linear operator T is an operator such that

i) the domain $D(T)$ of T is a vector space and the range $R(T)$ lies in a vector space over the same field,

ii) for all $x, y \in D(T)$ and scalars α,

$$T(x + y) = Tx + Ty$$

$$T(\alpha x) = \alpha T(x).$$

Definition 2.8. If $T(f) \geq 0$ for $f \geq 0$, then T is called a *positive operator.*

If T satisfies both linearity and positivity, then T is said to be a linear positive operator.

Lemma 2.1. *Linear positive operators are monotone increasing.That is;*

$$f \leq g \quad \rightarrow \quad T(f) \leq T(g).$$

Lemma 2.2. *If T is a linear positive operator, then $|T(f)| \leq T(|f|)$.*

2.2 Introduction to Bernstein Polynomials

Bernstein polynomials were introduced by Bernstein in 1912 to give the first constructive proof of the Weierstrass approximation theorem.

Definition 2.9. [1] Let $f \in C[0,1] = \{f : [0,1] \rightarrow \mathbb{R} \mid f \; is \; continuous\}$.
Then the following polynomials are called the n-th Bernstein polynomials;

$$B_n : C[0,1] \rightarrow P[0,1], \quad for \;\; f \in C[0,1],$$

$$(B_n f)(x) := \sum_{k=0}^{n} f\Big(\frac{k}{n}\Big) p_{k,n}(x), \quad 0 \leq x \leq 1$$

where $p_{k,n}(x) := \binom{n}{k} x^k (1-x)^{n-k} \geq 0$.
The Bernstein polynomials are connected with the theory of probability, with moment problems. Here $p_{k,n}(x)$ are the Binomial probabilities well known in the theory of probability.
$B_n f$ is a polynomial of degree n and it is also linear and positive.

Proof. (linearity) For $\alpha, \beta \in \mathbb{R}$, and $f, g \in AC[0,1]$,

$$B_n(\alpha f + \beta g, x) \;\; = \;\; \sum_{k=0}^{n} (\alpha f + \beta g)\Big(\frac{k}{n}\Big) p_{k,n}(x), \quad 0 \leq x \leq 1,$$

Since the domains of the functions f and g are same, we can seperate the functions and we have the equation as below

$$\begin{aligned} B_n(\alpha f+\beta g,x) &= \sum_{k=0}^{n}(\alpha f)\Big(\frac{k}{n}\Big)p_{k,n}(x)+\sum_{k=0}^{n}(\beta g)\Big(\frac{k}{n}\Big)p_{k,n}(x), \quad 0\le x\le 1, \\ &= \sum_{k=0}^{n}\alpha f\Big(\frac{k}{n}\Big)p_{k,n}(x)+\sum_{k=0}^{n}\beta g\Big(\frac{k}{n}\Big)p_{k,n}(x), \quad 0\le x\le 1, \\ &= \alpha\sum_{k=0}^{n} f\Big(\frac{k}{n}\Big)p_{k,n}(x)+\beta\sum_{k=0}^{n} g\Big(\frac{k}{n}\Big)p_{k,n}(x), \quad 0\le x\le 1, \\ &= \alpha(B_nf)(x)+\beta(B_ng)(x) \end{aligned}$$

Here we have

$$B_n(\alpha f+\beta g,x)=\alpha(B_nf)(x)+\beta(B_ng)(x).$$

Therefore, B_n is linear. □

Now, we will show the positivity property of Bernstein Polynomials.

Proof. (positivity) Since $\binom{n}{k}>0$, $x^k>0$ and $(1-x)^{n-k}>0$ for $0\le x\le 1$ and for $f(x)\ge 0$,

$$(B_nf)(x):=\sum_{k=0}^{n} f\Big(\frac{k}{n}\Big)p_{k,n}(x), \quad 0\le x\le 1 \quad is\ \ positive.$$

□

In conclusion, (B_nf) *is a linear positive operator.*
With the study of [17], we can give the main theorem which is an another form of Weierstrass Approximation theorem according to Bernstein. Berstein proved the same theorem for Bernstein operator which is defined on $C[0,1]$.

Theorem 2.4. *(Bernstein, 1912) For a function* $f\in[0,1]$,

$$lim_{n\to\infty}(B_nf)(x)=f(x)$$

holds at each point. The above relation holds uniformly on $[0,1]$.

Proof.

$$\begin{aligned}
\Big|(B_n f)(x) - f(x)\Big| &= \Big|\sum_{k=0}^{n} f\Big(\frac{k}{n}\Big)p_{k,n}(x) - f(x)\Big| \\
&= \Big|\sum_{k=0}^{n} f\Big(\frac{k}{n}\Big)p_{k,n}(x) - f(x)\sum_{k=0}^{n} p_{k,n}(x)\Big| \\
&= \Big|\sum_{k=0}^{n} \Big[f\Big(\frac{k}{n}\Big) - f(x)\Big]p_{k,n}(x)\Big| \\
&\leq \sum_{k=0}^{n} \Big|f\Big(\frac{k}{n}\Big) - f(x)\Big|p_{k,n}(x) \\
&= \sum_{|\frac{k}{n}-x|<\delta} \Big|f\Big(\frac{k}{n}\Big) - f(x)\Big|p_{k,n}(x) + \sum_{|\frac{k}{n}-x|\geq\delta} \Big|f\Big(\frac{k}{n}\Big) - f(x)\Big|p_{k,n}(x) \\
&\leq \varepsilon \sum_{|\frac{k}{n}-x|<\delta} p_{k,n}(x) + \sum_{|\frac{k}{n}-x|\geq\delta} \Big|f\Big(\frac{k}{n}\Big) - f(x)\Big|p_{k,n}(x) \\
&= \varepsilon + \sum_{|\frac{k}{n}-x|\geq\delta} \Big|f\Big(\frac{k}{n}\Big) - f(x)\Big|p_{k,n}(x).
\end{aligned}$$

Since f is continuous on the closed interval [0, 1], then from the Extreme value theorem, f is bounded;that is;

$$|f(x)| \leq M.$$

Hence,

$$\sum_{|\frac{k}{n}-x|\geq\delta} \Big|f\Big(\frac{k}{n}\Big) - f(x)\Big|p_{k,n}(x) \leq 2M \sum_{|\frac{k}{n}-x|\geq\delta} p_{k,n}(x).$$

We know that $\left|\frac{k}{n} - x\right| \geq \delta$ implies that $\frac{\left(\frac{k}{n}-x\right)^2}{\delta^2} \geq 1.$

When we write the inequality, we have

$$\begin{aligned}
2M \sum_{|\frac{k}{n}-x|\geq\delta} p_{k,n}(x) &\leq 2M \sum_{k=0}^{n} \frac{\left(\frac{k}{n} - x\right)^2}{\delta^2} p_{k,n}(x) \\
&= \frac{2M}{n^2\delta^2} \sum_{k=0}^{n} (k - nx)^2 p_{k,n}(x) \\
&= \frac{2M}{n^2\delta^2} \sum_{k=0}^{n} (k^2 - 2knx + n^2x^2) p_{k,n}(x) \qquad (1)
\end{aligned}$$

To calculate this, we need to calculate $(B_n 1)(x)$, $(B_n t)(x)$, $(B_n t^2)(x)$.

$$\begin{aligned}(B_n 1)(x) &= \sum_{k=0}^{n} 1.\binom{n}{k} x^k (1-x)^{n-k} \\ &= (1-x+x)^n = 1.\end{aligned}$$

Secondly we find $(B_n(t)x)$.

$$\begin{aligned}(B_n t)(x) &= \sum_{k=0}^{n} \frac{k}{n}\binom{n}{k} x^k (1-x)^{n-k} \\ &= \sum_{k=0}^{n} \frac{k}{n}\frac{n!}{(n-k)!k!} x^k (1-x)^{n-k} \\ &= \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} x^k (1-x)^{n-k} \\ &= x\sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} x^{k-1} (1-x)^{n-k} \\ &= x\sum_{k=0}^{n} \frac{(n-1)!}{(n-k-1)!k!} x^{k} (1-x)^{n-k-1} \\ &= x\sum_{k=0}^{n} \binom{n-1}{k} x^{k} (1-x)^{n-1-k} \\ &= x(1-x+x)^{n-1} = x.\end{aligned}$$

Finally, we calculate $(B_n t^2)(x)$.

$$(B_n t^2)(x) = \sum_{k=0}^{n} \Big(\frac{k}{n}\Big)^2 \binom{n}{k} x^k (1-x)^{n-k}$$

Here we have

$$\begin{aligned}(B_n t^2)(x) &= \sum_{k=0}^{n} \frac{k^2}{n^2}\frac{n!}{(n-k)!k!} x^k (1-x)^{n-k} \\ &= \sum_{k=1}^{n} \frac{k+1-1}{n}\frac{(n-1)!}{(n-k)!(k-1)!} x^k (1-x)^{n-k} \\ &= \sum_{k=1}^{n} \frac{k-1}{n}\frac{(n-1)!}{(n-k)!(k-1)!} x^k (1-x)^{n-k} + \sum_{k=1}^{n} \frac{1}{n}\frac{(n-1)!}{(n-k)!(k-1)!} x^k (1-x)^{n-k}.\end{aligned}$$

Lets say

$$I_1 = \sum_{k=1}^{n} \frac{k-1}{n} \frac{(n-1)!}{(n-k)!(k-1)!} x^k (1-x)^{n-k} \ \ and \ \ I_2 = \sum_{k=1}^{n} \frac{1}{n} \frac{(n-1)!}{(n-k)!(k-1)!} x^k (1-x)^{n-k}.$$

$$\begin{aligned}
I_1 &= x \sum_{k=1}^{n} \frac{(k-1)}{n} \frac{(n-1)!}{(n-k)!(k-1)!} x^{k-1} (1-x)^{n-k} \\
&= x \sum_{k=2}^{n} \frac{1}{n} \frac{(n-1)!}{(n-k)!(k-2)!} x^{k-1} (1-x)^{n-k} \\
&= x^2 \sum_{k=2}^{n} \frac{1}{n} \frac{(n-1)!}{(n-k)!(k-2)!} x^{k-2} (1-x)^{n-k} \\
&= x^2 \sum_{k=0}^{n+2} \frac{1}{n} \frac{(n-1)!}{(n-k-2)!k!} x^{k} (1-x)^{n-k-2} \\
&= x^2 \frac{(n-1)}{n} \sum_{k=0}^{n+2} \frac{(n-2)!}{(n-k-2)!k!} x^{k} (1-x)^{n-k-2} \\
&= x^2 \frac{(n-1)}{n} (1-x+x)^{(n-2)} = \frac{x^2(n-1)}{n}
\end{aligned}$$

$$\begin{aligned}
I_2 &= \sum_{k=1}^{n} \frac{1}{n} \frac{(n-1)!}{(n-k)!(k-1)!} x^k (1-x)^{n-k} \\
&= \frac{x}{n} \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} x^{k-1} (1-x)^{n-k} \\
&= \frac{x}{n} \sum_{k=0}^{n-1} \frac{(n-1)!}{(n-k-1)!k!} x^{k} (1-x)^{n-k-1} = \frac{x}{n}.
\end{aligned}$$

In conclusion,

$$(B_n t^2)(x) = \frac{x^2(n-1)}{n} + \frac{x}{n}.$$

Now, put this results into (1), we have

$$\begin{aligned}
\frac{2M}{n^2\delta^2} \sum_{k=0}^{n} (k^2 - 2knx + n^2x^2) p_{k,n}(x) &= \frac{1}{n^2\delta^2} \Big[n^2 \Big(\frac{n-1}{n} + \frac{x}{n} \Big) - 2nxnx + n^2x^2 \Big] \\
&= \frac{2M}{\delta^2} \Big[x^2 \Big(\frac{n-1}{n} - 1 \Big) + \frac{x}{n} \Big] \\
&= \frac{2Mx(1-x)}{n\delta^2}.
\end{aligned}$$

Hence,

$$\left|(B_n f)(x) - f(x)\right| \leq \frac{2Mx(1-x)}{n\delta^2}$$

If we take its max. value,since $0 \leq x(1-x) \leq \frac{1}{4}$,

$$max_{x\in[0,1]}\left|(B_n f)(x) - f(x)\right| \leq max_{x\in[0,1]}\frac{2Mx(1-x)}{n\delta^2} = \frac{2M}{4n\delta^2}$$

This implies that

$$lim_{n\to\infty}max_{x\in[0,1]}\left|(B_n f)(x) - f(x)\right| = 0.$$

$$\left|(B_n f)(x) - f(x)\right| \to 0 \quad uniformly \;\; as \;\; n \to \infty.$$

□

Another method that can be used to prove density is based on what is called Korovkin Theorem or the Bohman-Korovkin Theorem. A primitive form of this theorem was proved by Bohman(1952). His proof, and the main idea in his approach was a generalization of Bernstein's proof(1912). Korovkin, one year later (1953), proved the same theorem for integral type operators.Korovkin's original proof is infact based on positive singular integral operators.

Theorem 2.5. *(Bohman-Korovkin Theorem) Let* (L_n) *be a sequence of positive linear operators mapping* $C[a,b]$ *into itself. Assume that*

$$lim_{n\to\infty}L_n(x^i) = x^i, \qquad (i = 0, 1, 2, ...)$$

and the convergence is uniform on $[a,b]$. *Then*

$$lim_{n\to\infty}(L_n f)(x) = f(x) \quad uniformly \;\; on \;\; [a,b].$$

Proof. Let $f \in C[a,b]$ and f is uniformly continuous. Given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x_1 - x_2| < \delta$, then $|f(x_1) - f(x_2)| < \varepsilon$.

For each $y \in [a, b]$, set

$$P_u(x) = f(y) + \varepsilon + \frac{2||f||(x-y)^2}{\delta^2}$$

and

$$P_\ell(x) = f(y) - \varepsilon - \frac{2||f||(x-y)^2}{\delta^2}.$$

Since $|f(x) - f(y)| < \varepsilon$ for $|x - y| < \delta$, and $|f(x) - f(y)| < 2||f||\frac{(x-y)^2}{\delta^2}$, $|x - y| \geq \delta$. This shows that

$$P_\ell(x) \leq f(x) \leq P_u(x) \ \ for \ \ all \ \ x \in [a, b].$$

Since L_n is positive linear operator, then

$$(L_n P_\ell)(x) \leq (L_n f)(x) \leq (L_n P_u)(x).$$

Explicitely;

$$P_u(x) = f(y) + \varepsilon + \frac{2||f||y^2}{\delta^2} - \frac{4||f||y}{\delta^2}x + \frac{2||f||x^2}{\delta^2}$$

Since $lim_{n\to+\infty} L_n(x^i) = x^i$, $\ (i = 0, 1, 2)$ uniformly on $[a, b]$, then there exists an N such that for all $n \geq N$ and every choice of $y \in [a, b]$, we have

$$|(L_n P_u)(x) - P_u(x)| < \varepsilon \quad and \quad |(L_n P_\ell)(x) - P_\ell(x)| < \varepsilon.$$

Collecting these statements one has;

$$f(y) - \varepsilon < (L_n f)(y) < f(y) + \varepsilon$$

$$-\varepsilon < (L_n f)(y) - f(y) < \varepsilon$$

$$|(L_n f)(y) - f(y)| < \varepsilon$$

This completes the proof.

$\square$

The second main question in Approximation theory is the rate of convergence.

Definition 2.10. [3] Landau symbols: ($\mathcal{O}$- and $o-$relationships for series and integrals)

Using the notation now accepted in mathematical literature we write

$$u_n = o(v_n)$$

if $v_n > 0$ $(n = 0, 1, 2, ...)$ and $\frac{u_n}{v_n} \to 0$ at $n \to \infty$.

If $\frac{u_n}{v_n}$ is bounded, we write

$$u_n = \mathcal{O}(v_n).$$

If two positive constants A and B exist for which at sufficiently large n,

$$A \leq \frac{u_n}{v_n} \leq B,$$

then we can write

$$u_n \sim v_n,$$

and finally

$$u_n \approx v_n,$$

will signify

$$lim_{n\to\infty}\frac{u_n}{v_n} = 1.$$

(in this case it is usually said that u_n and v_n are asymptotically equal.)

The degree of approximation of a function f(x) defined on $[a, b]$ *by polynomials may be described in terms of its "Modulus of Continuity."*

Definition 2.11. Let $f \in C[a, b]$ and $\delta > 0$ be given. Then the modulus of continuity of f is given by;

$$\omega_{C[a,b]}(f;\delta) = max_{|x-y|<\delta}|f(x) - f(y)| \ \ for \ \ all \ \ a \leq x, y \leq b.$$

Properties of Modulus of Continuity in $C[a,b]$:

i) $\omega(f;\delta) \geq 0$ *(positivity),*

ii) *If* $\delta_1 \leq \delta_2$, *then* $\omega(f;\delta_1) \leq \omega(f;\delta_2)$ *(monotonity),*

iii) *Let* $m \in \mathbb{N}$, *then* $\omega(f;m\delta) \leq m\omega(f;\delta)$,

iv) *Let* $\lambda \in \mathbb{R}^+$, *then* $\omega(f;\lambda\delta) \leq (\lambda+1)\omega(f;\delta)$,

v) *lim* $\omega(f;\delta) = 0$ *when* $\delta \to 0^+$,

vi) $|f(t) - f(x)| \leq \omega(f;|t-x|)$,

vii) $|f(t) - f(x)| \leq \left(\frac{|t-x|}{\delta} + 1\right)\omega(f;\delta)$.

If $\omega(\delta)$ *is modulus of continuity of* $f \in C[0,1]$, *then for each polynomial* $P_n(x)$ *of degree* $\leq n$ *such that*

$$|P_n(x) - f(x)| < c.\omega\Big(\frac{1}{n}\Big), \quad c \ \ is \ \ a \ \ constant.$$

Theorem 2.6. *(Popoviciu) [5] If* $f(x)$ *is continuous and* $\omega(\delta)$ *is the modulus of continuity of* $f(x)$, *then*

$$|f(x) - (B_nf)(x)| < \frac{5}{4}\omega\Big(\frac{1}{\sqrt{n}}\Big).$$

Proof. For arbitrary x_1, x_2 in $[0,1]$ and a $\delta > 0$, we have

$$\begin{aligned}
|f(x) - (B_nf)(x)| &\leq \sum_{k=0}^{n} \Big|f\Big(\frac{k}{n}\Big) - f(x)\Big| p_{k,n}(x) \\
&\leq \sum_{k=0}^{n} \Big(\frac{|\frac{k}{n} - x|}{\delta} + 1\Big)\omega(f;\delta)p_{k,n}(x) \\
&= \omega(f;\delta) \sum_{k=0}^{n} \Big(\frac{|\frac{k}{n} - x|}{\delta} + 1\Big)p_{k,n}(x) \\
&= \omega(f;\delta) \sum_{k=0}^{n} \frac{|\frac{k}{n} - x|}{\delta} p_{k,n}(x) + \omega(f;\delta) \sum_{k=0}^{n} p_{k,n}(x) \\
&= \omega(f;\delta)\Big[\frac{1}{\delta} \sum_{k=0}^{n} \Big|\frac{k}{n} - x\Big| p_{k,n}(x) + 1\Big]
\end{aligned}$$

Now, we find

$$\begin{aligned}
\sum_{k=0}^{n}\Big|\frac{k}{n}-x\Big|p_{k,n}(x) &= \sum_{k=0}^{n}\Big|\frac{k}{n}-x\Big|p_{k,n}^{\frac{1}{2}}(x)p_{k,n}^{\frac{1}{2}}(x)\\
(by\ \ Cauchy-Schwartz\ \ inequality) &\leq \Big[\sum_{k=0}^{n}\Big(\frac{k}{n}-x\Big)^2\Big(p_{k,n}^{\frac{1}{2}}(x)\Big)^2\Big]^{\frac{1}{2}}\Big[\sum_{k=0}^{n}\Big(p_{k,n}^{\frac{1}{2}}(x)\Big)^2\Big]^{\frac{1}{2}}
\end{aligned}$$

Calculating the last inequality,

$$\begin{aligned}
\sum_{k=0}^{n}\Big|\frac{k}{n}-x\Big|p_{k,n}(x) &= \Big[\sum_{k=0}^{n}\Big(\frac{k}{n}-x\Big)^2 p_{k,n}(x)\Big]^{\frac{1}{2}}\Big[\sum_{k=0}^{n}p_{k,n}(x)\Big]^{\frac{1}{2}}\\
&= \Big[\frac{1}{n^2}\sum_{k=0}^{n}(k-nx)^2p_{k,n}(x)\Big]^{\frac{1}{2}}\\
&= \frac{1}{n}(nX)^{\frac{1}{2}}\\
(since\ \ 0\leq X=x(1-x)\leq\frac{1}{4}) &= \Big(\frac{1}{4n}\Big)^{\frac{1}{2}}\\
&= \frac{1}{2\sqrt{n}}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
|f(x)-(B_nf)(x)| &\leq \omega(f;\delta)\Big[\frac{1}{\delta}\sum_{k=0}^{n}\Big|\frac{k}{n}-x\Big|p_{n,k}(x)+1\Big]\\
&\leq \omega(f;\delta)\Big[\frac{1}{\delta 2\sqrt{n}}+1\Big].
\end{aligned}$$

If $\delta=\frac{1}{\sqrt{n}}$,

$$|f(x)-(B_nf)(x)|\leq\frac{3}{2}\omega\Big(f;\frac{1}{\sqrt{n}}\Big).$$

□

To define the Bernstein operators, Bernstein used the probability theory and for the Bernstein basis, that is,

$$p_{k,n}(x)=\binom{n}{k}x^k(1-x)^{n-k},$$

he used binomial distribution where the probability of an experiment consisting of n independent trials is

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

By using the Poisson distribution, Otto Szász defined the Szász-Mirakjan operators on $C[0,\infty)$ in 1950 as an another concept of Bernstein operators.

2.3 Introduction to Szász-Mirakjan Operators

A linear positive operator defined by $S_n : C[0,\infty) \to C[0,\infty)$

$$(S_n f)(x) := \sum_{k=0}^{\infty} f\Big(\frac{k}{n}\Big) s_{k,n}(x),$$

where $s_{k,n}(x) := e^{-nx}\frac{(nx)^k}{k!} \quad for \;\; x \geq 0$, is called Szász Operators.

Szász-Mirakjan operators are linear positive operators.

Proof. (linearity) For $\forall \alpha, \beta \in \mathbb{R}$ and $f, g \in C[0,1]$,

$$\begin{aligned}
(S_n(\alpha f + \beta g))(x) &= \sum_{k=0}^{\infty} (\alpha f + \beta g)\Big(\frac{k}{n}\Big) s_{k,n}(x) \\
&= \sum_{k=0}^{\infty} (\alpha f)\Big(\frac{k}{n}\Big) s_{k,n}(x) + \sum_{k=0}^{\infty} (\beta g)\Big(\frac{k}{n}\Big) s_{k,n}(x) \\
&= \sum_{k=0}^{\infty} \alpha f\Big(\frac{k}{n}\Big) s_{k,n}(x) + \sum_{k=0}^{\infty} \beta g\Big(\frac{k}{n}\Big) s_{k,n}(x) \\
&= \alpha \sum_{k=0}^{\infty} f\Big(\frac{k}{n}\Big) s_{k,n}(x) + \beta \sum_{k=0}^{\infty} g\Big(\frac{k}{n}\Big) s_{k,n}(x) \\
&= \alpha (S_n f)(x) + \beta (S_n g)(x).
\end{aligned}$$

Therefore, S_n is linear. □

Proof. (positivity) Since $k = 0, 1, 2, ..., n \in \mathbb{N}$ and $x \in [0,1]$, $e^{-nx}\frac{(nx)^k}{k!} \geq 0$ and if $f \geq 0$, $(S_n f) \geq 0$, (S_n) is a linear positive operator.

□

Theorem 2.7. *For $A \in \mathbb{R}^+$, $(S_n f)(x)$ is continuous on $[0, A]$ and $S_n f \rightrightarrows f$ at the positive semi-axes where f is bounded.That is, if $f \in C[0, A]$, then $S_n f \rightrightarrows f$, $x \in [0, A]$.*

Proof. To prove this theorem, we will use Bohman-Korovkin Theorem.
We will show that

i) $(S_n 1)(x) \rightrightarrows 1$,

ii) $(S_n t)(x) \rightrightarrows x$,

iii) $(S_n t^2)(x) \rightrightarrows x^2$,

Then, from the Korovkin theorem, we will say that

$$(S_n f)(x) \rightrightarrows f(x).$$

We need to find the value of $(S_n 1)(x)$, $(S_n t)(x)$, $(S_n t^2)(x)$.

$$(S_n 1)(x) = e^{-nx} \sum_{k=0}^{\infty} 1 \frac{(nx)^k}{k!} = e^{-nx} e^{nx} = 1.$$

$$\begin{aligned}
(S_n t)(x) &= e^{-nx} \sum_{k=0}^{\infty} \frac{k}{n} \frac{(nx)^k}{k!} \\
&= e^{-nx} \sum_{k=0}^{\infty} \frac{k}{n} \frac{n^k x^k}{k!} \\
&= e^{-nx} \sum_{k=1}^{\infty} \frac{n^{k-1} x^{k-1} x}{(k-1)!} \\
&= x e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} \\
&= x e^{-nx} e^{nx} = x.
\end{aligned}$$

$$\begin{aligned}
(S_n t^2)(x) &= e^{-nx} \sum_{k=0}^{\infty} \frac{k^2}{n^2} \frac{(nx)^k}{k!} \\
&= e^{-nx} \sum_{k=0}^{\infty} \frac{k^2}{n^2} \frac{n^k x^k}{k!}
\end{aligned}$$

From the last equality,

$$
\begin{aligned}
(S_n t^2)(x) &= e^{-nx}\sum_{k=1}^{\infty}\frac{k}{n}\frac{n^{k-1}x^{k-1}x}{(k-1)!}\\
&= e^{-nx}\sum_{k=1}^{\infty}\Big(\frac{k-1}{n}+\frac{1}{n}\Big)\frac{n^{k-1}x^{k-1}x}{(k-1)!}\\
&= e^{-nx}\Big[\sum_{k=1}^{\infty}\Big(\frac{k-1}{n}\Big)\frac{n^{k-1}x^{k-1}x}{(k-1)!}+\sum_{k=1}^{\infty}\frac{1}{n}\frac{n^{k-1}x^{k-1}x}{(k-1)!}\Big]\\
&= e^{-nx}\Big[\sum_{k=2}^{\infty}\frac{n^{k-2}x^{k-2}x^2}{(k-2)!}+\sum_{k=1}^{\infty}\frac{1}{n}\frac{n^{k-1}x^{k-1}x}{(k-1)!}\Big]\\
&= e^{-nx}\Big[x^2\sum_{k=0}^{\infty}\frac{n^k x^k}{k!}+\Big[\frac{x}{n}\sum_{k=0}^{\infty}\frac{n^k x^k}{k!}\Big]\\
&= e^{-nx}x^2e^{nx}+e^{-nx}\frac{x}{n}e^{nx}\\
&= x^2+\frac{x}{n}.
\end{aligned}
$$

Then,

$$|(S_n 1)(x)-1| = |1-1| = 0 < \frac{1}{n} \to 0$$

implies that $(S_n 1)(x) \rightrightarrows 1$.

$$|(S_n t)(x)-x| = |x-x| = 0 < \frac{1}{n} \to 0$$

implies that $(S_n t)(x) \rightrightarrows x$.

$$|(S_n t^2)(x)-x^2| = \Big|x^2+\frac{x}{n}-x^2\Big| = \Big|\frac{x}{n}\Big| \le \frac{A}{n} \to 0$$

implies that $(S_n t^2)(x) \rightrightarrows x^2$.

This completes the proof.

□

We showed the convergence of Szász-Mirakjan operators and secondly, we will show the convergence rate of this operator. For this, we will give the definition of modulus of continuity in the weighted spaces.

2.4 Modulus of Continuity in the Weighted Spaces

Definition 2.12. Let f be defined on the real line. The space of functions satisfying $|f(x)| \leq M_f \rho(x)$ is called $B_\rho(\mathbb{R}) - space$. That is;

$$B_\rho(\mathbb{R}) = \{f \;\; : \;\; |f(x)| \leq M_f \rho(x)\}.$$

Here, M_f is a constant number of f.

Definition 2.13. The space of continuous functions in $B_\rho(\mathbb{R}) - space$ is called the $C_\rho(\mathbb{R}) - function \;\; space$. That is;

$$C_\rho(\mathbb{R}) = \{f \;\; : \;\; f \in B_\rho(\mathbb{R}) \;\; and \;\; f \;\; is \;\; continuous\}.$$

We say $B_\rho(\mathbb{R}), C_\rho(\mathbb{R})$ are weighted spaces.

Definition 2.14. $C_{\rho,K_f}[0,\infty) = \{f : f \in C_\rho[0,\infty) \;\; and \;\; lim_{x\to\infty} \frac{f(x)}{\rho(x)} = K_f < \infty\}$

For the modulus of continuity in the weighted spaces, we investigate $|f(x+h)-f(x)|$. Since $f(x), f(x+h) \in C_\rho[\rho,\infty)$, then $|f(x)| \leq M_f \rho(x)$ and choosing $\rho(x) = 1 + x^2$, $\frac{|f(x)|}{1+x^2} \leq M_f$. ....(2.4.1)

Similarly,$|f(x+h)| \leq M_f \rho(x)$ implies that $\frac{|f(x+h)|}{1+(x+h)^2} \leq M_f$. ....(2.4.2)

By the triangle inequality,

$$\begin{aligned} |f(x+h) - f(x)| &\leq |f(x+h)| + |f(x)| \\ &= \frac{|f(x+h)|}{1+(x+h)^2}\Big[1 + (x+h)^2\Big] + \frac{|f(x)|}{1+x^2}(1+x^2). \end{aligned}$$

Using (2.4.1) and (2.4.2), we have

$$|f(x+h) - f(x)| \leq M_f\Big[(1 + (x+h)^2) + (1 + x^2)\Big] \quad .... \;\; (2.4.3)$$

For $\forall a, b \in \mathbb{R}$, $(a+b)^2 \leq 4(a^2 + b^2)$. For $a = x, b = h$, $(x+h)^2 \leq 4(x^2 + h^2)$.

Then,

$$
\begin{aligned}
(1+(x+h)^2)+(1+x^2) &\leq 1+4(x^2+h^2)+1+x^2 \\
&\leq 1+4(x^2+h^2)+1+x^2+3+h^2 \\
&= 5+5(x^2+h^2) \\
&= 5(1+x^2+h^2) \\
&\leq 5(1+x^2+h^2+x^2h^2) \\
&= 5(1+x^2)(1+h^2)
\end{aligned}
$$

Therefore, we have $(1+(x+h)^2)+(1+x^2) \leq 5(1+x^2)(1+h^2)$.

Using this inequality in (2.4.3),

$$|f(x+h)-f(x)| \leq M_f 5(1+x^2)(1+h^2).$$

This implies that since

$$|f(x+h)-f(x)| \leq C_f(1+x^2)(1+h^2),$$

then for $\forall f \in C_{\rho,K_f}[0,\infty)$, we have

$$\omega(f;\delta) = sup_{x\geq 0,|h|\leq\delta}\frac{|f(x+h)-f(x)|}{(1+x^2)(1+h^2)}.$$

Definition 2.15. Let $f \in C_{\rho,K_f}[0,\infty)$. Then the modulus of continuity of f on $C_{\rho,K_f}[0,\infty)$ is defined by;

$$\omega(f;\delta) = sup_{x\geq 0,|h|\leq\delta}\frac{|f(x+h)-f(x)|}{(1+x^2)(1+h^2)}.$$

Properties of Modulus of Continuity in the Weighted Spaces

i) $\omega(f;\delta) \geq 0$ (positivity),

ii) If $\delta_1 \leq \delta_2$, then $\omega(f;\delta_1) \leq \omega(f;\delta_2)$ (monotonity),

iii) Let $m \in \mathbb{N}$, then $\omega(f;m\delta) \leq 4m(1+\delta^2)\omega(f;\delta)$,

iv) Let $\lambda \in \mathbb{R}^+$, then $\omega(f;\lambda\delta) \leq 4(\lambda+1)(1+\delta^2)\omega(f;\delta)$, *v)* For $f \in C_{0,K_f}[0,\infty)$, $lim\omega(f;\delta) = 0$ when $\delta \to 0$,

vi) $|f(t) - f(x)| \leq (1 + x^2)(1 + (t - x)^2)\omega(f; |t - x|)$,

vii) $|f(t) - f(x)| \leq 4\Big(\frac{|t-x|}{\delta} + 1\Big)(1 + x^2)(1 + (t - x)^2)(1 + \delta^2)\omega(f;\delta)$.

Now, we will show the rate of convergence of Szász-Mirakjan operators with the modulus of continuity.

Theorem 2.8. *If $f \in C_{\rho,K_f}[0,\infty)$, then*

$$\|(S_n f) - f\|_{\rho^2} \leq 40\omega\Big(f; \frac{1}{\sqrt{n}}\Big).$$

Proof. Since $(S_n 1)(x) = 1$, from linearity,

$$\begin{aligned}
|(S_n f)(x) - f(x)| &= \Big|\sum_{k=0}^{\infty} f\Big(\frac{k}{n}\Big)s_{k,n}(x) - f(x)\sum_{k=0}^{\infty} s_{k,n}(x)\Big| \\
&= \Big|\sum_{k=0}^{\infty}\Big[f\Big(\frac{k}{n}\Big) - f(x)\Big]s_{k,n}(x)\Big| \\
&\leq \sum_{k=0}^{\infty}\Big|f\Big(\frac{k}{n}\Big) - f(x)\Big|s_{k,n}(x) \qquad \text{... (2.4.4)}
\end{aligned}$$

Choosing $t = \frac{k}{n}$ for the property (vii) of modulus of continuity,

$$\Big|f\Big(\frac{k}{n}\Big) - f(x)\Big| \leq 4\Big(\frac{|\frac{k}{n} - x|}{\delta} + 1\Big)(1 + x^2)\Big(1 + \Big(\frac{k}{n} - x\Big)^2\Big)(1 + \delta^2)\omega(f;\delta)$$

Using this inequality in (2.4.4),

$$\begin{aligned}
|(S_n f)(x) - f(x)| &\leq 4(1 + x^2)(1 + \delta^2)\omega(f;\delta)\sum_{k=0}^{\infty}\Big[\frac{|\frac{k}{n} - x|}{\delta} + 1\Big]\Big[1 + \Big(\frac{k}{n} - x\Big)^2\Big]s_{k,n}(x) \\
&= 4(1 + x^2)(1 + \delta^2)\omega(f;\delta)\Big\{\sum_{k=0}^{\infty} s_{k,n}(x) + \sum_{k=0}^{\infty}\Big(\frac{k}{n} - x\Big)^2 s_{k,n}(x) \\
&+ \frac{1}{\delta}\sum_{k=0}^{\infty}\Big|\frac{k}{n} - x\Big|s_{k,n}(x) + \frac{1}{\delta}\sum_{k=0}^{\infty}\Big|\frac{k}{n} - x\Big|\Big(\frac{k}{n} - x\Big)^2 s_{k,n}(x)\Big\}
\end{aligned}$$

Here,

$$
\begin{aligned}
A &= \sum_{k=0}^{\infty} \left|\frac{k}{n} - x\right| s_{k,n}(x) \\
B &= \sum_{k=0}^{\infty} \left|\frac{k}{n} - x\right| \left(\frac{k}{n} - x\right)^2 s_{k,n}(x)
\end{aligned}
$$

Starting from A, we can write

$$
A = \sum_{k=0}^{\infty} \left|\frac{k}{n} - x\right| \left(e^{-nx}\frac{(nx)^k}{k!}\right)^{\frac{1}{2}} \left(e^{-nx}\frac{(nx)^k}{k!}\right)^{\frac{1}{2}}
$$

Applying Cauchy-Schwarz inequality,

$$
\begin{aligned}
A &\leq \left(\sum_{k=0}^{\infty} \left(\frac{k}{n} - x\right)^2 e^{-nx}\frac{(nx)^k}{k!}\right)^{\frac{1}{2}} \left(\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!}\right)^{\frac{1}{2}} \\
&= \left(\sum_{k=0}^{\infty} \left(\frac{k}{n} - x\right)^2 e^{-nx}\frac{(nx)^k}{k!}\right)^{\frac{1}{2}} [(S_n 1)(x)]^{\frac{1}{2}} \\
&= \left(\sum_{k=0}^{\infty} \left(\frac{k}{n} - x\right)^2 e^{-nx}\frac{(nx)^k}{k!}\right)^{\frac{1}{2}}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\sum_{k=0}^{\infty} \left(\frac{k}{n} - x\right)^2 e^{-nx}\frac{(nx)^k}{k!} &= \sum_{k=0}^{\infty} \left[\frac{k^2}{n^2} + 2x\frac{k}{n} + x^2\right] s_{k,n}(x) \\
&= \sum_{k=0}^{\infty} \frac{k^2}{n^2} s_{k,n}(x) - 2x \sum_{k=0}^{\infty} \frac{k}{n} s_{k,n}(x) + x^2 \sum_{k=0}^{\infty} s_{k,n}(x) \\
&= (S_n t^2)(x) - 2x(S_n t)(x) + x^2(S_n 1)(x).
\end{aligned}
$$

Since we calculated this values before,

$$
\sum_{k=0}^{\infty} \left(\frac{k}{n} - x\right)^2 e^{-nx}\frac{(nx)^k}{k!} = x^2 + \frac{x}{n} - 2xx + x^2.1 = \frac{x}{n} \qquad ...(2.4.5)
$$

Therefore,

$$
A \leq \frac{\sqrt{x}}{\sqrt{n}} \qquad ...(2.4.6)
$$

Similarly, we can write

$$B = \sum_{k=0}^{\infty}\Big|\frac{k}{n} - x\Big|\Big(e^{-nx}\frac{(nx)^k}{k!}\Big)^{\frac{1}{2}}\Big(\frac{k}{n} - x\Big)^2\Big(e^{-nx}\frac{(nx)^k}{k!}\Big)^{\frac{1}{2}}$$

Applying Cauchy-Schwartz inequality,

$$B \leq \Big(\sum_{k=0}^{\infty}\Big(\frac{k}{n} - x\Big)^2 e^{-nx}\frac{(nx)^k}{k!}\Big)^{\frac{1}{2}}\Big(\sum_{k=0}^{\infty}\Big(\frac{k}{n} - x\Big)^4 e^{-nx}\frac{(nx)^k}{k!}\Big)^{\frac{1}{2}}$$

We know that

$$\sum_{k=0}^{\infty}\Big(\frac{k}{n} - x\Big)^4 e^{-nx}\frac{(nx)^k}{k!} = \frac{3x^2}{n^2} + \frac{x}{n^3}$$

Therefore,

$$B \leq \frac{\sqrt{x}}{\sqrt{n}}\Big(\frac{3x^2}{n^2} + \frac{x}{n^3}\Big)^{\frac{1}{2}} \qquad ...(2.4.7)$$

Using (2.4.5),(2.4.6) and (2.4.7),

$$|(S_n f)(x) - f(x)| \leq 4(1 + x^2)(1 + \delta^2)\omega(f;\delta)\Big[1 + \frac{x}{n} + \frac{1}{\delta}\frac{\sqrt{x}}{\sqrt{n}} + \frac{1}{\delta}\frac{\sqrt{x}}{\sqrt{n}}\Big(\frac{3x^2}{n^2} + \frac{x}{n^3}\Big)^{\frac{1}{2}}\Big]$$

Dividing both sides with $(1 + x^2)^2$,

$$\begin{aligned}
\frac{|(S_n f)(x) - f(x)|}{(1 + x^2)^2} &\leq 4(1 + \delta^2)\omega(f;\delta)\Big[\frac{1}{1 + x^2} + \frac{x}{1 + x^2}\frac{1}{n} + \frac{1}{\delta}\frac{1}{\sqrt{n}}\frac{\sqrt{x}}{1 + x^2} \\
&\quad + \frac{1}{\delta}\frac{1}{\sqrt{n}}\frac{\sqrt{x}}{1 + x^2}\Big(\frac{3x^2}{n^2} + \frac{x}{n^3}\Big)^{\frac{1}{2}}\Big] \\
&= 4(1 + \delta^2)\omega(f;\delta)\Big[\frac{1}{1 + x^2} + \frac{x}{1 + x^2}\frac{1}{n} + \frac{1}{\delta}\frac{1}{\sqrt{n}}\frac{\sqrt{x}}{1 + x^2} \\
&\quad + \frac{1}{\delta}\frac{1}{\sqrt{n}}\frac{1}{n}\frac{\sqrt{x}}{1 + x^2}\Big(3x^2 + \frac{x}{n}\Big)^{\frac{1}{2}}\Big] \\
&= 4(1 + \delta^2)\omega(f;\delta)\Big[\frac{1}{1 + x^2} + \frac{x}{1 + x^2} + \frac{1}{\delta}\frac{1}{\sqrt{n}}\frac{\sqrt{x}}{1 + x^2} \\
&\quad + \frac{1}{\delta}\frac{1}{\sqrt{n}}\frac{\sqrt{x}}{1 + x^2}(3x^2 + x)^{\frac{1}{2}}\Big] \qquad ...(2.4.8)
\end{aligned}$$

On the other hand,since

$$1 \leq 1 + x^2, \quad then \quad \frac{1}{1+x^2} \leq 1,$$

since

$$x \leq 1 + x^2, \quad then \quad \frac{x}{1+x^2} \leq 1,$$

and since

$$\sqrt{x} \leq x \leq 1 + x^2, \quad then \quad \frac{\sqrt{x}}{1+x^2} \leq 1.$$

Also,

$$\frac{\sqrt{x(3x^2+x)}}{1+x^2} \leq 2.$$

$$\frac{\sqrt{x}(3x^2+x)^{\frac{1}{2}}}{1+x^2} = \frac{\sqrt{x(3x^2+x)}}{1+x^2} = \frac{\sqrt{3x^3+x^2}}{\sqrt{(1+x^2)^2}} = \sqrt{\frac{3x^3}{(1+x^2)^2} + \frac{x^2}{(1+x^2)^2}}$$

$$\frac{x^3}{(1+x^2)^2} = \frac{x^2}{1+x^2}\frac{x}{1+x^2} < 1.1 = 1 \quad \Rightarrow \quad \frac{x^3}{(1+x^2)^2} < 1$$

and since

$$x^2 \leq 1 + x^2 < (1+x^2)^2 \quad \Rightarrow \quad \frac{x^3}{(1+x^2)^2} < 1,$$

then

$$\frac{\sqrt{x}(3x^2+x)^{\frac{1}{2}}}{1+x^2} \leq \sqrt{3.1+1} = 2.$$

Taking supremum for $x \geq 0$ into both sides in (2.4.8), we finally have

$$\|S_n f - f\|_{\rho^2} \leq 4(1+\delta^2)\omega(f;\delta)\Big(1 + 1 + \frac{1}{\delta}\frac{1}{\sqrt{n}}.1 + \frac{1}{\delta}\frac{1}{\sqrt{n}}.2\Big)$$

Choosing $\delta = \frac{1}{\sqrt{n}}$, we have

$$\begin{aligned} \|S_n f - f\|_{\rho^2} &\leq 4(1+\frac{1}{n})\omega(f;\frac{1}{\sqrt{n}})(1+1+1+2) \\ &\leq 40\omega(f;\frac{1}{\sqrt{n}}). \end{aligned}$$

□

CHAPTER 3

FUNCTIONS OF FINITE VARIATION

In this chapter, we introduce some basic facts of finite variation,$BV_{[a,b]} - space$, $TV_{[a,b]} - space$ and the relation between $TV_{[a,b]} - space$ and $AC_{[a,b]} - space$.

Definition 3.1. [2] A function $f(x)$, defined on the closed interval $[a, b]$ is said to be *increasing* if

$$f(x) \leq f(y) \quad for \quad x < y.$$

If $f(x) < f(y)$ for $x < y$, then $f(x)$ is said to be a $strictly\ \ increasing\ \ function$. Also, a function $f(x)$ is said to be $decreasing(strictly\ \ decreasing)$ if $f(x) \geq f(y)[f(x) > f(y)]$ for $x < y$. Increasing and decreasing functions are called $monotonic\ \ or\ \ monotone(strictly$ $monotonic)$.

Monotonic functions will always be considered finite.
Let $f(x)$ be an increasing function defined on $[a, b]$ and let

$$a \leq x_0 < b.$$

It is elementary and well-known fact that for any sequence of points $x_1, x_2, x_3, ...$ tending to x_0 and lying to the right of x_0,

$$x_n \to x_0, \quad x_n > x_0$$

the limit

$$lim_{n\to\infty} f(x_n)$$

exists and is finite.

3.1 Functions of Finite Variation

Let a function $f(x)$ be defined and finite on the interval $[a, b]$. Subdivide $[a, b]$ into parts by means of the points

$$x_0 = a < x_1 < x_2 < ... < x_n = b$$

and form the sum

$$V = \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|.$$

Definition 3.2. [2] The least upper bound of the set of all possible sums V is called the total variation of the function $f(x)$ on $[a, b]$ and is designated by $V_{[a,b]}[f]$.

If $V_{[a,b]}[f] < \infty$, then $f(x)$ is said to be a function of finite variation on $[a, b]$. We also say that $f(x)$ has finite variation on $[a, b]$.

Theorem 3.1. *[2] A monotonic function on* $[a, b]$ *has finite variation on* $[a, b]$.

Proof. It is sufficent to prove the theorem for an increasing function. If $f(x)$ is increasing on $[a, b]$, then all the differences $f(x_{k+1}) - f(x_k)$ are non-negative and

$$V = \sum_{k=0}^{n-1} \{f(x_{k+1}) - f(x_k)\} = f(b) - f(a).$$

This completes the proof. □

Definition 3.3. [2] A finite function $f(x)$ defined on $[a, b]$ is said to satisfy a Lipschitz condition if there exists a constant K such that for any two points x and y in $[a, b]$,

$$|f(x) - f(y)| \leq K|x - y|.$$

If the function $f(x)$ has a derivative $f'(x)$ at every point of $[a, b]$ and if $f'(x)$ is bounded, then, as clear from the Mean Value Theorem,

$$f(x) - f(y) = f'(z)(x - y) \qquad (x < z < y).$$

If $f(x)$ satisfies the Lipschitz condition, then

$$|f(x_{k+1}) - f(x_k)| \leq K(x_{k+1} - x_k),$$

then

$$V \leq K(b - a)$$

and $f(x)$ is a function of finite variation.

Theorem 3.2. *[2] Every function of finite variation in $[a, b]$ is bounded in $[a, b]$.*

Proof. Let a function f be defined and finite on the interval $[a, b]$. For $a \leq x \leq b$,

$$V = |f(x) - f(a)| + |f(b) - f(x)| \leq V_{[a,b]}[f]$$

$$|f(x) - f(a)| \leq V_{[a,b]}[f].$$

Therefore, by the inverse triangle inequality;

$$|f(x)| \leq V_{[a,b]}[f] + |f(a)|.$$

□

Theorem 3.3. *[2] The sum, difference, and product of two functions of finite variation are functions of finite variation.*

Theorem 3.4. *[2] If $f(x)$ and $g(x)$ are functions of finite variation and if $g(x) \geq \sigma > 0$, then the quotient $\frac{f(x)}{g(x)}$ is a function of finite variation.*

Theorem 3.5. *[2] Let a finite function $f(x)$ be defined on $[a, b]$ and let $a < c < b$. Then*

$$V_{[a,b]}[f] = V_{[a,c]}[f] + V_{[c,b]}[f].$$

Theorem 3.6. *[2] A function $f(x)$ defined and finite on $[a, b]$ is a function of finite variation if and only if it is representable as the difference of two increasing functions.*

Proof. The sufficiency of the condition follows from theorems (3.1) and (3.3). To prove, we set

$$\pi(x) = V_{[a,x]}[f] \qquad (a < x \leq b)$$

$$\pi(a) = 0.$$

By virtue of thm (3.5), the function $\pi(x)$ is an increasing function. Setting

$$v(x) = \pi(x) - f(x), \qquad (1)$$

we obtain another increasing function $v(x)$. In fact, if $a \leq x < y \leq b$, then, by Theorem (3.5),

$$v(y) = \pi(y) - f(y) = \pi(x) + V_{[x,y]}[f] - f(y)$$

and hence

$$v(y) - v(x) = V_{[x,y]}[f] - [f(y) - f(x)].$$

However, from the very definition of total variation, it is clear that

$$f(y) - f(x) \leq V_{[x,y]}[f],$$

so that

$$v(y) - v(x) \geq 0.$$

Therefore,$v(x)$ is an increasing function. It remains to write the equality (1) in the form

$$f(x) = \pi(x) - v(x),$$

in order to obtain the desired representation of $f(x)$. □

Corollary 3.1. *If a function $f(x)$ has finite variation on $[a, b]$, then at almost every* *point of* $[a,b]$ *the derivative* $f'(x)$ *exists, and is finite.*

Furthermore, $f'(x)$ *is a summable function on* $[a,b]$.

3.2 Continuous Functions of Finite Variation

Theorem 3.7. *[2] Let a function* $f(x)$ *of finite variation be defined on the closed interval* $[a,b]$. *If* $f(x)$ *is continuous at the point* x_0, *then the function*

$$\pi(x) = V_{[a,x]}[f]$$

is also continuous at x_0.

Proof. Suppose that $x_0 < b$. We shall show that $\pi(x)$ is continuous on the right at the point x_0. For this purpose, taking an arbitrary $\varepsilon > 0$, we subdivide the segment $[x_0, b]$ by means of the points

$$x_0 < x_1 < ... < x_n = b$$

so that

$$V = \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| > V_{[x_0,b]}[f] - \varepsilon \qquad (2)$$

Since the sum V only increases when new points are added, we may suppose that

$$|f(x_1) - f(x_0)| < \varepsilon.$$

It follows from (2) that

$$\begin{aligned} V_{[x_0,b]}[f] &< \varepsilon + \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| + \sum_{k=1}^{n-1} |f(x_{k+1}) - f(x_k)| \\ &\leq 2\varepsilon + V_{[x_1,b]}[f]. \end{aligned}$$

Hence

$$V_{[x_0,x_1]}[f] < 2\varepsilon,$$

and consequently

$$\pi(x_1) - \pi(x_0) < 2\varepsilon.$$

This implies that

$$\pi(x_0 + 0) - \pi(x_0) < 2\varepsilon.$$

Since ε is arbitrary, we have

$$\pi(x_0 + 0) = \pi(x_0).$$

It can be shown in like manner that $\pi(x_0 - 0) = \pi(x_0)$,i.e, that $\pi(x)$ is continuous on the left(if $x_0 > a$) at the point x_0. □

Corollary 3.2. *A continuous function of finite variation can be written as the difference of two continuous increasing functions.*

In fact, if $f(x)$ is a continuous function of finite variation defined on $[a, b]$, then both of its increasing components

$$\pi(x) = V_{[a,x]}[f] \quad \textit{and} \quad \nu(x) = \pi(x) - f(x)$$

are continuous.

Let a continuous function $f(x)$ be defined on $[a, b]$. Subdivide $[a, b]$ by means of the points

$$a = x_0 < x_1 < x_2 < ... < x_n = b \qquad [max(x_{k+1} - x_k) = \lambda]$$

and form the sums

$$V = \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|, \qquad \Omega = \sum_{k=0}^{n-1} \omega_k,$$

where ω_k designates the oscillation of the function $f(x)$ in the interval $[x_k - x_{k+1}]$.

Now, we give two spaces $BV[a, b]$ and $TV[a, b]$ which has the same elements, but the definitions of their norm are different. It is important to see.

3.3 BV[a,b]-Space

We define $BV[a,b] - space$ as

$$BV[a,b] = \{f : [a,b] \to \mathbb{R} \mid V_{[a,b]}[f] < +\infty\}$$

with the norm $\|f\|_{BV} = V_{[a,b]}[f] + |f(c)|, \ \ a \leq c \leq b.$

Now, we show the norm properties of $BV[a,b] - space.$

i) $\|f\|_{BV} \geq 0,$

Proof. We know that $\|f\|_{BV} = V_{[a,b]}[f] + |f(c)|, \ \ a \leq c \leq b.$

Since $V_{[a,b]}[f] = \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| \geq 0$ and $|f(c)| \geq 0$, then $\|f\|_{BV} \geq 0.$ □

ii) $\|f\|_{BV} = 0$ if and only if $f = 0,$

Proof. ($\Rightarrow$:)Let $\|f\|_{BV} = V_{[a,b]}[f] + |f(c)| = 0, \ \ a \leq c \leq b$. Since $V_{[a,b]}[f] \geq 0$ and $|f(c)| \geq 0$, then $V_{[a,b]}[f] = 0$ and $|f(c)| = 0$.

This means that $f(x_1) = f(x_2) = ... = f(c) = ... = f(x_n) = 0$, that is, f is a zero function.

($\Leftarrow$:) Let $f = 0$. This means that the value of f is zero for every partition on $[a,b]$.

Hence, $|f(c)| = 0$ and $V_{[a,b]}[f] = 0$.

This completes the proof. □

iii) $\|f+g\|_{BV} \leq \|f\|_{BV} + \|g\|_{BV},$

Proof.

$$\begin{aligned}
\|f+g\|_{BV} &= V_{[a,b]}[f+g] + |(f+g)(c)| \\
&= \sum_{k=0}^{n-1} \Big|(f+g)(x_{k+1}) - (f+g)(x_k)\Big| + \Big|(f+g)(c)\Big| \\
&= \sum_{k=0}^{n-1} \Big|f(x_{k+1}) + g(x_{k+1}) - f(x_k) - g(x_k)\Big| + \Big|f(c) + g(c)\Big|
\end{aligned}$$

By the triangle inequality, we have

$$\begin{aligned}
\|f+g\|_{BV} &\leq \sum_{k=0}^{n-1} \left|f(x_{k+1}) - f(x_k)\right| + \sum_{k=0}^{n-1} \left|g(x_{k+1}) - g(x_k)\right| + \left|f(c)\right| + \left|g(c)\right| \\
&= \|f\|_{BV} + \|g\|_{BV}.
\end{aligned}$$

□

iv) $\|\alpha f\|_{BV} = |\alpha|\|f\|_{BV}$ for any scalar α.

Proof.

$$\begin{aligned}
\|\alpha f\|_{BV} &= V_{[a,b]}[\alpha f] + |(\alpha f)(c)| \\
&= \sum_{k=0}^{n-1} \left|(\alpha f)(x_{k+1}) - (\alpha f)(x_k)\right| + \left|(\alpha f)(c)\right| \\
&= \sum_{k=0}^{n-1} \left|\alpha f(x_{k+1}) - \alpha f(x_k)\right| + \left|\alpha\right|\left|f(c)\right| \\
&= \sum_{k=0}^{n-1} \left|\alpha\right|\left|f(x_{k+1}) - f(x_k)\right| + \left|\alpha\right|\left|f(c)\right| \\
&= |\alpha| \sum_{k=0}^{n-1} \left[|f(x_{k+1}) - f(x_k)| + |f(c)|\right] \\
&= |\alpha|\|f\|_{BV}.
\end{aligned}$$

□

3.4 TV[a,b]-Space

We define $TV[a,b] - space$ as

$$TV[a,b] = \{f : [a,b] \to \mathbb{R} \;|\; f \in BV[a,b]\}$$

with the seminorm $\|f\|_{TV} = V_{[a,b]}[f]$.

Now, we show the seminorm properties of $TV[a,b] - space$.

i) $||f||_{TV} \geq 0$.

Proof. $||f||_{TV} = V_{[a,b]}[f] = \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| \geq 0$. □

ii) $||f||_{TV} = 0 \quad \Rightarrow \quad f = 0$ is not required.

Proof. Let $||f||_{TV} = V_{[a,b]}[f] = 0$. Since $V_{[a,b]}[f] \geq 0$, then

$f(x_1) = f(x_2) = ... = f(x_n)$.

This does not mean $f = 0$. □

iii) $||f + g||_{TV} \leq ||f||_{TV} + ||g||_{TV}$,

Proof.

$$\begin{aligned} ||f + g||_{TV} &= V_{[a,b]}[f + g] \\ &= \sum_{k=0}^{n-1} |(f + g)(x_{k+1}) - (f + g)(x_k)| \\ &= \sum_{k=0}^{n-1} |f(x_{k+1}) + g(x_{k+1}) - f(x_k) - g(x_k)| \end{aligned}$$

By the triangle inequality, we have

$$\begin{aligned} ||f + g||_{TV} &\leq \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| + \sum_{k=0}^{n-1} |g(x_{k+1}) - g(x_k)| \\ &= V_{[a,b]}[f] + V_{[a,b]}[g] \\ &= ||f||_{TV} + ||g||_{TV}. \end{aligned}$$

□

iv) $||\alpha f||_{TV} = |\alpha| ||f||_{TV}$ for any scalar α.

Proof.

$$\begin{aligned} ||\alpha f||_{TV} &= V_{[a,b]}[\alpha f] \\ &= \sum_{k=0}^{n-1} |(\alpha f)(x_{k+1}) - (\alpha f)(x_k)| \end{aligned}$$

Here we find

$$\begin{aligned}
||\alpha f||_{TV} &= \sum_{k=0}^{n-1} |\alpha f(x_{k+1}) - \alpha f(x_k)| \\
&= \sum_{k=0}^{n-1} |\alpha||f(x_{k+1}) - f(x_k)| \\
&= |\alpha| \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| \\
&= |\alpha| V_{[a,b]}[f] \\
&= |\alpha| ||f||_{TV}.
\end{aligned}$$

□

Convergence in the TV-norm is defined as;

Definition 3.4. [1] If the convergence in variation of $(T_w f)_{w\in J}$ where $J = \mathbb{N}\ \ or\ \ \mathbb{R}^+$ towards f, such as

$$lim_{w\to\infty} V_I[T_w f - f] = 0$$

holds for a sequence $(f_w)_{w\in J}$ in AC(I), then also $f \in AC(I)$ and

$$V_I[f_w - f] = \int_I |f'_w(u) - f'(u)| du.$$

3.5 Relation Between $AC[a, b]$ and $TV[a, b] - space$

Lemma 3.1. *[1] AC[a,b] is a closed subspace of TV[a,b] in the variation seminorm.*

Proof. Let $(f_n)_{(n\in\mathbb{N})}$ be a sequence of functions in AC[a,b] that converges in the *variation* seminorm to f. Then, given $\varepsilon > 0$, there exists a natural number n such that

$$V_{[a,b]}[f_n - f] = ||f_n - f||_{TV} < \varepsilon/2$$

As, $f_n \in AC[a, b]$, then there exists a $\delta > 0$ such that for every finite set $[a_i, b_i] : i = 1, 2, 3, .., k$

of non-overlapping intervals for which $\sum_{i=1}^{k}(b_i - a_i) < \delta$, we have

$$\sum_{i=1}^{k} |f_n(b_i) - f_n(a_i)| < \frac{\varepsilon}{2}$$

Hence, there follows

$$\begin{aligned}
\sum_{i=1}^{k} \Big|f(b_i) - f(a_i)\Big| &= \sum_{i=1}^{k} \Big|f(b_i) - f_n(b_i) + f_n(b_i) - f(a_i) - f_n(a_i) + f_n(a_i)\Big| \\
&= \sum_{i=1}^{k} \Big|(f - f_n)(b_i) - (f - f_n)(a_i)\Big| + \sum_{i=1}^{k} \Big|f_n(b_i) - f_n(a_i)\Big| \\
&\leq V_{a,b}[f - f_n] + \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

Therefore, $f \in AC[a, b]$ and so AC[a,b] is closed.

This completes the proof. □

3.6 The Stieltjes Integral

Let $f(x)$ and $g(x)$ be finite functions defined on the closed interval $[a, b]$. Subdivide $[a, b]$ into parts of means of the points

$$x_0 = a < x_1 < x_2 < ... < x_n = b,$$

choose a point ε_k in $[x_k, x_{k+1}]$ for $k = 0, 1, ..., n - 1$, and form the sum

$$\sigma = \sum_{k=0}^{n-1} f(\varepsilon_k)[g(x_{k+1}) - g(x_k)].$$

If the sum σ tends to a finite limit I as

$$\lambda = max(x_{k+1} - x_k) \to 0,$$

independently of both the method of subdivision and the choice of the points ε_k, this limit is called *the Stieltjes Integral of the function* $f(x)$ with respect to the function

$g(x)$ and designated by

$$\int_a^b f(x)dg(x) \ \ or \ \ (S)\int_a^b f(x)dg(x).$$

It is clear that the Riemann integral is a special case of the Stieltjes integral, obtained by setting $g(x) = x$.

Properties of Stieltjes Integral:

i) $\int_a^b \left[f_1(x) + f_2(x)\right]dg(x) = \int_a^b f_1(x)dg(x) + \int_a^b f_2(x)dg(x)$.

ii) $\int_a^b f(x)d\left[g_1(x) + g_2(x)\right] = \int_a^b f(x)dg_1(x) + \int_a^b f(x)dg_2(x)$

iii) If k and l are constants, then

$$\int_a^b kf(x)dlg(x) = kl\int_a^b f(x)dg(x)$$

iv) If $a < c < b$, and all three integrals involved in the equality

$$\int_a^b f(x)dg(x) = \int_a^c f(x)dg(x) + \int_c^b f(x)dg(x) \quad ...(*)$$

exists, then $(*)$ holds.

v) The existence of one of the integrals $\int_a^b f(x)dg(x)$ and $\int_a^b g(x)df(x)$ implies the existence of the other. In this case, the equality

$$\int_a^b f(x)dg(x) + \int_a^b g(x)df(x) = \left[f(x)g(x)\right]_a^b$$

holds, where

$$\left[f(x)g(x)\right]_a^b = f(b)g(b) - f(a)g(a)$$

Theorem 3.8. *[2] The integral*

$$\int_a^b f(x)dg(x)$$

exists if the function $f(x)$ is continuous and $g(x)$ is of finite variation on $[a,b]$.

Theorem 3.9. *[2] If the function $f(x)$ is continuous on $[a,b]$ and if the function $g(x)$ has a Riemann integrable derivative $g'(x)$ at every point of $[a,b]$, then*

$$(S)\int_a^b f(x)dg(x) = (R)\int_a^b f(x)g'(x)dx.$$

Theorem 3.10. *[2] If the function $f(x)$ is continuous on $[a,b]$ and $g(x)$ has finite variation on $[a,b]$, then*

$$\Big|\int_a^b f(x)dg(x)\Big| \leq M(f)V_{[a,b]}[g]$$

where $M(f) = max|f(x)|$.

Proof. Let f be continuous on $[a,b]$ and let g has finite variation on $[a,b]$.

$$\begin{aligned}
\Big|\int_a^b f(x)dg(x)\Big| &\leq \int_a^b |f(x)||dg(x)| \\
&\leq \int_a^b max|f(x)||dg(x)| \\
&= M(f)\int_a^b |dg(x)| \\
&= M(f)lim_{n\to\infty}\sum_{i=1}^{n} |g(x_{i+1}) - g(x_i)| \\
&= M(f)V_{[a,b]}[g].
\end{aligned}$$

□

Theorem 3.11. *[2] Let $g(x)$ be a function of finite variation defined on the closed interval $[a,b]$, and let $f_n(x)$ be a sequence of continuous functions on $[a,b]$, which converges uniformly to the (necesserily continuous) function $f(x)$. Then*

$$lim_{n\to\infty}\int_a^b f_n(x)dg(x) = \int_a^b f(x)dg(x).$$

CHAPTER 4

BERNSTEIN POLYNOMIALS

In this chapter, we show the variation detracting property of Bernstein polynomials. We show the convergence of Bernstein polynomials on the variation seminorm and its rate of convergence on the variation seminorm.

4.1 Approximation by Bernstein Polynomials

Consider the Bernstein polynomial of $f : [0,1] \to \mathbb{R}$ of degree n,

$$(B_n f)(x) := \sum_{k=0}^{n} f\Big(\frac{k}{n}\Big) p_{k,n}(x),$$

where $p_{k,n}(x) := \binom{n}{k} x^k (1-x)^{n-k} \geq 0$.

$$T_{r,n}(x) = \sum_{k=0}^{n} (k-nx)^r p_{k,n}(x), \quad (r \in \mathbb{N}_0)$$

Then for $X := x(1-x)$, there holds for all $x \in \mathbb{R}$ the identities

$$T_{r,n}(x) = \begin{cases} 1, & r=0 \\ 0, & r=1 \\ nX, & r=2 \\ nX(1-2x), & r=3 \\ 3(nX)^2 + (1-6X)nX, & r=4 \\ (10(nX)^2 + (1-12X)nX)(1-2x), & r=5 \\ 15(nX)^3 + (25-130X)(nX)^2 + (1-30X+120X^2)nX, & r=6 \\ (105(nX)^3 + (56-462X)(nX)^2 + (1-60X+360X^2)nX)(1-2x), & r=7. \end{cases}$$

We know that $$\frac{d}{dx}p_{k,n}(x) = \frac{(k-nx).p_{k,n}(x)}{X}.$$

There follow by differentiation the two fundamental representation for $(B_n f)'$,

$$(B_n f)'(x) = n\sum_{k=0}^{n-1}\left[f\Big(\frac{k+1}{n}\Big) - f\Big(\frac{k}{n}\Big)\right]p_{k,n-1}(x) \quad \ldots \ (3)$$

$$(B_n f)'(x) := X^{-1}\sum_{k=0}^{n} f\Big(\frac{k}{n}\Big)(k-nx)p_{k,n}(x)$$

Proposition 4.1. *[5] If $f \in TV[0,1]$, then*

$$V_{[a,b]}[B_n f] \leq V_{[a,b]}[f]. \qquad (4.1)$$

Proof. Observe that by the representation (3)

$$\begin{aligned}
V_{[a,b]}[B_n f] &= \int_0^1 |(B_n f)'(x)|dx \\
&= \int_0^1 \left|n\sum_{k=0}^{n-1}\left[f\Big(\frac{k+1}{n}\Big) - f\Big(\frac{k}{n}\Big)\right]\binom{n-1}{k}x^k(1-x)^{(n-1-k)}\right|dx \\
&\leq n\sum_{k=0}^{n-1}\left|f\Big(\frac{k+1}{n}\Big) - f\Big(\frac{k}{n}\Big)\right|\binom{n-1}{k}\int_0^1 x^k.(1-x)^{(n-1-k)}dx \\
&\leq V_{[a,b]}[f]
\end{aligned}$$

noting that the evaluation of the beta integral

$$n\binom{n-1}{k}\int_0^1 x^k(1-x)^{(n-1-k)}dx = n\binom{n-1}{k}B(k+1, n-k) = 1,$$

where the Beta-integral is defined as;

$$B(m+1, n+1) = \int_0^1 u^m(1-u)^n du$$

□

Proposition 4.2. *Let $f \in TV[0,1]$. There holds $f \in AC[0,1]$ if and only if*

$$lim_{n\to\infty} V_{[0,1]}[B_n f - f] = 0.$$

One would think that convergence in variation of the $B_n f$ *to* f would hold provided $f \in TV[0,1]$. But one needs the stronger class AC[0,1]. This surprising fact follows from the result that AC[0,1] is a closed subspace of TV[0,1]. Since the Bernstein polynomials $B_n f$ belongs to AC[0,1], the limit function f has to belong to AC[0,1],too. Corresponding results hold whenever a function f is approximated by a sequence $(f_n)_{(n\in\mathbb{N})} \subset AC$.

In addition to the property (4.1), the operator B_n is bounded with respect to the BV-norm, namely

$$||B_n f||_{BV} \le ||f||_{BV} := V_{[0,1]}[f] + |f(0)|.$$

This result follows from proposition (4.1), since $B_n f(0) = f(0)$. The operator B_n satisfying the inequality (4.1) will be called $variation\ \ detracting(or\ \ non{-}augmenting)$. Lorentz actually considered a modification of the Bernstein polynomials for $L^1(0,1) - functions$ in regard to proposition (4.2), one due to Kantorovich [7], namely the polynomials

$$(K_n f)(x) := \sum_{k=0}^{n} (n+1) p_{k,n}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(u)du \quad (0 \le x \le 1),$$

for which $(B_n f)'(x) = (K_{n-1} f')(x)$, provided $f \in AC[0,1]$(yielding that $f' \in L(0,1)$ and $f(x) = \int_0^x f'(u)du + f(0)$. Thus he basically proved that

$$V_{[0,1]}[B_n f - f] = \int_0^1 |(K_{n-1} f')(x) - f'(x)|dx \to 0 \quad as\ \ n \to \infty.$$

The Kantorovich operator K_n is also variation detracting as well as bounded with respect to the BV-norm.Indeed,

Proposition 4.3. *If* $f \in TV[0,1]$, *then*

$$V_{[0,1]}[K_n f] \leq V_{[0,1]}[f] \quad (n \in \mathbb{N}), \qquad (4.3)$$

$$\|K_n f\|_{BV} \leq 2\|f\|_{BV} \quad (n \in \mathbb{N}), \qquad (4.4).$$

Proof. Defining

$$F_{k,n} := (n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(u)du = \int_0^1 f\Big(\frac{v+k}{n+1}\Big)dv \quad (0 \leq k \leq n),$$

one has as in (3),

$$(K_n f)'(x) = n\sum_{k=0}^{n-1}(F_{k+1,n} - F_{k,n})p_{k,n-1}(x).$$

$$\begin{aligned} V_{[0,1]}[K_n f] &= \int_0^1 |(K_n f)'(x)|dx \\ &\leq n\sum_{k=0}^{n-1}|F_{k+1,n} - F_{k,n}|\binom{n-1}{k}\int_0^1 x^k(1-x)^{n-k-1}dx \\ &= \sum_{k=0}^{n-1}|F_{k+1,n} - F_{k,n}|. \end{aligned}$$

So let us show that $\sum_{k=0}^{n-1}|F_{k+1,n} - F_{k,n}| \leq V_{[0,1]}[f]$. With the abbreviation $v_{-1} := 0$ and $v_k := (v+k)/(n+1)$ for $0 \leq k \leq n$, and $v_{n+1} := 1$ we have

$$F_{k+1,n} - F_{k,n} = \int_0^1 (f(v_{k+1}) - f(v_k))dv \quad (0 \leq k \leq n),$$

and there follows

$$\begin{aligned} \sum_{k=0}^{n-1}|F_{k+1,n} - F_{k,n}| &\leq \int_0^1 \sum_{k=0}^{n-1}|f(v_{k+1}) - f(v_k)|dv \\ &\leq \int_0^1 \sum_{k=-1}^{n}|f(v_{k+1}) - f(v_k)|dv \\ &\leq V_{[0,1]}[f] \end{aligned}$$

Since the inner term of the foregoing integral has itself the estimate $V_{[0,1]}[f]$, noting that $(v_n)_{k=-1}^{n+1}$ is a particular partition of [0, 1]. This proves (4.3).
As to (4.4), we have

$$\begin{aligned}
||K_n f||_{BV} &= V_{[0,1]}[K_n f] + |(K_n f)(0)| \\
&\leq V_{[0,1]}[f] + |f(0)| + |(K_n f)(0) - f(0)| \\
&= ||f||_{BV} + |(K_n f)(0) - f(0)|
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|(K_n f)(0) - f(0)| &= \left|(n+1)\int_0^{\frac{1}{n+1}} (f(u) - f(0))du\right| \\
&\leq (n+1)\int_0^{\frac{1}{n+1}} \{|(f(u) - f(0)| + |f(1) - f(u)|\}du \\
&\leq V_{[0,1]}[f]
\end{aligned}$$

This establishes (4.4) and so the proposition.

□

We can also show the convergence of the Kantorovich operators in variation seminorm [13], but we do not concern with it in this thesis.

4.2 Convergence in Variation Seminorm

Theorem 4.1. *If f is bounded on [0,1] and $f'''(x)$ exists in a certain point $x \in [0,1]$, then*

$$lim_{n\to\infty} n\Big[(B_n f)'(x) - f'(x)\Big] = \frac{(1-2x)}{2}f''(x) + \frac{x(1-x)}{2}f'''(x).$$

Proof. For the proof of this theorem, we benefit from [14]. By Taylor's formula, we have

$$f\Big(\frac{k}{n}\Big) = f(x) + \Big(\frac{k}{n} - x\Big)f'(x) + \Big(\frac{k}{n} - x\Big)^2\frac{f''(x)}{2} + \Big(\frac{k}{n} - x\Big)^3\Big[\frac{1}{6}f'''(x) + h\Big(\frac{k}{n} - x\Big)\Big]$$

where $h(y) := h_x(y)$ is bounded for all y, $|h(y)| \leq M$ and converges to zero with y. So

$$\begin{aligned}
(B_n f)'(x) &:= X^{-1}. \sum_{k=0}^{n} f\Big(\frac{k}{n}\Big).(k-nx).p_{k,n}(x) \\
&= \frac{f(x)}{X}\sum_{k=0}^{n}(k-nx)p_{k,n}(x) + \frac{f'(x)}{X}\sum_{k=0}^{n}(k-nx)\Big(\frac{k}{n}-x\Big)p_{k,n}(x) \\
&+ \frac{f''(x)}{2X}\sum_{k=0}^{n}(k-nx)\Big(\frac{k}{n}-x\Big)^2 p_{k,n}(x) + \frac{f'''(x)}{6X}\sum_{k=0}^{n}(k-nx)\Big(\frac{k}{n}-x\Big)^3 p_{k,n}(x) \\
&+ \frac{1}{X}\sum_{k=0}^{n}(k-nx)\Big(\frac{k}{n}-x\Big)^3 h\Big(\frac{k}{n}-x\Big)p_{k,n}(x) \\
&= \frac{f'(x)}{nX}nX + \frac{f''x}{2Xn^2}nX(1-2x) + \frac{f'''(x)}{6Xn^3}\Big[3(nX)^2 + (1-6X)nX\Big] + R_n(x) \\
&= f'(x) + \frac{(1-2x)f''(x)}{2n} + \frac{Xf'''(x)}{2n} + \frac{(1-6X)f'''(x)}{6n^2} + R_n(x)
\end{aligned}$$

where the remainder is given by $R_n(x)$.

Firstly, we calculate the remainder;

$$\begin{aligned}
R_n(x) &= \frac{1}{n^3}\sum_{k=0}^{n}\frac{(k-nx)^4}{X}h\Big(\frac{k}{n}-x\Big)p_{k,n}(x) \\
&= \frac{1}{n^3}\Big\{\sum_{|\frac{k}{n}-x|<\delta} + \sum_{|\frac{k}{n}-x|\geq\delta}\Big\}\frac{(k-nx)^4}{X}h\Big(\frac{k}{n}-x\Big)p_{k,n}(x) \\
&= \sum_1 + \sum_2, \, say.
\end{aligned}$$

We know that

$$\frac{(1-6X)f'''(x)}{6n^2} \leq \frac{|f'''(x)|}{6n^2}.$$

So that it can be neglected.

So let us study only the Remainder $R_n(x)$.

For any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|h(y)| < \varepsilon$ for any $0 \leq y \leq \delta$.

Now, one has by the case $r = 4$ of $T_{r,n}(x)$, since $0 \leq X \leq \frac{1}{4}$, where $X := x(1-x)$, we calculate I_1 and I_2.

Lets start with I_1.

$$\begin{aligned}
\sum_1 &= \frac{1}{n^3}\sum_{|\frac{k}{n}-x|<\delta}\frac{(k-nx)^4}{X}h\Big(\frac{k}{n}-x\Big)p_{k,n}(x)\\
&\le \frac{\varepsilon}{n^3X}\sum_{|\frac{k}{n}-x|<\delta}(k-nx)^4 p_{k,n}(x)\\
&= \frac{\varepsilon}{n^3X}3(nX)^2+\frac{\varepsilon}{n^3X}(1-6X)nX\\
&= \frac{3\varepsilon n^2X+\varepsilon(1-6X)n}{n^3}\\
&= \frac{3\varepsilon X}{n}+\frac{\varepsilon}{n^2}(1-6X)\le\frac{2\varepsilon}{n}.
\end{aligned}$$

$$\begin{aligned}
\sum_2 &= \frac{1}{n^3}\sum_{|\frac{k}{n}-x|\ge\delta}\frac{(k-nx)^4}{X}h\Big(\frac{k}{n}-x\Big)p_{k,n}(x)\\
&\le \frac{1}{n^3}\sum_{k=0}^{n}\frac{(k-nx)^4}{X}\frac{(k-nx)^2}{n^2\delta^2}h\Big(\frac{k}{n}-x\Big)p_{k,n}(x)\\
&= \frac{M}{n^3X\delta^2}\sum_{k=0}^{n}\frac{(k-nx)^6}{n^2}p_{k,n}(x)
\end{aligned}$$

Then we have

$$\begin{aligned}
\sum_2 &= \frac{M}{n^5X\delta^2}\sum_{i=1}^{3}a_i(x)(nX)^i\\
&= \frac{M}{\delta^2}\sum_{i=1}^{3}\frac{a_i(x)(X)^{i-1}}{h^{5-i}}
\end{aligned}$$

where $a_3 := 15,\ \ |a_2(x)| := |25-130X| \le 25$ and $|a_1(x)| := |1-30X+120X^2| \le 16$.

collecting the results

$$n|R_n(x)| \le 2\varepsilon+\frac{56M}{\delta^2 n} < 3\varepsilon \quad for\ \ n \ge \frac{56M}{\varepsilon\delta^2}$$

This result shows that the error $(B_nf)'(x)-f'(x)$ is of order no better than $O\Big(\frac{1}{n}\Big)$ if, e.g., $f'''(x)\neq 0$, so even for very smooth functions.

$\square$

4.3 The Rate of Convergence in Variation Seminorm

The Jackson-type inequality for the approximation of a function g by the Bernstein polynomials $B_n g$ with respect to the variation seminorm reads

Proposition 4.4. *If* $g'' \in AC[0,1]$, *then*

$$V_{[0,1]}[B_n g - g] \leq \frac{c}{n}\Big\{V_{[0,1]}[g] + V_{[0,1]}[g'']\Big\} \quad (n \geq 3).$$

Proof. This time we make use of Taylor's formula with integral remainder term,

$$g(t) = g(x) + (t-x)g'(x) + (t-x)^2\frac{g''(x)}{2} + \frac{1}{6}\int_0^{t-x}(t-x-u)^2 g'''(x+u)du$$

As in proof of theorem (4.1),

$$(B_n g)'(x) = g'(x) + \frac{(1-2x)}{2n}g''(x) + (R_n g)(x),$$

the remainder now being

$$(R_n g)(x) := \frac{1}{6X}\sum_{k=0}^{n}\Big\{\int_0^{\frac{k}{n}-x}\Big(\frac{k}{n} - x - u\Big)^2 g'''(x+u)du(k-nx)p_{k,n}(x)\Big\}$$

Thus, since $|1-2x| \leq 1$,

$$\Big\|(B_n g)' - g'\Big\|_{L^1(0,1)} \leq \frac{1}{2n}\Big\|g''\Big\|_{L^1(0,1)} + \Big\|R_n g\Big\|_{L^1(0,1)} \quad (4.5)$$

Substituting $x + u = v$, and noting that $|\frac{k}{n} - v|^2 \leq |\frac{k}{n} - x|^2$ for $\frac{k}{n} \leq v \leq x$ and $x \leq v \leq \frac{k}{n}$, we obtain for the remainder term

$$\begin{aligned}
|(R_n g)(x)| &\leq \frac{1}{6Xn^2}\sum_{k=0}^{n}|k-nx|^3\Big|\int_x^{\frac{k}{n}} g'''(v)dv\Big|p_{k,n}(x)\\
&= \frac{1}{6Xn^2}\sum_{k=0}^{n}(k-nx)^3\int_x^{\frac{k}{n}}|g'''(v)|dv\,p_{k,n}(x)
\end{aligned}$$

Continuing with the inequality,

$$\begin{aligned}
\|R_n g\|_{L^1(0,1)} &\leq \frac{1}{6n^2}\int_0^1 \sum_{k=0}^{n} \frac{(k-nx)^3}{X} \int_x^{\frac{k}{n}} |g'''(v)| dv p_{k,n}(x) dx \\
&= \frac{1}{6n^2}\int_0^1 \sum_{k=0}^{n} \frac{(k-nx)^3}{X} \int_0^{\frac{k}{n}} |g'''(v)| dv p_{k,n}(x) dx - A_n g
\end{aligned}$$

Here we have added and then substracted term

$$\begin{aligned}
A_n g &:= \frac{1}{6n^2}\int_0^1 \sum_{k=0}^{n} \frac{(k-nx)^3}{X} \int_0^{x} |g'''(v)| dv p_{k,n}(x) dx \\
&= \frac{1}{6n^2}\int_0^1 n(1-2x) \int_0^{x} |g'''(v)| dv dx
\end{aligned}$$

which is in absolute value less than $\frac{1}{6n^2}\int_0^1 |g'''(v)| dv$.

Applying Fubini's theorem;

$$\|R_n g\|_{L^1(0,1)} \leq \frac{1}{6n^2}\sum_{k=0}^{n} \Big| \int_0^{\frac{k}{n}} |g'''(v)| dv \int_0^1 \frac{(k-nx)^3}{X} p_{k,n}(x) dx \Big| + |A_n g|.$$

The inner integral can be evaluated and estimated by;

$$\Big| \int_0^1 \frac{(k-nx)^3}{X} p_{k,n}(x) dx \Big| = \Big| \frac{2(2k-n)n}{(n+1)(n+2)} \Big| \leq \frac{2n^2}{(n+1)(n+2)},$$

and summing the former integral over all $k = 0, 1, 2, ..., n$, yields

$$\begin{aligned}
\|R_n g\|_{L^1(0,1)} &\leq \frac{1}{6n^2}\frac{2n^2(n+1)}{(n+1)(n+2)}\int_0^1 |g'''(v)| dv + |A_n g| \\
&= \frac{1}{3}\Big(\frac{1}{n+2} + \frac{1}{2n}\Big)\int_0^1 |g'''(v)| dv \\
&\leq \frac{1}{2n}\int_0^1 |g'''(v)| dv = \frac{1}{2n}\|g'''\|_{L^1(0,1)}
\end{aligned}$$

We need to give a definition to find the result,

Definition 4.1. [4] (Stein's Inequality) Let $f(x), f^1(x), f^2(x), ...f^{(n)}(x)$ be continuous and bounded on the interval $-\infty < x < +\infty$. Then:

$$||f^{(k)}(x)||_\infty^n \leq C_{k,n}||f(x)||_\infty^{n-k}||f^{(n)}(x)||_\infty^k$$

where $0 < k < n$, and $C_{k,n} = (K_{n-k})^n/(K_n)^{n-k}$, and where

$$K_i = 4/\pi \sum_{k=0}^{\infty}(-)^k/(2k+1)^{i+1}$$

for even i, while $K_i = 4/\pi \sum_{k=0}^{\infty} 1/(2k+1)^{i+1}$ for odd i.

Returning to proof, inserting this estimate for $||R_n g||_{L^1(0,1)}$ into (3.5), the proof finally follows by collecting the estimates, observing Stein's inequality,

$$||g''||_{L^1} \leq C\sqrt{||g'||_{L^1}||g'''||_{L^1}} \leq C(||g'||_{L^1} + ||g'''||_{L^1})$$

for a constant $C > 1$.

□

CHAPTER 5

SZÁSZ-MIRAKJAN OPERATORS

In this chapter, we show the variation detracting property of Szász - Mirakjan operators. We investigate the convergence of Szász-Mirakjan operators on the variation seminorm and its rate of convergence on the variation seminorm.

5.1 Approximation by the Szász-Mirakjan Operator

Consider the counterpart of the Bernstein operator on the positive real line, namely the above operator and its Kantorovich-type variant

$$(S_n f)(x) := \sum_{k=0}^{\infty} f\Big(\frac{k}{n}\Big) s_{k,n}(x) \quad (x \geq 0),$$

$$(S_n^* f)(x) := \sum_{k=0}^{\infty} \Big(n \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(u)du\Big) s_{k,n}(x) \quad (x \geq 0).$$

where $s_{k,n}(x) := exp(-nx)\frac{(nx)^k}{k!} \quad for \;\; x \geq 0$. Here

$$\begin{aligned}
(S_n f)'(x) &= -n \sum_{k=0}^{\infty} f\Big(\frac{k}{n}\Big) s_{k,n}(x) + \sum_{k=0}^{\infty} f\Big(\frac{k}{n}\Big)\frac{n^k k x^{k-1}}{k!} \\
&= -n \sum_{k=0}^{\infty} f\Big(\frac{k}{n}\Big) s_{k,n}(x) + \sum_{k=1}^{\infty} f\Big(\frac{k}{n}\Big)\frac{n^k x^{k-1}}{(k-1)!} \\
&= -n \sum_{k=0}^{\infty} f\Big(\frac{k}{n}\Big) s_{k,n}(x) + \sum_{k=0}^{\infty} f\Big(\frac{k+1}{n}\Big)\frac{n^k n x^k}{k!} \\
&= -n \sum_{k=0}^{\infty} f\Big(\frac{k}{n}\Big) s_{k,n}(x) + n \sum_{k=0}^{\infty} f\Big(\frac{k+1}{n}\Big)\frac{(nx)^k}{k!}
\end{aligned}$$

From the last equation, we have

$$
\begin{aligned}
(S_n f)'(x) &= n \sum_{k=0}^{\infty} \Big[f\Big(\frac{k+1}{n}\Big) - f\Big(\frac{k}{n}\Big) \Big] s_{k,n}(x) \\
&= \sum_{k=0}^{\infty} \Big[n \int_{\frac{k}{n}}^{\frac{k+1}{n}} f'(u) du \Big] s_{k,n}(x) = (S_n^* f')(x)
\end{aligned}
$$

provided $f \in AC[0,\infty)$, i.e, $f' \in L^1(0,\infty)$ with $f(x) = \int_0^x f'(u)du + f(0)$.

Proposition 5.1. *If $f \in TV[0,\infty)$, then*

$$
V_{[0,\infty)}[S_n f] \leq V_{[0,\infty)}[f]
$$

and

$$
\|S_n f\|_{BV[0,\infty)} \leq \|f\|_{BV[0,\infty)} := V_{[0,\infty)}[f] + |f(0)|.
$$

Thus the operators S_n are variation detracting and also bounded with respect to the $BV-norm$.

Proof.

$$
\begin{aligned}
V_{[0,\infty)}[S_n f] &= \int_0^{\infty} \Big| (S_n f)'(x) \Big| dx \\
&= \int_0^{\infty} \Big| n \sum_{k=0}^{\infty} \Big[f\Big(\frac{k+1}{n}\Big) - f\Big(\frac{k}{n}\Big) \Big] s_{k,n}(x) \Big| dx \\
&= \int_0^{\infty} \Big| n lim_{m\to\infty} \sum_{k=0}^{m} \Big[f\Big(\frac{k+1}{n}\Big) - f\Big(\frac{k}{n}\Big) \Big] s_{k,n}(x) \Big| dx \\
&\leq n lim_{m\to\infty} \sum_{k=0}^{m} \Big| f\Big(\frac{k+1}{n}\Big) - f\Big(\frac{k}{n}\Big) \Big| \int_0^{\infty} s_{k,n}(x) dx \\
&= \sum_{k=0}^{\infty} \Big| f\Big(\frac{k+1}{n}\Big) - f\Big(\frac{k}{n}\Big) \Big| n \int_0^{\infty} s_{k,n}(x) dx \\
&\leq V_{[0,\infty)}[f],
\end{aligned}
$$

noting that $\int_0^{\infty} s_{k,n}(x)dx = \frac{1}{n}$ for $k, n \in \mathbb{N}$, and $S_n f(0) = f(0)$.

□

Now, note that

$$\sum_{k=0}^{\infty}(k-nx)^r s_{k,n}(x) = \begin{cases} 1, & r=0 \\ 0, & r=1 \\ nx, & r=2 \\ nx, & r=3 \\ nx+3nx^2, & r=4. \end{cases}$$

5.2 Convergence in Variation Seminorm

As the study of [12],the Voronovskaja-type result in the Szász-Mirakjan situation,it states that if f is bounded in $\mathbb{R}_+$ and $f'''(x)$ exists at $x \in \mathbb{R}_+$, then

$$lim_{n\to\infty} n\Big[(S_n f)'(x) - f'(x)\Big] = f''(x) + \frac{x}{2}f'''(x).$$

Proof. By Taylor's formula, we have

$$f\Big(\frac{k}{n}\Big) = f(x) + \Big(\frac{k}{n}-x\Big)f'(x) + \Big(\frac{k}{n}-x\Big)^2\frac{f''(x)}{2} + \Big(\frac{k}{n}-x\Big)^3\Big[\frac{1}{6}f'''(x) + h\Big(\frac{k}{n}-x\Big)\Big]$$

where $h(y) := h_x(y)$ is bounded for all y, $|h(y)| \le M$ and converges to zero with y. So

$$\begin{aligned}
(S_n f)'(x) &= n\sum_{k=0}^{\infty}\Big[f\Big(\frac{k+1}{n}\Big) - f\Big(\frac{k}{n}\Big)\Big]s_{k,n}(x) \\
&= n\sum_{k=0}^{\infty}\Big[f(x) + \Big(\frac{k+1}{n}-x\Big)f'(x) + \Big(\frac{k+1}{n}-x\Big)^2\frac{f''(x)}{2} \\
&+ \Big(\frac{k+1}{n}-x\Big)^3\Big[\frac{1}{6}f'''(x) + h\Big(\frac{k+1}{n}-x\Big)\Big] \\
&- f(x) - \Big(\frac{k}{n}-x\Big)f'(x) - \Big(\frac{k}{n}-x\Big)^2\frac{f''(x)}{2} \\
&- \Big(\frac{k}{n}-x\Big)^3\Big[\frac{1}{6}f'''(x) + h\Big(\frac{k}{n}-x\Big)\Big]s_{k,n}(x) \\
&= n\sum_{k=0}^{\infty}\frac{1}{n}f'(x)s_{k,n}(x) + n\sum_{k=0}^{\infty}\Big[\frac{(k-nx)}{n^2}f''(x) + \frac{1}{2n^2}f''(x)\Big]s_{k,n}(x) \\
&+ n\sum_{k=0}^{\infty}\Big[\frac{3(k-nx)^2}{n^3} + \frac{3(k-nx)}{n^3} + \frac{1}{n^3}\Big]\frac{f'''(x)}{6}s_{k,n}(x) \\
&+ n\sum_{k=0}^{\infty}\Big(\frac{k+1}{n}-x\Big)^3 h\Big(\frac{k+1}{n}-x\Big)s_{k,n}(x) - n\sum_{k=0}^{\infty}\Big(\frac{k}{n}-x\Big)^3 h\Big(\frac{k}{n}-x\Big)s_{k,n}(x)
\end{aligned}$$

From the previous equality, we find

$$\begin{aligned}(S_n f)'(x) &= f'(x) + n\frac{1}{2n^2}f''(x) + n\sum_{k=0}^{\infty}\frac{3(k-nx)^2}{n^3}\frac{f'''(x)}{6}s_{k,n}(x)\\ &+ n\sum_{k=0}^{\infty}\frac{1}{n^3}\frac{f'''(x)}{6}s_{k,n}(x) + R_1(x) - R_2(x).\end{aligned}$$

That is;

$$(S_n f)'(x) = f'(x) + \frac{1}{2n}f''(x) + \frac{x}{2n}f'''(x) + \frac{1}{6n^2}f'''(x) + R_1(x) - R_2(x)$$

where the remainders are given by;

$$R_1(x) = n\sum_{k=0}^{\infty}\Big(\frac{k-nx}{n}+\frac{1}{n}\Big)^3 h\Big(\frac{k+1}{n}-x\Big)s_{k,n}(x)$$

$$R_2(x) = n\sum_{k=0}^{\infty}\Big(\frac{k-nx}{n}\Big)^3 h\Big(\frac{k}{n}-x\Big)s_{k,n}(x)$$

Let us study only the remainders R_1 *and* R_2.

$$\begin{aligned}R_1(x) &= n\sum_{k=0}^{\infty}\Big(\frac{k-nx}{n}+\frac{1}{n}\Big)^3 h\Big(\frac{k+1}{n}-x\Big)s_{k,n}(x)\\ &= n\Big\{\sum_{|\frac{k+1}{n}-x|<\delta} + \sum_{|\frac{k+1}{n}-x|\geq\delta}\Big\}\Big(\frac{k-nx}{n}+\frac{1}{n}\Big)^3 h\Big(\frac{k+1}{n}-x\Big)s_{k,n}(x)\\ &= \sum_1 + \sum_2, \quad say.\end{aligned}$$

For any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|h(y)| < \varepsilon$ for $0 \leq y \leq \delta$.

$$\begin{aligned}\sum_1 &= n\sum_{|\frac{k+1}{n}-x|<\delta}\Big(\frac{k-nx}{n}+\frac{1}{n}\Big)^3 h\Big(\frac{k+1}{n}-x\Big)s_{k,n}(x)\\ &\leq n\varepsilon\sum_{|\frac{k+1}{n}-x|<\delta}\frac{(k-nx)^3}{n^3}s_{k,n}(x) + n\varepsilon\sum_{|\frac{k+1}{n}-x|<\delta}\frac{3(k-nx)^2}{n^3}s_{k,n}(x)\\ &+ n\varepsilon\sum_{|\frac{k+1}{n}-x|<\delta}\frac{3(k-nx)}{n^3}s_{k,n}(x) + n\varepsilon\sum_{|\frac{k+1}{n}-x|<\delta}\frac{1}{n^3}s_{k,n}(x)\\ &= n\varepsilon\frac{nx}{n^3} + n\varepsilon\frac{3nx}{n^3} + n\varepsilon\frac{1}{n^3}\\ &= \varepsilon\frac{(4nx+1)}{n^2} \leq \tilde{\varepsilon}.\end{aligned}$$

Secondly, we calculate $\sum_2$.

$$\begin{aligned}
\sum_2 &= \sum_{|\frac{k+1}{n}-x|\geq\delta} \Big(\frac{k-nx}{n}+\frac{1}{n}\Big)^3 h\Big(\frac{k+1}{n}-x\Big)s_{k,n}(x)\\
&\leq n\sum_{k=0}^{\infty}\frac{(k-nx+1)^5}{n^5\delta^2}h\Big(\frac{k+1}{n}-x\Big)s_{k,n}(x)\\
&\leq n\sum_{k=0}^{\infty}\frac{(k-nx)^5}{n^5\delta^2}h\Big(\frac{k+1}{n}-x\Big)s_{k,n}(x)+n\sum_{k=0}^{\infty}\frac{5(k-nx)^4}{n^5\delta^2}h\Big(\frac{k+1}{n}-x\Big)s_{k,n}(x)\\
&+ n\sum_{k=0}^{\infty}\frac{10(k-nx)^3}{n^5\delta^2}h\Big(\frac{k+1}{n}-x\Big)s_{k,n}(x)+n\sum_{k=0}^{\infty}\frac{10(k-nx)^2}{n^5\delta^2}h\Big(\frac{k+1}{n}-x\Big)s_{k,n}(x)\\
&+ n\sum_{k=0}^{\infty}\frac{5(k-nx)}{n^5\delta^2}h\Big(\frac{k+1}{n}-x\Big)s_{k,n}(x)+n\sum_{k=0}^{\infty}\frac{1}{n^5\delta^2}h\Big(\frac{k+1}{n}-x\Big)s_{k,n}(x)\\
&= nM\frac{1}{n^5\delta^2}\Big[(nx)(1+10nx)+10(nx+3nx^2)+10nx+5nx\Big]\\
&\leq \tilde{\varepsilon}.
\end{aligned}$$

Now we can calculate $R_2(x)$.

$$\begin{aligned}
R_2(x) &= n\sum_{k=0}^{\infty}\frac{(k-nx)^3}{n^3}h\Big(\frac{k}{n}-x\Big)s_{k,n}(x)\\
&= n\Big\{\sum_{|\frac{k}{n}-x|<\delta}+\sum_{|\frac{k}{n}-x|\geq\delta}\Big\}\Big(\frac{k-nx}{n}\Big)^3 h\Big(\frac{k}{n}-x\Big)s_{k,n}(x)\\
&= \sum_3+\sum_4, \quad say.
\end{aligned}$$

Here we calculate $\sum_3$ and $\sum_4$.

$$\sum_3 = n\sum_{|\frac{k}{n}-x|<\delta}\frac{(k-nx)^3}{n^3}h\Big(\frac{k}{n}-x\Big)s_{k,n}(x)\leq n\varepsilon\frac{nx}{n^3}=\varepsilon\frac{x}{n}\leq\tilde{\varepsilon}.$$

$$\begin{aligned}
\sum_4 &= \sum_{|\frac{k}{n}-x|\geq\delta}\frac{(k-nx)^3}{n^3}h\Big(\frac{k}{n}-x\Big)s_{k,n}(x)\\
&\leq n\sum_{k=0}^{\infty}\frac{(k-nx)^5}{n^5\delta^2}h\Big(\frac{k}{n}-x\Big)s_{k,n}(x)\\
&= nM\frac{1}{n^5\delta^2}(nx)(1+10nx)\leq\bar{\varepsilon}
\end{aligned}$$

This shows that

$$lim_{n\to\infty} n\Big[(S_n f)'(x) - f'(x)\Big] = \frac{1}{2} f''(x) + \frac{x}{2} f'''(x).$$

□

5.3 The Rate of Convergence in Variation Seminorm

If $f \in AC[0,\infty)$, then one has by a general theorem on singular integrals that

$$V_{[0,\infty)}[S_n f - f] = \int_0^\infty |(S_n^* f')(x) - f'(x)| dx \to 0 \quad (n \to \infty).$$

$V_{[0,\infty)}[S_n f - f] \to 0$, since $S_n f \in AC[0,\infty)$ yields that $f \in AC[O,\infty)$.

Proof. As shown in [16], we prove the theorem in a similar way for Szász-Mirakjan operators.

By Taylor formula with integral remainder term;

$$f\Big(\frac{k}{n}\Big) = f(x) + \Big(\frac{k}{n} - x\Big) f'(x) + \Big(\frac{k}{n} - x\Big)^2 \frac{f''(x)}{2} + \frac{1}{2}\int_0^{\frac{k}{n}-x} \Big(\frac{k}{n} - x - u\Big)^2 f'''(x+u) du \quad ...(1)$$

If we apply the operator S_n to both sides of the equality (1), one has,

$$\begin{aligned}
(S_n f)'(x) &= n \sum_{k=0}^{\infty} \Big[f\Big(\frac{k+1}{n}\Big) - f\Big(\frac{k}{n}\Big)\Big] s_{k,n}(x) \\
&= n \sum_{k=0}^{\infty} \Big[f(x) + \Big(\frac{k+1}{n} - x\Big) f'(x) + \Big(\frac{k+1}{n} - x\Big)^2 \frac{f''(x)}{2} \\
&+ \frac{1}{2} \int_0^{\frac{k+1}{n}-x} \Big(\frac{k+1}{n} - x - u\Big)^2 f'''(x+u) du \\
&- f(x) - \Big(\frac{k}{n} - x\Big) f'(x) - \Big(\frac{k}{n} - x\Big)^2 \frac{f''(x)}{2} \\
&- \frac{1}{2} \int_0^{\frac{k}{n}-x} \Big(\frac{k}{n} - x - u\Big)^2 f'''(x+u) du \Big]\Big] s_{k,n}(x) \\
&= n \sum_{k=0}^{\infty} \Big[\frac{1}{n} f'(x) + \frac{(k-nx)}{n^2} f''(x) + \frac{1}{2n^2} f''(x) \Big] s_{k,n}(x) + R_n f(x).
\end{aligned}$$

Firstly, we will calculate $R_n(f)$.

Here $R_n(f)$ is as below;

$$\begin{aligned} R_n f(x) &= n\sum_{k=0}^{\infty}\frac{1}{2}\int_0^{\frac{k+1}{n}-x}\Big(\frac{k+1}{n}-x-u\Big)^2 f'''(x+u)du s_{k,n}(x) \\ &- n\sum_{k=0}^{\infty}\frac{1}{2}\int_0^{\frac{k}{n}-x}\Big(\frac{k}{n}-x-u\Big)^2 f'''(x+u)du s_{k,n}(x). \end{aligned}$$

Here, we have

$$(S_n f)'(x) - f'(x) = \frac{1}{2n}f''(x) + (R_n f)(x).$$

Since

$$V_{[I]}[f] = \int_{[I]} |f'(t)|dt,$$

in order to prove the theorem in convergence in variation seminorm, we need the L_1- *norm* of the difference between $(S_n f)'$ and f'.

$$\begin{aligned} \|(S_n f)' - f'\|_{L_1[0,\infty)} &\leq \frac{1}{2n}\int_0^{\infty}|f''(x)|dx + \|(R_n f)\|_{L_1[0,\infty)} \\ &= \frac{1}{2n}\|f''(x)\|_{L_1[0,\infty)} + \|(R_n f)\|_{L_1[0,\infty)}. \end{aligned}$$

Note that

$$\Big|\frac{k}{n}-v\Big|^2 \leq \Big|\frac{k}{n}-x\Big|^2$$

holds for $\frac{k}{n} \leq v \leq x$ and/or $x \leq v \leq \frac{k}{n}$. Substituting $x+u=v$ in the definition of $(R_n f)(x)$ and observing that $k-nx$ and $\int_x^{\frac{k}{n}}|...|dv$ have the same sign, we get for the remainder term

$$\begin{aligned} |(R_n f)(x)| &\leq \frac{1}{2n}\sum_{k=0}^{\infty}(k+1-nx)^2\int_x^{\frac{k+1}{n}}|f'''(v)|dv s_{k,n}(x) \\ &+ \frac{1}{2n}\sum_{k=0}^{\infty}(k-nx)^2\int_x^{\frac{k}{n}}|f'''(v)|dv s_{k,n}(x) \end{aligned}$$

Now, we can show the norm of $(R_nf)(x)$ in $L_{1[0,\infty)}$.

$$\begin{aligned}
\|R_nf\|_{L_1[0,\infty)} &= \frac{1}{2n}\int_0^\infty \sum_{k=0}^{\infty}(k+1-nx)^2\int_x^{\frac{k+1}{n}}|f'''(v)|dv s_{k,n}(x)dx \\
&+ \frac{1}{2n}\int_0^\infty \sum_{k=0}^{\infty}(k-nx)^2\int_x^{\frac{k}{n}}|f'''(v)|dv s_{k,n}(x)dx
\end{aligned}$$

Here, we say

$$\begin{aligned}
I_1 &= \frac{1}{2n}\int_0^\infty \sum_{k=0}^{\infty}(k+1-nx)^2\int_x^{\frac{k+1}{n}}|f'''(v)|dv s_{k,n}(x)dx \\
I_2 &= \frac{1}{2n}\int_0^\infty \sum_{k=0}^{\infty}(k-nx)^2\int_x^{\frac{k}{n}}|f'''(v)|dv s_{k,n}(x)dx.
\end{aligned}$$

Now, we start with I_1.

$$\begin{aligned}
I_1 &= \frac{1}{2n}\int_0^\infty \sum_{k=0}^{\infty}(k+1-nx)^2\int_0^{\frac{k+1}{n}}|f'''(v)|dv s_{k,n}(x)dx \\
&- \frac{1}{2n}\int_0^\infty \sum_{k=0}^{\infty}(k+1-nx)^2\int_0^{x}|f'''(v)|dv s_{k,n}(x)dx \\
&\leq \frac{1}{2n}\sum_{k=0}^{\infty}\Big|\int_0^{\frac{k+1}{n}}|f'''(v)|\Big[\int_0^\infty (k+1-nx)^2 s_{k,n}(x)dx\Big]dv\Big| \\
&+ \Big|\frac{1}{2n}\int_0^\infty \sum_{k=0}^{\infty}(k+1-nx)^2\int_0^{x}|f'''(v)|dv s_{k,n}(x)dx\Big|.
\end{aligned}$$

Since

$$\sum_{k=0}^{\infty}(k+1-nx)^2 s_{k,n}(x) = 1+nx,$$

then the second integral of the last inequality can be estimated as follows;

$$\begin{aligned}
\frac{1}{2n}\int_0^\infty \sum_{k=0}^{\infty}(k+1-nx)^2\int_0^{x}|f'''(v)|dv s_{k,n}(x)dx &= \frac{1}{2n}\int_0^\infty (1+nx)\int_0^x|f'''(v)|dvdx \\
(since\ \ 1+nx \leq n+nx) &\leq \frac{1}{2n}\int_0^\infty (n+nx)\int_0^x|f'''(v)|dvdx
\end{aligned}$$

From the last inequality, we have

$$\begin{aligned}
\frac{1}{2n}\int_0^\infty \sum_{k=0}^\infty (k+1-nx)^2 \int_0^x |f'''(v)|dv s_{k,n}(x)dx &= \frac{1}{2}\int_0^\infty (1+x)\int_0^x |f'''(v)|dvdx \\
&\leq \frac{1}{2}\int_0^\infty |f'''(v)|dv \\
&= \frac{1}{2}\|f'''\|_{L_1[0,\infty)}.
\end{aligned}$$

In addition,

$$\Big|\int_0^\infty (k+1-nx)^2 s_{k,n}(x)dx\Big| \leq \Big|\frac{\Gamma[2+k]}{nk!}\Big|$$

since $n > 0$ and $k \geq -1$, where $\Gamma(m) = (m-1)!$ is the gamma function. Here,

$$\Big|\int_0^\infty (k+1-nx)^2 s_{k,n}(x)dx\Big| \leq \Big|\frac{(k+1)!}{nk!}\Big| = \Big|\frac{k+1}{n}\Big| = \frac{k+1}{n}$$

since $k, n > 0$.

Thus, we have

$$\begin{aligned}
I_1 &\leq \Big(\frac{k+1}{n}\Big)\Big(\frac{1}{2n}\Big)\int_0^\infty |f'''(v)|dv + \frac{1}{2}\int_0^\infty |f'''(v)|dv \\
&= \Big(\frac{k+1}{2n^2} + \frac{1}{2}\Big)\int_0^\infty |f'''(v)|dv.
\end{aligned}$$

$$\begin{aligned}
I_2 &= \frac{1}{2n}\int_0^\infty \sum_{k=0}^\infty (k-nx)^2 \int_0^{\frac{k}{n}} |f'''(v)|dv s_{k,n}(x)dx \\
&- \frac{1}{2n}\int_0^\infty \sum_{k=0}^\infty (k-nx)^2 \int_0^x |f'''(v)|dv s_{k,n}(x)dx
\end{aligned}$$

If we take the absolute value,

$$\begin{aligned}
I_2 &\leq \frac{1}{2n}\sum_{k=0}^\infty \Big|\int_0^{\frac{k}{n}} |f'''(v)|\Big[\int_0^\infty (k-nx)^2 s_{k,n}(x)dx\Big]dv\Big| \\
&+ \Big|\frac{1}{2n}\int_0^\infty \sum_{k=0}^\infty (k-nx)^2 \int_0^x |f'''(v)|dv s_{k,n}(x)dx\Big|.
\end{aligned}$$

Since

$$\sum_{k=0}^{\infty}(k-nx)^2 s_{k,n}(x) = nx,$$

then the second integral of the last inequality can be estimated as follows;

$$\begin{aligned}
\frac{1}{2n}\int_0^{\infty}\sum_{k=0}^{\infty}(k-nx)^2\int_0^x |f'''(v)|dv s_{k,n}(x)dx &= \frac{1}{2n}\int_0^{\infty} nx\int_0^x |f'''(v)|dvdx \\
&= \frac{1}{2}\int_0^{\infty} x\int_0^x |f'''(v)|dvdx \\
&\leq \frac{1}{2}\int_0^{\infty} |f'''(v)|dv \\
&= \frac{1}{2}\|f'''\|_{L^1[0,\infty)}.
\end{aligned}$$

In addition,

$$\Big|\int_0^{\infty}(k-nx)^2 s_{k,n}(x)dx\Big| \leq \Big|\frac{[(2+k)\Gamma[1+k]]}{nk!}\Big|$$

since $n > 0$ and $k \geq -1$.

Here,

$$\Big|\int_0^{\infty}(k-nx)^2 s_{k,n}(x)dx\Big| \leq \Big|\frac{(2+k)k!}{nk!}\Big| = \Big|\frac{2+k}{n}\Big| = \frac{2+k}{n}.$$

since $k, n > 0$.

Thus, we have

$$\begin{aligned}
I_2 &\leq \Big(\frac{k+2}{n}\Big)\Big(\frac{1}{2n}\Big)\int_0^{\infty}|f'''(v)|dv + \frac{1}{2}\int_0^{\infty}|f'''(v)|dv \\
&= \Big(\frac{k+2}{2n^2}+\frac{1}{2}\Big)\int_0^{\infty}|f'''(v)|dv
\end{aligned}$$

Now,

$$\begin{aligned}
\|R_n f\|_{L_1[0,\infty)} &\leq \Big(\frac{k+1}{2n^2}+\frac{1}{2}\Big)\int_0^{\infty}|f'''(v)|dv + \Big(\frac{k+2}{2n^2}+\frac{1}{2}\Big)\int_0^{\infty}|f'''(v)|dv \\
&= \Big(\frac{2k+3}{2n^2}+1\Big)\int_0^{\infty}|f'''(v)|dv \\
&= \Big(\frac{2k+3}{2n^2}+1\Big)\|f'''\|_{L_1[0,\infty)}.
\end{aligned}$$

Then, we have;

$$\|(S_n f)' - f'\|_{L_1[0,\infty)} \leq \frac{1}{2n}\|f''(x)\|_{L_1[0,\infty)} + \frac{1}{2n}\|f'''(x)\|_{L_1[0,\infty)}.$$

In conclusion,

$$V_{[0,\infty)}[S_n f - f] \to 0 \quad for \quad n \to \infty.$$

$\square$

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